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Gérard Ben Arous, Cédric Gerbelot, Vanessa Piccolo. Langevin dynamics for high-dimensional optimization: the case of multi-spiked tensor PCA. 2024. hal-04671256v2

HAL Id: hal-04671256 https://hal.science/hal-04671256v2

Preprint submitted on 20 Dec 2024

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LANGEVIN DYNAMICS FOR HIGH-DIMENSIONAL OPTIMIZATION: THE CASE OF MULTI-SPIKED TENSOR PCA

GÉRARD BEN AROUS¹, CÉDRIC GERBELOT^{1,2}, AND VANESSA PICCOLO²

ABSTRACT. We study nonconvex optimization in high dimensions through Langevin dynamics, focusing on the multi-spiked tensor PCA problem. This tensor estimation problem involves recovering r hidden signal vectors (spikes) from noisy Gaussian tensor observations using maximum likelihood estimation. We study the number of samples required for Langevin dynamics to efficiently recover the spikes and determine the necessary separation condition on the signal-to-noise ratios (SNRs) for exact recovery, distinguishing the cases $p \geq 3$ and p = 2, where p denotes the order of the tensor. In particular, we show that the sample complexity required for recovering the spike associated with the largest SNR matches the well-known algorithmic threshold for the single-spike case, while this threshold degrades when recovering all r spikes. As a key step, we provide a detailed characterization of the trajectory and interactions of low-dimensional projections that capture the high-dimensional dynamics.

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Date: December 20, 2024.

²⁰²⁰ Mathematics Subject Classification. 68Q87, 62F30, 60G44, 60H30.

Key words and phrases. High-dimensional optimization, Multi-spiked tensor PCA, Langevin dynamics, Exact recovery, Subspace recovery.

1. INTRODUCTION

Understanding optimization and sampling in high dimensions is a central question across applied mathematics, theoretical physics, and computer science. The recent empirical success of deep learning methods in data science [39, 17] has challenged our understanding of the efficiency of gradient-based algorithms in navigating nonconvex landscapes. Such landscapes are often random functions composed of a signal component and a noise component. The noise typically introduces an exponential number of critical points (see, e.g., [2, 13, 48, 46]), potentially trapping gradient-based algorithms on exponential time scales. However, when the signal is sufficiently strong, a *topology trivialization transition* occurs, allowing optimization algorithms to efficiently converge within short time frames. One compelling reason for the surprising effectiveness of gradient-based optimization methods in high-dimensional, nonconvex problems stems from the inherent structure of many real-world data: the relevant information often lies on a manifold of much lower dimension than the ambient space. This underlying structure reduces the effective complexity of the problem, allowing models to sidestep the *curse of dimensionality* – the challenges that arise when dealing with high-dimensional data. Quantitatively describing how optimization dynamics exploit these properties to reach optimal regions of the landscape within polynomial time is therefore of fundamental importance.

In this context, tensor principal component analysis (PCA) emerges as a paradigmatic example to address these challenges. Tensor PCA involves estimating an unknown vector on the N-dimensional sphere from noisy Gaussian observations of a corresponding tensor. The problem was first introduced for matrices by Johnstone [36] and later extended to tensors by Richard and Montanari [47], where tensors of order 2 correspond to matrices. Recent work by the first author, in collaboration with Gheissari and Jagannath [10, 11], showed that for a broad class of problems, the high-dimensional dynamics of stochastic gradient descent (SGD) can be effectively reduced to low-dimensional, autonomous effective dynamics governed by a set of summary statistics. This builds on earlier work by the same authors [7, 8], where they studied Langevin dynamics for the single-spike tensor PCA problem and showed that the high-dimensional dynamics reduce to a one-dimensional dynamical system.

Building on these advances, our work extends the analysis to the *multi-spike setting*, providing deeper insights into the interactions between multiple summary statistics. Specifically, we consider the task of recovering r hidden orthogonal signal vectors lying on the sphere $\mathbb{S}^{N-1}(\sqrt{N})$ from noisy Gaussian tensor observations. We focus on recovering these vectors by computing the maximum likelihood estimator (MLE) and analyzing the sample complexity required for Langevin dynamics to achieve successful recovery, i.e., reaching the global minimizer of the loss landscape. To simplify the analysis, we assume the hidden vectors are orthogonal, reducing the MLE optimization problem to the Stiefel manifold, the set of $N \times r$ matrices whose columns vectors are orthogonal with norm \sqrt{N} . To study Langevin dynamics in this context, we employ stochastic analysis techniques originally developed for spin glass models. In particular, we extend the bounding flows method introduced in [7], which was used for the single-spike case [8]. Here, we extend this approach to include diffusions on the Stiefel manifold, providing a complete characterization of the resulting r^2 -dimensional dynamical systems. In particular, we observe a richer phenomenology in the multi-spike case compared to the single-spike scenario, largely due to the interaction of the trajectories of the correlations between the estimators and the spikes during recovery. These interactions introduce complex dynamical behavior, as the estimators compete to align with the hidden signal directions, influencing the recovery process.

Finally, tensor PCA is a prototypical example of statistical-to-computational gaps in high-dimensional estimation. These gaps arise when the statistical threshold for solving a problem is significantly lower than the computational threshold. The statistical properties of MLE are well understood [35, 20, 21]: beyond an order-one critical threshold, the MLE can effectively distinguish the signal from noise. However, computing the MLE is computationally challenging in practice. To address this challenge, various algorithmic approaches have been extensively studied to determine the sample complexity required for efficient recovery. Spectral and Sum-of-Squares methods have been shown to recover the hidden vector with sample complexity scaling as $N^{\frac{p-2}{2}}$ [31, 30, 38, 54, 12]. In contrast, local methods – such as gradient flow, Langevin dynamics [8], stochastic gradient descent (SGD) [9], tensor power iteration [34], and approximate message passing [47] – achieve recovery at the threshold of N^{p-2} .

We formalize the multi-rank spiked covariance and tensor model as follows. Let $p \ge 2$ and $r \ge 1$ be fixed integers. Suppose that we are given M i.i.d. observations \mathbf{Y}^{ℓ} of a rank-r p-tensor on \mathbb{R}^N of the form

$$\boldsymbol{Y}^{\ell} = N^{-\frac{p-1}{2}} \left(\sum_{i=1}^{r} \lambda_i \boldsymbol{v}_i^{\otimes p} + \boldsymbol{W}^{\ell} \right), \tag{1.1}$$

where each \boldsymbol{W}^{ℓ} is an independent sample of a Gaussian *p*-tensor with i.i.d. entries $W_{i_1,\ldots,i_p}^{\ell} \sim \mathcal{N}(0,1)$. The parameters $\lambda_1 \geq \ldots \geq \lambda_r \geq 0$ are the signal-to-noise ratios (SNRs) and $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r$ are unknown, orthogonal vectors lying on the *N*-dimensional unit sphere of radius \sqrt{N} , denoted by $\mathbb{S}^{N-1}(\sqrt{N})$. The task is to estimate the unknown signal vectors $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r \in \mathbb{S}^{N-1}(\sqrt{N})$ via empirical risk minimization. In particular, we choose the most common method for Gaussian variables, namely maximum likelihood estimation. This involves minimizing the empirical risk $\hat{\mathcal{R}}_{N,r}(\boldsymbol{X})$ defined by

$$\hat{\mathcal{R}}_{N,r}(\boldsymbol{X}) = \frac{1}{M} \sum_{\ell=1}^{M} \mathcal{L}_{N,r}(\boldsymbol{X}; \boldsymbol{Y}^{\ell}), \qquad (1.2)$$

over the constraint set $\mathcal{M}_{N,r} := \{ \boldsymbol{X} \in \mathbb{R}^{N \times r} : \boldsymbol{X}^\top \boldsymbol{X} = N \boldsymbol{I}_r \}$, known as the *normalized Stiefel manifold*. The loss function $\mathcal{L}_{N,r} : \operatorname{St}(N,r) \times (\mathbb{R}^N)^{\otimes p} \to \mathbb{R}$ is chosen as the log-likelihood function and results in

$$\mathcal{L}_{N,r}(\boldsymbol{X};\boldsymbol{Y}^{\ell}) = -\sum_{i=1}^{r} \lambda_{i} \langle \boldsymbol{Y}^{\ell}, \boldsymbol{x}_{i}^{\otimes p} \rangle = -N^{-\frac{p-1}{2}} \sum_{i=1}^{r} \lambda_{i} \langle \boldsymbol{W}^{\ell}, \boldsymbol{x}_{i}^{\otimes p} \rangle - \sum_{1 \leq i,j \leq r} N \lambda_{i} \lambda_{j} \left(m_{ij}^{(N)}(\boldsymbol{X}) \right)^{p}, \quad (1.3)$$

where $m_{ij}^{(N)}(\boldsymbol{X}) = N^{-\frac{1}{2}} \langle \boldsymbol{v}_i, \boldsymbol{x}_j \rangle$ denotes the *correlation* between \boldsymbol{v}_i and \boldsymbol{x}_j .

1.1. Our contributions

This article is part of a series of three companion papers that study high-dimensional optimization in the context of multi-spiked tensor PCA. In this work, we focus on Langevin dynamics, providing crucial results on which the other two papers, which address gradient flow [5] and online SGD [6], rely.

For $p \geq 3$, we determine the sample complexity and the conditions that the SNRs must satisfy for Langevin dynamics to achieve *exact recovery* of the r spikes. Specifically, we establish the conditions under which $\langle \boldsymbol{v}_i, \boldsymbol{x}_i \rangle = 1 - o(1)$ with high probability in the large-N limit (see Definition 1.2). In our companion papers [6, 5], we extend this analysis to online SGD and gradient flow in the more general case where no assumptions are made about the separation of SNRs. In that case, we determine the number of samples required to recover a *permutation of the spikes*, as exact recovery is not always guaranteed. Recovering a permutation of the spikes means achieving $\langle \boldsymbol{v}_{\sigma(i)}, \boldsymbol{x}_i \rangle = 1 - o(1)$, for some permutation $\sigma \in S_r$ that we characterize, with high probability in the large-N limit. A key challenge in analyzing Langevin dynamics stems from controlling the Brownian motion, which complicates the analysis when the SNRs are not sufficiently separated. We remark that the same separation condition on the SNRs is also required to ensure exact recovery (i.e. the identity permutation) in the cases of gradient flow and online SGD. For p = 2, we consider two distinct cases: one where the SNRs are sufficiently separated, and another where all SNRs are equal.

The estimates derived here for Langevin dynamics play a crucial role in the results of our companion papers [6, 5]. Specifically, the proofs presented in this work provide a complete description of the continuous-time population dynamics. In [6, 5], the analysis is carried out up to a point where it relies directly on our findings for Langevin dynamics.

Finally, we rigorously characterize the *sequential elimination phenomenon* (see Definition 1.8), which governs the recovery process. This interesting phenomenon provides new insights into how the hidden signal vectors are progressively recovered.

1.2. Model and dynamics

To enable the application of Langevin dynamics, we introduce the Hamiltonian by scaling the empirical risk by \sqrt{M} . Let $H: \mathcal{M}_{N,r} \to \mathbb{R}$ denote the Hamiltonian given by

$$H(\boldsymbol{X}) = H_0(\boldsymbol{X}) - \sum_{1 \le i,j \le r} N \sqrt{M} \lambda_i \lambda_j \left(m_{ij}^{(N)}(\boldsymbol{X}) \right)^p, \qquad (1.4)$$

where

$$H_0(\boldsymbol{X}) = N^{-\frac{p-1}{2}} \sum_{i=1}^r \lambda_i \langle \boldsymbol{W}, \boldsymbol{x}_i^{\otimes p} \rangle.$$
(1.5)

The noise Hamiltonian H_0 is a centered Gaussian process with covariance of the form

$$\mathbb{E}\left[H_0(\boldsymbol{X})H_0(\boldsymbol{Y})\right] = \frac{1}{N}\sum_{1 \le i,j \le r} \lambda_i \lambda_j \left(\frac{\langle \boldsymbol{x}_i, \boldsymbol{y}_j \rangle}{N}\right)^p$$

Note that minimizing the empirical risk (1.2) is equivalent, in law, to minimizing the Hamiltonian (1.4), while reducing the running time by a factor of \sqrt{M} .

We define Langevin dynamics with Hamiltonian H as follows. For $\beta \in (0, \infty)$ and an initial point $\mathbf{X}_0 \in \mathcal{M}_{N,r}$, we let $\mathbf{X}_t^{\beta} \in \mathcal{M}_{N,r}$ solve the stochastic differential equation given by

$$d\boldsymbol{X}_{t}^{\beta} = \sqrt{2}d\boldsymbol{B}_{t} - \beta\nabla H(\boldsymbol{X}_{t}^{\beta})dt, \qquad (1.6)$$

where $\boldsymbol{B}_t \in \mathbb{R}^{N \times r}$ stands for the Brownian motion on $\mathcal{M}_{N,r}$ and ∇ denotes the Riemannian gradient on $\mathcal{M}_{N,r}$. Specifically, for any function $f: \mathcal{M}_{N,r} \to \mathbb{R}$, the Riemannian gradient is given by

$$\nabla f(\boldsymbol{X}) = \hat{\nabla} f(\boldsymbol{X}) - \frac{1}{2N} \boldsymbol{X} \left(\boldsymbol{X}^{\top} \hat{\nabla} f(\boldsymbol{X}) + \hat{\nabla} f(\boldsymbol{X})^{\top} \boldsymbol{X} \right),$$
(1.7)

where $\hat{\nabla}$ denotes the Euclidean gradient. The generator L_{β} of this process is given by

$$L_{\beta} = \Delta - \beta \langle \nabla H, \hat{\nabla} \cdot \rangle, \tag{1.8}$$

where Δ denotes the Laplace-Beltrami operator on $\mathcal{M}_{N,r}$ (endowed with the Euclidean metric) and is given according to [55] by

$$\Delta f(\boldsymbol{X}) = \hat{\Delta} f(\boldsymbol{X}) - \frac{N-1}{N} \operatorname{Tr}(\boldsymbol{X}^{\top} \hat{\nabla} f(\boldsymbol{X})) - \frac{1}{N} \sum_{i,k=1}^{r} \sum_{j,\ell=1}^{N} (x_i)_j (x_k)_\ell \frac{\partial^2}{\partial (x_i)_j \partial (x_k)_\ell} f(\boldsymbol{X}), \quad (1.9)$$

for any function $f: \mathcal{M}_{N,r} \to \mathbb{R}$. Here, $\hat{\Delta}$ denotes the Euclidean Laplace operator. Similarly, we denote by $L_{0,\beta}$ the generator of Langevin dynamics related to the noise Hamiltonian H_0 , i.e.,

$$L_{0,\beta} = \Delta - \beta \langle \nabla H_0, \hat{\nabla} \cdot \rangle. \tag{1.10}$$

Remark 1.1. Langevin dynamics reduces to gradient flow when $\beta = \infty$. Accordingly, the results under Langevin dynamics presented in this article also hold under gradient flow dynamics.

1.3. Main results

Our main results address the sample complexity required to efficiently recover the unknown orthogonal vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r \in \mathbb{S}^{N-1}(\sqrt{N})$. We assume the SNRs satisy $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$ and are constants of order-1. In this subsection, we present the results in their asymptotic form. A nonasymptotic formulation with explicit constants and convergence rates is provided later in Section 3.

We consider the Langevin dynamics process $(\boldsymbol{X}_{t}^{\beta})_{t\geq 0}$, initialized from the uniform distribution $\mu_{N\times r}$ on the normalized Stiefel manifold $\mathcal{M}_{N,r}$. The probability measure $\mu_{N\times r}$ is the unique probability measure on $\mathcal{M}_{N,r}$ that is invariant under both the left- and right- orthogonal transformations. Sampling from this distribution is straightforward: if $\boldsymbol{Z} \in \mathbb{R}^{N\times r}$ is a Gaussian random matrix filled with i.i.d. standard normal entries, then the matrix $\boldsymbol{Z} \left(\frac{1}{N} \boldsymbol{Z}^{\top} \boldsymbol{Z}\right)^{-1/2}$ is uniformly distributed on $\mathcal{M}_{N,r}$ [22]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space on which the noise *p*-tensor \boldsymbol{W} is defined and let $\mathbb{Q}_{\boldsymbol{X}_{0}}$ denote the law of the Langevin dynamics $\boldsymbol{X}_{t}^{\beta}$, initialized at \boldsymbol{X}_{0} .

More precisely, our results determine both the sample complexity and the separation condition on the SNRs required to ensure *exact recovery* of the spikes. We formally define exact recovery in the context of Langevin dynamics as follows.

Definition 1.2 (Exact recovery). We say that Langevin dynamics *exactly recovers* the r unknown signal vectors with rate ξ if, for every $\varepsilon > 0$ and every $1 \le i \le r$, there exists T_0 such that for every $T \ge T_0$,

$$\lim_{N \to \infty} \int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}_0} \left(\min_{t \in [T_0,T]} m_{ii}^{(N)}(\boldsymbol{X}_t^\beta) \ge 1 - \varepsilon \right) \mathrm{d}\mu_{N \times r} = \xi, \quad \mathbb{P}\text{-a.s.}$$

In contrast, when focusing only on the recovery of the leading spike, we introduce the following definition.

Definition 1.3 (Strong recovery of the leading spike). We say that Langevin dynamics strongly recovers the leading spike v_1 with rate ξ if, for every $\varepsilon > 0$, there exists T_0 such that for every $T \ge T_0$,

$$\lim_{N\to\infty}\int_{\mathcal{M}_{N,r}}\mathbb{Q}_{\boldsymbol{X}_0}\left(\min_{t\in[T_0,T]}m_{11}^{(N)}(\boldsymbol{X}_t^\beta)\geq 1-\varepsilon\right)\mathrm{d}\mu_{N\times r}=\xi,\quad\mathbb{P}\text{-a.s.}$$

We are now in a position to present our main results, starting with the case $p \ge 3$. Our first result concerns the strong recovery of the leading spike.

Theorem 1.4 (Recovery of the leading spike for $p \ge 3$). Fix any $p \ge 3$ and $\beta \in (0, \infty)$. If for every $\eta > 1$,

$$\lambda_1 > c(\eta)\lambda_2$$

for $c(\eta) = C(\eta\sqrt{\log(\eta)})^{p-2} > 0$ and C and absolute constant, and if $M = N^{\alpha}$ with $\alpha > p-2$, then Langevin dynamics strongly recovers the spike v_1 with rate $\xi = 1 - \frac{1}{n}$.

More detail on the separation condition $c(\eta)$ is provided in Proposition 3.5. In particular, to ensure the recovery of the leading spike with probability close to 1, the SNR λ_1 needs to be separated from λ_2 by a large but *N*-independent factor. The sample complexity required for recovering the leading spike v_1 is of order $N^{p-2+\delta}$ for any $\delta > 0$, consistent with the well-known result for the single-spike case [8]. While it is conceivable to reduce the factor N^{δ} to a polylogarithmic factor by improving our proof estimates, this would require substantial additional effort and is left as an open question.

Our second result concerns the exact recovery of all spikes.

Theorem 1.5 (Exact recovery for $p \ge 3$). Fix any $p \ge 3$ and $\beta \in (0, \infty)$. If for every $\eta > 1$ and every $1 \le i \le r - 1$,

$$\lambda_i > c(\eta)\lambda_{i+1},$$

with $c(\eta)$ as in the previous Theorem, and if $M \gtrsim N^{p-1}$, then Langevin dynamics exactly recovers all r spikes with rate $\xi = 1 - \frac{1}{n}$.

To achieve exact recovery of all spikes, the total number of samples M must scale as N^{p-1} , and the SNRs must satisfy the separation condition $c(\eta)$ previously introduced in Theorem 1.4. The discrepancy in sample complexity between the recovery of the first spike and subsequent spikes can be explained as follows. At initialization, the estimator X_0 and the noise tensor W are independent. Under this independence, the typical scale of the noise generator $L_{0,\beta}m_{ij}^{(N)}(X)$ is of order $\frac{1}{\sqrt{N}}$, compared to an upper bound of order 1 without this independence. This reduces the required sample complexity for estimating the signal component of the generator from N^{p-1} to N^{p-2} , provided this initialization property can be effectively leveraged over an extended time frame for recovering the first direction. However, once the first spike is recovered, this advantageous scaling of the noise generator can no longer be exploited by our proof method. As a result, the sample complexity for subsequent spike recovery aligns with the order-one bound of the noise generator. Further details and a formal discussion of this phenomenon are provided in Section 2.

Remark 1.6. At present, we do not have any refutation results to rule out a sample complexity between N^{p-2} and N^{p-1} for exact recovery of all spikes. Therefore, it remains an open question whether the N^{p-1} complexity is optimal. In our companion paper on the online SGD algorithm [6], we achieve the sharper N^{p-2} complexity threshold for the recovery of all directions. The key difference lies in the sampling dynamics: in the online SGD setting, a different sample \mathbf{W}^{ℓ} is used at each step, and this sample is independent of the estimator from the previous iteration. This independence allows the beneficial scaling of the noise generator to be exploited at every time step, thereby achieving the sharp threshold for the sample complexity required to recover directions beyond the first.

Remark 1.7. When the order $p \geq 3$ of the tensor is even, each estimator recovers $\operatorname{sgn}(m_{ii}^{(N)}(X_0))v_i$. This implies that if the correlation at initialization is positive, then x_i recovers v_i ; otherwise, x_i recovers $-v_i$. In contrast, when p is odd, each estimator x_i recovers v_i for all correlations that are positive at initialization. Correlations that are negative at initialization are trapped at the equator and do not grow. The phenomenology of spike recovery is richer than described in Theorem 1.5. Specifically, the correlations $\{m_{ii}^{(N)}(\boldsymbol{X}_{t}^{\beta})\}_{i=1}^{r}$ reach a macroscopic threshold one by one. Once a correlation $m_{ii}^{(N)}$ reaches a sufficiently large threshold, all correlations $m_{ij}^{(N)}$ and $m_{ji}^{(N)}$ for $j \neq i$ begin to decrease and reach a very low level. The fact that the off-diagonal correlations become negligible allows the next diagonal correlation $m_{i+1\,i+1}^{(N)}$ to grow to a macroscopic level. We refer to this phenomenon as *sequential elimination*, which we define as follows.

Definition 1.8 (Sequential elimination). We say that the correlations $\{m_{ii}^{(N)}\}_{i=1}^r$ follow a sequential elimination if there exist r stopping times $T_1 \leq \cdots \leq T_r$ such that for every $i \in [r]$, every $\varepsilon, \varepsilon' > 0$, and $T \geq T_i$,

$$|m_{ii}^{(N)}(\boldsymbol{X}_T^\beta)| \ge 1 - \varepsilon \quad \text{and} \quad |m_{ij}^{(N)}(\boldsymbol{X}_T^\beta)| \le \varepsilon', |m_{ji}^{(N)}(\boldsymbol{X}_T^\beta)| \le \varepsilon' \text{ for every } j \ne i.$$

We have the following result, which serves as a foundation for Theorem 1.5.

Theorem 1.9 (Sequential recovery of the spikes). If $M \gtrsim N^{p-1}$, then the correlations $\{m_{ii}^{(N)}(\boldsymbol{X}_t^{\beta})\}_{i=1}^r$ follow a sequential elimination with \mathbb{P} -probability 1 in the large-N limit.

We now consider the case p = 2. In this setting, we obtain similar results to those for $p \ge 3$; however, the separation condition among the SNRs is significantly less restrictive, as an order-1 factor suffices for exact recovery. Our first main result for p = 2 shows that Langevin dynamics strongly recovers the leading spike with a sample complexity of order N^{δ} for any $\delta > 0$.

Theorem 1.10 (Recovery of the leading spike for p = 2). Fix p = 2 and any $\beta \in (0, \infty)$. Let $\lambda_1 = \lambda_2(1 + \kappa_1)$ for an order-1 constant $\kappa_1 > 0$. If $M = N^{\delta}$ for any $\delta > 0$, then Langevin dynamics strongly recovers the spike v_1 with rate $\xi = 1$.

Compared to Theorem 1.4 for $p \ge 3$, Theorem 1.10 shows that the separation condition required for the SNRs to ensure strong recovery of the leading spike is an order-1 factor. Exact recovery of all spikes is possible, as soon as the SNRs are separated by order-1 factors and the number of samples scales as N^{ξ} , where $\xi \in (0, 1)$ depends on the ratio between the signal sizes.

Theorem 1.11 (Exact recovery for p = 2). Fix p = 2 and any $\beta \in (0, \infty)$. Let $\lambda_i = \lambda_{i+1}(1 + \kappa_i)$ for every $1 \le i \le r - 1$ and order-1 constants $\kappa_i > 0$. If $M \ge N^{1 - \frac{\lambda_r^2}{\lambda_1^2} + \delta}$ for any $\delta > 0$, then Langevin dynamics exactly recovers all r spikes with rate $\xi = 1$.

As with $p \ge 3$, for p = 2, we also observe a difference in sample complexity between recovering the first spike and the subsequent ones. However, unlike the results for $p \ge 3$, once the first direction is recovered, the subsequent correlations $m_{ij}^{(N)}$ scale up to $N^{\frac{\delta_{ij}}{2}}$, where $\delta_{ij} \in (0,1)$ depends on the ratio between the SNRs. This scaling can then be exploited to show that the sample complexity required to recover the directions beyond the first one degrades only by a factor δ_{ij} rather than by a factor 1. For a more detailed explanation, we direct the reader to Section 2. Unlike Theorem 1.5, for p = 2, exact recovery of all spikes is possible as soon as the SNRs differ by order-1 factors.

For p = 2, we also study the Langevin dynamics in the special case where the signal sizes are all equal. In this scenario, the Hamiltonian H becomes invariant under both right and left rotations. As a result, the problem shifts from recovering each individual spike to recovering the subspace spanned by v_1, \ldots, v_r . In particular, we study the evolution of the eigenvalues $\theta_1, \ldots, \theta_r$ of the matrix $\boldsymbol{G} = \boldsymbol{M}^\top \boldsymbol{M}$, where $\boldsymbol{M} = (m_{ij}^{(N)}(\boldsymbol{X}))_{1 \leq i,j \leq r}$. This is due to the fact that

$$\|\boldsymbol{X}\boldsymbol{X}^{\top} - \boldsymbol{V}\boldsymbol{V}^{\top}\|_{\mathrm{F}} = 2(r - \mathrm{Tr}(\boldsymbol{G})),$$

meaning that finding an estimator $X \in \mathcal{M}_{N,r}$ such that $\|XX^{\top} - VV^{\top}\|_{\mathrm{F}} = o(1)$ is equivalent to ensuring that the eigenvalues of G converge to 1 - o(1) with high probability.

The following result provides the number of samples required for recovery of the correct subspace.

Theorem 1.12 (Subspace recovery). Fix any $\beta \in (0, \infty)$. Let p = 2 and $\lambda_1 = \cdots = \lambda_r$. If $M = N^{\delta}$ for any $\delta > 0$, then Langevin dynamics recovers the subspace spanned by the r spikes, that is, for every $\varepsilon > 0$, there exists T_0 such that for every $T \ge T_0$,

$$\lim_{N\to\infty}\int_{\mathcal{M}_{N,r}}\mathbb{Q}_{\boldsymbol{X}_0}\left(\min_{t\in[T_0,T]}\theta_{\min}(\boldsymbol{X}_t^\beta)\geq 1-\varepsilon\right)\mathrm{d}\mu_{N\times r}=1,\quad\mathbb{P}\text{-a.s.}$$

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There is a third regime of SNRs that remains unexplored in this paper as well as in our companion article [6] on online SGD. This regime concerns SNRs that are separated by a sufficiently small ε -factor, where ε may depend on N. The study of this regime is left for future work.

1.4. Related works

The multi-spiked tensor PCA problem belongs to a broader class of models where maximum likelihood estimation leads to a loss function that comprises both a noise term and a signal term. In many problems of statistical inference and learning, the deterministic high-dimensional dynamics of the signal part can be studied through the evolution of the *sufficient statistics* of the problem, which evolve autonomously. The simplest example of this fact is the single-spiked spherical tensor PCA model. In this case, the sufficient statistic is the correlation $m = \frac{1}{N} \langle \boldsymbol{v}, \boldsymbol{x} \rangle$, whose evolution under population gradient flow reduces to the one-dimensional ODE $\dot{m} \propto m^{p-1}(1-m^2)$ [8]. The notion of dimensionality reduction in certain random dynamical systems arising from gradient-based optimization has been recently formalized under generic conditions in [10, 11]. This framework introduces autonomous quantities known as summary statistics, of which sufficient statistics are a subset. The evolution of summary statistics under gradient-based dynamics poses several challenges. First, the effects of Gaussian noise (and Brownian motion in the case of Langevin dynamics) must be controlled over sufficiently long timescales to ensure that the events of interest, such as achieving a nontrivial correlation with the hidden signal, can occur. Second, the system of autonomous equations governing the evolution of the summary statistics must be simple enough to allow for precise quantitative analysis. Finally, the projection of the dynamics onto the space of summary statistics should retain as much of the regularity and structure of the original high-dimensional system as possible. In statistical physics, low-dimensional quantities that facilitate the study of a random high-dimensional system in closed form are commonly referred to as order parameters, and we will use this terminology for the following discussion.

In statistical physics of disordered systems, classical examples of signal-plus-noise Hamiltonians are planted versions of spin glass Hamiltonians [44, 56]. In this context, the out-of-equilibrium dynamics of the order parameters are typically studied using low-dimensional integro-differential equations involving two-time autocorrelation functions. This approach, originally proposed in [52, 53, 23, 24, 18] to model aging phenomena in spin glasses, is commonly referred to as *dynamical mean-field theory* (DMFT). Recently, DMFT has been adapted in the statistical physics of learning to analyze Langevin dynamics in high-dimensional estimation problems. Examples include perceptron models [1], mixed matrix-tensor estimation [50, 51, 49], Gaussian mixture classification [42], and phase retrieval [43]. In these works, the resulting integro-differential equations are solved numerically to obtain phase diagrams for the order parameters over order-one timescales. However, a key challenge of this approach lies in the complexity of the integro-differential equations, which often hinders a complete analytical study.

On a rigorous level, DMFT equations have been proven for mean-field spin glasses using large deviations techniques [29, 3, 4] and more recently for inference problems using iterative Gaussian conditioning [19, 28] and random matrix theory in [16, 15, 41] for instances of spiked matrix models. In these works, quantitative statements (e.g. recovery thresholds) are only obtained due to simplifications specific to quadratic models, and extending these results to a multi-spike tensor model appears challenging. Furthermore, DMFT approaches for statistical estimation problems are always proven in the proportional limit of dimension and number of samples one order one timescales, making it difficult to study problems that require polynomial sample complexity or running time. Finally, these proofs also require an $\mathcal{O}(1)$ (akin to a warm start, see the discussion in [41]) initial correlation between the estimator and the hidden signal, whereas this correlation is typically of order $\mathcal{O}(\frac{1}{\sqrt{N}})$ when using uninformative initializations.

In statistical learning, the traditional approach to controlling noisy dynamics involves establishing a single uniform convergence bound (over the input space) for the empirical risk, or gradient thereof, and then directly analyzing the population dynamics under this control. Recently, this approach has been applied to problems involving low-dimensional population dynamics [25, 14]. Its main advantages include simplicity, robustness across different time horizons, sample complexity regimes, and initialization scales, as well as the tractability of the systems of ordinary differential equations (ODEs) derived from the noiseless problem. However, this method has significant limitations, particularly its inability to achieve sharp bounds. Specifically, it fails to leverage the independence between the initial condition and the noise tensor discussed earlier, resulting in a loss of a factor $N^{1(2)}$ in the recovery threshold.

To address these shortcomings, we adopt an alternative approach, first introduced in [7] for Langevin dynamics in spin glasses and subsequently applied to the single-spiked tensor PCA problem in [8], where it was used to prove the conjectured recovery threshold for first-order methods proposed in [47]. In this work, we extend this method to the multi-spiked tensor PCA problem, showing that it overcomes the deficiencies of traditional approaches outlined above. It is worth noting, however, that while results obtained using DMFT are asymptotically exact, the bounding flows method provides inequalities rather than exact characterizations of the studied trajectories. The main difference between our work and [8] lies in the increased complexity of the resulting dynamical system, which involves the summary statistics $\left\{m_{ij}^{(N)}\right\}_{1\leq i,j\leq r}$ as well as the time-dependent bounds for the quantities $\left\{\|L_{0,\beta}m_{ij}^{(N)}\|_{\infty}\right\}_{1\leq i,j\leq r}$.

We conclude this section by noting that the multi-spiked tensor PCA problem also serves as a benchmark for studying nonconvex optimization in high dimensions, a topic of significant importance in machine learning theory. For a detailed discussion of the relevance of our work in this context and additional references on this subject, we refer the reader to the related works section of our companion paper [6].

1.5. Overview

An overview of the paper is given as follows. In Section 2, we outline the proofs of our main results, providing a roadmap for the key arguments. The nonasymptotic formulation of our main results, including explicit constants and convergence rates, is presented in Section 3. Preliminary results necessary for the proofs are discussed in Section 4. The detailed proofs of our main results are provided in Sections 5 through 7. These sections address the cases $p \geq 3$, p = 2 with distinct SNRs, and p = 2 with equal SNRs, respectively. Finally, Appendix A provides concentration properties of the uniform measure on the normalized Stiefel manifold $\mathcal{M}_{N,r}$.

Acknowledgements. G.B. and C.G. acknowledge the support of NSF grant DMS-2134216. V.P. acknowledges the support of the ERC Advanced Grant LDRaM No. 884584.

2. Outline of proofs

In this section, we outline our proofs, focusing on the rank-2 spiked tensor model for simplicity. We first address the recovery of all spikes and then focus on subspace recovery, by assuming p = 2 and $\lambda_1 = \lambda_2$. The core of the proof revolves around the simultaneous control of the noise and signal part of the dynamics. Once the noise is controlled, we describe the population dynamics, which is similar to the discrete time dynamical system obtained in our companion paper [6]. The main difference between these two works lies in the control of the noise : in the case of Langevin dynamics, the drift induced by the Gaussian noise in the Hamiltonian is predictable (as opposed to a martingale noise in SGD), and more involved tools are needed to control the resulting correlations. Moreover, the effect of Brownian motion must be considered, which leads to a condition on the separation of the SNRs.

2.1. Full recovery of spikes

We focus on the evolution of the correlations $m_{ij}^{(N)}(\boldsymbol{X}_t^{\beta})$ under Langevin dynamics. We assume an initial random start with a completely uniformative prior, specifically the uniform distribution on the normalized Stiefel manifold $\mathcal{M}_{N,r}$. As a consequence, all correlations $m_{ij}^{(N)}(\boldsymbol{X}_0)$ have the typical scale of order $N^{-\frac{1}{2}}$ at initialization. For simplicity, we assume that all correlations are positive at initialization in the following discussion. Moreover, we write $m_{ij}(t) \coloneqq m_{ij}^{(N)}(\boldsymbol{X}_t^{\beta})$ to simplify notation slightly. According to (1.6), the evolution equation for $m_{ij}(t)$ under Langevin dynamics is given by

$$dm_{ij}(t) = L_\beta m_{ij}(t)dt + dM_t^{m_{ij}},\tag{2.1}$$

where $M_t^{m_{ij}} = \sqrt{2} \int_0^t \langle \nabla m_{ij}(s), d\mathbf{B}_s \rangle$ denotes the martingale part of the evolution. The generator L_β is given according to (1.8):

$$L_{\beta}m_{ij}(t)dt = \Delta m_{ij}(t) - \beta \langle \nabla H, \hat{\nabla} m_{ij} \rangle.$$

Recall that

$$H(\boldsymbol{X}) = H_0(\boldsymbol{X}) - \sum_{1 \le i,j \le r} N \sqrt{M} \lambda_i \lambda_j \left(m_{ij}(\boldsymbol{X}) \right)^p,$$

where H_0 is given by (1.5). A straightforward computation shows that

$$L_{\beta}m_{ij} = L_{0,\beta}m_{ij} + \beta\sqrt{M}p\lambda_i\lambda_jm_{ij}^{p-1} - \beta\sqrt{M}\frac{p}{2}\sum_{1\le k,\ell\le r}\lambda_km_{kj}m_{k\ell}m_{i\ell}\left(\lambda_jm_{kj}^{p-2} + \lambda_\ell m_{k\ell}^{p-2}\right),$$

where $L_{0,\beta}$ is the generator induced by the noise part (see (1.10)). The population generator consists of two terms: the first term $p\lambda_i\lambda_j m_{ij}^{p-1}$ represents the drift and dominates the dynamics, particularly near initialization, while the second term $\frac{p}{2}\sum_{1\leq k,\ell\leq r}\lambda_k m_{kj}m_{k\ell}m_{i\ell}\left(\lambda_j m_{kj}^{p-2} + \lambda_\ell m_{k\ell}^{p-2}\right)$ is the "correction" term due to the constraint of the estimator \boldsymbol{X} being on the orthogonal manifold. Analogous to the analysis for online SGD, the main challenge lies in balancing the noise and signal terms. At small scales, such as near initialization, the population drift predominates over the correction term and the generator $L_\beta m_{ij}$ can be approximated by

$$L_{\beta}m_{ij}(t) \approx \Delta m_{ij}(t) - \beta \langle \nabla H_0, \hat{\nabla} m_{ij}(t) \rangle + \beta \sqrt{M} p \lambda_i \lambda_j m_{ij}^{p-1}(t).$$

For the correlations m_{ij} to increase, the drift $p\lambda_i\lambda_j m_{ij}^{p-1}$ at initialization needs to be larger than the noise part. At initialization, the correlations m_{ij} typically scale as $N^{-\frac{1}{2}}$, so that m_{ij}^{p-1} typically scales as $N^{\frac{p-1}{2}}$. The noise generator $L_{0,\beta}m_{ij} = \Delta m_{ij}(t) - \beta \langle \nabla H_0, \hat{\nabla} m_{ij}(t) \rangle$ with $\Delta m_{ij}(t) = -\frac{N-1}{N}m_{ij}(t)$ according to (1.9). The term $-\beta \langle \nabla H_0, \hat{\nabla} m_{ij}(t) \rangle$ is also of order $N^{-\frac{1}{2}}$ at initialization, thus a sample complexity of order N^{p-2} is sufficient for the population drift to dominate over the noise influence. Under this sample complexity, the dynamics during this first phase can described by the following simple stochastic differential equation:

$$dm_{ij}(t) \approx \left(\beta \sqrt{M} p \lambda_i \lambda_j m_{ij}^{p-1} - \frac{N-1}{N} m_{ij}(t)\right) dt + dM_t^{m_{ij}}.$$
(2.2)

At this point, we need to show that the drift term in the population dynamics keeps dominating the noise generator $L_{0,\beta}m_{ij}$ over a sufficiently large time scale, allowing m_{ij} to escape mediocrity. This is exactly what is achieved by the bounding flows method of [7, 8], providing a time dependent bound for the noise processes based on estimates of a Sobolev-type norm of $H_0(\mathbf{X})$, while retaining the regularity of the initial condition. Once the first correlation reaches a critical threshold, the correction term in the population generator, which was negligible during the initial phase of dynamics, becomes relevant and the correlations begin to interact with each other. As explained below in the description of the population dynamics, the values of the SNRs and of the correlations at initialization lead to an ordering of the correlations at the microscopic scale that is crucial for the analysis of the recovery process. However, in the presence of Brownian motion, the fluctuations due to the martingale part $M_t^{m_{ij}}$ of the dynamics are comparable to the values of the correlations near initialization. In particular, unless the SNRs are sufficiently separated, we are unable to guarantee that this ordering is stable.

Analysis of the population dynamics. The evolution of correlations under the population dynamics of gradient flow is close to the behavior described for online SGD in [6, Section 2]. For the sake of completeness and the reader's understanding, we provide a brief explanation of this population analysis below.

Assume first $p \ge 3$. The solution to the ODE (2.2) shows that the correlations m_{ij} are well approximated by

$$m_{ij}(t) \approx m_{ij}(0) \left(1 - \beta \sqrt{M} \lambda_i \lambda_j p(p-2) m_{ij}(0)^{p-2} t\right)^{-\frac{1}{p-2}},$$
 (2.3)

where $m_{ij}(0) = \frac{\gamma_{ij}}{\sqrt{N}}$ for some order-1 constant $\gamma_{ij} > 0$. From (2.3), it follows that the typical time for m_{ij} to reach a macroscopic threshold $\varepsilon > 0$ is given by

$$T_{\varepsilon}^{(ij)} \approx \frac{1 - \left(\frac{\gamma_{ij}}{\varepsilon\sqrt{N}}\right)^{p-2}}{\beta\sqrt{M}\lambda_i\lambda_j p(p-2)\gamma_{ij}^{p-2}} N^{\frac{p-2}{2}}.$$

In particular, the first correlation to become macroscopic is the one associated with the largest value of $\lambda_i \lambda_j \gamma_{ij}^{p-2}$, where $(\gamma_{ij})_{1 \leq i,j \leq 2}$ approximately follow independent standard normal distributions. Recall that the true evolution of $m_{ij}(t)$ corresponds to a perturbation of 2.3, whose sensitivity to the initial condition crucially determines the ordering of the correlations near initialization. We show that, using the sample complexity M, the perturbation due to $L_{0,\beta}$ becomes negligible as soon as $M \gtrsim N^{p-2}$ for some constant that is made explicit in the proof. To obtain a uniform control of the martingale part $M_t^{m_{ij}}$ over the time interval required to use the bounding flow method, we use Doob's maximal inequality to compare it with the initialization, which perturbes the ordering of the quantities $\lambda_i \lambda_j \gamma_{ij}^{p-2}$. To compensate this perturbation, we impose a separation condition on the $\{\lambda_i\}_{i=1}^r$. In our companion paper on gradient flow, we show that, due to the absence of the Brownian motion, a sharper analysis can be carried out without any separation assumption on the SNRs. We leave the elaboration of a more refined argument at finite temperature for future work.

Once the first correlation m_{ij} reaches some macroscopic value $\varepsilon > 0$, we can show that the other correlations are still of order $\frac{1}{\sqrt{N}}$. Next, we describe the evolution of m_{12}, m_{21} and m_{22} during this time interval. We first note that the correction part of the population generator, i.e.,

$$\sum_{1 \le k, \ell \le 2} \lambda_k m_{kj} m_{k\ell} m_{i\ell} (\lambda_j m_{kj}^{p-2} + \lambda_\ell m_{k\ell}^{p-2}),$$

becomes dominant in the evolution of the correlations m_{12}, m_{21} , as soon as m_{11} reaches the microscopic threshold $N^{-\frac{p-2}{2p}}$. Therefore, until m_{11} reaches this threshold value, m_{12} and m_{21} are non-decreasing and they begin to decrease from this point onward. On the other hand, the correction part of the population generator may become dominant in the evolution of m_{22} , as soon as m_{11} reaches another microscopic threshold of order $N^{-\frac{p-3}{2(p-1)}}$, leading to a potential decrease of m_{22} . Through a careful analysis, we show that this potential decrease is at most of order $\frac{\log(N)}{N}$, so that m_{22} remains stable at the scale $\frac{1}{\sqrt{N}}$ during the ascending phase of m_{11} . Once m_{12}, m_{21} become sufficiently small, we show that the evolution of m_{22} can still be described by (2.3), thus ensuring the recovery of the second direction. The phenomenon we observe here is referred to as sequential elimination phenomenon, as introduced in Definition 1.8. The correlations increase sequentially, one after another. When the first correlation m_{11} exceeds a certain threshold, the correlations sharing a row and column index, i.e., m_{12} and m_{21}) start decreasing until they become sufficiently small to be negligible, thereby enabling the subsequent correlation m_{22} to increase and reach a macroscopic value. We refer to Figure 9 in our companion paper [6] to an illustration of this phenomenon.

We now consider p = 2. In this case, the solution to (2.2) is given by

$$m_{ij} \approx m_{ij}(0) \exp\left(2\lambda_i \lambda_j t\right),$$

$$(2.4)$$

with $m_{ij}(0) = \gamma_{ij} N^{-\frac{1}{2}}$. This implies that the typical time for m_{ij} to reach a macroscopic threshold $\varepsilon > 0$ is given by

$$T_{\varepsilon}^{(ij)} \approx \frac{\frac{1}{2}\log(N) - \log\left(\frac{\gamma_{ij}}{\varepsilon}\right)}{2\lambda_i\lambda_j}$$

Compared to the case where $p \geq 3$, where the quantity $\lambda_i \lambda_j \gamma_{ij}^{p-2}$ determines the correlation that first escapes mediocrity, we observe here that the initialization γ_{ij} has less influence and the differences in time for the correlations to reach a macroscopic threshold are dominated by the separation between the SNRs, which is less pronounced as the γ_{ij} 's are not in the denominator. When $\lambda_1 > \lambda_2$, a sequential elimination phenomenon akin to the one observed for $p \geq 3$ occurs. However, there is an important difference compared to the tensor case: once m_{11} reaches the macroscopic threshold $\varepsilon > 0$, the correlations m_{12}, m_{21} and m_{22} scale as $N^{-\delta_1/2}, N^{-\delta_1/2}$ and $N^{-\delta_2/2}$, respectively, where $\delta_1 = 1 - \frac{\lambda_2}{\lambda_1}$ and $\delta_2 = 1 - \frac{\lambda_2^2}{\lambda_1^2}$. We then show that m_{11} must reach approximately the critical value $(\frac{\lambda_2}{\lambda_1})^{1/2}$ for m_{12}, m_{21} to decrease. This value is achieved in an order one time once m_{11} has reached ε , while m_{12}, m_{21} still require a time of order $\log(N)$ to escape their scale of $N^{-\frac{\delta_1}{2}}$. In addition, we show the stability of m_{22} during this first phase by controlling that it remains increasing throughout the ascending phase of m_{11} and until m_{12}, m_{21} begin to decrease. Note that, since m_{22} is of order $N^{-\delta_2/2}$ as m_{11} reaches ε , the sample complexity for the recovery of the second direction degrades by a factor $N^{1-\frac{\lambda_r^2}{\lambda_1^2}}$, as opposed to \sqrt{N} in the tensor case.

