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# MULTIVARIATE MULTIPLICATIVE FUNCTIONS OF UNIFORM RANDOM VECTORS IN LARGE INTEGER DOMAINS

ZAKHAR KABLUCHKO, OLEKSANDR MARYNYCH, AND KILIAN RASCHEL

**ABSTRACT.** For a wide class of sequences of integer domains  $\mathcal{D}_n \subset \mathbb{N}^d$ ,  $n \in \mathbb{N}$ , we prove distributional limit theorems for  $F(X_1^{(n)}, \dots, X_d^{(n)})$ , where  $F$  is a multivariate multiplicative function and  $(X_1^{(n)}, \dots, X_d^{(n)})$  is a random vector with uniform distribution on  $\mathcal{D}_n$ . As a corollary, we obtain limit theorems for the greatest common divisor and least common multiple of the random set  $\{X_1^{(n)}, \dots, X_d^{(n)}\}$ . This generalizes previously known limit results for  $\mathcal{D}_n$  being either a discrete cube or a discrete hyperbolic region.

## 1. INTRODUCTION

Let  $F : \mathbb{N}^d \rightarrow \mathbb{C}$  be an arithmetic function of  $d \geq 1$  integer arguments, with  $\mathbb{N} = \{1, 2, 3, \dots\}$ . A standard problem in analytic number theory is the estimation of the multivariate sum

$$\sum_{x_1=1}^{n_1} \cdots \sum_{x_d=1}^{n_d} F(x_1, \dots, x_d)$$

for large values of  $(n_1, \dots, n_d) \in \mathbb{N}^d$ . A particular instance of this problem consists in establishing existence of the so-called mean value of  $F$ , which is defined via

$$(1) \quad M(f) := \lim_{n_1, \dots, n_d \rightarrow \infty} \frac{1}{n_1 \cdots n_d} \sum_{x_1=1}^{n_1} \cdots \sum_{x_d=1}^{n_d} F(x_1, \dots, x_d).$$

In the probabilistic language, (1) may be recast as follows. Let  $(U_1^{(n_1)}, \dots, U_d^{(n_d)})$  be a random vector defined on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and which has the uniform distribution on the finite rectangular set

$$(2) \quad \mathcal{R}_{n_1, \dots, n_d} := \left( \prod_{i=1}^d [1, n_i] \right) \cap \mathbb{N}^d.$$

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Then, with  $\mathbb{E}$  denoting the expectation with respect to  $\mathbb{P}$ ,

$$(3) \quad M(F) = \lim_{n_1, \dots, n_d \rightarrow \infty} \mathbb{E}F(U_1^{(n_1)}, \dots, U_d^{(n_d)}).$$

A general result on existence of  $M(F)$  is due to Ushiroya [21].

A multivariate arithmetic function  $F : \mathbb{N}^d \rightarrow \mathbb{C}$  is called multiplicative, see [20, 21, 22], if

$$F(1, \dots, 1) = 1 \quad \text{and} \quad F(m_1 n_1, \dots, m_d n_d) = F(m_1, \dots, m_d) F(n_1, \dots, n_d),$$

for all  $(m_1, \dots, m_d) \in \mathbb{N}^d$  and  $(n_1, \dots, n_d) \in \mathbb{N}^d$  such that

$$\text{GCD}(m_1 \cdots m_d, n_1 \cdots n_d) = 1.$$

A specialization of Ushiroya's results from [21] to a multiplicative function  $F$  implies that under a mild summability assumption on  $F$ , the mean value  $M(F)$  exists and is equal to

$$(4) \quad M(F) := \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)^d \sum_{i_1=0}^{\infty} \cdots \sum_{i_d=0}^{\infty} \frac{F(p^{i_1}, \dots, p^{i_d})}{p^{i_1 + \dots + i_d}},$$

where  $\mathcal{P}$  stands for the set of prime numbers.

In the last years, there has been a lot of activity around various generalizations and extensions of the aforementioned results. In a probabilistic direction, one may ask about the asymptotic behavior of *distributions* of the random variable  $F(U_1^{(n_1)}, \dots, U_d^{(n_d)})$ , as  $n_1, \dots, n_d \rightarrow \infty$  in (2). This question has been addressed in [4] for a particular choice of  $F$ , namely, for  $F(x_1, \dots, x_d) = G(\text{LCM}(x_1, \dots, x_d))$ , with  $G$  being a univariate multiplicative arithmetic function. The univariate case  $d = 1$  is the classical Erdős-Wintner theorem, see [11], which provides necessary and sufficient conditions for the distributional convergence of  $F(U_1^{(n)})$  as  $n \rightarrow \infty$ . In another, more analytic direction, the rectangular domains  $\mathcal{R}_{n_1, \dots, n_d}$  in (2) are replaced by more sophisticated domains of summation  $\mathcal{D}_n \subset \mathbb{N}^d$ , which grow to  $\mathbb{N}^d$  as  $n \rightarrow \infty$ . In particular, in the recent work [17], the case of *spherical* summation over the regions

$$\mathcal{S}_n := \{(x_1, \dots, x_d) \in \mathbb{N}^d : x_1^2 + \dots + x_d^2 \leq n\},$$

has been analyzed, whereas the papers [14, 15, 16] were devoted to the study of summation over *hyperbolic* regions

$$\mathcal{H}_n := \{(x_1, \dots, x_d) \in \mathbb{N}^d : x_1 \cdots x_d \leq n\}$$

and their generalizations. A surprising phenomenon revealed in the cited works is that the mean value  $M(F)$  given by (4) is universal for rectangular, spherical and hyperbolic domains.

More specifically, let  $\mathcal{D}_n$  be either  $\mathcal{R}_{n,\dots,n}$ ,  $\mathcal{S}_n$  or  $\mathcal{H}_n$ . For every  $n \in \mathbb{N}$ , let  $(X_1^{(n)}, \dots, X_d^{(n)})$  be a random vector defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  and having the uniform distribution on  $\mathcal{D}_n$ , that is,

$$\mathbb{P}\{(X_1^{(n)}, \dots, X_d^{(n)}) = (i_1, \dots, i_d)\} = \frac{1}{\#\mathcal{D}_n}, \quad (i_1, \dots, i_d) \in \mathcal{D}_n,$$

where  $\#\mathcal{D}_n$  denotes the cardinality of  $\mathcal{D}_n$ . Then, under the same summability assumption on  $F$  as in Ushiroya's result, we have

$$\begin{aligned} (5) \quad \lim_{n \rightarrow \infty} \mathbb{E}F(X_1^{(n)}, \dots, X_d^{(n)}) &= \lim_{n \rightarrow \infty} \frac{1}{\#\mathcal{D}_n} \sum_{(x_1, \dots, x_d) \in \mathcal{D}_n} F(x_1, \dots, x_d) \\ &= M(F) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)^d \sum_{i_1=0}^{\infty} \dots \sum_{i_d=0}^{\infty} \frac{F(p^{i_1}, \dots, p^{i_d})}{p^{i_1 + \dots + i_d}}. \end{aligned}$$

The purpose of the present paper is two-fold. First, we shall provide a probabilistic explanation which lies in the core of (5), by providing sufficient conditions on  $F$  for the distributional convergence of  $F(X_1^{(n)}, \dots, X_d^{(n)})$  as  $n \rightarrow \infty$ . Second, we shall do this not only for the three types of regions mentioned before, but for a quite general class of integer domains  $\mathcal{D}_n$  satisfying mild assumptions.

The paper is organized as follows. In Section 2, we formulate our standing assumptions on  $\mathcal{D}_n$  and present our main results, which are distributional limit theorems for  $F(X_1^{(n)}, \dots, X_d^{(n)})$ . The proofs are collected in Section 3. In Section 4, we provide various examples of domains  $\mathcal{D}_n$  satisfying our standing assumptions. In particular, the aforementioned domains  $\mathcal{R}_{n_1, \dots, n_d}$ ,  $\mathcal{S}_n$  and  $\mathcal{H}_n$  are covered. In Section 5 we discuss how to construct new domains satisfying our conditions, using standard set-theoretic operations. Some auxiliary results are collected in Appendix A.

Throughout the paper we use the following standard notation:  $\xrightarrow{w}$  denotes the convergence in distribution (weak convergence of probability measures);  $\text{Int}(A)$ ,  $\text{cl}(A)$  and  $\partial A$  are the topological interior, closure and boundary of a set  $A \subset \mathbb{R}^d$ , respectively;  $a(n) \sim b(n)$ ,  $n \rightarrow \infty$ , means that  $\lim_{n \rightarrow \infty} (a(n)/b(n)) = 1$ .

