



HAL
open science

Finite-dimensional pseudofinite groups of small dimension, without CFSG

Ulla Karhumäki, Frank Olaf Wagner

► **To cite this version:**

Ulla Karhumäki, Frank Olaf Wagner. Finite-dimensional pseudofinite groups of small dimension, without CFSG. 2024. hal-04671528v1

HAL Id: hal-04671528

<https://hal.science/hal-04671528v1>

Preprint submitted on 15 Aug 2024 (v1), last revised 12 Dec 2024 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

FINITE-DIMENSIONAL PSEUDOFINITE GROUPS OF SMALL DIMENSION, WITHOUT CFSG

ULLA KARHUMÄKI AND FRANK O. WAGNER

ABSTRACT. Any simple pseudofinite group G is known to be isomorphic to a (twisted) Chevalley group over a pseudofinite field. This celebrated result mostly follows from the work of Wilson in 1995 and heavily relies on the classification of finite simple groups (CFSG). It easily follows that G is finite-dimensional with additive and fine dimension and, in particular, that if $\dim(G) = 3$ then G is isomorphic to $\mathrm{PSL}_2(F)$ for some pseudofinite field F . We describe pseudofinite finite-dimensional groups when the dimension is fine, additive and ≤ 3 and, in particular, show that the classification $G \cong \mathrm{PSL}_2(F)$ is independent from CFSG.

1. INTRODUCTION

Pseudofinite groups (resp. fields) are infinite groups (fields) which satisfy every first-order property that is true in all finite groups (fields). In [1], Ax characterised pseudofinite fields in purely algebraic terms. Using Ax's result and the classification of finite simple groups (CFSG), Wilson proved that a simple pseudofinite group is elementarily equivalent to a (twisted) Chevalley group over a pseudofinite field [18].

A structure is *supersimple of finite SU-rank* if its definable sets (in some monster model) are equipped with a notion of dimension, called *SU-rank*, taking integer values. It is well known [4] that a pure pseudofinite field F is supersimple of SU-rank 1. Hence, using Wilson's classification, the theory of any simple pseudofinite group G is supersimple of finite SU-rank [12, Theorem 4.1]. In particular, it is easy to see [6, Proposition 6.1] that a simple pseudofinite group G with $\mathrm{SU}(G) = 3$ is isomorphic to $\mathrm{PSL}_2(F)$ for some pseudofinite field F . However, this observation heavily relies on CFSG.

A more general notion was introduced by the second author in [17, Definition 1.1]: A structure is *finite-dimensional* if there is a dimension function \dim from the collection of all interpretable sets (again, in some monster model) to $\mathbb{N} \cup \{-\infty\}$. In particular, supersimple structures of finite SU-rank are finite-dimensional, where the dimension is SU-rank. In this case the dimension is *additive* and *fine* (see Section 2.3). Therefore, our main result below implies that the classification from the above, $G \cong \mathrm{PSL}_2(F)$, can be proven without using CFSG and thus solves the problem proposed in [6, p. 3, Question (3)] and in [12, p. 171].

Theorem 1.1 (Without CFSG). *Let G be a pseudofinite non-abelian definably simple finite-dimensional group with additive and fine dimension. If $\dim(G) = 3$, then $G \cong \mathrm{PSL}_2(F)$ for some pseudofinite field F .*

The first author is funded by the Finnish Science Academy grant no. 338334. The second author would like to acknowledge the support of the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19677451).

Theorem 1.1 is a corollary of a more general result (Theorem 3.7) which implies that a pseudofinite finite-dimensional group G with additive and fine dimension and with $\dim(G) = 3$ is either soluble-by-finite, or has a finite normal subgroup Z so that G/Z has a definable subgroup of finite index isomorphic to $\mathrm{PSL}_2(F)$ for some pseudofinite field F .

Let G be a finite-dimensional pseudofinite group and assume that the dimension is additive and fine. To prove Theorem 1.1 we need to understand the structure of G when its dimension is 1 or 2. It is easy to observe (see Section 2.3) that G satisfies the icc^0 -condition: In G , any chain of intersections of uniformly definable subgroups, each having infinite index in its predecessor, has finite length, bounded by some n_φ depending only on the defining formula φ . In particular, G satisfies the icc^0 -condition for centralisers, called the $\widetilde{\mathfrak{M}}_c$ -condition, and therefore it follows from [17, Corollaries 4.14 and 5.2] that if $\dim(G) = 1$ then G is finite-by-abelian-by-finite, and if $\dim(G) = 2$ then G is soluble-by-finite. However, only the result in the case $\dim(G) = 1$ is proven independently of CFSG. Indeed, to the authors knowledge, there is no published result without CFSG which describes the structure of G when $\dim(G) = 2$. The second author has proven such result without CFSG and the preprint where the proof appears is available at [16]. In this paper, we give a different proof. Namely, using similar arguments in dimension 2 and 3 (Section 3.2), we show that if G is not soluble-by-finite, then in dimension 2 we get a contradiction whereas in dimension 3 we may identify $\mathrm{PSL}_2(F)$ as in Theorem 3.7. Thus, in dimension 2 we recover the following:

Theorem 1.2 (Without CFSG, cf. [16]). *Let G be a finite-dimensional pseudofinite group with additive and fine dimension. If $\dim(G) = 2$, then G is soluble-by-finite. Moreover, if G is not finite-by-abelian-by-finite, then there is a definable normal subgroup N of G with $\dim(N) = 1$.*

Our results suggest the following question.

Question 1.3. Let G be a non-abelian definably simple pseudofinite group. Can one show, independently of CFSG, that G is finite-dimensional with additive and fine dimension?

We prove our results in Section 3 and give all necessary background results in Section 2. In Section 2.1 we discuss which results from finite group theory are needed in our proofs.

2. BACKGROUND RESULTS

Let G be a group. We use the following standard terminology and notation:

- If the largest soluble (resp. nilpotent) normal subgroup of G exists, then it is denoted by $\mathrm{Rad}(G)$ (resp. $\mathrm{Fitt}(G)$) and called the *soluble radical* (resp. *Fitting subgroup*) of G . Note that if $\mathrm{Rad}(G)$ or $\mathrm{Fitt}(G)$ exists, then it is definable [11, Theorem 1.1].
- The set of involutions of G is denoted by $I(G)$.
- If P and Q are properties of groups then G is called *P -by- Q* if G has a normal subgroup N so that N has the property P and G/N has the property Q . Note that *P -by- $(Q$ -by- $R)$* implies *$(P$ -by- $Q)$ -by- R* , but may be strictly stronger. However, finite-by-abelian-by-finite is unambiguous: If F is finite normal in A which is normal of finite index in G and A/F is abelian, then A' is finite characteristic in

A , whence normal in G , so (finite-by-abelian)-by-finite implies finite-by-(abelian-by-finite). Moreover, $C_G(A')$ is a normal subgroup of finite index in G and $C_A(A')$ is nilpotent of class at most 2, so G is nilpotent-by-finite. Similarly, (finite-by-nilpotent)-by-finite equals nilpotent-by-finite, and (finite-by-soluble)-by-finite equals soluble-by-finite (the nilpotency class or the derived length may go up by 1).

2.1. Needed results from finite group theory. As one suspects, the Feit-Thompson Theorem is used in our results.

Theorem 2.1 (Feit-Thompson [7]). *A finite group without involutions is soluble.*

The *rank* of a finite group G is the smallest cardinality of a generating set, and the *2-rank* of G , denoted by $m_2(G)$, is the largest rank of an elementary abelian 2-subgroup. We use the classification of finite simple non-abelian groups of 2-rank 2 whose proof can be found in [8, Section 1].

Theorem 2.2 (See e.g. [8, page 6]). *Let G be a non-abelian finite simple group with $m_2(G) \leq 2$. Then $m_2(G) = 2$ and G is isomorphic to one of the following groups*

$$\mathrm{PSL}_2(q), \mathrm{PSL}_3(q), \mathrm{PSU}_3(q) \text{ for odd } q, \mathrm{PSU}_3(4), A_7 \text{ or } M_{11}.$$

Theorem 2.2 is used to recognise a simple pseudofinite group over a pseudofinite field F of characteristic $\neq 2$. For the case of $\mathrm{char}(F) = 2$ we use Bender's result below. A *strongly embedded* subgroup of a finite group G is a proper subgroup E of even order such that $E \cap E^g$ has odd order whenever $g \in G \setminus E$.