2.2. Subspace recovery

As explained in our companion paper on SGD, the multi-spike matrix model becomes isotropic when the SNRs are equal, since any rotation of the hidden directions is a global minimizer. Thus, rather than focusing on the correlations m_{ij} individually, we apply our bounding flow directly at the level of the eigenvalues of the matrix $\boldsymbol{G} = \boldsymbol{M}^{\top} \boldsymbol{M}$. In particular, fluctuations of the initial condition (at the microscopic scale) have little effect in this case, so that the martingale part can be easily controlled. We then show that, when the noisy dynamics is sufficiently close to the population dynamics, the matrix \boldsymbol{G} follows a constant coefficient algebraic Ricatti equation, i.e.,

$$\dot{\boldsymbol{G}}(t) \approx \lambda^2 \boldsymbol{G}(t) (\mathbf{I}_r - \boldsymbol{G}(t)),$$

where its eigenvalues $\theta_1, \ldots, \theta_r$ evolve according to

$$\dot{\theta}_i(t) \approx \lambda^2 \theta_i(t) \left(1 - \theta_i(t)\right).$$

Since these equations are coordinate-wise autonomous and have monotone solutions, it is easy to prove efficient recovery once the estimator has exited the maximum entropy region of the landscape.

3. Main results

Here, we present our main results of Subsection 1.3 in nonasymptotic form. We first introduce natural conditions that the initial data must satisfy in order to efficiently recover the spikes. This conditions are then shown to be satisfied by the uniform measure $\mu_{N\times r}$ on $\mathcal{M}_{N,r}$.

The first condition is on the regularity of the noise generator $L_{0,\beta}$.

Definition 3.1 (Condition 0 at level n). For every $\gamma_0 > 0$, $n \ge 1$, we let $C_0(n, \gamma_0)$ denote the sequence of events given by

$$\mathcal{C}_0(n,\gamma_0) = \bigcap_{k=0}^{n-1} \left\{ \boldsymbol{X} \in \mathcal{M}_{N,r} \colon |L_{0,\beta}^k m_{ij}^{(N)}(\boldsymbol{X})| \le \frac{\gamma_0}{\sqrt{N}} \text{ for every } 1 \le i,j \le r \right\}.$$

We then say that a sequence of random probability measures $\mu \in \mathcal{P}(\mathcal{M}_{N,r})$ satisfies Condition 0 at level n if for every $\gamma_0 > 0$,

$$\mu\left(\mathcal{C}_0(n,\gamma_0)^{\mathbf{c}}\right) \le Ce^{-c\gamma_0^2}$$

for absolute constants C, c.

Definition 3.2 (Condition 0 at level ∞). For every $\gamma_0 > 0$ and T > 0, we let $\mathcal{C}_0(T, \gamma_0)$ denote the sequence of events given by

$$\mathcal{C}_0^{\infty}(T,\gamma_0) = \left\{ \boldsymbol{X} \in \mathcal{M}_{N,r} \colon \sup_{t \leq T} |e^{tL_{0,\beta}} L_{0,\beta} m_{ij}^{(N)}(\boldsymbol{X})| \leq \frac{\gamma_0}{\sqrt{N}} \text{ for every } 1 \leq i,j \leq r \right\},\$$

where e^{tL_0} denotes the semigroup induced by L_0 . We then say that a sequence of random probability measures $\mu \in \mathcal{P}(\mathcal{M}_{N,r})$ weakly satisfies *Condition 0 at level* ∞ if for every $\gamma_0, T > 0$,

$$\mu\left(\mathcal{C}_0^{\infty}(T,\gamma_0)^{\mathrm{c}}\right) \le C\sqrt{N}Te^{-c\gamma_0^2},$$

for absolute constants C, c.

The second condition ensures that the initial correlation is on the typical scale $\Theta(N^{-\frac{1}{2}})$.

Definition 3.3 (Condition 1). For every $\gamma_1 > \gamma_2 > 0$, we let $C_1(\gamma_1, \gamma_2)$ denote the sequence of events given by

$$\mathcal{C}_1(\gamma_1,\gamma_2) = \left\{ \boldsymbol{X} \in \mathcal{M}_{N,r} \colon \frac{\gamma_2}{\sqrt{N}} \le m_{ij}^{(N)}(\boldsymbol{X}) < \frac{\gamma_1}{\sqrt{N}} \text{ for every } 1 \le i,j \le r \right\}.$$

We say that a sequence of random probability measures $\mu \in \mathcal{P}(\mathcal{M}_{N,r})$ satisfies *Condition 1* if for every $\gamma_1 > \gamma_2 > 0$,

$$\mu \left(C_1(\gamma_1, \gamma_2)^c \right) \le C_1 e^{-c_1 \gamma_1^2} + C_2 e^{-c_2 \gamma_2 \sqrt{N}} + C_3 \gamma_2,$$

for absolute constants C_1, c_1, C_2, c_2, C_3 .

The most natural initialization is the uniform measure $\mu_{N\times r}$ on $\mathcal{M}_{N,r}$. We claim that

Lemma 3.4. The uniform measure $\mu_{N\times r}$ on $\mathcal{M}_{N,r}$ weakly satisfies Condition 0 at level ∞ and satisfies Condition 1.

The proof of Lemma 3.4 is deferred to Appendix A.

We are now in the position to state our main results in nonasymptotic form. Our first result shows that for $p \ge 3$, the critical threshold required to achieve strong recovery of the first spike v_1 must scale as $\sqrt{M} \sim N^{\frac{p-2}{2}}$.

Proposition 3.5 (Recovery of leading spike for $p \geq 3$). Fix any $\beta \in (0, \infty)$ and $p \geq 3$. Consider a sequence of initializations $\mu_0 \in \mathcal{P}(\mathcal{M}_{N,r})$ which satisfies Condition 0 at level n and Condition 1. Then, the following holds: For every $n \geq 1$, $\gamma_0 > 0$, $\gamma_1 > \gamma_2 > 0$ and $\varepsilon > 0$, there exists $C_0 \in (0, \frac{1}{2})$ such that if $\lambda_1 > \frac{1+C_0}{1-C_0} \left(\frac{3\gamma_1}{\gamma_2}\right)^{p-2} \lambda_2$ and $\sqrt{M} \gtrsim \frac{(n+2)\gamma_0}{\beta p \lambda_r^2 C_0 \gamma_2^{p-1}} N^{\frac{p-1}{2} - \frac{n}{2(n+1)}}$, there exists $T_0 \gtrsim \frac{1}{(n+2)\gamma_0} N^{-\frac{1}{2(n+1)}}$ such that for every $T > T_0$ and N sufficiently large,

$$\int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t \in [T_0,T]} m_{11}(\boldsymbol{X}_t^\beta) \ge 1 - \varepsilon \right) d\mu_0(\boldsymbol{X}) \ge 1 - \eta(n,\gamma_0,\gamma_1,\gamma_2),$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$, where $\eta = \eta(n, \gamma_0, \gamma_1, \gamma_2)$ is given by

$$\begin{split} \eta &= C_1 e^{-c_1 \gamma_0^2} + C_2 e^{-c_2 \gamma_1^2} + C_3 \gamma_2 + C_4 e^{-c_4 \gamma_2 \sqrt{N}} + K_1 e^{-\gamma_0^3 (n+2)N^{\frac{2(n+1)}{2(n+1)}}/K_1} \\ &+ r^2 K_2 e^{-\gamma_2^2 \gamma_0 (n+2)N^{\frac{1}{2(n+1)}}/K_2} + K_3 e^{-\gamma_0 (n+2)N^{\frac{2p-1}{2(p-1)} - \frac{n}{2(n+1)}}/K_3} + K_4 e^{-N\varepsilon^2/(K_2 T)} \end{split}$$

Our second main result shows that the critical threshold for achieving exact recovery of all r spikes v_1, \ldots, v_r must scale as $\sqrt{M} \sim N^{\frac{p-1}{2}}$.

Proposition 3.6 (Exact recovery of all spikes for $p \ge 3$). Let $\beta \in (0, \infty)$, $p \ge 3$ and $\lambda_1 \ge \cdots \ge \lambda_r > 0$. Consider a sequence of initializations $\mu_0 \in \mathcal{P}(\mathcal{M}_{N,r})$ which satisfies Condition 1. For every $\varepsilon > 0$, we let $R(\varepsilon)$ denote

$$R(\varepsilon) = \left\{ \boldsymbol{X} \colon m_{ii}(\boldsymbol{X}) \ge 1 - \varepsilon \ \forall i \in [r] \text{ and } m_{ij}(\boldsymbol{X}), m_{ji}(\boldsymbol{X}) \lesssim \log(N)^{-\frac{1}{2}} N^{-\frac{p-1}{4}} \ \forall j \neq i \right\}.$$
(3.1)

Then, the following holds: For every $\gamma_1 > 1 > \gamma_2 > 0$ and $\varepsilon > 0$, there exist $\Lambda = \Lambda(n, p, \beta, \{\lambda_i\}_{i=1}^r) > 0$ and $C_0 \in (0, \frac{1}{2})$ such that if $\lambda_i > \frac{1+C_0}{1-C_0} \left(\frac{3\gamma_1}{\gamma_2}\right)^{p-2} \lambda_{i+1}$ for every $1 \le i \le r-1$ and $\sqrt{M} \gtrsim \frac{1}{\beta p \lambda_r^2 C_0 \gamma_2^{p-1}} N^{\frac{p-1}{2}}$, there exists $T_0 \gtrsim \frac{1}{\sqrt{N}}$ such that for every $T > T_0$ and N sufficiently large,

$$\int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t \in [T_0,T]} \boldsymbol{X}_t^{\beta} \in R(\varepsilon) \right) d\mu_0(\boldsymbol{X}) \ge 1 - \eta(\gamma_1,\gamma_2).$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$, where $\eta = \eta(\gamma_1, \gamma_2)$ is given by

$$\eta = C_1 e^{-c_1 \gamma_1^2} + C_2 \gamma_2 + C_3 e^{-c_3 \gamma_2 \sqrt{N}} + r^3 K_1 e^{-\gamma_2^2 \sqrt{N}/K_1} + r K_2 e^{-N^{\frac{p+1}{2(p-1)}}/K_2} + r K_3 e^{-\varepsilon^2 N/(K_3 T)}$$

According to the event of strong recovery given by (3.1), it is important to note that after successfully recovering the first spike, recovery of the subsequent spikes is possible provided the correlations m_{ij} for $i \neq j$ decrease below the initial scale $\Theta(N^{-\frac{1}{2}})$. More precisely, we observe that the off-diagonal correlations m_{ij} must reach at least the threshold $\log(N)^{-\frac{1}{2}}N^{-\frac{p-1}{4}}$ to efficiently solve the recovery problem.

We now present our main results for the matrix PCA problem, i.e., when p = 2. We begin with the case where the SNRs are separated by constants of order 1. Similar to the previous case, we identify two different algorithmic thresholds for the sample complexity: one threshold required for efficient recovery of the first spike and another for recovery of all spikes. Our first main result shows that strong recovery of v_1 is achievable when M is of order 1.

Proposition 3.7 (Recovery of leading spike for p = 2). Let $\beta \in (0, \infty)$, p = 2, and $\lambda_i = \lambda_{i+1}(1 + \kappa_i)$ for every $1 \leq i \leq r-1$ and $\kappa_i > 0$. Consider a sequence of initializations $\mu_0 \in \mathcal{P}(\mathcal{M}_{N,r})$ which satisfies Condition 0 at level n and Condition 1. Then, the following holds: For every $n \geq 1$, $\gamma_0 > 0$, $\gamma_1 > 1 > \gamma_2 > 0$ and $\varepsilon > 0$, there exists $C_0 \in (0, \frac{1}{2})$ such that for every $\sqrt{M} \gtrsim \frac{(n+2)\gamma_0}{\beta\lambda_r^2 C_0 \gamma_2} N^{\frac{1}{2(n+1)}}$ and $\log(N) \gtrsim 2\frac{\kappa_1+1}{\kappa_1-2}\log(3\gamma_1/2) + 2\log(\varepsilon)$, there exists $T_0 \gtrsim \frac{1}{(n+2)\gamma_0}\log(N)N^{-\frac{1}{2(n+1)}}$ such that for every $T > T_0$,

$$\int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t \in [T_0,T]} m_{11}(\boldsymbol{X}_t^\beta) \ge 1 - \varepsilon \right) d\mu_0(\boldsymbol{X}) \ge 1 - \eta(n,\gamma_0,\gamma_1,\gamma_2),$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$, where $\eta = \eta(n, \gamma_0, \gamma_1, \gamma_2)$ is given by

$$\eta = C_1 e^{-c_1 \gamma_0^2} + C_2 e^{-c_2 \gamma_1^2} + C_3 \gamma_2 + C_4 e^{-c_4 \gamma_2 \sqrt{N}} + K_1 e^{-\gamma_2^2 (n+2)\gamma_0 N^{\frac{1}{2(n+1)}}/K_1} + K_2 e^{-N\varepsilon^2/(K_2 T)}$$

Our second main result shows that strong recovery of all spikes is possible, provided M is of order N^{δ} for some $\delta \in (0, 1)$ which depends on the ratio between the SNRs..

Proposition 3.8 (Exact recovery of all spikes for p = 2). Let $\beta \in (0, \infty)$, p = 2, and $\lambda_i = \lambda_{i+1}(1 + \kappa_i)$ for every $1 \leq i \leq r-1$ and $\kappa_i > 0$. Let κ denote $\kappa = \min_{1 \leq i \leq r-1} \kappa_i$. Consider a sequence of initializations $\mu_0 \in \mathcal{P}(\mathcal{M}_{N,r})$ which satisfies Condition 0 at level n and Condition 1. For every $\varepsilon > 0$ and $C_0 \in (0, 1)$, we let $R(\varepsilon, C_0)$ denote

$$R(\varepsilon, C_0) = \left\{ \boldsymbol{X} \colon m_{ii}(\boldsymbol{X}) \ge 1 - \varepsilon \ \forall i \in [r] \text{ and } m_{k\ell}(\boldsymbol{X}) \lesssim N^{-\frac{1}{2} \left(1 - \frac{1 - C_0}{1 + C_0} \frac{\lambda_r^2}{\lambda_1^2} \right)} \ \forall k, \ell \in [r], k \neq \ell \right\}.$$
(3.2)

Then, the following holds: For every $n \ge 1$ and $\gamma_1 > 1 > \gamma_2 > 0$, there exist $\varepsilon_0 > 0$ and $c_0 \in (0, \frac{1}{2} \land \frac{\kappa}{2+\kappa})$ such that for every $\varepsilon < \varepsilon_0$, $C_0 < c_0$, if $\sqrt{M} \gtrsim \frac{(n+2)\gamma_0\gamma_1}{\beta\lambda_r^2 C_0\gamma_2} N^{\frac{1}{2}\left(\frac{1+C_0}{1-C_0}-\frac{\lambda_r^2}{\lambda_1^2}\right)}$ and $\log(N) \gtrsim \frac{1}{1+\kappa} \log\left(\frac{1}{\kappa}\right)$, there exists $T_0 \gtrsim \frac{1}{(n+2)\gamma_0\gamma_1} \log(N) N^{-\frac{1}{2}\left(\frac{1+C_0}{1-C_0}-\frac{\lambda_r^2}{\lambda_1^2}\right)}$ such that for every $T > T_0$, $\int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\mathbf{X}} \left(\inf_{t\in[T_0,T]} \mathbf{X}_t^\beta \in R(\varepsilon, C_0)\right) d\mu_0(\mathbf{X}) \ge 1 - \eta(n, \gamma_0, \gamma_1, \gamma_2),$ with \mathbb{P} -probability at least $1 - \exp(-KN)$, where $\eta = \eta(n, \gamma_0, \gamma_1, \gamma_2)$ is given by

$$\eta = C_1 e^{-c_1 \gamma_0^2} + C_2 e^{-c_2 \gamma_1^2} + C_3 \gamma_2 + C_4 e^{-c_4 \gamma_2 \sqrt{N}} + K_1 e^{-N \varepsilon^2 / (K_1 T)} + K_2 e^{-N^{\frac{1}{2}(1+\delta)} / (K_2 T)}$$

Our final main result in this section concerns strong recovery when p = 2 and all signal strengths are equal, i.e., $\lambda_1 = \cdots = \lambda_r \equiv \lambda > 0$. In this case, the random landscape H is invariant under rotations. Therefore, rather than precisely characterizing the recovery of each planted signal by analyzing the evolution of the correlations as in the previous cases, we focus here on *subspace recovery*, meaning recovery of the correct subspace shared by the signal vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$. Specifically, we can look at the distance between the orthogonal projections $\boldsymbol{X}\boldsymbol{X}^{\top}$ and $\boldsymbol{V}\boldsymbol{V}^{\top}$, where $\boldsymbol{V} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r] \in \mathcal{M}_{N,r}$ and $\boldsymbol{X} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r] \in \mathcal{M}_{N,r}$. This can be quantified by

$$\frac{1}{N^2} \| \boldsymbol{X} \boldsymbol{X}^\top - \boldsymbol{V} \boldsymbol{V}^\top \|_{\mathrm{F}}^2 = 2 \left(r - \mathrm{Tr} \left(\boldsymbol{M}^\top \boldsymbol{M} \right) \right),$$

where M denotes the correlation matrix defined by $M = \frac{1}{N} V^{\top} X$. To analyze this, we study the behavior of the eigenvalues $\theta_1, \ldots, \theta_r$ of $G = M^{\top} M \in \mathbb{R}^{r \times r}$ under Langevin dynamics, as stated in the following theorem. To this end, we need to ensure that the eigenvalues of G at initialization are on the typical scale $\Theta(N-1)$.

Definition 3.9 (Condition 1'). For every $\gamma_1 > \gamma_2 > 0$, we let $C'_1(\gamma_1, \gamma_2)$ denote the sequence of events given by

$$\mathcal{C}_1'(\gamma_1, \gamma_2) = \left\{ \boldsymbol{X} \in \mathcal{M}_{N,r} \colon \frac{\gamma_2}{N} \le \theta_i(\boldsymbol{G}(\boldsymbol{X})) < \frac{\gamma_1}{N} \text{ for every } i \in [r] \right\}.$$

We say that $\mu \in \mathcal{P}(\mathcal{M}_{N,r})$ satisfies *Condition* 1' if for every $\gamma_1 > \gamma_2 > 0$,

$$\mu\left(\mathcal{C}_{1}'(\gamma_{1},\gamma_{2})^{c}\right) \leq C_{1}e^{-c_{1}\gamma_{1}^{2}} + C_{2}\gamma_{2} + C_{3}e^{-c_{2}\gamma_{2}\sqrt{N}}.$$

Lemma A.1 ensures that the uniform measure $\mu_{N \times r}$ on $\mathcal{M}_{N,r}$ satisfies Condition 1'.

Proposition 3.10 (Subspace recovery for p = 2). Let $\beta \in (0, \infty)$, p = 2, and $\lambda_1 = \cdots = \lambda_r \equiv \lambda > 0$. Consider a sequence of initializations $\mu_0 \in \mathcal{P}(\mathcal{M}_{N,r})$ which satisfies Condition 0 at level n and Condition 1'. Then, the following holds: For every $n \ge 1$ and $\gamma_1 > 1 > \gamma_2 > 0$, there exist $\varepsilon_0 > 0$ and $c_0 \in (0, \frac{1}{3r})$ such that for every $\varepsilon < \varepsilon_0$, $C_0 < c_0$, $\sqrt{M} \gtrsim \frac{(n+2)\gamma_0^2\gamma_1}{\beta\lambda^2 C_0\gamma_2}N^{\delta(n,C_0)}$, there exists $T_0 \gtrsim \frac{1+C_0^2}{(n+2)\gamma_0^2\gamma_1}\log(N)N^{-\delta(n,C_0)}$ such that for every $T > T_0$,

$$\int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t \in [T_0,T]} \theta_{\min} \left(\boldsymbol{X}_t^{\beta} \right) \ge 1 - \varepsilon \right) d\mu_0(\boldsymbol{X}) \ge 1 - \eta(n,\gamma_0,\gamma_1,\gamma_2),$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$, where $\delta(n, C_0) = \frac{1}{2(n+1)} \vee \frac{2C_0 r}{1-C_0 r}$ and $\eta = \eta(n, \gamma_0, \gamma_1, \gamma_2)$ is given by

$$\begin{split} \eta &= C_1 e^{-c_1 \gamma_0^2} + C_2 e^{-c_2 \gamma_1^2} + C_3 \gamma_2 + C_4 e^{-c_4 \gamma_2 \sqrt{N}} + K_1 e^{-\gamma_0^4 \gamma_1 (n+2)N^{\delta(n,C_0)}/(K_1 \log(N))} \\ &+ K_2 e^{-\gamma_0^2 \gamma_2 (n+2)N^{\delta(n,C_0)}/K_2} + K_3 e^{-\gamma_0^2 \gamma_1 (n+2)N^{1-\frac{4C_0 r}{1+C_0 r} + \delta(n,C_0)}/(K_3 C_0^2 \log(N))} + K_4 e^{-N\varepsilon^2/(K_4 T)}. \end{split}$$

Propositions 3.5 and 3.6 are proved in Section 5. Propositions 3.7 and 3.8 are proved in Section 6. Finally, the proof of Proposition 3.10 is provided in Section 7. Our asymptotic results presented in Subsection 1.3 follow straightforwardly from the above results and using the definitions of the initial conditions.

4. Preliminary results

In this section, we present preliminary results necessary for proving the main results in Section 3. Specifically, we study the regularity of the Hamiltonian $H_0: \mathcal{M}_{N,r} \to \mathbb{R}$ and introduce the bounding flows method from [7], which plays a crucial role in deriving estimates for $\|L_{0,\beta}m_{ij}^{(N)}\|_{\infty}$. Additionally, we present the evolution equations governing the correlations $\{m_{ij}^{(N)}\}_{1\leq i,j\leq r}$ as well as the equations satisfied by the entries of $\mathbf{M}^{\top}\mathbf{M}$, where $\mathbf{M} = (m_{ij}^{(N)})_{1\leq i,j\leq r}$. The results of this section generalize those presented in [8]. To simplify notation, we will omit explicit reference to N in $m_{ij}^{(N)}(\mathbf{X})$ and instead write $m_{ij}(\mathbf{X})$.

4.1. Regularity of noise, ladder relations, and bounding flows method

Recall the Hamiltonian $H_0: \mathcal{M}_{N,r} \to \mathbb{R}$ given by

$$H_0(\boldsymbol{X}) = N^{-\frac{p-1}{2}} \sum_{i=1}^r \lambda_i \langle \boldsymbol{W}, \boldsymbol{x}_i^{\otimes p} \rangle,$$

where $\boldsymbol{W} \in (\mathbb{R}^N)^{\otimes p}$ is an order-*p* tensor with i.i.d. entries $W_{i_1,\ldots,i_p} \sim \mathcal{N}(0,1)$ and $\mathcal{M}_{N,r}$ is the normalized Stiefel manifold. Similar to the works [7, 8], we work with the \mathcal{G} -norm which is motivated by the homogeneous Sobolev norm.

Definition 4.1 (*G*-norm on $\mathcal{M}_{N,r}$). A function $F: \mathcal{M}_{N,r} \to \mathbb{R}$ is in the space $\mathcal{G}^k(\mathcal{M}_{N,r})$ if

$$|F||_{\mathcal{G}^k} \coloneqq \sum_{0 \le \ell \le k} N^{\ell/2} |||\nabla^{\ell} F|_{\mathrm{op}}||_{L^{\infty}(\mathcal{M}_{N,r})} < \infty,$$

where $|\nabla^{\ell} F|_{\text{op}}(\mathbf{X})$ denote the natural operator norm when $\nabla^{\ell} F$ is viewed as an ℓ -form acting on the ℓ -fold product of the tangent space $T_{\mathbf{X}}\mathcal{M}_{N,r}$.

We emphasize that this definition is a generalization of the \mathcal{G} -norm defined for functions on $\mathbb{S}^{N-1}(\sqrt{N})$ introduced by [7]. We then have the following important estimate for the \mathcal{G} -norm of H_0 . Lemma 4.2 (Regularity of H_0). For every n, there exist $C_1 = C_1(p,n), C_2 = C_2(p,n) > 0$ such that $\mathbb{P}\left(\|H_0\|_{Cn} > C_1\left(\sum_{i=1}^r \lambda_i\right)N\right) < \exp\left(-C_2\frac{(\sum_{i=1}^r \lambda_i)^2}{N}N\right).$

$$\mathbb{P}\left(\|H_0\|_{\mathcal{G}^n} \ge C_1\left(\sum_{i=1}\lambda_i\right)N\right) \le \exp\left(-C_2\frac{(\sum_{i=1}\lambda_i)}{\sum_{i=1}^r\lambda_i^2}N\right).$$

educes to of [7, Theorem 4.3] for the case $r = 1$. The proof of Lemma

Lemma 4.2 reduces to of [7, Theorem 4.3] for the case r = 1. The proof of Lemma 4.2 follows the same strategy used to prove Theorem 4.3 of [7] and one can therefore mimic the same arguments, and it is thus left. We next present the ladder relations which will be useful to bound $||L_{0,\beta}m_{ij}||_{\infty}$, where we recall from (1.10) that the generator $L_{0,\beta}$ is of the form $L_{0,\beta} = \Delta - \beta \langle \nabla H_0, \hat{\nabla} \cdot \rangle$.

Lemma 4.3 (Ladder relations). For every linear operator L acting on the space of smooth functions $F: \mathcal{M}_{N,r} \to \mathbb{R}$ and every integer numbers $n \ge m \ge 1$, we define

$$||L||_{\mathcal{G}^n \to \mathcal{G}^m} = \sup_{F \in \mathcal{C}(\mathcal{M}_{N,r})} \frac{||LF||_{\mathcal{G}^m}}{||F||_{\mathcal{G}^n}}$$

Then, for every $n \ge 1$ there exists c(n) such that for every N and r,

$$\|\Delta\|_{\mathcal{G}^n \to \mathcal{G}^{n-2}} \le r \tag{4.1}$$

$$\|\langle \nabla G, \nabla \cdot \rangle\|_{\mathcal{G}^n \to \mathcal{G}^{n-1}} \le \frac{c(n)}{N} \|G\|_{\mathcal{G}^n}.$$
(4.2)

Proof. We first prove (4.1). Since $\Delta F = \operatorname{Tr}(\nabla^2 F)$ and $\nabla \operatorname{Tr}(F) = \operatorname{Tr}(\nabla F)$, we have that

$$\nabla^{\ell} \Delta F \big|_{\text{op}} = \big| \nabla^{\ell} \operatorname{Tr}(\nabla^{2} F) \big|_{\text{op}} = \big| \operatorname{Tr}(\nabla^{\ell+2} F) \big|_{\text{op}}.$$

Moreover, since dim $(\mathcal{M}_{N,r}) = Nr - \frac{r(r+1)}{2} \leq Nr$, we have that $\left| \operatorname{Tr}(\nabla^{\ell+2}F) \right|_{\mathrm{op}} \leq Nr \left| \nabla^{\ell+2}F \right|_{\mathrm{op}}$, thus

$$\|\Delta F\|_{\mathcal{G}^{n-2}} = \sum_{\ell=0}^{n-2} N^{\ell/2} \||\nabla^{\ell} \Delta F|_{\rm op}\|_{\infty} \le r \sum_{\ell=0}^{n-2} N^{(\ell+2)/2} \||\nabla^{\ell+2} \Delta F|_{\rm op}\|_{\infty} \le r \|F\|_{\mathcal{G}^n},$$

as desired. Next, we show (4.2). By the general Leibniz rule and Cauchy-Schwarz, we have that

$$\begin{aligned} \|\langle \nabla G, \nabla F \rangle \|_{\mathcal{G}^{n-1}} &= \sum_{\ell=0}^{n-1} N^{\ell/2} \sum_{k=0}^{\ell} \binom{\ell}{k} \||\langle \nabla^{k+1}G, \nabla^{\ell-k+1}F \rangle|_{\mathrm{op}} \|_{L^{\infty}} \\ &\leq \sum_{0 \leq k \leq \ell \leq n-1} N^{\ell/2} \binom{\ell}{k} \||\nabla^{k+1}G|_{\mathrm{op}} \|_{\infty} \||\nabla^{\ell-k+1}F|_{\mathrm{op}} \|_{\infty} \\ &\leq \frac{c(n)}{N} \|G\|_{\mathcal{G}^{n}} \|F\|_{\mathcal{G}^{n}}, \end{aligned}$$

where the inequality in the last line follows by definition of the \mathcal{G} -norm.

Using the ladder relations from Lemma 4.3, we can provide an estimate for $||L_{0,\beta}m_{ij}||_{\infty}$ for every $1 \leq i, j \leq r$. According to Lemma 4.2, for every $n \geq 1$, there exist $K = K(p, n, \{\lambda_i\}_{i=1}^r)$ and $C = C(p, n, \{\lambda_i\}_{i=1}^r)$ such that $||H_0||_{\mathcal{G}^n} \leq CN$, with \mathbb{P} -probability at least $1 - \exp(-KN)$. Moreover, a simple computation gives $||m_{ij}||_{\mathcal{G}^n} \leq c(n)$. Therefore, there exists a constant $\Lambda = \Lambda(p, n, \{\lambda_i\}_{i=1}^r, r, \beta)$ such that

$$\|L_{0,\beta}m_{ij}\|_{\infty} \le \|\Delta m_{ij}\|_{\infty} + \beta \|\langle \nabla H_0, \hat{\nabla} m_{ij} \rangle\|_{\infty} \le r \|m_{ij}\|_{\mathcal{G}^2} + \frac{\beta}{N} \|H_0\|_{\mathcal{G}^1} \|m_{ij}\|_{\mathcal{G}^1} \le \Lambda$$
(4.3)

with \mathbb{P} -probability at least $1 - \exp(-KN)$. This estimate turns out to be suboptimal. Therefore, we next introduce the "bounding flows" method which provides a significantly more precise approach to estimation. The bounding flows method, originally introduced in [7] and applied in [8], was used to derive estimates for the evolution of functions under Langevin dynamics on $\mathbb{S}^{N-1}(\sqrt{N})$. In this work, we extend the method to provide more accurate bounds for the evolution of functions under Langevin dynamics on $\mathcal{M}_{N,r}$. In particular, the following result generalizes Theorem 5.3 of [8].

Lemma 4.4 (Bounding flows method on $\mathcal{M}_{N,r}$). For every $\gamma > 0$, we let I_{γ} denote the interval $I_{\gamma} = \left[-\frac{\gamma}{\sqrt{N}}, \frac{\gamma}{\sqrt{N}}\right]$. Let $D \subset \mathcal{M}_{N,r}$, L be the infinitesimal generator of an Ito process \mathbf{X}_t , $F: D \to \mathbb{R}$ be smooth, and $\mathbf{X}_0 \in D$ with exit time \mathcal{T}_{D^c} . Moreover, suppose that the following is satisfied for some $n \geq 1$:

- (1) L is a differential operator of the form $L = L_0 + \sum_{1 \le i,j \le r} a_{ij}(\mathbf{X}) A_{ij}$, where (a) $A_{ij} = \langle \nabla \psi_{ij}, \hat{\nabla} \cdot \rangle$ for some function $\psi_{ij} \in C^{\infty}$ with $\|\psi_{ij}\|_{\mathcal{G}^1} \leq c_1 N$; (b) $a_{ij} \in C(\mathcal{M}_{N,r});$
 - (c) $L_0 = \Delta + \langle \nabla U, \hat{\nabla} \cdot \rangle$ for some $U \in C^{\infty}$ with $||U||_{\mathcal{G}^{2n}} \leq c_2(n)N$.
- (2) The function F is smooth with $||F||_{\mathcal{G}^{2n}} \leq c_3(n)$.
- (3) There exists $\gamma > 0$ such that \mathbf{X}_0 satisfies $L_0^k f(\mathbf{X}_0) \in I_{\gamma}$ for every $0 \le k \le n-1$.
- (4) There exist $\varepsilon \in (0,1)$ and $T_0^{(ij)} > 0$, possibly depending on ε , such that for any $t \leq \mathcal{T}_{D^c} \wedge T_0^{(ij)}$,

$$\int_0^t |a_{ij}(\boldsymbol{X}_s)| ds \leq \varepsilon |a_{ij}(\boldsymbol{X}_t)|$$

Then, there exists a constant $K_1 > 0$ depending only on c_1, c_2, c_3 and γ such that, for every $T_0 > 0$,

$$|F(\boldsymbol{X}_t)| \le K_1 \left(\frac{\gamma}{\sqrt{N}} \sum_{k=0}^{n-1} t^k + t^n + \frac{1}{1-\varepsilon} \sum_{1 \le i,j \le r} \int_0^t |a_{ij}(\boldsymbol{X}_s)| ds \right)$$
(4.4)

for every $t \leq \mathcal{T}_{D^c} \wedge \min_{1 \leq i,j \leq r} T_0^{(ij)} \wedge T_0$ with $\mathbb{Q}_{\boldsymbol{X}}$ -probability at least $1 - K_2 \exp\left(-\frac{\gamma^2}{K_2 T_0}\right)$. If instead of item (1)(c), item (1)(c') is satisfied, where

(c') $L_0 = \langle \nabla U, \hat{\nabla} \cdot \rangle$ for some $U \in C^{\infty}$ such that $||U||_{\mathcal{G}^{2n}} \leq c_2 N$,

then (4.4) holds deterministically for every $t \leq \mathcal{T}_{D^c} \wedge \min_{1 \leq i,j \leq r} T_0^{(ij)} \wedge T_0$. If instead of item (3), item (3') is satisfied, where

(3') There exist $T_1, \gamma > 0$ such that X_0 satisfies $e^{tL_0}F(X_0) \in I_{\gamma}$ for every $t < T_1$,

then (4.4) holds for every $t \leq \mathcal{T}_{D^c} \wedge \min_{1 \leq i,j \leq r} T_0^{(ij)} \wedge T_0 \wedge T_1 \wedge 1$, with $\mathbb{Q}_{\boldsymbol{X}}$ -probability at least 1 - 1 $K_2 \exp\left(-\frac{\gamma^2}{K_2 T_0}\right).$

Proof. We mimic the proof of [8, Theorem 5.3]. We claim that the function F can be expanded as

$$F(t) = F(0) + M_t^F + \sum_{k=1}^{n-1} \int_0^t \cdots \int_0^{t_{k-1}} \left(L_0^k F(0) + M_{t_k}^{L_0^k F} \right) dt_k \cdots dt_1 + \int_0^t \cdots \int_0^{t_{n-1}} L_0^n F(t_n) dt_n \cdots dt_1 + \sum_{1 \le i,j \le r} \sum_{k=1}^n \int_0^t \cdots \int_0^{t_{k-1}} a_{ij}(t_k) A_{ij} L_0^{k-1} F(t_k) dt_k \cdots dt_1.$$
(4.5)

The proof is by induction on n. The claim is verified for n = 1 by Lemma 4.5 since

$$F(t) = F(0) + M_t^F + \int_0^t LF(s)ds = F(0) + M_t^F + \int_0^t L_0F(s)ds + \sum_{1 \le i,j \le r} \int_0^t a_{ij}(s)A_{ij}F(s)ds.$$

Assume that the result holds in the *n*th case. By definition of $M^{L_0^n F}$ we find that

$$L_0^n F(t_n) = L_0^n F(0) + M^{L_0^n F} + \int_0^{t_n} L L_0^n F(t_{n+1}) dt_{n+1}$$

= $L_0^n F(0) + M^{L_0^n F} + \int_0^{t_n} L_0^{n+1} F(t_{n+1}) dt_{n+1} + \sum_{1 \le i,j \le r} \int_0^{t_n} a_{ij}(t_{n+1}) A_{ij} L_0^n F(t_{n+1}) dt_{n+1}.$

Therefore, for the (n+1)st case we use the above equality to expand the second-to-last term as

$$\int_{0}^{t} \cdots \int_{0}^{t_{n-1}} L_{0}^{n} F(t_{n}) dt_{n} \cdots dt_{1}$$

$$= \int_{0}^{t} \cdots \int_{0}^{t_{n-1}} \left(L_{0}^{n} F(0) + M_{t_{n}}^{L_{0}^{n} F} \right) dt_{n} \cdots dt_{1} + \int_{0}^{t} \cdots \int_{0}^{t_{n-1}} \int_{0}^{t_{n}} L_{0}^{n+1} F(t_{n+1}) dt_{n+1} dt_{n} \cdots dt_{1}$$

$$+ \sum_{1 \le i, j \le r} \int_{0}^{t} \cdots \int_{0}^{t_{n-1}} \int_{0}^{t_{n}} a_{ij}(t_{n+1}) A_{ij} L_{0}^{n} F(t_{n+1}) dt_{n+1} dt_{n} \cdots dt_{1}.$$

Combining the terms yields the desired expression by induction. We next bound the absolute values in (4.5) term-by-term. We start with the first line of (4.5) and we claim that for every $1 \le k \le n-1$,

$$\|L_0^k F\|_{\mathcal{G}^{2n-2k}} \le (r+c(n)c_2)^k \|F\|_{\mathcal{G}^{2n}} \le (r+c(n)c_2)^k c_3.$$
(4.6)

The second inequality in (4.6) follows by assumption (2). We next prove the first inequality in (4.6) by induction on k. The claim is verified for k = 1 by Lemma 4.3 since

$$\|L_0F\|_{\mathcal{G}^{2n-2}} \le r\|F\|_{\mathcal{G}^{2n}} + \frac{2^n - 1}{N} \|U\|_{\mathcal{G}^{2n}} \|F\|_{\mathcal{G}^{2n}} \le (r + (2^n - 1)c_2) \|F\|_{\mathcal{G}^{2n}}$$

where we used assumption (1c). Assume that (4.6) holds in the kth case. Then, in the (k + 1)th case we have that

$$\begin{split} \|L_0^{k+1}F\|_{\mathcal{G}^{2n-2(k+1)}} &\leq r \|L_0^kF\|_{\mathcal{G}^{2n-2k}} + \frac{2^n - 1}{N} \|U\|_{\mathcal{G}^{2n-2k}} \|L_0^kF\|_{\mathcal{G}^{2n-2k}} \\ &\leq (r + (2^n - 1)c_2) \|L_0^kF\|_{\mathcal{G}^{2n-2k}} \\ &\leq (r + (2^n - 1)c_2)(r + (2^n - 1)c_2)^k \|F\|_{\mathcal{G}^{2n}}, \end{split}$$

where the last inequality follows by induction hypothesis. This proves the claim (4.6). It then follows that

$$\||\nabla_{\mathcal{M}_{N,r}}L_0^k F|_{\mathrm{op}}\|_{\infty} \le N^{-1/2} \|L_0^k F\|_{\mathcal{G}^{2n-2k}} \le CN^{-1/2},$$

where the first inequality follows by Definition 4.1 and the second inequality by (4.6). Therefore, by Lemma 4.7 there exists a universal constant $K_2 > 0$ such that for every $\gamma, T_0 > 0$ and every N,

$$\mathbb{Q}_{\boldsymbol{X}}\left(\sup_{t\in[0,T_0]}|M_t^{L_0^kF}|\leq\frac{\gamma}{\sqrt{N}}\right)\geq 1-K_2\exp(-\gamma^2/(K_2T_0))$$

For every $0 \le k \le n-1$, we then bound $|L_0^k F(0)|$ by $\gamma N^{-\frac{1}{2}}$ using assumption (3). We therefore have that the first line of (4.5) is bounded by

$$\sum_{k=0}^{n-1} \int_0^t \cdots \int_0^{t_{k-1}} \left(\left| L_0^k F(0) \right| + \left| M_{t_k}^{L_0^k F} \right| \right) dt_k \cdots dt_1 \le \frac{2\gamma}{\sqrt{N}} \sum_{k=0}^{n-1} \frac{t^k}{k!} \le \frac{2\gamma}{\sqrt{N}} \sum_{k=0}^{n-1} t^k,$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1-K_2 \exp(-\gamma^2/(K_2T_0))$. Since $|L_0^n F| = ||L_0^n F||_{\infty} \leq C(n, r, c_2, c_3)$ by (4.6), the second line of (4.5) is bounded by

$$\int_0^t \cdots \int_0^{t_{n-1}} |L_0^n F(t_n)| dt_n \cdots dt_1 \le C \frac{t^n}{n!} \le C' t^n.$$

We turn to the last line of (4.5). According to Lemma 4.3 and (4.6), it follows that

$$|A_{ij}L_0^{k-1}F| = ||A_{ij}L_0^{k-1}F||_{\infty} \le \frac{1}{N} ||\psi_{ij}||_{\mathcal{G}^1} ||L_0^{k-1}F||_{\mathcal{G}^1} \le \tilde{C}(n, r, c_1, c_2, c_3).$$

Therefore, using assumption (4), we bound the last line of (4.5) by

$$\sum_{1 \le i,j \le r} \sum_{k=1}^{n} \int_{0}^{t} \cdots \int_{0}^{t_{k-1}} |a_{ij}(t_k)| dt_k \cdots dt_1 \le \sum_{1 \le i,j \le r} \sum_{k=1}^{n} \varepsilon^{k-1} \int_{0}^{t} |a_{ij}(s)| ds \le \frac{1}{1-\varepsilon} \sum_{1 \le i,j \le r} \int_{0}^{t} |a_{ij}(s)| ds.$$

Choosing $K = 2 \lor C' \lor \tilde{C}$ gives the desired estimate.

To prove the result under assumption (3'), we begin by noting that the third term in (4.5) can be rewritten as

$$\sum_{k=1}^{n-1} \int_0^t \cdots \int_0^{t_{k-1}} L_0^k F(0) dt_k \cdots dt_1 = \sum_{k=1}^{n-1} L_0^k F(0) \frac{t^k}{k!},$$

and for every $t \in (0, 1)$, an order *n* Taylor expansion gives:

$$e^{tL_0}F(0) = \sum_{k=1}^{n-1} L_0^k F(0) \frac{t^k}{k!} + L_0^n F(\tilde{t}) \frac{t^n}{n!}$$
(4.7)

for some $\tilde{t} \in [0, t]$, so that the following estimate holds according to the ladder relations (see Lemma 4.3)

$$|e^{tL_0}F(0) - \sum_{k=1}^{n-1} L_0^k F(0) \frac{t^k}{k!}| \le ||L_0^n F||_{\infty} \frac{t^n}{n!}.$$
(4.8)

4.2. Evolution equations for the correlations and eigenvalues

We are interested in the evolution of the correlations m_{ij} under Langevin dynamics. In particular, we let $m_{ij}(\mathbf{X}_t^{\beta})$ denote the evolution of \mathbf{X}_t^{β} according to (1.6).

Lemma 4.5 (Itô's formula in $\mathbb{R}^{N \times r}$). Let X_t solve the following $\mathbb{R}^{N \times r}$ -valued SDE:

$$d\boldsymbol{X}_t = a(\boldsymbol{X}_t)dt + b\,d\boldsymbol{B}_t, t > 0$$

with initial solution $\mathbf{X}_0 \in \mathbb{R}^{N \times r}$. Here, $(\mathbf{B}_t)_{t \geq 0}$ is a $\mathbb{R}^{N \times r}$ -valued Brownian motion, $a: \mathbb{R}^{N \times r} \to \mathbb{R}^{N \times r}$ is a measurable function, and b is a constant. Suppose that F is a C^2 function from some open domain $D \subset \mathbb{R}^{N \times r}$ into \mathbb{R} . Suppose that almost surely, $\mathbf{X}_t \in D$ for all $t \geq 0$. Then, the process $(F(\mathbf{X}_t))_{t \geq 0}$ satisfies

$$F(\boldsymbol{X}_t) = F(\boldsymbol{X}_0) + M_t^F + \int_0^t LF(\boldsymbol{X}_s) ds,$$

where $M_t^F = b \int_0^t \langle \hat{\nabla} F(\boldsymbol{X}_s), d\boldsymbol{B}_s \rangle$ is a local martingale such that $[M_t^F] \leq b^2 t \| \hat{\nabla} F \|_{L^{\infty}}^2$ and $LF(\boldsymbol{X}_t) = \frac{b^2}{2} \hat{\Delta} F(\boldsymbol{X}_t) + \langle \hat{\nabla} F(\boldsymbol{X}_t), \hat{\nabla} a(\boldsymbol{X}_t) \rangle.$

For every $1 \leq i, j \leq r$, the correlations m_{ij} are smooth functions from $\mathcal{M}_{N,r} \subset \mathbb{R}^{N \times r}$ to \mathbb{R} . Thus, according to the Itô's formula, we have that

$$m_{ij}(\boldsymbol{X}_{t}^{\beta}) = m_{ij}(\boldsymbol{X}_{0}^{\beta}) + M_{t}^{m_{ij}} + \int_{0}^{t} L_{\beta}m_{ij}(\boldsymbol{X}_{s}^{\beta})ds$$

where $M_t^{m_{ij}} = \sqrt{2} \int_0^t \langle \nabla m_{ij}(\boldsymbol{X}_s^{\beta}), d\boldsymbol{B}_s \rangle$ and $L_{\beta}m_{ij}(\boldsymbol{X}_t^{\beta}) = \Delta m_{ij}(\boldsymbol{X}_t^{\beta}) - \beta \langle \nabla H(\boldsymbol{X}_t^{\beta}), \hat{\nabla} m_{ij}(\boldsymbol{X}_t^{\beta}) \rangle$, as stated by (1.8). An explicit computation of the generator gives the the following evolution equations for $\{m_{ij}(\boldsymbol{X}_t^{\beta})\}_{1 \leq i,j \leq r}$. We write $m_{ij}(t) = m_{ij}(\boldsymbol{X}_t^{\beta})$ to simplify notation slightly.