## 2. MAIN RESULTS

**2.1. Preliminaries.** Throughout the paper, we assume that  $F$  is a multivariate multiplicative arithmetic function of  $d \geq 2$  variables. Every multivariate multiplicative function is completely determined by its values on the powers of primes. More precisely, let  $\lambda_p(n)$  denote the power

of prime  $p \in \mathcal{P}$  in the prime decomposition of  $n \in \mathbb{N}$ . Then

$$x_i = \prod_{p \in \mathcal{P}} p^{\lambda_p(x_i)}, \quad i = 1, \dots, d,$$

implies

$$F(x_1, \dots, x_d) = \prod_{p \in \mathcal{P}} F(p^{\lambda_p(x_1)}, \dots, p^{\lambda_p(x_d)}).$$

The crucial observation for everything to follow is the representation for  $M(F)$  in (4) via independent geometric random variables. Let  $(\mathcal{G}_1(p), \dots, \mathcal{G}_d(p))_{p \in \mathcal{P}}$  be an array of mutually independent random variables with geometric distributions

$$\mathbb{P}\{\mathcal{G}_k(p) \geq j\} = \frac{1}{p^j}, \quad j \in \mathbb{N}_0, \quad p \in \mathcal{P}, \quad k = 1, \dots, d,$$

where  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . Then

$$M(F) = \mathbb{E} \left( \prod_{p \in \mathcal{P}} F(p^{\mathcal{G}_1(p)}, \dots, p^{\mathcal{G}_d(p)}) \right).$$

The main result of our paper gives sufficient conditions on  $F$  which ensure the convergence in distribution

$$(6) \quad F(X_1^{(n)}, \dots, X_d^{(n)}) = \prod_{p \in \mathcal{P}} F(p^{\lambda_p(X_1^{(n)})}, \dots, p^{\lambda_p(X_d^{(n)})}) \xrightarrow[n \rightarrow \infty]{w} \prod_{p \in \mathcal{P}} F(p^{\mathcal{G}_1(p)}, \dots, p^{\mathcal{G}_d(p)}) =: F_\infty,$$

for a general class of integer domains  $\mathcal{D}_n$ , which we are now going to introduce.

Let  $(\mathcal{D}_n)_{n \in \mathbb{N}}$  be a sequence of finite, non-empty subsets of  $\mathbb{N}^d$ . Assume that for every fixed  $c \in \mathbb{Z}^d$ , where  $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ , the following condition is fulfilled:

$$(7) \quad \lim_{n \rightarrow \infty} \frac{\#((\mathcal{D}_n + c) \cap \mathcal{D}_n)}{\#\mathcal{D}_n} = 1.$$

Note that (7) is equivalent to saying that for all  $c \in \mathbb{Z}^d$ ,

$$\lim_{n \rightarrow \infty} \frac{\delta_n(c)}{\#\mathcal{D}_n} = 0,$$

where, denoting  $\Delta$  the symmetric difference of two sets,

$$(8) \quad \delta_n(c) := \#(\mathcal{D}_n \Delta (\mathcal{D}_n + c)).$$

Condition (7) is known in the literature as the regular growth condition; see Chapter 3 in [5]. Several equivalent versions of (7) can be found in Appendix A below.

**2.2. Convergence of prime powers to geometric laws.** Our first main result states that, solely under assumption (7), the array of random vectors  $(\lambda_p(X_1^{(n)}), \dots, \lambda_p(X_d^{(n)}))_{p \in \mathcal{P}}$  converges in distribution to an array of independent geometric variables, thereby providing the first evidence supporting (6).

**Theorem 2.1.** *Assume that (7) holds. Then*

$$\left( \lambda_p(X_1^{(n)}), \dots, \lambda_p(X_d^{(n)}) \right)_{p \in \mathcal{P}} \xrightarrow[n \rightarrow \infty]{w} (\mathcal{G}_1(p), \dots, \mathcal{G}_d(p))_{p \in \mathcal{P}},$$

in the space  $(\mathbb{R}^d)^\infty$  endowed with the product topology.

*Remark 2.2.* In the rectangular case  $\mathcal{D}_n = \mathcal{R}_{n_1, \dots, n_d}$ , Theorem 2.1 is well known in probabilistic number theory and has a long history, see, for instance, Eqs. (2.5)–(2.7) in [19] and [2]. Note that in this case, the components  $X_1^{(n)}, \dots, X_d^{(n)}$  are independent and  $X_j^{(n)}$  has the uniform distribution on  $\{1, \dots, n_j\}$ , for every  $j = 1, \dots, d$ .

**2.3. Limit theorems for  $F$ .** We start with finding conditions ensuring a.s. finiteness of  $F_\infty$  in (6). Recall that we assume  $d \geq 2$ . According to Eq. (20) in [4] (or just by an appeal to the Borel-Cantelli lemma), we have

$$\sum_{p \in \mathcal{P}} \mathbb{1}_{\{\sum_{k=1}^d \mathcal{G}_k(p) \geq 2\}} < \infty \quad \text{a.s.}$$

Furthermore, because  $F$  is multiplicative,  $F(1, 1, \dots, 1) = 1$ . Thus, a.s. finiteness of  $F_\infty$  is equivalent to the a.s. convergence of the product

$$\widehat{F}_\infty := \prod_{p \in \mathcal{P} : \sum_{k=1}^d \mathcal{G}_k(p) = 1} F(p^{\mathcal{G}_1(p)}, \dots, p^{\mathcal{G}_d(p)}).$$

For  $i = 1, \dots, d$ , put

$$F_i(x) := \log F(1, \dots, 1, x, 1, \dots, 1),$$

where  $x \in \mathbb{N}$  on the right-hand side is on the  $i$ -th position and  $\log$  is the principal branch of the logarithm (a branch which satisfies  $\log(1) = 0$  and has a branch cut along  $(-\infty, 0]$ ). We assume that for all  $i = 1, \dots, d$ , there are only finitely many  $p \in \mathcal{P}$  such that  $F(1, \dots, 1, p, 1, \dots, 1)$  falls inside the branch cut. Otherwise, we stipulate that the series diverges. Thus, the a.s. convergence of  $\widehat{F}_\infty$ , hence of  $F_\infty$ , is equivalent to the a.s. convergence of the series

$$(9) \quad \sum_{p \in \mathcal{P}} \left( \sum_{i=1}^d F_i(p) \mathbb{1}_{\{\mathcal{G}_i(p)=1, \mathcal{G}_j(p)=0 \text{ for } j \neq i\}} \right),$$

comprised of independent random variables. An application of Kolmogorov's three series theorem immediately yields the following:

**Proposition 2.3.** *The infinite product  $F_\infty$  converges a.s. if and only if the following series converge for every  $A > 0$ :*

$$(10) \quad \sum_{p \in \mathcal{P}} \frac{1}{p} \sum_{i=1}^d \mathbb{1}_{\{|F_i(p)| > A\}}, \quad \sum_{p \in \mathcal{P}} \frac{1}{p} \sum_{i=1}^d F_i(p) \mathbb{1}_{\{|F_i(p)| \leq A\}}, \quad \sum_{p \in \mathcal{P}} \frac{1}{p} \sum_{i=1}^d |F_i(p)|^2 \mathbb{1}_{\{|F_i(p)| \leq A\}}.$$

It is clear that the convergence of the three series (10) is a necessary condition for (6). Proving (6) under (10) alone seems to be a very difficult task, even for simple regions  $\mathcal{D}_n$  as  $\mathcal{R}_{n, \dots, n}$ .

In this paper, we restrict our attention to a subclass of multivariate multiplicative functions satisfying (10). Namely, we shall assume that, for all  $i = 1, \dots, d$ ,

$$(11) \quad \sum_{p \in \mathcal{P}} \frac{1}{p} \mathbb{1}_{\{|F_i(p)| > A\}} < \infty \quad \text{and} \quad \sum_{p \in \mathcal{P}} \frac{1}{p} |F_i(p)| \mathbb{1}_{\{|F_i(p)| \leq A\}} < \infty.$$

It is obvious that (11) implies (10). The difference between conditions (10) and (11) is that (11) is necessary and sufficient for the a.s. *absolute* convergence of the series (9), whereas under (10) the a.s. convergence of the series (9) is, in general, only conditional.