Theorem 2.3 (Bender [2]). *Let G be a finite group containing a strongly embedded subgroup. Assume that $m_2(G) > 1$ and $\mathrm{Rad}(G) = 1$. Then G has a normal subgroup which is isomorphic to one of the following groups*

$$\mathrm{PSL}_2(2^n), \mathrm{Sz}(2^{2n-1}), \mathrm{PSU}_3(2^n), \text{ for } n \geq 2.$$

2.2. Commensurability and almost operators. Let G be a group, and $H, K \leq G$. The group H is said to be *almost contained* in K , written $H \lesssim K$, if $H \cap K$ has finite index in H . The subgroups $H, K \leq G$ are *commensurable* if both $H \lesssim K$ and $K \lesssim H$. This is denoted by $H \simeq K$. If H and K are not commensurable then we write $H \not\simeq K$. A family \mathcal{H} of subgroups of G is *uniformly commensurable* if there is $n \in \mathbb{N}$ so that $|H_1 : H_1 \cap H_2| < n$ for all $H_1, H_2 \in \mathcal{H}$. Likewise, $K \leq G$ is uniformly commensurable to \mathcal{H} if and only if \mathcal{H} is uniformly commensurable and K is commensurable to some (equiv. any) group in \mathcal{H} . The following result is due to Schlichting but the formulation we give here can be found in [15, Theorem 4.2.4].

Theorem 2.4 (Schlichting's Theorem). *Let G be a group and \mathcal{H} a uniformly commensurable family of subgroups of G . Then there is a subgroup N of G which is uniformly commensurable with all members of \mathcal{H} and is invariant under all automorphisms of G which fix \mathcal{H} setwise.*

Definition 2.5. Let G be a group, and $H, K \leq G$. We define the following subgroups:

- (1) $\tilde{N}_K(H) = \{k \in K : H \simeq H^k\}$ is the *almost normaliser* of H in K .
- (2) $\tilde{C}_K(H) = \{k \in K : H \lesssim C_H(k)\}$ is the *almost centraliser* of H in K .
- (3) $\tilde{Z}(G) = \tilde{C}_G(G)$ is the *almost centre* of G .

The commensurability is *uniform* in $\tilde{N}_K(H)$ (resp. in $\tilde{C}_K(H)$) if there is some $m \in \mathbb{N}$ so that if $H \simeq H^k$ then $|H : H \cap H^k| < m$ (resp. if $H \lesssim C_H(k)$ then $|H : C_H(k)| < m$).

Notice that given a group G and definable subgroups H, K of G so that the commensurability is uniform in $\tilde{N}_K(H)$ (resp. in $\tilde{C}_K(H)$), then $\tilde{N}_K(H)$ (resp. $\tilde{C}_K(H)$) is a definable subgroup of G .

If $\tilde{N}_G(H) = G$ we say that H is *almost normal* in G .

Fact 2.6 (Hempel [9, Theorem 2.17]). *Let G be a group and $H, K \leq G$ be definable subgroups so that the commensurability is uniform in $\tilde{C}_K(H)$ and in $\tilde{C}_H(K)$. Then $H \lesssim \tilde{C}_H(K)$ if and only if $K \lesssim \tilde{C}_K(H)$.*

2.3. Finite-dimensional groups. A group G is called *finite-dimensional* if there is a dimension function \dim from the collection of all interpretable sets in models of $\text{Th}(G)$ to $\mathbb{N} \cup \{-\infty\}$ such that, for any formula $\phi(x, y)$ and interpretable sets X and Y , the following hold:

- (1) Invariance: If $a \equiv a'$ then $\dim(\phi(x, a)) = \dim(\phi(x, a'))$.
- (2) Algebraicity: If $X \neq \emptyset$ is finite then $\dim(X) = 0$, and $\dim(\emptyset) = -\infty$.
- (3) Union: $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.
- (4) Fibration: If $f : X \rightarrow Y$ is an interpretable map such that $\dim(f^{-1}(y)) \geq d$ for all $y \in Y$ then $\dim(X) \geq \dim(Y) + d$.

The dimension of a tuple a of elements over a set B is defined as

$$\dim(a/B) := \inf\{\dim(\phi(x)) : \phi \in \text{tp}(a/B)\}.$$

We say that the dimension is

- *additive* if $\dim(a, b/C) = \dim(a/b, C) + \dim(b/C)$ holds for any tuples a and b and for any set C ; and
- *fine* if $\dim(X) = 0$ implies that X is finite.

In this paper we work with a finite-dimensional group with additive and fine dimension. Below we list those properties of such groups which will be used repeatedly throughout the paper. For any further details on (finite-)dimensional groups we refer to [17].

Fact 2.7 (Lascar equality). *Let G be a finite-dimensional group with additive and fine dimension and $H \leq G$ be a definable subgroup. Then $\dim(G) = \dim(H) + \dim(G/H)$.*

Proof. The map $G \rightarrow G/H$ has fibres of dimension $\dim(H)$. Hence, by the additivity, we get $\dim(G) = \dim(H) + \dim(G/H)$ [17, Remark 1.4]. \square

Lemma 2.8. *Let G be a finite-dimensional group with additive and fine dimension. Then G satisfies the chain condition on interscutions of uniformly definable subgroups, icc^0 :*

icc^0 : Given a family \mathcal{H} of uniformly definable subgroups of G , there is $m < \omega$ so that there is no sequence $\{H_i : i \leq m\} \subset \mathcal{H}$ with $|\bigcap_{i < j} H_i : \bigcap_{i \leq j} H_i| \geq m$ for all $j \leq m$.

(If \mathcal{H} is the family of centralisers of elements, this is called the $\tilde{\mathfrak{M}}_c$ -condition.)

Proof. Assume to the contrary that G does not satisfy the icc^0 . Then, by compactness, the condition

$$|\bigcap_{i < m} H_i : \bigcap_{i \leq m} H_i| \geq m \text{ for all } m < \omega$$

is a consistent first-order condition on the parameters needed to define the groups $(H_i : i < \omega)$. So there is a family $\{H_i : i \leq \dim(G)\} \subset \mathcal{H}$ so that

$$G > H_0 > H_1 \cap H_0 > \dots > \bigcap_{i \leq \dim(G)} H_i$$

is a descending chain of length $\dim(G) + 2$ of definable subgroups of G , each having infinite index in its predecessor, contradicting Fact 2.7. \square

Corollary 2.9. *Let G be an icc^0 group. If $H \leq G$ is definable, then $\tilde{N}_G(H)$ is also definable.*

Proof. We need to show that the commensurability is uniform for conjugates of H . So suppose not. Let m be given by the icc^0 -condition, and choose a maximal chain

$$H > H \cap H^{g_0} > \dots > H \cap \bigcap_{i=0}^k H^{g_i}$$

with every group of finite index at least m in its predecessor. Then $k < m$. Now, if there were $g \in G$ so that $\infty > |H : H \cap H^g| > m |H : H \cap \bigcap_{i=0}^k H^{g_i}|$, then

$$|H \cap \bigcap_{i=0}^k H^{g_i} : H \cap \bigcap_{i=0}^k H^{g_i} \cap H^g| > m,$$

contradicting the maximality of k . Since $|H^g : H^g \cap H| = |H : H \cap H^{g^{-1}}|$, this finishes the proof. \square

Fact 2.10 ([17, Lemma 4.4]). *Let G be an $\tilde{\mathfrak{M}}_c$ -group. If $H \leq G$ is definable, then $\tilde{C}_G(H)$ is also definable.*

Recall that if in a group G there is a finite bound on the size of its conjugacy classes, then the derived subgroup G' is finite. In particular, in any $\tilde{\mathfrak{M}}_c$ -group the almost centre $\tilde{Z}(G)$ is finite-by-abelian.

Fact 2.11 (Hempel [9, Theorem 4.7.]). *Let G be an $\tilde{\mathfrak{M}}_c$ -group. Then the Fitting subgroup $\text{Fitt}(G)$ exists and is therefore definable.*

For an arbitrary group H denote by $R(H)$ the group generated by all soluble normal subgroups of H ; note that if $R(H)$ is soluble then $R(G) = \text{Rad}(H)$. To prove our results we need the following fact.

