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BOGOMOLOV-GIESEKER INEQUALITY FOR LOG TERMINAL KÄHLER THREEFOLDS

by

Henri Guenancia & Mihai Păun

with an appendix by Frédéric Campana, Andreas Höring and Thomas Peternell

Abstract. — In this article we are concerned with two main themes revolving around compact Kähler spaces with log terminal singularities. The first one is the orbifold version of the Bogomolov-Gieseker inequality for stable \mathbb{Q} -sheaves on threefolds while the second one has to do with the construction of canonical metrics on such spaces and their properties.

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1. Introduction

1.1. Bogomolov-Gieseker inequality. — Let (X, ω_X) be a compact Kähler manifold of dimension n and let \mathcal{F} be a vector bundle of rank r which is stable with respect to the Kähler class $[\omega_X]$. By Donaldson-Uhlenbeck-Yau theorem [Don85, UY86], \mathcal{F} admits an Hermite-Einstein metric with respect to ω_X , hence the Bogomolov-Gieseker discriminant

$$\Delta(\mathcal{F}) := 2rc_2(\mathcal{F}) - (r-1)c_1(\mathcal{F})^2$$

satisfies

$$(1.1) \quad \Delta(\mathcal{F}) \cdot [\omega_X]^{n-2} \geq 0.$$

Moreover, the equality case happens if and only if \mathcal{F} is projectively flat. The inequality (1.1) was generalized by Bando and Siu [BS94] to the case where \mathcal{F} is merely a coherent reflexive sheaf as they proved the existence of an "admissible" Hermite-Einstein metric

for \mathcal{F} on the locus where it is locally free. A simpler, alternative argument amounts to appealing to Rossi's desingularization result [Ros68] and using the fact that stability is an open condition. Note that when X is projective and $[\omega_X] = c_1(H)$ is the class of an ample line bundle, then one is reduced to the statement on a surface for which the inequality was first proved, cf [MP97] and references therein.

In this article, we are interested in generalizations of (1.1) when X is allowed to be singular. A thorny issue that arises when dealing with singular varieties is the definition of Chern classes of coherent sheaves. More precisely, several notions exist that are not equivalent, already when $\mathcal{F} = T_X$. When X is normal and has finite quotient singularities in codimension two (i.e. X is an orbifold away from a set of codimension at least three), one can rely on the notions of orbifold Chern classes to define the first two Chern numbers

$$c_1(\mathcal{F})^2 \cdot [\omega_X]^{n-2} \quad \text{and} \quad c_2(\mathcal{F}) \cdot [\omega_X]^{n-2}$$

in a satisfying way provided that \mathcal{F} is an orbifold bundle in codimension two - sometimes referred to as \mathcal{F} being a \mathbb{Q} -sheaf. This goes back to Mumford when X is projective and $[\omega_X] = c_1(L)$ for some line bundle L [Mum83] relying on the existence of suitable Galois covers while Graf and Kirschner [GK20] gave a purely analytic definition which they showed coincides with Mumford's when X is algebraic and $[\omega_X] \in \text{NS}_{\mathbb{Q}}(X)$.

A folklore result established by [GKKP11] asserts that normal complex varieties with log terminal singularities have quotient singularities in codimension two, hence it makes sense to consider the first two Chern numbers associated to a \mathbb{Q} -sheaf on such a variety. Our main theorem goes as follows.

Theorem A. — *Let (X, ω_X) be a compact Kähler variety of dimension three with log terminal singularities. Let \mathcal{F} be a reflexive \mathbb{Q} -sheaf on X which is stable with respect to $[\omega_X]$. Then the Bogomolov-Gieseker inequality holds for \mathcal{F} , i.e.*

$$\Delta(\mathcal{F}) \cdot [\omega_X] \geq 0.$$

In the equality case, $\mathcal{F}|_{X_{\text{reg}}}$ is a projectively flat vector bundle.

Theorem A confirms a conjecture by Campana-Höring-Peternell. Actually the main motivation for Theorem A is the proof of the abundance conjecture for Kähler threefolds. We refer to [CHP23], and more particularly to the appendix of this paper for a complete discussion of the relevance of our result in this framework.

In the equality case, we also get that the \mathbb{P}^{r-1} bundle $\mathbb{P}(\mathcal{F}|_{X_{\text{reg}}})$ extends to a locally trivial projective bundle after a finite quasi-étale cover of X , cf Corollary 7.4.

• **Comparison with earlier work.**

◦ When X is smooth in codimension two (of any dimension), then the Chern numbers can be computed on a resolution of X and the Bogomolov-Gieseker inequality is a rather simple consequence of its version for manifolds, cf e.g. [CGG22, Proposition 3.4] and references therein.

◦ When X is projective and $[\omega_X] = c_1(H)$ for some ample divisor H , then one is reduced to the classic inequality for stable orbifold bundles on an orbifold surface.

◦ When X admits an orbifold resolution (e.g. when X is projective [LT19]), [CGN⁺23] have showed that the inequality holds, relying on the construction of singular HE metrics.

◦ When X is Kähler of any dimension, [Ou24] has showed the existence of a partial orbifold resolution, and hence the inequality, cf. [CGN⁺23].

◦ For closely related results and references we refer to the interesting recent articles [CW24] and [Che22].

• **A few words about the proof of Theorem A.** Our arguments are relying on results and ideas from [CGN⁺23] (see Section 5 of *loc. cit.*), i.e. using an "admissible" Hermite-Einstein metric for \mathcal{F} to compute its Chern numbers. The key difference is that here we are able to analyze further the geometric properties of (X, ω_θ) , where ω_θ is a singular Kähler metric on X having orbifold singularities in the complement of an open subset containing the finite set of non-quotient singularities of X , thanks to the key Proposition C below, itself relying on [GPSS22]. The existence of a metric $h_{\mathcal{F}}$ on \mathcal{F} which verifies the Hermite-Einstein equation with respect to ω_θ together with important estimates is a result from [CGN⁺23]. In addition, here we establish the crucial intersection properties of the current $\Delta(\mathcal{F}, h_{\mathcal{F}})$ which lead to the proof of Theorem A. Also, most of the new arguments we supply in the last section hold in higher dimension. The main reason why Theorem A is formulated and proved in dimension three only is explained in Remark 7.1. We refer to the few lines above of Section 7.2 for a more detailed plan of the proof of Theorem A.

1.2. Canonical metrics on Kähler spaces with log terminal singularities. — Let (X, ω_X) be a compact Kähler space with log terminal singularities. A natural question is whether one can construct positive currents of the form $\omega := \omega_X + dd^c \varphi$ that reflect the singularities of X . For instance, one would like ω to be an orbifold Kähler metric near any quotient singularity of X , in the sense that the pull-back of ω in any local, smooth uniformizing chart should be a Kähler metric. Coming up with explicit candidates ω seems to be a hard task, cf. Section 6 for a few comments and very particular results in this direction.

Since X is klt, K_X is a \mathbb{Q} -line bundle and given any smooth hermitian metric on K_X , one can construct a positive measure μ_h which is a smooth volume form on X_{reg} and puts no mass on X_{sing} , cf Section 2. Next, it has been showed by [EGZ09] that up to renormalizing μ_h , there is a unique positive current $\omega = \omega_X + dd^c \varphi$ with $\varphi \in \text{PSH}(X, \omega_X) \cap L^\infty(X)$ solving

$$(1.2) \quad \omega^n = \mu_h.$$

Moreover, $\omega|_{X_{\text{reg}}}$ is a Kähler metric. For instance, if $c_1(K_X) = 0$ and the Chern curvature of h vanishes, then $\omega|_{X_{\text{reg}}}$ is Ricci-flat.

In general, the geometry of (X_{reg}, ω) near X_{sing} remains quite mysterious, and the very first question to address is the behavior of ω near quotient singularities. The classification of klt surface singularities shows that they are all quotient singularities. Capitalizing on that result and using slicing arguments, [GKKP11] have proved that there exists a subset $Z \subset X$ of codimension at least three, and such that $X \setminus Z$ admits at most

quotient singularities. Our next main result shows that $\omega|_{X \setminus Z}$ is indeed an orbifold Kähler metric.

Theorem B. — *Let (X, ω_X) be a compact Kähler space with klt singularities, let h be a smooth hermitian metric on K_X and let $\omega := \omega_X + dd^c \varphi$ solve (1.2). Then, there exists a subset $Z \subset X$ of codimension at least three such that $X \setminus Z$ has at most quotient singularities and $\omega|_{X \setminus Z}$ is an orbifold Kähler metric.*

It is still unclear whether ω is an orbifold Kähler metric in the neighborhood of *any* quotient singularity. However, this holds when $\dim X = 3$ as we show in Corollary 5.3.

• **Comparison with earlier work.**

◦ When X is projective, ω has been showed to be an orbifold Kähler metric near any quotient singularity using the existence of an orbifold resolution [LT19]. We provide in Section 5.1 a much more direct proof of that fact relying only on Artin’s approximation theorem.

◦ Using the existence of a partial orbifold resolution [Ou24] in the analytic setting and the arguments in [LT19], one could also derive an alternative proof of Theorem B.

• **Connection with the work of Guo-Phong-Song-Sturm [GPSS22].** As explained above, very few general results are known about the geometry of (X_{reg}, ω) . Recently though, Guo-Phong-Song-Sturm studied the properties of Green kernels of compact Kähler manifolds in the impressive series of articles [GPSS22, GPSS23], which have their origin in [CC21, GPT23] and ultimately enabled them to derive diameter bounds for (X_{reg}, ω) . From our point of view, the strength of the results in *loc. cit.* is that they are established without any sort of assumption on the Ricci curvature, i.e. no C^2 bounds for the volume element is required, not even one-sided.

As we explain in Section 3, their estimates allow one to construct and finely control Green’s functions for the incomplete manifold (X_{reg}, ω) . However, a roadblock to further exploit these functions seems to be related to validity of the following property.