In order to prove (6) under (11), we shall impose a mild additional assumption on  $\mathcal{D}_n$ . For  $i = 1, \dots, d$  and  $a \in \mathbb{N}$ , put

$$\mathcal{Z}_i(a) := \{(x_1, \dots, x_d) \in \mathbb{Z}^d : x_i \text{ is divisible by } a\}.$$

As we shall see below in Lemma 3.1, solely under assumption (7), one has

$$(12) \quad \lim_{n \rightarrow \infty} \frac{\#(\mathcal{D}_n \cap \mathcal{Z}_i(a) \cap \mathcal{Z}_j(b))}{\#\mathcal{D}_n} = \frac{1}{ab},$$

for every fixed  $a, b \in \mathbb{N}$  and  $i, j = 1, \dots, d$ ,  $i \neq j$ . However, we shall need a further assumption that refines the above limit relation, providing a kind of uniformity in (12). Namely, we assume that there exists  $K > 0$  such that for all  $i, j = 1, \dots, d$ ,  $i \neq j$ ,  $a, b \in \mathbb{N}$  and  $n \in \mathbb{N}$ ,

$$(13) \quad \frac{\#(\mathcal{D}_n \cap \mathcal{Z}_i(a) \cap \mathcal{Z}_j(b))}{\#\mathcal{D}_n} \leq \frac{K}{ab}.$$

Recall that  $(X_1^{(n)}, \dots, X_d^{(n)})$  is a random vector picked uniformly at random from  $\mathcal{D}_n$ . Below is our main result.

**Theorem 2.4.** *Assume that  $F : \mathbb{N}^d \rightarrow \mathbb{C}$  is a multiplicative arithmetic function such that conditions (11) hold. Let  $\mathcal{D}_n$ ,  $n \in \mathbb{N}$ , be a sequence of subsets of  $\mathbb{N}^d$  such that (7) and (13) hold. Then*

$$F(X_1^{(n)}, \dots, X_d^{(n)}) \xrightarrow[n \rightarrow \infty]{w} \prod_{p \in \mathcal{P}} F(p^{\mathcal{G}_1(p)}, \dots, p^{\mathcal{G}_d(p)}).$$

Examples of integer domains satisfying (7) and (13) will be presented in Section 4.

The following functions  $F$

$$\mathbb{N}^d \ni (x_1, \dots, x_d) \mapsto \text{GCD}(x_1, \dots, x_d) \quad \text{and} \quad \mathbb{N}^d \ni (x_1, \dots, x_d) \mapsto \frac{\text{LCM}(x_1, \dots, x_d)}{x_1 \cdots x_d}$$

are multiplicative and satisfy  $F_i(x) \equiv 0$  for every  $i = 1, \dots, d$ . Thus, Theorem 2.4 is applicable, leading to the following corollaries.

**Corollary 2.5.** *Assume that (7) and (13) hold. Then*

$$\text{GCD}(X_1^{(n)}, \dots, X_d^{(n)}) \xrightarrow[n \rightarrow \infty]{w} \prod_{p \in \mathcal{P}} p^{\min_{k=1, \dots, d} \mathcal{G}_k(p)}.$$

The limiting random variable has the following distribution

$$(14) \quad \mathbb{P} \left\{ \prod_{p \in \mathcal{P}} p^{\min_{k=1, \dots, d} \mathcal{G}_k(p)} = j \right\} = \frac{1}{\zeta(d)} \frac{1}{j^d}, \quad j \in \mathbb{N},$$

where  $\zeta$  is the Riemann zeta function.

**Corollary 2.6.** *Assume that (7) and (13) hold. Then*

$$\frac{\text{LCM}(X_1^{(n)}, \dots, X_d^{(n)})}{X_1^{(n)} \cdots X_d^{(n)}} \xrightarrow[n \rightarrow \infty]{w} \prod_{p \in \mathcal{P}} p^{\max_{k=1, \dots, d} \mathcal{G}_k(p) - \sum_{k=1}^d \mathcal{G}_k(p)}.$$

*Remark 2.7* (Bibliographic comments). Below is a comparison of our results with the existing ones.

CASE  $\mathcal{D}_n = \mathcal{R}_{n, \dots, n}$ . In this case Corollaries 2.5 and 2.6 are known, with Corollary 2.5 having a long history. The fact that two independent random integers picked uniformly at random from  $\{1, \dots, n\}$  are asymptotically co-prime with probability  $1/\zeta(2) = 6/\pi^2$ , that is

$$\lim_{n \rightarrow \infty} \mathbb{P}\{\text{GCD}(X_1^{(n)}, X_2^{(n)}) = 1\} = \frac{6}{\pi^2}$$

goes back to Dirichlet [10], and generalizations of this relation to  $d > 2$  integers are due to Cesàro [6, 7]. To the best of our knowledge, Corollary 2.5 is due to Christopher [8], see also [9]. Formula (14) follows from the following chain of equalities. For  $s < d - 1$ , by Euler's product formula

$$\mathbb{E} \left( \prod_{p \in \mathcal{P}} p^{\min_{k=1, \dots, d} \mathcal{G}_k(p)} \right)^s = \prod_{p \in \mathcal{P}} \mathbb{E} p^{s \min_{k=1, \dots, d} \mathcal{G}_k(p)} = \prod_{p \in \mathcal{P}} \left( 1 - \frac{1}{p^d} \right) \frac{1}{1 - p^{s-d}} = \frac{\zeta(d-s)}{\zeta(d)} = \frac{1}{\zeta(d)} \sum_{j=1}^d \frac{j^s}{j^d}.$$

Corollary 2.6 can be extracted from Theorem 2.1 in [18] and is given explicitly in Remark 2.4 in [4]. Further pointers to literature related to Corollaries 2.5 and 2.6 in case  $\mathcal{D}_n = \mathcal{R}_{n, \dots, n}$

can be found in the introduction [4] and in the survey [13]. In [4] a version of Theorem 2.4 was proved assuming that  $F(x_1, \dots, x_d) = G(\text{LCM}(x_1, \dots, x_d))$  for some univariate multiplicative function  $G : \mathbb{N} \rightarrow \mathbb{C}$ . Asymptotics of moments accompanying the aforementioned distributional convergences have been derived in [18, 20, 21].

CASE  $\mathcal{D}_n = \mathcal{H}_n$  (AND MORE GENERAL HYPERBOLIC REGIONS, SEE EXAMPLE 4.5 BELOW). In this case, Corollaries 2.5 and 2.6 can be found in Theorems 3.5 and 3.7 in [14]. The corresponding asymptotics of moments has been derived in [15, 16].

CASE  $\mathcal{D}_n = \mathcal{S}_n$ . The distributional convergence is completely new. The asymptotics of moments has been analyzed in [17].

### 3. PROOF OF THE MAIN RESULTS

**3.1. Proof of Theorem 2.1.** We first need an auxiliary lemma.

**Lemma 3.1.** *Fix  $m_1, \dots, m_d \in \mathbb{N}$  and  $j_k \in \{0, \dots, m_k - 1\}$ ,  $k = 1, \dots, d$ . Put*

$$\mathcal{D}_n^{(j_1, m_1, \dots, j_d, m_d)} := \{(i_1, \dots, i_d) \in \mathcal{D}_n : i_k \equiv j_k \pmod{m_k} \text{ for all } k = 1, \dots, d\}.$$

If (7) holds, then

$$(15) \quad \lim_{n \rightarrow \infty} \frac{\#\mathcal{D}_n^{(j_1, m_1, \dots, j_d, m_d)}}{\#\mathcal{D}_n} = \frac{1}{m_1 \cdots m_d}.$$

*Proof.* Note that

$$(16) \quad \mathcal{D}_n = \bigcup_{j_1=0}^{m_1-1} \cdots \bigcup_{j_d=0}^{m_d-1} \mathcal{D}_n^{(j_1, m_1, \dots, j_d, m_d)},$$

and the sets on the right-hand side are pairwise disjoint. Furthermore,

$$\mathcal{D}_n^{(j_1, m_1, \dots, j_d, m_d)} = \mathcal{D}_n \cap (j_1 + m_1\mathbb{Z}, \dots, j_d + m_d\mathbb{Z}) = (j_1, \dots, j_d) + (\mathcal{D}_n - (j_1, \dots, j_d)) \cap (m_1\mathbb{Z}, \dots, m_d\mathbb{Z}).$$

Thus,

$$\begin{aligned} & \left| \#\mathcal{D}_n^{(0, m_1, \dots, 0, m_d)} - \#\mathcal{D}_n^{(j_1, m_1, \dots, j_d, m_d)} \right| \\ &= \left| \#(\mathcal{D}_n \cap (m_1\mathbb{Z}, \dots, m_d\mathbb{Z})) - \#((\mathcal{D}_n - (j_1, \dots, j_d)) \cap (m_1\mathbb{Z}, \dots, m_d\mathbb{Z})) \right| \\ &\leq \#((\mathcal{D}_n \cap (m_1\mathbb{Z}, \dots, m_d\mathbb{Z})) \Delta ((\mathcal{D}_n - (j_1, \dots, j_d)) \cap (m_1\mathbb{Z}, \dots, m_d\mathbb{Z}))) \\ &\leq \#(\mathcal{D}_n \Delta (\mathcal{D}_n - (j_1, \dots, j_d))), \end{aligned}$$

and we have proved that (with  $\delta_n$  introduced in (8))

$$(17) \quad \left| \#\mathcal{D}_n^{(0, m_1, \dots, 0, m_d)} - \#\mathcal{D}_n^{(j_1, m_1, \dots, j_d, m_d)} \right| \leq \delta_n(-(j_1, \dots, j_d)).$$