Lemma 4.6 (Evolution equations for m_{ij}). For every $1 \le i, j \le r$, the evolution equation for m_{ij} is given by

$$dm_{ij}(t) = L_\beta m_{ij}(t)dt + dM_t^{m_{ij}}$$

where

$$L_{\beta}m_{ij} = L_{0,\beta}m_{ij} + \beta\sqrt{M}p\lambda_i\lambda_jm_{ij}^{p-1} - \beta\sqrt{M}\frac{p}{2}\sum_{1\le k,\ell\le r}\lambda_km_{kj}m_{k\ell}m_{i\ell}\left(\lambda_jm_{kj}^{p-2} + \lambda_\ell m_{k\ell}^{p-2}\right),$$

and

$$L_{0,\beta}m_{ij} = \Delta m_{ij} - \beta \langle \nabla H_0, \hat{\nabla} m_{ij} \rangle.$$

Proof. According to (1.8), we have that

$$L_{\beta}m_{ij} = \Delta m_{ij} - \beta \langle \nabla H, \hat{\nabla} m_{ij} \rangle = L_{0,\beta}m_{ij} + \beta \langle \nabla \Phi, \hat{\nabla} m_{ij} \rangle,$$

where $\Phi(\mathbf{X}) = N\sqrt{M} \sum_{1 \le i,j \le r} \lambda_i \lambda_j m_{ij}^p(\mathbf{X})$. The Riemannian gradient of Φ on the manifold $\mathcal{M}_{N,r}$ is given by

$$\nabla \Phi(\boldsymbol{X}) = \hat{\nabla} \Phi(\boldsymbol{X}) - \frac{1}{2N} \boldsymbol{X} \left(\boldsymbol{X}^{\top} \hat{\nabla} \Phi(\boldsymbol{X}) + \hat{\nabla} \Phi(\boldsymbol{X})^{\top} \boldsymbol{X} \right).$$

An explicit computation gives that

$$(\nabla \Phi(\boldsymbol{X}))_{j} = \nabla_{\boldsymbol{x}_{j}} \Phi(\boldsymbol{X}) - \frac{1}{2N} \sum_{i=1}^{r} \left(\boldsymbol{X}^{\top} \hat{\nabla} \Phi(\boldsymbol{X}) + \hat{\nabla} \Phi(\boldsymbol{X})^{\top} \boldsymbol{X} \right)_{ij} \boldsymbol{x}_{i}$$

$$= \sqrt{M} \sum_{k=1}^{r} p \lambda_{k} \lambda_{j} m_{kj}^{p-1} \boldsymbol{v}_{k} - \sqrt{M} \frac{p}{2} \sum_{k=1}^{r} \sum_{\ell=1}^{r} \lambda_{k} m_{kj} m_{k\ell} \left(\lambda_{j} m_{kj}^{p-2} + \lambda_{\ell} m_{k\ell}^{p-2} \right) \boldsymbol{x}_{\ell}.$$

Since $\nabla_{\boldsymbol{x}_j} m_{ij} = \frac{\boldsymbol{v}_i}{N}$ and $\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = N \delta_{ij}$, we obtain that

$$\langle \nabla \Phi, \hat{\nabla} m_{ij} \rangle = \sqrt{M} p \lambda_i \lambda_j m_{ij}^{p-1} - \sqrt{M} \frac{p}{2} \sum_{1 \le k, \ell \le r} \lambda_k m_{i\ell} m_{kj} m_{k\ell} \left(\lambda_j m_{kj}^{p-2} + \lambda_\ell m_{k\ell}^{p-2} \right).$$

We also recall a classical estimate to bound the martingale part of the evolution of a function under Langevin dynamics, based on the Doob's maximal inequality.

Lemma 4.7 (Sub-Gaussian tail bound for local martingales). Let $F: \mathcal{M}_{N,r} \to \mathbb{R}$ be a smooth function such that $||F||_{\mathcal{G}^1} \leq K$ and let $F(t) = F(\mathbf{X}_t^\beta)$ denote its evolution under Langevin dynamics (1.6). Then, for every $\varepsilon, T > 0$ and every N, we have that

$$\sup_{\boldsymbol{X}_0 \in \mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}_0} \left(\sup_{t \in [0,T]} \left| M_t^F \right| \ge \varepsilon \right) \le 2 \exp\left(-\frac{N\varepsilon^2}{4K^2T} \right).$$

Proof. According to Lemma 4.5, M_t^F is a local martingale and its quadratic variation $[M_t^F]$ satisfies

$$\left[M_{t}^{F}\right] \leq 2t \|\nabla F\|_{L^{\infty}}^{2} \leq \frac{2t}{N} \|F\|_{\mathcal{G}^{1}}^{2} \leq \frac{2K^{2}t}{N},$$

where the second inequality follows by definition of the \mathcal{G}^1 -norm. For every $\lambda \geq 0$, $Z_t^{\lambda} = \exp(\lambda M_t^F - \lambda^2 [M_t^F]/2)$ is a positive super-martingale such that $Z_0^{\lambda} = 1$ and $\sup_{\mathbf{X}_0} \mathbb{E}_{\mathbb{Q}_{\mathbf{X}_0}} \left[Z_t^{\lambda} \right] \leq 1$ for all $t \geq 0$. For all $t, \lambda, \varepsilon \geq 0$, we then have that

$$\begin{split} \sup_{\boldsymbol{X}_{0}} \mathbb{Q}_{\boldsymbol{X}_{0}} \left(\sup_{t \in [0,T]} M_{t}^{F} \geq \varepsilon \right) &\leq \sup_{\boldsymbol{X}_{0}} \mathbb{Q}_{\boldsymbol{X}_{0}} \left(\sup_{t \in [0,T]} X_{t}^{\lambda} \geq e^{\lambda \varepsilon - \frac{\lambda^{2}}{2} \frac{2K^{2}T}{N}} \right) \\ &\leq \sup_{\boldsymbol{X}_{0}} \mathbb{E}_{\mathbb{Q}_{\boldsymbol{X}_{0}}} \left[Z_{0}^{\lambda} \right] e^{\lambda^{2} \frac{K^{2}T}{N} - \lambda \varepsilon} = e^{\lambda^{2} \frac{K^{2}T}{N} - \lambda \varepsilon}, \end{split}$$

where we used the maximal inequality for the super-martingale Z^{λ} for the last inequality. Since the above inequality holds for all $\lambda \geq 0$, we can take $\lambda = \varepsilon N/(2K^2T)$ and obtain that

$$\sup_{\mathbf{X}_0} \mathbb{Q}_{\mathbf{X}_0} \left(\sup_{t \in [0,T]} M_t^F \ge \varepsilon \right) \le \exp\left(-\frac{N\varepsilon^2}{4K^2T}\right).$$

Proceeding similarly for -M we get

$$\sup_{\boldsymbol{X}_0} \mathbb{Q}_{\boldsymbol{X}_0} \left(\sup_{t \in [0,T]} (-M_t^F) \ge \varepsilon \right) \le \exp\left(-\frac{N\varepsilon^2}{4K^2T} \right),$$

yielding the desired result by union bound.

In particular, since $||m_{ij}||_{\mathcal{G}^n} \leq c(n)$, it follows from Lemma 4.7 that there is a universal constant K > 0 such that for every $\gamma, T > 0$ and every N,

$$\sup_{\mathbf{X}_{0}} \mathbb{Q}_{\mathbf{X}_{0}} \left(\sup_{t \in [0,T]} \left| M_{t}^{m_{ij}} \right| \geq \frac{\gamma}{\sqrt{N}} \right) \leq 2 \exp \left(-\frac{\gamma^{2}}{KT} \right).$$

$$(4.9)$$

In addition to the evolution of the correlations m_{ij} under Langevin dynamics, we are also interested in the evolution of the matrix-valued function $G(\mathbf{X}) = \mathbf{M}^{\top} \mathbf{M}$, where $\mathbf{M} = (m_{ij})_{1 \leq i,j \leq r}$ is the correlation matrix. The latter is needed to study the case p = 2 and $\lambda_1 = \cdots = \lambda_r$. Let $\mathbf{G} \colon \mathcal{M}_{N,r} \to \mathbb{R}^{r \times r}$ denote

$$\boldsymbol{G}(\boldsymbol{X}) = \boldsymbol{M}^{\top} \boldsymbol{M} = \frac{1}{N^2} \boldsymbol{X}^{\top} \boldsymbol{V} \boldsymbol{V}^{\top} \boldsymbol{X}, \qquad (4.10)$$

where each entry G_{ij} is a smooth function from $\mathcal{M}_{N,r}$ to \mathbb{R} such that $G_{ij}(\mathbf{X}) = \sum_{k=1}^{r} m_{ki}(\mathbf{X}) m_{kj}(\mathbf{X})$. Note that $G_{ij} = G_{ji}$ for every $i, j \in [r]$, that is, $\mathbf{G}(\mathbf{X})$ is a symmetric matrix.

Lemma 4.8 (Evolution equation for G_{ij}). For every $1 \le i, j \le r$, the evolution equation for G_{ij} is given by

$$dG_{ij}(t) = L_{\beta}G_{ij}(t)dt + dM_t^{G_{ij}}$$

where $M_t^{G_{ij}} = \sqrt{2} \int_0^t \langle \nabla G_{ij}(s), d\mathbf{B}_s \rangle$ is the martingale part of the evolution, and the generator $L_\beta G_{ij}(t)$ is given by $L_\beta G_{ij}(t) = L_{0,\beta} G_{ij}(t) + \hat{L}_\beta G_{ij}(t)$ and is of the form

$$L_{0,\beta}G_{ij} = \sum_{k=1}^{\prime} \left(m_{ki}L_{0,\beta}m_{kj} + m_{kj}L_{0,\beta}m_{ki} \right) + \frac{2}{N} \left(r\delta_{ij} - G_{ij} \right), \tag{4.11}$$

and

$$\hat{L}_{\beta}G_{ij} = 4\beta\sqrt{M}\lambda^2(G_{ij} - (\boldsymbol{G}^2)_{ij}).$$
(4.12)

Moreover, the quadratic variation of $M_t^{G_{ij}}$ satisfies

$$[M_t^{G_{ij}}] \le \frac{2r}{N} \int_0^t \|\boldsymbol{M}(s)\|_{\text{op}}^2 ds.$$
(4.13)

Proof. For every $i, j \in [r]$, the summary statistic $G_{ij} \colon \mathcal{M}_{N,r} \to \mathbb{R}$ is a smooth function, thus by Itô's Lemma 4.5 we have that

$$G_{ij}(\boldsymbol{X}_t) = G_{ij}(\boldsymbol{X}_0) + M_t^{G_{ij}} + \int_0^t L_\beta G_{ij}(\boldsymbol{X}_s) ds$$

where $M_t^{G_{ij}} = \sqrt{2} \int_0^t \langle \nabla G_{ij}(\boldsymbol{X}_s), d\boldsymbol{B}_s \rangle$ is a local martingale and $L_\beta G_{ij}(\boldsymbol{X})$ is given by $L_\beta G_{ij}(\boldsymbol{X}) = L_{0,\beta}G_{ij}(\boldsymbol{X}) + \beta \langle \nabla \Phi(\boldsymbol{X}), \hat{\nabla} G_{ij}(\boldsymbol{X}) \rangle$ with $\Phi(\boldsymbol{X}) = H_0(\boldsymbol{X}) - H(\boldsymbol{X})$. By an explicit computation as done in the proof of Lemma 4.6, we then find that the population generator $\hat{L}_\beta = \beta \langle \nabla \Phi(\boldsymbol{X}), \hat{\nabla} \cdot \rangle$ satisfies

$$\hat{L}_{\beta}G_{ij} = 4\beta\sqrt{M}\lambda^2(G_{ij} - (\boldsymbol{G}^2)_{ij})$$

Moreover, according to [33, Proposition 1.3.1] the quadratic variation $[M_t^{G_{ij}}]$ of $M_t^{G_{ij}}$ satisfies

$$[M_t^{G_{ij}}] = \int_0^t \|\nabla G_{ij}(s)\|_{\mathbf{F}}^2 ds \le \int_0^t \|\hat{\nabla} G_{ij}(s)\|_{\mathbf{F}}^2 ds$$

where we recall that ∇ denotes the Euclidean gradient. Since $\nabla_{\boldsymbol{x}_k} m_{ij}(\boldsymbol{X}) = \frac{\boldsymbol{v}_i}{N} \delta_{kj}$, we have that

$$\|\hat{\nabla}G_{ij}\|_{\mathrm{F}}^{2} = \mathrm{Tr}((\hat{\nabla}G_{ij})^{\top}(\hat{\nabla}G_{ij})) = \frac{2}{N}\sum_{k=1}^{r}(m_{ki}^{2} + m_{kj}^{2}) = \frac{2}{N}(G_{ii} + G_{jj}) \le \frac{2}{N}\mathrm{Tr}(\boldsymbol{G}).$$

In particular, we find that

$$[M_t^{G_{ij}}] \le \frac{2}{N} \int_0^t \operatorname{Tr}(\boldsymbol{G}(s)) ds = \frac{2}{N} \int_0^t \|\boldsymbol{M}(s)\|_{\mathrm{F}}^2 ds \le \frac{2r}{N} \int_0^t \|\boldsymbol{M}(s)\|_{\mathrm{op}}^2 ds,$$

as desired.

We readily obtain the following corollary.

Corollary 4.9 (Evolution equation for G). The evolution equation for G is given by

$$\boldsymbol{G}(t) = \boldsymbol{G}(0) + M_t^{\boldsymbol{G}} + \int_0^t \boldsymbol{L}_{\beta} \boldsymbol{G}(s) ds,$$

where $M_t^{\boldsymbol{G}} = (M_t^{G_{ij}})_{ij} \in \mathbb{R}^{r \times r}$ and $\boldsymbol{L}_{\beta}\boldsymbol{G}(t) = (L_{\beta}G_{ij}(t))_{ij} \in \mathbb{R}^{r \times r}$. In particular, $\boldsymbol{L}_{\beta}\boldsymbol{G}$ is of the form $\boldsymbol{L}_{\beta}\boldsymbol{G} = \boldsymbol{L}_{0,\beta}\boldsymbol{G} + \hat{\boldsymbol{L}}_{\beta}\boldsymbol{G}$, where

$$\boldsymbol{L}_{0,\beta}\boldsymbol{G} = \left(\boldsymbol{L}_{0,\beta}\boldsymbol{M}\right)^{\top}\boldsymbol{M} + \boldsymbol{M}^{\top}\left(\boldsymbol{L}_{0,\beta}\boldsymbol{M}\right) + \frac{2}{N}\left(r\boldsymbol{I}_{r} - \boldsymbol{G}\right), \qquad (4.14)$$

and

$$\hat{\boldsymbol{L}}_{\beta}\boldsymbol{G} = 4\beta\sqrt{M}\lambda^2(\boldsymbol{G} - \boldsymbol{G}^2).$$
(4.15)

Similarly to the sub-Gaussian tail bound for the martingale part of the evolution of the correlations m_{ij} given by (4.9) obtained by the Doob's maximal inequality (see Lemma 4.7), we have the following result.

Lemma 4.10. For every $\gamma > 0$, T > 0 and every positive, increasing, and deterministic function f(t) verifying

$$f(t) \ge \int_0^t \|\boldsymbol{M}(s)\|_{\rm op}^2 ds,$$

for all $t \in [0, T]$, we have that

$$\sup_{\boldsymbol{X}_{0} \in \mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}_{0}} \left(\sup_{t \in [0,T]} \| M_{t}^{\boldsymbol{G}} \|_{\mathrm{op}} \geq \frac{\gamma}{\sqrt{N}} f(T)^{1/2} \right) \leq r(r+1) e^{-\gamma^{2}/(4r^{2})}$$

Proof. We first note that $||M_t^G||_{\text{op}} \leq r \max_{1 \leq i,j \leq} |M_t^{G_{ij}}|$, where $M_t^{G_{ij}}$ is a local martingale given by Lemma 4.8 such that its quadratic variation satisfies (4.13), i.e., for every $i, j \in [r]$

$$[M_t^{G_{ij}}] \le \frac{2r}{N} \int_0^t \|\boldsymbol{M}(s)\|_{\text{op}}^2 ds \le \frac{2r}{N} f(t)$$

For every $\lambda \geq 0$, we now introduce the exponential martingale $Z_t^{\lambda} = \exp(\lambda M_t^{G_{ij}} - \lambda^2 [M_t^{G_{ij}}]/2)$. The same reasoning used in the proof of Lemma 4.7 gives that

$$\sup_{\boldsymbol{X}_0} \mathbb{Q}_{\boldsymbol{X}_0} \left(\sup_{t \in [0,T]} |M_t^{G_{ij}}| \ge \varepsilon \right) \le 2 \exp\left(-\frac{N\varepsilon^2}{4rf(T)}\right)$$

for every $i, j \in [r]$. From this, we then have that

$$\begin{split} \sup_{\boldsymbol{X}_{0}} \mathbb{Q}_{\boldsymbol{X}_{0}} \left(\sup_{t \in [0,T]} \| M_{t}^{\boldsymbol{G}} \|_{\mathrm{op}} \geq \varepsilon \right) &\leq \sup_{\boldsymbol{X}_{0}} \mathbb{Q}_{\boldsymbol{X}_{0}} \left(\sup_{t \in [0,T]} \max_{1 \leq i,j \leq r} | M_{t}^{\boldsymbol{G}_{ij}} | \geq \frac{\varepsilon}{r} \right) \\ &\leq \frac{r(r+1)}{2} \sup_{\boldsymbol{X}_{0}} \mathbb{Q}_{\boldsymbol{X}_{0}} \left(\sup_{t \in [0,T]} | M_{t}^{\boldsymbol{G}_{11}} | \geq \frac{\varepsilon}{r} \right) \\ &\leq r(r+1) \exp\left(-\frac{N\varepsilon^{2}}{4r^{2}f(T)}\right). \end{split}$$

Choosing $\varepsilon = \frac{\gamma}{\sqrt{N}} f(T)^{1/2}$ gives the desired result, since f is an increasing function by assumption. \Box

4.3. Comparison inequalities

We first report Lemma 5.1 of [8] that provides simple comparison inequalities for functions.

Lemma 4.11 (Bounds on functions). Let $\gamma > 0$ with $\gamma \neq 1$, c > 0, and $f \in C_{loc}([0,T))$ with f(0) > 0. (a) Suppose that there exists T such that f satisfies the integral inequality

$$f(t) \ge a + \int_0^t c f^{\gamma}(s) ds, \tag{4.16}$$

for every $t \leq T$ and some a > 0. Then, for $t \geq 0$ satisfying $(\gamma - 1)ca^{\gamma - 1}t < 1$, we have that

$$f(t) \ge a \left(1 - (\gamma - 1)ca^{\gamma - 1}t\right)^{-\frac{1}{\gamma - 1}}$$

- (b) If the integral inequality (4.16) holds in reverse, i.e., if $f(t) \leq a + \int_0^t c f^{\gamma}(s) ds$, then the corresponding upper bound holds.
- (c) If $\gamma > 1$, then $T \leq t_*$, where $t_* = ((\gamma 1)ca^{\gamma 1})^{-1}$ is called the blow-up time.
- (d) If (4.16) holds with $\gamma = 1$, then the Grönwall's inequality gives $f(t) \ge a \exp(ct)$.

Lemma 4.11 will be useful in providing bounds on the correlation functions $m_{ij}^{(N)}$ for both $p \geq 3$ and p = 2. For subspace recovery when p = 2, we need to establish comparison inequalities for the eigenvalues of the matrix-valued function G defined by (4.10). The following lemma is a standard result on the rate of change of eigenvalues for matrix differential equations, as found e.g. in [27, 45]. For clarity and completeness, we include a brief proof. **Lemma 4.12.** Let G be an entry-wise continuously differentiable matrix-valued function defined on $[0, \infty)$ and taking values in the space of symmetric positive semi-definite matrices of size $r \times r$. Let $\theta_i(t)$ denote the *i*th eigenvalue of G(t) and $u_i(t)$ be the corresponding normalized eigenvector. Then, for every $i \in [r]$, $\theta_i(t)$ and $u_i(t)$ are continuously differentiable and satisfy

$$\dot{\theta}_i(t) = \boldsymbol{u}_i(t)^{\top} \dot{\boldsymbol{G}}(t) \boldsymbol{u}_i(t).$$

Proof. For the regularity of the eigenvalues and eigenvectors of parameter-dependent symmetric matrices, we refer to [37, Chapter 2, Sections 5 and 6]. For every $i \in [r]$, we have that

$$(\boldsymbol{G}(t) - \theta_i(t)\boldsymbol{I}_r) \,\boldsymbol{u}_i(t) = 0.$$

Differentiating with respect to t gives

$$\left(\dot{\boldsymbol{G}}(t) - \dot{\theta}_i(t)\boldsymbol{I}_r\right)\boldsymbol{u}_i(t) + \left(\boldsymbol{G}(t) - \theta_i(t)\boldsymbol{I}_r\right)\dot{\boldsymbol{u}}_i(t) = 0.$$

Since $\dot{\boldsymbol{u}}_i(t) \in \mathbb{R}^r$, it can be expressed as a linear combination of the orthonormal basis $\{\boldsymbol{u}_1(t), \ldots, \boldsymbol{u}_r(t)\}$, i.e.,

$$\dot{\boldsymbol{u}}_i(t) = \sum_{1 \le j \le r} \alpha_{ij}(t) \boldsymbol{u}_j(t),$$

for a collection of coefficients $\alpha_{ij}(t)$. Therefore, we have that

$$\left(\dot{\boldsymbol{G}}(t) - \dot{\theta}_i(t)\boldsymbol{I}_r\right)\boldsymbol{u}_i(t) + \sum_{1 \le j \le r} \alpha_{ij}(t) \left(\boldsymbol{G}(t) - \theta_i(t)\boldsymbol{I}_r\right)\boldsymbol{u}_j(t) = 0.$$

By expanding the second term we have that

$$\sum_{1 \le j \le r} \alpha_{ij}(t) \left(\boldsymbol{G}(t) - \theta_i(t) \boldsymbol{I}_r \right) \boldsymbol{u}_j(t) = \sum_{1 \le j \le r} \alpha_{ij}(t) \left(\theta_j(t) - \theta_i(t) \right) \boldsymbol{u}_j(t) = \sum_{\substack{1 \le j \le r \\ j \ne i}} \alpha_{ij}(t) \left(\theta_j(t) - \theta_i(t) \right) \boldsymbol{u}_j(t).$$

Multiplying from the left by \boldsymbol{u}_i^{\top} the term

$$\left(\dot{\boldsymbol{G}}(t) - \dot{\boldsymbol{\theta}}_{i}(t)\boldsymbol{I}_{r}\right)\boldsymbol{u}_{i}(t) + \sum_{\substack{1 \leq j \leq r \\ j \neq i}} \alpha_{ij}(t) \left(\boldsymbol{\theta}_{j}(t) - \boldsymbol{\theta}_{i}(t)\right)\boldsymbol{u}_{j}(t) = 0$$

It.

gives the desired result.

In the case of Langevin dynamics however, the matrix-valued function G(t) is not differentiable and we need an integral comparison inequality, as stated in the following lemma.

Lemma 4.13. Assume that the matrix-valued function $A \in \mathbb{R}^{r \times r}$ satisfies

$$\begin{cases} \frac{d}{dt} \mathbf{A}(t) = a \mathbf{G}(t) (\mathbf{I}_r - \mathbf{G}(t)) \\ \mathbf{A}(0) = \mathbf{G}(0) \end{cases}$$

for a > 0. Then, for every $i \in [r]$ and every $t \ge 0$, we have that

$$\theta_i(0) + a \int_0^t (\theta_{\min}(s) - \theta_{\max}^2(s)) ds \le \mu_i (\mathbf{A}(t)) \le \theta_i(0) + a \int_0^t (\theta_{\max}(s) - \theta_{\min}^2(s)) ds,$$
(4.17)

where $\mu_i(\mathbf{A}(t))$ denotes the *i*th eigenvalue of $\mathbf{A}(t)$. Moreover, for every $i \in [r]$ and every $t \ge 0$ such that $\theta_{\max}(t) \le \frac{1}{2}$, we have that

$$\theta_i(0) + a \int_0^t (\theta_{\min}(s) - \theta_{\min}^2(s)) ds \le \mu_i (\mathbf{A}(t)) \le \theta_i (\mathbf{G}(0)) + a \int_0^t (\theta_{\max}(s) - \theta_{\max}^2(s)) ds.$$
(4.18)

Proof. We show the lower bound. The upper bound is obtained in a similar fashion. If $u_i^A(t)$ denotes the *i*th normalized eigenvector of A(t), by Lemma 4.12 we have that

$$\dot{\mu}_{i} \left(\boldsymbol{A}(t) \right) = \left(\boldsymbol{u}_{i}^{\boldsymbol{A}}(t) \right)^{\top} \frac{d}{dt} \boldsymbol{A}(t) \boldsymbol{u}_{i}^{\boldsymbol{A}}(t)$$
$$= a \left(\boldsymbol{u}_{i}^{\boldsymbol{A}}(t) \right)^{\top} \boldsymbol{G}(t) (\boldsymbol{I}_{r} - \boldsymbol{G}(t)) \boldsymbol{u}_{i}^{\boldsymbol{A}}(t)$$
$$\geq a \lambda_{\min} \left(\boldsymbol{G}(t) - \boldsymbol{G}^{2}(t) \right),$$

where the inequality follows from the fact that $\theta_{\min}(\mathbf{M}) = \min_{\mathbf{u}: \|\mathbf{u}\|=1} \mathbf{u}^{\top} \mathbf{M} \mathbf{u}$ for any symmetric matrix \mathbf{M} . Since \mathbf{G} is positive semi-definite, the eigenvalues of $\mathbf{G} - \mathbf{G}^2$ are given by $\theta_i - \theta_i^2$ for every $i \in [r]$, where $\theta_1, \ldots, \theta_r$ denote the eigenvalues of \mathbf{G} . In particular, we have that

$$\lambda_{\min} \left(\boldsymbol{G}(t) - \boldsymbol{G}(t)^2 \right) = \min_{1 \le i \le r} \left(\theta_i(t) - \theta_i^2(t) \right).$$

From this, it follows that

$$\min_{1 \le i \le r} \left(\theta_i(t) - \theta_i^2(t) \right) \ge \theta_{\min}(t) - \theta_{\max}^2(t),$$

yielding

$$\mu_i(\mathbf{A}(t)) \ge \mu_i(\mathbf{A}(0)) + a \int_0^t \left(\theta_{\min}(s) - \theta_{\max}^2(s)\right) ds = \theta_i(0) + a \int_0^t \left(\theta_{\min}(s) - \theta_{\max}^2(s)\right) ds,$$

thus proving (4.17). The lower bound in (4.18) can be established similarly by noting that the function $x \mapsto x - x^2$ is increasing on $[0, \frac{1}{2}]$. We therefore have that

$$\min_{1 \le i \le r} \left(\theta_i(t) - \theta_i^2(t) \right) = \theta_{\min}(t) - \theta_{\min}^2(t),$$

provided $\theta_{\max}(t) \leq \frac{1}{2}$.

5. Proofs for $p \geq 3$

This section is devoted to the proof of Propositions 3.5 and 3.6.

5.1. Recovery of leading spike

We begin by showing Proposition 3.5. The first step is to show that weak recovery of the first spike is achieved, provided \sqrt{M} scales as $N^{\frac{p-2}{2}}$, as stated in the following lemma.

Lemma 5.1. Let $\beta \in (0, \infty)$, $p \geq 3$ and $\lambda_i = \lambda_{0,i}\lambda_{i+1}$ for every $1 \leq i \leq r-1$ and $\lambda_{0,i} \geq 1$. Consider a sequence of initializations $\mu_0 \in \mathcal{P}(\mathcal{M}_{N,r})$. Let $\varepsilon_N = N^{-\frac{p-2}{2(p-1)}}$ and let $\mathcal{T}_{\varepsilon_N}^{(11)}$ denote the hitting time of the set $\{\mathbf{X} : m_{11}(\mathbf{X}) \geq \varepsilon_N\}$. Then, the following holds: For every $n \geq 1, \gamma_0 > 0, \gamma_1 > 1 > \gamma_2 > 0$, there exists $C_0 \in (0, \frac{1}{2})$ such that if $\lambda_{0,1} > \frac{1+C_0}{1-C_0} \left(\frac{3\gamma_1}{\gamma_2}\right)^{p-2}$ and $\sqrt{M} \gtrsim \frac{(n+2)\gamma_0}{\beta p \lambda_r^2 C_0 \gamma_2^{p-1}} N^{\frac{p-1}{2} - \frac{n}{2(n+1)}}$, then for N sufficiently large we have that

$$\int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{\varepsilon_{N}}^{(11)} \gtrsim \frac{1}{(n+2)\gamma_{0}N^{\frac{1}{2(n+1)}}} \right) \mathbf{1} \{ \mathcal{C}_{0}(n,\gamma_{0}) \cap \mathcal{C}_{1}(\gamma_{1},\gamma_{2}) \} d\mu_{0}(\boldsymbol{X})$$

$$\leq K_{1} e^{-\gamma_{0}^{3}(n+2)N^{\frac{1}{2(n+1)}}/K_{1}} + r^{2}K_{2} e^{-\gamma_{2}^{2}\gamma_{0}(n+2)N^{\frac{1}{2(n+1)}}/K_{2}},$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$.

The strategy used to prove weak recovery of the first spike is the following. We first show that, since λ_1 is much larger than the other SNRs, the correlation m_{11} is the first correlation to reach the microscopic threshold $\tilde{\varepsilon}_N = \tilde{\gamma} N^{-\frac{1}{2}}$, where $\tilde{\gamma}$ is a sufficiently large constant of order 1 in N. Therefore, despite a potentially lower initial value $m_{11}(0) < \max_{i,j} m_{ij}(0)$, the strength of λ_1 ensures that m_{11} exceeds $\tilde{\varepsilon}_N$ before the other correlations do. Furthermore, we show that m_{11} keeps increasing and becomes the first correlation to cross the critical threshold $\varepsilon_N = N^{-\frac{p-2}{2(p-1)}}$. Given the algorithmic threshold $M \sim N^{p-1-\frac{n}{n+1}}$, there is no control on the stability of the evolution of the other correlations.

Now, strong recovery of the first spike immediately follows from the weak recovery result, as stated in the following result.

Lemma 5.2. Let $\beta \in (0,\infty)$, $p \geq 3$ and $\varepsilon_N = N^{-\frac{p-2}{2(p-1)}}$. Then, for every $\varepsilon > 0$ and $\sqrt{M} \gtrsim N^{\frac{p-1}{2} - \frac{n}{2(n+1)}}$, there exists $T_0 > \frac{1}{(n+2)\gamma_0} N^{-\frac{1}{2(n+1)}}$ such that for all $T \geq T_0$ and N sufficiently large,

$$\inf_{\mathbf{X}: \ m_{11}(\mathbf{X}) \ge \varepsilon_N} \mathbb{Q}_{\mathbf{X}} \left(\inf_{t \in [T_0, T]} m_{11}(\mathbf{X}_t^\beta) \ge 1 - \varepsilon \right) \\
\ge 1 - K_1 e^{-\gamma_0 (n+2)N^{\frac{2p-1}{2(p-1)} - \frac{n}{2(n+1)}/K_1} - 2K_2 e^{-N\varepsilon^2/(K_2T)},$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$.

Having Lemmas 5.1 and 5.2 at hand, Proposition 3.5 follows by the strong Markov property.

Proof of Proposition 3.5. We note that

$$\int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t \in [T_0,T]} m_{11}(\boldsymbol{X}_t^{\beta}) \ge 1 - \varepsilon \right) d\mu_0(\boldsymbol{X})$$

$$\geq \inf_{\boldsymbol{X} : m_{11}(\boldsymbol{X}) \ge \varepsilon_N} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t \in [T_0,T]} m_{11}(\boldsymbol{X}_t^{\beta}) \ge 1 - \varepsilon \right) \times \int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{\varepsilon_N}^{(11)} \le \frac{1}{(n+2)\gamma_0 N^{\frac{1}{2(n+1)}}} \right) d\mu_0(\boldsymbol{X})$$

where we first conditioned the $\mathbb{Q}_{\mathbf{X}}$ -probability inside the integral by the event $\mathcal{T}_{\varepsilon_N}^{(11)} \leq \frac{1}{(n+2)\gamma_0 N^{\frac{1}{2(n+1)}}}$ and then used the strong Markov property. We estimate the first term according to Lemma 5.2 and the second term by

$$\begin{split} &\int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{\varepsilon_{N}}^{(11)} \geq \frac{1}{(n+2)\gamma_{0}N^{\frac{1}{2(n+1)}}} \right) d\mu_{0}(\boldsymbol{X}) \\ &\leq \mu_{0}(\mathcal{C}_{0}(n,\gamma_{0})^{c}) + \mu_{0}(\mathcal{C}_{1}(\gamma_{1},\gamma_{2})^{c}) \\ &+ \int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{\varepsilon_{N}}^{(11)} \geq \frac{1}{(n+2)\gamma_{0}N^{\frac{1}{2(n+1)}}} \right) \mathbf{1} \{ \mathcal{C}_{0}(n,\gamma_{0}) \cap \mathcal{C}_{1}(\gamma_{1},\gamma_{2}) \} d\mu_{0}(\boldsymbol{X}). \end{split}$$

Combining Lemma 5.1 with Definitions 3.1 and 3.3 gives the estimate for the second term. Proposition 3.5 then follows straightforwardly.

It remains to prove the two intermediate results, namely Lemmas 5.1 and 5.2. We begin with the proof of Lemma 5.1.

Proof of Lemma 5.1. We let $\mathcal{A} = \mathcal{A}(n, \gamma_0, \gamma_1, \gamma_2)$ denote the event

$$\mathcal{A}(n,\gamma_0,\gamma_1,\gamma_2) = \left\{ \boldsymbol{X}_0 \sim \mu \colon \boldsymbol{X}_0 \in \mathcal{C}_0(n,\gamma_0) \cap \mathcal{C}_1(\gamma_1,\gamma_2) \right\},\$$

where we recall that $C_0(n, \gamma_0)$ and $C_1(\gamma_1, \gamma_2)$ are given by Definitions 3.1 and 3.3, respectively. We first note that on $C_1(\gamma_1, \gamma_2)$, for every $i, j \in [r]$ there exists $\gamma_{ij} \in (\gamma_2, \gamma_1)$ such that $m_{ij}(0) = \gamma_{ij}N^{-\frac{1}{2}}$. For some $T_0^{(ij)} > 0$ to be chosen later, we then define the event $\mathcal{A}^{(ij)} = \mathcal{A}^{(ij)}(n, \gamma_0, \gamma_1, \gamma_2, T_0^{(ij)})$ by

$$\mathcal{A}^{(ij)}(n,\gamma_0,\gamma_1,\gamma_2,T_0^{(ij)}) = \mathcal{A}(n,\gamma_0,\gamma_1,\gamma_2) \cap \left\{ \sup_{t \in [0,T_0^{(ij)}]} |M_t^{m_{ij}}| \le \frac{\gamma_2}{2\sqrt{N}} \right\},$$

where we recall that according to Lemma 4.7 and (4.9), there exists an absolute constant $K_1 > 0$ such that

$$\sup_{\boldsymbol{X}} \mathbb{Q}_{\boldsymbol{X}} \left(\sup_{t \in [0, T_0^{(ij)}]} |M_t^{m_{ij}}| \ge \frac{\gamma_2}{2\sqrt{N}} \right) \le K_1 \exp\left(-\frac{\gamma_2^2}{4K_1 T_0^{(ij)}}\right).$$

Moreover, for every $i, j \in [r]$, we let $\mathcal{T}_{L_{0,\beta}}^{(ij)}$ denote the hitting time of the set

$$\left\{\boldsymbol{X}: |L_{0,\beta}m_{ij}(\boldsymbol{X})| > C_0\beta\sqrt{M}p\lambda_i\lambda_j m_{ij}^{p-1}(\boldsymbol{X})\right\},\,$$

where $C_0 \in (0, \frac{1}{2})$ is a constant which does not depend on N. We note that on $\mathcal{C}_0(n, \gamma_0)$,

$$|L_{0,\beta}m_{ij}(\boldsymbol{X}_0)| \leq \frac{\gamma_0}{\sqrt{N}} \leq C_0\beta\sqrt{M}p\lambda_i\lambda_j \left(\frac{\gamma_2}{\sqrt{N}}\right)^{p-1}$$

provided $\sqrt{M} \geq \frac{\gamma_0}{C_0 \beta p \lambda_i \lambda_j \gamma_2^{p-1}} N^{\frac{p-2}{2}}$, which certainly holds by assumption. Therefore, on the event $C_0(n, \gamma_0)$, we have that

$$|L_{0,\beta}m_{ij}(\boldsymbol{X}_0)| \le C_0\beta\sqrt{M}p\lambda_i\lambda_jm_{ij}^{p-1}(\boldsymbol{X}_0),$$

and by continuity of the process \boldsymbol{X}_t , $\mathcal{T}_{L_{0,\beta}}^{(ij)} > 0$. We also introduce the hitting time $\mathcal{T}_{L_{0,\beta}}$ of the set

$$\left\{ \boldsymbol{X} \colon \sup_{1 \le k, \ell \le r} |L_{0,\beta} m_{k\ell}(\boldsymbol{X})| > C_0 \beta \sqrt{M} p \lambda_1^2 m_{11}^{p-1}(\boldsymbol{X}) \right\}.$$

We also have that $\mathcal{T}_{L_{0,\beta}} > 0$ and we note that $\mathcal{T}_{L_{0,\beta}} \leq \mathcal{T}_{L_{0,\beta}}^{(11)}$

In the following, we fix $i, j \in [r]$ and place ourselves on the event $\mathcal{A}^{(ij)}$. We consider the threshold $\varepsilon_N = N^{-\frac{p-2}{2(p-1)}}$ and we denote by $\mathcal{T}_{\varepsilon_N}^{(ij)}$ the hitting time for the set $\{\mathbf{X} : m_{ij}(\mathbf{X}) \ge \varepsilon_N\}$. Recalling the generator expansion given by Lemma 4.6, i.e.,

$$L_{\beta}m_{ij} = L_{0,\beta}m_{ij} + \beta\sqrt{M}p\lambda_i\lambda_j m_{ij}^{p-1} - \beta\sqrt{M}\frac{p}{2}\sum_{1\le k,\ell\le r}\lambda_k m_{i\ell}m_{kj}m_{k\ell}(\lambda_j m_{kj}^{p-2} + \lambda_\ell m_{k\ell}^{p-2}),$$

we have that

$$(1 - C_0)\beta\sqrt{M}p\lambda_i\lambda_j m_{ij}^{p-1}(t) \le L_\beta m_{ij}(t) \le (1 + C_0)\beta\sqrt{M}p\lambda_i\lambda_j m_{ij}^{p-1}(t),$$

for every $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)}$. Since the evolution equation for m_{ij} is given by

$$m_{ij}(t) = m_{ij}(0) + M_t^{m_{ij}} + \int_0^t Lm_{ij}(s)ds$$

we obtain the integral inequality

$$\frac{\gamma_{ij}}{2\sqrt{N}} + (1 - C_0)\beta\sqrt{M}p\lambda_i\lambda_j \int_0^t m_{ij}^{p-1}(s)ds \le m_{ij}(t) \le \frac{3\gamma_{ij}}{2\sqrt{N}} + (1 + C_0)\beta\sqrt{M}p\lambda_i\lambda_j \int_0^t m_{ij}^{p-1}(s)ds, \quad (5.1)$$

which holds for $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k,\ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)} \wedge T_0^{(ij)}$. Applying items (a) and (b) of Lemma 4.11, we then have the comparison inequality

$$\ell_{ij}(t) \le m_{ij}(t) \le u_{ij}(t),\tag{5.2}$$

for $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)} \wedge T_0^{(ij)}$, where the functions ℓ_{ij} and u_{ij} are given by

$$\ell_{ij}(t) = \frac{\gamma_{ij}}{2\sqrt{N}} \left(1 - (1 - C_0)\beta\sqrt{M}p(p-2)\lambda_i\lambda_j \left(\frac{\gamma_{ij}}{2\sqrt{N}}\right)^{p-2} t \right)^{-\frac{1}{p-2}},\tag{5.3}$$

and

$$u_{ij}(t) = \frac{3\gamma_{ij}}{2\sqrt{N}} \left(1 - (1+C_0)\beta\sqrt{M}p(p-2)\lambda_i\lambda_j \left(\frac{3\gamma_{ij}}{2\sqrt{N}}\right)^{p-2} t \right)^{-\frac{1}{p-2}},$$
(5.4)

respectively. We then let $T_{\ell,\varepsilon_N}^{(ij)}$ denote the time such that $\ell_{ij}(T_{\ell,\varepsilon_N}^{(ij)}) = \varepsilon_N$, i.e.,

$$T_{\ell,\varepsilon_N}^{(ij)} = \frac{1 - \left(\frac{\gamma_{ij}}{2}\right)^{p-2} N^{-\frac{p-2}{2(p-1)}}}{(1 - C_0)\beta\sqrt{M}p(p-2)\lambda_i\lambda_j \left(\frac{\gamma_{ij}}{2\sqrt{N}}\right)^{p-2}}.$$
(5.5)

Similarly, we let $T_{u,\varepsilon_N}^{(ij)}$ satisfy $u_{ij}(T_{u,\varepsilon_N}^{(ij)}) = \varepsilon_N$, i.e.,

$$T_{u,\varepsilon_N}^{(ij)} = \frac{1 - \left(\frac{3\gamma_{ij}}{2}\right)^{p-2} N^{-\frac{p-2}{2(p-1)}}}{(1+C_0)\beta\sqrt{M}p(p-2)\lambda_i\lambda_j \left(\frac{3\gamma_{ij}}{2\sqrt{N}}\right)^{p-2}}.$$
(5.6)

On the event $\mathcal{A}^{(ij)}$, we therefore have that $T_{u,\varepsilon_N}^{(ij)} \leq \mathcal{T}_{\varepsilon_N}^{(ij)} \leq T_{\ell,\varepsilon_N}^{(ij)}$. We then choose $T_0^{(ij)} = T_{\ell,\varepsilon_N}^{(ij)} > 0$ and note that

$$T_0^{(ij)} \lesssim rac{1}{(n+2)\gamma_0 N^{rac{1}{2(n+1)}}} < 1,$$

where we bounded $T_{\ell,\varepsilon_N}^{(ij)}$ according to the assumption that $\sqrt{M} \gtrsim \frac{(n+2)\gamma_0}{\beta p \lambda_r^2 C_0 \gamma_2^{p-1}} N^{\frac{p-1}{2} - \frac{n}{2(n+1)}}$. By assumption we also have that $\lambda_1 \geq \lambda_0 \lambda_2$ and $\lambda_2 \geq \cdots \geq \lambda_r$, where λ_0 satisfies

$$\lambda_0 = c(\lambda_0) \frac{1 + C_0}{1 - C_0} \left(\frac{3\gamma_1}{\gamma_2}\right)^{p-2},$$
(5.7)

for some constant $c(\lambda_0) > 1$. It then follows that $T_0^{(11)} = \min_{1 \le k, \ell \le r} T_0^{(k\ell)}$ since for every $i, j \in I$ $[r], (i, j) \neq (1, 1),$

$$T_{\ell,\varepsilon_N}^{(11)} \le \frac{1 - \left(\frac{\gamma_{11}}{2}\right)^{p-2} N^{-\frac{p-2}{2(p-1)}}}{c(\lambda_0)(1+C_0)\beta\sqrt{M}p(p-2)\lambda_i\lambda_j \left(\frac{\gamma_{11}}{2\sqrt{N}}\right)^{p-2} \left(\frac{3\gamma_1}{\gamma_2}\right)^{p-2}} \le T_{u,\varepsilon_N}^{(ij)},$$

provided $N > \left(\frac{c(\lambda_0)}{c(\lambda_0)-1}\right)^{\frac{2(p-1)}{p-2}} \left(\frac{3\gamma_1}{2}\right)^{2(p-1)}$. Our goal is to show that $\mathcal{T}_{\varepsilon_N}^{(11)} = \min_{1 \le k, \ell \le r} \mathcal{T}_{\varepsilon_N}^{(k\ell)}$ and that $\mathcal{T}_{\varepsilon_N}^{(11)} \le \mathcal{T}_{L_{0,\beta}}$ since $\mathcal{T}_{L_{0,\beta}} \le \mathcal{T}_{L_{0,\beta}}^{(11)}$. To this end, in order to get an estimate for $L_{0,\beta}m_{ij}$ for every $i, j \in [r]$, we wish to apply Lemma 4.4 to the function $F_{ij}(\mathbf{X}) = L_{0,\beta}m_{ij}(\mathbf{X})$. We see that if we let $\psi_{k\ell}(\mathbf{X}) = \langle \mathbf{v}_k, \mathbf{x}_\ell \rangle, \ a_{k\ell}(\mathbf{X}) = \beta p \sqrt{M} \lambda_k \lambda_\ell m_{k\ell}^{p-1}(\mathbf{X})$ and $U = H_0$, then condition (1) is satisfied with P-probability at least $1 - \exp(-KN)$ for every $n \ge 1$ according to Lemma 4.2. Condition (2) is easily verified since the function F_{ij} is smooth and for every $n \geq 1$ it holds that $||F_{ij}||_{\mathcal{G}^{2n}} \leq \Lambda$ with \mathbb{P} -probability at least $1 - \exp(-KN)$ according to (4.3). Item (3) holds by assumption on the initial data, namely the event $\mathcal{C}_0(n, \gamma_0)$. It therefore remains to check condition (4). For every $i, j \in [r]$ we note that on the event $\mathcal{A}^{(ij)}$,

$$\int_{0}^{t} |a_{ij}(s)| ds \leq \frac{1}{1 - C_0} \left(m_{ij}(t) - \frac{\gamma_{ij}}{2\sqrt{N}} \right) \leq \frac{1}{1 - C_0} m_{ij}(t),$$
(5.8)

for every $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)} \wedge T_0^{(11)}$, where we used the lower bound in the integral inequality (5.1). We then observe that at time t = 0, for every $\xi > 0$ we have that

$$\xi\beta\sqrt{M}p\lambda_i\lambda_j\left(\ell_{ij}(0)\right)^{p-1} = \xi\beta\sqrt{M}p\lambda_i\lambda_j\left(\frac{\gamma_{ij}}{2\sqrt{N}}\right)^{p-1} \ge C\xi(n+2)\gamma_0N^{-\frac{n}{2(n+1)}} \ge \ell_{ij}(0),$$

where we used that by assumption $\sqrt{M} \geq C \frac{(n+2)\gamma_0}{\beta p \lambda_r^2 C_0 \gamma_2^{p-1}} N^{\frac{p-1}{2} - \frac{n}{2(n+1)}}$ for some constant C > 0. Since by (5.2) the function $m_{ij}(t)$ is lower bounded by $\ell_{ij}(t)$ for every $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)} \wedge \mathcal{T}_0^{(11)}$ and since $\ell_{ij}(t)$ is an increasing function satisfying the above inequality, we therefore obtain that

$$m_{ij}(t) \le \xi \beta \sqrt{M} p \lambda_i \lambda_j m_{ij}^{p-1}(t),$$

so that from (5.8) it follows that

$$\int_{0}^{t} |a_{ij}(s)| ds \le \frac{1}{1 - C_0} m_{ij}(t) \le \frac{\xi}{1 - C_0} \beta \sqrt{M} p \lambda_i \lambda_j m_{ij}^{p-1}(t),$$

for $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)} \wedge T_0^{(11)}$. Choosing $\xi = (1 - C_0)/2$ yields condition (4) with $\varepsilon = 1/2$. We can therefore apply Lemma 4.4: for every $i, j \in [r]$, there exists K > 0 such that on the event $\cap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$,

$$|L_{0,\beta}m_{ij}(t)| \le K\left(\frac{\gamma_0}{\sqrt{N}}\sum_{k=0}^{n-1} t^k + t^n + 2\sum_{1\le k,\ell\le r} \int_0^t |a_{k\ell}(s)|ds\right),\tag{5.9}$$

for every $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)} \wedge T_0^{(11)}$, with $\mathbb{Q}_{\boldsymbol{X}}$ -probability at least $1 - 1 \leq k \leq r$ $K_2 \exp\left(-\gamma_0^2/(K_2 T_0^{(11)})\right)$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$.