Theorem 2.12. *Let G be a finite-dimensional group with additive and fine dimension. Then the soluble radical $\text{Rad}(G)$ exists and is therefore definable.*

Proof. We prove the claim by induction on the dimension. If $\dim(G) = 0$ then G is finite and $R(G) = \text{Rad}(G)$. Assume that the claim holds when $\dim(G) = n - 1$ and let $\dim(G) = n$. Now, by Fact 2.11, the Fitting subgroup $\text{Fitt}(G)$ exists. Note that $R(\text{Fitt}(G)) = \text{Rad}(\text{Fitt}(G)) = \text{Fitt}(G)$. If $\text{Fitt}(G)$ is infinite then $\dim(G/\text{Fitt}(G)) < n$ and $R(G/\text{Fitt}(G)) = \text{Rad}(G/\text{Fitt}(G))$ by the inductive

assumption. Since $\text{Fitt}(G)$ is a normal and soluble subgroup of G , we get that $R(G) = \text{Rad}(G)$.

Now, we may assume that $\text{Fitt}(G)$ is finite. Set $G_0 := C_G(\text{Fitt}(G))$ and $Z := Z(\text{Fitt}(G))$. Then G_0 is a normal finite index subgroup of G . If $AZ/Z \leq G_0/Z$ is an abelian normal subgroup then $AZ \leq \text{Fitt}(G_0) \leq \text{Fitt}(G) \cap G_0 = Z$. So G_0/Z has no non-trivial abelian normal subgroups and hence $R(G_0) = \text{Rad}(G_0) = Z$. Now $R(G) \cap G_0 = R(G_0)$ and since $|R(G) : R(G) \cap G_0| \leq |G : G_0|$ is finite, $R(G)$ is finite, whence soluble. \square

2.4. Pseudofinite groups. We denote by \mathcal{L}_{gr} the language of groups.

Definition 2.13. A *pseudofinite* group is an infinite group which satisfies every first-order sentence of \mathcal{L}_{gr} that is true of all finite groups.

Note that by Łoś' Theorem, an infinite group (resp. \mathcal{L} -structure) is pseudofinite if and only if it is elementarily equivalent to a non-principal ultraproduct of finite groups (\mathcal{L} -structures) of increasing orders, see [12].

Typical examples of pseudofinite groups are torsion-free divisible abelian groups, infinite extraspecial groups of exponent $p > 2$ and rank n and (twisted) Chevalley groups over pseudofinite fields (recall that *using* CFSG, Wilson [18] proved that a simple pseudofinite group is elementarily equivalent to a (twisted) Chevalley group $X(F)$ over a pseudofinite field F . Further, as explained in [12], results by Point [13] allow one to conclude that a simple group is pseudofinite if and only if it is elementarily equivalent to $X(F)$; Ryten [14, Chapter 5] has generalised this by showing that 'elementarily equivalent' can be strengthened to 'isomorphic'. Note that replacing \equiv by \cong is done without further use of CFSG).

We will use the following theorem repeatedly, often without referring to it.

Theorem 2.14 (Wagner [17, Corollary 4.14]). *Let G be a finite-dimensional pseudofinite group with additive and fine dimension. If $\dim(G) = 1$, then G is finite-by-abelian-by-finite.*

Remark 2.15. Let G be a pseudofinite finite-dimensional group with additive and fine dimension. If $\dim(G) = 1$, then, by the above, $\tilde{Z}(G)$ is of finite index in G and the commutator group $\tilde{Z}(G)'$ is finite. So G contains a definable infinite (finite and characteristic)-by-abelian characteristic subgroup.

One of the earliest (1955) results on centralisers of involutions in finite groups states that, up to isomorphism, there are only a finite number of finite simple groups with a given centraliser of an involution. This was shown by Brauer and Fowler [3]. The following result (which, like the result by Brauer and Fowler, is independent of CFSG) provides an alternative proof for this fact (see [10, Corollary 2.5]):

Fact 2.16 (Hempel-Palacín [10, Lemma 2.3]). *Let G be a pseudofinite group and assume that $\tilde{Z}(G) = 1$. Then the centraliser $C_G(i)$ is infinite for any $i \in I(G)$.*

3. PROOFS OF OUR RESULTS

Before starting our proofs, we still need some definitions and observations.

A group G is called *definably simple* if it has no proper non-trivial definable normal subgroup. It is *semisimple* if it has no non-trivial abelian normal subgroup. Note that if A is a non-trivial normal abelian subgroup and $1 \neq a \in A$, then $Z(C_G(a^G))$ is a definable non-trivial normal abelian subgroup, so semisimplicity is

the same as definable semisimplicity. Clearly, a semisimple group has trivial soluble radical. Moreover, if G is semisimple and N is a normal subgroup of finite index, then N is also semisimple: If A were a normal abelian subgroup of N , the finitely many G -conjugates of A would generate a nilpotent subgroup normal in G , whose centre would be a non-trivial abelian normal subgroup of G .

The *socle* $\text{Soc}(G)$ of a finite group G is the subgroup generated by all minimal normal non-trivial subgroups.

A definable subgroup H of G is *strongly embedded* if H has involutions, but $H \cap H^g$ does not for any $g \in G \setminus H$. By Loś' Theorem, if $G = \prod_{i \in I} G_i / \mathcal{U}$ is pseudofinite, then H is strongly embedded if and only if H_i is strongly embedded in G_i for almost all i .

3.1. Useful lemmas. We shall call an infinite group *almost simple* if it is not abelian-by-finite and has no definable normal subgroup of infinite index.

Lemma 3.1. *An almost simple icc^0 -group G is semisimple. Moreover, $\tilde{Z}(G)$ is trivial, and if H is an infinite definable subgroup of infinite index, then $\tilde{N}_G(H) < G$. If H is a definable infinite soluble-by-finite subgroup, then $\tilde{N}_G(H)$ has infinite index in G .*

Proof. If A is an abelian normal subgroup of G and $1 \neq a \in A$, then $Z(C_G(a^G))$ is a definable abelian normal subgroup. It has either finite index or smaller dimension than G , contradicting almost simplicity in both cases.

Now $\tilde{Z}(G)$ is finite normal in G , whence trivial, so $\tilde{Z}(G)$ is abelian normal, and must be trivial as well.

If an infinite definable infinite index subgroup H is almost normal in G , then by Theorem 2.4 it is commensurable with a definable normal subgroup N of G , again contradicting almost simplicity as commensurability implies that N is also of infinite index in G .

Finally suppose H is definable infinite and soluble-by-finite. If $\tilde{N}_G(H)$ is of finite index in G then

$$N = \bigcap_{g \in G} \tilde{N}_G(H)^g$$

is a normal subgroup of finite index in G . By Theorem 2.4 there is \bar{H} normal in $\tilde{N}_G(H)$ and commensurable with the soluble-by-finite group H . Now \bar{H} is again soluble-by-finite, as is $\bar{H} \cap N$. The product of the finitely many G -conjugates of $\bar{H} \cap N$ is a definable subgroup S which is again soluble-by-finite, and normal in G . Hence $\text{Rad}(S)$ is a non-trivial definable normal soluble subgroup of G , contradicting semisimplicity. \square

We shall in particular apply Lemma 3.1 when G is finite-dimensional and $\dim(H) = 1$.

Lemma 3.2. *Let $G \equiv \prod_{i \in I} G_i / \mathcal{U}$ be a non-abelian semisimple pseudofinite finite-dimensional group with additive and fine dimension. Then G has involutions. Assume further that $\dim(G) \leq 3$ and that at least one of the following holds.*

- (1) $m_2(G) \leq 2$.
- (2) G contains a definable strongly embedded subgroup.

Then $\dim(G) = 3$ and G has a definable normal subgroup of finite index isomorphic to $\text{PSL}(F)$ where F is a pseudofinite field.

Proof. Assume first that $G = \prod_{i \in I} G_i / \mathcal{U}$. We shall argue modulo \mathcal{U} even if only implicitly.

By semisimplicity $C_{G_i}(\text{Soc}(G_i)) = 1$; thus $G_i \hookrightarrow \text{Aut}(\text{Soc}(G_i))$. Now the non-trivial socle $\text{Soc}(G_i)$ of the finite group G_i is a direct product of non-abelian simple finite groups; the Feit-Thompson Theorem then implies that $\text{Soc}(G_i)$ has involutions for almost all i , and so does G .