Plugging this into (16) yields

$$\left| \#\mathcal{D}_n - m_1 \cdots m_d \#\mathcal{D}_n^{(0, m_1, \dots, 0, m_d)} \right| \leq \sum_{j_1=0}^{m_1-1} \cdots \sum_{j_d=0}^{m_d-1} \delta_n(-(j_1, \dots, j_d)).$$

Dividing both sides by  $\#\mathcal{D}_n$  and sending  $n \rightarrow \infty$  implies (15) for  $j_1 = \cdots = j_d = 0$ . Using the estimate (17), we obtain (15) for arbitrary  $j_1, \dots, j_d$ .  $\square$

*Proof of Theorem 2.1.* Fix pairwise distinct prime numbers  $p_1, \dots, p_m \in \mathcal{P}$ , nonnegative integers  $j_{k,t}$ ,  $k = 1, \dots, d$ ,  $t = 1, \dots, m$ , and write

$$\begin{aligned} & \mathbb{P}\{\lambda_{p_t}(X_k^{(n)}) \geq j_{k,t} \text{ for all } k = 1, \dots, d \text{ and } t = 1, \dots, m\} \\ &= \mathbb{P}\{X_k^{(n)} \text{ is divisible by } p_t^{j_{k,t}} \text{ for all } k = 1, \dots, d \text{ and } t = 1, \dots, m\} \\ &= \mathbb{P}\{X_k^{(n)} \text{ is divisible by } \prod_{t=1}^m p_t^{j_{k,t}} =: \mu_k \text{ for all } k = 1, \dots, d\} \\ &= \frac{1}{\#\mathcal{D}_n} \sum_{i_1=1}^{\infty} \cdots \sum_{i_d=1}^{\infty} \mathbb{1}\{(i_1, \dots, i_d) \in \mathcal{D}_n : i_k \equiv 0 \pmod{\mu_k}, k = 1, \dots, d\}. \end{aligned}$$

By Lemma 3.1 applied with  $m_k = \mu_k$  and  $j_k = 0$ ,  $k = 1, \dots, d$ , we see that the right-hand side converges to  $(\mu_1 \cdots \mu_d)^{-1}$  as  $n \rightarrow \infty$ . It remains to note that

$$\frac{1}{\mu_1 \cdots \mu_d} = \prod_{k=1}^d \prod_{t=1}^m \frac{1}{p_t^{j_{k,t}}} = \mathbb{P}\{\mathcal{G}_k(p_t) \geq j_{k,t} \text{ for all } k = 1, \dots, d \text{ and } t = 1, \dots, m\}.$$

The proof of Theorem 2.1 is complete.  $\square$

**3.2. Proof of Theorem 2.4.** Fix a large positive constant  $M$  and note that

$$\begin{aligned} F(X_1^{(n)}, \dots, X_d^{(n)}) &= \prod_{p \in \mathcal{P}} F(p^{\lambda_p(X_1^{(n)})}, \dots, p^{\lambda_p(X_d^{(n)})}) \\ &= \left( \prod_{p \in \mathcal{P}, p \leq M} F(p^{\lambda_p(X_1^{(n)})}, \dots, p^{\lambda_p(X_d^{(n)})}) \right) \left( \prod_{p \in \mathcal{P}, p > M} F(p^{\lambda_p(X_1^{(n)})}, \dots, p^{\lambda_p(X_d^{(n)})}) \right) =: Y_1(M, n) Y_2(M, n). \end{aligned}$$

By Theorem 2.1, one has

$$Y_1(M, n) \xrightarrow[n \rightarrow \infty]{w} \prod_{p \in \mathcal{P}, p \leq M} F(p^{\mathcal{G}_1(p)}, \dots, p^{\mathcal{G}_d(p)}).$$

Furthermore, the right-hand side of the latter converges a.s. to  $F_\infty$  as  $M \rightarrow \infty$ , which is a.s. finite. According to Theorem 3.2 in [1], it remains to check that for every fixed  $\varepsilon > 0$ ,

$$(18) \quad \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\{|Y_2(M, n) - 1| \geq \varepsilon\} = 0.$$

Note that

$$(19) \quad \mathbb{P}\{|Y_2(M, n) - 1| \geq \varepsilon\} \leq \mathbb{P}\left\{\text{for all } p \in \mathcal{P}, p > M, \sum_{i=1}^d \lambda_p(X_i^{(n)}) \leq 1, |Y_2(M, n) - 1| \geq \varepsilon\right\} \\ + \mathbb{P}\left\{\text{for some } p \in \mathcal{P}, p > M, \sum_{i=1}^d \lambda_p(X_i^{(n)}) \geq 2\right\}.$$

The second term in (19) can be estimated as follows:

$$\begin{aligned} & \mathbb{P}\{\text{for some } p \in \mathcal{P}, p > M, \sum_{i=1}^d \lambda_p(X_i^{(n)}) \geq 2\} \\ & \leq \mathbb{P}\{\text{there exist } p \in \mathcal{P}, p > M \text{ and } i = 1, \dots, d \text{ such that } \lambda_p(X_i^{(n)}) \geq 2\} \\ & \quad + \mathbb{P}\{\text{there exist } p \in \mathcal{P}, p > M \text{ and } i, j = 1, \dots, d, i \neq j \text{ such that } \lambda_p(X_i^{(n)}) \geq 1, \lambda_p(X_j^{(n)}) \geq 1\} \\ & = \mathbb{P}\{\text{there exist } p \in \mathcal{P}, p > M \text{ and } i = 1, \dots, d \text{ such that } p^2 \text{ divides } X_i^{(n)}\} \\ & \quad + \mathbb{P}\{\text{there exist } p \in \mathcal{P}, p > M \text{ and } i, j = 1, \dots, d, i \neq j \text{ such that } p \text{ divides } X_i^{(n)} \text{ and } X_j^{(n)}\} \\ & \leq \sum_{i=1}^d \sum_{p \in \mathcal{P}, p > M} \mathbb{P}\{p^2 \text{ divides } X_i^{(n)}\} + \sum_{i, j=1, i \neq j}^d \sum_{p \in \mathcal{P}, p > M} \mathbb{P}\{p \text{ divides } X_i^{(n)} \text{ and } X_j^{(n)}\} \\ & = \sum_{i=1}^d \sum_{p \in \mathcal{P}, p > M} \frac{\#(\mathcal{D}_n \cap \mathbb{Z}_i(p^2))}{\#\mathcal{D}_n} + \sum_{i, j=1, i \neq j}^d \sum_{p \in \mathcal{P}, p > M} \frac{\#(\mathcal{D}_n \cap \mathbb{Z}_i(p) \cap \mathbb{Z}_j(p))}{\#\mathcal{D}_n}. \end{aligned}$$

The double limit ( $n \rightarrow \infty, M \rightarrow \infty$ ) of the first term is equal to zero by an appeal to (13) with  $a = p^2$  and  $b = 1$ , since

$$\lim_{M \rightarrow \infty} \sum_{p \in \mathcal{P}, p > M} \frac{1}{p^2} = 0.$$

Similarly, the double limit of the second term is equal to zero by an appeal to (13) with  $a = b = p$ .