Property (P). X_{sing} admits a family of cut-off functions (ρ_δ) such that there exist a constants $\varepsilon > 0$ independent of δ satisfying

$$\limsup_{\delta \rightarrow 0} \int_X (|\nabla \rho_\delta|_\omega^{2+\varepsilon} + |\Delta_\omega \rho_\delta|^{1+\varepsilon}) \omega^n = 0.$$

One of our main technical results, which is at the heart of the proof of both Theorem A and Theorem B is the following twofold statement:

Proposition C. — *Let (X, ω, Z) as in Theorem B.*

(C.1) *Assume that Property (P) is satisfied. There exists $p > 0$ such that if a non-negative function $f \in C^2(X_{\text{reg}})$ satisfies $f + (\Delta_\omega f)_- \in L^p$, then $f \in L^\infty(X_{\text{reg}})$.*

(C.2) *Property (P) is satisfied locally near each point $x \in X_{\text{sing}} \setminus Z$.*

(C.1) is proved in Proposition 3.3 and relies crucially on [GPSS22]. Another application similar to (C.1) is provided in Proposition 4.7 and has to do with the strict positivity of ω across X_{sing} .

(C.2) is embedded in the proof of Theorem B (i.e. Theorem 5.2 in the body of the text) and the key input is that ω is strictly positive on $X \setminus Z$, cf Proposition 4.3. The latter proposition relies on an iterated use of Chern-Lu formula and the fact that the functorial resolution of singularities of $X \setminus Z$ has nonpositive discrepancies.

We believe that the results of [GPSS22] combined with the statements above will be useful for exploring further the geometry of compact Kähler spaces with log-terminal singularities.

• **A few words about the proof of Theorem B.** The proof of Theorem B follows from Proposition C, combined with techniques from MA theory. We proceed by approximation on a non-singular model of X (which is rather standard), but what makes the problem interesting here is that even if the Ricci curvature of ω is bounded from below by a negative multiple of a Kähler metric ω_X , this is not the case for the sequence of approximating metrics. Nevertheless, a careful choice of the approximating sequence combined with Chern-Lu inequality will establish the strict positivity of ω on compact sets contained in the complement of a set Z of codimension at least three. From this point on we work directly on $(X \setminus Z, \omega)$ and consider the squared norm f of a local orbifold holomorphic 1-form. It is easy to see that $f \in L^1$. Since the Ricci curvature of ω is under control, one infers an inequality $\Delta_\omega f \geq -Cf$, which we then upgrade to find a similar lower bound for $\Delta_\omega f^\varepsilon$ where ε small enough so that one can apply Proposition C.

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2. Canonical metrics on log terminal Kähler spaces

Let X be a compact normal Kähler space.

Definition 2.1. — A Kähler form on X is a Kähler form ω_X on X_{reg} such that ω_X extends to a Kähler form on \mathbb{C}^N given any local embedding $X \xrightarrow[\text{loc}]{\hookrightarrow} \mathbb{C}^N$.

Assume that the rank one reflexive sheaf $mK_X := ((\det \Omega_X^1)^{\otimes m})^{**}$ is locally free for some integer $m \geq 1$. It makes sense to consider a smooth hermitian metric h on K_X as well as a local generator σ of mK_X . Then

$$\mu_h := i^{m^2} \frac{(\sigma \wedge \bar{\sigma})^{\frac{1}{m}}}{|\sigma|_{h^{\otimes m}}^{2/m}}$$

defines a positive measure on X_{reg} which is independent of the choice of m or σ . We extend it to X trivially. We have $\mu_h(X) < +\infty$ if and only if X has log terminal singularities. From now on, we assume that X has log terminal singularities.

Given a Kähler form ω_X , we set $V := \int_X \omega_X^n = [\omega_X]^n$ and rescale h so that $\mu_h(X) = V$. We consider the Monge-Ampère equation

$$(MA) \quad (\omega_X + dd^c \varphi)^n = \mu_h,$$

for $\varphi \in \text{PSH}(X, \omega_X) \cap L^\infty(X)$ normalized so that $\sup_X \varphi = 0$.

It was proved in [EGZ09] that (MA) admits a unique solution φ . Moreover, the positive current $\omega := \omega_X + dd^c \varphi$ induces a Kähler metric on X_{reg} , satisfying

$$(2.1) \quad \text{Ric } \omega = -\Theta_h(K_X).$$

The resolution of (MA) goes as follows. Let $\pi : \widehat{X} \rightarrow X$ be a log resolution of singularities of X and let $E = \sum_{i \in I} E_i$ be the exceptional divisor of π . One can write $K_{\widehat{X}} = \pi^* K_X + \sum a_i E_i$ for some $a_i \in \mathbb{Q}$ and the klt condition translates into $a_i > -1$ for every i . Let s_i be a section of $\mathcal{O}_{\widehat{X}}(E_i)$ cutting out E_i and let h_i be a smooth hermitian metric on $\mathcal{O}_{\widehat{X}}(E_i)$. We write $|s_i|^2 := |s_i|_{h_i}^2$. There exists a smooth volume form $dV_{\widehat{X}}$ on \widehat{X} such that the pull back measure $\pi^* \mu_h$ (which is well-defined) satisfies $\pi^* \mu_h = \prod_{i \in I} |s_i|^{2a_i} dV_{\widehat{X}}$. Solving (MA) is then equivalent to solving the degenerate complex Monge-Ampère equation

$$(2.2) \quad (\pi^* \omega_X + dd^c \psi)^n = \prod_{i \in I} |s_i|^{2a_i} dV_{\widehat{X}},$$

where $\pi^* \omega_X$ is semipositive but degenerate along the exceptional locus of π . Let $\omega_{\widehat{X}}$ be a Kähler form on \widehat{X} . For any $\varepsilon > 0$, the form $\pi^* \omega_X + \varepsilon \omega_{\widehat{X}}$ is Kähler, with volume $V_\varepsilon = [\pi^* \omega_X + \varepsilon \omega_{\widehat{X}}]^n$. By Yau's theorem [Yau75], there exists a unique function $\psi_\varepsilon \in \text{PSH}(\widehat{X}, \pi^* \omega_X + \varepsilon \omega_{\widehat{X}}) \cap C^\infty(\widehat{X})$ solution of

$$(2.3) \quad (\pi^* \omega_X + \varepsilon \omega_{\widehat{X}} + dd^c \psi_\varepsilon)^n = c_\varepsilon \prod_{i \in I} (|s_i|^2 + \varepsilon^2)^{a_i} dV_{\widehat{X}},$$

such that $\sup_{\widehat{X}} \psi_\varepsilon = 0$, where $c_\varepsilon = V_\varepsilon / V$. We set

$$\omega_\varepsilon := \pi^* \omega_X + \varepsilon \omega_{\widehat{X}} + dd^c \psi_\varepsilon.$$

A key feature of this approximation is that

$$(2.4) \quad \exists p > 1, C > 0; \quad \left\| \frac{\omega_\varepsilon^n}{dV_{\widehat{X}}} \right\|_{L^p(dV_{\widehat{X}})} \leq C.$$

One of the consequences of (2.4) is that there exists $C > 0$ independent of ε such that $\|\psi_\varepsilon\|_\infty \leq C$. Moreover, given any compact subset $K \Subset X_{\text{reg}}$ and any integer $k \geq 1$, there exists $C_{K,k}$ such that $\|\psi_\varepsilon\|_{C^k(K)} \leq C_{K,k}$. From this one deduces easily that ω_ε converges to a current $\pi^* \omega$ such that ω is a Kähler metric on X_{reg} and the convergence $\omega_\varepsilon \rightarrow \pi^* \omega$ is locally smooth on $\pi^{-1}(X_{\text{reg}})$.

3. Construction and properties of Green's functions

With the notations from the previous section, let $x \in \widehat{X}$ and let G_x^ε be the Green function associated to $(\widehat{X}, \omega_\varepsilon)$. That it, it is the unique solution of

$$\Delta_{\omega_\varepsilon} G_x^\varepsilon = \delta_x - \frac{1}{V_\varepsilon}, \quad \text{and} \quad \sup_{\widehat{X}} G_x^\varepsilon = -1.$$

The function G_x^ε belongs to $W^{1,p}(\widehat{X})$ for any $p < \frac{2n}{2n+1}$, and for any function $f \in C^\infty(X)$, one has

$$f(x) = \int_X f \omega_\varepsilon^n + \int_X G_x^\varepsilon \Delta_{\omega_\varepsilon} f \omega_\varepsilon^n.$$

It also implies that if f is merely in $W^{1,q}$, then $f(x) = \int_X f \omega_\varepsilon^n - \int_X \langle \nabla G_x^\varepsilon, \nabla f \rangle \omega_\varepsilon^n$.

The following result is a consequence of the recent breakthrough result recently obtained by Guo-Phong-Song-Sturm [[GPSS22](#)]:

Theorem 3.1. — *There exists $C, \delta > 0$ independent of ε such that for any $x \in \widehat{X}$*

$$\int_X (-G_x^\varepsilon)^{1+\delta} \omega_\varepsilon^n + \int_X |\nabla G_x^\varepsilon|^{1+\delta} \omega_\varepsilon^n \leq C.$$

Moreover, for any $\beta > 0$, there exists $C_\beta > 0$ independent of ε such that for any $x \in \widehat{X}$

$$\int_X \frac{|\nabla G_x^\varepsilon|^2}{(-G_x^\varepsilon)^{1+\beta}} \omega_\varepsilon^n \leq C_\beta.$$

As we have seen above, the Kähler metrics ω_ε converge locally smoothly to a Kähler metric $\pi^* \omega$ on $\widehat{X} \setminus E$. In what follows, we identify $\widehat{X} \setminus E$ with X_{reg} and $\pi^* \omega$ with ω . From now on, we fix $x \in X_{\text{reg}}$.

We have uniform $W_{\text{loc}}^{1,1+\delta}$ bounds for G_x^ε on the manifold X_{reg} in the usual sense. Therefore, up to using cut-off functions of X_{sing} and a standard diagonal argument, one can apply Kondrachov compactness theorem [[GT77](#), Theorem 7.22] in order to extract a subsequential limit $G_x \in L_{\text{loc}}^1(X_{\text{reg}})$ of G_x^ε when $\varepsilon \rightarrow 0$ for the L_{loc}^1 topology. Thanks to Fatou lemma, it will satisfy

$$(G.1) \quad \text{esssup}_{X_{\text{reg}}} G_x \leq -1.$$

$$(G.2) \quad \Delta_\omega G_x = \delta_x - \frac{1}{V} \text{ weakly on } X_{\text{reg}}.$$

$$(G.3) \quad \int_{X_{\text{reg}}} (-G_x)^{1+\delta} \omega^n + \int_{X_{\text{reg}}} |\nabla G_x|^{1+\delta} \omega^n \leq C$$

$$(G.4) \quad \text{For every } \beta > 0, \int_X \frac{|\nabla G_x|^2}{(-G_x)^{1+\beta}} \omega^n \leq C_\beta.$$

The constants C, C_β above are independent of the point $x \in X_{\text{reg}}$. Note that (G.2) relies heavily on the locally smooth convergence of ω_ε to ω on X_{reg} .