Assume now that $\dim(G) \leq 3$. If (1) holds, then, by Theorem 2.2, each simple factor in the direct product $\text{Soc}(G_i)$ is of 2-rank 2 and hence $\text{Soc}(G_i)$ is simple. Further, again by Theorem 2.2, $\text{Soc}(G_i) = X(q_i)$ where $X \in \{\text{PSL}_2, \text{PSL}_3, \text{PSU}_3\}$ and q_i is odd. If (1) does not hold and (2) holds, then Theorem 2.3 implies that G_i has a normal subgroup $X(q_i)$ where $X \in \{\text{PSL}_2, \text{PSU}_3, \text{Sz}\}$ and q_i is a power of 2. Note that since $|G_i| \leq |\text{Aut}(X(q_i))|$, the q_i grow without a bound when i varies. Therefore, in either of the two cases, by [5], there is $x_i \in X(q_i)$ so that $X(q_i) = x_i^{X(q_i)} x_i^{X(q_i)}$. (The result in [5] states that Thompson's Conjecture holds for finite simple (twisted) Chevalley groups $X(q)$, provided that $q > 8$. This does not use CFSG. Note however that while Thompson's Conjecture is known to hold for 'almost all' finite simple groups (see [5, Introduction]), this more general result uses a case-by-case analysis provided by CFSG.) Since $X(q_i) \trianglelefteq G_i$ we see that $X(q_i) = x_i^{G_i} x_i^{G_i}$, and there is a definable normal subgroup $N = \prod_{i \in I} X(q_i) / \mathcal{U} = X(F)$, where $F = \prod_{i \in I} \mathbb{F}_{q_i} / \mathcal{U}$ is a pseudofinite field and $X \in \{\text{PSL}_2, \text{PSL}_3, \text{PSU}_3, \text{Sz}\}$. As $\dim(N) \leq 3$, we have $\dim(N) = 3$ and $X = \text{PSL}_2$ by [6, Proposition 6.1].

Now the field F and the linear structure $X(F)$ are definable in G . It follows that the result also holds for any group elementarily equivalent to G (see [14, Chapter 5]). \square

Lemma 3.3. *Let G be a pseudofinite finite-dimensional group with additive and fine dimension. Assume that $\dim(G) \leq 3$, and that there is a definable subgroup $H < G$ with $\dim(H) = 2$ which is not almost normalised by G . Then either $\tilde{Z}(H)$ is finite, or $\dim(\tilde{Z}(G)) \geq 1$.*

Proof. Consider $x \in G \setminus \tilde{N}_G(H)$. Then $\dim(H \cap H^x) \geq 1$, as otherwise $\dim(HH^x) = 4 > \dim(G)$. Also $\dim(H \cap H^x) < 2$ by the choice of x . So $\dim(H \cap H^x) = 1$ and $H \cap H^x$ is finite-by-abelian-by-finite. Thus $\tilde{Z}(H \cap H^x) \simeq H \cap H^x$.

Now assume that $\tilde{Z}(H)$ is infinite. Since $H \lesssim \tilde{C}_H(\tilde{Z}(H))$ (Fact 2.6), we have $L = \tilde{Z}(H \cap H^x) \cap \tilde{C}_H(\tilde{Z}(H)) \simeq \tilde{Z}(H \cap H^x)$. Let $y \in L$. Then $C_H(y)$ almost contains both $\tilde{Z}(H)$ and $\tilde{Z}(H \cap H^x)$. Now, if $\tilde{Z}(H) \cap \tilde{Z}(H \cap H^x)$ was finite then $\dim(C_H(y)) = 2$ and $y \in \tilde{Z}(H)$. Thus $\tilde{Z}(H \cap H^x) \lesssim L \lesssim \tilde{Z}(H)$. If $\tilde{Z}(H) \cap \tilde{Z}(H \cap H^x)$ is infinite, then also $\tilde{Z}(H \cap H^x) \lesssim \tilde{Z}(H)$ since $\dim(\tilde{Z}(H \cap H^x)) = 1$. Similarly $\tilde{Z}(H \cap H^x) \lesssim \tilde{Z}(H^x)$. So $\tilde{Z}(H \cap H^x) \lesssim \tilde{Z}(H) \cap \tilde{Z}(H^x) \leq \tilde{Z}(G)$, and $\dim(\tilde{Z}(G)) \geq 1$. \square

3.2. The identification lemma. In this section we prove Lemma 3.6 which plays a key role in the proof of both Theorem 1.2 and Theorem 1.1. We use similar arguments in dimension 2 and 3, but in dimension 2 this will lead to a contradiction, while in dimension 3 we shall identify $\text{PSL}_2(F)$. However, the case of dimension 3 is more complicated due to the possible existence of definable proper subgroups of dimension 2. So we start with the analysis of such subgroups.

Lemma 3.4. *Let G be an almost simple pseudofinite finite-dimensional group with additive and fine dimension. Assume that $\dim(G) = 3$ and that all definable subgroups of dimension 2 are soluble-by-finite. If L is a definable subgroup of dimension 2, then:*

- (1) $B = \tilde{N}_G(L)$ is a maximal definable subgroup of dimension 2. Moreover, $B = N_G(B) = \tilde{N}_G(B)$; if $Z = \tilde{Z}(B)$ then Z is finite, and B/Z is a Frobenius group with Frobenius kernel U/Z and Frobenius complement T/Z , where $U = \tilde{C}_G(U)$ and $T = \tilde{C}_B(T)$ have dimension 1 and are finite-by-abelian.
- (2) There is a pseudofinite field F such that $U/Z \cong F^+$ and T/Z embeds into T^\times as a subgroup of finite index. In particular T/Z is abelian, and T has only finitely many elements of any given order.
- (3) If $C = U_C T_C$ is another (maximal) subgroup of dimension 2 and $Z_C := \tilde{Z}(C) \neq 1$, then $U^g \not\leq T_C$ for any $g \in G$.
- (4) If $g \in G \setminus B$ then $U \cap U^g = 1$. If B contains involutions, then either B is strongly embedded, or there is an involution in $T \setminus Z$ and no involution in $U \setminus Z$.
- (5) Suppose $Z \neq 1$. If $g \in G \setminus B$ then $B \cap B^g$ is a finite index subgroup of T^u for some $u \in U$. Moreover, if $x \in B$ with $\dim(C_G(x)) = 2$ then $x \in Z$.

Proof. Put $N = \tilde{N}_G(L)$. Then $\dim(N) = 2$ by Lemma 3.1. If K is any group commensurable with H then $K \leq \tilde{N}_G(K) = \tilde{N}_G(L)$. This shows maximality of N , as well as $N = \tilde{N}_G(N) = N_G(N)$.

Since $\tilde{Z}(G)$ is trivial and N is not almost normal in G , Lemma 3.3 yields that $Z := \tilde{Z}(N)$ is finite. But $\tilde{Z}(N)$ contains all finite normal subgroups of N . As N is soluble-by-finite by assumption, there is $a \in N \setminus \tilde{Z}(N)$ such that $\langle a^N \rangle / \tilde{Z}(N)$ is abelian; it is infinite since $a \notin \tilde{Z}(N)$. Then $C_N(a^N / \tilde{Z}(N))$ is an infinite definable subgroup containing a^N ; if it were of dimension 2 then $a^N \subseteq Z$, a contradiction. Thus $\dim(C_N(a^N / \tilde{Z}(N))) = 1$. Define:

$$(1) U := \{x \in N : C_N(x) \succeq C_N(a^N / \tilde{Z}(N))\} = \tilde{C}_N(C_N(a^N / \tilde{Z}(N))).$$

Clearly U only depends on the commensurability class of $C_N(a^N / \tilde{Z}(N))$. If $\dim(U) = 2$, then $N \lesssim U = \tilde{C}_N(C_N(a^N / \tilde{Z}(N)))$, so $C_N(a^N / \tilde{Z}(N)) \lesssim \tilde{C}_N(N) = Z$ which is finite, a contradiction. Hence $U \simeq C_N(a^N / \tilde{Z}(N))$, and $U = \tilde{Z}(U)$ is finite-by-abelian. Clearly $N \leq N_G(U) \leq \tilde{N}_G(U)$, and we have equality by maximality and almost simplicity of G .