In order to deal with the first summand in (19), we first observe that on the event

$$\left\{\text{for all } p \in \mathcal{P}, p > M, \sum_{i=1}^d \lambda_p(X_i^{(n)}) \leq 1\right\},$$

we may pass to the logarithm of  $Y_2(M, n)$ . Thus, it suffices to prove that, for every  $\varepsilon > 0$ ,

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left\{\text{for all } p \in \mathcal{P}, p > M, \sum_{i=1}^d \lambda_p(X_i^{(n)}) \leq 1, \left|\sum_{p \in \mathcal{P}, p > M} \log F(p^{\lambda_p(X_1^{(n)})}, \dots, p^{\lambda_p(X_d^{(n)})})\right| \geq \varepsilon\right\} = 0.$$

Introduce, for  $n \in \mathbb{N}$ ,  $i = 1, \dots, d$  and  $p \in \mathcal{P}$ , the events

$$C_{n,i,p} := \{\lambda_p(X_i^{(n)}) = 1, \lambda_p(X_j^{(n)}) = 0, j \neq i\},$$

and note that  $C_{n,i,p} \cap C_{n,j,p} = \emptyset$  as soon as  $i \neq j$ . On the event  $C_{n,i,p}$ , we have

$$\log F(p^{\lambda_p(X_1^{(n)})}, \dots, p^{\lambda_p(X_d^{(n)})}) = F_i(p)$$

and, therefore, it suffices to show that, for every fixed  $\varepsilon > 0$ ,

$$(20) \quad \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left\{ \left| \sum_{p \in \mathcal{P}, p > M} \sum_{i=1}^d F_i(p) \mathbb{1}_{C_{n,i,p}} \right| \geq \varepsilon \right\} = 0.$$

Fix some  $A > 0$  and note that, for every  $\varepsilon > 0$ ,

$$\begin{aligned} & \mathbb{P} \left\{ \left| \sum_{p \in \mathcal{P}, p > M} \sum_{i=1}^d F_i(p) \mathbb{1}_{\{|F_i(p)| > A, C_{n,i,p}\}} \right| \geq \varepsilon \right\} \\ & \leq \mathbb{P} \{ \text{for some } p \in \mathcal{P} \text{ and } i = 1, \dots, d, |F_i(p)| > A \text{ and } C_{n,i,p} \text{ holds} \} \\ & \leq \sum_{p \in \mathcal{P}, p > M} \sum_{i=1}^d \mathbb{1}_{\{|F_i(p)| > A\}} \mathbb{P}\{C_{n,i,p}\} \leq \sum_{p \in \mathcal{P}, p > M} \sum_{i=1}^d \mathbb{1}_{\{|F_i(p)| > A\}} \mathbb{P}\{\lambda_p(X_i^{(n)}) \geq 1\} \\ & = \sum_{p \in \mathcal{P}, p > M} \sum_{i=1}^d \mathbb{1}_{\{|F_i(p)| > A\}} \frac{\#(\mathcal{D}_n \cap \mathcal{Z}_i(p))}{\#\mathcal{D}_n} \leq K \sum_{p \in \mathcal{P}, p > M} \frac{1}{p} \sum_{i=1}^d \mathbb{1}_{\{|F_i(p)| > A\}}, \end{aligned}$$

where we used (13) with  $a = p$  and  $b = 1$  for the last passage. The right-hand side converges to zero as  $M \rightarrow \infty$ , in view of the first relation in (10). So, in order to prove (20), we need to check that

$$(21) \quad \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left\{ \left| \sum_{p \in \mathcal{P}, p > M} \sum_{i=1}^d F_i(p) \mathbb{1}_{\{|F_i(p)| \leq A, C_{n,i,p}\}} \right| \geq \varepsilon \right\} = 0.$$

This is accomplished by an appeal to Markov's inequality as follows:

$$\begin{aligned} & \mathbb{P} \left\{ \left| \sum_{p \in \mathcal{P}, p > M} \sum_{i=1}^d F_i(p) \mathbb{1}_{\{|F_i(p)| \leq A, C_{n,i,p}\}} \right| \geq \varepsilon \right\} \leq \frac{1}{\varepsilon} \sum_{p \in \mathcal{P}, p > M} \sum_{i=1}^d |F_i(p)| \mathbb{1}_{\{|F_i(p)| \leq A\}} \mathbb{P}\{C_{n,i,p}\} \\ & \leq \frac{1}{\varepsilon} \sum_{p \in \mathcal{P}, p > M} \sum_{i=1}^d |F_i(p)| \mathbb{1}_{\{|F_i(p)| \leq A\}} \mathbb{P}\{\lambda_p(X_i^{(n)}) \geq 1\} \\ & = \frac{1}{\varepsilon} \sum_{p \in \mathcal{P}, p > M} \sum_{i=1}^d |F_i(p)| \mathbb{1}_{\{|F_i(p)| \leq A\}} \frac{\#(\mathcal{D}_n \cap \mathcal{Z}_i(p))}{\#\mathcal{D}_n} \\ & \leq \frac{K}{\varepsilon} \sum_{p \in \mathcal{P}, p > M} \frac{1}{p} \sum_{i=1}^d |F_i(p)| \mathbb{1}_{\{|F_i(p)| \leq A\}}, \end{aligned}$$

where we have utilized (13) with  $a = p$  and  $b = 1$  for the last inequality. The proof of Theorem 2.4 is complete, since the right-hand side converges to zero, as  $M \rightarrow \infty$ , by the second relation in (11).

#### 4. EXAMPLES OF SUITABLE INTEGER DOMAINS

In this section we provide a series of examples of domains  $\mathcal{D}_n$  that satisfy (7) and (13). In particular, we show that  $\mathcal{R}_{n_1, n_2, \dots, n_d}$  in (2),  $\mathcal{S}_n$  and  $\mathcal{H}_n$  mentioned in the introduction, are all admissible. Thus, under assumption (11) on  $F$ , the distributional convergence (6) holds true for all domains listed below.

##### 4.1. Sublevels of monotone functions.

**Proposition 4.1.** *Assume that  $f : [1, \infty)^d \rightarrow \mathbb{R}$  is a coordinate-wise nondecreasing function such that, for every  $j = 1, \dots, d$ ,*

$$\lim_{x_j \rightarrow \infty} f(x_1, \dots, x_d) = \infty,$$

provided  $x_i \geq 1$ ,  $i \neq j$ , are fixed. Put

$$\mathcal{D}_n := \mathcal{D}_n^f = \{(x_1, \dots, x_d) \in \mathbb{N}^d : f(x_1, \dots, x_d) \leq n\}$$

and

$$\mathcal{D}_{n,i} := \mathcal{D}_{n,i}^f = \{(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \in \mathbb{N}^{d-1} : f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_d) \leq n\},$$

for  $i = 1, \dots, d$ . If, for every  $i = 1, \dots, d$ ,

$$(22) \quad \lim_{n \rightarrow \infty} \frac{\#\mathcal{D}_{n,i}}{\#\mathcal{D}_n} = 0,$$

then the sequence  $\mathcal{D}_n$ ,  $n \in \mathbb{N}$ , satisfies (7) and (13).

*Proof.* Let us first verify (7). According to Proposition A.2 in Appendix A, it is sufficient to check (7) for  $c = e_i$ ,  $i = 1, \dots, d$ , where  $e_1, \dots, e_d$  denotes the standard basis of  $\mathbb{R}^d$ . Note that  $\mathcal{D}_n \setminus (\mathcal{D}_n + e_i) = \mathcal{D}_{n,i}$ . Thus, (22) yields that for  $i = 1, \dots, d$ ,

$$\lim_{n \rightarrow \infty} \frac{\#(\mathcal{D}_n \setminus (\mathcal{D}_n + e_i))}{\#\mathcal{D}_n} = 0.$$

It remains to check that for  $i = 1, \dots, d$ ,

$$(23) \quad \lim_{n \rightarrow \infty} \frac{\#((\mathcal{D}_n + e_i) \setminus \mathcal{D}_n)}{\#\mathcal{D}_n} = 0.$$

Without loss of generality, we shall do this for  $i = 1$ . Note that

$$(\mathcal{D}_n + e_1) \setminus \mathcal{D}_n = \{(x_1, \dots, x_d) \in \mathbb{N}^d : x_1 \geq 2, f(x_1 - 1, x_2, \dots, x_d) \leq n, f(x_1, \dots, x_d) > n\}.$$

For every fixed collection  $(x_2, \dots, x_d) \in \mathbb{N}^{d-1}$  and  $n \in \mathbb{N}$ , there exists at most one  $x_1 \geq 2$ ,  $x_1 \in \mathbb{N}$ , such that

$$f(x_1 - 1, x_2, \dots, x_d) \leq n \quad \text{and} \quad f(x_1, \dots, x_d) > n,$$

since  $f$  is monotone in  $x_1$ . Therefore,

$$\begin{aligned} \#((\mathcal{D}_n + e_1) \setminus \mathcal{D}_n) &= \sum_{x_2=1}^{\infty} \cdots \sum_{x_d=1}^{\infty} \mathbb{1}_{\{\text{there exists } x_1 \geq 2 \text{ such that } f(x_1-1, x_2, \dots, x_d) \leq n, f(x_1, \dots, x_d) > n\}} \\ &\leq \sum_{x_2=1}^{\infty} \cdots \sum_{x_d=1}^{\infty} \mathbb{1}_{\{\text{there exists } x_1 \geq 2 \text{ such that } f(x_1-1, x_2, \dots, x_d) \leq n\}} = \sum_{x_2=1}^{\infty} \cdots \sum_{x_d=1}^{\infty} \mathbb{1}_{\{f(1, x_2, \dots, x_d) \leq n\}} \\ &= \#\mathcal{D}_{n,1}. \end{aligned}$$

This proves (23) for  $i = 1$ .