Given the estimate (5.9), in the remaining of the proof we show that m_{11} is the first correlation to reach ε_N and that $\mathcal{T}_{\varepsilon_N}^{(11)} \leq \mathcal{T}_{L_{0,\beta}}$. To this end, we first introduce an intermediate threshold $\tilde{\varepsilon}_N = \tilde{\gamma} N^{-\frac{1}{2}}$ for a sufficiently large constant $\tilde{\gamma}$ of order 1 and show that $\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)} \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}}$, where $\mathcal{T}_{\tilde{\varepsilon}_N}^{(ij)}$ denotes the hitting time for the set $\{\boldsymbol{X}: m_{ij}(\boldsymbol{X}) \geq \tilde{\varepsilon}_N\}$. In particular, we choose γ to solve the equation

$$c(\lambda_0) \ge \frac{(2\gamma)^{p-2}}{(2\gamma)^{p-2} - (3\gamma_1)^{p-2}}, \quad \text{so that } \gamma \ge \frac{3}{2} \left(\frac{c(\lambda_0)}{c(\lambda_0) - 1}\right)^{\frac{1}{p-2}} \gamma_1.$$
(5.10)

Since the integral and comparison inequalities (5.1) and (5.2) are still valid for $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge min_{1\leq k,\ell\leq r} \mathcal{T}_{\tilde{\varepsilon}_N}^{(k\ell)} \wedge T_0^{(ij)}$, we can define $T_{\ell,\tilde{\varepsilon}_N}^{(ij)}$ and $T_{u,\tilde{\varepsilon}_N}^{(ij)}$ similarly to (5.5) and (5.6), and we see that for every $(i,j) \neq (1,1)$,

$$T_{\ell,\tilde{\varepsilon}_N}^{(11)} = \frac{1 - \left(\frac{\gamma_{11}}{2\gamma}\right)^{p-2}}{(1 - C_0)\beta\sqrt{M}p(p-2)\lambda_1^2 \left(\frac{\gamma_{11}}{2\sqrt{N}}\right)^{p-2}} \le \frac{1}{c(\lambda_0)(1 + C_0)\beta\sqrt{M}p(p-2)\lambda_i\lambda_j \left(\frac{3\gamma_{ij}}{2\sqrt{N}}\right)^{p-2}} \le \frac{1 - \left(\frac{3\gamma_{ij}}{2\gamma}\right)^{p-2}}{(1 + C_0)\beta\sqrt{M}p(p-2)\lambda_i\lambda_j \left(\frac{3\gamma_{ij}}{2\sqrt{N}}\right)^{p-2}} = T_{u,\tilde{\varepsilon}_N}^{(ij)}$$

where we used (5.7) and (5.10). In order to deduce that on the event $\bigcap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$, it holds that $\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)} = \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\tilde{\varepsilon}_N}^{(k\ell)}$, we need to show that $\min_{1 \leq k, \ell \leq r} \mathcal{T}_{\tilde{\varepsilon}_N}^{(k\ell)} \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}}$. Since the estimate (5.9) holds for every $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\tilde{\varepsilon}_N}^{(k\ell)} \wedge \mathcal{T}_0^{(ij)}$, it thus suffices to show that each term in the sum (5.9) is upper bounded by $\frac{C_0 \beta \sqrt{M_P \lambda_i \lambda_j}}{n+2} m_{ij}^{p-1}(t)$ for every $i, j \in [r]$.

(i) We start by observing that according to the lower bound in (5.2), for $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge$ $\min_{1 \le k, \ell \le r} \mathcal{T}_{\tilde{\varepsilon}_N}^{(k\ell)} \wedge T_0^{(11)}$

$$\frac{C_0\beta\sqrt{M}p\lambda_i\lambda_j}{n+2}m_{ij}^{p-1}(t) \geq \frac{C_0\beta\sqrt{M}p\lambda_i\lambda_j}{n+2}\ell_{ij}^{p-1}(t) \geq \frac{C_0\beta\sqrt{M}p\lambda_i\lambda_j}{n+2}\ell_{ij}^{p-1}(0),$$

so that for every $0 \le k \le n-1$,

$$\frac{C_0\beta\sqrt{M}p\lambda_i\lambda_j}{n+2}\left(\frac{\gamma_{ij}}{2\sqrt{N}}\right)^{p-1} \ge C\frac{\gamma_0}{N^{\frac{n}{2(n+1)}}} \ge K\frac{\gamma_0}{\sqrt{N}}t^k.$$

provided $t^k = \mathcal{O}(1)$, which certainly holds for all $k \le n-1$ and $t \le T_0^{(11)}$ since $T_0^{(11)} < 1$. (ii) A sufficient condition to control the second term is given by $F(t) \le G(t)$, where $F(t) = Kt^n$ and $G(t) = \frac{C_0\beta\sqrt{M}p\lambda_i\lambda_j}{n+2}\ell_{ij}^{p-1}(t)$. By an easy computation, we have that for any $k \le n$

$$F^{(k)}(t) = Kn(n-1)\cdots(n-k+1)t^{n-k}$$

and

$$G^{(k)}(t) = \frac{C_0 \beta \sqrt{M} p \lambda_i \lambda_j \prod_{i=1}^k \left(\frac{p-1}{p-2} + (i-1)\right)}{n+2} \left(\frac{\gamma_{ij}}{2\sqrt{N}}\right)^{p-1} \left(\frac{1}{t_*^{(ij)}}\right)^k \left(1 - \frac{t}{t_*^{(ij)}}\right)^{-\left(\frac{p-1}{p-2} + k\right)}$$

where $t_*^{(ij)}$ denotes the blow-up time of ℓ_{ij} which is given by

$$t_*^{(ij)} = \left[(1 - C_0)\beta\sqrt{M}p(p-2)\lambda_i\lambda_j \left(\frac{\gamma_{ij}}{2\sqrt{N}}\right)^{p-2} \right]^{-1}$$

For $k \leq n-1$, it holds that $G^{(k)}(0) \geq 0 = F^{(k)}(0)$, and for k = n we have that

$$G^{(n)}(t) \ge \frac{(\sqrt{M}\beta p\lambda_i\lambda_j)^{n+1}C_0(1-C_0)^n}{n+2} \left(\frac{\gamma_{ij}}{2\sqrt{N}}\right)^{p-1+n(p-2)} \left(1-\frac{t}{t_*^{(ij)}}\right)^{-\binom{p-1}{p-2}+n} \\\gtrsim C_0(1-C_0)^n(n+2)^n\gamma_0^{n+1} \\\ge Kn! = F^{(n)}(t),$$

provided $t/t_*^{(ij)}$ is of order 1 for any $t \leq T_0^{(11)}$, which certainly holds since $T_0^{(11)} \leq t_*^{(11)} < t_*^{(ij)}$.

(iii) We control the last term as follows. According to the integral inequality (5.1), on the event $\cap_{1 \le k, \ell \le r} \mathcal{A}^{(k\ell)}$ we have that

$$2\sum_{1\leq k,\ell\leq r}\int_0^t |a_{k\ell}(s)|ds \leq \frac{2r^2}{1-C_0}\max_{1\leq k,\ell\leq r}\{m_{k\ell}(t)\} \leq \frac{2r^2}{1-C_0}\tilde{\varepsilon}_N = \frac{2r^2}{1-C_0}\frac{\gamma}{\sqrt{N}},$$

for $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)} \wedge \mathcal{T}_0^{(11)}$. Moreover, it follows from (5.2) that $C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq i \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq r \leq C_0 \beta \sqrt{M} p \lambda_i \lambda_i \quad p=1 \leq r \leq C_0 \beta \sqrt{M} p$

$$\frac{C_0\beta\sqrt{Mp\lambda_i\lambda_j}}{n+2}m_{ij}^{p-1}(t) \ge \frac{C_0\beta\sqrt{Mp\lambda_i\lambda_j}}{n+2}\ell_{ij}^{p-1}(t) \ge \frac{C_0\beta\sqrt{Mp\lambda_i\lambda_j}}{n+2}\ell_{ij}^{p-1}(0)$$

for $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)} \wedge \mathcal{T}_0^{(11)}$. Since by assumption on \sqrt{M} we have that

$$\frac{C_0\beta\sqrt{M}p\lambda_i\lambda_j}{n+2}\left(\frac{\gamma_{ij}}{2\sqrt{N}}\right)^{p-1} \ge C\frac{\gamma_0}{(1-C_0)N^{\frac{n}{2(n+1)}}} \ge K\frac{2r^2}{1-C_0}\frac{\gamma_0}{\sqrt{N}}$$

for some constant C > 0, the desired bound then follows. On the event $\bigcap_{1 \le k, \ell \le r} \mathcal{A}^{(k\ell)}$, we therefore have that $\min_{1 \le k, \ell \le r} \mathcal{T}^{(k\ell)}_{\varepsilon_N} \le \min_{1 \le k, \ell \le r} \mathcal{T}^{(k\ell)}_{L_{0,\beta}} \wedge \mathcal{T}_{L_{0,\beta}}$ and it follows that

$$\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)} = \min_{1 \le k, \ell \le r} \mathcal{T}_{\varepsilon_N}^{(k\ell)},$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1-K_2 \exp\left(-\gamma_0^2/(K_2 T_0^{(11)})\right)$ and with \mathbb{P} -probability at least $1-\exp(-KN)$. Now, according to the generator expansion $L_{\beta}m_{ij}$ given by Lemma 4.6 we see that at $t = \mathcal{T}_{\tilde{\varepsilon}_N}^{(11)}$

$$L_{\beta}m_{11}(\mathcal{T}_{\tilde{\varepsilon}_{N}}^{(11)}) \ge (1 - C_{0})\beta\sqrt{M}p\lambda_{1}^{2}m_{11}^{p-1}(\mathcal{T}_{\tilde{\varepsilon}_{N}}^{(11)}),$$

and for any $(i, j) \neq (1, 1)$

$$L_{\beta}m_{ij}(\mathcal{T}_{\tilde{\varepsilon}_{N}}^{(11)}) \leq C_{0}\beta\sqrt{M}p\lambda_{1}^{2}m_{11}^{p-1}(\mathcal{T}_{\tilde{\varepsilon}_{N}}^{(11)}) + \beta\sqrt{M}p\lambda_{i}\lambda_{j}m_{ij}^{p-1}(\mathcal{T}_{\tilde{\varepsilon}_{N}}^{(11)}),$$

yielding

$$L_{\beta}m_{ij}(\mathcal{T}^{(11)}_{\varepsilon_N}) \leq C_0\beta\sqrt{M}p\lambda_1^2 \left(\frac{\gamma}{\sqrt{N}}\right)^{p-1} + \beta\sqrt{M}p\lambda_i\lambda_j \left(\frac{\gamma}{\sqrt{N}}\right)^{p-1} \leq \left(C_0 + \frac{1}{\lambda_0}\right)\beta\sqrt{M}p\lambda_1^2 \left(\frac{\gamma}{\sqrt{N}}\right)^{p-1},$$

where we used the fact that $\lambda_i \lambda_j \leq \lambda_1^2 / \lambda_0^2 < \lambda_1^2 / \lambda_0$. According to (5.7) we then have that $\frac{1}{\lambda_0} \leq \frac{1}{3c(\lambda_0)} \frac{1-C_0}{1+C_0} \leq 1-2C_0$ for some $C_0 < 1/2$, thus $C_0 + \frac{1}{\lambda_0} \leq 1-C_0$ and it follows that

$$L_{\beta}m_{ij}(\mathcal{T}^{(11)}_{\tilde{\varepsilon}_N}) \leq L_{\beta}m_{11}(\mathcal{T}^{(11)}_{\tilde{\varepsilon}_N}).$$

As a consequence, on the event $\bigcap_{1 \le k, \ell \le r} \mathcal{A}^{(k\ell)}$, since $m_{11}(\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)}) > m_{ij}(\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)})$ and $L_{\beta}m_{11}(\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)}) \ge L_{\beta}m_{ij}(\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)})$ for every $(i, j) \ne (1, 1)$, we obtain that $m_{11}(t) > m_{ij}(t)$ for $\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)} \le t \le \min_{1 \le k, \ell \le r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \land$ $\mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)} \wedge T_0^{(11)}$, ensuring that

$$\mathcal{T}_{\varepsilon_N}^{(11)} = \min_{1 \le k, \ell \le r} \mathcal{T}_{\varepsilon_N}^{(k\ell)},$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp\left(-\gamma_0^2/(K_2 T_0^{(11)})\right)$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. It therefore remains to show that on the event $\cap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$, $\mathcal{T}_{\varepsilon_N}^{(11)} \leq \mathcal{T}_{L_{0,\beta}}$ with high $\mathbb{Q}_{\mathbf{X}^-}$ and \mathbb{P}_{1} -probability. We recall that the estimate (5.9) holds for $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \mathcal{T}_{\varepsilon_N}^{(11)} \wedge \mathcal{T}_{0}^{(11)}$ with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp\left(-\gamma_0^2/(K_2 T_0^{(11)})\right)$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. Therefore, since each $a_{k\ell}(t) \leq a_{11}(t)$ for $\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)} \leq t \leq \min_{1 \leq k,\ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}}^{(11)} \wedge \mathcal{T}_{0}^{(11)}$, we obtain that on the event $\bigcap_{1 \le k, \ell \le r} \mathcal{A}^{(k\ell)}$,

$$|L_{0,\beta}m_{ij}(t)| \le K\left(\frac{\gamma_0}{\sqrt{N}}\sum_{k=0}^{n-1} t^k + t^n + 2r^2 \int_0^t |a_{11}(s)|ds\right),\tag{5.11}$$

for $\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)} \leq t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \mathcal{T}_{\varepsilon_N}^{(11)} \wedge \mathcal{T}_0^{(11)}$, with $\mathbb{Q}_{\boldsymbol{X}}$ -probability at least $1 - K_2 \exp\left(-\gamma_0^2/(K_2 T_0^{(11)})\right)$ and with P-probability at least $1 - \exp(-KN)$. In the same way as before, we can show that by the

assumption on \sqrt{M} each term in the sum (5.11) is bounded above by $\frac{C_0\beta\sqrt{M}p\lambda_1^2}{n+2}m_{11}^{p-1}(t)$ for every $\mathcal{T}_{\varepsilon_N}^{(11)} \leq t \leq \min_{1\leq k,\ell\leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \mathcal{T}_{\varepsilon_N}^{(11)} \wedge T_0^{(11)}$, ensuring that $\mathcal{T}_{\varepsilon_N}^{(11)} \leq \mathcal{T}_{L_{0,\beta}}$ with high $\mathbb{Q}_{\mathbf{X}}$ - and \mathbb{P} -probability. In particular, on the event $\mathcal{C}_0(n,\gamma_0) \cap \mathcal{C}_1(\gamma_1,\gamma_2)$ we have that

$$\mathcal{T}_{\varepsilon_N}^{(11)} \le T_0^{(11)} \lesssim \frac{1}{(n+2)\gamma_0 N^{\frac{1}{2(n+1)}}} < 1$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1-K_2 \exp\left(-\gamma_0^3(n+2)N^{\frac{1}{2(n+1)}}/K_2\right)-K_2r^2 \exp\left(-\gamma_2^2\gamma_0(n+2)N^{\frac{1}{2(n+1)}}/(4K_2)\right)$ and with \mathbb{P} -probability at least $1-\exp(-KN)$, which completes the proof of Lemma 5.1.

We finally provide the proof of the second intermediate result, namely Lemma 5.2.

Proof of Lemma 5.2. Let $\varepsilon > 0$ be sufficiently small and let $\mathcal{T}_{\varepsilon}^{(11)}$ denote the first hitting time of the set $\{\mathbf{X} : m_{11}(\mathbf{X}) \ge \varepsilon\}$. We first show that from any initial data satisfying $m_{11}(\mathbf{X}_0) \in [\varepsilon_N, \varepsilon)$, we have that $\mathcal{T}_{\varepsilon}^{(11)} \le 1$ with high $\mathbb{Q}_{\mathbf{X}}$ - and \mathbb{P} -probability.

We have from Lemma 4.6 that $L_{\beta}m_{11}$ satisfies

$$L_{\beta}m_{11}(t) \geq -\|L_{0,\beta}m_{11}\|_{\infty} + \beta\sqrt{M}p\lambda_{1}^{2}m_{11}^{p-1}(t)(1-r^{2}m_{11}^{2}(t)) \geq -\|L_{0,\beta}m_{11}\|_{\infty} + \beta\sqrt{M}p\lambda_{1}^{2}m_{11}^{p-1}(t) > 0$$

for $t \leq \mathcal{T}_{\varepsilon_{N}/2}^{(11)} \wedge \mathcal{T}_{\varepsilon}^{(11)}$, provided $\varepsilon < 1/r$. Moreover, we also have for $t \leq \mathcal{T}_{\varepsilon_{N}/2}^{(11)} \wedge \mathcal{T}_{\varepsilon}^{(11)}$,

$$||L_{0,\beta}m_{11}||_{\infty} \le \Lambda \le C_0 \beta \sqrt{M} p \lambda_1^2 m_{11}^{p-1}(t)$$

for some $C_0 \in (0, \frac{1}{2})$, with \mathbb{P} -probability at least $1 - \exp(-KN)$. Therefore, for every $t \leq \mathcal{T}_{\varepsilon_N/2}^{(11)} \wedge \mathcal{T}_{\varepsilon}^{(11)}$, we obtain that

$$L_{\beta}m_{11}(t) \ge (1 - C_0)\beta\sqrt{M}p\lambda_1^2 m_{11}^{p-1}(t) > 0,$$

yielding

$$m_{11}(t) \ge \frac{\varepsilon_N}{2} \left(1 - (1 - C_0)\beta\sqrt{M}p(p-2)\lambda_1^2 \left(\frac{\varepsilon_N}{2}\right)^{p-2} t \right)^{-\frac{1}{p-2}}.$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-N^{\frac{1}{p-1}}/(4K_2(\mathcal{T}_{\varepsilon_N/2}^{(11)} \wedge \mathcal{T}_{\varepsilon}^{(11)})))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. Since on the interval $[0, \mathcal{T}_{\varepsilon_N/2}^{(11)} \wedge \mathcal{T}_{\varepsilon}^{(11)}], m_{11}(t)$ is lower bounded by an increasing function which is positive at t = 0, we deduce that $\mathcal{T}_{\varepsilon}^{(11)} < \mathcal{T}_{\varepsilon_N/2}^{(11)}$ and by an easy computation we have that

$$\mathcal{T}_{\varepsilon}^{(11)} \leq \frac{1 - \left(\frac{1}{2\varepsilon N^{\frac{p-2}{2(p-1)}}}\right)^{p-2}}{(1 - C_0)\beta\sqrt{M}p(p-2)\lambda_1^2 \left(\frac{N^{-\frac{p-2}{2(p-1)}}}{2}\right)^{p-2}} \lesssim \frac{1}{(n+2)\gamma_0 N^{\frac{2p-3}{2(p-1)} - \frac{n}{2(n+1)}}} < 1.$$

We then obtain that

$$\inf_{\mathbf{X}: \ m_{11}(\mathbf{X})\in[\varepsilon_N,\varepsilon)} \mathbb{Q}_{\mathbf{X}}\left(\mathcal{T}_{\varepsilon}^{(11)} \le 1\right) \ge 1 - K_2 \exp\left(-(n+2)\gamma_0 N^{\frac{2p-1}{2(p-1)} - \frac{n}{2(n+1)}}/(4K_2)\right)$$
(5.12)

with \mathbb{P} -probability at least $1 - \exp(-KN)$. By the strong Markov property for X_t , it remains to consider the case that $m_{11}(X_0) \ge \varepsilon$ for some $\varepsilon > 0$ sufficiently small. According to Lemma 4.1 of [8] there exists T_0 such that for every $T \ge T_0$,

$$\inf_{\mathbf{X}: m_{11}(\mathbf{X}) \ge \varepsilon} \mathbb{Q}_{\mathbf{X}} \left(\inf_{t \in [T_0, T]} m_{11}(t) \ge 1 - \varepsilon \right) \ge 1 - 2K_2 \exp\left(-N\varepsilon^2/(K_2T)\right)$$
(5.13)

with \mathbb{P} -probability at least $1 - \exp(-KN)$. Combining (5.12) and (5.13) yields the desired statement. \Box

5.2. Recovery of all spikes

Now, we prove Proposition 3.6 on the recovery of all spikes. The proof of Proposition 3.6 proceeds through r steps, each focusing on the recovery of a new correlation m_{ii} . For every $\varepsilon > 0$, we consider the following events

$$\begin{split} E_1(\varepsilon) &= R_1(\varepsilon) \cap \left\{ \boldsymbol{X} \colon m_{ij}(\boldsymbol{X}) \in \Theta(N^{-\frac{1}{2}}) \text{ for } i, j \neq 1 \right\}, \\ E_2(\varepsilon) &= R_1(\varepsilon) \cap R_2(\varepsilon) \cap \left\{ \boldsymbol{X} \colon m_{ij}(\boldsymbol{X}) \in \Theta(N^{-\frac{1}{2}}) \text{ for } i, j \neq 1, 2 \right\}, \\ & \dots \\ E_{r-1}(\varepsilon) &= \cap_{1 \leq i \leq r-1} R_i(\varepsilon) \cap \left\{ \boldsymbol{X} \colon m_{rr}(\boldsymbol{X}) \in \Theta(N^{-\frac{1}{2}}) \right\}, \\ E_r(\varepsilon) &= \cap_{1 \leq i \leq r-1} R_i(\varepsilon) \cap \left\{ \boldsymbol{X} \colon m_{rr}(\boldsymbol{X}) \geq 1 - \varepsilon \right\}, \end{split}$$

where $R_i(\varepsilon)$ denotes the set of strong recovery of the spike v_i :

$$R_i(\varepsilon) = \left\{ \boldsymbol{X} \colon m_{ii}(\boldsymbol{X}) \ge 1 - \varepsilon \text{ and } m_{ij}(\boldsymbol{X}), m_{ji}(\boldsymbol{X}) \lesssim \log(N)^{-\frac{1}{2}} N^{-\frac{p-1}{4}} \quad \forall j \neq i \right\}.$$
(5.14)

We note that the set $E_r(\varepsilon)$ corresponds to $R(\varepsilon)$, which is defined in (3.1). Looking at the *r* events defined earlier, we note once a correlation m_{ii} reaches a macroscopic threshold ε , all correlations m_{ij} and m_{ji} for $j \neq i$ decrease below $\log(N)^{-\frac{1}{2}}N^{-\frac{p-1}{4}}$. This is crucial to ensure the recovery of the subsequent correlation $m_{i+1,i+1}$.

The following result shows that attaining the event E_1 starting from a random initialization that meets Condition 1 is possible, provided M is of order N^{p-1} . According to Proposition 3.5 and Lemma 5.1, this suggests that the complexity threshold N^{p-2} ensures the recovery of the first direction, but is not sufficient to ensure the stability of the other directions.

Lemma 5.3. Let $\beta \in (0, \infty)$, $p \geq 3$ and $\lambda_i = \lambda_{0,i}\lambda_{i+1}$ for every $1 \leq i \leq r-1$ and $\lambda_{0,i} \geq 1$. Consider a sequence of initializations $\mu_0 \in \mathcal{P}(\mathcal{M}_{N,r})$. Then, the following holds: For every $\gamma_1 > 1 > \gamma_2 > 0$ and $\varepsilon > 0$, there exist $\Lambda = \Lambda(p, n, \beta, \{\lambda_i\}_{i=1}^r) > 0$ and $C_0 \in (0, \frac{1}{2})$ such that if $\lambda_{0,1} > \frac{1+C_0}{1-C_0} \left(\frac{3\gamma_1}{\gamma_2}\right)^{p-2}$ and $\sqrt{M} \gtrsim \frac{\Lambda}{\beta p \lambda_c^2 C_0 \gamma_2^{p-1}} N^{\frac{p-1}{2}}$, then for N sufficiently large,

$$\int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{E_1} \gtrsim \frac{1}{\sqrt{N}} \right) \mathbf{1} \{ \mathcal{C}_1(\gamma_1, \gamma_2) \} d\mu_0(\boldsymbol{X}) \\ \leq r^2 K_1 e^{-\gamma_2^2 \sqrt{N}/K_1} + K_2 e^{-N^{\frac{p+1}{2(p-1)}}/K_2} + K_3 e^{-\varepsilon^2 N^{3/2}/K_3},$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$.

Once event E_1 is attained, reaching event E_2 is straightforward, and so on. Having the (k-1)st event $E_{k-1}(\varepsilon)$ at hand, we now show that achieving $E_k(\varepsilon)$ is possible.

Lemma 5.4. Let $\beta \in (0, \infty)$, $p \geq 3$, and $\lambda_i = \lambda_{0,i}\lambda_{i+1}$ for every $1 \leq i \leq r-1$ and $\lambda_{0,i} \geq 1$. Then, the following holds: For every $\gamma_1 > 1 > \gamma_2 > 0$ and $\varepsilon > 0$, there exist $\Lambda = \Lambda(p, n, \beta, \{\lambda_i\}_{i=1}^r) > 0$ and $C_0 \in (0, \frac{1}{2})$ such that if $\lambda_{0,k} > \frac{1+C_0}{1-C_0} \left(\frac{3\gamma_1}{\gamma_2}\right)^{p-2}$ and $\sqrt{M} \gtrsim \frac{\Lambda}{\beta_p \lambda_r^2 C_0 \gamma_2^{p-1}} N^{\frac{p-1}{2}}$, then there exists $T_k > T_{k-1}$ (with $T_0 = \mathcal{T}_{E_1}$) such that for every $T > T_k$ and N sufficiently large,

$$\inf_{\mathbf{X}_{0}\in E_{k-1}(\varepsilon)} \mathbb{Q}_{\mathbf{X}_{0}} \left(\inf_{t\in[T_{k},T]} \mathbf{X}_{t}^{\beta} \in E_{k}(\varepsilon) \right)$$

$$\geq 1 - (r - (k-1))^{2} K_{1} e^{-\gamma_{2}^{2}\sqrt{N}/K_{1}} - K_{2} e^{-N^{\frac{p+1}{2(p-1)}}/K_{2}} - K_{3} e^{-\varepsilon^{2}N/(K_{3}T)},$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$.

Having Lemmas 5.3 and 5.4 at hand, we now are in the position to provide the proof of Proposition 3.6.

Proof of Proposition 3.6. By the strong Markov property, we have that

$$\int_{\mathcal{M}_{N\times r}} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t\in[T_0,T]} \boldsymbol{X}_t^{\beta} \in R(\varepsilon) \right) d\mu_0(\boldsymbol{X})$$

$$\geq \inf_{\boldsymbol{X}\in E_1(\varepsilon)} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t\in[T_0,T]} \boldsymbol{X}_t^{\beta} \in R(\varepsilon) \right) \times \int_{\mathcal{M}_{N\times r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{E_1(\varepsilon)} \lesssim \frac{1}{\sqrt{N}} \right) d\mu_0(\boldsymbol{X})$$

Again by the strong Markov property, we have that

$$\inf_{\mathbf{X}\in E_1(\varepsilon)} \mathbb{Q}_{\mathbf{X}} \left(\inf_{t\in[T_0,T]} \mathbf{X}_t^{\beta} \in R(\varepsilon) \right) \geq \inf_{\mathbf{X}\in E_1(\varepsilon)} \mathbb{Q}_{\mathbf{X}} \left(\inf_{t\in[T_0,T]} \mathbf{X}_t^{\beta} \in E_2(\varepsilon) \right) \times \inf_{\mathbf{X}\in E_2(\varepsilon)} \mathbb{Q}_{\mathbf{X}} \left(\inf_{t\in[T_0,T]} \mathbf{X}_t^{\beta} \in R(\varepsilon) \right)$$
Applying the strong Markov property recursively, we find that

$$\int_{\mathcal{M}_{N\times r}} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t\in[T_0,T]} \boldsymbol{X}_t^{\beta} \in R(\varepsilon) \right) d\mu_0(\boldsymbol{X})$$

$$\geq \prod_{k=2}^r \inf_{\boldsymbol{X}\in E_{k-1}(\varepsilon)} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t\in[T_k,T]} \boldsymbol{X}_t^{\beta} \in E_k(\varepsilon) \right) \times \int_{\mathcal{M}_{N\times r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{E_1(\varepsilon)} \lesssim \frac{1}{\sqrt{N}} \right) d\mu_0(\boldsymbol{X})$$

The first term is bounded according to Lemma 5.4. For the second term we first have by the strong Markov property,

$$\begin{split} &\int_{\mathcal{M}_{N\times r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{E_{1}(\varepsilon)} \gtrsim \frac{1}{\sqrt{N}} \right) d\mu_{0}(\boldsymbol{X}) \\ &\leq \mu_{0}(\mathcal{C}_{1}(\gamma_{1}, \gamma_{2})^{\mathrm{c}}) + \int_{\mathcal{M}_{N\times r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{E_{1}(\varepsilon)} \gtrsim \frac{1}{\sqrt{N}} \right) \mathbf{1} \{ \mathcal{C}_{1}(\gamma_{1}, \gamma_{2}) \} d\mu_{0}(\boldsymbol{X}) \end{split}$$

We then bound $\mu_0(\mathcal{C}_1(\gamma_1,\gamma_2)^c)$ according to Definition 3.3 and the integral according to Lemma 5.3. Proposition 3.6 then follows straightforwardly.

We therefore need to prove the two intermediate results, namely Lemmas 5.3 and 5.4.

Proof of Lemma 5.3. The proof of this lemma is divided into four different parts, which we briefly describe. First, we show that m_{11} is the first correlation to reach the microscopic threshold $\varepsilon_N = N^{-\frac{p-2}{2(p-1)}}$, while the other correlations are still in $\Theta(N^{-\frac{1}{2}})$. This is due to the fact that by assumption the parameter λ_1^2 is much larger than $\lambda_i \lambda_j$. As already seen in Lemma 5.2, the microscopic threshold ε_N is sufficient to ensure strong recovery of the first spike v_1 . Next, we see that as m_{11} crosses the threshold $N^{-\frac{p-2}{2p}}$, the correlations m_{i1} and m_{1i} for $i \neq 1$ start decreasing from $\Theta(N^{-\frac{1}{2}})$ since the evolution equation at this point is approximately given by $\dot{m}_{i1} \approx -\lambda_1^2 m_{11}^p m_{i1}$ and $\dot{m}_{1i} \approx -\lambda_1^2 m_{11}^p m_{1i}$, respectively. In particular, we show that such correlations will reach the lower threshold $N^{-\frac{p-1}{4}} \log(N)^{-\frac{1}{2}}$ in a microscopic time. Finally, we study the evolution of the correlations m_{ij} for $i, j \neq 1$ as m_{11} crosses ε_N which is approximately given by $\dot{m}_{ij} \approx \lambda_i \lambda_j m_{ij}^{p-1} - r^2 \lambda_1^2 m_{11}^{p-1} m_{i1} m_{1j}$. In particular, we show that such correlations can undergo a decrease, which is at most by a constant, and thus globally we have that m_{ij} remain on the scale $\Theta(N^{-\frac{1}{2}})$.

Step 1: Evolution of the correlations until the first correlation reaches $\varepsilon_N = N^{-\frac{p-2}{2(p-1)}}$. On the initial event $C_1(\gamma_1, \gamma_2)$ given by Definition 3.3, for every $i, j \in [r]$ there exists $\gamma_{ij} \in (\gamma_2, \gamma_1)$ such that $m_{ij}(0) = \gamma_{ij} N^{-\frac{1}{2}}$. For some $T_0^{(ij)} > 0$ to be chosen later, we then define the event $\mathcal{A}^{(ij)} =$ $\mathcal{A}^{(ij)}(\gamma_1,\gamma_2,T_0^{(ij)})$ by

$$\mathcal{A}^{(ij)}(\gamma_1, \gamma_2, T_0^{(ij)}) = \mathcal{C}_1(\gamma_1, \gamma_2) \cap \left\{ \sup_{t \in [0, T_0^{(ij)}]} |M_t^{m_{ij}}| \le \frac{\gamma_2}{2\sqrt{N}} \right\},$$

where we recall that according to (4.9), there exists a constant $K_2 > 0$ such that

$$\sup_{\boldsymbol{X}} \mathbb{Q}_{\boldsymbol{X}} \left(\sup_{t \in [0, T_0^{(ij)}]} |M_t^{m_{ij}}| \ge \frac{\gamma_2}{2\sqrt{N}} \right) \le K_2 \exp\left(-\frac{\gamma_2^2}{4K_2 T_0^{(ij)}}\right)$$

In the following, we fix $i, j \in [r]$ and place ourselves on the event $\mathcal{A}^{(ij)}$. Let $\varepsilon_N = N^{-\frac{p-2}{2(p-1)}}$ and let $\mathcal{T}_{\varepsilon_N}^{(ij)}$ denote the hitting time of the set $\{\mathbf{X} : m_{ij}(\mathbf{X}) \geq \varepsilon_N\}$. Given the generator expansion by Lemma 4.6, i.e.,

$$L_{\beta}m_{ij} = L_{0,\beta}m_{ij} + \beta\sqrt{M}p\lambda_i\lambda_j m_{ij}^{p-1} - \beta\sqrt{M}\frac{p}{2}\sum_{1\le k,\ell\le r}\lambda_k m_{i\ell}m_{kj}m_{k\ell}(\lambda_j m_{kj}^{p-2} + \lambda_\ell m_{k\ell}^{p-2}),$$

we see that

$$-\|L_{0,\beta}m_{ij}\|_{\infty}+\beta\sqrt{M}p\lambda_i\lambda_jm_{ij}^{p-1}(t)\leq L_{\beta}m_{ij}(t)\leq \|L_{0,\beta}m_{ij}\|_{\infty}+\beta\sqrt{M}p\lambda_i\lambda_jm_{ij}^{p-1}(t),$$

for $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)}$. Moreover, according to Lemma 4.2 and especially to (4.3), we have that $\|L_{0,\beta}m_{ij}\|_{\infty} \leq \Lambda$ for some constant $\Lambda = \Lambda(\beta, p, n, \{\lambda_i\}_{i=1}^r)$, with \mathbb{P} -probability at least $1 - \exp(-KN)$. We then observe that for $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)}$,

$$C_0\beta\sqrt{M}p\lambda_i\lambda_j m_{ij}^{p-1}(t) \ge C\frac{\Lambda}{\gamma_2^{p-1}}N^{\frac{p-1}{2}}m_{ij}^{p-1}(t) \ge \Lambda$$

for some constant $C_0 \in (0, \frac{1}{2})$, where we used the facts that $\sqrt{M} \ge C \frac{\Lambda}{\beta p \lambda_r^2 C_0 \gamma_2^{p-1}} N^{\frac{p-1}{2}}$ and that $m_{ij}(t) \ge \gamma_2 N^{-\frac{1}{2}}$. We therefore obtain the integral inequality given by

$$\frac{\gamma_{ij}}{2\sqrt{N}} + (1 - C_0)\beta\sqrt{M}p\lambda_i\lambda_j \int_0^t m_{ij}^{p-1}(s)ds \le m_{ij}(t) \le \frac{3\gamma_{ij}}{2\sqrt{N}} + (1 + C_0)\beta\sqrt{M}p\lambda_i\lambda_j \int_0^t m_{ij}^{p-1}(s)ds,$$

for $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)} \wedge T_0^{(ij)}$, with \mathbb{P} -probability at least $1 - \exp(-KN)$. Lemma 4.11 then yields the comparison inequality

$$\ell_{ij}(t) \le m_{ij}(t) \le u_{ij}(t)$$

for $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)} \wedge T_0^{(ij)}$, where the functions ℓ_{ij} and u_{ij} are given by

$$\ell_{ij}(t) = \frac{\gamma_{ij}}{2\sqrt{N}} \left(1 - (1 - C_0)\beta\sqrt{M}p(p-2)\lambda_i\lambda_j \left(\frac{\gamma_{ij}}{2\sqrt{N}}\right)^{p-2} t \right)^{-\frac{1}{p-2}}$$

and

$$u_{ij}(t) = \frac{3\gamma_{ij}}{2\sqrt{N}} \left(1 - (1+C_0)\beta\sqrt{M}p(p-2)\lambda_i\lambda_j \left(\frac{3\gamma_{ij}}{2\sqrt{N}}\right)^{p-2} t \right)^{-\frac{1}{p-2}}$$

respectively. We then define $T_{\ell,\varepsilon_N}^{(ij)}$ by $\ell_{ij}(T_{\ell,\varepsilon_N}^{(ij)}) = \varepsilon_N$, i.e.,

$$T_{\ell,\varepsilon_N}^{(ij)} = \frac{1 - (\gamma_{ij}/2)^{p-2} N^{-\frac{p-2}{2(p-1)}}}{(1 - C_0)\beta\sqrt{M}p(p-2)\lambda_i\lambda_j \left(\frac{\gamma_{ij}}{2\sqrt{N}}\right)^{p-2}}$$

Similarly, we let $T_{u,\varepsilon_N}^{(ij)}$ denote the time such that $u_{ij}(T_{u,\varepsilon_N}^{(ij)}) = \varepsilon_N$, i.e.,

$$T_{u,\varepsilon_N}^{(ij)} = \frac{1 - (3\gamma_{ij}/2)^{p-2} N^{-\frac{p-2}{2(p-1)}}}{(1+C_0)\beta\sqrt{M}p(p-2)\lambda_i\lambda_j \left(\frac{3\gamma_{ij}}{2\sqrt{N}}\right)^{p-2}}$$

We note that on the event $\mathcal{A}^{(ij)}$, $T^{(ij)}_{u,\varepsilon_N} \leq \mathcal{T}^{(ij)}_{\varepsilon_N} \leq T^{(ij)}_{\ell,\varepsilon_N}$. We then choose $T^{(ij)}_0 = T^{(ij)}_{\ell,\varepsilon_N}$. Since by assumption $\lambda_1 \geq \lambda_0 \lambda_i$, where $\lambda_0 > 1$ is given by

$$\lambda_0 = c(\lambda_0) \frac{1 + C_0}{1 - C_0} \left(\frac{3\gamma_1}{\gamma_2}\right)^{p-2}$$

for some constant $c(\lambda_0) > 1$, we see that for every $(i, j) \neq (1, 1)$

$$T_{\ell,\varepsilon_N}^{(11)} \le \frac{1 - \left(\frac{\gamma_{11}}{2}\right)^{p-2} N^{-\frac{1}{2(p-1)}}}{c(\lambda_0)(1+C_0)\beta\sqrt{M}p(p-2)\lambda_i\lambda_j \left(\frac{\gamma_{ij}}{2\sqrt{N}}\right)^{p-2} \left(\frac{3\gamma_1}{\gamma_2}\right)^{p-2}} \le T_{u,\varepsilon_N}^{(ij)}$$

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provided $N > \left(\frac{c(\lambda_0)}{c(\lambda_0)-1}\right)^{\frac{2(p-1)}{p-2}} \left(\frac{3}{2}\gamma_1\right)^{2(p-1)}$. As a consequence, on the event $\cap_{1 \le k, \ell \le r} \mathcal{A}^{(k\ell)}$, we have that $\mathcal{T}_{\varepsilon_N}^{(11)} = \min_{1 \le k, \ell \le r} \mathcal{T}_{\varepsilon_N}^{(k\ell)}$

with \mathbb{P} -probability at least $1 - \exp(-KN)$, that is, m_{11} is the first correlation that reaches the threshold ε_N . We therefore have that on the event $\bigcap_{1 \le k, \ell \le r} \mathcal{A}^{(k\ell)}$,

$$\mathcal{T}_{\varepsilon_N}^{(11)} \le T_0^{(11)} \lesssim \frac{1}{\sqrt{N}}$$

with \mathbb{P} -probability at least $1 - \exp(-\mathcal{K}N)$. Furthermore, we observe that as the correlation $m_{11}(t)$ exceeds ε_N , the other correlations are still on the scale $\Theta(N^{-\frac{1}{2}})$. Indeed, since $\mathcal{T}_{\varepsilon_N}^{(11)} \leq T_{\ell,\varepsilon_N}^{(11)}$ and u_{ij} is a monotone increasing function, on the event $\mathcal{A}^{(11)} \cap \mathcal{A}^{(ij)}$ we can upper bound $m_{ij}(\mathcal{T}_{\varepsilon_N}^{(11)})$ by $u_{ij}(\mathcal{T}_{\varepsilon_N}^{(11)}) \leq u_{ij}(\mathcal{T}_{\ell,\varepsilon_N}^{(11)})$ and we find that

$$u_{ij}(T_{\ell,\varepsilon_N}^{(11)}) = \frac{3\gamma_{ij}}{2\sqrt{N}} \left(1 - \frac{\lambda_i\lambda_j}{\lambda_1^2} \frac{1+C_0}{1-C_0} \left(\frac{3\gamma_{ij}}{\gamma_{11}}\right)^{p-2} \left(1 - (\gamma_{11}/2)^{p-2} N^{-\frac{p-2}{2(p-1)}} \right) \right)^{-\frac{1}{p-2}} \\ \leq \frac{3\gamma_{ij}}{2\sqrt{N}} \left(1 - \frac{\lambda_i\lambda_j}{\lambda_1^2} \frac{\lambda_0}{c(\lambda_0)} \right)^{-\frac{1}{p-2}} = \frac{3\gamma_{ij}}{2\sqrt{N}} \left(\frac{c(\lambda_0)}{c(\lambda_0) - 1} \right)^{\frac{1}{p-2}}.$$
(5.15)

Therefore, on the event $\bigcap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$, we have that $m_{ij}(\mathcal{T}_{\varepsilon_N}^{(11)}) = \gamma'_{ij}N^{-\frac{1}{2}}$ for some constant $\gamma'_{ij} > 0$ for every $(i, j) \neq (1, 1)$.