Now N/U is 1-dimensional, whence finite-by-abelian-by-finite. For any $x_0 \in \tilde{C}_N(N/U) \setminus U$ the orbit $(x_0 U)^N$ in N/U is finite. Then $\dim(x_0^N) = \dim(U) = 1$ and $C_N(x_0)$ has dimension 1. Note that $x_0 \notin U$ implies finiteness of $C_N(x_0) \cap U$. So we can define the following:

$$(2) T := \{x \in N : C_N(x) \succeq C_N(x_0)\} = \tilde{C}_N(C_N(x_0)).$$

$$(3) B := UT.$$

Note that $\tilde{N}_G(B) = \tilde{N}_G(N) = N$, as B has finite index in N . As $\dim(C_N(x_0)U) = 2$, the index of $C_N(x_0)U$ in N is finite, so

$$Z = \tilde{Z}(N) = \tilde{C}_N(C_N(x_0)U) = \tilde{C}_N(C_N(x_0)) \cap \tilde{C}_N(C_N(a^N / \tilde{Z}(N))) = T \cap U.$$

Thus $\dim(T) = 1$; it follows that $T \simeq C_N(x_0)$ and $T = \tilde{C}_N(T)$ is finite-by-abelian. Moreover, any N -conjugate of T which is commensurable with T must be equal to T , and $\tilde{N}_N(T) = N_N(T)$. For any $g \in N \setminus N_N(T)$ we have $T \cap T^g = Z$, since $C_N(h)$ has dimension 2 for any $h \in T \cap T^g$. Moreover $N_B(T) = N_U(T) \times_Z T$; it cannot have dimension 2 as otherwise it would be finite-by-abelian-by-finite, contradicting finiteness of $\tilde{Z}(N)$. Hence $N_U(T)$ is finite and $N_U(T) \leq \tilde{C}_U(T) = Z$. Thus $N_B(T) = T$.

Put $A := U/Z$, $H := T/Z$ and $\hat{B} := B/Z$. Clearly $A \cap H = 1$ and $\tilde{Z}(\hat{B}) = 1$, so $\hat{B} = A \rtimes H$ and $H \cap H^g = 1$ for any $g \in \hat{B} \setminus H$. This means that \hat{B} is a pseudofinite Frobenius group with a definable Frobenius kernel A . Therefore, by the structure of finite Frobenius groups, the conjugates of H cover $\hat{B} \setminus A$, and for any definable subgroup K of \hat{B} with $K \cap A = 1$ and $AK = \hat{B}$ there is $a \in A$ with $K = H^a$.

Now, let $M \leq N$ be definable of dimension 1 with $M \cap U$ finite. As above $T_1 := \tilde{C}_N(M)$ is of dimension 1, commensurable with M , satisfies $T_1 \cap U = Z$, and $N_{UT_1}(T_1) = T_1$. Then $\hat{B}_1 = UT_1/Z \leq \hat{B}$ is also a Frobenius group with Frobenius kernel A . Since $H_1 = H \cap \hat{B}_1$ satisfies $H_1 \cap A = 1$ and $AH_1 = \hat{B}_1$, there is $u \in U$ with $H_1^u = T_1/Z$. Thus $T^u \simeq (T \cap B_1)^u = T_1 \simeq M$, so $T_1 = \tilde{C}_N(M) = \tilde{C}_N(T^u) = \tilde{C}_N(T)^u = T^u$. In particular, any definable subgroup of N of dimension 1 is commensurable either with U or with a U -conjugate of T . Moreover, for any $g \in N$ there is $u \in U$ with $T^g = T^u$, so $N = U N_N(T)$. This shows (1), once we know that $N_N(T) = T$ (which is proven below), since that implies $N = B$.

We say that a subset X of A is *almost H -invariant* if there is some finite subset $H_0 \subseteq H$ so that for any $h \in H$ we have $X^h \subseteq \bigcup_{h_0 \in H_0} X^{h_0}$. Any finite almost H -invariant subset of A is almost central in \hat{B} and thus trivial. So, as A is (finite and characteristic)-by-abelian, it is abelian.

Now, let $R := \text{End}_H(A)$ be the ring of endomorphisms of A generated by H . Let $r \in R \setminus \{0\}$. We show that r is an automorphism of A : Since H is finite-by-abelian, both $\ker(r)$ and $\text{im}(r)$ are almost H -invariant definable subgroups of A and hence either finite, and thus trivial, or finite index subgroups of the 1-dimensional group A . Now $\ker(r)$ cannot have finite index in A as otherwise $\text{im}(r)$ is finite, whence trivial, so we would get $r = 0$. So any $r \in R \setminus \{0\}$ is injective; since a definable injective map from a pseudofinite group to itself must be surjective, R acts on A by automorphisms. In particular it is invertible. By [17, Proposition 3.6 or Corollary 3.10] the field of fractions F of R is an interpretable skew field; $A \cong F^+$ and $H \hookrightarrow F^\times$. Since $\dim(H) = \dim(A) = \dim(F)$, the image of H has finite index in F^\times . As F is pseudofinite and any finite skew field is commutative by Wedderburn's Little Theorem, F^\times is commutative and contains only finitely many elements in each finite order; clearly the same holds for H .

Finally, suppose $g \in N_N(T)$. Then g induces an automorphism σ of F of finite order via conjugation: If $r \in R$ is given by $x \mapsto \prod_i x^{h_i}$, then $\sigma(r)$ is given by $x \mapsto \prod_i x^{h_i^g}$, and extends to F by $\sigma(r^{-1}r') = \sigma(r)^{-1}\sigma(r')$. Let F_0 be the fixed field of σ . Then $[F : F_0] = o(\sigma)$ and $1 = \dim(F) = [F : F_0] \dim(F_0) \geq [F : F_0]$, as $\dim(F_0) \geq 1$. Hence σ fixes F , so $g \in C_N(T/Z) \leq \tilde{C}_N(T) = T$. This shows (2), as well as $N_N(T) = T$ and $N = B$. So we have proven (1) and (2).

From now on, we think of U and T as the ‘unipotent’ and the ‘semisimple’ parts of B , respectively.

In order to show (3), assume that there is $g \in G$ so that $U^g \simeq T_C$. Then $\tilde{N}_G(T_C) = \tilde{N}_G(U^g) = B^g$. Therefore $T_C \leq B^g$; since T_C is finite-by-abelian, $T_C \leq \{x \in B^g : U^g \lesssim C_{B^g}(x)\} = U^g$. By Part (2), T_C/Z_C contains only finitely many elements of any given order. But T_C has finite index in U^g , and U^g is finite-by-abelian. Therefore U^g only contains finitely many elements of any given order.

Let $1 \neq z \in Z_C$ be of order ℓ and put $\Omega_\ell = \{t \in U^g : o(t) = \ell\}$. Then Ω_ℓ is finite and contains z . So $C_G(\Omega_\ell) \simeq N_G(\Omega_\ell) \geq B^g$. This implies $z \in Z_C \cap Z^g$, whence $C \simeq C_G(z) \simeq B^g$ and $U^g \simeq T_C = \tilde{N}_C(T_C) \simeq \tilde{N}_{B^g}(U^g) = B^g$, a contradiction.

To show (4), consider $g \in G \setminus B$. Suppose that there is non-trivial $y \in U \cap U^g$, and put $C = C_G(y)$. Then C almost contains U and U^g and must be of dimension 2. Clearly $y \in Z_C = \tilde{Z}(C)$, so neither U nor U^g can be commensurable with T_C (the semisimple part of C) by Part (3). Hence both are commensurable with U_C , so $U \simeq U^g$ and $g \in \tilde{N}_G(U) = B$, a contradiction. In particular $Z \cap Z^g \leq U \cap U^g = 1$.

Next, note if $\text{char}(F) \neq 2$ then $I(U \setminus Z) = \emptyset$, and if $\text{char}(F) = 2$ then $I(T \setminus Z) = \emptyset$. In the latter case, since the U -conjugates of $T \setminus Z$ cover $B \setminus U$, we have $I(B) = I(U)$, so $I(B \cap B^g) \leq U \cap U^g = 1$, and B is strongly embedded.