We shall now prove that (13) holds, for all  $i, j = 1, \dots, d$ , with  $K = 1$ . For notational simplicity, we shall do this only for  $i = 1$  and  $j = 2$ . The monotonicity of  $f$  implies that, for all  $a, b \in \mathbb{N}$ ,

$$\begin{aligned} \#\mathcal{D}_n &= \sum_{j=0}^{a-1} \sum_{k=0}^{b-1} \left( \sum_{x_1=1}^{\infty} \sum_{x_2=1}^{\infty} \cdots \sum_{x_d=1}^{\infty} \mathbb{1}_{\{f(ax_1-j, bx_2-k, x_3, \dots, x_d) \leq n\}} \right) \\ &\geq ab \sum_{x_1=1}^{\infty} \sum_{x_2=1}^{\infty} \cdots \sum_{x_d=1}^{\infty} \mathbb{1}_{\{f(ax_1, bx_2, x_3, \dots, x_d) \leq n\}} \\ &= ab \#(\mathcal{D}_n \cap \mathbb{Z}_1(a) \cap \mathbb{Z}_2(b)). \end{aligned}$$

The proof of Proposition 4.1 is complete.  $\square$

Proposition 4.1 yields the following explicit examples.

**Example 4.2** (Rectangular domains). *Let  $f_1, \dots, f_d : [1, \infty) \rightarrow [1, \infty)$  be strictly increasing continuous functions. Putting  $f(x_1, \dots, x_d) := \max(f_1^{-1}(x_1), \dots, f_d^{-1}(x_d))$ , we obtain*

$$\mathcal{D}_n = \mathcal{R}_{f_1(n), \dots, f_d(n)} = ([1, f_1(n)] \times \cdots \times [1, f_d(n)]) \cap \mathbb{N}^d.$$

*Condition (22) is fulfilled if  $\lim_{x \rightarrow \infty} f_i(x) = \infty$ , for every  $i = 1, \dots, d$ .*

**Example 4.3** (Tetrahedral domains). *Let  $a_1, \dots, a_d > 0$  be fixed positive real numbers. The sequence of tetrahedral sets*

$$\mathcal{D}_n = \mathcal{T}_n := \{(x_1, \dots, x_d) \in \mathbb{N}^d : a_1 x_1 + \cdots + a_d x_d \leq n\}$$

*satisfies (7) and (13). Indeed,*

$$\#\mathcal{T}_n \sim \frac{1}{d! a_1 \cdots a_d} n^d, \quad n \rightarrow \infty,$$

whereas, for  $i = 1, \dots, d$ ,

$$\#\mathcal{T}_{n,i} \sim \frac{a_i}{(d-1)!a_1 \cdots a_d} n^{d-1}, \quad n \rightarrow \infty.$$

Thus, Proposition 4.1 is applicable.

**Example 4.4** (Hyperbolic domains). Let  $f(x_1, \dots, x_d) = x_1 \cdots x_d$ . Then the sequence of sets

$$\mathcal{D}_n = \mathcal{H}_n := \{(x_1, \dots, x_d) \in \mathbb{N}^d : x_1 \cdots x_d \leq n\}$$

satisfies (7) and (13). Indeed, according to Proposition 4.1 in [14].

$$\#\mathcal{D}_n \sim \frac{n \log^{d-1} n}{(d-1)!}, \quad n \rightarrow \infty,$$

and, for every  $i = 1, \dots, d$ ,

$$\#\mathcal{D}_{n,i} \sim \frac{n \log^{d-2} n}{(d-2)!}, \quad n \rightarrow \infty.$$

Thus, Proposition 4.1 is applicable.

**Example 4.5** (Further hyperbolic domains). Fix  $2 \leq \ell \leq d$ . Define the  $\ell$ -th standard symmetric polynomial in  $d$  variables by

$$f(x_1, \dots, x_d) = P_\ell(x_1, \dots, x_d) := \sum_{1 \leq i_1 < \dots < i_\ell \leq d} x_{i_1} \cdots x_{i_\ell}.$$

The associated domain is

$$\mathcal{D}_n = \mathcal{H}_{\ell,d}(n) := \{(x_1, \dots, x_d) \in \mathbb{N}^d : P_\ell(x_1, \dots, x_d) \leq n\}.$$

Example 4.4 corresponds to the particular case  $\ell = d$ . If now  $2 \leq \ell < d$ , then Proposition 4.4 in [14] entails that  $\#\mathcal{D}_n \sim C(d, \ell)n^{d/\ell}$ , for some positive constant  $C(d, \ell) > 0$ . Furthermore, by symmetry  $\#\mathcal{D}_{n,i} = \#\mathcal{D}_{n,1}$  for all  $i = 1, \dots, d$ , and

$$\begin{aligned} \mathcal{D}_{n,1} &= \{(x_2, \dots, x_d) \in \mathbb{N}^d : P_\ell(1, x_2, \dots, x_d) \leq n\} \\ &= \{(x_2, \dots, x_d) \in \mathbb{N}^d : P_\ell(x_2, \dots, x_d) + P_{\ell-1}(x_2, \dots, x_d) \leq n\} \\ &\subset \{(x_2, \dots, x_d) \in \mathbb{N}^d : P_\ell(x_2, \dots, x_d) \leq n\} = \mathcal{H}_{\ell,d-1}(n). \end{aligned}$$

Thus,  $\mathcal{D}_{n,1} \subseteq \mathcal{H}_{\ell,d-1}(n)$  and thereupon  $\#\mathcal{D}_{n,1} \leq \#\mathcal{H}_{\ell,d-1}(n)$ . If  $\ell < d-1$ , then

$$\#\mathcal{H}_{\ell,d-1}(n) \sim C(d-1, \ell)n^{(d-1)/\ell}, \quad n \rightarrow \infty,$$

whereas if  $\ell = d-1$ ,

$$\#\mathcal{H}_{\ell,d-1}(n) = \#\mathcal{H}_{d-1,d-1}(n) \sim \frac{n \log^{d-2} n}{(d-2)!}, \quad n \rightarrow \infty.$$

In both cases  $\lim_{n \rightarrow \infty} \#\mathcal{H}_{\ell, d-1}(n)/\#\mathcal{D}_n = 0$ . Summarizing, Proposition 4.1 is applicable to  $\mathcal{D}_n = \mathcal{H}_{\ell, d}(n)$ .

#### 4.2. Dilations of a convex body.

**Proposition 4.6.** *Let  $\mathcal{D} \subset [0, \infty)^d$  be a compact convex set with nonempty interior and  $a_n, n \in \mathbb{N}$ , be a sequence of positive numbers such that  $\lim_{n \rightarrow \infty} a_n = \infty$ . Then, the following sequence of sets satisfies (7) and (13):*

$$\mathcal{D}_n := a_n \mathcal{D} \cap \mathbb{N}^d.$$

*Proof.* For the proof of (7), we shall use Proposition A.3. Put

$$V := \mathcal{D} \cap (0, \infty)^d, \quad V_n := a_n V = a_n \mathcal{D} \cap (0, \infty)^d,$$

and note that  $\mathcal{D}_n = V_n \cap \mathbb{N}^d$ . Let us check that (30) holds for the sequence  $V_n$ . First of all, since  $\mathcal{D}$  is compact, convex and has a non-empty interior, it holds

$$\mathcal{D} = \text{cl}(\text{Int}(\mathcal{D})) = \text{cl}(\text{Int}(\mathcal{D}) \cap (0, \infty)^d) = \text{cl}(\text{Int}(\mathcal{D} \cap (0, \infty)^d)) = \text{cl}(V),$$

and, thereupon,

$$\partial V = \text{cl}(V) \setminus \text{Int}(V) = \text{cl}(V) \setminus \text{Int}(\mathcal{D}) = \mathcal{D} \setminus \text{Int}(\mathcal{D}) = \partial \mathcal{D}.$$

Further, observe that  $\text{Vol}(V) > 0$  and, denoting  $B_\varepsilon^d(0)$  the ball  $\{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1^2 + \dots + x_d^2 < \varepsilon\}$  and  $A \oplus B := \{x + y : x \in A, y \in B\}$  the Minkowski addition,

$$(24) \quad \frac{\text{Vol}(\partial V_n \oplus B_\varepsilon^d(0))}{\text{Vol}(V_n)} = \frac{\text{Vol}(a_n(\partial V \oplus B_{\varepsilon/a_n}^d(0)))}{\text{Vol}(a_n V)} = \frac{\text{Vol}(\partial V \oplus B_{\varepsilon/a_n}^d(0))}{\text{Vol}(V)} = \frac{\text{Vol}(\partial \mathcal{D} \oplus B_{\varepsilon/a_n}^d(0))}{\text{Vol}(V)}.$$