Lemma 3.2. — *The function G_x is smooth on $X_{\text{reg}} \setminus \{x\}$.*

Proof. — On $X_{\text{reg}} \setminus \{x\}$, we have $\Delta_\omega G_x = -\frac{1}{V}$. Since ω is a smooth Kähler metric on X_{reg} , we can apply [[Agm60](#), Theorem 7.1] inductively to G_x and its derivatives since $G_x \in L^{1+\delta}$ for some $\delta > 0$. \square

Proposition 3.3 (L^∞ -estimate). — Assume that X_{sing} admits a family of cut-off functions (ρ_δ) such that $\int_X (|\nabla \rho_\delta|_\omega^{2+\varepsilon} + |\Delta_\omega \rho_\delta|^{1+\varepsilon}) \omega^n$ converges to zero for some $\varepsilon > 0$, when $\delta \rightarrow 0$. There exists $p_0 > 0$ such that for any non-negative function $f \in C^2(X_{\text{reg}})$ satisfying

- $f \in L^{p_0}(X, \omega^n)$.
- $\Delta_\omega f \geq -g$ on X_{reg} for some $g \in L^{p_0}(X, \omega^n)$.

Then $\sup_{X_{\text{reg}}} f < +\infty$.

Proof. — Up to decreasing $\varepsilon > 0$, we can assume that

$$\int_X \left(|G_x|^{1+\varepsilon} + |\nabla G_x|^{1+\varepsilon} + \frac{|\nabla G_x|^2}{(-G_x)^{1+\varepsilon}} \right) \omega^n \leq C,$$

for some C independent of $x \in X_{\text{reg}}$. Let p (resp. p') be the conjugate exponent of $1 + \varepsilon$ (resp. $1 + \varepsilon/2$). We assume that $p_0 \geq \max\{p, 2p'\}$. Fix $x \in X_{\text{reg}}$. We claim that

$$(3.1) \quad f(x) \leq \int_X f \omega^n + \|G_x\|_{L^{1+\varepsilon}} \|g\|_{L^{p_0}}.$$

Let M be the RHS of (3.1). Since f is continuous on X_{reg} , the claim will follow if one can prove that for any smooth non-negative function μ with compact support on X_{reg} , we have

$$(3.2) \quad \int_X \mu(x) f(x) \omega^n \leq M \int_X \mu(x) \omega^n$$

The function $\rho_\delta f$ is C^2 and compactly supported on X_{reg} , so we have

$$(3.3) \quad \rho_\delta f(x) = \int_X \rho_\delta f \omega^n + \int_X G_x \Delta_\omega (f \rho_\delta) \omega^n.$$

Since $\Delta_\omega (f \rho_\delta) = f \Delta_\omega \rho_\delta + \rho_\delta \Delta_\omega f + 2\langle \nabla f, \nabla \rho_\delta \rangle_\omega$ and

$$\int_X G_x \langle \nabla f, \nabla \rho_\delta \rangle_\omega \omega^n = - \int_X G_x f \Delta_\omega \rho_\delta \omega^n - \int_X f \langle \nabla G_x, \nabla \rho_\delta \rangle_\omega \omega^n$$

we find that

$$(3.4) \quad \int_X G_x \Delta_\omega (f \rho_\delta) \omega^n = \int_X \rho_\delta G_x \Delta_\omega f \omega^n - 2 \int_X f \langle \nabla G_x, \nabla \rho_\delta \rangle_\omega \omega^n - \int_X G_x f \Delta_\omega \rho_\delta \omega^n.$$

Clearly, one has

$$(3.5) \quad \int_X \rho_\delta G_x \Delta_\omega f \omega^n \leq \|G_x\|_{L^{1+\varepsilon}} \|g\|_{L^{p_0}}$$

Next, Cauchy-Schwarz inequality yields

$$\begin{aligned} \left| \int_X f \langle \nabla G_x, \nabla \rho_\delta \rangle_\omega \omega^n \right| &\leq \left(\int_X f^2 G_x^{1+\varepsilon} |\nabla \rho_\delta|_\omega^2 \omega^n \right)^{\frac{1}{2}} \left(\int_{X_{\text{reg}}} \frac{|\nabla G_x|^2}{G_x^{1+\varepsilon}} \omega^n \right)^{\frac{1}{2}} \\ &\leq C \left(\int_X f^2 G_x^{1+\varepsilon} |\nabla \rho_\delta|_\omega^2 \omega^n \right)^{\frac{1}{2}} \end{aligned}$$

By Cauchy-Schwarz again, the symmetry of $G(\cdot, \cdot)$ yields

$$\begin{aligned} \left| \int_X \mu(x) \omega^n \int_X f \langle \nabla G_x, \nabla \rho_\delta \rangle \omega^n \right| &\leq CV^{\frac{1}{2}} \left(\int_X \mu(x)^2 \omega^n \int_X f^2(y) G_y^{1+\varepsilon}(x) |\nabla \rho_\delta(y)|_\omega^2 \omega^n \right)^{\frac{1}{2}} \\ &\leq CV (\sup_X \mu^2) \sup_{y \in X_{\text{reg}}} \|G_y\|_{L^{1+\varepsilon}}^{1/2} \int_X f^2(y) |\nabla \rho_\delta(y)|_\omega^2 \omega^n \\ &\leq CV (\sup_X \mu^2) \sup_{y \in X_{\text{reg}}} \|G_y\|_{L^{1+\varepsilon}}^{1/2} \|f\|_{L^{2p'}} \|\nabla \rho_\delta\|_{L^{2+\varepsilon}} \end{aligned}$$

and thus

$$(3.6) \quad \limsup_{\delta \rightarrow 0} \int_X \mu(x) \omega^n \left| \int_X f \langle \nabla G_x, \nabla \rho_\delta \rangle \omega^n \right| = 0.$$

Finally, we have similarly

$$\begin{aligned} \left| \int_X \mu(x) \omega^n \int_X f G_x \Delta_\omega \rho_\delta \omega^n \right| &= \left| \int_X f(y) \Delta_\omega \rho_\delta(y) \omega^n \int_X \mu(x) G_y(x) \omega^n \right| \\ &\leq \sup_X \mu \sup_{y \in X_{\text{reg}}} \|G_y\|_{L^1} \|f\|_{L^{2p}} \|\Delta_\omega \rho_\delta\|_{L^{1+\varepsilon}} \end{aligned}$$

and thus

$$(3.7) \quad \limsup_{\delta \rightarrow 0} \int_X \mu(x) \omega^n \left| \int_X f G_x \Delta_\omega \rho_\delta \omega^n \right| = 0.$$

Recall that given (3.3), all we need to conclude is to show that

$$\limsup_{\delta \rightarrow 0} \int_X \mu(x) \omega^n \int_X G_x \Delta_\omega (f \rho_\delta) \omega^n \leq M \int_X \mu(x) \omega^n.$$

This follows from (3.4), (3.6) and (3.7), hence the proposition is proved. \square

Remark 3.4. — As the proof shows, the above statement is local. That is, let $x \in X$ and let U be a neighborhood of x such that $U_{\text{sing}} = X_{\text{sing}} \cap U$ admits a family of cut-off functions (ρ_δ) such that $\lim_{\delta \rightarrow 0} \int_U (|\nabla \rho_\delta|_\omega^{2+\varepsilon} + |\Delta_\omega \rho_\delta|^{1+\varepsilon}) \omega^n = 0$ for some $\varepsilon > 0$. Then the conclusion of the lemma holds for any $f \in \mathcal{C}^2(U)$ with $\text{Supp}(f) \subset U$.

Remark 3.5. — Since the support of the singular set has measure zero with respect to ω^n , Hölder inequality shows that the existence of $\varepsilon > 0$ such that $\int_X (|\nabla \rho_\delta|_\omega^{2+\varepsilon} + |\Delta_\omega \rho_\delta|^{1+\varepsilon}) \omega^n$ converges to zero when $\delta \rightarrow 0$ is equivalent to the existence of $\varepsilon' > 0$ such that $\int_X (|\nabla \rho_\delta|_\omega^{2+\varepsilon'} + |\Delta_\omega \rho_\delta|^{1+\varepsilon'}) \omega^n$ remains bounded as $\delta \rightarrow 0$.

We provide below a complementary result to Proposition 3.3; it will however not be used later in the article.

Proposition 3.6. — Assume that X_{sing} admits a family of cut-off functions (ρ_δ) such that $\int_X |\nabla \rho_\delta|_\omega^{2+\varepsilon} \omega^n$ converges to zero for some $\varepsilon > 0$, when $\delta \rightarrow 0$. Let f be a non-negative Lipschitz function on X_{reg} satisfying

- $f \in L^p(X, \omega^n)$ for some $p > 0$.
- $\sup_{X_{\text{reg}}} |\nabla f|_\omega < +\infty$.

Then $\sup_{X_{\text{reg}}} f < +\infty$.