Step 2: Recovery of the first spike. Next, we study the evolution of $m_{11}(t)$ for $t \geq \mathcal{T}_{\varepsilon_N}^{(11)}$. Let $\varepsilon \in (0,1)$ be sufficiently small. According to the generator expansion given by Lemma 4.6, for every $\mathcal{T}_{\varepsilon_N}^{(11)} \leq t \leq \mathcal{T}_{\varepsilon}^{(11)} \wedge \mathcal{T}_{\varepsilon_N/2}^{(11)}$ we obtain that

$$L_{\beta}m_{11}(t) \geq -\|L_{0,\beta}m_{11}\|_{\infty} + \beta\sqrt{M}p\lambda_{1}^{2}m_{11}^{p-1}(t)(1-m_{11}^{2}(t))$$

$$\geq -\|L_{0,\beta}m_{11}\|_{\infty} + \beta\sqrt{M}p\lambda_{1}^{2}m_{11}^{p-1}(t) > 0.$$

Since $||L_{0,\beta}m_{11}||_{\infty} \leq \Lambda$ with \mathbb{P} -probability at least $1 - \exp(-KN)$ and since by assumption $\sqrt{M} \geq C \frac{\Lambda}{\beta p \lambda_r^2 C_0 \gamma_2^{p-1}} N^{\frac{p-1}{2}}$ for some constant C > 0, we then have that $L_{\beta}m_{11}$ is bounded below by

$$L_{\beta}m_{11}(t) \ge (1 - C_0)\beta \sqrt{Mp\lambda_1^2 m_{11}^{p-1}(t)} > 0$$

for $\mathcal{T}_{\varepsilon_N}^{(11)} \leq t \leq \mathcal{T}_{\varepsilon}^{(11)} \wedge \mathcal{T}_{\varepsilon_N/2}^{(11)}$, with P-probability at least $1 - \exp(-KN)$. According to Lemma 4.11, this then yields

$$m_{11}(t) \ge \frac{\varepsilon_N}{2} \left(1 - (1 - C_0)\beta\sqrt{M}p(p-2)\lambda_1^2 \left(\frac{\varepsilon_N}{2}\right)^{p-2} (t - \mathcal{T}_{\varepsilon_N}^{(11)}) \right)^{-\frac{1}{p-2}}$$

for every $\mathcal{T}_{\varepsilon_N}^{(11)} \leq t \leq \mathcal{T}_{\varepsilon}^{(11)} \wedge \mathcal{T}_{\varepsilon_N/2}^{(11)}$, with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-N^{\frac{1}{p-1}}/(4K_2(\mathcal{T}_{\varepsilon}^{(11)} \wedge \mathcal{T}_{\varepsilon_N/2}^{(11)})))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. In particular, we have that $m_{11}(t)$ is bounded below by a monotone increasing function which is positive at $t = \mathcal{T}_{\varepsilon_N}^{(11)}$. We therefore deduce that $\mathcal{T}_{\varepsilon}^{(11)} < \mathcal{T}_{\varepsilon_N/2}^{(11)}$ and by an easy computation we have that

$$\mathcal{T}_{\varepsilon}^{(11)} - \mathcal{T}_{\varepsilon_N}^{(11)} \le \frac{1 - \left(\frac{1}{2\varepsilon N^{\frac{p-2}{2(p-1)}}}\right)^{p-2}}{(1 - C_0)\beta\sqrt{M}p(p-2)\lambda_1^2 \left(\frac{N^{-\frac{p-2}{2(p-1)}}}{2}\right)^{p-2}} \lesssim \frac{1}{N^{\frac{2p-3}{2(p-1)}}} < 1,$$
(5.16)

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-N^{\frac{1}{p-1}}/(4K_2\mathcal{T}_{\varepsilon}^{(11)}))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. As a consequence, on the event $\cap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$, we find that

$$\mathcal{T}_{\varepsilon}^{(11)} = \mathcal{T}_{\varepsilon_N}^{(11)} + (\mathcal{T}_{\varepsilon}^{(11)} - \mathcal{T}_{\varepsilon_N}^{(11)}) \lesssim \frac{1}{\sqrt{N}},$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1-K_2 \exp(-N^{\frac{p+1}{2(p-1)}}/(4K_2))$ and with \mathbb{P} -probability at least $1-\exp(-KN)$. This proves weak recovery of the first spike. In order to show strong recovery, we proceed in a similar way. We note from Lemma 4.6 that $L_{\beta}m_{11}$ satisfies

$$L_{\beta}m_{11}(t) \geq -\|L_{0,\beta}m_{11}\|_{\infty} + 2^{-p+1}\beta\sqrt{M}p\lambda_1^2\varepsilon^{p-1}(2\varepsilon - \varepsilon^2),$$

for all $\mathcal{T}_{\varepsilon}^{(11)} \leq t \leq \mathcal{T}_{\varepsilon/2}^{(11)} \wedge \mathcal{T}_{1-\varepsilon}^{(11)}$. It therefore follows that

$$m_{11}(t) \ge \frac{\varepsilon}{2} + \frac{c}{2^{p-1}} \beta \sqrt{M} p \lambda_1^2 \varepsilon^{p-1} (2\varepsilon - \varepsilon^2) (t - \mathcal{T}_{\varepsilon}^{(11)})$$

for every $\mathcal{T}_{\varepsilon}^{(11)} \leq t \leq \mathcal{T}_{\varepsilon/2}^{(11)} \wedge \mathcal{T}_{1-\varepsilon}^{(11)}$, with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-\varepsilon^2 N / (4K_2(\mathcal{T}_{\varepsilon/2}^{(11)} \wedge \mathcal{T}_{1-\varepsilon}^{(11)})))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. As a consequence, we have that $\mathcal{T}_{1-\varepsilon}^{(11)} < \mathcal{T}_{\varepsilon/2}^{(11)}$ and we find that

$$\mathcal{T}_{1-\varepsilon}^{(11)} - \mathcal{T}_{\varepsilon}^{(11)} \leq \frac{(1-3\varepsilon/2)2^{p-1}}{c\beta\sqrt{M}p\lambda_1^2\varepsilon^{p-1}} \lesssim \frac{1}{N^{\frac{p-1}{2}}} < \frac{1}{\sqrt{N}}$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1-K_2 \exp(-\varepsilon N/(4K_2\mathcal{T}_{1}^{(11)}))$ and with \mathbb{P} -probability at least $1-\exp(-KN)$. In particular, we see that on the event $\bigcap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$

$$\mathcal{T}_{1-\varepsilon}^{(11)} = \mathcal{T}_{\varepsilon}^{(11)} + (\mathcal{T}_{1-\varepsilon}^{(11)} - \mathcal{T}_{\varepsilon}^{(11)}) \lesssim \frac{1}{\sqrt{N}}$$

with $\mathbb{Q}_{\boldsymbol{X}}$ -probability at least $1 - K_2 \exp(-N^{\frac{p+1}{2(p-1)}}/(4K_2)) - K_2 \exp(-\varepsilon^2 N/(4K_2 \mathcal{T}_{1-\varepsilon}^{(11)}))$ and with \mathbb{P} probability at least $1 - \exp(-KN)$.

Step 3: Evolution of $m_{i1}(t)$ and $m_{1i}(t)$ for $i \neq 1$ as $t \geq \mathcal{T}_{\varepsilon_N}^{(11)}$. Now, we study the evolution of $m_{i1}(t)$ and $m_{1i}(t)$ for $i \neq 1$ as $t \geq \mathcal{T}_{\varepsilon_N}^{(11)}$. According to Step 1, as m_{11} crosses the threshold $\varepsilon_N = N^{-\frac{p-2}{2(p-1)}}$, the correlations m_{i1} and m_{1i} are on the scale $\Theta(N^{-\frac{1}{2}})$. We first observe that for $t \geq \mathcal{T}_{\varepsilon_N}^{(11)}$ the generators $L_\beta m_{1i}$ and $L_\beta m_{i1}$ are bounded above by

$$L_{\beta}m_{i1}(t) \leq \|L_{0,\beta}m_{i1}\|_{\infty} + \beta\sqrt{M}p\lambda_{1}\lambda_{i}m_{i1}^{p-1}(t) - \beta\sqrt{M}p\lambda_{1}^{2}m_{11}^{p}(t)m_{i1}(t),$$

$$L_{\beta}m_{1i}(t) \leq \|L_{0,\beta}m_{1i}\|_{\infty} + \beta\sqrt{M}p\lambda_{1}\lambda_{i}m_{1i}^{p-1}(t) - \beta\sqrt{M}p\lambda_{1}^{2}m_{11}^{p}(t)m_{1i}(t).$$
(5.17)

In particular, we note that for $\mathcal{T}_{\varepsilon_N}^{(11)} \leq t \leq \mathcal{T}_{\varepsilon_N}^{(11)} \wedge \mathcal{T}_{\varepsilon_N}^{(i1)}$,

$$L_{\beta}m_{i1}(t) \le \|L_{0,\beta}m_{i1}\|_{\infty} + \beta\sqrt{M}p\lambda_{1}\lambda_{i}m_{i1}^{p-1}(t) \le (1+C_{0})\beta\sqrt{M}p\lambda_{1}\lambda_{i}m_{i1}^{p-1}(t),$$

and similarly for $\mathcal{T}_{\varepsilon_N}^{(11)} \leq t \leq \mathcal{T}_{N^{-\frac{p-2}{2p}}}^{(11)} \wedge \mathcal{T}_{\varepsilon_N}^{(1i)}$,

$$L_{\beta}m_{1i}(t) \le \|L_{0,\beta}m_{1i}\|_{\infty} + \beta\sqrt{M}p\lambda_1\lambda_i m_{1i}^{p-1}(t) \le (1+C_0)\beta\sqrt{M}p\lambda_1\lambda_i m_{1i}^{p-1}(t),$$

for some constant $C_0 \in (0, \frac{1}{2})$, with P-probability at least $1 - \exp(-KN)$. In particular, we can extend the argument of Step 1 and bound $m_{1i}(t), m_{i1}(t)$ at time $t = \mathcal{T}_{N^{-\frac{p-2}{2p}}}^{(11)}$ by (5.15) so that we have that $m_{1i}(\mathcal{T}_{N^{-\frac{p-2}{2p}}}^{(11)}) = \gamma_{1i}' N^{-\frac{1}{2}} \text{ and } m_{i1}(\mathcal{T}_{N^{-\frac{p-2}{2p}}}^{(11)}) = \gamma_{i1}' N^{-\frac{1}{2}} \text{ with high } \mathbb{Q}_{\mathbf{X}^{-}} \text{ and } \mathbb{P}\text{-probability.}$ We now study the evolution of m_{i1} and m_{1i} as $t \geq \mathcal{T}_{N^{-\frac{p-2}{2p}}}^{(11)}$ for every $i \neq 1$. We note from (5.17) that

as m_{11} exceeds $N^{-\frac{p-2}{2p}}$.

$$L_{\beta}m_{i1}(t) \le \|L_{0,\beta}m_{i1}\|_{\infty} - c\beta\sqrt{M}p\lambda_1^2 m_{11}^p(t)m_{i1}(t),$$

and m_{i1} start decreasing from $\Theta(N^{-\frac{1}{2}})$. The same holds for the correlations m_{1i} for $i \neq 1$. We therefore introduce the stopping time $\mathcal{T}_{lower}^{(i1)}$ given by

$$\mathcal{T}_{\text{lower}}^{(i1)} = \inf\left\{t \colon m_{i1}(t) \lesssim \frac{1}{\sqrt{\log(N)}N^{\frac{p-1}{4}}}\right\},\tag{5.18}$$

$$L_{\beta}m_{i1}(t) \leq \Lambda - c\beta\sqrt{M}p\lambda_1^2 m_{11}^p(t)m_{i1}(t) \leq -\frac{c}{2}\beta\sqrt{M}p\lambda_1^2\varepsilon^p m_{i1}(t),$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$ provided $\sqrt{M} \geq \frac{\Lambda}{\frac{C}{2}\beta p \lambda_1^2 \varepsilon^p m_{i1}(t)}$, which certainly holds since $\log(N)^{-\frac{1}{2}} N^{-\frac{p-1}{4}} \lesssim m_{i1}(t) \lesssim N^{-\frac{1}{2}}$. By the Grönwall's inequality (see item (d) of Lemma 4.11) we then have on the event $\cap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$,

$$m_{i1}(t) \leq \frac{3\gamma_{i1}'}{2\sqrt{N}} - \frac{c}{2}\beta\sqrt{M}p\lambda_1^2\varepsilon^p \int_{\mathcal{T}_{\varepsilon}^{(11)}}^t m_{i1}(s)ds \leq \frac{3\gamma_{i1}'}{2\sqrt{N}}\exp\left(-\frac{c}{2}\beta\sqrt{M}p\lambda_1^2\varepsilon^p(t-\mathcal{T}_{\varepsilon}^{(11)})\right),$$

for $\mathcal{T}_{\varepsilon}^{(11)} \leq t \leq \mathcal{T}_{\text{lower}}^{(i1)} \wedge \mathcal{T}_{\varepsilon_N}^{(i1)}$, with \mathbb{P} -probability at least $1 - \exp(-KN)$. Since m_{i1} is upper bounded by a decreasing function for every $\mathcal{T}_{\varepsilon}^{(11)} \leq t \leq \mathcal{T}_{\text{lower}}^{(i1)} \wedge \mathcal{T}_{\varepsilon_N}^{(i1)}$, it then follows that $\mathcal{T}_{\text{lower}}^{(i1)} < \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon_N}^{(i1)}$. Therefore, we have that

$$\mathcal{T}_{\text{lower}}^{(i1)} - \mathcal{T}_{\varepsilon}^{(11)} \le \frac{2}{c\beta\sqrt{M}p\lambda_1^2\varepsilon^p}\log(\sqrt{\log(N)}N^{\frac{p-3}{4}}) \lesssim \log(N)N^{-\frac{p-1}{2}},\tag{5.19}$$

ensuring that on the event $\bigcap_{1 \le k, \ell \le r} \mathcal{A}^{(k\ell)}$,

$$\mathcal{T}_{\text{lower}} = \mathcal{T}_{\varepsilon}^{(11)} + (\mathcal{T}_{\text{lower}} - \mathcal{T}_{\varepsilon}^{(11)}) \lesssim \mathcal{T}_{\varepsilon}^{(11)} + \log(N)N^{-\frac{p-1}{2}} < \frac{1}{\sqrt{N}},$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1-K_2 \exp(-N^{\frac{p+1}{2(p-1)}}/(4K_2))$ and with \mathbb{P} -probability at least $1-\exp(-KN)$.

Step 4: Evolution of $m_{ij}(t)$ for $i, j \neq 1$ as $t \geq \mathcal{T}_{\varepsilon_N}^{(11)}$. It remains to study the evolution of $m_{ij}(t)$ for every $i, j \neq 1$ as $t \geq \mathcal{T}_{\varepsilon_N}^{(11)}$. We first observe that for $t \geq \mathcal{T}_{\varepsilon_N}^{(11)}$, according to Lemma 4.6 the generator $L_{\beta}m_{ij}$ is bounded above by

$$L_{\beta}m_{ij}(t) \le \|L_{0,\beta}m_{ij}\|_{\infty} + \beta\sqrt{M}p\lambda_i\lambda_j m_{ij}^{p-1}(t) - \frac{1}{2}\beta\sqrt{M}p\lambda_1^2 m_{11}^{p-1}(t)m_{i1}(t)m_{1j}(t).$$
(5.20)

In particular, we note that for $\mathcal{T}_{\varepsilon_N}^{(11)} \leq t \leq \mathcal{T}_{N^{-\frac{p-3}{2(p-1)}}}^{(11)} \wedge \mathcal{T}_{\varepsilon_N}^{(ij)}$,

$$L_{\beta}m_{ij}(t) \le \|L_{0,\beta}m_{ij}\|_{\infty} + \beta\sqrt{M}p\lambda_i\lambda_j m_{ij}^{p-1}(t) \le (1+C_0)\beta\sqrt{M}p\lambda_i\lambda_j m_{ij}^{p-1}(t),$$

for some $C_0 \in (0, \frac{1}{2})$, with \mathbb{P} -probability at least $1 - \exp(-KN)$. In particular, as argued in Step 3, we can extend the argument given by (5.15) so that we have that $m_{ij}\left(\mathcal{T}_{N^{-\frac{p-3}{2(p-1)}}}^{(11)}\right) = \gamma'_{ij}N^{-\frac{1}{2}}$ with high $\mathbb{Q}_{\mathbf{X}}$ - and \mathbb{P} -probability.

We now fix $i, j \neq 1$ and study the evolution of m_{ij} for $t \geq \mathcal{T}_{N^{-\frac{p-3}{2(p-1)}}}^{(11)}$. We first note that as $t \geq \mathcal{T}_{N^{-\frac{p-3}{2(p-1)}}}^{(11)}$, the term $\lambda_1^2 m_{11}^{p-1}(t) m_{i1}(t) m_{1j}(t)$ in (5.20) may be larger than $\lambda_i \lambda_j m_{ij}^{p-1}(t)$ and may lead to a decrease in m_{ij} . Recalling the stopping time $\mathcal{T}_{\text{lower}}$ introduced by (5.18), we see that if $\mathcal{T}_{N^{-\frac{p-2}{2p}}}^{(11)} \leq \mathcal{T}_{\text{lower}} \leq \mathcal{T}_{N^{-\frac{p-3}{2(p-1)}}}^{(11)}$, then we still have the estimate

$$-\|L_{0,\beta}m_{ij}\|_{\infty} + \beta\sqrt{M}p\lambda_i\lambda_j m_{ij}^{p-1}(t) \le L_{\beta}m_{ij}(t) \le \|L_{0,\beta}m_{ij}\|_{\infty} + \beta\sqrt{M}p\lambda_i\lambda_j m_{ij}^{p-1}(t),$$
(5.21)

for every $\mathcal{T}_{N^{-\frac{p-3}{2(p-1)}}}^{(11)} \leq t \leq \mathcal{T}_{\varepsilon}^{(11)} \wedge \mathcal{T}_{\varepsilon_{N}}^{(ij)}$, and the evolution m_{ij} keeps increasing. Otherwise, if $\mathcal{T}_{\text{lower}} > \mathcal{T}_{N^{-\frac{p-3}{2(p-1)}}}^{(11)}$, then $L_{\beta}m_{ij}(t)$ is bounded according to

$$L_{\beta}m_{ij}(t) \geq -\|L_{0,\beta}m_{ij}\|_{\infty} - r^{2}\beta\sqrt{M}p\lambda_{1}^{2}m_{11}^{p-1}(t)m_{i1}^{p-1}(t)m_{1j}^{p-1}(t) L_{\beta}m_{ij}(t) \leq \|L_{0,\beta}m_{ij}\|_{\infty} - \frac{1}{2}\beta\sqrt{M}p\lambda_{1}^{2}m_{11}^{p-1}(t)m_{i1}^{p-1}(t)m_{1j}^{p-1}(t),$$
(5.22)

for $\mathcal{T}_{N^{-\frac{p-3}{2(p-1)}}}^{(11)} \leq t \leq \mathcal{T}_{\text{lower}} \wedge \mathcal{T}_{\varepsilon}^{(11)} \wedge \mathcal{T}_{\varepsilon_{N}}^{(ij)}$, thus the evolution of m_{ij} on this time interval is decreasing. We therefore need to quantify the decrease of m_{ij} between $\mathcal{T}_{N^{-\frac{p-3}{2(p-1)}}}^{(11)}$ and $\mathcal{T}_{\text{lower}}$. We may assume without loss of generality that $\mathcal{T}_{\varepsilon}^{(11)} < \mathcal{T}_{\text{lower}}$ in order to have the maximal decrease of m_{ij} . It is then sufficient to show that the decrease of m_{ij} between $\mathcal{T}_{N^{-\frac{p-3}{2(p-1)}}}^{(11)}$ and $\mathcal{T}_{\varepsilon}^{(11)}$ is by a constant, thus ensuring that as m_{11} exceeds ε the correlations m_{ij} are still on a scale $\Theta(N^{-\frac{1}{2}})$. We claim that we have shown that $m_{ij}(\mathcal{T}_{\varepsilon}^{(11)}) = \gamma_{ij}'' N^{-\frac{1}{2}}$ for some constant $\gamma_{ij}'' > 0$. Then, we see from (5.22) that

$$L_{\beta}m_{ij}(t) \ge -\|L_{0,\beta}m_{ij}\|_{\infty} - r^{2}\beta\sqrt{M}p\lambda_{1}^{2}m_{11}^{p-1}(t)m_{i1}(t)m_{1j}(t) \ge -Cr^{2}\beta\sqrt{M}p\lambda_{1}^{2}(1-\varepsilon)^{p-1}N^{-1},$$

for every $\mathcal{T}_{\varepsilon}^{(11)} \leq t \leq \mathcal{T}_{\text{lower}} \wedge \mathcal{T}_{1-\varepsilon}^{(11)}$, with \mathbb{P} -probability at least $1 - \exp(-KN)$. We therefore obtain that

$$m_{ij}(t) \ge m_{ij}(\mathcal{T}_{\varepsilon}^{(11)}) + M_t^{m_{ij}} - Cr^2\beta\sqrt{M}p\lambda_1^2(1-\varepsilon)^{p-1}N^{-1}(t-\mathcal{T}_{\varepsilon}^{(11)})$$

for $\mathcal{T}_{\varepsilon}^{(11)} \leq t \leq \mathcal{T}_{\text{lower}} \wedge \mathcal{T}_{1-\varepsilon}^{(11)}$, yielding

$$m_{ij}(\mathcal{T}_{\text{lower}}) \geq \frac{\gamma_{ij}''}{2\sqrt{N}} - Cr^2 \beta \sqrt{M} p \lambda_1^2 (1-\varepsilon)^{p-1} N^{-1} \left(\mathcal{T}_{\text{lower}} - \mathcal{T}_{\varepsilon}^{(11)} \right)$$
$$\geq \frac{\gamma_{ij}''}{2\sqrt{N}} - C' (1-\varepsilon)^{p-1} \frac{\log(N)}{\sqrt{N}\sqrt{N}} \approx \frac{c'}{\sqrt{N}},$$

on the event $\bigcap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$, with \mathbb{P} -probability at least $1 - \exp(-KN)$, where we used the bound given by (5.19).

It therefore remains to prove the claim that $m_{ij}(\mathcal{T}_{\varepsilon}^{(11)}) = \gamma_{ij}^{\prime\prime} N^{-\frac{1}{2}}$. We first note that there is nothing to prove for p = 3 since m_{ij} may decrease once m_{11} is macroscopic since m_{ij} start decreasing as m_{11} exceeds $N^{-\frac{p-3}{2(p-1)}} \approx \varepsilon$. We therefore consider $p \geq 4$. In particular, we observe that for every $\nu \in (0, \frac{p-3}{2(p-1)}]$ and every $\delta < \nu$, the time difference $\mathcal{T}_{N^{-\nu+\delta}}^{(11)} - \mathcal{T}_{N^{-\nu}}^{(11)}$ is bounded above by

$$\mathcal{T}_{N^{-\nu+\delta}}^{(11)} - \mathcal{T}_{N^{-\nu}}^{(11)} \le 2\left(T_{\ell,N^{-\nu+\delta}}^{(11)} - T_{\ell,N^{-\nu}}^{(11)}\right) = \frac{2}{(1-C_0)\beta\sqrt{M}p(p-2)\lambda_1^2}N^{\nu(p-2)}\left(1-N^{-\delta(p-2)}\right),$$

where we used (5.16). We will use the above estimate to provide a bound on the correlation $m_{ij}(t)$ for $\mathcal{T}_{N^{-\nu}}^{(11)} \leq t \leq \mathcal{T}_{N^{-\nu+\delta}}^{(11)}$. According to (5.22), for every $\mathcal{T}_{N^{-\nu}}^{(11)} \leq t \leq \mathcal{T}_{N^{-\nu+\delta}}^{(11)}$, we bound $L_{\beta}m_{ij}$ below by

$$L_{\beta}m_{ij}(t) \geq -\|L_{0,\beta}m_{ij}\|_{\infty} - r^{2}\beta\sqrt{M}p\lambda_{1}^{2}m_{11}^{p-1}(t)m_{i1}(t)m_{1j}(t)$$

$$\geq -\Lambda - r^{2}\beta\sqrt{M}p\lambda_{1}^{2}N^{-(\nu-\delta)(p-1)-1}$$

$$\geq -2r^{2}\beta\sqrt{M}p\lambda_{1}^{2}N^{-(\nu-\delta)(p-1)-1},$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$, provided $\sqrt{M} \ge \frac{\Lambda}{\beta p r^2 \lambda_1^2} N^{(\nu-\delta)(p-1)+1}$ which certainly holds since by assumption $(\nu - \delta)(p-1) + 1 \le \frac{p-1}{2}$. We therefore obtain that

$$m_{ij}(t) \ge m_{ij}(\mathcal{T}_{N^{-\nu}}^{(11)}) + M_t^{m_{ij}} - r^2 \beta \sqrt{M} p \lambda_1^2 N^{-(\nu-\delta)(p-1)-1} \left(t - \mathcal{T}_{N^{-\nu}}^{(11)}\right)$$

for every $\mathcal{T}_{N^{-\nu}}^{(11)} \leq t \leq \mathcal{T}_{N^{-\nu+\delta}}^{(11)}$, yielding

$$m_{ij}(\mathcal{T}_{N^{-\nu+\delta}}^{(11)}) \geq \frac{1}{2}m_{ij}(\mathcal{T}_{N^{-\nu}}^{(11)}) - r^2\beta\sqrt{M}p\lambda_1^2N^{-(\nu-\delta)(p-1)-1}\left(\mathcal{T}_{N^{-\nu+\delta}}^{(11)} - \mathcal{T}_{N^{-\nu}}^{(11)}\right)$$

$$\geq \frac{1}{2}m_{ij}(\mathcal{T}_{N^{-\nu}}^{(11)}) - C\frac{N^{\delta(p-1)} - N^{\delta}}{NN^{\nu}}.$$
(5.23)

We see that if $\nu = \frac{p-3}{2(p-1)}$ and $\delta < \frac{\nu}{p-1}$, then

$$m_{ij}(\mathcal{T}_{N^{-\nu+\delta}}^{(11)}) \ge \frac{\gamma'_{ij}}{2\sqrt{N}} - \frac{C}{\sqrt{N}} \frac{N^{\delta(p-1)} - N^{\delta}}{N^{\nu+\frac{1}{2}}} \ge \frac{C}{\sqrt{N}}.$$

In particular, we can divide the interval $[\mathcal{T}_{N^{-\frac{(p-3)}{2(p-1)}}}^{(11)}, \mathcal{T}_{\varepsilon}^{(11)}]$ into a finite number $n(\delta)$ of small intervals and by iterating the argument above until $N^{-\nu+\delta} \approx \varepsilon$ we can show that at each step the decrease of m_{ij} is at most by a constant. This ensures that at $t = \mathcal{T}_{\varepsilon}^{(11)}$ the correlation $m_{ij}(t) \in \Theta(N^{-\frac{1}{2}})$.

Step 5: Conclusion of the proof. We therefore see that the hitting time $\mathcal{T}_{E_1(\varepsilon)} = \inf \{ X \colon X \in E_1(\varepsilon) \}$ is given by

$$\mathcal{T}_{E_1(\varepsilon)} = \mathcal{T}_{1-\varepsilon}^{(11)} \vee \mathcal{T}_{\text{lower}}.$$

In particular, according to the Steps 2 and 3 we have that on the event $\bigcap_{1 \le k, \ell \le r} \mathcal{A}^{(k\ell)}$,

$$\mathcal{T}_{E_1(\varepsilon)} = \mathcal{T}_{1-\varepsilon}^{(11)} \lor \mathcal{T}_{\text{lower}} \lesssim \frac{1}{\sqrt{N}},$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-N^{\frac{p+1}{2(p-1)}}/(4K_2)) - K_2 \exp(-\varepsilon^2 N^{3/2}/(4K_2))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. We then choose $T_0^{(11)} = T_{\ell,\varepsilon_N}^{(11)}$ and $T_0^{(ij)} = \mathcal{T}_{\text{lower}}$ for every $(i, j) \neq (1, 1)$. In this way, we have that on the initial event $\mathcal{C}_1(\gamma_1, \gamma_2)$,

$$\mathcal{T}_{E_1(\varepsilon)} \lesssim \frac{1}{\sqrt{N}},$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least

$$1 - r^{2}K_{2}\exp(-\gamma_{2}^{2}\sqrt{N}/(4K_{2})) - K_{2}\exp(-N^{\frac{p+1}{2(p-1)}}/(4K_{2})) - K_{2}\exp(-\varepsilon^{2}N^{3/2}/(4K_{2})), \qquad (5.24)$$

with P-probability at least $1 - \exp(-KN)$ thus completing the proof.

and with \mathbb{P} -probability at least $1 - \exp(-KN)$ thus completing the proof.

It remains to prove Lemma 5.4.

Proof of Lemma 5.4. We prove the statement for k = 2 since the proof will be identical for the other cases. Let $\varepsilon > 0$ and assume that $X_0 \in E_1(\varepsilon)$. We first show that the evolution of the correlations m_{11} and m_{1i}, m_{i1} for $i \neq 1$ are stable for all $t \geq 0$. This can indeed be easily see from the generator expansion given by Lemma 4.6. In particular, we have that

$$L_{\beta}m_{11}(t) \ge -\|L_{0,\beta}m_{11}\|_{\infty} + \beta\sqrt{M}p\lambda_1^2m_{11}^{p-1}(t)(1-m_{11}^2(t))$$

for every $t \ge 0$. Using similar arguments to that used for Step 2 of the proof of Lemma 5.3 we can show that $m_{11}(t)$ will stay above $1 - \varepsilon$ for all $t \ge 0$ with high $\mathbb{Q}_{\mathbf{X}^{-}}$ and \mathbb{P} -probability. Similarly, we see that

$$L_{\beta}m_{1i}(t) \le \|L_{0,\beta}m_{1i}\|_{\infty} - c\beta\sqrt{Mp\lambda_1^2}m_{11}^p(t)m_{1i}(t)$$

for all $t \ge 0$. As done in Step 3 of the proof of Lemma 5.3 we can show that $m_{1i}(t)$ and $m_{i1}(t)$ will not increase and therefore will stay below the threshold $1/(\sqrt{\log(N)}N^{\frac{p-1}{4}})$ for every $t \ge 0$, with high \mathbb{Q}_{X} and \mathbb{P} -probability.

We therefore consider the evolution of the correlations m_{ij} for $i, j \neq 1$. Since $X_0 \in E_1(\varepsilon)$ we have that $m_{ij}(0) = \gamma_{ij} N^{-\frac{1}{2}}$ for some order-1 constant $\gamma_{ij} > 0$. Let $\varepsilon_N = N^{-\frac{p-2}{2(p-1)}}$. By the generator expansion from Lemma 4.6, i.e.,

$$L_{\beta}m_{ij} = L_{0,\beta}m_{ij} + \beta\sqrt{M}p\lambda_i\lambda_j m_{ij}^{p-1} - \beta\sqrt{M}\frac{p}{2}\sum_{1\le k,\ell\le r}\lambda_k m_{i\ell}m_{kj}m_{k\ell}(\lambda_j m_{kj}^{p-2} + \lambda_\ell m_{k\ell}^{p-2}),$$

we see that for every $i, j \neq 1$,

$$-\|L_{0,\beta}m_{ij}\|_{\infty} + \beta\sqrt{M}p\lambda_i\lambda_j m_{ij}^{p-1}(t) \le L_{\beta}m_{ij}(t) \le \|L_{0,\beta}m_{ij}\|_{\infty} + \beta\sqrt{M}p\lambda_i\lambda_j m_{ij}^{p-1}(t),$$

for all $t \leq \min_{2 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)}$. Indeed, the terms associated with m_{11} in the generator expansion are also accompanied by m_{i1} and m_{1i} which make that globally they are small compared to the term $\beta \sqrt{M} p \lambda_i \lambda_j m_{ij}^{p-1}$, for N sufficiently large. We can therefore proceed exactly as done in the proof of

Lemma 5.3 by mimicking all the four steps. This shows that there exists $T_2 > \mathcal{T}_{E_1(\varepsilon)}$ such that for all $T > T_2$,

$$\inf_{\mathbf{X}_0 \in E_1(\varepsilon)} \mathbb{Q}_{\mathbf{X}_0} \left(\inf_{t \in [T_2, T]} \mathbf{X}_t^\beta \in E_2(\varepsilon) \right) \\
\geq 1 - (r-1)^2 K_2 \exp(-\gamma_2^2 \sqrt{N}/(4K_2)) - K_2 \exp(-N^{\frac{p+1}{2(p-1)}}/(4K_2)) - K_2 \exp(-\varepsilon^2 N^{3/2}/(4K_2)),$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$ provided N is sufficiently large.

6. Proofs for p = 2 and distinct SNRs

This subsection is devoted to the proofs of Propositions 3.7 and 3.8.

6.1. Recovery of leading spike

The proof of Proposition 3.7 follows a similar structure to the proof of Proposition 3.5. We begin to state the result on weak recovery of the first spike.

Lemma 6.1. Let $\beta \in (0, \infty)$, p = 2, $\lambda_i = \lambda_{i+1}(1 + \kappa_i)$ for every $1 \le i \le r-1$ and $\kappa_i > 0$. Consider a sequence of initializations $\mu_0 \in \mathcal{P}(\mathcal{M}_{N,r})$. For every $\varepsilon > 0$, we let $\mathcal{T}_{\varepsilon}^{(11)}$ denote the hitting time of the set $\{\mathbf{X} : m_{11}(\mathbf{X}) \ge \varepsilon\}$. Then, the following holds: For every $n \ge 1, \gamma_0 > 0, \gamma_1 > 1 > \gamma_2 > 0$, there exist $\varepsilon_0 > 0$ and $c_0 \in (0, \frac{1}{2} \land \frac{\kappa_1}{2 + \kappa_1})$ such that for every $\varepsilon < \varepsilon_0$, $C_0 < c_0$, $\sqrt{M} \gtrsim \frac{(n+2)\gamma_0}{\beta\lambda_r^2 C_0 \gamma_2} N^{\frac{1}{2(n+1)}}$, and $\log(N) \ge 2\log(3\gamma_1/2) + 2\log(3\gamma_1/\gamma_2) \frac{1+C_0}{(1-C_0)(1+\kappa_1)-(1+C_0)} - 2\log(\varepsilon)$ we have that

$$\int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\mathbf{X}} \left(\mathcal{T}_{\varepsilon}^{(11)} \gtrsim \frac{\log(N)}{(n+2)\gamma_0 N^{\frac{1}{2(n+1)}}} \right) \mathbf{1} \{ \mathcal{C}_0(n,\gamma_0) \cap \mathcal{C}_1(\gamma_1,\gamma_2) \} d\mu_0(\mathbf{X}) \\ \leq K_1 e^{-\gamma_0^3(n+2)N^{\frac{1}{2(n+1)}}/(K_1 \log(N))} + r^2 K_2 e^{-\gamma_2^2 \gamma_0(n+2)N^{\frac{1}{2(n+1)}}/(4K_2 \log(N))},$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$.

As for $p \ge 3$, strong recovery of the first spike follows straightforwardly from the weak recovery result, as stated in the following lemma.

Lemma 6.2. Let $\beta \in (0, \infty)$, p = 2, $\lambda_i = \lambda_{i+1}(1 + \kappa_i)$ for every $1 \le i \le r - 1$ and $\kappa_i > 0$. Then, for every $\varepsilon > 0$, there exists $T_0 \gtrsim \frac{1}{(n+2)\gamma_0} \log(N) N^{-\frac{1}{2(n+1)}}$ such that for all $T \ge T_0$ and N large enough,

$$\inf_{\boldsymbol{X}: \ m_{11}(\boldsymbol{X}) \ge \varepsilon} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t \in [T_0, T]} m_{11}(\boldsymbol{X}_t^\beta) \ge 1 - \varepsilon \right) \ge 1 - 2K_1 e^{-N\varepsilon^2/(K_2T)}$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$.

The proof of Lemma 6.2 follows the same strategy used to prove [8, Lemma 4.1] and is therefore left to the reader. Proposition 3.7 follows the same approach of the proof of Proposition 3.5 by combining Lemmas 6.1 and 6.2 via the strong Markov property. It then remains to prove Lemma 6.1, for which we will mimic the proof of Lemma 5.1.

Proof of Lemma 6.1. We let $\mathcal{A} = \mathcal{A}(n, \gamma_0, \gamma_1, \gamma_2)$ denote the event

$$\mathcal{A}(n,\gamma_0,\gamma_1,\gamma_2) = \{ \boldsymbol{X}_0 \sim \mu \colon \boldsymbol{X}_0 \in \mathcal{C}_0(n,\gamma_0) \cap \mathcal{C}_1(\gamma_1,\gamma_2) \}$$

where $C_0(n, \gamma_0)$ and $C_1(\gamma_1, \gamma_2)$ are given in Definitions 3.1 and 3.3, respectively. We note that on $C_1(\gamma_1, \gamma_2)$, for every $i, j \in [r]$ there exists $\gamma_{ij} \in (\gamma_2, \gamma_1)$ such that $m_{ij}(0) = \gamma_{ij} N^{-\frac{1}{2}}$. For some $T_0^{(ij)} > 0$ to be chosen later, we then define the event $\mathcal{A}^{(ij)} = \mathcal{A}^{(ij)}(n, \gamma_0, \gamma_1, \gamma_2, T_0^{(ij)})$ by

$$\mathcal{A}^{(ij)}(n,\gamma_0,\gamma_1,\gamma_2,T_0^{(ij)}) = \mathcal{A}(n,\gamma_0,\gamma_1,\gamma_2) \cap \left\{ \sup_{t \in [0,T_0^{(ij)}]} |M_t^{m_{ij}}| \le \frac{\gamma_2}{2\sqrt{N}} \right\},$$

where we recall that according to Lemma 4.7 and (4.9), there exists a constant $K_2 > 0$ such that

$$\sup_{\boldsymbol{X}} \mathbb{Q}_{\boldsymbol{X}} \left(\sup_{t \in [0, T_0^{(ij)}]} |M_t^{m_{ij}}| \ge \frac{\gamma_2}{2\sqrt{N}} \right) \le K_2 \exp\left(-\frac{\gamma_2^2}{4K_2 T_0^{(ij)}}\right)$$

Moreover, for every $i, j \in [r]$, we let $\mathcal{T}_{L_{0,\beta}}^{(ij)}$ denote the hitting time of the set

$$\left\{ \boldsymbol{X} : |L_{0,\beta}m_{ij}(\boldsymbol{X})| > 2C_0\beta\sqrt{M}\lambda_i\lambda_jm_{ij}(\boldsymbol{X}) \right\}$$

where $C_0 \in (0, \frac{1}{2})$ is a constant which does not depend on N. We note that on $\mathcal{C}_0(n, \gamma_0)$,

$$|L_{0,\beta}m_{ij}(\boldsymbol{X}_0)| \leq \frac{\gamma_0}{\sqrt{N}} \leq 2C_0\beta\sqrt{M}\lambda_i\lambda_j\frac{\gamma_2}{\sqrt{N}}$$

provided $\sqrt{M} \geq \frac{\gamma_0}{2C_0\beta\lambda_i\lambda_j\gamma_2}$, which certainly holds by assumption. Therefore, on the event $\mathcal{C}_0(n,\gamma_0)$, we have that

$$|L_{0,\beta}m_{ij}(\boldsymbol{X}_0)| \le 2C_0\beta\sqrt{M\lambda_i\lambda_j}m_{ij}(\boldsymbol{X}_0)$$

and by continuity of the process \boldsymbol{X}_t , $\mathcal{T}_{L_{0,\beta}}^{(ij)} > 0$. We also introduce the hitting time $\mathcal{T}_{L_{0,\beta}}$ of the set

$$\left\{ \boldsymbol{X}: \sup_{1 \leq k, \ell \leq r} |L_{0,\beta} m_{k\ell}(\boldsymbol{X})| > 2C_0 \beta \sqrt{M} \lambda_1^2 m_{11}(\boldsymbol{X}) \right\}.$$

We also have that $\mathcal{T}_{L_{0,\beta}} > 0$ and we note that $\mathcal{T}_{L_{0,\beta}} \leq \mathcal{T}_{L_{0,\beta}}^{(11)}$. In the following, we fix $i, j \in [r]$ and place ourselves on the event $\mathcal{A}^{(ij)}$. For every $\varepsilon > 0$, we denote by $\mathcal{T}_{\varepsilon}^{(ij)}$ the hitting time for the set $\{\mathbf{X}: m_{ij}(\mathbf{X}) \geq \varepsilon\}$. Recalling the generator expansion given by Lemma 4.6, i.e.,

$$L_{\beta}m_{ij}(t) = L_{0,\beta}m_{ij}(t) + 2\beta\sqrt{M}\lambda_i\lambda_jm_{ij}(t) - \beta\sqrt{M}\sum_{1\le k,\ell\le r}\lambda_k(\lambda_j+\lambda_\ell)m_{i\ell}(t)m_{kj}(t)m_{k\ell}(t),$$

we have that

$$2(1-C_0)\beta\sqrt{M}\lambda_i\lambda_j m_{ij}(t) \le L_\beta m_{ij}(t) \le 2(1+C_0)\beta\sqrt{M}\lambda_i\lambda_j m_{ij}(t)$$

for every $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)}$. According to the evolution equation for m_{ij} , namely $m_{ij}(t) = m_{ij}(0) + M_t^{m_{ij}} + \int_0^t Lm_{ij}(s) ds$, we obtain the integral inequality

$$\frac{\gamma_{ij}}{2\sqrt{N}} + 2(1-C_0)\beta\sqrt{M}\lambda_i\lambda_j \int_0^t m_{ij}(s)ds \le m_{ij}(t) \le \frac{3\gamma_{ij}}{2\sqrt{N}} + 2(1+C_0)\beta\sqrt{M}\lambda_i\lambda_j \int_0^t m_{ij}(s)ds, \quad (6.1)$$

which holds for $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)} \wedge T_{0}^{(ij)}$. Grönwall's inequality (see item (d) of Lemma 4.11) then gives that

$$\ell_{ij}(t) \le m_{ij}(t) \le u_{ij}(t), \tag{6.2}$$

for every $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k,\ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)} \wedge T_{0}^{(ij)}$, where the functions ℓ_{ij} and u_{ij} are given by

$$\ell_{ij}(t) = \frac{\gamma_{ij}}{2\sqrt{N}} \exp\left(2(1-C_0)\beta\sqrt{M}\lambda_i\lambda_j t\right),\,$$

and

$$u_{ij}(t) = \frac{3\gamma_{ij}}{2\sqrt{N}} \exp\left(2(1+C_0)\beta\sqrt{M}\lambda_i\lambda_j t\right),\,$$

respectively. We then define $T_{\ell,\varepsilon}^{(ij)}$ by $\ell_{ij}(T_{\ell,\varepsilon}^{(ij)}) = \varepsilon$, i.e.,

$$T_{\ell,\varepsilon}^{(ij)} = \frac{\log(\varepsilon\sqrt{N}) - \log(\gamma_{ij}/2)}{2(1 - C_0)\beta\sqrt{M}\lambda_i\lambda_j}.$$
(6.3)

Similarly, we let $T_{u,\varepsilon}^{(ij)}$ satisfy $u_{ij}(T_{u,\varepsilon}^{(ij)}) = \varepsilon$, so that

$$T_{u,\varepsilon}^{(ij)} = \frac{\log(\varepsilon\sqrt{N}) - \log(3\gamma_{ij}/2)}{2(1+C_0)\beta\sqrt{M}\lambda_i\lambda_j}.$$
(6.4)

We note that on the event $\mathcal{A}^{(ij)}, T^{(ij)}_{u,\varepsilon_N} \leq \mathcal{T}^{(ij)}_{\varepsilon_N} \leq T^{(ij)}_{\ell,\varepsilon_N}$. We then choose $T^{(ij)}_0 = T^{(ij)}_{\ell,\varepsilon} > 0$. We then note that

$$\begin{split} T_{\ell,\varepsilon}^{(11)} &\leq \frac{\log(\varepsilon\sqrt{N}) - \log(\gamma_2/2)}{2(1-C_0)\beta\sqrt{M}\lambda_i\lambda_j\prod_{k=1}^{i-1}(1+\kappa_k)\prod_{\ell=1}^{j-1}(1+\kappa_\ell)} \\ &\leq \frac{\log(\varepsilon\sqrt{N}) - \log(\gamma_2/2)}{2(1-C_0)\beta\sqrt{M}\lambda_i\lambda_j(1+\kappa_1)} \\ &\leq \frac{\log(\varepsilon\sqrt{N}) - \log(3\gamma_1/2)}{2(1+C_0)\beta\sqrt{M}\lambda_i\lambda_j} \leq T_{u,\varepsilon}^{(ij)}, \end{split}$$

where for the first two inequalities we used the fact that by definition $\lambda_1 = \lambda_i (1 + \kappa_1) \cdots (1 + \kappa_{i-1})$ and that $\lambda_1^2 \geq \lambda_i \lambda_j (1+\kappa_1)^2 \geq \lambda_i \lambda_j (1+\kappa_1)$ for every $i \geq 2$, and for the last inequality we used the fact that by assumption,

$$\log(\varepsilon\sqrt{N}) \ge \log\left(\frac{3\gamma_1}{2}\right) + \log\left(\frac{3\gamma_1}{\gamma_2}\right) \frac{(1+C_0)}{(1-C_0)(1+\kappa_1) - (1+C_0)}.$$

We need to choose $C_0 < \frac{1}{2} \land \frac{\kappa_1}{2+\kappa_1}$ in order to ensure that $(1 - C_0)(1 + \kappa_1) - (1 + C_0) > 0$. Since

 $T_{\ell,\varepsilon}^{(11)} \leq T_{u,\varepsilon}^{(ij)} \text{ for every } (i,j) \neq (1,1), \text{ we deduce that } T_0^{(11)} = \min_{1 \leq k, \ell \leq r} T_0^{(k\ell)}.$ Our goal is to show that $\mathcal{T}_{\varepsilon}^{(11)} = \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)}$ and that $\mathcal{T}_{\varepsilon}^{(11)} \leq \mathcal{T}_{L_{0,\beta}} \leq \mathcal{T}_{L_{0,\beta}}^{(11)}.$ As done in the proof of Lemma 5.1 for $p \geq 3$, to get an estimate for $L_{0,\beta}m_{ij}$ for every $i, j \in [r]$, we wish to apply Lemma 4.4 to the function $F_{ij}(\mathbf{X}) = L_{0,\beta}m_{ij}(\mathbf{X}).$ Conditions (1)-(3) are easily verified using the same arguments of the proof of Lemma 5.1. To see condition (4), we note that for every $i, j \in [r]$, on the event $\mathcal{A}^{(ij)}$,

$$\int_{0}^{t} |a_{ij}(s)| ds \leq \frac{1}{1 - C_0} \left(m_{ij}(t) - \frac{\gamma_{ij}}{2\sqrt{N}} \right) \leq \frac{1}{1 - C_0} m_{ij}(t), \tag{6.5}$$

for every $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)} \wedge \mathcal{T}_{0}^{(11)}$, where we used the lower bound in the integral inequality (6.1) and recall that $a_{ij}(t) = 2\beta \sqrt{M} \lambda_i \lambda_j m_{ij}(t)$. We then observe that at time t = 0, for every $\xi > 0$ we have that

$$\xi 2\beta \sqrt{M}\lambda_i \lambda_j \ell_{ij}(0) = \xi \beta \sqrt{M}\lambda_i \lambda_j \frac{\gamma_{ij}}{\sqrt{N}} \ge C\xi(n+2)\gamma_0 N^{-\frac{n}{2(n+1)}} \ge \ell_{ij}(0)$$

where we used that $\sqrt{M} \geq C \frac{n+2}{\beta \lambda_r^2 C_0(1-C_0)\gamma_2} \gamma_0 N^{\frac{1}{2(n+1)}}$ for some constant C > 0. Since by (5.2) the function $m_{ij}(t)$ is lower bounded by $\ell_{ij}(t)$ for every $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)} \wedge \mathcal{T}_0^{(11)}$ and since $\ell_{ij}(t)$ is an increasing function satisfying the above inequality, we therefore obtain that

$$m_{ij}(t) \le 2\xi\beta\sqrt{M\lambda_i\lambda_j}m_{ij}(t)$$

so that from (6.5) it follows that

$$\int_{0}^{t} |a_{ij}(s)| ds \le \frac{1}{1 - C_0} m_{ij}(t) \le \frac{\xi}{1 - C_0} 2\beta \sqrt{M} \lambda_i \lambda_j m_{ij}(t),$$

for $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k,\ell \leq r} \mathcal{T}_{\varepsilon_N}^{(k\ell)} \wedge T_0^{(11)}$. Choosing $\xi = (1 - C_0)/2$ yields condition (4) with $\varepsilon = 1/2$. From Lemma 4.4 it follows that there exists K > 0 such that on the event $\bigcap_{1 \leq k,\ell \leq r} \mathcal{A}^{(k\ell)}$, for every $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)} \wedge \mathcal{T}_{0}^{(11)}$, it holds that

$$|L_{0,\beta}m_{ij}(t)| \le K\left(\frac{\gamma_0}{\sqrt{N}}\sum_{k=0}^{n-1} t^k + t^n + 2\sum_{1\le k,\ell\le r} \int_0^t |a_{k\ell}(s)|ds\right),\tag{6.6}$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-\gamma_0^2/(K_2 T_0^{(11)}))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. We then proceed in the same way as in the proof of Lemma 5.1. We first introduce an intermediate

threshold $\tilde{\varepsilon}_N = \tilde{\gamma} N^{-\frac{1}{2}}$ for a sufficiently large constant $\tilde{\gamma}$ of order 1. In particular, we choose γ to verify the inequality

$$\log(\gamma) \ge \frac{\kappa + 1}{\kappa - 2} \log\left(\frac{3\gamma_1}{2}\right). \tag{6.7}$$

We also let $\mathcal{T}_{\tilde{\varepsilon}_N}^{(ij)}$ denote the hitting time for the set $\{\mathbf{X}: m_{ij}(\mathbf{X}) \geq \tilde{\varepsilon}_N\}$. Since the integral and comparison inequalities (6.1) and (6.2) are still valid for $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k,\ell \leq r} \mathcal{T}_{\tilde{\varepsilon}_N}^{(k\ell)} \wedge \mathcal{T}_0^{(ij)}$, we can define $\mathcal{T}_{\ell,\tilde{\varepsilon}_N}^{(ij)}$ and $\mathcal{T}_{u,\tilde{\varepsilon}_N}^{(ij)}$ as in (6.3) and (6.4). We then see that for every $(i, j) \neq (1, 1)$,

$$T_{\ell,\tilde{\varepsilon}_N}^{(11)} = \frac{\log(\gamma) - \log(\gamma_{11}/2)}{2(1 - C_0)\beta\sqrt{M}\lambda_1^2} \le \frac{\log(\gamma)}{2(1 - C_0)\beta\sqrt{M}\lambda_i\lambda_j(1 + \kappa)} \le \frac{\log(\gamma) - \log(3\gamma_1/2)}{2(1 + C_0)\beta\sqrt{M}\lambda_i\lambda_j} \le T_{u,\tilde{\varepsilon}_N}^{(ij)},$$

where we used that $\lambda_1 = \lambda_2(1+\kappa) \geq \lambda_i(1+\kappa)$ for every $3 \leq i \leq r$ and (6.7). In order to deduce that, on the event $\bigcap_{1\leq k,\ell\leq r} \mathcal{A}^{(k\ell)}, \ \mathcal{T}^{(11)}_{\tilde{\varepsilon}_N} = \min_{1\leq k,\ell\leq r} \mathcal{T}^{(k\ell)}_{\tilde{\varepsilon}_N}$, we need to show that $\min_{1\leq k,\ell\leq r} \mathcal{T}^{(k\ell)}_{\tilde{\varepsilon}_N} \leq \min_{1\leq k,\ell\leq r} \mathcal{T}^{(k\ell)}_{L_{0,\beta}} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \mathcal{T}_{L_{0,\beta}}$. It therefore suffices to show that for every $i, j \in [r]$, each term in the sum (6.6) is upper bounded by $\frac{2C_0\beta\sqrt{M}\lambda_i\lambda_j}{n+2}m_{ij}(t)$ for every $t \leq \min_{1\leq k,\ell\leq r} \mathcal{T}^{(k\ell)}_{L_{0,\beta}} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1\leq k,\ell\leq r} \mathcal{T}^{(k\ell)}_{\tilde{\varepsilon}_N} \wedge \mathcal{T}^{(11)}_{0}$.