Since  $\mathcal{D}$  is a compact convex set, its boundary  $\partial \mathcal{D}$  is  $(d-1)$ -rectifiable subset of  $\mathbb{R}^d$ , that is, can be represented as the image of a Lipschitz function<sup>1</sup>  $h$  defined on a bounded subset of  $\mathbb{R}^{d-1}$  and taking values in  $\mathbb{R}^d$ . Thus, by Theorem 3.2.39 in [12],

$$\lim_{n \rightarrow \infty} a_n \text{Vol}(\partial \mathcal{D} \oplus B_{\varepsilon/a_n}^d(0)) = 2\varepsilon \mathcal{H}_{d-1}(\partial \mathcal{D}) < \infty,$$

where  $\mathcal{H}_{d-1}$  is the  $(d-1)$ -dimensional Hausdorff measure in  $\mathbb{R}^d$ . Summarizing, we have shown that the right-hand side of (24) converges to zero as  $n \rightarrow \infty$ .

For the proof of (13), we employ Proposition A.4 from Appendix A.

For  $i = 1, \dots, d$ , put

$$m_i(\mathcal{D}) := \inf\{x_i \geq 0 : (x_1, \dots, x_i, \dots, x_d) \in \mathcal{D}\},$$

$$M_i(\mathcal{D}) := \sup\{x_i \geq 0 : (x_1, \dots, x_i, \dots, x_d) \in \mathcal{D}\},$$

<sup>1</sup>As  $h$  one can take, for example, the function  $\partial B_R(0) \ni x \mapsto \pi_{\mathcal{D}}(x)$ , where  $R > 0$  is such that  $\mathcal{D} \subseteq B_R(0)$  and  $\pi_{\mathcal{D}}(x)$  is a unique closest to  $x$  point in  $\mathcal{D}$  (metric projection on  $\mathcal{D}$ ).

and note that  $0 \leq m_i(\mathcal{D}) < M_i(\mathcal{D}) < \infty$ . Here the second inequality is strict since  $\mathcal{D}$  has a non-empty interior; the last inequality follows from the compactness of  $\mathcal{D}$ . Proposition A.4 is applicable with the rectangle

$$\Pi_n := \left( \bigtimes_{i=1}^d \left[ \lfloor a_n m_i(\mathcal{D}) \rfloor, \lceil a_n M_i(\mathcal{D}) \rceil \right] \right) \cap \mathbb{N}^d.$$

By construction

$$a_n \mathcal{D} \subset a_n \left( \bigtimes_{i=1}^d \left[ m_i(\mathcal{D}), M_i(\mathcal{D}) \right] \right) \subset \left( \bigtimes_{i=1}^d \left[ \lfloor a_n m_i(\mathcal{D}) \rfloor, \lceil a_n M_i(\mathcal{D}) \rceil \right] \right).$$

It remains to note that as  $n \rightarrow \infty$ ,

$$(25) \quad \#\Pi_n \sim a_n^d \prod_{i=1}^d (M_i(\mathcal{D}) - m_i(\mathcal{D})),$$

and also

$$(26) \quad \liminf_{n \rightarrow \infty} \frac{\#\mathcal{D}_n}{a_n^d} > 0,$$

which is a consequence of the fact that  $\mathcal{D}$  has a non-empty interior and, therefore, contains a small  $d$ -dimensional cube in the interior. Relations (25) and (26) imply

$$\limsup_{n \rightarrow \infty} \frac{\#\Pi_n}{\#\mathcal{D}_n} < \infty.$$

The proof of Proposition 4.6 is complete. □

**Example 4.7** (Spherical domains). Put  $\mathcal{B} := \{(x_1, \dots, x_d) \in [0, \infty)^d : x_1^2 + \dots + x_d^2 \leq 1\}$ . Then the sequence of discrete balls

$$\mathcal{D}_n = \mathcal{S}_n := \sqrt{n} \mathcal{B} \cap \mathbb{N}^d = \{(x_1, \dots, x_d) \in \mathbb{N}^d : x_1^2 + \dots + x_d^2 \leq n\}$$

satisfies (7) and (13) by Proposition 4.6.

Truncated cones, such as Weyl chambers, also satisfy (7) and (13).

**Example 4.8** (Truncated Weyl chambers). Let  $\mathcal{A} := \{(x_1, \dots, x_d) \in [0, \infty)^d : x_1 \leq \dots \leq x_d \leq 1\}$ . Then the sequence of sets

$$\mathcal{D}_n = \mathcal{A}_n := n \mathcal{A} \cap \mathbb{N}^d = \{(x_1, \dots, x_d) \in \mathbb{N}^d : x_1 \leq \dots \leq x_d \leq n\}$$

satisfies (7) and (13) by Proposition 4.6.

## 5. SET-THEORETIC OPERATIONS PRESERVING PROPERTIES (7) AND (13)

In this section we discuss stability properties of sets satisfying (7) and (13) with respect to the standard set-theoretic operations.

First, immediately from the definitions, one obtains the following.

**Proposition 5.1.** *Let  $\mathcal{D}_n^{(1)}$  and  $\mathcal{D}_n^{(2)}$  be two sequences of sets satisfying (7) and (13). Then the sequence  $\mathcal{D}_n := \mathcal{D}_n^{(1)} \cup \mathcal{D}_n^{(2)}$  satisfies (7) and (13).*

As far as intersections and differences of sets are concerned, additional assumptions ensuring that the resulting sets are not small have to be imposed. The following holds true.

**Proposition 5.2.** *Let  $\mathcal{D}_n^{(1)}$  and  $\mathcal{D}_n^{(2)}$  be two sequences of sets satisfying (7). Suppose further that*

$$(27) \quad \mathcal{D}_n^{(2)} \subset \mathcal{D}_n^{(1)} \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{\#\mathcal{D}_n^{(2)}}{\#\mathcal{D}_n^{(1)}} \in [0, 1).$$

*Then the sequence  $\mathcal{D}_n := \mathcal{D}_n^{(1)} \setminus \mathcal{D}_n^{(2)}$  satisfies (7). Moreover, if  $\mathcal{D}_n^{(1)}$  satisfies (13), then so does  $\mathcal{D}_n$ .*

*Proof.* Using the inclusion  $(A \setminus B) \Delta (C \setminus D) \subseteq (A \Delta C) \cup (B \Delta D)$  we obtain, for every fixed  $c \in \mathbb{Z}^d$ ,

$$\begin{aligned} \frac{\#(\mathcal{D}_n \Delta (\mathcal{D}_n + c))}{\#\mathcal{D}_n} &= \frac{\#((\mathcal{D}_n^{(1)} \setminus \mathcal{D}_n^{(2)}) \Delta ((\mathcal{D}_n^{(1)} + c) \setminus (\mathcal{D}_n^{(2)} + c)))}{\#\mathcal{D}_n} \\ &\leq \frac{\#(\mathcal{D}_n^{(1)} \Delta (\mathcal{D}_n^{(1)} + c)) \#\mathcal{D}_n^{(1)}}{\#\mathcal{D}_n^{(1)} \#\mathcal{D}_n} + \frac{\#(\mathcal{D}_n^{(2)} \Delta (\mathcal{D}_n^{(2)} + c)) \#\mathcal{D}_n^{(2)}}{\#\mathcal{D}_n^{(2)} \#\mathcal{D}_n}. \end{aligned}$$

In view of (27),

$$(28) \quad 0 \leq \limsup_{n \rightarrow \infty} \frac{\#\mathcal{D}_n^{(2)}}{\#\mathcal{D}_n} \leq \limsup_{n \rightarrow \infty} \frac{\#\mathcal{D}_n^{(1)}}{\#\mathcal{D}_n} = \limsup_{n \rightarrow \infty} \frac{\#\mathcal{D}_n^{(1)}}{\#\mathcal{D}_n^{(1)} - \#\mathcal{D}_n^{(2)}} < \infty,$$

and we see that  $\mathcal{D}_n$  satisfies (7).