Proof. — Up to decreasing $\varepsilon > 0$, we can assume that $\|f\|_{L^p} + \|G_x\|_{L^{1+\varepsilon}} + \|\nabla G_x\|_{L^{1+\varepsilon}} \leq C$ for some C independent of $x \in X_{\text{reg}}$. Let q be the conjugate exponent of $1 + \varepsilon/2$. Up to replacing f with $f + 1$, one can assume that $f \geq 1$. In particular, one has $\sup_{X_{\text{reg}}} |\nabla f^\gamma|_\omega < +\infty$ for any $1 \geq \gamma > 0$. Up to replacing f with f^γ for small γ , one can assume without loss of generality that $f \in L^p(X, \omega^n)$ for $p \geq 2q$. Next, we write

$$(3.8) \quad \rho_\delta f(x) = \int_X \rho_\delta f \omega^n - \int_X \langle \nabla G_x, \nabla(f\rho_\delta) \rangle \omega^n.$$

In particular, one gets

$$\rho_\delta f(x) \leq \|f\|_{L^1} + \|\nabla G_x\|_{L^1} \cdot \sup_{X_{\text{reg}}} |\nabla f| + \int_X f |\langle \nabla G_x, \nabla \rho_\delta \rangle| \omega^n.$$

Using the same argument as in the proof of Proposition 3.3, we would be done if one could prove that for any smooth non-negative function μ with compact support on X_{reg} , one has

$$\limsup_{\delta \rightarrow 0} \int_X \mu(x) \omega^n \int_X f |\langle \nabla G_x, \nabla \rho_\delta \rangle| \omega^n = 0.$$

Since $f \in L^{2q}$, the proof of (3.6) applies verbatim to show the above identity, which ends the proof of the proposition. \square

Remark 3.7. — The great news in this framework is that a very general Sobolev inequality was recently established in [GPSS23]: it represents an important result, which for sure will have a crucial impact on differential geometry of singular spaces. It would be very interesting to see if one can obtain a more general version of Proposition 3.3 (*zum Beispiel* by imposing less restrictive conditions on the family of cutoff functions) via Sobolev inequality.

4. Strict positivity

4.1. Strict positivity in codimension two. —

Definition 4.1. — Let (X, ω_X) be a normal Kähler space and let ω be a positive $(1, 1)$ -current on X . One says that ω is strictly positive if given any compact subset $K \subset X$, there exists $\varepsilon = \varepsilon(K) > 0$ such that $\omega \geq \varepsilon \omega_X$ weakly on K .

It is proved in [GGZ23] that if X is compact and $\dim X = 3$, then the solution ω of (MA) is strictly positive. The result is also proved when X has isolated smoothable singularities. We will give a weaker version of that statement holding in any dimension. In order to do this, we need to recall the following result, due to [GKKP11, Proposition 9.3]. It is stated in the algebraic setting, but the same proof applies in the analytic setting, cf e.g. [GK20].

Theorem 4.2. — *Let X be a compact Kähler space with log terminal singularities. There exists a proper analytic subset $Z \subset X$ of codimension at least three such that each point $x \in X \setminus Z$ is at worse a quotient singularity of X . Moreover, if $x \in X_{\text{sing}} \setminus Z$, we have $(X, x) \simeq S \times (\mathbb{C}^{n-2}, 0)$ where S is a germ of surface with an isolated quotient singularity.*

Relying on the structure theorem above, one can prove the following:

Proposition 4.3. — *Let (X, ω_X) be a compact Kähler space with log terminal singularities, and let Z as in Theorem 4.2. The current ω solving (MA) is strictly positive on $X \setminus Z$.*

Remark 4.4. — We will actually prove that ω is strictly positive over the locus where the functorial resolution is has nonpositive discrepancies. We will explain in the end of the proof why that locus contains $\widehat{X} \setminus Z$.

Proof. — Let $\pi : \widehat{X} \rightarrow X$ be a log resolution of X and let $E = \sum_{i \in I} E_i$ be the exceptional divisor of π . Write $K_{\widehat{X}} = \pi^* K_X + \sum a_i E_i$ and let s_i be a section of $\mathcal{O}_{\widehat{X}}(E_i)$ cutting out E_i and let h_i be a smooth hermitian metric on $\mathcal{O}_{\widehat{X}}(E_i)$. We write $|s_i|^2 := |s_i|_{h_i}^2$. There exists a smooth volume form $dV_{\widehat{X}}$ on \widehat{X} such that the pull back measure $\pi^* \mu_h$ (which is well-defined) satisfies $\pi^* \mu_h = \prod_{i \in I} |s_i|^{2a_i} dV_{\widehat{X}}$. Let $\omega_{\widehat{X}}$ be a Kähler form on \widehat{X} . For any $t > 0$, the form $\pi^* \omega_X + t\omega_{\widehat{X}}$ is Kähler, with volume $V_t = [\pi^* \omega_X + t\omega_{\widehat{X}}]^n$. For any $\varepsilon > 0$, Yau's theorem provides a smooth function $\varphi_{t,\varepsilon} \in \text{PSH}(\widehat{X}, \pi^* \omega_X + t\omega_{\widehat{X}}) \cap C^\infty(\widehat{X})$ solution of

$$(4.1) \quad (\pi^* \omega_X + t\omega_{\widehat{X}} + dd^c \varphi_{t,\varepsilon})^n = c_t \prod_{i \in I} (|s_i|^2 + \varepsilon^2)^{a_i} dV_{\widehat{X}},$$

such that $\sup_{\widehat{X}} \varphi_{t,\varepsilon} = 0$, where $c_t = V_t/V$. We set $\omega_{t,\varepsilon} := \pi^* \omega_X + t\omega_{\widehat{X}} + dd^c \varphi_{t,\varepsilon}$. The metrics $\omega_{t,\varepsilon}$ are also defined as currents when t or ε vanishes. To simplify the notation, we write $\omega_t := \omega_{t,0}$. It is well-known that there exists C independent of t, ε such that

$$\|\varphi_{t,\varepsilon}\|_\infty \leq M.$$

Next, we set $u := \sum_{a_i \geq 0} a_i \log |s_i|^2$. We have the following result

Claim 4.5. — For every $t > 0$, there exists $C_t > 0$ such that for any $\varepsilon > 0$, one has

$$\omega_t \geq C_t^{-1} e^u (\pi^* \omega_X + t\omega_{\widehat{X}}).$$

Proof of the claim. — Set $\widehat{\omega}_t := \pi^* \omega_X + t\omega_{\widehat{X}}$, and let B_t be an upper bound for the bisectional curvature of $\widehat{\omega}_t$. Recall that for each i , we have

$$\Theta_{h_i}(E_i) + dd^c \log(|s_i|^2 + \varepsilon^2) \geq -\frac{\varepsilon^2}{|s_i|^2 + \varepsilon^2} \Theta_{h_i}(E_i),$$

and that $-\text{Ric}(dV_{\widehat{X}}) = \pi^* \Theta_h(K_X) + \sum_{i \in I} a_i \Theta_{h_i}(E_i)$. Let $K > 0$ (independent of t) such that $\Theta_h(K_X) \leq K\omega_X$.

Next, let $A_t > 0$ such that for each $i \in I$, one has $\pm \Theta_{h_i}(E_i) \leq A_t \widehat{\omega}_t$. Therefore, we have $\log(|s_i|^2 + \varepsilon^2) \in \text{PSH}(\widehat{X}, A_t \widehat{\omega}_t)$ for every i . Finally, set $u_\varepsilon := \sum_{a_i \geq 0} a_i \log(|s_i|^2 + \varepsilon^2)$;

clearly $u_\varepsilon \leq D$ for D independent of ε . We have

$$\begin{aligned} \text{Ric } \omega_{t,\varepsilon} &= -\pi^* \Theta_h(K_X) - \sum_{i \in I} a_i (\Theta_{h_i}(E_i) + dd^c \log(|s_i|^2 + \varepsilon^2)) \\ &\geq -(K + A_t \sum_{i \in I} |a_i|) \widehat{\omega}_t - dd^c u_\varepsilon. \end{aligned}$$

Thanks to [Gue16, Proposition 2.1], there exists a constant C_t depending on M, K, A_t, B_t and D such that $\omega_{t,\varepsilon} \geq C_t^{-1} e^{u_\varepsilon} \widehat{\omega}_t$. Since $\omega_{t,\varepsilon}$ converges smoothly to ω_t outside of the support of E , we get the desired inequality on that open set. The inequality remains valid across E since none of the two currents charge pluripolar sets. \square

It follows from the claim that $\omega_t \geq C_t^{-1} e^u \pi^* \omega_X$. We now aim to show that we can choose C_t uniformly in t .

Claim 4.6. — There exists $C > 0$ such that

$$\pi^* \omega \geq C^{-1} e^u \pi^* \omega_X.$$

Proof of the claim. — For $\varepsilon > 0$, let $H_t := \log \text{tr}_{\omega_t}(\pi^* \omega_X) + u$. Since $\pi^* \omega_X \leq \widehat{\omega}_t$, the claim above shows that H_t is bounded above, although not uniformly in t . From now on, we work on $\widehat{X} \setminus E$ which we identify with X_{reg} . Let $\psi \in \text{PSH}(X, \omega_X)$ be a smooth function on X_{reg} such that $\psi \rightarrow -\infty$ near X_{sing} . Then for any $\delta > 0$, $H_t + \delta \psi$ attains its maximum in X_{reg} . We have

$$\text{Ric } \omega_t = -\Theta_h(K_X) \geq -K \omega_X,$$

cf also (2.1) and the bisectional curvature of ω_X is bounded above by a constant B since curvature decreases when passing to holomorphic submanifolds. By Chern-Lu inequality, we have on X_{reg}

$$\Delta_{\omega_t}(H_t + \delta \psi) \geq -(K + 2B + \delta) \text{tr}_{\omega_t} \omega_X,$$

and since $\Delta_{\omega_t}(-\varphi_t) \geq \text{tr}_{\omega_t} \omega_X - n$, we get

$$\Delta_{\omega_t}(H_t + \delta \psi - E \varphi_t) \geq \text{tr}_{\omega_t} \omega_X - nE,$$

with $E = K + 2B + 2$. A standard application of the maximum principle yields

$$H_t + \delta \psi \leq \log(nE) + D + \delta \sup_X \psi + 2EM.$$

Since this holds for any $\delta > 0$, we can pass to the limit and find that H_t is uniformly bounded above when $t \rightarrow 0$. The claim follows by letting $t \rightarrow 0$ and using a similar argument as in the earlier claim. \square

For every $x \in X_{\text{sing}} \setminus Z$, we have $(X, x) \simeq S \times (\mathbb{C}^{n-2}, 0)$ where S is a germ of surface with an isolated log terminal singularity. Now, the minimal resolution of S has nonpositive discrepancies by the negativity lemma. Since the functorial resolution commutes with smooth morphism (hence the morphism $S \times \mathbb{C}^{n-2} \rightarrow S$) we see that the discrepancies of any exceptional divisor lying above $X \setminus Z$ are nonpositive. In particular, the function u is locally bounded on $\pi^{-1}(X \setminus Z)$ and the proposition follows from the claim above. \square

4.2. Strict positivity under the existence of suitable cut-off functions. — The main result of this section is to show that if X_{sing} admits a family of cut-off functions with good integrability properties, then ω dominates a Kähler form.