To show (5), assume $Z \neq 1$ and consider $g \in G \setminus B$. As $\dim(B) = \dim(B^g) = 2$, $\dim(G) = 3$ and $B \not\cong B^g$, we have $\dim(B \cap B^g) = 1$. Hence $B \cap B^g$ is commensurable with U or a U -conjugate of T , and with U^g or a U^g -conjugate of T^g . Clearly $U \simeq B \cap B^g \simeq U^g$ is impossible, as $g \notin \tilde{N}_G(U) = B$. Part (3) excludes the mixed case, so $B \cap B^g \simeq T^u$ for some $u \in U$. Hence

$$B \cap B^g \leq \tilde{N}_B(B \cap B^g) = \tilde{N}_B(T^u) = \tilde{N}_B(T)^u = T^u.$$

Finally, consider $x \in B$ with $\dim(C_G(x)) = 2$. Note that $x \in \tilde{Z}(C_G(x))$, which is finite by Lemma 3.3. So the order $o(x)$ is finite. If $x \in U$ then $x \in U \cap U^c$ for any $c \in C_G(x)$, so $c \in B$ and $C_G(x) \leq B$. Thus $x \in Z$.

Now suppose $x \in T \setminus Z$. Then $T \simeq C_B(x)$; by Parts (1) and (3) we have $C_G(x) \simeq U_x T_x$, where the unipotent part U_x is not almost contained in B and the semisimple part T_x is commensurable to T . For any $u_x \in C_{U_x}(x) \setminus B$ we have $x \in B \cap B^{u_x}$ and $x \in (T \cap T^{u_x}) \setminus (Z \cup Z^{u_x})$. But the U -conjugates of $T \setminus Z$ are disjoint and there are $u, u' \in U$ with $T^u \supseteq B \cap B^{u_x} \leq T^{u' u_x}$. It follows that $u, u' \in N_U(T) = Z$, so $T \simeq T^{u_x}$ and $u_x \in \tilde{N}_G(T)$. Thus $\tilde{N}_{U_x}(T_x) = \tilde{N}_{U_x}(T)$ is infinite, a contradiction. As the conjugates of T cover $B \setminus U$, we are done.

This finishes the proof of the lemma. \square

From now on, we use the following notation.

Notation. Let G be a group. For $g \in G$ we denote $C_g := \tilde{C}_G(C_G(g))$. Note that if G is finite-dimensional with additive and fine dimension and if $\dim(C_G(g)) = 1$, then $C_G(g) \lesssim C_g$ by Theorem 2.14.

Lemma 3.5. Let G be an almost simple pseudofinite finite-dimensional group with additive and fine dimension. Assume that $\dim(G) \leq 3$. If $g \in G$ satisfies $\dim(C_G(g)) = 1$, then $C_g \simeq C_G(g)$ and $\tilde{N}_G(C_g) = N_G(C_g)$. Moreover, if $i, j \in N_G(C_g) \setminus C_g$ are distinct involutions with $\dim(C_G(i)) = \dim(C_G(j)) = 1$, then $j \in iC_g$.

Proof. Note that $C_G(g)$ is finite-by-abelian-by-finite, so $C_G(g) \lesssim C_g$. Suppose that C_g had finite index in G . Since $C_g \lesssim \tilde{C}_G(C_G(g))$, by Fact 2.6 we have $C_G(g) \lesssim \tilde{C}_G(C_g) = \tilde{Z}(G)$, contradicting triviviality of $\tilde{Z}(G)$. So $1 \leq \dim(C_g) < \dim(G) \leq 3$.

Suppose $\dim(C_g) = 2$. As $C_G(g) \lesssim \tilde{Z}(C_g)$ by Fact 2.6, we get $\tilde{N}_G(C_g) = G$ by Lemma 3.3, contradicting almost normality. So $\dim(C_g) = 1$ and $C_g \simeq C_G(g)$, whence $\tilde{N}_G(C_g) = N_G(C_g)$.

Now, let $i, j \in N_G(C_g) \setminus C_g$ be distinct involutions with

$$\dim(C_G(i)) = \dim(C_G(j)) = 1.$$

Let $Z \leq A$ be definable characteristic subgroups of C_g so that Z is finite and A/Z is infinite and abelian (exist by Remark 2.15). By the above and our assumptions, $\dim(C_g) = 1$, $\dim(C_G(i)) = 1$ and $C_G(i) \not\leq C_G(g)$; hence $C_A(i)$ is finite, as is $C_{A/Z}(i)$. Then the subgroup of A/Z inverted by iZ , say B_i/Z , contains $[r, A]Z/Z$ and thus is 1-dimensional. Similar observations can be made for the group B_j/Z of elements of A/Z inverted by jZ ; so $(B_i \cap B_j)/Z$ is infinite. Now ijZ centralises $(B_i \cap B_j)/Z$; as Z is finite there is $z \in Z$ such that $ijx_\ell j i = x_\ell z$ for infinitely many $x_\ell \in B_i \cap B_j$. Hence there is $1 \neq x_0 \in B_i \cap B_j$ so that ij centralises $x_\ell x_0^{-1}$ and $C_{B_i \cap B_j}(ij)$ is infinite. Thus $\dim(C_{C_G(g)}(ij)) = 1$ and $ij \in C_g$, yielding $j \in iC_g$. \square

Lemma 3.6. *Let G be an almost simple pseudofinite finite-dimensional group with additive and fine dimension. Assume that $\dim(G) \leq 3$, and that, if $\dim(G) = 3$, all definable subgroups of dimension 2 are soluble-by-finite. Then $\dim(G) = 3$ and G has a definable subgroup of finite index isomorphic to $\mathrm{PSL}_2(F)$, where F is a pseudofinite field.*

Proof. Note first that G is semisimple and has trivial almost centre by Lemma 3.1; moreover it has dimension at least 2. If $\dim(G) = 2$ then all centralisers of non-trivial elements have dimension at most 1 by Theorem 2.4. Note that we may assume that $m_2(G) > 2$, that G has no strongly embedded definable subgroup and that G has involutions (Lemma 3.2). Further, by Fact 2.16 all involutions of G have infinite centralisers.

Claim 1. There is an involution $i \in G$ with $\dim(C_G(i)) = 1$, and for any such involution there is an involution $k \in N_G(C_i) \setminus C_i$. Moreover, if there is an involution $j \in G$ with $\dim(C_G(j)) = 2$, then we can choose $k \in C_G(i) \setminus C_i$.

Proof of claim. Note first that if $i \in G$ is an involution with $\dim(C_G(i)) = 1$, then $\dim(\tilde{N}_G(C_i)) = 1$. For otherwise, setting $H = \tilde{N}_G(C_i)$ and $B = \tilde{N}_G(H)$ we have $\dim(H) = \dim(B) = 2$. As $\dim(\tilde{N}_B(C_i)) = 2$ we must have $C_i \simeq U$ in the notation of Lemma 3.4, so $i \in C_i = \tilde{C}_G(C_i) = \tilde{C}_G(U) = U$. Moreover $\dim(C_G(i)) = 1$ implies $i \notin \tilde{Z}(B)$. But then B is strongly embedded by Lemma 3.4(4), a contradiction.

Next, suppose that $\dim(C_G(i)) = 1$ for all involutions $i \in G$, and fix some involution i . Put $H = \tilde{N}_G(C_G(i))$. Then $C_G(i) \lesssim C_i \leq H$ are all commensurable, so $H = \tilde{N}_G(H)$. Since H is not strongly embedded, there is $g \in G \setminus H$ and an involution $k \in H \cap H^g$; if $k \in C_i \cap C_i^g$ then $C_G(k)$ almost contains C_i and C_i^g ; as $g \notin H = \tilde{N}_G(C_i)$ the intersection $C_i \cap C_i^g$ is finite, so $C_G(k)$ has dimension 2, a contradiction. If $k \notin C_i$ we are done, otherwise $k \notin C_i^g$ and $k^{g^{-1}} \notin C_i$ is as required.

Now suppose that $j \in G$ is an involution with $\dim(C_G(j)) = 2$. Note that this implies $\dim(G) = 3$, as $\tilde{Z}(G) = 1$. We take $L = C_G(j)$ in Lemma 3.4, so $B = \tilde{N}_G(L)$ and $Z = \tilde{Z}(B)$. Then $j \in Z \leq B$, so by Lemma 3.4(4) there is an

involution $i \in T \setminus Z$, and we have $\dim(C_G(i)) = 1$ by Lemma 3.4(5). This shows the first part of the claim.