If  $\mathcal{D}_n^{(1)}$  satisfies (13), then, for every  $a, b \in \mathbb{N}$  and  $i, j = 1, \dots, d$ ,  $i \neq j$ , it holds that for all  $n \in \mathbb{N}$ ,

$$\frac{\#(\mathcal{D}_n \cap \mathbb{Z}_i(a) \cap \mathbb{Z}_j(b))}{\#\mathcal{D}_n} \leq \frac{\#(\mathcal{D}_n^{(1)} \cap \mathbb{Z}_i(a) \cap \mathbb{Z}_j(b)) \#\mathcal{D}_n^{(1)}}{\#\mathcal{D}_n^{(1)} \#\mathcal{D}_n} \leq \frac{K}{ab} \sup_{n \in \mathbb{N}} \frac{\#\mathcal{D}_n^{(1)}}{\#\mathcal{D}_n} =: \frac{K'}{ab},$$

where we used (28) for the last passage. □

With minimal changes, the above proof leads to the following.

**Proposition 5.3.** *Let  $\mathcal{D}_n^{(1)}$  and  $\mathcal{D}_n^{(2)}$  be two sequences of sets satisfying (7). Suppose further that*

$$\limsup_{n \rightarrow \infty} \frac{\#(\mathcal{D}_n^{(1)} \cup \mathcal{D}_n^{(2)})}{\#(\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)})} < \infty.$$

Then the sequence  $\mathcal{D}_n := \mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)}$  satisfies (7). Moreover, if  $\mathcal{D}_n^{(1)}$  or  $\mathcal{D}_n^{(2)}$  satisfies (13), then  $\mathcal{D}_n$  satisfies (13) as well.

#### APPENDIX A. ON THE REGULAR GROWTH CONDITION FOR DISCRETE DOMAINS

The following definition can be found on p. 173 in [5].

**Definition A.1.** A sequence of finite sets  $\mathcal{D}_n \subset \mathbb{Z}^d$  is said to be regularly growing to infinity if as  $n \rightarrow \infty$ ,

$$(29) \quad \#\mathcal{D}_n \rightarrow \infty \quad \text{and} \quad \frac{\#(\mathcal{D}_n^1 \setminus \mathcal{D}_n)}{\#\mathcal{D}_n} \rightarrow 0,$$

where for  $A \subset \mathbb{Z}^d$  and  $p \in \mathbb{N}$ , we denote by

$$A^p := \{x = (x_1, \dots, x_d) \in \mathbb{Z}^d : \text{dist}(x, A) \leq p\},$$

and  $\text{dist}$  is the supremum metric on  $\mathbb{Z}^d$ .

**Proposition A.2.** Assume that  $\mathcal{D}_n \subset \mathbb{Z}^d$  is a sequence of finite sets and  $\#\mathcal{D}_n \rightarrow \infty$  as  $n \rightarrow \infty$ . The following statements are equivalent:

- (i) Condition (7) holds for all  $c \in \mathbb{Z}^d$ .
- (ii) Condition (7) holds for  $c = \pm e_k$ ,  $k = 1, \dots, d$ .
- (iii) Condition (7) holds for  $c = e_k$ ,  $k = 1, \dots, d$ .
- (iv) The sequence  $\mathcal{D}_n$  is regularly growing.

*Proof.* Condition (i) trivially implies condition (ii), and (ii) clearly implies (iii). The fact that (iii)  $\implies$  (ii) follows from

$$\#((\mathcal{D}_n - e_k) \Delta \mathcal{D}_n) = \#(((\mathcal{D}_n - e_k) \Delta \mathcal{D}_n) + e_k) = \#(\mathcal{D}_n \Delta (\mathcal{D}_n + e_k)) = \#((\mathcal{D}_n + e_k) \Delta \mathcal{D}_n).$$

We now prove that (ii)  $\implies$  (iv). Note that

$$\mathcal{D}_n^1 = \bigcup_{k=1}^d (\mathcal{D}_n \pm e_k).$$

Thus,

$$\frac{\#(\mathcal{D}_n^1 \setminus \mathcal{D}_n)}{\#\mathcal{D}_n} \leq \sum_{k=1}^d \frac{\#((\mathcal{D}_n \pm e_k) \setminus \mathcal{D}_n)}{\#\mathcal{D}_n} \leq \sum_{k=1}^d \frac{\#((\mathcal{D}_n \pm e_k) \Delta \mathcal{D}_n)}{\#\mathcal{D}_n}.$$

The right-hand side converges to 0, since by (7) every summand converges to 0.

We proceed to the proof of (iv) $\implies$ (i). Assume that (29) holds and fix  $c \in \mathbb{Z}^d$ . Using the inclusion  $A \setminus B \subset (A \setminus C) \cup (C \setminus B)$  which holds for any sets  $A, B, C$ , we conclude that

$$(\mathcal{D}_n + c) \Delta \mathcal{D}_n \subset \bigcup_j \left( (\mathcal{D}_n + u_j) \setminus (\mathcal{D}_n + v_j) \right),$$

where the union is finite and for every index  $j$ ,  $u_j - v_j = \pm e_{k_j}$  for some  $k_j \in \{1, \dots, d\}$ . Since  $\#\mathcal{D}_n = \#(\mathcal{D}_n + x)$  for every  $x \in \mathbb{Z}^d$ , it suffices to check that, for every  $j$ ,

$$\lim_{n \rightarrow \infty} \frac{\# \left( (\mathcal{D}_n + u_j) \setminus (\mathcal{D}_n + v_j) \right)}{\#(\mathcal{D}_n + v_j)} = 0,$$

but this follows from the inclusion  $(\mathcal{D}_n + u_j) = (\mathcal{D}_n + v_j \pm e_{k_j}) \subset (\mathcal{D}_n + v_j)^1$  and the fact that if (29) holds for a sequence  $\mathcal{D}_n$ , it also holds for the shifted sequence  $\mathcal{D}_n + x$ , for every fixed  $x \in \mathbb{Z}^d$ .  $\square$

The following result is a combination of Proposition A.2 and Lemma 1.5 in [5]. In some cases, it is useful for checking (29).

**Proposition A.3.** *Assume that  $V_n, n \in \mathbb{N}$ , is a sequence of bounded measurable subsets of  $\mathbb{R}^d$  satisfying the so-called van Hove condition, meaning that for every  $\varepsilon > 0$*

$$(30) \quad \lim_{n \rightarrow \infty} \frac{\text{Vol}(\partial V_n \oplus B_\varepsilon^d(0))}{\text{Vol}(V_n)} = 0,$$

where  $\partial V_n$  is the topological boundary of  $V_n$ . Then the sequence  $\mathcal{D}_n := V_n \cap \mathbb{Z}^d$  satisfies (29).

Our last auxiliary result provides sufficient conditions for (13). It has been used in the proof of Proposition 4.6.

**Proposition A.4.** *Assume that there exist two sequences  $(s_1(n), \dots, s_d(n))_{n \in \mathbb{N}}$  and  $(c_1(n), \dots, c_d(n))_{n \in \mathbb{N}}$  of nonnegative integers such that the rectangle*

$$\Pi_n := \left( \bigtimes_{i=1}^d [c_i(n), c_i(n) + s_i(n)] \right) \cap \mathbb{N}^d$$

satisfies

$$(31) \quad \#\mathcal{D}_n \subset \Pi_n \quad \text{and} \quad \bar{C} := \sup_{n \in \mathbb{N}} \frac{\#\Pi_n}{\#\mathcal{D}_n} < \infty.$$

Then (13) holds. More generally, if (13) holds with  $\mathcal{D}_n$  replaced by some set  $\Pi_n$  which satisfies (31), then (13) holds for  $\mathcal{D}_n$ .

*Proof.* Fix  $i, j = 1, \dots, d$ ,  $i \neq j$ . If (31) holds, then for all  $n \in \mathbb{N}$  and all  $a, b \in \mathbb{N}$  it holds

$$\frac{\#(\mathcal{D}_n \cap \mathcal{Z}_i(a) \cap \mathcal{Z}_j(b))}{\#\mathcal{D}_n} \leq \frac{\#(\Pi_n \cap \mathcal{Z}_i(a) \cap \mathcal{Z}_j(b))}{\#\mathcal{D}_n} \leq \bar{C} \frac{\#(\Pi_n \cap \mathcal{Z}_i(a) \cap \mathcal{Z}_j(b))}{\#\Pi_n}.$$

Since  $i \neq j$ , we obtain

$$\frac{\#(\Pi_n \cap \mathcal{Z}_i(a) \cap \mathcal{Z}_j(b))}{\#\Pi_n} \leq \frac{1}{s_i(n)+1} \left\lfloor \frac{s_i(n)+1}{a} \right\rfloor \frac{1}{s_j(n)+1} \left\lfloor \frac{s_j(n)+1}{b} \right\rfloor \leq \frac{1}{ab}$$

and the desired estimate holds true with  $K = \bar{C}$ .  $\square$

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