(i) We start by observing that on the event $\cap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$,

$$\frac{2C_0\beta\sqrt{M}\lambda_i\lambda_j}{n+2}m_{ij}(t) \ge \frac{2C_0\beta\sqrt{M}\lambda_i\lambda_j}{n+2}\ell_{ij}(t) \gtrsim \gamma_0 N^{-\frac{n}{2(n+1)}}\exp((n+2)\gamma_0 N^{\frac{1}{2(n+1)}}t),$$

for $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\tilde{\varepsilon}_N}^{(k\ell)} \wedge T_0^{(11)}$, where we used the assumption $\sqrt{M} \gtrsim \frac{(n+2)\gamma_0}{\beta \lambda_r^2 C_0(1-C_0)\gamma_2} N^{\frac{1}{2(n+1)}}$. We then see that for every $0 \leq k \leq n-1$,

$$\exp((n+2)\gamma_0 N^{\frac{1}{2(n+1)}}t) \ge \frac{(n+2)^k \gamma_0^k N^{\frac{\kappa}{2(n+1)}}}{k!} t^k \gtrsim t^k$$

for $t \leq T_0^{(11)} < \log(\varepsilon \sqrt{N})/((n+2)\gamma_0 N^{\frac{1}{2(n+1)}})$. We therefore have that for every $0 \leq k \leq n-1$, $2C_0\beta_0 \sqrt{M}\lambda_1\lambda_2$, $\gamma_0 = 1$

$$\frac{2C_0\beta\sqrt{M\lambda_i\lambda_j}}{n+2}m_{ij}(t)\gtrsim K\frac{\gamma_0}{\sqrt{N}}t^k$$

(ii) We next observe that a sufficient condition to control the second term is given by $F(t) = Kt^n \leq \frac{2C_0\beta\sqrt{M}\lambda_i\lambda_j}{n+2}\ell_{ij}(t) = G(t)$. An explicit computation shows that for every $k \leq n$,

$$F^{(k)}(t) = Kn(n-1)\cdots(n-k+1)t^{n-k}$$

and

$$G^{(k)}(t) = \frac{C_0 \beta \sqrt{M} \lambda_i \lambda_j}{n+2} \frac{\gamma_{ij}}{\sqrt{N}} \left(2(1-C_0) \beta \sqrt{M} \lambda_i \lambda_j \right)^k \exp(2(1-C_0) \beta \sqrt{M} \lambda_i \lambda_j t).$$

For $k \leq n-1$, it holds that $F^{(k)}(0) = 0 \leq G^{(k)}(0)$, and for k = n we have that

$$G^{(n)}(t) = \frac{\left(\beta\sqrt{M}\lambda_{i}\lambda_{j}\right)^{n+1}\gamma_{ij}C_{0}2^{n}(1-C_{0})^{n}}{(n+2)\sqrt{N}}\exp(2(1-C_{0})\beta\sqrt{M}\lambda_{i}\lambda_{j}t)$$
$$\gtrsim \frac{2^{n}(n+2)^{n}\gamma_{0}^{n+1}}{\gamma_{2}^{n}C_{0}^{n}(1-C_{0})}\exp((n+2)\gamma_{0}N^{\frac{1}{2(n+1)}}t)$$
$$\geq Kn! = F^{(n)}(t),$$

which gives the bound for the second term.

(iii) To bound the last term, we note from (6.1) that on the event $\bigcap_{1 \le k, \ell \le r} \mathcal{A}^{(k\ell)}$,

$$2\sum_{1\leq k,\ell\leq r}\int_0^t |a_{k\ell}(s)|ds \leq \frac{2r^2}{1-C_0}\max_{1\leq k,\ell\leq r}\{m_{k\ell}(t)\} \leq \frac{2r^2}{1-C_0}\tilde{\varepsilon}_N = \frac{2r^2}{1-C_0}\frac{\gamma}{\sqrt{N}},$$

for $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\tilde{\varepsilon}_N}^{(k\ell)} \wedge T_0^{(11)}$. Then, since by the assumption on \sqrt{M} we have that

$$\frac{C_0\beta\sqrt{M}\lambda_i\lambda_j}{n+2}\frac{\gamma_{ij}}{\sqrt{N}}\gtrsim \frac{\gamma_0}{1-C_0}N^{-\frac{n}{2(n+1)}}\geq K\frac{2r^2}{1-C_0}\frac{\gamma}{\sqrt{N}}$$

it follows that

$$\frac{2C_0\beta\sqrt{M}\lambda_i\lambda_j}{n+2}m_{ij}(t) \ge \frac{2C_0\beta\sqrt{M}\lambda_i\lambda_j}{n+2}\ell_{ij}(0) \gtrsim K\frac{2r^2}{1-C_0}\frac{\gamma}{\sqrt{N}}$$

for $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\tilde{\varepsilon}_N}^{(k\ell)} \wedge T_0^{(11)}$. This shows the last bound.

We therefore have that, on the event $\bigcap_{1 \le k, \ell \le r} \mathcal{A}^{(k\ell)}$, $\min_{1 \le k, \ell \le r} \mathcal{T}^{(k\ell)}_{\tilde{\varepsilon}_N} \le \min_{1 \le k, \ell \le r} \mathcal{T}^{(k\ell)}_{L_{0,\beta}} \wedge \mathcal{T}_{L_{0,\beta}}$ so that it follows

$$\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)} = \min_{1 \le k, \ell \le r} \mathcal{T}_{\tilde{\varepsilon}_N}^{(k\ell)},$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-\gamma_0^2/(K_2 T_0^{(11)}))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. Now, according to the generator expansion $L_\beta m_{ij}$ given by Lemma 4.6 we note that at $t = \mathcal{T}_{\varepsilon_N}^{(11)}$,

$$L_{\beta}m_{11}(\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)}) \ge 2(1-C_0)\lambda_1^2 m_{11}(\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)})$$

and for every $(i, j) \neq (1, 1)$,

$$Lm_{ij}(\mathcal{T}^{(11)}_{\tilde{\varepsilon}_N}) \leq 2C_0\beta\sqrt{M}\lambda_1^2m_{11}(\mathcal{T}^{(11)}_{\tilde{\varepsilon}_N}) + 2\beta\sqrt{M}\lambda_i\lambda_jm_{ij}(\mathcal{T}^{(11)}_{\tilde{\varepsilon}_N}).$$

In particular, since $\lambda_i \lambda_j \leq \lambda_1^2/(1+\kappa)$ we have that

$$Lm_{ij}(\mathcal{T}^{(11)}_{\tilde{\varepsilon}_N}) \le 2C_0\beta\sqrt{M}\lambda_1^2\frac{\gamma}{\sqrt{N}} + 2\beta\sqrt{M}\frac{\lambda_1^2}{1+\kappa}\frac{\gamma}{\sqrt{N}} = 2\left(C_0 + \frac{1}{1+\kappa}\right)\beta\sqrt{M}\lambda_1^2\frac{\gamma}{\sqrt{N}}.$$

By (6.7) we then see that

$$C_0 + \frac{1}{1+\kappa} \le C_0 + \frac{1-C_0}{1+C_0} \frac{\log(\gamma) - \log(3\gamma_1/2)}{\log(\gamma)} \le 1 - C_0$$

provided $C_0 < 1/2$ is sufficiently small. We therefore have that for every $(i, j) \neq (1, 1)$,

$$L_{\beta}m_{ij}(\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)}) \le L_{\beta}m_{11}(\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)})$$

As a result, on the event $\bigcap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$, we have that for every $(i, j) \neq (1, 1), m_{11}(\mathcal{T}^{(11)}_{\bar{\varepsilon}_N}) > m_{ij}(\mathcal{T}^{(11)}_{\bar{\varepsilon}_N})$ and since $L_{\beta}m_{11}(\mathcal{T}^{(11)}_{\bar{\varepsilon}_N}) \geq L_{\beta}m_{ij}(\mathcal{T}^{(11)}_{\bar{\varepsilon}_N})$, hence

$$m_{11}(t) > m_{ij}(t),$$

for every $\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)} \leq t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)} \wedge \mathcal{T}_0^{(11)}$, with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-\gamma_0^2/(K_2 T_0^{(11)}))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. In particular, we deduce that on the event $\cap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$,

$$\mathcal{T}_{\varepsilon}^{(11)} = \min_{1 \le k, \ell \le r} \mathcal{T}_{\varepsilon}^{(k\ell)},$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-\gamma_0^2/(K_2 T_0^{(11)}))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. It remains to show that $\mathcal{T}_{\varepsilon}^{(11)} \leq \mathcal{T}_{L_{0,\beta}}^{(11)}$ with high $\mathbb{Q}_{\mathbf{X}}$ - and \mathbb{P} -probability. To this end, we want to use

the estimate (6.6). Since each $a_{k\ell}(t) \leq a_{11}(t)$ for $\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)} \leq t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \mathcal{T}_{\varepsilon}^{(11)} \wedge \mathcal{T}_{0}^{(11)}$, it follows from (6.6) that on the event $\cap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$, for every $\mathcal{T}_{\tilde{\varepsilon}_N}^{(11)} \leq t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \mathcal{T}_{\varepsilon}^{(11)} \wedge \mathcal{T}_{\varepsilon}^{(11)}$,

$$|L_{0,\beta}m_{ij}(t)| \le K\left(\frac{\gamma_0}{\sqrt{N}}\sum_{k=0}^{n-1} t^k + t^n + 2r^2 \int_0^t |a_{11}(s)|ds\right),\tag{6.8}$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1-K_2 \exp\left(-\gamma_0^2/(K_2T_0^{(11)})\right)$ and with \mathbb{P} -probability at least $1-\exp(-KN)$. In the same way as before, we can show that each term in the sum (6.8) is upper bounded by $\frac{2C_0\beta\sqrt{M}\lambda_1^2}{n+2}m_{11}(t)$ for $\mathcal{T}_{\varepsilon_N}^{(11)} \leq t \leq \min_{1\leq k,\ell\leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \mathcal{T}_{L_{0,\beta}} \wedge \mathcal{T}_{\varepsilon}^{(11)} \wedge T_0^{(11)}$, ensuring that $\mathcal{T}_{\varepsilon}^{(11)} \leq \mathcal{T}_{L_{0,\beta}}$. It then follows that on the event $\mathcal{C}_0(n,\gamma_0) \cap \mathcal{C}_1(\gamma_1,\gamma_2)$ we have that

$$\mathcal{T}_{\varepsilon}^{(11)} \le T_0^{(11)} \lesssim \frac{\log(N)}{(n+2)\gamma_0 N^{\frac{1}{2(n+1)}}}$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least

$$1 - K_2 \exp\left(-\gamma_0^3(n+2)N^{\frac{1}{2(n+1)}}/(K_2\log(N))\right) - K_2 r^2 \exp\left(-\gamma_2^2 \gamma_0(n+2)N^{\frac{1}{2(n+1)}}/(4K_2\log(N))\right)$$

and \mathbb{P} -probability at least $1 - \exp(-KN)$, which completes the proof of Lemma 6.1.

6.2. Recovery of all spikes

In the following, we focus on the proof of exact recovery of all spikes when p = 2 and the SNRs are sufficiently separated. We first recall the event of strong recovery (3.2) that we wish to achieve from random initializations that meet Condition 0 at level n and Condition 1. For every $\varepsilon > 0$ and $C_0 \in (0, \frac{1}{2})$, $R(\varepsilon, C_0)$ is defined by

$$R(\varepsilon, C_0) = \left\{ \boldsymbol{X} \colon m_{ii}(\boldsymbol{X}) \ge 1 - \varepsilon \ \forall i \in [r] \text{ and } m_{k\ell}(\boldsymbol{X}) \lesssim N^{-\frac{1}{2} \left(1 - \frac{1 - C_0}{1 + C_0} \frac{\lambda_r^2}{\lambda_1^2}\right)} \ \forall k, \ell \in [r], k \neq \ell \right\}.$$

Similar to the proof of Proposition 3.6, we proceed through several steps, each focusing on establishing the weak recovery of a correlation. To this end, we first need to introduce some notations. For every $1 \le k \le r-1$ and every $k \le i, j \le r$, we denote by $\delta_{ij}^{(k)}$ and $\xi_{ij}^{(k)}$ the parameters given by

$$\delta_{ij}^{(k)} = 1 - \frac{1+C_0}{1-C_0} \frac{\lambda_i \lambda_j}{\lambda_k^2} \quad \text{and} \quad \xi_{ij}^{(k)} = 1 - \frac{1-C_0}{1+C_0} \frac{\lambda_i \lambda_j}{\lambda_k^2}$$

We see that $\delta_{ij}^{(k)} < \xi_{ij}^{(k)}$, $\delta_{ij}^{(k)} = \delta_{ji}^{(k)}$ and $\xi_{ij}^{(k)} = \xi_{ji}^{(k)}$ for every i, j, k. Moreover, we have that $0 < \xi_{kk}^{(k)} \le \xi_{ij}^{(k)}$ for every $i, j \ge k$. On the other hand, we have that $\delta_{kk}^{(k)} < 0 < \delta_{k,k+1}^{(k)} \le \delta_{ij}^{(k)}$ for every $i, j \ge k + 1$, provided $C_0 < \frac{1}{2} \land \frac{\kappa_k}{2+\kappa_k}$. Since we want to ensure that this condition holds for all k, we choose $C_0 < \frac{1}{2} \land \frac{\kappa}{2+\kappa}$ with $\kappa = \min_{1 \le k \le r-1} \kappa_k$. For every $\varepsilon > 0$ and $C_0 \in (0, \frac{1}{2} \land \frac{\kappa}{2+\kappa})$, we then consider the following sets:

$$E_{1}(\varepsilon, C_{0}) = W_{1}(\varepsilon, C_{0}) \cap \left\{ \boldsymbol{X} : N^{-\frac{\xi_{ij}^{(1)}}{2}} \lesssim m_{ij}(\boldsymbol{X}) \lesssim N^{-\frac{\delta_{ij}^{(1)}}{2}} \quad \forall 2 \le i, j \le r \right\},$$

$$E_{2}(\varepsilon, C_{0}) = S_{1}(\varepsilon, C_{0}) \cap W_{2}(\varepsilon, C_{0}) \cap \left\{ \boldsymbol{X} : N^{-\frac{\xi_{ij}^{(2)}}{2}} \lesssim m_{ij}(\boldsymbol{X}) \lesssim N^{-\frac{\delta_{ij}^{(2)}}{2}} \quad \forall 3 \le i, j \le r \right\},$$

$$\dots$$

$$E_{r-1}(\varepsilon, C_0) = \bigcap_{1 \le i \le r-2} S_i(\varepsilon, C_0) \cap W_{r-1}(\varepsilon, C_0) \cap \left\{ \mathbf{X} : N^{-\frac{1}{2}} \le m_{rr}(\mathbf{X}) \le N^{-\frac{1}{2}} \right\}$$
$$E_r(\varepsilon, C_0) = \bigcap_{1 \le i \le r-1} S_i(\varepsilon, C_0) \cap W_r(\varepsilon, C_0),$$

where $W_i(\varepsilon, C_0)$ and $S_i(\varepsilon, C_0)$ denote the set of weak and strong recovery of the spike v_i , i.e.,

$$W_{i}(\varepsilon, C_{0}) = \left\{ \boldsymbol{X} \colon m_{ii}(\boldsymbol{X}) \geq \varepsilon \text{ and } m_{ij}(\boldsymbol{X}), m_{ji}(\boldsymbol{X}) \lesssim N^{-\frac{\delta_{ij}^{(i)}}{2}} \quad \forall j > i \right\},$$
$$S_{i}(\varepsilon, C_{0}) = \left\{ \boldsymbol{X} \colon m_{ii}(\boldsymbol{X}) \geq 1 - \varepsilon \text{ and } m_{ij}(\boldsymbol{X}), m_{ji}(\boldsymbol{X}) \lesssim N^{-\frac{\xi_{rr}^{(1)}}{2}} \quad \forall j > i \right\}.$$

With these notations, achieving $R(\varepsilon, C_0)$ from $E_r(\varepsilon, C_0)$ means showing that weak recovery of m_{rr} implies strong recovery.

The first step consists in showing that the estimator m_{11} becomes macroscopic, provided a sample complexity of order N^{δ} for some $\delta \in (0, 1)$ which depends on the ratio between the signal sizes.

Lemma 6.3. Let $\beta \in (0, \infty)$, p = 2, and $\lambda_i = \lambda_{i+1}(1+\kappa_i)$ for every $1 \le i \le r-1$ and $\kappa_i > 0$. Consider a sequence of initializations $\mu \in \mathcal{P}(\mathcal{M}_{N,r})$. For every $\varepsilon > 0$ and $C_0 \in \left(0, \frac{1}{2} \land \frac{\kappa_1}{2+\kappa_1}\right)$, we consider $E_1(\varepsilon, C_0)$ and we let \mathcal{T}_{E_1} denote the first hitting time of this set. Then, the following holds: For every $n \ge 1, \gamma_0 > 0$ and $\gamma_1 > \gamma_2 > 0$ there exist $\varepsilon_0 > 0$ and $c_0 \in \left(0, \frac{1}{2} \land \frac{\kappa_1}{2+\kappa_1}\right)$ such that for every $\varepsilon < \varepsilon_0$, $C_0 < c_0$, if $\sqrt{M} \gtrsim 1$

 $\frac{(n+2)\gamma_0\gamma_1}{\beta C_0\lambda_r^2\gamma_2}N^{\frac{1}{2}\left(\frac{1+C_0}{1-C_0}-\frac{\lambda_r^2}{\lambda_1^2}\right)} and if \log(N) \ge 2\log(3\gamma_1/2) + 2\log(3\gamma_1/\gamma_2)\frac{1+C_0}{(1-C_0)(1+\kappa_1)-(1+C_0)} - 2\log(\varepsilon), we have that$

$$\begin{split} &\int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{E_{1}} \gtrsim \frac{\log(N)N^{-\frac{1}{2} \left(\frac{1+C_{0}}{1-C_{0}} - \frac{\lambda_{r}^{2}}{\lambda_{1}^{2}} \right)}}{(n+2)\gamma_{0}\gamma_{1}} \right) \mathbf{1} \left\{ \mathcal{C}_{0}(n,\gamma_{0}) \cap \mathcal{C}_{1}(\gamma_{1},\gamma_{2}) \right\} d\mu_{0}(\boldsymbol{X}) \\ &\lesssim K_{1} e^{-\gamma_{0}^{3}\gamma_{1}(n+2)N^{\frac{1}{2} \left(\frac{1+C_{0}}{1-C_{0}} - \frac{\lambda_{r}^{2}}{\lambda_{1}^{2}} \right)} / (K_{1}\log(N))} + r^{2}K_{2} e^{-\gamma_{2}^{2}\gamma_{0}\gamma_{1}(n+2)N^{\frac{1}{2} \left(\frac{1+C_{0}}{1-C_{0}} - \frac{\lambda_{r}^{2}}{\lambda_{1}^{2}} \right)} / (K_{2}\log(N))}, \end{split}$$

with \mathbb{P} -probability $1 - \exp(-KN)$.

Having the first step at hand, we next show that x_1 achieves strong recovery with v_1 and that x_2 achieves weak recovery with v_2 , i.e., we show that from the event E_1 we obtain the event E_2 . In general, having the event E_k with $1 \le k \le r-1$ at hand, we can reach the event E_{k+1} , as stated by the following lemma.

Lemma 6.4. Let $\beta \in (0,\infty)$, p = 2, and $\lambda_i = \lambda_{i+1}(1+\kappa_i)$ for every $1 \le i \le r-1$ and $\kappa_i > 0$. Let $\kappa = \min_{1\le i\le k+1}\kappa_i$. Then, the following holds: For every $n \ge 1$ and $\gamma_1 > \gamma_2 > 0$ there exist $\varepsilon_0 > 0$, $c_0 \in (0, \frac{1}{2} \land \frac{\kappa}{2+\kappa})$, and $\Lambda = \Lambda(p, n, \beta, \{\lambda_i\}_{i=1}^r)$ such that for every $\varepsilon < \varepsilon_0$, $C_0 < c_0$, if $\sqrt{M} \ge \frac{\Lambda}{\beta C_0 \lambda_r^2 \gamma_2} N^{\frac{1}{2} \left(\frac{1+C_0}{1-C_0} - \frac{\lambda_r^2}{\lambda_1^2}\right)}$, and if $\log(N) \ge \frac{1}{(1+\kappa_k)^2} \log(\frac{1}{\kappa_k})$, there exists $T_k > T_{k-1}$ (with $T_0 = \mathcal{T}_{E_1}$) such that for every $T > T_k$,

$$\inf_{\boldsymbol{X}\in E_k(\varepsilon,C_0)} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t\in[T_k,T]} \boldsymbol{X}_t^{\beta} \in E_{k+1}(\varepsilon,C_0) \right) \ge 1 - K_1 e^{-N\varepsilon^2/(K_1T)} - K_2 e^{-N^{\frac{1}{2}(1+\delta)}/(K_2T)}$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$.

The last phase is to show that strong recovery of the last spike follows straightforwardly from $E_r(\varepsilon, C_0)$.

Lemma 6.5. Let $\beta \in (0, \infty)$, p = 2, and $\lambda_i = \lambda_{i+1}(1 + \kappa_i)$ for every $1 \le i \le r-1$ and $\kappa_i > 0$. Then, for every $\varepsilon > 0$ and $\sqrt{M} \gtrsim \frac{\Lambda}{\beta C_0 \lambda_r^2 \gamma_2} N^{\frac{1}{2} \left(\frac{1+C_0}{1-C_0} - \frac{\lambda_r^2}{\lambda_1^2}\right)}$, there exists an order time T_r such that for every $T \ge T_r$,

$$\inf_{\mathbf{X}\in E_r(\varepsilon,C_0)} \mathbb{Q}_{\mathbf{X}}\left(\inf_{t\in[T_r,T]} \mathbf{X}_t^\beta \in R(\varepsilon,C_0)\right) \ge 1 - K_1 \exp(-N\varepsilon^2/(K_1T)),\tag{6.9}$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$.

Having Lemmas 6.3, 6.4, and 6.5 at hand, we now provide the proof of Proposition 3.8 using the strong Markov property.

Proof of Proposition 3.8. By the strong Markov property, we have that

$$\begin{split} &\int_{\mathcal{M}_{N\times r}} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t\in[T_0,T]} \boldsymbol{X}_t^{\beta} \in R(\varepsilon,C_0) \right) d\mu_0(\boldsymbol{X}) \\ &\geq \inf_{\boldsymbol{X}\in E_1(\varepsilon,C_0)} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t\in[T_0,T]} \boldsymbol{X}_t^{\beta} \in R(\varepsilon,C_0) \right) \times \int_{\mathcal{M}_{N\times r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{E_1(\varepsilon,C_0)} \lesssim \frac{\log(N)N^{-\frac{1}{2}\left(\frac{1+C_0}{1-C_0}-\frac{\lambda_r^2}{\lambda_1^2}\right)}{(n+2)\gamma_0\gamma_1} \right) d\mu_0(\boldsymbol{X}) \end{split}$$

Applying the strong Markov property recursively to the first term, we have that

$$\inf_{\mathbf{X}\in E_{1}(\varepsilon,C_{0})} \mathbb{Q}_{\mathbf{X}} \left(\inf_{t\in[T_{0},T]} \mathbf{X}_{t}^{\beta} \in R(\varepsilon,C_{0}) \right) \\
\geq \inf_{\mathbf{X}\in E_{1}(\varepsilon,C_{0})} \mathbb{Q}_{\mathbf{X}} \left(\inf_{t\in[T_{0},T]} \mathbf{X}_{t}^{\beta} \in E_{2}(\varepsilon,C_{0}) \right) \times \inf_{\mathbf{X}\in E_{2}(\varepsilon,C_{0})} \mathbb{Q}_{\mathbf{X}} \left(\inf_{t\in[T_{0},T]} \mathbf{X}_{t}^{\beta} \in R(\varepsilon,C_{0}) \right) \\
\geq \cdots \\
\geq \prod_{k=1}^{r-1} \inf_{\mathbf{X}\in E_{k}(\varepsilon,C_{0})} \mathbb{Q}_{\mathbf{X}} \left(\inf_{t\in[T_{0},T]} \mathbf{X}_{t}^{\beta} \in E_{k+1}(\varepsilon,C_{0}) \right) \times \inf_{\mathbf{X}\in E_{r}(\varepsilon,C_{0})} \mathbb{Q}_{\mathbf{X}} \left(\inf_{t\in[T_{0},T]} \mathbf{X}_{t}^{\beta} \in R(\varepsilon,C_{0}) \right)$$

so that we can use Lemmas 6.4 and 6.5 to bound the above factors. It remains to estimate the integral term. We write

$$\begin{split} &\int_{\mathcal{M}_{N\times r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{E_{1}(\varepsilon,C_{0})} \lesssim \frac{\log(N)N^{-\frac{1}{2}\left(\frac{1+C_{0}}{1-C_{0}}-\frac{\lambda_{r}^{2}}{\lambda_{1}^{2}}\right)}{(n+2)\gamma_{0}\gamma_{1}} \right) d\mu_{0}(\boldsymbol{X}) \\ &\leq \mu_{0}(\mathcal{C}_{0}(n,\gamma_{0})^{c}) + \mu_{0}(\mathcal{C}_{1}(\gamma_{1},\gamma_{2})^{c}) \\ &\quad + \int_{\mathcal{M}_{N\times r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{E_{1}(\varepsilon,C_{0})} \lesssim \frac{\log(N)N^{-\frac{1}{2}\left(\frac{1+C_{0}}{1-C_{0}}-\frac{\lambda_{r}^{2}}{\lambda_{1}^{2}}\right)}{(n+2)\gamma_{0}\gamma_{1}} \right) \mathbf{1}\{\mathcal{C}_{0}(n,\gamma_{0}) \cap \mathcal{C}_{1}(\gamma_{1},\gamma_{2})\} d\mu_{0}(\boldsymbol{X}), \end{split}$$

so that we bound $\mu_0(\mathcal{C}_0(n,\gamma_0)^c)$ and $\mu_0(\mathcal{C}_1(\gamma_1,\gamma_2)^c)$ according to Definitions 3.1 and 3.3, and the integral according to Lemma 6.3. Proposition 3.8 then follows straightforwardly.

It therefore remains to prove all the auxiliary results stated at the beginning of this subsubsection.

Proof of Lemma 6.3. The proof follows the same approach used for the proof of Lemma 6.1 and consists in studying the evolution of the correlations until $\max_{1 \le k, \ell \le r} m_{k\ell}$ reaches the macroscopic threshold $\varepsilon > 0$. We let $\mathcal{A} = \mathcal{A}(n, \gamma_0, \gamma_1, \gamma_2)$ denote the initial event

$$\mathcal{A}(n,\gamma_0,\gamma_1,\gamma_2) = \{ \boldsymbol{X}_0 \sim \mu \colon \boldsymbol{X}_0 \in \mathcal{C}_0(n,\gamma_0) \cap \mathcal{C}_1(\gamma_1,\gamma_2) \}$$

We note that on $C_1(\gamma_1, \gamma_2)$, for every $i, j \in [r]$ there exists $\gamma_{ij} \in (\gamma_2, \gamma_1)$ such that $m_{ij}(0) = \gamma_{ij} N^{-\frac{1}{2}}$. For some $T_0^{(ij)} > 0$ to be chosen later, we then define the event $\mathcal{A}^{(ij)} = \mathcal{A}^{(ij)}(n, \gamma_0, \gamma_1, \gamma_2, T_0^{(ij)})$ by

$$\mathcal{A}^{(ij)}(n,\gamma_0,\gamma_1,\gamma_2,T_0^{(ij)}) = \mathcal{A}(n,\gamma_0,\gamma_1,\gamma_2) \cap \left\{ \sup_{t \in [0,T_0^{(ij)}]} |M_t^{m_{ij}}| \le \frac{\gamma_2}{2\sqrt{N}} \right\},$$

where we recall that according to Lemma 4.7 and (4.9), there exists a constant $K_2 > 0$ such that

$$\sup_{\boldsymbol{X}} \mathbb{Q}_{\boldsymbol{X}} \left(\sup_{t \in [0, T_0^{(ij)}]} |M_t^{m_{ij}}| \ge \frac{\gamma_2}{2\sqrt{N}} \right) \le K_2 \exp\left(-\frac{\gamma_2^2}{4K_2 T_0^{(ij)}}\right).$$

Moreover, for every $i, j \in [r]$ we let $\mathcal{T}_{L_{0,\beta}}^{(ij)}$ denote the hitting time of the set

$$\left\{\boldsymbol{X}: |L_{0,\beta}m_{ij}(\boldsymbol{X})| > 2C_0\beta\sqrt{M}\lambda_i\lambda_jm_{ij}(\boldsymbol{X})\right\},\,$$

for some order 1 constant $C_0 \in (0, 1/2)$. We note that on $\mathcal{C}_0(n, \gamma_0)$,

$$|L_{0,\beta}m_{ij}(\boldsymbol{X}(0))| \leq \frac{\gamma_0}{\sqrt{N}} \leq 2C_0\beta\sqrt{M}\lambda_i\lambda_j\frac{\gamma_2}{\sqrt{N}},$$

provided $\sqrt{M} \geq \frac{\gamma_0}{2C_0\beta\lambda_i\lambda_j\gamma_2}$, which certainly holds by assumption. Therefore, on the initial event \mathcal{A} , we have that $|L_{0,\beta}m_{ij}(0)| \leq 2C_0\beta\sqrt{M}\lambda_i\lambda_jm_{ij}(0)$, thus by continuity of the process \mathbf{X}_t it follows that $\mathcal{T}_{L_{0,\beta}}^{(ij)} > 0$ for every $i, j \in [r]$.

In the following, we fix $i, j \in [r]$ and place ourselves on the event $\mathcal{A}^{(ij)}$. For every $\varepsilon > 0$, we denote by $\mathcal{T}_{\varepsilon}^{(ij)}$ the hitting time for the set $\{\mathbf{X} : m_{ij}(\mathbf{X}) \geq \varepsilon\}$. Recalling the generator expansion given by Lemma 4.6, i.e.,

$$L_{\beta}m_{ij}(t) = L_{0,\beta}m_{ij}(t) + 2\beta\sqrt{M}\lambda_i\lambda_jm_{ij}(t) - \beta\sqrt{M}\sum_{1\le k,\ell\le r}\lambda_k(\lambda_j+\lambda_\ell)m_{i\ell}(t)m_{kj}(t)m_{k\ell}(t),$$

we have that

$$2(1-C_0)\beta\sqrt{M}\lambda_i\lambda_jm_{ij}(t) \le L_\beta m_{ij}(t) \le +2(1+C_0)\beta\sqrt{M}\lambda_i\lambda_jm_{ij}(t)$$

for every $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \min_{1 \leq k,\ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)}$, provided ε is sufficiently small. We then obtain the integral inequality

$$\frac{\gamma_{ij}}{2\sqrt{N}} + 2(1-C_0)\beta\sqrt{M}\lambda_i\lambda_j \int_0^t m_{ij}(s)ds \le m_{ij}(t) \le \frac{3\gamma_{ij}}{2\sqrt{N}} + 2(1+C_0)\beta\sqrt{M}\lambda_i\lambda_j \int_0^t m_{ij}(s)ds, \quad (6.10)$$

for all $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)} \wedge T_{0}^{(ij)}$. Furthermore, by the Grönwall's inequality (see item (d) of Lemma 4.11) we have that

$$\frac{\gamma_{ij}}{2\sqrt{N}}\exp\left(2(1-C_0)\beta\sqrt{M}\lambda_i\lambda_jt\right) = \ell_{ij}(t) \le m_{ij}(t) \le u_{ij}(t) = \frac{3\gamma_{ij}}{2\sqrt{N}}\exp\left(2(1+C_0)\beta\sqrt{M}\lambda_i\lambda_jt\right),\tag{6.11}$$

for every $t \leq \mathcal{T}_{L_{0,\beta}}^{(ij)} \wedge \min_{1 \leq k,\ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)} \wedge T_{0}^{(ij)}$. We then define $T_{\ell,\varepsilon}^{(ij)}$ by $\ell_{ij}(T_{\ell,\varepsilon}^{(ij)}) = \varepsilon$, i.e.,

$$T_{\ell,\varepsilon}^{(ij)} = \frac{\log(\varepsilon\sqrt{N}) - \log(\gamma_{ij}/2)}{2(1 - C_0)\beta\sqrt{M}\lambda_i\lambda_j}.$$
(6.12)

Similarly, we let $T_{u,\varepsilon}^{(ij)}$ satisfy $u_{ij}(T_{u,\varepsilon}^{(ij)}) = \varepsilon$, so that

$$T_{u,\varepsilon}^{(ij)} = \frac{\log(\varepsilon\sqrt{N}) - \log(3\gamma_{ij}/2)}{2(1+C_0)\beta\sqrt{M}\lambda_i\lambda_j}.$$
(6.13)

We note that on the event $\mathcal{A}^{(ij)}, T^{(ij)}_{u,\varepsilon} \leq \mathcal{T}^{(ij)}_{\varepsilon} \leq T^{(ij)}_{\ell,\varepsilon}$. We then choose $T^{(ij)}_0 = T^{(ij)}_{\ell,\varepsilon} > 0$. Moreover, we observe that

$$T_{\ell,\varepsilon}^{(11)} \le \frac{\log(\varepsilon\sqrt{N}) - \log(\gamma_2/2)}{2(1-C_0)\beta\sqrt{M}\lambda_i\lambda_j(1+\kappa_1)} \le \frac{\log(\varepsilon\sqrt{N}) - \log(3\gamma_1/2)}{2(1+C_0)\beta\sqrt{M}\lambda_i\lambda_j} \le T_{u,\varepsilon}^{(ij)},$$

where we used the fact that by assumption $\log(\varepsilon\sqrt{N}) \geq \log(3\gamma_1/2) + \log(3\gamma_1/\gamma_2) \frac{1+C_0}{(1-C_0)(1+\kappa_1)-(1+C_0)}$. Since $T_{\ell,\varepsilon}^{(11)} \leq T_{u,\varepsilon}^{(ij)}$ for every $(i,j) \neq (1,1)$, we also deduce that $T_0^{(11)} = \min_{1 \leq k, \ell \leq r} T_0^{(k\ell)}$. In order to deduce that $\mathcal{T}_{\varepsilon}^{(11)} = \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)}$, we have to show that on the event $\bigcap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$ it holds that $\min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)} \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)}$ with high $\mathbb{Q}_{\mathbf{X}^-}$ and \mathbb{P} -probability. To this end, we wish to apply Lemma 4.4 to the function $F_{ij}(\mathbf{X}) = L_{0,\beta}m_{ij}(\mathbf{X})$. We refer the reader to the proof of Lemma 6.1 for a proof of conditions (1)-(4). We therefore have that there exists a constant K > 0 such that on the event $\bigcap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$,

$$|L_{0,\beta}m_{ij}(t)| \le K\left(\frac{\gamma_0}{\sqrt{N}}\sum_{k=0}^{n-1}t^k + t^n + 2\sum_{1\le k,\ell\le r}\int_0^t |a_{k\ell}(s)|ds\right),\tag{6.14}$$

for $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)} \wedge \mathcal{T}_{0}^{(11)}$, with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-\gamma_0^2/(K_2 T_0^{(11)}))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. Our goal is to show that, on the event $\cap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$, $\min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)} \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)}$ so that we have that $\mathcal{T}_{\varepsilon}^{(11)} = \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)}$. It therefore suffices to show that for every $i, j \in [r]$, each term in the sum (6.14) is upper bounded by $\frac{2C_0 \beta \sqrt{M} \lambda_i \lambda_j}{n+2} m_{ij}(t)$ for every $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)} \wedge \mathcal{T}_{0}^{(11)}$. (i) We start by observing that on the event $\bigcap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$,

$$\frac{2C_0\beta\sqrt{M}\lambda_i\lambda_j}{n+2}m_{ij}(t) \ge \frac{2C_0\beta\sqrt{M}\lambda_i\lambda_j}{n+2}\ell_{ij}(t) = \frac{C_0\beta\sqrt{M}\lambda_i\lambda_j}{n+2}\frac{\gamma_{ij}}{\sqrt{N}}\exp(2(1-C_0)\beta\sqrt{M}\lambda_i\lambda_jt),$$

for $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)} \wedge T_{0}^{(11)}$. We then see that for every $0 \leq k \leq n-1$,

$$\exp(2(1-C_0)\beta\sqrt{M\lambda_i\lambda_j}t) \ge \frac{(2(1-C_0)\beta\sqrt{M\lambda_i\lambda_j})^k}{k!}t^k$$

Therefore, a sufficient condition for the first estimate to hold is given by

$$\frac{K\gamma_0}{\sqrt{N}}t^k \le \frac{C_0\beta\sqrt{M\lambda_i\lambda_j}}{n+2}\frac{\gamma_{ij}}{\sqrt{N}}\frac{(2(1-C_0)\beta\sqrt{M\lambda_i\lambda_j})^k}{k!}t^k.$$

which implies that

$$\sqrt{M} \ge \frac{1}{\beta \lambda_i \lambda_j} \left(\frac{K \gamma_0 (n+2)k!}{2^k C_0 (1-C_0)^k \gamma_{ij}} \right)^{\frac{1}{k+1}}$$

for every $0 \le k \le n-1$, which certainly holds by assumption.

(ii) We next observe that a sufficient condition to control the second term is given by $F(t) = Kt^n \leq \frac{2C_0\beta\sqrt{M}\lambda_i\lambda_j}{n+2}\ell_{ij}(t) = G(t)$. An explicit computation shows that for every $k \leq n$,

$$F^{(k)}(t) = Kn(n-1)\cdots(n-k+1)t^{n-k}$$

and

$$G^{(k)}(t) = \frac{C_0 \beta \sqrt{M} \lambda_i \lambda_j}{n+2} \frac{\gamma_{ij}}{\sqrt{N}} \left(2(1-C_0) \beta \sqrt{M} \lambda_i \lambda_j \right)^k \exp(2(1-C_0) \beta \sqrt{M} \lambda_i \lambda_j t).$$

For $k \leq n-1$, it holds that $F^{(k)}(0) = 0 \leq G^{(k)}(0)$, and for k = n we have that

$$G^{(n)}(t) = \frac{\left(\beta\sqrt{M}\lambda_{i}\lambda_{j}\right)^{n+1}\gamma_{ij}C_{0}2^{n}(1-C_{0})^{n}}{(n+2)\sqrt{N}}\exp(2(1-C_{0})\beta\sqrt{M}\lambda_{i}\lambda_{j}t)$$

$$\gtrsim \frac{2^{n}(n+2)^{n}\gamma_{0}^{n+1}}{\gamma_{2}^{n}C_{0}^{n}(1-C_{0})}\exp((n+2)\gamma_{0}N^{\frac{1}{2(n+1)}}t)$$

$$\geq Kn! = F^{(n)}(t),$$

which gives the bound for the second term.

(iii) To bound the last term, we note from (6.10) that on the event $\bigcap_{1 \le k, \ell \le r} \mathcal{A}^{(k\ell)}$,

$$2\sum_{1\leq k,\ell\leq r} \int_0^t |a_{k\ell}(s)| ds \leq \frac{2r^2}{1-C_0} \max_{1\leq k,\ell\leq r} m_{k\ell}(t) \leq \frac{2r^2}{1-C_0} \max_{1\leq k,\ell\leq r} u_{k\ell}(t),$$

for $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)} \wedge \mathcal{T}_{0}^{(11)}$. Therefore a sufficient condition for the last estimate is given by

$$\frac{2r^2}{1-C_0}\max_{1\le k,\ell\le r}u_{k\ell}(t)\le \frac{2C_0\beta\sqrt{M}\lambda_i\lambda_j}{n+2}\ell_{ij}(t),$$

which implies that

$$\sqrt{M} \ge \frac{3r^2(n+2)}{C_0(1-C_0)\beta\lambda_i\lambda_j\gamma_{ij}} \max_{1\le k,\ell\le r} \{\gamma_{k\ell}\exp(2\beta\sqrt{M}\left((1+C_0)\lambda_k\lambda_\ell - (1-C_0)\lambda_i\lambda_j\right)t)\},\$$

for $t \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)} \wedge \min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)} \wedge T_{0}^{(11)}$. Since the right-hand side is increasing, the maximum value of this function is attained at $T_{0}^{(11)}$. We therefore obtain that

,

$$\sqrt{M} \ge \frac{3r^2(n+2)\gamma_1}{C_0(1-C_0)\beta\lambda_i\lambda_j\gamma_{ij}} \left(\frac{2\varepsilon\sqrt{N}}{\gamma_{11}}\right)^{\frac{1+\varepsilon_0}{1-C_0}-\frac{\gamma_i\gamma_j}{\lambda_1^2}}$$

which holds by assumption.

We therefore have that, on the event $\bigcap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$, $\min_{1 \leq k, \ell \leq r} \mathcal{T}_{\varepsilon}^{(k\ell)} \leq \min_{1 \leq k, \ell \leq r} \mathcal{T}_{L_{0,\beta}}^{(k\ell)}$, so that

$$\min_{1 \le k, \ell \le r} \mathcal{T}_{\varepsilon}^{(k\ell)} = \mathcal{T}_{\varepsilon}^{(11)} \lesssim \frac{\log(N)N^{-\frac{1}{2}\left(\frac{1+C_0}{1-C_0} - \frac{\lambda_r^2}{\lambda_1^2}\right)}}{(n+2)\gamma_0\gamma_1}$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1-K_2 \exp(-\gamma_0^3 \gamma_1(n+2)N^{\frac{1}{2}\left(\frac{1+C_0}{1-C_0}-\frac{\lambda_r^2}{\lambda_1^2}\right)}/(K_2 \log(N)))$ and with \mathbb{P} -probability at least $1-\exp(-KN)$.