Let $i \in G$ be any involution with $\dim(C_G(i)) = 1$ and put $N = \tilde{N}_G(C_i)$. Consider the conjugacy class j^G , which has dimension 1. Recall that in any finite group two involutions are either conjugate or there is an involution commuting with both of them; by Loś's Theorem this also holds in G . Clearly i and j^g are not conjugate for any $g \in G$, so for each $g \in G$ there is an involution $k_g \in C_G(i) \cap C_G(j^g)$.

We want to show that $k_g \notin C_i$ for some $g \in G$. So suppose otherwise. Now either $\dim(C_G(k_1)) = 2$, and we can take $B = \tilde{N}_G(C_G(k_1))$. Then $i \in C_G(k_1) \leq B$, so $C_B(i)$ is infinite, as is $C_i \cap B$. If $C_i \cap B \simeq U$, then $U = \tilde{C}_G(U) = \tilde{C}_G(C_i) = C_i$ contains an involution i with $\dim(C_G(i)) = 1$, so $i \notin Z$ and B is strongly embedded, a contradiction. Hence $C_i \cap B = \tilde{C}_G(C_i) \cap B = \tilde{C}_G(T) \cap B = T$.

Otherwise $\dim(C_G(k_1)) = 1$, so $C_i = C_{k_1}$ and we can take $B = \tilde{N}_G(C_G(j))$. The same argument (with k_1 and j instead of i and k_1) yields that $C_i \cap B = T$. But C_i is finite-by-abelian, T has finite index in C_i and T contains only finitely many involutions (Lemma 3.4(2)). Hence C_i only contains finitely many involutions, and there is an involution $k \in C_i$ which commutes with infinitely many distinct conjugates j^g .

Fix $j' \in C_G(k) \cap j^G$, put $B' = \tilde{N}_G(C_G(j'))$ and $Z' = \tilde{Z}(B') \ni j'$, and let U' and T' denote the unipotent and a semisimple subgroups of B' . Then $k \in C_G(j') \leq B'$. Put $X = \{x \in G : k \in B'^x\}$. Then $X \cap j^G \supseteq C_G(k) \cap j^G$ is infinite; since $B' \cap j^G \subseteq Z'$ is finite, there is $x_0 \in X \setminus B'$, and we may choose $T' \leq B'$ so that $k \in B' \cap B'^{x_0} \leq T'$ (recall that $\tilde{N}_G(B') = B'$). As $Z' \cap B' \cap B'^{x_0} = 1$ we have $k \notin Z'$, and T' is the unique U' -conjugate of B' containing k . Hence T' does not depend on x_0 . But for any $x \in X$ there is $u \in U'$ such that $B' \cap B'^x \leq T'^u$, so clearly $T'^u = T'$. As k is in some U'^x -conjugate of T'^x , there is $u' \in U'$ with $k \in T'^{u'x} \leq B'^x$, so $T' \geq B' \cap B'^x \leq T'^{u'x}$, whence $T' \simeq T'^{u'x}$ and $u'x \in \tilde{N}_G(T')$. Moreover, if $x \in C_G(k)$ then we can take $u' = 1$. Hence $C_G(k) \cap j^G \subseteq \tilde{N}_G(T')$. It follows that $\tilde{N}_G(T')$ cannot have dimension 2, since this would imply $C_G(k) \cap j^G \subseteq \tilde{Z}(\tilde{N}_G(T'))$ which is finite. Therefore $\dim(\tilde{N}_G(T')) = 1$.

Put $B^\sharp = \bigcap_{g \in \tilde{N}_G(T')} (T')^g$, a normal subgroup of $\tilde{N}_G(T')$ of finite index contained in all B'^x for $x \in X$. Now $k \notin Z'$, so $C_G(k) \simeq T'$ and $C_G(k) \lesssim B^\sharp$. Hence there is fixed $j'^{x'} \in C_G(k)$ and infinitely many distinct $j'^{x_\ell} \in C_G(k)$ so that $j'^{x'} = j'^{x_\ell} t_\ell$ for some $t_\ell \in B^\sharp$. Now $j'^{x'} \in B'^{x'}$, whence $k \in C_G(j'^{x'}) \leq B'^{x'}$ and $x' \in X$. Also $t_\ell \in B^\sharp \leq B'^{x'}$, whence $j'^{x_\ell} \in B'^{x'}$. So $j'^{x_\ell} \in \tilde{Z}(B'^{x'})$ by Lemma 3.4(5), contradicting finiteness of $\tilde{Z}(B'^{x'})$. This shows that there is an involution $k \in C_G(i) \setminus C_i$ commuting with i . \square

We now fix an involution $i \in G$ with $\dim(C_G(i)) = 1$ and put $N = N_G(C_i)$.

Claim 2. An involution $k \in N \setminus C_i$ centralises a unique involution $\ell \in C_i$, and $\dim(C_G(k)) = \dim(C_G(\ell)) = 1$.

Proof of claim. In a finite group N , if C is normal in N and both C and $N \setminus C$ contain involutions, then for any involution $k \in N \setminus C$ a Sylow 2-subgroup P containing k must intersect C non-trivially, as all Sylow 2-subgroups are conjugate. Then $Z(P) \cap C$ is non-trivial by nilpotency of P , and C contains an involution j

commuting with k . By Łoś' Theorem, the same holds for $N > C_i$, and there is an involution $\ell \in C_i$ commuting with k .

Next we show $\dim(C_G(k)) = \dim(C_G(\ell)) = 1$. This is clear if $\dim(G) = 2$, so suppose $\dim(G) = 3$.

Assume first that $\dim(C_G(k)) = 2$. Then $\dim(\tilde{N}_G(C_G(k))) = 2$ by almost simplicity of G . If $\dim(C_G(\ell)) = 2$, then, as $\ell \in C_G(k) \leq \tilde{N}_G(C_G(k))$, we get $\ell \in \tilde{Z}(N_G(C_G(k)))$. Hence $C_G(i) \lesssim C_G(\ell) \simeq C_G(k)$, contradicting $k \notin C_i$. If $\dim(C_G(\ell)) = 1$, then $C_G(i) \simeq C_G(\ell) \lesssim C_G(k)$, again a contradiction. Thus $\dim(C_G(k)) = 1$.

Next, suppose $\dim(C_G(\ell)) = 2$ and put $B'' = \tilde{N}_G(C_G(\ell))$. Then $i, k \in C_\ell \leq B''$, so $C_G(i)$ and $C_G(k)$ are commensurable to two different B'' -conjugates of the semisimple part T'' of B'' . As $k \notin \tilde{Z}(B'')$, this implies $k \notin N$, since $\tilde{N}_{B''}(T'') = T''$. Hence $\dim(C_G(\ell)) = 1$.

Finally, we show uniqueness of ℓ . So assume that k fixes another involution $\ell' \in C_i$. Then $\dim(C_G(\ell')) = 1$; by Lemma 3.5 we have $\ell' \in \ell C_k$, so $\ell \ell' \in C_k$ and $C_G(k) \lesssim C_G(\ell \ell')$. Moreover $C_G(i) \lesssim C_G(\ell \ell')$. So $C_G(\ell \ell')$ is 2-dimensional. If $\dim(G) = 2$ this is a contradiction; if $\dim(G) = 3$ we put $B''' = \tilde{N}_G(C_G(\ell \ell'))$. Then $C_G(k)$ and $C_G(\ell) \simeq C_G(i)$ are commensurable to two different B''' -conjugates of the semisimple part T''' of B''' , which again implies $k \notin N$. Thus k centralises a unique involution in C_i . \square

Claim 3. Any two involutions x, y in C_i with $\dim(C_G(x)) = \dim(C_G(y)) = 1$ commute.

Proof of claim. Let $k_x, k_y \in N \setminus C_i$ be involutions commuting with x, y respectively. Now C_i is normal in N , so y and k_x are not N -conjugate, and there is an involution $z \in N$ commuting with both. If $z \in C_i$ then, as it is centralised by k_x , we have $z = x$, so x and y commute. If $z \in N \setminus C_i$ then either $z = k_x$, or zk_x is an involution in C_i centralised by k_x , whence $zk_x = x$. In either case z commutes with x and with y , whence $x = y$. So x and y commute. \square

Claim 4. If $x, y, z \in C_i$ are three distinct involutions with

$$\dim(C_G(x)) = \dim(C_G(y)) = \dim(C_G(z)) = 1,$$

then $xyz = 1$.