Proposition 4.7. — *Let (X, ω_X) be a compact Kähler space with log terminal singularities, and let $\omega = \omega_X + dd^c \varphi$ be the current solving (MA). Assume that X_{sing} admits a family of cut-off functions (ρ_δ) such that $\limsup_{\delta \rightarrow 0} \int_X (|\nabla \rho_\delta|_\omega^{2+\varepsilon} + |\Delta_\omega \rho_\delta|^{1+\varepsilon}) \omega^n = 0$ for some $\varepsilon > 0$. Then, there exists $C > 0$ such that*

$$\omega \geq C^{-1} \omega_X.$$

Proof. — First, we observe that there exists $A, B > 0$ such that

$$\text{Ric } \omega = -\Theta_h(K_X) \geq -A\omega_X, \quad \text{and} \quad \text{Bisec } \omega_X \leq B.$$

Then, Chern-Lu inequality [Rub14, Proposition 7.2] applied to the identity map $(X_{\text{reg}}, \omega) \rightarrow (X_{\text{reg}}, \omega_X)$ yields on X_{reg}

$$\Delta_\omega \log \text{tr}_\omega(\omega_X) \geq -(A + 2B) \text{tr}_\omega(\omega_X).$$

We set $C := A + 2B$. Next, we have $\Delta_\omega(-\varphi) = -n + \text{tr}_\omega(\omega_X)$ so that, if we set $H := e^{-C\varphi} \text{tr}_\omega(\omega_X)$ we have

$$\Delta_\omega \log H \geq -nC.$$

This shows that $H\Delta_\omega H \geq |\partial H|^2 - nCH^2$, hence

$$(H + 1)^2 \Delta_\omega \log(H + 1) = (H + 1) \Delta_\omega H - |\partial H|^2 \geq -nCH(H + 1)$$

so that

$$(4.2) \quad \Delta_\omega \log(H + 1) \geq -nC.$$

Now, $H \in L^1(\omega^n)$ since

$$\int_X H \omega^n = \int_X ne^{-C\varphi} \omega_X \wedge \omega^{n-1} \leq ne^{C\|\varphi\|_\infty} [\omega_X]^n.$$

This implies that

$$(4.3) \quad \forall p > 0, \quad \log(H + 1) \in L^p(\omega^n).$$

Combining (4.2), (4.3) and Proposition 3.3, this yields that $\sup_{X_{\text{reg}}} H < +\infty$, which ends the proof of the proposition. \square

Remark 4.8. — Given Remark 3.4, it would be desirable to have a local version of the above result but it does not seem to be a straightforward consequence of the above proof.

5. Orbifold regularity

Recall that if X is a Kähler space with quotient singularities, then an orbifold Kähler metric on X is a closed positive $(1, 1)$ current ω with local potentials on X such that $\omega|_{X_{\text{reg}}}$ is a smooth Kähler form and such that for any local uniformizing chart $p : V \rightarrow U$ where V is smooth, $p^*(\omega|_{U_{\text{reg}}})$ extends to a smooth Kähler metric on V .

5.1. The projective case. — Let (X, ω_X) be a compact Kähler space with log terminal singularities and let $X_{\text{orb}} \subset X$ be the euclidean open subset of X where X has quotient singularities. Next, let $\omega = \omega_X + dd^c \varphi$ solving (MA).

Theorem 5.1 ([LT19]). — Assume that X is projective. Then, $\omega|_{X_{\text{orb}}}$ is an orbifold Kähler metric.

The proof relies on the existence of a partial desingularization $\mu : \widehat{X} \rightarrow X$ such that \widehat{X} has only quotient singularities and μ is isomorphic over X_{orb} . The construction of μ relies on deep compactification results for stacks and is highly non-canonical, making its generalization to the complex analytic case difficult. A weaker version where μ is only isomorphic in codimension two over X has been proved in the analytic case in [Ou24].

We provide below a much more elementary proof of Theorem 5.1 relying solely on Artin's approximation theorem [Art69]. Recall that Artin's result implies that X_{orb} admits a covering in the étale topology by quasi-projective varieties which are finite quasi-étale quotients of smooth quasi-projective varieties.

Alternative proof of Theorem 5.1. — Let $x \in X_{\text{orb}}$. By [Art69, Corollary 2.6], there exists an étale map $i : U \rightarrow X$ such that $x \in i(U)$ and a finite, quasi-étale Galois cover $p : V \rightarrow U$ with V quasi-projective smooth. By Zariski's main theorem, the quasi-finite map $i \circ p : V \rightarrow X$ is the composition of an open immersion $V \hookrightarrow \overline{V}$ into a normal variety \overline{V} and a finite morphism $f : \overline{V} \rightarrow X$. In summary, we have the following diagram

$$\begin{array}{ccccc} & & \overline{V} & & \\ & & \uparrow & \searrow f & \\ & & V & \xrightarrow{p} & U \xrightarrow{i} X \end{array}$$

Of course, f is ramified in codimension one in general so that \overline{V} need not be klt, but by construction f is quasi-étale over a neighborhood of x .

Let $g : W \rightarrow \overline{V}$ be the functorial resolution of \overline{V} (so that g is isomorphic over $V \subset \overline{V}_{\text{reg}}$) and let $\pi := f \circ g : W \rightarrow X$ be the composition, i.e.

$$\begin{array}{ccc} W & & \\ g \downarrow & \searrow \pi & \\ \overline{V} & \xrightarrow{f} & X \end{array}$$

One can write $K_W = \pi^* K_X + E$ where $E = \sum_{i \in I} a_i E_i$ is a \mathbb{Q} -divisor supported on the codimension one component of the non-étale locus of π , i.e. the union of $\text{Exc}(g)$ and the strict transform by g of the branching divisor of f . In particular, $g(\text{Supp}(E)) \subset \overline{V} \setminus V$. Note that E need not have SNC support at this point. But one can consider a log resolution $W' \rightarrow W$ of (W, E) . Then the induced maps $\pi' : W' \rightarrow X$ and $g' : W' \rightarrow \overline{V}$ satisfy $K_{W'} = \pi'^* K_X + E'$ and $g'(\text{Supp}(E')) \subset \overline{V} \setminus V$. So without loss of generality, one

can assume that $\text{Supp}(E)$ has simple normal crossings and

$$(5.1) \quad g(\text{Supp}(E)) \subset \bar{V} \setminus V.$$

If Ω is a local trivialization of mK_X defined on an open set $S \subset X$, then $\pi^*\Omega|_{S_{\text{reg}}}$ extends to a meromorphic section of mK_W over $\pi^{-1}(S)$ with a pole of order $-ma_i$ along $E_i|_{\pi^{-1}(S)}$. This shows that $\pi^*\mu_h$ is of the form $\prod_{i \in I} |s_i|_{h_i}^{2a_i} \omega_W^n$ where s_i is a section cutting out E_i , h_i is a smooth hermitian metric on $\mathcal{O}_W(E_i)$ and ω_W is a Kähler form on W . Set $\psi_+ := \sum_{a_i > 0} a_i \log |s_i|^2$ and $\psi_- := \sum_{a_i < 0} (-a_i) \log |s_i|^2$; these are quasi-psh functions on W such that

$$(5.2) \quad (\pi^*\omega_X + dd^c \pi^*\varphi)^n = e^{\psi_+ - \psi_-} \omega_W^n.$$

Moreover, since E has snc support and $\int_W e^{\psi_+ - \psi_-} \omega_W^n = \int_X d\mu_h < +\infty$, we have

$$(5.3) \quad \exists p > 1, \quad e^{-\psi_-} \in L^p(\omega_W^n).$$

Next, since f is finite the class $[f^*\omega_X]$ is Kähler by e.g. [GK20, Proposition 3.5]. In particular, the ample locus of $[\pi^*\omega_X] = [g^*f^*\omega_X]$ satisfies

$$(5.4) \quad \text{Amp}([\pi^*\omega_X]) \supset W \setminus \text{Exc}(g).$$

Actually both sets are equal but we don't need it. Combining [BBE⁺19, Theorem B.1], (5.3), (5.4), we see that $\pi^*\omega$ is a Kähler metric away from $\text{Supp}(E) \cup \text{Exc}(g)$. Since g is isomorphic over V and $g^{-1}(V) \cap \text{Supp}(E) = \emptyset$ by (5.1), this shows that $f^*\omega|_V$ is a Kähler metric, i.e. ω is an orbifold Kähler metric near x . \square

5.2. The Kähler case. —

Theorem 5.2. — *Let (X, ω_X) be a compact Kähler space with log terminal singularities and let ω solving (MA). There exists a proper analytic subset $Z \subset X$ of codimension at least three such that $X \setminus Z$ has at most quotient singularities and $\omega|_{X \setminus Z}$ is an orbifold Kähler metric.*

Proof. — Let Z as in Theorem 4.2. Up to enlarging Z , one can assume that $X_{\text{sing}} \setminus Z$ is either empty or smooth of dimension $n - 2$. In the first case, the theorem is trivial since ω is a genuine Kähler metric on X_{reg} . So we work in the second case from now on. Next, we can assume that $n \geq 3$, otherwise X itself has quotient singularities and the result is well-known.

Step 0. *Properties of the uniformizing Galois cover.*

The statement is local, so we fix $x \in X \setminus Z$ as well as two open euclidean neighborhoods $U \Subset U'$ of x such that $U' \Subset X \setminus Z$ and there exists a surjective, finite, quasi-étale Galois cover $p : V' \rightarrow U'$ such that V' is smooth. One can identify V' with a neighborhood of the origin in \mathbb{C}^n so that $G = \text{Gal}(p)$ becomes a subgroup of $\text{GL}(n, \mathbb{C})$ which does not contain any reflexion. The analytic set

$$W' := p^{-1}(U'_{\text{sing}}) = \bigcup_{g \in G \setminus \{e\}} \text{Fix}(g)$$

is therefore a finite union of linear subspaces of dimension $n - 2$. Recall from [GKKP11, Proposition 9.3] that p is totally branched over U'_{sing} ; since we ensured that the latter is smooth, its inverse image $p^{-1}(U'_{\text{sing}})$ must be irreducible, hence it is smooth, too.