Furthermore, on the event $\cap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$, we have from (6.11) that $m_{ij}(\mathcal{T}_{\varepsilon}^{(11)}) \leq u_{ij}(\mathcal{T}_{\varepsilon}^{(11)})$. Since the function u_{ij} is monotone increasing, we can upper bound $m_{ij}(\mathcal{T}_{\varepsilon}^{(11)})$ by $u_{ij}(\mathcal{T}_{\ell,\varepsilon}^{(11)})$. It therefore follows from (6.11) and (6.12) that

$$u_{ij}(T_{\ell,\varepsilon}^{(11)}) = \frac{3\gamma_{ij}}{2\sqrt{N}} \exp\left(2(1+C_0)\beta\sqrt{M\lambda_i\lambda_j}T_{\ell,\varepsilon}^{(11)}\right) = C_1^{(ij)}N^{-\frac{1}{2}\left(1-\frac{1+C_0}{1-C_0}\frac{\lambda_i\lambda_j}{\lambda_1^2}\right)},$$

where $C_1^{(ij)} = \frac{3}{2} \gamma_{ij} \left(\frac{2\varepsilon}{\gamma_{11}}\right)^{\frac{1+C_0}{1-C_0} \frac{\lambda_i \lambda_j}{\lambda_1^2}}$. Since $C_0 < \frac{1}{2} \wedge \frac{\kappa_1}{2+\kappa_1}$, we have that

$$\frac{1+C_0}{1-C_0}\frac{\lambda_i\lambda_j}{\lambda_1^2} = \frac{1+C_0}{1-C_0}\frac{1}{\prod_{k=1}^{i-1}(1+\kappa_k)\prod_{\ell=1}^{j-1}(1+\kappa_\ell)} \le \frac{1+C_0}{1-C_0}\frac{1}{1+\kappa_1} \le 1.$$

Similarly, on the event $\bigcap_{1 \leq k, \ell \leq r} \mathcal{A}^{(k\ell)}$, we have that $m_{ij}(\mathcal{T}_{\varepsilon}^{(11)}) \geq \ell_{ij}(\mathcal{T}_{\varepsilon}^{(11)})$. Since ℓ_{ij} is a monotone increasing function, we can bound $m_{ij}(\mathcal{T}_{\varepsilon}^{(11)})$ below by $\ell_{ij}(\mathcal{T}_{u,\varepsilon}^{(11)})$. It therefore follows from (6.11) and (6.13) that

$$\ell_{ij}(T_{u,\varepsilon}^{(11)}) = \frac{\gamma_{ij}}{2\sqrt{N}} \exp\left(2(1-C_0)\beta\sqrt{M}\lambda_i\lambda_j T_{u,\varepsilon}^{(11)}\right) = C_2^{(ij)} N^{-\frac{1}{2}\left(1-\frac{1-C_0}{1+C_0}\frac{\lambda_i\lambda_j}{\lambda_1^2}\right)},$$

where $C_2^{(ij)} = \frac{\gamma_{ij}}{2} \left(\frac{2\varepsilon}{3\gamma_{11}}\right)^{\frac{1-C_0}{1+C_0}\frac{\lambda_i\lambda_j}{\lambda_1^2}}.$ This completes the proof of Lemma 6.3.

Proof of Lemma 6.4. We show Lemma 6.4 for k = 1. We now assume that $X_0 \in E_1(\varepsilon, C_0)$. First, we show that there is a sufficiently large threshold $\varepsilon' > \varepsilon$ such that when m_{11} reaches ε' , then the correlations m_{1k} and m_{k1} for $2 \le k \le r$ begin to decrease. Moreover, we show that this occurs before m_{1k} and m_{k1} are too large to result in a decrease in m_{ij} for every $2 \le i, j \le r$. Subsequently, we show that m_{1k} and m_{k1} decrease below a certain threshold, allowing m_{22} to become macroscopic.

Step 1: Evolution of $m_{11}(t)$ for $t \leq \mathcal{T}_{\varepsilon'}^{(11)}$ for every $\varepsilon' > \varepsilon$. Let $\varepsilon' \in (\varepsilon, 1)$. For every $i, j \neq 1$ and $\delta > 0$, we consider the stopping times $\mathcal{T}_{ij}^{(1)} \mathcal{T}_{ij}^{(1)} \mathcal{T}_{ij$

$$\begin{split} L_{\beta}m_{11}(t) &\geq -\|L_{0,\beta}m_{11}\|_{\infty} + 2\beta\sqrt{M}\lambda_{1}^{2}m_{11}(t)\left(1 - m_{11}^{2}(t) - (r^{2} - 1)\max_{2 \leq i,j \leq r}\{m_{i1}(t)m_{1j}(t)\}\right) \\ &\geq -\|L_{0,\beta}m_{11}\|_{\infty} + 2\beta\sqrt{M}\lambda_{1}^{2}m_{11}(t)\left(1 - m_{11}^{2}(t) - (r^{2} - 1)N^{-(\delta_{12} - \delta)}\right) \\ &\geq -\|L_{0,\beta}m_{11}\|_{\infty} + \beta\sqrt{M}\lambda_{1}^{2}m_{11}(t)\left(1 - m_{11}^{2}(t)\right), \end{split}$$

for every $t \leq \mathcal{T}_{\varepsilon'}^{(11)} \wedge \mathcal{T}_{\varepsilon/2}^{(11)} \wedge \min_{2 \leq i,j \leq r} \left\{ \mathcal{T}_{N-\frac{\delta_{1j}^{(1)} - \delta}{2}}^{(1j)} \wedge \mathcal{T}_{N-\frac{\delta_{1j}^{(1)} - \delta}{2}}^{(i1)} \wedge \mathcal{T}_{N-\frac{\delta_{1j}^{(1)} - \delta}{2}}^{(ij)} \right\}$, where the last inequality holds provided $N > \left(\frac{2(r^2 - 1)}{1 - (\varepsilon')^2}\right)^{\frac{1}{\delta_{12}^{(1)} - \delta}}$. Therefore, we have that

$$L_{\beta}m_{11}(t) \ge -\Lambda + \beta\sqrt{M}\lambda_1^2m_{11}(t)(1-m_{11}^2(t)) \ge \frac{1}{2}\beta\sqrt{M}\lambda_1^2m_{11}(t)(1-m_{11}^2(t)) > 0,$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$, provided $\sqrt{M} \geq \frac{2\Lambda}{\beta \lambda_1^2 m_{11}(t)(1-m_{11}^2(t))}$, which certainly holds by assumption. Recalling that

$$m_{11}(t) = m_{11}(0) + M_t^{m_{11}} + \int_0^t L_\beta m_{11}(s) ds,$$

we obtain the integral inequality given by

$$m_{11}(t) \ge \frac{\varepsilon}{2} + \frac{1}{2}\beta\sqrt{M}\lambda_1^2 \int_0^t m_{11}(s)(1 - m_{11}^2(s))ds,$$
(6.15)

for every $t \leq \mathcal{T}_{\varepsilon'}^{(11)} \wedge \mathcal{T}_{\varepsilon/2}^{(11)} \wedge \min_{2 \leq i,j \leq r} \left\{ \mathcal{T}_{\varepsilon}^{(1j)} \wedge \mathcal{T}_{\varepsilon}^{(11)} \wedge \mathcal{T}_{\varepsilon/2}^{(11)} \wedge \mathcal{T}_{\varepsilon/2}^{(11)} \wedge \mathcal{T}_{\varepsilon/2}^{(11)} \wedge \mathcal{T}_{\varepsilon/2}^{(11)} \wedge \mathcal{T}_{\varepsilon/2}^{(11)} \right\}, \text{ with } \mathbb{Q}_{\mathbf{X}} \text{-probability at}$

least $1 - K_1 \exp\left(-\varepsilon^2 N/(4K_1(\mathcal{T}_{\varepsilon'}^{(11)} \wedge \mathcal{T}_{\varepsilon/2}^{(11)}))\right)$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. Since the function $x \mapsto x(1-x^2)$ is locally Lipschitz on (ε, ∞) for every $\varepsilon > 0$, we have that the equation

$$\begin{cases} \dot{f} = af(1 - f^2) \\ f(0) = b \end{cases}$$

admits as unique solution $f(t) = e^{at} \left(e^{2at} + \frac{1}{b^2} - 1 \right)^{-\frac{1}{2}}$. It then follows from (6.15) that

$$m_{11}(t) \ge \frac{\exp(\beta\sqrt{M\lambda_1^2}t/2)}{\sqrt{\exp(\beta\sqrt{M\lambda_1^2}t) + \frac{4}{\varepsilon^2} - 1}},$$

for every $t \leq \mathcal{T}_{\varepsilon'}^{(11)} \wedge \mathcal{T}_{\varepsilon/2}^{(11)} \wedge \min_{2 \leq i,j \leq r} \left\{ \mathcal{T}_{\frac{\delta_{1j}^{(1)} - \delta}{2}}^{(1j)} \wedge \mathcal{T}_{\frac{\delta_{1j}^{(1)} - \delta}{2}}^{(i1)} \wedge \mathcal{T}_{\frac{\delta_{1j}^{(1)} - \delta}{2}}^{(i1)} \wedge \mathcal{T}_{\frac{\delta_{1j}^{(1)} - \delta}{2}}^{(i1)} \right\}$ and we deduce that $\mathcal{T}_{\varepsilon'}^{(11)} \leq \mathcal{T}_{\varepsilon/2}^{(11)}$. Moreover, we have the following estimate for $\mathcal{T}_{\varepsilon'}^{(11)}$, namely

$$\mathcal{T}_{\varepsilon'}^{(11)} \le \frac{1}{\beta\sqrt{M}\lambda_1^2} \log\left(\frac{(\varepsilon')^2}{1-(\varepsilon')^2} \frac{4-\varepsilon^2}{\varepsilon^2}\right),\tag{6.16}$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_1 \exp\left(-\varepsilon^2 N/(4K_1\mathcal{T}_{\varepsilon'}^{(11)})\right)$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$.

Step 2: Evolution of $m_{i1}(t)$, $m_{1j}(t)$ and $m_{ij}(t)$ for every $t \leq \mathcal{T}_{\varepsilon'}^{(11)}$. We first consider the correlations m_{i1} and m_{1j} for $i, j \neq 1$ and observe that

$$L_{\beta}m_{i1}(t) \leq \|L_{0,\beta}m_{i1}\|_{\infty} + 2\beta\sqrt{M}\lambda_{1}\lambda_{i}m_{i1}(t) - \beta\sqrt{M}\sum_{1\leq k,\ell\leq r}\lambda_{k}(\lambda_{1}+\lambda_{\ell})m_{k1}(t)m_{i\ell}(t)m_{k\ell}(t)$$
$$\leq \Lambda + 2\beta\sqrt{M}\lambda_{1}\left(\lambda_{i}-\lambda_{1}m_{11}^{2}(t)\right)m_{i1}(t),$$

and similarly,

$$L_{\beta}m_{1j}(t) \leq \|L_{0,\beta}m_{1j}\|_{\infty} + 2\beta\sqrt{M}\lambda_{1}\lambda_{j}m_{1j}(t) - \beta\sqrt{M}\sum_{1\leq k,\ell\leq r}\lambda_{k}(\lambda_{j}+\lambda_{\ell})m_{kj}(t)m_{1\ell}(t)m_{k\ell}(t)$$
$$\leq \Lambda + \beta\sqrt{M}\lambda_{1}\left(2\lambda_{j}-(\lambda_{1}+\lambda_{j})m_{11}^{2}(t)\right)m_{1j}(t),$$

for every $t \leq \mathcal{T}_{\varepsilon'}^{(11)} \wedge \min_{2 \leq i,j \leq r} \left\{ \mathcal{T}_{N-\frac{\delta_{1j}^{(1)}-\delta}{2}}^{(1j)} \wedge \mathcal{T}_{N-\frac{\delta_{1j}^{(1)}-\delta}{2}}^{(i1)} \wedge \mathcal{T}_{N-\frac{\delta_{ij}^{(1)}-\delta}{2}}^{(ij)} \right\}$, with \mathbb{P} -probability at least $1 - \exp(-KN)$. We then note that the correlation m_{i1} starts decreasing as soon as

$$m_{11}^2 > \frac{\lambda_i}{\lambda_1} = \frac{1}{\prod_{\ell=1}^{i-1} (1+\kappa_\ell)},$$

so that as m_{11}^2 exceeds $\frac{1}{1+\kappa_1}$, all correlations m_{21}, \ldots, m_{r1} are decreasing. In the same way, we see that the correlation m_{1i} starts decreasing as soon as

$$m_{11}^2 > \frac{2\lambda_j}{\lambda_1 + \lambda_j} = \frac{2}{1 + \prod_{\ell=1}^{j-1} (1 + \kappa_\ell)}$$

so that the condition $m_{11}^2 > \frac{2}{2+\kappa_1}$ ensures that m_{12}, \ldots, m_{1r} are decreasing. Since $\frac{1}{1+\kappa_1} < \frac{2}{2+\kappa_1}$, we have that as soon as $m_{11}^2 = \frac{2}{2+\kappa_1} + \omega$ for $0 < \omega < \frac{\kappa_1}{2+\kappa_1}$, all correlations m_{i1} and m_{1j} are decreasing. Without loss of generality, we may assume that $\omega = \frac{1}{2} \frac{\kappa_1}{2+\kappa_1}$, and we set

$$(\varepsilon')^2 = \frac{4+\kappa_1}{2(2+\kappa_1)}.$$

According to (6.16), we have that

$$\mathcal{T}_{\varepsilon'}^{(11)} \le \frac{1}{\beta\sqrt{M}\lambda_1^2} \log\left(\frac{4+\kappa_1}{\kappa_1}\frac{4-\varepsilon^2}{\varepsilon^2}\right) \lesssim N^{-\frac{1}{2}\left(\frac{1+C_0}{1-C_0}-\frac{\lambda_r^2}{\lambda_1^2}\right)} \log\left(\frac{4+\kappa_1}{\kappa_1}\frac{4-\varepsilon^2}{\varepsilon^2}\right)$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp\left(-\varepsilon^2 N/(4K_2\mathcal{T}_{\varepsilon'}^{(11)})\right)$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. Since $\mathcal{T}_{\varepsilon'}^{(11)}$ is of order 1, the correlations m_{1i} and m_{i1} do not change scales during this interval (as it requires a time of order $\log(N)$ to do so) so that they are still upper bounded in a scale $N^{-\frac{\delta_{1i}^{(1)}}{2}}$.

It remains to consider the evolution of the correlations m_{ij} for $2 \leq i, j \leq r$ during $[0, \mathcal{T}_{\varepsilon'}^{(11)}]$. We aim to show that during this interval the functions m_{ij} remain increasing. To carry out this analysis, we let $T_1^{(ij)}$ denote the hitting time for the set

$$\left\{\boldsymbol{X}: 2r^2\lambda_1^2m_{11}(\boldsymbol{X})m_{i1}(\boldsymbol{X})m_{1j}(\boldsymbol{X}) > C_0\lambda_i\lambda_jm_{ij}(\boldsymbol{X})\right\}.$$

Our goal is to show that $\mathcal{T}_{\varepsilon'}^{(11)} \leq \min_{2 \leq k, \ell \leq r} T_1^{(k\ell)}$. To this end, we first observe that

$$L_{\beta}m_{ij}(t) \geq -\|L_{0,\beta}m_{ij}\|_{\infty} + 2\beta\sqrt{M}\lambda_{i}\lambda_{j}m_{ij}(t) - \beta\sqrt{M}\sum_{1\leq k,\ell\leq r}\lambda_{k}(\lambda_{j}+\lambda_{\ell})m_{kj}(t)m_{i\ell}(t)m_{k\ell}(t)$$
$$\geq -\Lambda + 2\beta\sqrt{M}\lambda_{i}\lambda_{j}m_{ij}(t) - 2r^{2}\beta\sqrt{M}\lambda_{1}^{2}m_{1j}(t)m_{i1}(t)m_{11}(t)$$
$$\geq 2(1-C_{0})\beta\sqrt{M}\lambda_{i}\lambda_{j}m_{ij}(t),$$

for every $t \leq \mathcal{T}_{\varepsilon'}^{(11)} \wedge T_1^{(ij)}$, with \mathbb{P} -probability at least $1 - \exp(-KN)$, provided $\sqrt{M} \geq \frac{\Lambda}{C_0 \beta \lambda_i \lambda_j m_{ij}(t)}$, which certainly holds by assumption. From this, we obtain that

$$m_{ij}(t) \ge m_{ij}(0) + M_t^{m_{ij}} + 2(1 - C_0)\beta\sqrt{M}\lambda_i\lambda_j \int_0^t m_{ij}(s)ds$$
$$\ge \frac{N^{-\frac{\xi_{ij}^{(1)}}{2}}}{2} + 2(1 - C_0)\beta\sqrt{M}\lambda_i\lambda_j \int_0^t m_{ij}(s)ds$$
$$\ge \frac{N^{-\frac{\xi_{ij}^{(1)}}{2}}}{2} \exp\left(2(1 - C_0)\beta\sqrt{M}\lambda_i\lambda_jt\right),$$

for every $t \leq \mathcal{T}_{\varepsilon'}^{(11)} \wedge \mathcal{T}_1^{(ij)}$, with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-N^{\frac{1-C_0}{1+C_0}\frac{\lambda_i\lambda_j}{\lambda_1^2}}/4K_2\mathcal{T}_{\varepsilon'}^{(11)})$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$, where the last inequality follows from item (d) of Lemma 4.11. We denote by ℓ_{ij} the function given by

$$\ell_{ij}(t) = \frac{N^{-\frac{\xi_{ij}^{(1)}}{2}}}{2} \exp\left(2(1-C_0)\beta\sqrt{M\lambda_i\lambda_j}t\right).$$

Furthermore, we always have the brutal upper bound on the generator $L_{\beta}m_{ij}$, i.e., for every $i, j \in [r]$ and $t \geq 0$ it holds that

$$L_{\beta}m_{ij}(t) \le \Lambda + 2\beta\sqrt{M}\lambda_i\lambda_j m_{ij}(t) \le 2(1+C_0)\beta\sqrt{M}\lambda_i\lambda_j m_{ij}(t),$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$. Therefore, we find that

$$m_{ij}(t) \le \frac{m_{ij}(0)}{2} + 2(1+C_0)\beta\sqrt{M}\lambda_i\lambda_j \int_0^t m_{ij}(s)ds \le \frac{m_{ij}(0)}{2}\exp\left(2(1+C_0)\beta\sqrt{M}\lambda_i\lambda_jt\right),$$

for $t \leq \mathcal{T}_{\varepsilon'}^{(11)} \wedge T_1^{(ij)}$, with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-m_{ij}(0)^2 N/(4K_2 \mathcal{T}_{\varepsilon'}^{(11)}))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. We denote by u_{ij} the function given by

$$u_{ij}(t) = \frac{m_{ij}(0)}{2} \exp\left(2(1+C_0)\beta\sqrt{M}\lambda_i\lambda_j t\right).$$

A sufficient condition to show that $\mathcal{T}_{\varepsilon'}^{(11)} \leq \min_{2 \leq k, \ell \leq r} T_1^{(k\ell)}$ is therefore given by $\mathcal{T}_{\varepsilon'}^{(11)} \leq \min_{2 \leq k, \ell \leq r} \tilde{T}_1^{(k\ell)}$, where $\tilde{T}_1^{(ij)}$ denotes the hitting time for the set

$$\left\{\boldsymbol{X}: 2r^2\lambda_1^2u_{11}(\boldsymbol{X})u_{i1}(\boldsymbol{X})u_{1j}(\boldsymbol{X}) > C_0\lambda_i\lambda_j\ell_{ij}(\boldsymbol{X})\right\}.$$

The inequality defining the above event can be written as

$$2r^2\lambda_1^2\varepsilon'\frac{N^{-\delta_{1i}^{(1)}/2}}{2}\frac{N^{-\delta_{1j}^{(1)}/2}}{2}e^{2(1+C_0)\beta\sqrt{M}\lambda_1(\lambda_i+\lambda_j)t} > C_0\lambda_i\lambda_j\frac{N^{-\xi_{ij}^{(1)}/2}}{2}e^{2(1-C_0)\beta\sqrt{M}\lambda_i\lambda_jt},$$

yielding an explicit expression for $\tilde{T}_1^{(ij)}$, i.e.,

$$\tilde{T}_{1}^{(ij)} = \frac{\frac{1}{2} \left(\delta_{1i}^{(1)} + \delta_{1j}^{(1)} - \xi_{ij}^{(1)} \right) \log(N) + \log(C_0 \lambda_i \lambda_j / (r^2 \varepsilon' \lambda_1^2))}{2\beta \sqrt{M} \left((1 + C_0) \lambda_1 (\lambda_i + \lambda_j) - (1 - C_0) \lambda_i \lambda_j \right)}$$

We note that the term $\delta_{1i}^{(1)} + \delta_{1j}^{(1)} - \xi_{ij}^{(1)}$, which is given by

$$\delta_{1i}^{(1)} + \delta_{1j}^{(1)} - \xi_{ij}^{(1)} = 1 + \frac{1 - C_0}{1 + C_0} \frac{\lambda_i \lambda_j}{\lambda_1^2} - \frac{1 + C_0}{1 - C_0} \frac{\lambda_i + \lambda_j}{\lambda_1}$$

is positive, provided C_0 sufficiently small. In particular, if $\kappa = \min_{1 \le i \le r-1} \kappa_i$ is sufficiently large then we can easily found $C_0 < \frac{1}{2} \land \frac{\kappa}{1+\kappa}$ such that $\delta_{1i}^{(1)} + \delta_{1j}^{(1)} - \xi_{ij}^{(1)} > 0$. Otherwise, we need to take C_0 sufficiently close to zero. Therefore, we can find $c_0 = c_0(\kappa)$ such that for every $C_0 < c_0$ the quantity $\delta_{1i}^{(1)} + \delta_{1j}^{(1)} - \xi_{ij}^{(1)}$ is in (0,1). In addition, we see that $\tilde{T}_1^{(22)} = \min_{2 \le k, \ell \le r} \tilde{T}_1^{(k\ell)}$. A sufficient condition for $\mathcal{T}_{\varepsilon'}^{(11)} \le \min_{2 \le k, \ell \le r} \tilde{T}_1^{(k\ell)}$ is given by

$$\log(N) \ge \frac{4}{2\delta_{12}^{(1)} - \xi_{22}^{(1)}} \frac{1}{(1+\kappa_1)^2} \left(2(1+C_0)(1+\kappa_1) - (1-C_0)\right) \log\left(\frac{4+\kappa_1}{\kappa_1} \frac{4-\varepsilon^2}{\varepsilon^2}\right),$$

and we can take

$$\operatorname{og}(N) \gtrsim \frac{4}{1+\kappa_1} \log\left(\frac{4+\kappa_1}{\kappa_1} \frac{4-\varepsilon^2}{\varepsilon^2}\right)$$

Under this condition, we have that $\mathcal{T}_{\varepsilon'}^{(11)} \leq \tilde{T}_1^{(ij)} \leq T_1^{(ij)}$ for all $2 \leq i, j \leq r$ with $\mathbb{Q}_{\mathbf{X}}$ -probability at least

$$1 - (r-1)^2 K_2 \exp\left(-N^{\frac{1-C_0}{1+C_0}\frac{\lambda_2^2}{\lambda_1^2}}/(4K_2\mathcal{T}_{\varepsilon'}^{(11)})\right) - 2(r-1)K_2 \exp\left(-N^{\frac{1+C_0}{1-C_0}\frac{\lambda_2}{\lambda_1}}/(4K_2\mathcal{T}_{\varepsilon'}^{(11)})\right)$$

and with \mathbb{P} -probability at least $1 - \exp(-KN)$. We deduce that m_{ij} does not become decreasing during $[0, \mathcal{T}_{\varepsilon'}^{(11)}]$ and remains at least of order $N^{-\frac{\xi_{ij}^{(11)}}{2}}$.

Step 3: Evolution of all $m_{ij}(t)$ for $\mathcal{T}_{\varepsilon'}^{(11)} \leq t \leq \mathcal{T}_{E_2}$. The last step focuses on the evolution of the correlations m_{ij} during the time interval $\mathcal{T}_{\varepsilon'}^{(11)} \leq t \leq \mathcal{T}_{E_2}$, where we remind that $(\varepsilon')^2 = \frac{4+\kappa_1}{2(2+\kappa_1)}$.

We start with the correlation $m_{11}(t)$ and note that for every $\mathcal{T}_{\varepsilon'}^{(11)} \leq t \leq \mathcal{T}_{\varepsilon'/2}^{(11)} \wedge \mathcal{T}_{1-\varepsilon}^{(11)}$, it holds that

$$L_{\beta}m_{11}(t) \ge -\|L_{0,\beta}m_{11}\|_{\infty} + 2\beta\sqrt{M}\lambda_1^2m_{11}(t)\left(1 - r^2m_{11}^2(t)\right)$$

We can then proceed using Lemma 6.2 to show that m_{11} reaches $1 - \varepsilon$ with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-N\varepsilon^2/(K_2\mathcal{T}_{1-\varepsilon}^{(11)}))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$.

We next focus on $m_{1i}(t)$ and $m_{i1}(t)$, and observe that for every $\mathcal{T}_{\varepsilon'}^{(11)} \leq t \leq \mathcal{T}_{E_2}$, it holds that

$$L_{\beta}m_{i1}(t) \le \|L_{0,\beta}m_{i1}\|_{\infty} - 2\beta\sqrt{M}\lambda_{1}^{2}\frac{\kappa_{1}}{2+\kappa_{1}}m_{i1}(t) \le -\beta\sqrt{M}\lambda_{1}^{2}\frac{\kappa_{1}}{2+\kappa_{1}}m_{i1}(t),$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$, provided $\sqrt{M} \geq \frac{\Lambda(2+\kappa_1)}{\beta \lambda_1^2 \kappa_1 m_{i1}(t)}$, which certainly holds since $\sqrt{M} \gtrsim N^{\frac{\xi_{rr}^{(1)}}{2}}$. By Grönwall's inequality it then follows that

$$\begin{split} n_{i1}(t) &\leq \frac{1}{2} m_{i1}(\mathcal{T}_{\varepsilon'}^{(11)}) - \frac{\kappa_1}{2 + \kappa_1} \beta \sqrt{M} \lambda_1^2 \int_{\mathcal{T}_{\varepsilon'}^{(11)}}^t m_{i1}(s) ds \\ &\leq \frac{1}{2} m_{i1}(\mathcal{T}_{\varepsilon'}^{(11)}) e^{-\frac{\kappa_1}{2 + \kappa_1} \beta \sqrt{M} \lambda_1^2 (t - \mathcal{T}_{\varepsilon'}^{(11)})} \\ &\leq \frac{N^{-\frac{\delta_{1i}^{(1)}}{2}}}{2} e^{-\frac{\kappa_1}{2 + \kappa_1} \beta \sqrt{M} \lambda_1^2 (t - \mathcal{T}_{\varepsilon'}^{(11)})}, \end{split}$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_3 \exp\left(-N^{\frac{1+C_0}{1-C_0}\frac{\lambda_2}{\lambda_1}}/(4K_3\mathcal{T}_{E_2})\right)$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. This shows that the correlations m_{i1} descend to a value of order $N^{-\frac{\xi_{rr}^{(1)}}{2}}$ in a time of order $\frac{(\xi_{rr}^{(1)}-\delta_{i1}^{(1)})(2+\kappa_1)}{2\kappa_1\beta\sqrt{M\lambda_1^2}}\log(N)$.

We finally look at the correlations m_{ij} for $2 \le i, j \le r$ and show that m_{22} is the second correlation to become macroscopic. We see that

$$L_{\beta}m_{ij}(t) \geq -\Lambda + 2\beta\sqrt{M}\lambda_{i}\lambda_{j}m_{ij}(t) - \beta\sqrt{M}\sum_{1\leq k,\ell\leq r}\lambda_{k}(\lambda_{j}+\lambda_{\ell})m_{kj}(t)m_{i\ell}(t)m_{k\ell}(t)$$
$$\geq -\Lambda + 2\beta\sqrt{M}\lambda_{i}\lambda_{j}m_{ij}(t) - 2r^{2}\beta\sqrt{M}\lambda_{1}^{2}m_{1j}(t)m_{i1}(t)m_{11}(t)$$
$$\geq -\Lambda + 2\beta\sqrt{M}\left(\lambda_{i}\lambda_{j}m_{ij}(t) - r^{2}\lambda_{1}^{2}(1-\varepsilon)N^{-\delta_{rr}}\right)$$
$$\geq 2(1-C_{0})\beta\sqrt{M}\lambda_{i}\lambda_{j}m_{ij}(t).$$

By Grönwall's inequality (i.e., Lemma 4.11) we have that

$$\frac{N^{-\xi_{ij}^{(1)}/2}}{2} \exp(2(1-C_0)\beta\sqrt{M\lambda_i\lambda_j}t) \le m_{ij}(t) \le \frac{N^{-\delta_{ij}^{(1)}/2}}{2} \exp(2(1+C_0)\beta\sqrt{M\lambda_i\lambda_j}t),$$

with high \mathbb{Q}_{X} - and \mathbb{P} -probability. It then follows that

$$\mathcal{T}_{\varepsilon}^{(22)} \leq \frac{\log(2\varepsilon N^{\xi_{22}^{(1)}/2})}{2(1-C_0)\beta\sqrt{M}\lambda_2^2}$$

Moreover, we see that for every $i, j \neq 1, (i, j) \neq (2, 2)$,

$$m_{ij}(\mathcal{T}_{\varepsilon}^{(22)}) \leq \frac{N^{-\delta_{ij}^{(1)}/2}}{2} \left(2\varepsilon N^{\xi_{22}^{(1)}/2}\right)^{\frac{1+C_0}{1-C_0}\frac{\lambda_i\lambda_j}{\lambda_2^2}} \lesssim N^{-\frac{1}{2}\delta_{ij}^{(2)}},$$

and similarly

$$m_{ij}(\mathcal{T}_{\varepsilon}^{(22)}) \geq \frac{N^{-\xi_{ij}^{(1)}/2}}{2} \left(2\varepsilon N^{\delta_{22}^{(1)}/2}\right)^{\frac{1-C_0}{1+C_0}\frac{\lambda_i\lambda_j}{\lambda_2^2}} \gtrsim N^{-\frac{1}{2}\xi_{ij}^{(2)}}.$$

This completes the proof of Lemma 6.4 for k = 1. We proceed analogously for $k \ge 2$.

7. Proofs for p = 2 and equal SNRs

This section is devoted to the proof of Proposition 3.10. We recall that when p = 2 and the SNRs are all equal, the problem becomes isotropic. Consequently, we focus on the recovery of the eigenvalues $\theta_1, \ldots, \theta_r$ of $\mathbf{G} = \mathbf{M}^\top \mathbf{M}$ under Langevin dynamics. We refer to Subsection 4.2 of Section 4 for details on the evolution equations for the matrix-valued function \mathbf{G} .

To prove Proposition 3.10, we first show that weak recovery of the largest eigenvalue θ_{max} of G is achieved, provided a number of samples of order 1.

Lemma 7.1. Let $\beta \in (0, \infty)$, p = 2, and $\lambda_1 = \cdots = \lambda_r \equiv \lambda > 0$. Consider a sequence of initializations $\mu_0 \in \mathcal{P}(\mathcal{M}_{N,r})$. For every $\varepsilon > 0$ and every $C_0 \in (0, \frac{1}{r})$ we define the set $E = E(\varepsilon, C_0)$ by

$$E(\varepsilon, C_0) = \left\{ \boldsymbol{X} : \theta_{\max}(\boldsymbol{X}) \ge \varepsilon \text{ and } \theta_{\min}(\boldsymbol{X}) \gtrsim N^{-\frac{2C_0 r}{1+C_0 r}} \right\},\$$

where θ_i denotes the *i*th eigenvalue of \mathbf{G} . We let \mathcal{T}_E denote the hitting time of $E = E(\varepsilon, C_0)$. Then, the following holds: For every $n \ge 1$, $\gamma_0 > 0$, $\gamma_1 > \gamma_2 > 0$, there exists $\varepsilon_0 > 0$ and $c_0 \in (0, \frac{1}{3r})$ such that for every $\varepsilon < \varepsilon_0$, $C_0 < c_0$, $\sqrt{M} \gtrsim \frac{(n+2)\gamma_0^2\gamma_1}{\beta C_0\lambda^2\gamma_2} N^{\frac{1}{2(n+1)}\vee \frac{2C_0r}{1-C_0r}}$,

$$\int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{E} \gtrsim \frac{\log(N) N^{-\left(\frac{1}{2(n+1)} \vee \frac{2C_{0}r}{1-C_{0}r}\right)}}{(n+2)\gamma_{0}^{2}\gamma_{1}} \right) \mathbf{1} \{ \mathcal{C}_{0}(n,\gamma_{0}) \cap \mathcal{C}_{1}'(\gamma_{1},\gamma_{2}) \} d\mu_{0}(\boldsymbol{X})$$
$$\lesssim K_{1} e^{-\gamma_{1}\gamma_{0}^{4}(n+2)N^{\frac{1}{2(n+1)} \vee \frac{2C_{0}r}{1-C_{0}r}/(K_{1}\log(N))}} + K_{2} e^{-\gamma_{0}^{2}\gamma_{2}(n+2)N^{\frac{1}{2(n+1)} \vee \frac{2C_{0}r}{1-C_{0}r}/K_{2}}},$$

with \mathbb{P} -probability at least $1 - \exp(-KN)$.

Proof. We let $\mathcal{A}_0 = \mathcal{A}_0(n, \gamma_0, \gamma_1, \gamma_2)$ denote the initial event given by

$$\mathcal{A}_0(n,\gamma_0,\gamma_1,\gamma_2) = \{ \boldsymbol{X}_0 \sim \mu \colon \boldsymbol{X}_0 \in \mathcal{C}_0(n,\gamma_0) \cap \mathcal{C}_1'(\gamma_1,\gamma_2) \}$$

We note that on $C'_1(\gamma_1, \gamma_2)$, for every $i \in [r]$ there exists $\gamma_{ii} \in (\gamma_2, \gamma_1)$ such that $\theta_i(0) = \gamma_{ii}N^{-1}$. Moreover, for $\gamma > 0$ and $T_0 > 0$ to be chosen later, we also define the event $\mathcal{A} = \mathcal{A}(n, \gamma_0, \gamma_1, \gamma_2, \gamma, T_0)$ by

$$\mathcal{A} = \mathcal{A}_0(n, \gamma_0, \gamma_1, \gamma_2) \cap \left\{ \sup_{t \in [0, T_0]} \|M_t^{\boldsymbol{G}}\|_{\text{op}} \le \frac{\gamma}{\sqrt{N}} f(T_0)^{1/2} \right\}$$

for some positive, monotone increasing function f to be determined later such that $f(t) \ge \int_0^t ||\mathbf{M}(s)||_{\text{op}}^2$ for every $t \in [0, T_0]$. We then introduce the hitting time $\mathcal{T}_{\mathbf{G}}$ for the set

$$\left\{t: \|M_t^G\|_{\mathrm{op}} > \min_{1 \le i \le r} 2C_0 r\beta \sqrt{M}\lambda^2 \int_0^t \theta_i(s) ds\right\},\$$

for some $C_0 \in (0, \frac{1}{2r})$. We note that since $||M_0^G||_{\text{op}} = 0$, we have that $\mathcal{T}_G > 0$. We also introduce the hitting time $\mathcal{T}_{\mathbf{L}_{0,\beta}}$ for the set

$$\left\{ \boldsymbol{X} : \|\boldsymbol{L}_{0,\beta}\boldsymbol{G}(\boldsymbol{X})\|_{\mathrm{op}} > 2C_0\beta\sqrt{M}\lambda^2\theta_{\mathrm{min}}(\boldsymbol{X}) \right\}.$$

According to (4.14) we have the bound

$$\|\boldsymbol{L}_{0,\beta}\boldsymbol{G}\|_{\rm op} \le 2\|\boldsymbol{M}\|_{\rm op}\|L_{0,\beta}\boldsymbol{M}\|_{\rm op} + \frac{2r}{N} + \frac{2}{N}\|\boldsymbol{G}\|_{\rm op},\tag{7.1}$$

so that under the event \mathcal{A} ,

$$\begin{split} \|\boldsymbol{L}_{0,\beta}\boldsymbol{G}(\boldsymbol{X}_{0})\|_{\text{op}} &\leq 2r \max_{1 \leq i,j \leq r} |m_{ij}(\boldsymbol{X}_{0})| \max_{1 \leq i,j \leq r} |L_{0,\beta}m_{ij}(\boldsymbol{X}_{0})| + \frac{2r}{N} + \frac{2r}{N} \sum_{k=1}^{r} \max_{1 \leq i,j \leq r} |m_{ki}(\boldsymbol{X}_{0})m_{kj}(\boldsymbol{X}_{0})| \\ &\leq \frac{2r^{2}\gamma_{0}^{2}}{N} + \frac{2r}{N} + \frac{2r^{2}\gamma_{0}^{2}}{N} \\ &\leq 2C_{0}\beta\sqrt{M}\lambda^{2}\frac{\gamma_{2}}{N}, \end{split}$$

provided $\sqrt{M} \gtrsim \frac{r\gamma_0^2}{C_0\beta\lambda^2\gamma_2}$ which certainly holds by assumption. This implies that $\mathcal{T}_{L_{0,\beta}} > 0$.

For every $\varepsilon > 0$, we then let $\mathcal{T}_{\varepsilon}^{(i)}$ denote the hitting time for the set $\{\mathbf{X}: \theta_i(\mathbf{X}) \geq \varepsilon\}$. We recall the evolution equation for \mathbf{G} under Langevin dynamics given by Corollary 4.9, i.e.,

$$\boldsymbol{G}(t) = \boldsymbol{G}(0) + M_t^{\boldsymbol{G}} + \int_0^t \boldsymbol{L}_{0,\beta} \boldsymbol{G}(s) ds + \int_0^t \hat{\boldsymbol{L}}_{\beta} \boldsymbol{G}(s) ds$$

where $\hat{L}_{\beta}G$ satisfies $\hat{L}_{\beta}G = 4\beta\sqrt{M}\lambda^2 G(I_r - G)$. For all $t \geq 0$ and $i \in [r]$, we obtain by Weyl's inequality that

$$\left|\theta_{i}\left(t\right)-\mu_{i}\left(\boldsymbol{G}(0)+4\beta\sqrt{M}\lambda^{2}\int_{0}^{t}\boldsymbol{G}(s)(\boldsymbol{I}_{r}-\boldsymbol{G}(s))ds\right)\right|\leq \|\boldsymbol{M}_{t}^{\boldsymbol{G}}+\int_{0}^{t}\boldsymbol{L}_{0,\beta}\boldsymbol{G}(s)\|_{\mathrm{op}}ds,$$

where $\mu_i(\mathbf{A})$ stands for the *i*th eigenvalue of \mathbf{A} . Moreover, by the triangle inequality and the fact that $\|\int_0^t \mathbf{L}_{0,\beta} \mathbf{G}(s) ds\|_{\text{op}} \leq r \int_0^t \|\mathbf{L}_{0,\beta} \mathbf{G}(s)\|_{\text{op}} ds$ it then follows that

$$\left|\theta_i\left(t\right) - \mu_i\left(\boldsymbol{G}(0) + 4\beta\sqrt{M}\lambda^2 \int_0^t \boldsymbol{G}(s)(\boldsymbol{I}_r - \boldsymbol{G}(s))ds\right)\right| \le \|\boldsymbol{M}_t^{\boldsymbol{G}}\|_{\mathrm{op}} + r\int_0^t \|\boldsymbol{L}_{0,\beta}\boldsymbol{G}(s)\|_{\mathrm{op}}ds.$$

According to Lemma 4.13, we then have that

$$\theta_i(t) \le \theta_i(0) + 4\beta\sqrt{M}\lambda^2 \int_0^t \left(\theta_{\max}(s) - \theta_{\min}^2(s)\right) ds + \|M_t^{\boldsymbol{G}}\|_{\mathrm{op}} + r \int_0^t \|\boldsymbol{L}_{0,\beta}\boldsymbol{G}(s)\|_{\mathrm{op}} ds,$$

$$\theta_i(t) \ge \theta_i(0) + 4\beta\sqrt{M}\lambda^2 \int_0^t \left(\theta_{\min}(s) - \theta_{\max}^2(s)\right) ds - \|M_t^{\boldsymbol{G}}\|_{\mathrm{op}} - r \int_0^t \|\boldsymbol{L}_{0,\beta}\boldsymbol{G}(s)\|_{\mathrm{op}} ds,$$

for all $t \ge 0$ and $i \in [r]$. Then, on the event \mathcal{A} , for every $i \in [r]$ we have that

$$\frac{\gamma_{ii}}{N} + 4(1 - C_0 r)\beta\sqrt{M}\lambda^2 \int_0^t \theta_{\min}(s)ds \le \theta_i(t) \le \frac{\gamma_{ii}}{N} + 4(1 + C_0 r)\beta\sqrt{M}\lambda^2 \int_0^t \theta_{\max}(s)ds,$$
(7.2)

for all $t \leq \mathcal{T}_{\mathbf{L}_{0,\beta}} \wedge \mathcal{T}_{\mathbf{G}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)}$. Given the integral inequality (7.2) we then obtain from Lemma 4.11 the following comparison inequality,

$$\ell(t) \le \theta_{\min}(t) \le \theta_i(t) \le \theta_{\max}(t) \le u(t), \tag{7.3}$$

for every $i \in [r]$ and every $t \leq \mathcal{T}_{L_{0,\beta}} \wedge \mathcal{T}_{G} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)}$, where the functions ℓ and u are given by

$$\ell(t) = \frac{\gamma_2}{N} \exp(4(1 - C_0 r)\beta\sqrt{M}\lambda^2 t),$$

$$u(t) = \frac{\gamma_1}{N} \exp(4(1 + C_0 r)\beta\sqrt{M}\lambda^2 t).$$

We then let $T_{\ell,\varepsilon}$ denote the time such that $\ell(T_{\ell,\varepsilon}) = \varepsilon$, i.e.,

$$T_{\ell,\varepsilon} = \frac{\log(N) - \log(\gamma_2/\varepsilon)}{4\beta\sqrt{M}\lambda^2(1 - C_0 r)}$$

Similarly, we let $T_{u,\varepsilon}$ be such that $u(T_{u,\varepsilon}) = \varepsilon$, i.e.,

$$T_{u,\varepsilon} = \frac{\log(N) - \log(\gamma_1/\varepsilon)}{4\beta\sqrt{M}\lambda^2(1+C_0r)}$$

We note that on the event \mathcal{A} , $T_{u,\varepsilon} \leq \mathcal{T}_{\varepsilon}^{(i)} \leq T_{\ell,\varepsilon}$ for every $i \in [r]$. Moreover, since the functions ℓ and u are monotone increasing, we can bound $\theta_i(\min_{1 \leq j \leq r} \mathcal{T}_{\varepsilon}^{(j)})$ for every $i \in [r]$ by

$$\gamma_2 \left(\frac{\varepsilon}{\gamma_1}\right)^{\frac{1-C_0 r}{1+C_0 r}} N^{-\frac{2C_0 r}{1+C_0 r}} = \ell\left(T_{u,\varepsilon}\right) \le \theta_i \left(\min_{1\le j\le r} \mathcal{T}_{\varepsilon}^{(j)}\right) \le u\left(T_{\ell,\varepsilon}\right) = \gamma_1 \left(\frac{\varepsilon}{\gamma_2}\right)^{\frac{1+C_0 r}{1-C_0 r}} N^{\frac{2C_0 r}{1-C_0 r}}.$$

In the following, we choose $T_0 = T_{\ell,\varepsilon}$.

Now, we wish to show that on the event \mathcal{A} , $\mathcal{T}_{G} \wedge \mathcal{T}_{L_{0,\beta}} \geq \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)}$. We first show that on the event \mathcal{A} , $\mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0 \leq \mathcal{T}_{G}$. We observe that

$$\|\boldsymbol{M}\|_{\mathrm{op}}^2 = \|\boldsymbol{M}^\top \boldsymbol{M}\|_{\mathrm{op}} = \theta_{\mathrm{max}},$$

so that from (7.3) we have that

$$\int_{0}^{t} \|\boldsymbol{M}(s)\|_{\text{op}}^{2} ds \leq \frac{\gamma_{1}}{N} \int_{0}^{t} e^{4(1+C_{0}r)\beta\sqrt{M}\lambda^{2}s} ds = \frac{\gamma_{1}}{N} \frac{e^{4(1+C_{0}r)\beta\sqrt{M}\lambda^{2}t} - 1}{4(1+C_{0}r)\beta\sqrt{M}\lambda^{2}}$$

for every $t \leq \mathcal{T}_{\mathbf{L}_{0,\beta}} \wedge \mathcal{T}_{\mathbf{G}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)}$. In particular, the function $f(t) = \frac{\gamma_1}{N} \frac{e^{4(1+C_0r)\beta\sqrt{M\lambda^2t}-1}}{4(1+C_0r)\beta\sqrt{M\lambda^2}}$ is non-negative, monotone increasing for all t > 0, and bounds $\int_0^t \|\mathbf{M}(s)\|_{\text{op}}^2 ds$ by above for every $t \leq \mathcal{T}_{\mathbf{L}_{0,\beta}} \wedge \mathcal{T}_{\mathbf{G}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)}$. In order to prove that on the event \mathcal{A} , $\mathcal{T}_{\mathbf{L}_{0,\beta}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0 \leq \mathcal{T}_{\mathbf{G}}$, it suffices to show that there is $\gamma > 0$ such that

$$\frac{\gamma}{\sqrt{N}}\sqrt{f(t)} \le \frac{\gamma_2 C_0 r}{2(1 - C_0 r)N} \left(e^{4(1 - C_0 r)\beta\sqrt{M}\lambda^2 t} - 1 \right),\tag{7.4}$$

for every $t \leq \mathcal{T}_{\boldsymbol{L}_{0,\beta}} \wedge \mathcal{T}_{\boldsymbol{G}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0$. Indeed, according to the comparison inequality (7.3), the right-hand side of (7.4) is a lower bound for the desired quantity, i.e.,

$$\frac{\gamma_2 C_0 r}{2(1-C_0 r)N} \left(e^{4(1-C_0 r)\beta\sqrt{M}\lambda^2 t} - 1 \right) = 2C_0 r\beta\sqrt{M}\lambda^2 \frac{\gamma_2}{N} \int_0^t e^{4(1-C_0 r)\beta\sqrt{M}\lambda^2 s} ds$$
$$\leq 2C_0 r\beta\sqrt{M}\lambda^2 \int_0^t \theta_{\min}(s) ds,$$

for every $t \leq \mathcal{T}_{\mathbf{L}_{0,\beta}} \wedge \mathcal{T}_{\mathbf{G}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)}$. It is easily noticed that (7.4) is verified at t = 0 for every $\gamma > 0$. Moreover, the inequality (7.4) holds at $t = T_0$, provided $C_0 r \leq 1/3$ and

$$\gamma^2 \le \frac{\gamma_2^2 C_0^2 r^2 \beta \sqrt{M} \lambda^2}{\gamma_1}$$

Since both sides of (7.4) are increasing functions and the inequality is satisfied at t = 0 and at $t = T_0$, the estimate (7.4) holds for every $t \leq \mathcal{T}_{\boldsymbol{L}_{0,\beta}} \wedge \mathcal{T}_{\boldsymbol{G}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0$. This implies that $\mathcal{T}_{\boldsymbol{L}_{0,\beta}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0 \leq \mathcal{T}_{\boldsymbol{G}}$, provided $C_0 r \leq 1/3$ and $\gamma^2 \leq \frac{\gamma_2^2 C_0^2 r^2 \beta \sqrt{M} \lambda^2}{\gamma_1}$.