Proof of claim. Let $k \in N \setminus C_i$ be an involution commuting with x . Then y and y^k are distinct commuting involutions in C_i . Therefore $yy^k = (yk)^2$ is an involution in C_i fixed by k , whence $(yk)^2 = x$. Similarly, $(zk)^2 = x$, and $k^y = k^z = xk$. So yz is an involution in C_i commuting with k , whence $yz = x$ and $xyz = 1$. \square

Claim 5. $\dim(C_G(j)) = 1$ for any involution $j \in G$.

Proof of claim. Suppose, as in the proof of Claim 1, that j is an involution with $\dim(C_G(j)) = 2$. Note that we may assume that $i, j \in B$ commute (where B is again as in Claim 1, whence as in Lemma 3.4): If not, then there is a third involution $z \in C_G(j) \cap C_G(i)$. If $z \in \tilde{Z}(B)$ we can replace j by z . Otherwise $z \in T \setminus Z$, and we can replace i by z . Note that $ij \in T \setminus \tilde{Z}(B)$, so $\dim(C_G(ij)) = 1$.

Consider $k \in C_G(i) \setminus C_i$ (exists by Claim 1). Then $B \cap B^k \simeq T$, and $\ell = j^k \in (C_i \cap C_G(i)) \setminus B$. Then i, ij and $i\ell$ are distinct involutions in C_i with $\dim(C_G(i)) =$

$\dim(C_G(ij)) = \dim(C_G(i\ell)) = 1$. Then, by the above, $ijjil = 1$ whence $j = i\ell$, but $C_G(j)$ is 2-dimensional, a contradiction. \square

Now, since $m_2(G) > 2$ there are three pairwise commuting distinct involutions x, y, z with $xyz \neq 1$. By Claim 4 and 5 we cannot have $C_x = C_y = C_z$. If $C_x = C_y \neq C_z$, then $z \in N_G(C_x) \setminus C_x$ centralises two distinct involutions $x, y \in C_x$, contradicting Claim 2. If $C_x \neq C_y \neq C_z \neq C_x$, then $y, z \in N_G(C_x)$, so $y \in zC_x$ by Lemma 3.5, and $yz \in C_x$ is an involution centralised by y , whence $yz = x$ again by Claim 2, contradicting our hypothesis.

This finishes the proof. \square

3.3. Proof of Theorem 1.2.

Theorem 1.2. *Let G be a finite-dimensional pseudofinite group with additive and fine dimension. If $\dim(G) = 2$ then G is soluble-by-finite. Moreover, if G is not finite-by-abelian-by-finite, then there is a definable normal subgroup N of G with $\dim(N) = 1$.*

Proof. Suppose first that G has a definable normal subgroup N of dimension 1. Then N is finite-by-abelian-by-finite. In fact $\tilde{Z}(N)'$ is finite normal in G , and $C_N(\tilde{Z}(N)')$ is a normal 2-nilpotent subgroup of G of dimension 1. The quotient $\overline{G} = G/C_N(\tilde{Z}(N)')$ is also of dimension 1, whence finite-by-abelian-by-finite. Put $\overline{S} = C_{\overline{G}}(\tilde{Z}(\overline{G})')$. Then \overline{S} is 2-nilpotent of finite index in \overline{G} , and its preimage S is soluble of finite index in G . Thus G is soluble-by-finite and has a definable normal subgroup of dimension 1.

Now suppose that there is no definable normal subgroup of dimension 1. Every finite normal subgroup of G is contained in $\tilde{Z}(G)$, which is finite-by-abelian.

If $\dim(\tilde{Z}(G)) = 2$ we are done and G is finite-by-abelian-by-finite. So suppose that $\tilde{Z}(G)$ is finite. Then $\overline{G} = G/\tilde{Z}(G)$ has no non-trivial finite normal subgroup. Replacing G by \overline{G} , we may thus assume that G is almost simple. So we may apply Lemma 3.6, which implies that G has dimension 3, a contradiction. \square

3.4. Proof of Theorem 1.1. In fact, we shall prove something stronger:

Theorem 3.7 (Without CFSG). *Let G be a pseudofinite finite-dimensional group with additive and fine dimension. If $\dim(G) = 3$, then either G is soluble-by-finite, or $\tilde{Z}(G)$ is finite and $G/\tilde{Z}(G)$ has a definable subgroup of finite index isomorphic to $\mathrm{PSL}_2(F)$ where F is a pseudofinite field.*

Proof. Suppose first that G has a definable normal subgroup N with $1 \leq \dim(N) \leq 2$. Then $1 \leq \dim(G/N) \leq 2$, so by Theorem 1.2 both N and G/N are soluble-by-finite. Put $S = \mathrm{Rad}(N)$, which is characteristic in N and hence normal in G . Then $C_G(N/S)$ is a definable normal soluble subgroup of finite index in G .

So suppose that G has no definable normal subgroup of dimension 1 or 2. If $\dim(\tilde{Z}(G)) = 3$, then G is finite-by-abelian-by-finite. Otherwise $\tilde{Z}(G)$ is finite and contains all normal finite subgroups. Replacing G by $G/\tilde{Z}(G)$, we may assume that G is almost simple. By Theorem 1.2 all definable subgroups of dimension 2 are soluble-by-finite. So we finish by Lemma 3.6. \square

REFERENCES

- [1] James Ax. The Elementary Theory of Finite Fields. *Annals of Mathematics*, 88(2):239–271, 1968.
- [2] Helmut Bender. Transitive gruppen gerader ordnung, in denen jede involuotion genau einen punkt festläbetat. *Journal of Algebra*, 17(4):527–554, 1971.
- [3] Richard Brauer and Kimberley A. Fowler. On Groups of Even Order. *Annals of Mathematics*, 62(2):565–583, 1955.
- [4] Zoé Chatzidakis, Lou van den Dries, and Angus Macintyre. Definable sets over finite fields. *J. Reine Angew. Math.*, 427:107–135, 1992.
- [5] Erich W. Ellers and Nikolai Gordeev. On the conjectures of J. Thompson and O. Ore. *Transactions of the American Mathematical Society*, 350(9):3657–3641, 1998.
- [6] Richard Elwes, Eric Jaligot, Dugald Macpherson, and Mark Ryten. Groups in supersimple and pseudofinite theories. *Proceedings of the London Mathematical Society*, 103(6):1049–1082, 2011.
- [7] Walter Feit and John G. Thompson. Solvability of groups of odd order. *Pacific Journal of Mathematics*, 13(3):775–787, 1963.
- [8] Daniel Gorenstein. *The Classification of Finite Simple Groups*. The University Series in Mathematics. Springer, 1983.
- [9] Nadja Hempel. Almost group theory. *J. Algebra*, 556:169–224, 2020.
- [10] Nadja Hempel and Daniel Palacín. Centralizers in pseudo-finite groups. *Journal of Algebra*, 569:258–275, 2021.
- [11] Abderezak Ould Houcine. A remark on the definability of the fitting subgroup and the soluble radical. *Mathematical Logic Quarterly*, 59(1-2):62–65, 2013.
- [12] Dugald Macpherson. Model theory of finite and pseudofinite groups. *Archive for Mathematical Logic*, 57:159–184, 2018.
- [13] Françoise Point. Ultraproducts and Chevalley groups. *Archive for Mathematical Logic*, 38(6):355–372, 1999.
- [14] Mark J. Ryten. *Model Theory of Finite Difference Fields and Simple Groups*. PhD thesis, University of Leeds, 2007.
- [15] Frank O. Wagner. *Simple Theories*. Kluwer Academic Publishers, Dordrecht, NL, 2000.
- [16] Frank O. Wagner. Pseudofinite \tilde{m}_c -groups. <https://hal.archives-ouvertes.fr/hal01235178v1/document>, 2015.
- [17] Frank O. Wagner. Dimensional groups and fields. *The Journal of Symbolic Logic*, 85(3):918–936, 2020.
- [18] John S. Wilson. On simple pseudofinite groups. *Journal of the London Mathematical Society*, 51(2):471–490, 1995.

UNIVERSITY OF HELSINKI

Email address: ukarhumaki@gmail.com

UNIVERSITÉ CLAUDE BERNARD LYON 1; CNRS; INSTITUT CAMILLE JORDAN

Email address: wagner@math.univ-lyon1.fr