Without loss of generality, we can assume that $W' = (z_1 = z_2 = 0) \cap V'$ and the functions z_1, z_2 are G -invariant.

Finally, we set $V := p^{-1}(U)$ and $W := W' \cap V$.

Step 1. Construction of cut-off functions.

First, we construct a cut-off for U . Let χ be a non-negative smooth function on X such that $\chi \equiv 1$ on U and $\text{Supp}(\chi) \subset U'$, and satisfying $|\nabla\chi|_{\omega_X}^2 \leq C(n)\chi$ and $|dd^c\chi|_{\omega_X} \leq C(n)$.

It follows from Proposition 4.3 that there exists $C_0 = C(U') > 0$ such that

$$(5.5) \quad \omega \geq C_0^{-1}\omega_X \quad \text{on } U'.$$

This implies that

$$(5.6) \quad |\nabla\chi|_{\omega}^2 \leq C_1\chi, \quad \text{and} \quad |dd^c\chi|_{\omega} \leq C_1.$$

with $C_1 = C(n)C_0$.

Finally, observe that since p is quasi-étale, one has $p^*K_{U'} = K_{V'}$ so that the reflexive pull-back of any (local) trivialization of (a multiple of) $K_{U'}$ is a (local) trivialization of (a multiple of) $K_{V'}$. In particular, $p^*\omega^n$ is a smooth volume form on V' .

Next, we construct a family of cut-off functions for W' . Recall that the functions z_i ($i = 1, 2$) are G -invariant, hence one can find a holomorphic function f_i on U' such that $z_i = p^*f_i$. We define $\psi := \log(|f_1|^2 + |f_2|^2)$ on U' . A simple computation shows that

$$(5.7) \quad \pm dd^c\psi + d\psi \wedge d^c\psi \leq C_2e^{-\psi}\omega_X \leq C_0C_2e^{-\psi}\omega$$

where the second inequality follows from (5.5). Now, let $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing smooth function such that $\zeta \equiv 0$ on $(-\infty, -0]$ and $\zeta \equiv 1$ on $[1, +\infty)$. We define $\rho_\delta(x) = \zeta(\psi(x) + \delta^{-1})$ so that $\rho_\delta(x) = 1$ iff $\psi(x) \geq 1 - \delta^{-1}$ and $\rho_\delta(x) = 0$ iff $\psi(x) \leq -\delta^{-1}$. Thanks to (5.7), one finds

$$(5.8) \quad |\Delta_\omega\psi| + |\nabla\rho_\delta|_{\omega}^2 \leq C_3e^{-\psi}.$$

Since $p^*\omega^n$ is a smooth (non-vanishing) volume form on V' , we infer that for any $p < 2$, we have

$$(5.9) \quad \int_U (|\Delta_\omega\psi|^p + |\nabla\rho_\delta|_{\omega}^{2p})\omega^n \leq \frac{C_4}{|G|} \int_V \frac{dV_{\mathbb{C}^n}}{(|z_1|^2 + |z_2|^2)^p} \leq \frac{C_5}{2-p}.$$

Step 2. Bounding the norm of orbifold forms.

Let us set $\widehat{\omega} := p^*\omega$; this is positive current on V' with bounded potentials which defines a Kähler metric outside W' satisfying

$$\text{Ric } \widehat{\omega} \geq -C_6\widehat{\omega}$$

thanks to (5.5).

Let η be a non-vanishing holomorphic 1-form on V' , and let $f := |\eta|_{p^*\omega}^2$ which is a smooth function on $V' \setminus W'$ which is in $L^1(V, dV_{\mathbb{C}^n})$ since $p^*\omega^n \simeq dV_{\mathbb{C}^n}$ hence

$$(5.10) \quad \int_V f dV_{\mathbb{C}^n} \approx \int_V \eta \wedge \bar{\eta} \wedge p^*\omega^{n-1} < +\infty$$

and the last integral is finite e.g. since ω has bounded potentials. We have

$$\Delta_{\hat{\omega}} f = |\nabla \eta|_{\hat{\omega}}^2 + \langle \text{Ric}^{\sharp}(\eta), \eta \rangle_{\hat{\omega}}$$

where ∇ is the Chern connection induced from ω on Ω_V^1 and Ric^{\sharp} is the endomorphism of Ω_V^1 induced by $\text{Ric}_{\hat{\omega}}$. In particular, we get

$$(5.11) \quad \Delta_{\hat{\omega}} f \geq |\nabla \eta|_{\hat{\omega}}^2 - C_6 f$$

Given $\varepsilon \in (0, 1)$, we have

$$\partial \bar{\partial} f^{\varepsilon} = \varepsilon f^{\varepsilon-1} \partial \bar{\partial} f - \varepsilon(1-\varepsilon) f^{\varepsilon-2} \partial f \wedge \bar{\partial} f$$

Since $\partial f = \langle \nabla \eta, \eta \rangle_{\hat{\omega}}$, we find by Cauchy-Schwarz

$$\text{tr}_{\hat{\omega}}(i \partial f \wedge \bar{\partial} f) = |\partial f|_{\hat{\omega}}^2 \leq |\nabla \eta|_{\hat{\omega}}^2 \cdot |\eta|_{\hat{\omega}}^2$$

hence (5.11) implies

$$(5.12) \quad \Delta_{\hat{\omega}} f^{\varepsilon} \geq \varepsilon f^{\varepsilon-1} |\nabla \eta|_{\hat{\omega}}^2 - \varepsilon C_6 f^{\varepsilon}.$$

We now want to estimate

$$\Delta_{\hat{\omega}}(\chi f^{\varepsilon}) = \chi \Delta f^{\varepsilon} + \langle \nabla \chi, \nabla f^{\varepsilon} \rangle_{\hat{\omega}} + f^{\varepsilon} \Delta_{\hat{\omega}} \chi.$$

By Cauchy-Schwarz and the AM-GM inequality, we get for any $\delta > 0$

$$\begin{aligned} |\langle \nabla \chi, \nabla f^{\varepsilon} \rangle_{\hat{\omega}}| &\leq \varepsilon f^{\varepsilon-1} |\nabla \chi|_{\hat{\omega}} \cdot |\nabla \eta|_{\hat{\omega}} \cdot |\eta|_{\hat{\omega}} \\ &\leq \delta \varepsilon f^{\varepsilon-1} |\nabla \chi|_{\hat{\omega}}^2 |\nabla \eta|_{\hat{\omega}}^2 + \delta^{-1} \varepsilon f^{\varepsilon}. \end{aligned}$$

Choosing $\delta = 1/2C_1$, we get from (5.6)

$$(5.13) \quad \begin{aligned} \Delta_{\hat{\omega}}(\chi f^{\varepsilon}) &\geq \frac{1}{2} \chi \varepsilon f^{\varepsilon-1} |\nabla \eta|_{\hat{\omega}}^2 - C_7 \tilde{\chi} f^{\varepsilon} \\ &\geq -C_7 \tilde{\chi} f^{\varepsilon} \end{aligned}$$

with $C_7 = \varepsilon C_6 + C_1$, and where $\tilde{\chi}$ a cutoff function which is equal to 1 on $\text{Supp}(\chi)$ and whose support is included in U' . The function $\sum_{g \in G} g \cdot \chi f^{\varepsilon}$ is a positive G -invariant function on V that dominates χf^{ε} and which is in $L^{\frac{1}{\varepsilon}}$ thanks to (5.10). It descends to a function on X supported on U' that satisfies the hypotheses of Proposition 3.3 (recall that we have good cut-off functions, cf (5.9) and Remark 3.5), hence it is bounded, and so is f . Varying the holomorphic form η , we see that $p^*(\omega|_U)$ dominates a Kähler form. Since $p^*(\omega^n|_U)$ is a smooth volume form, we infer that $p^*(\omega|_U)$ is quasi-isometric to a Kähler form on V . By ellipticity, this is sufficient to infer that $\omega|_U$ is an orbifold Kähler metric. \square

If X is a Kähler space with log terminal singularities, we denote by X_{orb} the largest euclidean open subset of X such that any point $x \in X_{\text{orb}}$ has a neighborhood U_x with finite quotient singularities. By Theorem 4.2, we know that $X \setminus X_{\text{orb}}$ has codimension at least three. It is expected (and known in the projective case, cf § 5.1) that the solution ω of (MA) is an orbifold Kähler metric on the whole X_{orb} . The proof of Theorem 5.2 would provide the result if one could show that ω is strictly positive on X_{orb} . If $\dim X = 3$, ω is strictly positive thanks to (a minor twist of) [GGZ23, Theorem 5.3], so the result follows in that case. Actually, we provide below a different argument relying exclusively on the statement of Theorem 5.2.

Corollary 5.3. — *Let X be a compact Kähler space with log terminal singularities. Assume that $\dim X = 3$. Then the current ω solving (MA) is an orbifold Kähler metric on X_{orb} .*

Proof. — Let Z be the analytic subspace of codimension at least three in the statement of Theorem 5.2. Since $\dim X = 3$, Z is a finite collection of points. Now, let $x \in X_{\text{orb}}$. If $x \notin Z$, then we know that ω is an orbifold Kähler metric near x . So we can assume that $x \in Z$. There exists a small neighborhood U of x and a uniformizing chart $p : V \rightarrow U$ with $V \subset \mathbb{C}^n$ smooth such that $\bar{U} \cap Z = \{x\}$ and $p^{-1}(x) = \{0\}$ is a singleton. In particular, Theorem 5.2 implies that $\hat{\omega} := p^*(\omega|_U)$ is a Kähler metric on $V \setminus \{0\}$. Up to shrinking U , one can assume $\omega|_U = dd^c \phi$ has a potential and that $\hat{\omega}$ extends to a Kähler metric in a neighborhood of ∂V . Since p is quasi-étale, we know that $\hat{\omega}^3 = p^* d\mu_h$ is a smooth volume form on \bar{V} . Therefore, $\hat{\phi} := p^* \phi$ is smooth on $\bar{V} \setminus \{0\}$, and solves the Dirichlet equation

$$\begin{cases} (dd^c \hat{\phi})^3 = \hat{\omega}^3 & \text{on } V, \\ \hat{\phi}|_{\partial V} = p^* \phi|_U. \end{cases}$$

It is standard that the above equation has a unique solution ψ such that $dd^c \psi$ is a smooth Kähler metric on V . Hence so is $\hat{\omega}$, which proves the corollary. \square

6. A direct construction

Given a compact Kähler space (X, ω_X) with quotient singularities, it is quite straightforward to construct a metric

$$\omega_o := \omega_X + dd^c \varphi$$

compatible with the local uniformizations, i.e. an orbifold Kähler metric.