Next, we show that $\mathcal{T}_{\mathbf{L}_{0,\beta}} \geq \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0$. To this end, we wish to apply the bounding flow method in order to estimate the operator norm $\|\mathbf{L}_{0,\beta}\mathbf{G}\|_{\text{op}}$. We first recall from (7.1) that

$$\|\boldsymbol{L}_{0,\beta}\boldsymbol{G}\|_{\mathrm{op}} \leq 2\|\boldsymbol{M}\|_{\mathrm{op}}\|\boldsymbol{L}_{0,\beta}\boldsymbol{M}\|_{\mathrm{op}} + \frac{2r}{N} + \frac{2}{N}\|\boldsymbol{G}\|_{\mathrm{op}}$$

Since $\|\boldsymbol{M}\|_{\text{op}}^2 = \theta_{\max}(\boldsymbol{G})$, we can bound $\|\boldsymbol{M}\|_{\text{op}}$ via (7.3). Moreover, we will use the bound $\|\boldsymbol{G}\|_{\text{op}} \leq r^2 \max_{1 \leq i,j,k \leq r} |m_{ki}m_{kj}| \leq r^2$ for the last term. It remains to bound $\|\boldsymbol{L}_{0,\beta}\boldsymbol{M}\|_{\text{op}}$ and for this purpose we will use Lemma 4.4 with a different control on the last term. We direct the reader to the proof of Lemma 6.1 for more details on the application of Lemma 4.4. We now consider condition (4) of Lemma 4.4. Since we do not have comparison inequalities for the m_{ij} , in order to verify item (4) of Lemma 4.4 we need to relate the control of the drift part of the expansion in Lemma 4.4 to the quantity $\|\boldsymbol{M}\|_{\text{op}}$. Recall the expression for the third term in the expansion (4.5) which is bounded by

$$\sum_{1 \le i,j \le r} \sum_{k=1}^{n} \int_{0}^{t} \cdots \int_{0}^{t_{k-1}} |a_{ij}(t_k)| dt_k \cdots dt_1 = 2\beta \sqrt{M} \lambda^2 \sum_{k=1}^{n} \int_{0}^{t} \cdots \int_{0}^{t_{k-1}} \|\boldsymbol{M}(t_k)\|_{\ell_1} dt_k \cdots dt_1$$
$$\leq 2\beta \sqrt{M} \lambda^2 r^{3/2} \sum_{k=1}^{n} \int_{0}^{t} \cdots \int_{0}^{t_{k-1}} \|\boldsymbol{M}(t_k)\|_{\mathrm{op}} dt_k \cdots dt_1$$

where we used that $a_{ij}(s) = 2\beta \sqrt{M} \lambda^2 m_{ij}(s)$ and we used the equivalence of norms $\|\boldsymbol{M}\|_{\ell_1} \leq r^{3/2} \|\boldsymbol{M}\|_{\text{op}}$ (see e.g. [32]). Our goal is therefore to prove that

$$\int_0^t \|\boldsymbol{M}(s)\|_{\rm op} ds \le \epsilon \|\boldsymbol{M}(t)\|_{\rm op},\tag{7.5}$$

for some $\epsilon \in (0, 1)$, ensuring that the third term in the expansion (4.5) is bounded by

$$2\beta\sqrt{M}\lambda^{2}r^{3/2}\sum_{k=1}^{n}\int_{0}^{t}\cdots\int_{0}^{t_{k-1}}\|\boldsymbol{M}(t_{k})\|_{\mathrm{op}}dt_{k}\cdots dt_{1} \leq 2\beta\sqrt{M}\lambda^{2}r^{3/2}\sum_{k=1}^{n}\epsilon^{k-1}\int_{0}^{t}\|\boldsymbol{M}(s)\|_{\mathrm{op}}ds$$
$$\leq 2\beta\sqrt{M}\lambda^{2}r^{3/2}\frac{1}{1-\epsilon}\int_{0}^{t}\|\boldsymbol{M}(s)\|_{\mathrm{op}}ds.$$

A sufficient condition for (7.5) to hold is given by

$$\sqrt{\frac{\gamma_1}{N}} \frac{e^{2(1+C_0r)\beta\sqrt{M}\lambda^2t} - 1}{2(1+C_0r)\beta\sqrt{M}\lambda^2} \le \epsilon \sqrt{\frac{\gamma_2}{N}} e^{2(1-C_0r)\beta\sqrt{M}\lambda^2t}$$

for every $t \leq \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0$, where we used (7.3). The above inequality is easily verified at t = 0 and holds at $t = T_0$ with $\epsilon = 1/2$, provided $\sqrt{M} \gtrsim \frac{\sqrt{\gamma_1}}{\beta \lambda^2 \sqrt{\gamma_2}} N^{\frac{C_0 r}{1 - C_0 r}}$, which holds by assumption.

We therefore have according to Lemma 4.4 that on the event \mathcal{A} ,

$$\|\boldsymbol{L}_{0,\beta}\boldsymbol{M}(t)\|_{\mathrm{op}} \le r \max_{1 \le i,j \le r} |L_{0,\beta}m_{ij}(t)| \le Kr \left(\frac{\gamma_0}{\sqrt{N}} \sum_{k=1}^{n-1} t^k + t^n + 4\beta\sqrt{M}\lambda^2 r^{3/2} \int_0^t \|\boldsymbol{M}(s)\|_{\mathrm{op}} ds\right),$$

for every $t \leq \mathcal{T}_{\mathbf{L}_{0,\beta}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0$, with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-\gamma_0^2/(K_2 T_0))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. From this we obtain that, on the event \mathcal{A} ,

$$\|\boldsymbol{L}_{0,\beta}\boldsymbol{G}(t)\|_{\rm op} \le 2Kr\|\boldsymbol{M}(t)\|_{\rm op} \left(\frac{\gamma_0}{\sqrt{N}}\sum_{k=1}^{n-1}t^k + t^n + 4\beta\sqrt{M}\lambda^2r^{3/2}\int_0^t\|\boldsymbol{M}(s)\|_{\rm op}ds\right) + \frac{2r}{N} + \frac{2r^2}{N}, \quad (7.6)$$

for every $t \leq \mathcal{T}_{\mathbf{L}_{0,\beta}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0$, with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-\gamma_0^2/(K_2T_0))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. In order to deduce that $\mathcal{T}_{\mathbf{L}_{0,\beta}} \geq \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0$, it suffices to show that each term in the right-hand-side of (7.6) is bounded above by $\frac{2C_0\beta\sqrt{M\lambda^2}}{n+2}\theta_{\min}(t)$ for all $t \leq \mathcal{T}_{\mathbf{L}_{0,\beta}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0$. The reasoning is similar to the one used for the proof of Lemma 6.1.

(i) We first see that according to the lower bound in (7.3), we have that

$$\frac{2C_0\beta\sqrt{M\lambda^2}}{n+2}\theta_{\min}(t) \ge \frac{2C_0\beta\sqrt{M\lambda^2}}{n+2}\frac{\gamma_2}{N}\exp(4(1-C_0r)\beta\sqrt{M\lambda^2}t),$$

for $t \leq \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0$. Similarly, according to the upper bound in (7.3), we have that

$$\|\boldsymbol{M}(t)\|_{\rm op} = \sqrt{\theta_{\rm max}(t)} \le \frac{\sqrt{\gamma_1}}{\sqrt{N}} \exp(2(1+C_0 r)\beta\sqrt{M}\lambda^2 t),$$

for every $t \leq \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0$. Therefore, a sufficient condition for the first term is given by

$$2Kr\frac{\gamma_0}{\sqrt{N}}\frac{\sqrt{\gamma_1}}{\sqrt{N}}t^k e^{2(1+C_0r)\beta\sqrt{M}\lambda^2t} + \frac{2r(r+1)}{N} \le \frac{2C_0\beta\sqrt{M}\lambda^2}{n+2}\frac{\gamma_2}{N}e^{4(1-C_0r)\beta\sqrt{M}\lambda^2t},$$

for every $0 \le k \le n-1$. We observe that the inequality holds at t = 0. For t > 0 it is equivalent to verify that

$$\frac{C_0\beta\sqrt{M\lambda^2\gamma_2}}{n+2}e^{2(1-3C_0r)\beta\sqrt{M}\lambda^2t} \ge Cr\gamma_0\sqrt{\gamma_1}t^k$$

for some constant C which may depend on r. We then see that for every $0 \le k \le n-1$,

$$\begin{aligned} \frac{C_0 \beta \sqrt{M} \lambda^2 \gamma_2}{n+2} e^{2(1-3C_0 r)\beta \sqrt{M} \lambda^2 t} &\gtrsim \gamma_0 \gamma_1 r N^{\frac{1}{2(n+1)} \vee \frac{2C_0 r}{1-C_0 r}} e^{2(1-3C_0 r)(n+2)\gamma_0 \gamma_1 r} \\ &\gtrsim (n+2)^k \frac{t^k}{k!} \gamma_0^{k+1} \gamma_1^{k+1} r^{k+1} N^{\frac{1}{2(n+1)} \vee \frac{2C_0 r}{1-C_0 r}} \\ &\gtrsim r \gamma_0 \sqrt{\gamma_1} t^k, \end{aligned}$$

where the first inequality follows by assumption on \sqrt{M} . This ensures that the first term is controlled as desired for every $t \leq \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0$ and every $0 \leq k \leq n-1$. (ii) Similar to item (i), a sufficient condition for the second term is given by

$$2Kr\frac{\sqrt{\gamma_1}}{\sqrt{N}}t^n e^{2(1+C_0r)\beta\sqrt{M}\lambda^2t} + \frac{2r(r+1)}{N} \le \frac{2C_0\beta\sqrt{M}\lambda^2}{n+2}\frac{\gamma_2}{N}e^{4(1-C_0r)\beta\sqrt{M}\lambda^2t}$$

for all $t \leq \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0$. The bound holds at t = 0 and for t > 0 it is equivalent to verify that

$$\frac{C_0\beta\sqrt{M}\lambda^2}{n+2}\frac{\gamma_2}{\sqrt{N}}e^{2(1-3C_0r)\beta\sqrt{M}\lambda^2t} \ge Cr\sqrt{\gamma_1}t^n$$

for some constant C which may depend on r. Proceeding as done in the proof of Lemma 6.1 to verify item (ii), gives that the control holds provided

$$\sqrt{M} \gtrsim \frac{(r\sqrt{\gamma_1}(n+2))^{\frac{1}{n+1}}}{\beta \lambda^2 C_0^{\frac{1}{n+1}} \gamma_2^{\frac{1}{n+1}}} N^{\frac{1}{2(n+1)}},$$

which holds by assumption.

(iii) Finally, we wish to show that

$$8Kr^{5/2}\beta\sqrt{M}\lambda^2 \|\boldsymbol{M}(t)\|_{\rm op} \int_0^t \|\boldsymbol{M}(s)\|_{\rm op}ds + \frac{2r(r+1)}{N} \le \frac{2C_0\beta\sqrt{M}\lambda^2}{n+2}\theta_{\rm min}(t),$$

for every $t \leq \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0$. To this end, a sufficient condition is given by

$$4Kr^{5/2}\frac{\gamma_1}{N}e^{2(1+C_0r)\beta\sqrt{M}\lambda^2t}\frac{e^{2(1+C_0r)\beta\sqrt{M}\lambda^2t}-1}{1+C_0r} + \frac{2r(r+1)}{N} \le \frac{2C_0\beta\sqrt{M}\lambda^2}{n+2}\frac{\gamma_2}{N}e^{4(1-C_0r)\beta\sqrt{M}\lambda^2t},$$

for every $t \leq \mathcal{T}_{\mathbf{L}_{0,\beta}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0$. We easily verify that the inequality holds at t = 0 and at $t = T_0$, provided

$$\sqrt{M} \gtrsim \frac{r^{5/2} \gamma_1(n+2)}{C_0 \beta \lambda^2 \gamma_2} N^{\frac{2C_0 r}{1-C_0 r}}.$$

Therefore, we have that the third term is controlled for every $t \leq \mathcal{T}_{L_{0,\beta}} \wedge \min_{1 \leq i \leq r} \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0$.

We therefore have that on the event \mathcal{A} , $\min_i \mathcal{T}_{\varepsilon}^{(i)} \wedge T_0 \leq \mathcal{T}_{\mathbf{L}_{0,\beta}}$ with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - K_2 \exp(-\gamma_0^2/(K_2T_0))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. In particular, under the event $\mathcal{C}_0(n, \gamma_0) \cap \mathcal{C}'_1(\gamma_1, \gamma_2)$ we have that

$$\min_{1 \le i \le r} \mathcal{T}_{\varepsilon}^{(i)} \le T_0 \lesssim \frac{\log(N)}{(n+2)\gamma_0^2 \gamma_1 N^{\frac{1}{2(n+1)} \vee \frac{2C_0 r}{1-C_0 r}}},$$

with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1-K_2 \exp(-\gamma_1 \gamma_0^4 (n+2) N^{\frac{1}{2(n+1)} \vee \frac{2C_0 r}{1-C_0 r}}/(K_2 \log(N))) - r(r+1) \exp(-\gamma^2/(4r^2))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. This completes the proof of Lemma 7.1. \Box

It remains to prove Proposition 3.10.

Proof of Proposition 3.10. We recall from Lemma 7.1 the set $E(\varepsilon, C_0)$ for every $\varepsilon > 0$ and $C_0 \in (0, \frac{1}{r})$ given by

$$E(\varepsilon, C_0) = \left\{ \boldsymbol{X} \colon \theta_{\max}(\boldsymbol{X}) \ge \varepsilon \text{ and } \theta_{\min}(\boldsymbol{X}) \gtrsim N^{-\frac{2C_0 r}{1+C_0 r}} \right\}$$

Moreover, \mathcal{T}_E denotes the hitting time of the set $E = E(\varepsilon, C_0)$. For every $T > T_0 \gtrsim \frac{\log(N)}{(n+2)\gamma_0^2 \gamma_1 N^{\frac{1}{2(n+1)} \vee \frac{2C_0 r}{1-C_0 r}}}$, we have by the strong Markov property that

$$\begin{split} &\int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t \in [T_0,T]} \theta_{\min}(\boldsymbol{X}_t^{\beta}) \ge 1 - \varepsilon \right) d\mu_0(\boldsymbol{X}) \\ &\geq \inf_{\boldsymbol{X} \in E(\varepsilon,C_0)} \mathbb{Q}_{\boldsymbol{X}} \left(\inf_{t \in [T_0,T]} \theta_{\min}(\boldsymbol{X}_t^{\beta}) \ge 1 - \varepsilon \right) \times \int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_E \lesssim \frac{\log(N)}{(n+2)\gamma_0^2 \gamma_1 N^{\frac{1}{2(n+1)} \vee \frac{2C_0 r}{1-C_0 r}}} \right) d\mu_0(\boldsymbol{X}). \end{split}$$

We estimate the integral according to Lemma 7.1 since it holds that

$$\begin{split} &\int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{E} \gtrsim \frac{\log(N)}{(n+2)\gamma_{0}^{2}\gamma_{1}N^{\frac{1}{2(n+1)}\vee\frac{2C_{0}r}{1-C_{0}r}}} \right) d\mu_{0}(\boldsymbol{X}) \\ &\leq \mu_{0}(\mathcal{C}_{0}(n,\gamma_{0})^{c}) + \mu_{0}(\mathcal{C}_{1}'(\gamma_{1},\gamma_{2})^{c}) \\ &+ \int_{\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}} \left(\mathcal{T}_{E} \lesssim \frac{\log(N)}{(n+2)\gamma_{0}^{2}\gamma_{1}N^{\frac{1}{2(n+1)}\vee\frac{2C_{0}r}{1-C_{0}r}}} \right) \mathbf{1} \{ \mathcal{C}_{0}(n,\gamma_{0}) \cap \mathcal{C}_{1}'(\gamma_{1},\gamma_{2}) \} d\mu_{0}(\boldsymbol{X}) \end{split}$$

Therefore, it remains to estimate the probability of strong recovery of all eigenvalues starting from $X_0 \in E(\varepsilon, C_0)$. It follows from Corollary 4.9 and Weyl's inequality that for every $i \in [r]$,

$$\left|\theta_i(t) - \mu_i \left(\boldsymbol{G}(0) + 4\beta\sqrt{M}\lambda^2 \int_0^t \boldsymbol{G}(s)(\boldsymbol{I}_r - \boldsymbol{G}(s))ds\right)\right| \le \|\boldsymbol{M}_t^{\boldsymbol{G}}\|_{\mathrm{op}} + r \int_0^t \|\boldsymbol{L}_{0,\beta}\boldsymbol{G}(s)\|_{\mathrm{op}}ds,$$

where $\mu_i(\mathbf{A})$ stands for the *i*th eigenvalue of \mathbf{A} . Now, for every $\varepsilon' > \varepsilon > 0$ such that $\varepsilon \leq \frac{1}{2}$, according to Lemma 4.13 and especially to (4.18), we have that

$$\mu_i \left(\boldsymbol{G}(0) + 4\beta\sqrt{M}\lambda^2 \int_0^t \boldsymbol{G}(s)(\boldsymbol{I}_r - \boldsymbol{G}(s))ds \right) \ge \theta_i(0) + 4\beta\sqrt{M}\lambda^2 \int_0^t \theta_{\min}(s)\left(1 - \theta_{\min}(s)\right)ds,$$

for every $t \leq \mathcal{T}_{\varepsilon'}^{(\max)} \wedge \mathcal{T}_{\varepsilon}^{(\min)}$. In particular, we assume that $\theta_{\max} \in (\varepsilon, \varepsilon')$. We then obtain that

$$\theta_{\min}(t) \ge \theta_{\min}(0) + 4\beta\sqrt{M}\lambda^2 \int_0^t \theta_{\min}(s)ds - \|M_t^G\|_{\text{op}} - r\int_0^t \|\boldsymbol{L}_{0,\beta}\boldsymbol{G}(s)\|_{\text{op}}ds$$

for $t \leq \mathcal{T}_{\varepsilon'}^{(\max)} \wedge \mathcal{T}_{\varepsilon}^{(\min)}$. According to (7.1), the operator norm of $L_{0,\beta}G$ is bounded by

$$\|\boldsymbol{L}_{0,\beta}\boldsymbol{G}\|_{\rm op} \le 2\|\boldsymbol{M}\|_{\rm op}\|L_{0,\beta}\boldsymbol{M}\|_{\rm op} + \frac{2r}{N} + \frac{2}{N}\|\boldsymbol{G}\|_{\rm op} \le 2r^2 \max_{i,j}\|L_{0,\beta}m_{ij}\|_{\infty} + \frac{2r(r+1)}{N}.$$

We recall from Lemma (4.3) that for every $n \ge 1$, there exists a constant $\Lambda = \Lambda(p, n, \{\lambda_i\}_{i=1}^2)$ such that $\|L_{0,\infty}m_{ij}\|_{\infty} \le \Lambda$ with \mathbb{P} -probability at least $1 - \exp(-KN)$. It then follows that

$$r \int_0^t \|\boldsymbol{L}_{0,\beta} \boldsymbol{G}(s)\|_{\text{op}} ds \le 4C_0 r \beta \sqrt{M} \lambda^2 \int_0^t \theta_{\min}(s) ds,$$

since

$$4C_0 r\beta\sqrt{M}\lambda^2 \int_0^t \theta_{\min}(s)ds \ge 4C_0 r\beta\sqrt{M}\lambda^2 t N^{-\frac{2C_0 r}{1+C_0 r}}$$
$$> 4rt(n+2)\gamma_0^2\gamma_1 N^{\frac{4C_0^2 r^2}{1-C_0^2 r^2}}$$
$$> 2rt\left(r^2\Lambda + \frac{r(r+1)}{N}\right) \ge \|\boldsymbol{L}_{0,\beta}\boldsymbol{G}\|_{\text{op}}.$$

Furthermore, since $\int_0^t \|\boldsymbol{M}(s)\|_{\text{op}}^2 ds \leq r^2 t$, it follows from Lemma 4.10 with $f(t) = r^2 t$ that

$$\sup_{\boldsymbol{X}\in\mathcal{M}_{N,r}} \mathbb{Q}_{\boldsymbol{X}}\left(\sup_{t\in[0,T]} \|\boldsymbol{M}_{t}^{\boldsymbol{G}}\|_{\mathrm{op}} \geq \gamma\right) \leq r(r+1)e^{-N\gamma^{2}/(4r^{4}T)}$$

Combining the above results yields

$$\theta_{\min}(t) \ge \frac{1}{2} N^{-\frac{2C_0 r}{1+C_0 r}} + 4(1-C_0 r)\beta \sqrt{M}\lambda^2 \int_0^t \theta_{\min}(s) ds,$$

for every $t \leq \mathcal{T}_{\varepsilon'}^{(\max)} \wedge \mathcal{T}_{\varepsilon}^{(\min)}$, with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - r(r+1)\exp(-N^{1-\frac{4C_0r}{1+C_0r}}/(8r^4\mathcal{T}_{\varepsilon}^{(\min)}))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. Grönwall's inequality from Lemma 4.11 implies that

$$\theta_{\min}(t) \ge \frac{1}{2} N^{-\frac{2C_0 r}{1+C_0 r}} \exp(4(1-C_0 r)\beta \sqrt{M}\lambda^2 t),$$

for every $t \leq \mathcal{T}_{\varepsilon'}^{(\max)} \wedge \mathcal{T}_{\varepsilon}^{(\min)}$, with $\mathbb{Q}_{\mathbf{X}}$ -probability at least $1 - r(r+1)\exp(-N^{1-\frac{4C_0r}{1+C_0r}}/(8r^4\mathcal{T}_{\varepsilon}^{(\min)}))$ and with \mathbb{P} -probability at least $1 - \exp(-KN)$. A simple computation shows that

$$\mathcal{T}_{\varepsilon}^{(\min)} \lesssim \frac{C_0^2 \log(N)}{(n+2)\gamma_0^2 \gamma_1 N^{\frac{1}{2(n+1)} \vee \frac{2C_0 r}{1-C_0 r}}}.$$

Showing strong recovery of θ_{\min} from weak recovery is straightforward and we direct the reader to the proof of Lemma 5.2.

Appendix A. Concentration properties of the uniform measure on the Stiefel manifold

The concentration and anti-concentration properties of the uniform measure $\mu_{N\times r}$ on $\mathcal{M}_{N,r}$ were studied in [5]. In particular, there we proved that $\mu_{N\times r}$ satisfies both Conditions 1 and 2. We now need to present that Condition 1 is satisfied for the eigenvalues.

Lemma A.1. If $\mathbf{X} \sim \mu_{N \times r}$ and $\mathbf{G} = \mathbf{M}^{\top} \mathbf{M}$ with $\mathbf{M} = \frac{1}{N} \mathbf{V}^{\top} \mathbf{X}$, then there exist constants C(r), C'(r) and c(r) > 0 such that for every t > 0,

$$\mu_{N \times r} \left(\theta_{\max}(\boldsymbol{G}) > \frac{r}{N} + t \right) \le C(r) \exp(-c(r)Nt),$$

and for every $t \in (0, \frac{1}{2})$,

$$\mu_{N \times r} \left(\theta_{\min}(\boldsymbol{G}) < \frac{t}{N} \right) \le C'(r)\sqrt{t} + C(r) \exp\left(-c(r)N^{1/4}t\right).$$

Proof. Since \boldsymbol{M} is a square matrix, it is equivalent to study the operator norm of $\boldsymbol{M}\boldsymbol{M}^{\top}$. Using the characterization $\boldsymbol{X} = \boldsymbol{Z} \left(\frac{1}{N} \boldsymbol{Z}^{\top} \boldsymbol{Z}\right)^{-1/2}$ for $\boldsymbol{X} \sim \mu_{N \times r}$ given by [22], we then have that

$$\boldsymbol{M}\boldsymbol{M}^{\top} = \left(\frac{1}{N}\boldsymbol{V}^{\top}\boldsymbol{Z}\right) \left(\frac{1}{N}\boldsymbol{Z}^{\top}\boldsymbol{Z}\right)^{-1} \left(\frac{1}{N}\boldsymbol{Z}^{\top}\boldsymbol{V}\right).$$

We now introduce $\tilde{M} = \frac{1}{\sqrt{N}} V^{\top} Z$ and note that $\tilde{M} \in \mathbb{R}^{r \times r}$ has i.i.d. standard normal entries. We therefore rewrite the previous identity as

$$oldsymbol{M}oldsymbol{M}^{ op} = rac{1}{N} ilde{oldsymbol{M}} ilde{oldsymbol{M}}^{ op} + \left(rac{1}{N}oldsymbol{V}^{ op}oldsymbol{Z}
ight) \left(\left(rac{1}{N}oldsymbol{Z}^{ op}oldsymbol{Z}
ight)^{-1} - oldsymbol{I}_r
ight) \left(rac{1}{N}oldsymbol{Z}^{ op}oldsymbol{V}
ight).$$

In particular, we have that

$$\begin{split} & \mu_{N \times r} \left(\|\boldsymbol{M}\boldsymbol{M}^{\top} - \frac{r}{N} \boldsymbol{I}_{r}\|_{\mathrm{op}} > t \right) \\ &= \mu_{N \times r} \left(\|\frac{1}{N} \tilde{\boldsymbol{M}} \tilde{\boldsymbol{M}}^{\top} - \frac{r}{N} \boldsymbol{I}_{r} + \frac{1}{N} (\boldsymbol{V}^{\top} \boldsymbol{Z}) \left(\left(\frac{1}{N} \boldsymbol{Z}^{\top} \boldsymbol{Z}\right)^{-1} - \boldsymbol{I}_{r} \right) \frac{1}{N} (\boldsymbol{Z}^{\top} \boldsymbol{V})\|_{\mathrm{op}} > t \right) \\ &\leq \mu_{N \times r} \left(\|\frac{1}{N} \tilde{\boldsymbol{M}}^{\top} \tilde{\boldsymbol{M}} - \frac{r}{N} \boldsymbol{I}_{r}\|_{\mathrm{op}} > \frac{t}{2} \right) + \mu_{N \times r} \left(\|\frac{1}{N} (\boldsymbol{V}^{\top} \boldsymbol{Z}) \left(\left(\frac{1}{N} \boldsymbol{Z}^{\top} \boldsymbol{Z}\right)^{-1} - \boldsymbol{I}_{r} \right) \frac{1}{N} (\boldsymbol{Z}^{\top} \boldsymbol{V})\|_{\mathrm{op}} > \frac{t}{2} \right) \\ &\leq \mu_{N \times r} \left(\|\frac{1}{N} \tilde{\boldsymbol{M}}^{\top} \tilde{\boldsymbol{M}} - \frac{r}{N} \boldsymbol{I}_{r}\|_{\mathrm{op}} > \frac{t}{2} \right) + \mu_{N \times r} \left(\|\frac{1}{N} \tilde{\boldsymbol{M}}^{\top} \tilde{\boldsymbol{M}}\|_{\mathrm{op}} \| \left(\frac{1}{N} \boldsymbol{Z}^{\top} \boldsymbol{Z}\right)^{-1} - \boldsymbol{I}_{r}\|_{\mathrm{op}} > \frac{t}{2} \right). \end{split}$$

According to equation (A.1) in [5], we find that

$$\mu_{N \times r} \left(\|\frac{1}{N} \tilde{\boldsymbol{M}}^{\top} \tilde{\boldsymbol{M}} - \frac{r}{N} \boldsymbol{I}_{r} \|_{\text{op}} > \frac{t}{2} \right) = \mu_{N \times r} \left(\|\frac{1}{r} \tilde{\boldsymbol{M}}^{\top} \tilde{\boldsymbol{M}} - \boldsymbol{I}_{r} \|_{\text{op}} > \frac{Nt}{2r} \right)$$
$$\leq 2 \exp \left(-r \left(\frac{1}{c} \left(\frac{Nt}{2r} \right)^{1/2} - 1 \right)^{2} \right),$$

The remaining term is handled in a similar fashion to what is done in the proof of [5, Lemma A.1].

For the lower bound on the smallest eigenvalue of G, we first note that by Weyl's inequality we have that

$$heta_{\min}(\boldsymbol{G}) \geq \lambda_{\min}\left(\frac{1}{N}\tilde{\boldsymbol{M}}\tilde{\boldsymbol{M}}^{\top}\right) - \lambda_{\max}\left(\left(\frac{1}{N}\boldsymbol{V}^{\top}\boldsymbol{Z}\right)\left(\left(\frac{1}{N}\boldsymbol{Z}^{\top}\boldsymbol{Z}\right)^{-1} - \boldsymbol{I}_{r}\right)\left(\frac{1}{N}\boldsymbol{Z}^{\top}\boldsymbol{V}\right)\right).$$

Letting a and b denote the two terms in the r.h.s. of the above expression, we have for every t > 0,

$$\mathbb{P}\left(\theta_{\min}(\boldsymbol{G}) \geq \frac{t}{N}\right) \geq \mathbb{P}\left(a-b \geq \frac{t}{N}\right) \geq \mathbb{P}\left(a \geq \frac{2t}{N} \cap b \leq \frac{t}{N}\right),$$

so that

$$\mathbb{P}\left(\theta_{\min}(\boldsymbol{G}) < \frac{t}{N}\right) \leq \mathbb{P}\left(a < \frac{2t}{N} \cup b > \frac{t}{N}\right) \leq \mathbb{P}\left(a < \frac{2t}{N}\right) + \mathbb{P}\left(b > \frac{t}{N}\right)$$

We first focus on $\mathbb{P}(a < 2t/N)$. The matrix $\tilde{M}\tilde{M}^{\top}$ is a Wishart matrix, thus the distribution of its smallest eigenvalue is given by Theorem 3.1 of [26]. If $f_{\theta_{\min}}$ denotes its density function, for every

 $t \in (0, 1)$, there exists a constant C'(r) such that

$$f_{\theta_{\min}}(t) \le C'(r) \frac{1}{\sqrt{t}}$$

Therefore, for every $t \in (0, 1/2)$ it follows that

$$\mathbb{P}\left(\lambda_{\min}\left(\frac{1}{N}\tilde{\boldsymbol{M}}\tilde{\boldsymbol{M}}^{\top}\right) \leq \frac{2t}{N}\right) \leq 2C'(r)\sqrt{t}.$$

Turning to the second term $\mathbb{P}(b > t/N)$, we see that

$$\lambda_{\max}\left(\left(\frac{1}{N}\boldsymbol{V}^{\top}\boldsymbol{Z}\right)\left(\left(\frac{1}{N}\boldsymbol{Z}^{\top}\boldsymbol{Z}\right)^{-1}-\boldsymbol{I}_{r}\right)\left(\frac{1}{N}\boldsymbol{Z}^{\top}\boldsymbol{V}\right)\right)$$

$$\leq \|\frac{1}{N}\boldsymbol{V}^{\top}\boldsymbol{Z}\|_{\mathrm{op}}^{2}\|\left(\frac{1}{N}\boldsymbol{Z}^{\top}\boldsymbol{Z}\right)^{-1}\|_{\mathrm{op}}\|\frac{1}{N}\boldsymbol{Z}^{\top}\boldsymbol{Z}-\boldsymbol{I}_{r}\|_{\mathrm{op}}.$$

For every $t_1, t_2, t_3 \ge 0$ we then have that

$$\mathbb{P}\left(\|\frac{1}{N}\boldsymbol{V}^{\top}\boldsymbol{Z}\|_{\mathrm{op}}^{2} > t_{1}\right) = \mathbb{P}\left(\|\frac{1}{N}\boldsymbol{V}^{\top}\boldsymbol{Z}\|_{\mathrm{op}} > \sqrt{t_{1}}\right) \leq 2\exp\left(-\left(\frac{\sqrt{t_{1}N}}{C} - 2\sqrt{r}\right)^{2}\right),$$
$$\mathbb{P}\left(\|\left(\frac{1}{N}\boldsymbol{Z}^{\top}\boldsymbol{Z}\right)^{-1}\|_{\mathrm{op}} > \frac{1}{t_{2}}\right) \leq 2\exp\left(-\left(\frac{\sqrt{N}}{C}(1 - t_{2}) - \sqrt{r}\right)^{2}\right),$$
$$\mathbb{P}\left(\|\frac{1}{N}\boldsymbol{Z}^{\top}\boldsymbol{Z} - \boldsymbol{I}_{r}\|_{\mathrm{op}} > \sqrt{t_{3}}\right) \leq 2\exp\left(-\left(\frac{\sqrt{t_{3}N}}{C} - \sqrt{r}\right)^{2}\right).$$

Choosing $t_1 = \frac{\sqrt{t}}{\sqrt{2}N^{3/4}}$, $\sqrt{t_3} = \frac{\sqrt{t}}{\sqrt{2}N^{1/4}}$ and $t_2 = \frac{1}{2}$, we deduce that there exist constants C(r), c(r) such that $\mathbb{P}(b > t/N) \le C(r) \exp\left(-c(r)N^{1/4}t\right)$.

$$(b > t/N) \le C(r) \exp\left(-c(r)N^{1/4}t\right).$$

This completes the proof.

It therefore remains to prove the following concentration estimate which ensures that $\mu_{N\times r}$ weakly satisfies Condition 0 at level ∞ .

Lemma A.2. For every T > 0 and every $1 \le i, j \le r$, there exist $C_1, C_2 > 0$ depending only on $p, r, \{\lambda_i\}_{i=1}^r$ such that for every $\gamma > 0$,

$$\mu_{N \times r} \left(\sup_{t \le T} |e^{tL_{0,\beta}} L_{0,\beta} m_{ij}^{(N)}(\boldsymbol{X})| \ge \gamma \right) \le C_1 N T \exp\left(-C_2 \gamma^2 N\right),$$

with \mathbb{P} -probability at least $1 - \mathcal{O}(e^{-KN})$.

The proof of the Lemma A.2 follows similar arguments to those made for gradient flow in our companion [5]. In turn, we use similar ideas to those used to prove [8, Theorem 6.2]. In the following, we let $\hat{\mathbf{X}}_t$ denote the Langevin dynamics generated by $L_{0,\beta}$ (see (1.10)). The first step is to establish the rotational invariance properties of this dynamics. For every $\mathbf{X} \in \mathcal{M}_{N,r}$ we let $R_{\mathbf{X}}^N: T_{\mathbf{X}}\mathcal{M}_{N,r} \to \mathcal{M}_{N,r}$ denote the polar retraction defined by

$$R_{\boldsymbol{X}}^{N}(\boldsymbol{U}) = (\boldsymbol{X} + \boldsymbol{U}) \left(\boldsymbol{I}_{r} + \frac{1}{N} \boldsymbol{U}^{\top} \boldsymbol{U} \right)^{-1/2}$$

which verifies $\left(R_{\boldsymbol{X}}^{N}(\boldsymbol{U})\right)^{\top}R_{\boldsymbol{X}}^{N}(\boldsymbol{U}) = N\boldsymbol{I}_{r}.$

Lemma A.3. For every $t \ge 0$, if $\hat{\mathbf{X}}_0 \sim \mu_{N \times r}$, then $\hat{\mathbf{X}}_t$ and $R_{\mathbf{X}}^N \left(\nabla H_0(\hat{\mathbf{X}}_t) \right)$ are elements of $\mathcal{M}_{N,r}$ that are invariant under left rotations.

Proof. A similar argument to that used in proving [5, Lemma A.5] applies, upon noting that Brownian motion is invariant under left rotations. \Box

We now provide the proof of Lemma A.2.

Proof of Lemma A.2. Let $\mathbb{E}_{\mathbb{Q}_{X}}$ denote the expectation with respect to the law of the Langevin dynamics \hat{X}_{t} generated by $L_{0,\beta}$ started at X. By definition of the semigroup of the noise process, it holds for every $1 \leq i, j \leq r$,

$$e^{tL_0}L_0m_{ij}^{(N)}(\hat{X}_0) = \mathbb{E}_{\mathbb{Q}_{\hat{X}_0}}\left[L_{0,\beta}m_{ij}^{(N)}(\hat{X}_t)\right].$$

By definition (1.10) and of (1.9) we have that

$$L_{0,\beta}m_{ij}^{(N)}(\hat{\boldsymbol{X}}_t) = \Delta m_{ij}^{(N)}(\hat{\boldsymbol{X}}_t) - \frac{\beta}{N} \langle \nabla H_0(\hat{\boldsymbol{X}}_t), [\boldsymbol{v}_i]_j \rangle = -\frac{N-1}{N^2} \langle \hat{\boldsymbol{X}}_t, [\boldsymbol{v}_i]_j \rangle - \frac{\beta}{N} \langle \nabla H_0(\hat{\boldsymbol{X}}_t), [\boldsymbol{v}_i]_j \rangle,$$

where $[\boldsymbol{v}_i]_j = [\boldsymbol{0}, \dots, \boldsymbol{0}, \boldsymbol{v}_i, \boldsymbol{0}, \dots, \boldsymbol{0}] \in \mathbb{R}^{N \times r}$ denotes the matrix with all zero columns except for the *j*th column, which is \boldsymbol{v}_i . Let $\boldsymbol{H} \in \mathbb{R}^{r \times r}$ be a matrix sampled from the Haar measure on O(r). For every $t \geq 0$, we introduce

$$\begin{aligned} \boldsymbol{Z}_t &= R_{\hat{\boldsymbol{X}}_0}^N (\nabla H_0(\hat{\boldsymbol{X}}_t)) \boldsymbol{H}, \\ \tilde{\boldsymbol{Z}}_t &= \hat{\boldsymbol{X}}_t \boldsymbol{H}. \end{aligned}$$

According to Lemma A.3 and by definition of the Haar measure, both Z_t and \hat{Z}_t belong to $\mathcal{M}_{N,r}$ and are invariant under left and right rotations. Since this property uniquely characterizes the invariant measure on $\mathcal{M}_{N,r}$, we deduce that both Z_t and \hat{Z}_t are distributed according to $\mu_{N\times r}$. The remainder of the proof follows a similar approach to that used in [5, Lemma A.4]. Specifically, we establish uniform-in-time deviation inequalities over one-dimensional projections of Z_t and \hat{Z}_t by discretizing the interval [0, T]and relating these inequalities to the quantities appearing in $L_{0,\beta}m_{ij}^{(N)}(\hat{X}_t)$. In light of Lemma 4.2, for constants $\Gamma = \Gamma(p, \{\lambda_i\}_{i=1}^r, r)$ and $K = K(p, \{\lambda_i\}_{i=1}^r, r)$ we have that the event

$$\mathcal{E} = \{ \|H_0\|_{\mathcal{G}^2} \ge \Gamma N \}$$

holds with \mathbb{P} -probability at most exp(-KN). We direct the reader to Definition 4.1 for a definition of the \mathcal{G}^n -norm on $\mathcal{M}_{N,r}$. According to Definition 4.1, we also notice that, under the event \mathcal{E}^c ,

$$\|\nabla^2 H_0(\boldsymbol{X})\|_{\mathrm{op}}\|_{\infty} \leq \Gamma.$$

For every $0 \le s \le t$ and every $1 \le i, j \le r$, we now focus on controlling the increments

$$rac{1}{N} \langle ilde{m{Z}}_t - ilde{m{Z}}_s, [m{v}_i]_j
angle \quad ext{and} \quad rac{1}{N} \langle m{Z}_t - m{Z}_s, [m{v}_i]_j
angle$$

Defining the function $\tilde{m}_{ij}^{(N)}(\hat{X}_t) \coloneqq \frac{1}{N} \langle \tilde{\mathbf{Z}}_t, [\boldsymbol{v}_i]_j \rangle$, Itô's formula gives

$$\frac{1}{N} \langle \tilde{\boldsymbol{Z}}_t - \tilde{\boldsymbol{Z}}_s, [\boldsymbol{v}_i]_j \rangle = \frac{1}{N} \int_s^t \langle \boldsymbol{H}^\top [\boldsymbol{v}_i]_j, d\boldsymbol{B}_u \rangle + \int_s^t L_{0,\beta} \tilde{m}_{ij}^{(N)}(\hat{\boldsymbol{X}}_u) d\boldsymbol{u}_j \rangle$$

where we remind that B_t is a Brownian motion on the normalized Stiefel manifold $\mathcal{M}_{N,r}$. Using the Burkholder-Davis-Gundy inequality (see e.g. [40, Theorem 5.16]) to bound the martingale part, we obtain

$$\left| \mathbb{E}_{\mathbb{Q}_{\hat{\mathbf{X}}_0}} \left[\frac{1}{N} \langle \tilde{\mathbf{Z}}_t - \tilde{\mathbf{Z}}_s, [\mathbf{v}_i]_j \rangle \right] \right| \le r\sqrt{t-s} + \|L_{0,\beta} \tilde{m}_{ij}^{(N)}\|_{\infty} (t-s).$$

From Lemma 4.3 on the ladder relations, the norm $\|L_{0,\beta}\tilde{m}_{ij}^{(N)}\|_{\infty}$ is bounded by a constant depending on $\beta, p, \{\lambda_i\}_{i\leq r}$. Turning to the second increment, we introduce the shorthand $U_t = \nabla H_0\left(\hat{X}_t\right)$. Recall the following identity from [5, Lemma A.4]:

$$\|\mathbf{Z}_t - \mathbf{Z}_s\|_{\mathrm{F}} \leq \frac{r}{\sqrt{N}} \|\hat{\mathbf{X}}_0 + \mathbf{U}_t\|_{\mathrm{F}} \|\mathbf{U}_t^\top \mathbf{U}_t - \mathbf{U}_s^\top \mathbf{U}_s\|_{\mathrm{F}}^{1/2} + r \|\mathbf{U}_t - \mathbf{U}_s\|_{\mathrm{F}}.$$

Applying Itô's formula to the vector field \boldsymbol{U}_t , we obtain that

$$\boldsymbol{U}_t - \boldsymbol{U}_s = \int_s^t \nabla^2 H_0(d\boldsymbol{B}_u, .) + \int_s^t L_{0,\beta} \nabla H_0(\hat{\boldsymbol{X}}_u) du,$$

where we used the shorthand $L_{0,\beta}\nabla H_0(\hat{X}_u)$ to denote the vector field on $\mathcal{M}_{N,r}$ whose coordinates are given by, for any $1 \leq k \leq N, 1 \leq \ell \leq r$,

$$[L_{0,\beta}\nabla H_0(\hat{\boldsymbol{X}}_u)]_{k\ell} = L_{0,\beta}[\nabla H_0(\hat{\boldsymbol{X}}_u)]_{k,\ell}.$$

We then reach

$$\mathbb{E}_{\mathbb{Q}_{\tilde{\mathbf{X}}_0}}\left[\|\boldsymbol{U}_t - \boldsymbol{U}_s\|_{\mathrm{F}}\right] \leq \left(\int_s^t \|\nabla^2 H_0\|_{\mathrm{F}}^2 ds\right)^{1/2} + \|L_{0,\beta}\nabla H_0\|_{\infty}(t-s)$$
$$\leq \sqrt{Nr}\Gamma\sqrt{t-s} + \|L_{0,\beta}\nabla H_0\|_{\infty}(t-s),$$

where we used Burkholder-Davis-Gundy inequality again to bound the martingale part. Now, to bound $||L_{0,\beta}\nabla H_0||_{\infty}$, we write

$$\begin{aligned} \|L_{0,\beta}\nabla H_0\|_{\infty} &\leq \||\nabla^2 H_0|_{\mathrm{op}}\|_{\infty} \||\nabla H_0|_{\mathrm{op}}\|_{\infty} + \sqrt{Nr} \left(\sup_{1 \leq i \leq N, 1 \leq j \leq r} \|\Delta \frac{\partial H_0}{\partial x_{ij}}\|_{\infty}^2\right)^{1/2} \\ &\leq \Gamma^2 \sqrt{N} + \Gamma N \sqrt{r}, \end{aligned}$$

where we used the ladder relation (4.1) in the last line. Similarly,

$$\boldsymbol{U}_{t}^{\top}\boldsymbol{U}_{t} - \boldsymbol{U}_{s}^{\top}\boldsymbol{U}_{s} = 2\int_{s}^{t}\nabla^{2}H_{0}\left(d\boldsymbol{B}_{u},\nabla H_{0}(\hat{\boldsymbol{X}}_{u})\right) + 2\int_{s}^{t}L_{0,\beta}\nabla H_{0}(\hat{\boldsymbol{X}}_{u})^{\top}\nabla H_{0}(\hat{\boldsymbol{X}}_{u})d\boldsymbol{u}_{s}$$

so that, applying Burkholder-Davis-Gundy inequality again for the martingale part, we reach

$$\mathbb{E}_{\mathbb{Q}_{\mathbf{X}_0}}\left[\|\boldsymbol{U}_t^{\top}\boldsymbol{U}_t - \boldsymbol{U}_s^{\top}\boldsymbol{U}_s\|_{\mathrm{F}}\right] \leq 2\Gamma^2 \sqrt{N}\sqrt{t-s} + 2\|L_{0,\beta}\nabla H_0(\hat{\boldsymbol{X}}_u)^{\top}\nabla H_0(\hat{\boldsymbol{X}}_u)\|_{\infty}(t-s).$$

Finally, to bound $||L_{0,\beta}\nabla H_0(\hat{X}_u)^{\top}\nabla H_0(\hat{X}_u)||_{\infty}$, we use the ladder relation (4.1) again to control the Laplacian (applied coordinate-wise):

$$\begin{aligned} \|L_{0,\beta} \nabla H_0(\hat{\boldsymbol{X}}_u)^\top \nabla H_0(\hat{\boldsymbol{X}}_u)\|_{\infty} &\leq \|\nabla^2 H_0(\nabla H_0, \nabla H_0)\|_{\infty} + Nr \left(\sup_{1 \leq i \leq N, 1 \leq j \leq r} \|\Delta \left(\frac{\partial H_0}{\partial x_{ij}}\right)^2\|_{\infty} \right)^{1/2} \\ &\leq \Gamma^3 N + \Gamma Nr \end{aligned}$$

To conclude the proof, we follow the approach outlined in the proof of [5, Lemma A.4] for gradient flow dynamics. Specifically, we discretize the interval [0,T] with a sufficiently fine partition to handle the polynomial increments. We then apply exponential concentration inequalities for one-dimensional projections of matrices sampled from the invariant measure on $\mathcal{M}_{N,r}$ and conclude with a union bound

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