In the more general context we are considering in this article, X admits log-terminal singularities and as we have seen in Theorem 5.2, we can construct a metric with this property in the complement of an analytic set of co-dimension at least three. But the metric obtained in Theorem 5.2 is far from being explicit, given that it is the solution of a MA equation. In this section we show that in dimension three and for a very particular class of singularities an explicit orbifold Kähler metric can be obtained.

More precisely, we show that the following holds.

Theorem 6.1. — *Let (X, ω_X) be a 3-dimensional compact Kähler space with log-terminal singularities, and let $Z \subset X$ be the maximal subset with the properties in Theorem 4.2. We assume that $X_{\text{sing}} \setminus Z$ consists of (non-necessarily isolated) ordinary double points only. Then there exists an "explicit" closed positive $(1, 1)$ current ω with local potentials on X with the following properties.*

- *It is a non-singular Kähler metric when restricted to X_{reg} , the set of regular points of X .*
- *It is an orbifold Kähler metric at each point of $X_0 \cap X_{\text{sing}}$, where $X_0 = X \setminus Z$.*
- *Let $\pi : \hat{X} \rightarrow X$ be a resolution of singularities of X , and consider*

$$\mathcal{J}_\pi := \pi^* \omega^3 / \omega_{\hat{X}}^3$$

where $\omega_{\hat{X}}$ is a (Kähler) metric on the non-singular model \hat{X} of X . Then \mathcal{J}_π is in $L^p(\hat{X}, \omega_{\hat{X}})$ for some $p > 1$.

We will only point out next the main ideas of the proof, and leave some of the -rather tedious- calculations to the interested readers. The main observation is the following simple fact (which unfortunately seems to be specific to the case of A_1 -singularities).

Claim 6.2. — The $(1,1)$ -form

$$\omega_0 := dd^c(|u|^4 + |v|^4 + |uv|^2)^{\frac{1}{2}}$$

is equivalent to the Euclidean metric in \mathbb{C}^2 near the origin, i.e. there exists a positive constant $C > 0$ such that

$$C^{-1}\omega_{\text{euc}} \leq \omega_0 \leq C\omega_{\text{euc}}$$

for all $z \in \mathbb{C}^2 \setminus \{0\}$ such that $|z| \ll 1$.

This can be verified by a direct computation; more generally, let (f_ℓ) be a finite set of holomorphic functions defined locally near $0 \in \mathbb{C}^n$, such that $\{0\} = \bigcap_\ell \{f_\ell = 0\}$. Fix a parameter $0 < \alpha < 1$, and set

$$\varphi(z) := \left[\sum_\ell |f_\ell(z)|^2 \right]^\alpha = \psi^\alpha, \quad \text{with } \psi = \sum_\ell |f_\ell(z)|^2 \geq 0.$$

Using that $\log \psi$ is psh we notice that $\psi dd^c \psi - d\psi \wedge d^c \psi \geq 0$, hence

$$\alpha^2 \frac{dd^c \psi}{\psi^{1-\alpha}} \leq dd^c \varphi = \alpha \frac{dd^c \psi}{\psi^{1-\alpha}} - \alpha(1-\alpha) \frac{d\psi \wedge d^c \psi}{\psi^{2-\alpha}} \leq \alpha \frac{dd^c \psi}{\psi^{1-\alpha}}.$$

It follows that the behaviour of $dd^c \varphi$ near 0 is comparable to that of $\frac{dd^c \psi}{\psi^{1-\alpha}}$. In particular, this settles Claim 6.2.

Next we have to "guess" a way to recover the orbifold metric from the equation

$$Z_1 Z_2 - Z_3^2 = 0$$

of $(S,0) \times \mathbb{C} \subset \mathbb{C}^4$. This is easy: if we denote by $f(Z) = Z_1 Z_2 - Z_3^2$, then

$$(6.1) \quad \omega_0 + dd^c |z|^2 \simeq \iota^*(dd^c |\nabla f| + C\omega_{\text{euc}})$$

where we use the notations

$$|\nabla f| := \left(\sum_{i=1}^4 |f_{,Z_i}|^2 \right)^{\frac{1}{2}}, \quad \omega_{\text{euc}} := dd^c \left(\sum |Z_i|^2 \right)$$

as well as

$$\iota(u, v, z) := (u^2, v^2, uv, z).$$

This leads to the solution: the metric required by Theorem 6.1 is constructed by gluing the potentials in (6.1) by a partition of unity. Concretely, consider a local embedding

$$(X, x) \hookrightarrow (\mathbb{C}^N, 0)$$

of X near an arbitrary point $x_0 \in X$ and denote by

$$\mathcal{I}_x = (f_1, \dots, f_k)$$

the ideal corresponding to the analytic subset in $(\mathbb{C}^N, 0)$.

We define the function

$$\varphi_x := \left(\sum \left| \frac{\partial f_i}{\partial z_j} \right|^2 \right)^{\frac{1}{2}}$$

and then the next assertions hold true:

- On the overlapping sets corresponding to two points x_1, x_2 , we have

$$C^{-1} \leq \frac{\varphi_{x_1}}{\varphi_{x_2}} \leq C$$

for some positive constant C .

- If x_1 is an orbifold singularity, and if we choose the embedding

$$(u, v, z) \rightarrow (u^2, v^2, uv, z)$$

then $dd^c \varphi_{x_1}$ is the same as ω_0 in the Claim 6.2.

- If x_1 is an orbifold singularity, then on overlapping sets we have

$$C^{-1} dd^c \varphi_{x_2} \leq dd^c \varphi_{x_1} \leq C dd^c \varphi_{x_2}$$

for any other point x_2 (orbifold or not).

The metric ω is obtained by gluing together a finite subset of potentials (φ_{x_i}) by the familiar formula

$$\omega := C\omega_X + dd^c \left(\sum_i \theta_i^2 \varphi_{x_i} \right)$$

where $C \gg 0$ is a large enough constant and (θ_i) is a partition of unity subordinate to the definition domains of (φ_{x_i}) . The verification of the items above consists in a direct computation -which we skip-, using the compatibility relations between the ideals \mathcal{I}_{x_i} corresponding to the local embeddings of X mentioned above.

Remark 6.3. — This method seems to break down already for A_2 singularities. Indeed, the natural potential to consider would be

$$\phi := (|u|^6 + |v|^6 + |uv|^4)^{\frac{1}{3}}$$

given by the pull-back of the norm of the gradient of

$$(x, y, z) \rightarrow xy - z^3$$

raised to the appropriate power. But then its dd^c is not positive definite.

7. The Bogomolov-Gieseker inequality

7.1. Geometric setup and statement of the main result. — Let (X, ω_X) be a compact Kähler threefold with log-terminal singularities. According to Theorem 4.2, there exists a finite set of points $\{p_1, \dots, p_k\} \subset X$, such that in the complement of this set, the space X has at worst quotient singularities.

We equally consider a coherent, reflexive sheaf \mathcal{F} of rank r on X , such that the following are satisfied.

- (a) For each point $x \in X \setminus \{p_1, \dots, p_k\}$ there exists an open subset $x \in V \subset X$ containing x together with a finite quasi-étale Galois cover $p : B \rightarrow V$ with B smooth such that the bi-dual of the sheaf $p^*\mathcal{F}$ is a vector bundle. In other words, \mathcal{F} is a so-called \mathbb{Q} -sheaf when restricted to $X \setminus \{p_1, \dots, p_k\}$.
- (b) The sheaf \mathcal{F} is $[\omega_X]$ -stable.

Cut-off functions. Recall that the metric ω_X is assumed to be locally dd^c -exact. Therefore, for each $i = 1, \dots, k$ let $U_i \Subset U'_i \Subset U''_i \subset X$ be a collection of open neighborhoods of $p_i \in X$ such that $U''_i \hookrightarrow \mathbb{C}^{N_i}$ admits an embedding in an Euclidean space. Let us define $U := \cup_i U_i$ (resp. $U' := \cup_i U'_i$ and $U'' := \cup_i U''_i$) and let us consider φ_p be a potential of ω_X on U'' , i.e. $\omega_X|_{U''} = dd^c \varphi_p$.

We let θ (resp. θ') be a smooth cut-off function on X with support in U (resp. U''), such that $\theta = 1$ on $\frac{1}{2}U$ and $\text{Supp}(\theta) \subset U$ (resp. $\theta' = 1$ on U' and $\text{Supp}(\theta') \subset U''$).

Next, we set $\phi := -\theta'^2 \varphi_p$ so that

$$(7.1) \quad \omega_X + dd^c \phi|_{U'} = 0$$

That is to say, locally near each of the points p_i the form $\omega_X + dd^c \phi$ vanishes, whereas far from p_i it coincides with the metric ω_X .

Remark 7.1. — The result in [GKKP11] holds without any restriction on the dimension of X , and shows that X has at worst quotient singularities in the complement of a set of codimension at least three. But as soon as $\dim(X) \geq 4$, this means that we have to exclude not only points, but also positive dimensional subsets of X so that the resulting space is locally an orbifold. Assume that C is a one-dimensional analytic space, contained in the non-quotient singularities of X . It would be very interesting to decide whether there exists an open subset $U \subset X$ together with a smooth 3-form Ψ defined on U such that

$$\omega_X^2 + d\Psi|_U = 0, \quad \Psi_x = 0$$

for all $x \in C$. We refer to the arguments at the end of the current section for the motivation.

Given the property (a) above, one can construct a Hermitian orbifold metric h on $\mathcal{F}|_{X \setminus U}$. For each $i \geq 1$ we denote by $\theta_i(\mathcal{F}, h)$ the Chern-Weil forms induced by the metric h , and if r is the generic rank of \mathcal{F} , let

$$(7.2) \quad \Delta(\mathcal{F}, h) := 2r\theta_2(\mathcal{F}, h) - (r-1)\theta_1(\mathcal{F}, h)^2$$

be the corresponding discriminant (2, 2)-form. We notice that by (7.1) the integral

$$(7.3) \quad \int_{X_{\text{reg}}} \Delta(\mathcal{F}, h) \wedge (\omega_X + dd^c \phi)$$

is convergent, where $X_{\text{reg}} \subset X$ is the set of regular points of our space X . Moreover, the above quantity only depends on \mathcal{F} and the cohomology class $[\omega_X] \in H^2(X, \mathbb{R})$, cf e.g. [GK20, § 5]. That is, it is independent of the particular choice of $U \Subset U'$, h or ϕ as long as $h|_{X \setminus U}$ is an orbifold metric on $\mathcal{F}|_{X \setminus U}$ and (7.1) holds.

Therefore, it is legitimate to set

$$(7.4) \quad \Delta(\mathcal{F}) \cdot [\omega_X] := \int_{X_{\text{reg}}} \Delta(\mathcal{F}, h) \wedge (\omega_X + dd^c \phi)$$

The main result we will establish here states as follows.

Theorem 7.2 (Bogomolov-Gieseker inequality). — *Let (X, ω_X) be a 3-dimensional compact Kähler space with at most klt singularities, and let \mathcal{F} be a coherent, reflexive sheaf on X such that the properties (a) and (b) above are satisfied. Then we have*

$$\Delta(\mathcal{F}) \cdot [\omega_X] \geq 0.$$

We remark that this settles a conjecture proposed by Campana-Höring-Peternell in [CHP23]. Theorem 7.2 was proved in [CGN⁺23] under the additional assumption that X admits a *partial resolution*. Here our arguments are purely analytic. Other than a few important technicalities, the main new input here are the results in the previous sections, especially Proposition 3.3.

Remark 7.3. — It is very likely that the Bogomolov-Gieseker inequality $\Delta(\mathcal{F}) \cdot [\omega_X] \geq 0$ holds in the more general case of an $[\omega_X]$ -semistable sheaf \mathcal{F} . A possible approach would be to use the Jordan-Hölder filtration and elementary Chern classes computations, as in [CGG22, § 3.B] (see also the references therein).

In case the inequality in Theorem 7.2 is an equality, i.e. when $\Delta(\mathcal{F}) \cdot [\omega_X] = 0$, then \mathcal{F} enjoys some additional properties thanks to the result below.

Corollary 7.4 (Equality case). — *Let $(X, \omega_X, \mathcal{F})$ as in Theorem 7.2 above. Assume that $\Delta(\mathcal{F}) \cdot [\omega_X] = 0$. Then the following holds.*

- $\mathcal{F}|_{X_{\text{reg}}}$ is locally free and projectively flat.
- There exists a finite, quasi-étale Galois cover $p : Y \rightarrow X$ such that $\mathbb{P}(p^*(\mathcal{F}|_{X_{\text{reg}}}))$ extends to a locally trivial \mathbb{P}^{r-1} -bundle over Y .

Remark 7.5. — In the second item, one cannot expect that $p^{[*]}\mathcal{F}$ itself be locally free in general. Indeed, if $X = \{xy = zt\} \subset \mathbb{C}^4$ is a cone over a smooth quadric surface, then any quasi-étale cover of X is étale. Next, it is well-known that the Weil divisor $D = (x = z = 0)$ is not Cartier, i.e. $L := \mathcal{O}_X(D)$ is not locally free. Now consider $\mathcal{F} := L^{\oplus 2}$ which is projectively flat on X_{reg} but not locally free even after finite (quasi-)étale covers. One can easily globalize the example.

The rest of this section is organized as follows.

- In § 7.2, we construct a Kähler metric $\omega_\theta \in [C\omega_X]$ for some large constant $C > 0$. The metric ω_θ is compatible with the orbifold structure in a complement of an open subset U containing the set $\{p_1, \dots, p_k\}$ and such that locally near each of the p_i it is equal to $(C + 1)\omega_X$. To our greatest regret, we cannot use directly the metric constructed in the previous sections, for reasons which will appear clearly in the proof. Anyway, the main results in [GPSS22] will be used at this point in order to show that the metric ω_θ verifies the necessary hypotheses so that its corresponding Green kernel has the properties (G.i), for $i = 1, \dots, 4$ listed in Section 3.

- In § 7.3, we use [CGN⁺23] to construct a HE metric $h_{\mathcal{F}}$ on \mathcal{F} with respect to ω_{θ} , which satisfies a few estimates.
- In § 7.4, we refine these estimates to show that $h_{\mathcal{F}}$ is quasi-isometric to an orbifold metric away from the set $\{p_1, \dots, p_k\}$.
- In § 7.5, we capitalize on the results from the previous section to show that the current $\Delta(\mathcal{F}, h_{\mathcal{F}})$ is $\bar{\partial}$ -closed away from the set $\{p_1, \dots, p_k\}$. This is the key property needed in order to deal with the boundary terms which appear in the evaluation of the Bogomolov-Gieseker discriminant (7.3).
- In § 7.6, we complete the proof of Theorem 7.2.
- In § 7.7, we give the proof of Corollary 7.4.

7.2. Construction of an interpolating metric and its main properties. — Let us pick once and for all a suitably normalized smooth hermitian metric h_{K_X} on K_X , and consider the function $\varphi_{\text{orb}} \in \text{PSH}(X, \omega_X) \cap L^{\infty}(X)$ obtained by solving the Monge-Ampère

$$(\omega_X + dd^c \varphi_{\text{orb}})^3 = \mu_{h_{K_X}},$$

cf § 2. The current

$$\omega := \omega_X + dd^c \varphi_{\text{orb}}$$

has been showed in Theorem 5.2 to satisfy the crucial property of being an orbifold Kähler metric on $X \setminus \{p_1, \dots, p_k\}$. The metric we are considering in this section is defined by the expression

$$(7.5) \quad \omega_{\theta} := C\omega_X + dd^c (\theta^2 \varphi_p + (1 - \theta)^2 \varphi_{\text{orb}})$$

where $C > 0$ is a large enough positive constant.

The main result of this section is Proposition 7.6 which leads to the crucial Corollary 7.9 further below. Proposition 7.6 describes some fine geometric properties of ω_{θ} . Although the first item below is not very difficult to check, the second one requires a fair amount of work. The rationale is that ω_{θ} is constructed by gluing two metrics (ω_X and ω) with significantly different behavior, hence the analysis of the Monge-Ampère operator of smooth approximants of ω_{θ} becomes quite tricky.

Proposition 7.6. — *The $(1, 1)$ -form ω_{θ} defined in (7.5) satisfies the following:*

- (i) *It is quasi-isometric with the Hermitian metric $\omega_X + (1 - \theta)^2 \omega$, so in particular, it is positive definite.*
- (ii) *For any $x \in X_{\text{reg}}$, there exists a Green function G_x for ω_{θ} satisfying the conditions (G.1)-(G.4) from § 3.*

Before discussing the arguments for this statement, a quick observation:

Remark 7.7. — It is also the case that the manifold $(X_{\text{reg}}, \omega_{\theta})$ satisfies the Sobolev inequality: there exist constants $C > 0$ and $q > 1$ such that for any smooth function f with compact support in X_{reg} we have

$$(7.6) \quad \|f\|_{L^{2q}(X, \omega_{\theta})}^2 \leq C \left(\|f\|_{L^2(X, \omega_{\theta})}^2 + \|df\|_{L^2(X, \omega_{\theta})}^2 \right).$$

Since we do not really need this result, we will not prove it, but it can certainly be established along the arguments used for Proposition 7.6. We generously leave the details to the interested readers.

Proof of Proposition 7.6. — We prove each item separately.

Proof of (i). We simply do a direct calculation: the terms

$$\varphi_p dd^c \theta^2 + \varphi_{\text{orb}} dd^c (1 - \theta)^2$$

of the derivative of the potential defining ω_θ are absorbed by $C\omega_X$, since φ_{orb} is bounded. The next ones

$$\theta^2 dd^c \varphi_p + (1 - \theta)^2 dd^c \varphi_{\text{orb}} = (\theta^2 - (1 - \theta)^2) \omega_X + (1 - \theta)^2 \omega$$

to which we add $C\omega_X$ are quasi-isometric to the $\omega_X + (1 - \theta)^2 \omega$, and finally the terms of type

$$C\omega_X + 2\theta d\theta \wedge d^c \varphi_p - 2(1 - \theta) d\theta \wedge d^c \varphi_{\text{orb}}$$

are also seen to be controlled by $\omega_X + (1 - \theta)^2 \omega$, given that we have

$$d\varphi_{\text{orb}} \wedge d^c \varphi_{\text{orb}} \leq C\omega$$

on the support of $d\theta$, as one sees easily e.g. in local uniformizing charts where ω becomes a smooth Kähler metric.

Proof of (ii). It would be sufficient to show that there exists a family of smooth Kähler metrics ω_ε on a desingularization $\pi : \widehat{X} \rightarrow X$ such that

1. When $\varepsilon \rightarrow 0$, ω_ε converges to $\pi^* \omega_\theta$ weakly on \widehat{X} and locally smoothly on $\pi^{-1}(X_{\text{reg}})$.
2. There exists $p > 1$ and a smooth volume form $dV_{\widehat{X}}$ on \widehat{X} such that $\frac{\omega_\varepsilon^3}{dV_{\widehat{X}}}$ is uniformly bounded in $L^p(\widehat{X}, dV_{\widehat{X}})$.

Indeed, it follows from [GPSS22] (and the subsequent refinements [GPSS24, GT24, Vu24] which removed the nondegeneracy assumption) that if we can approximate $\pi^* \omega_\theta$ with the sequence $(\omega_\varepsilon)_{\varepsilon > 0}$ enjoying the properties 1. and 2. above, then Theorem 3.1 holds true for the family of metrics $(\omega_\varepsilon)_{\varepsilon > 0}$, and as a corollary, Proposition 7.6 (ii) above follows (see the beginning of Section 3). We proceed in two steps.

Step 1. Density of the measure $(\pi^* \omega_\theta + \varepsilon \widehat{\omega})^3$. The arguments we supply next are general, but we will only consider the $\dim = 3$ case here. Consider $\pi : \widehat{X} \rightarrow X$ a resolution of singularities of X . We fix a reference metric $\widehat{\omega}$ on \widehat{X} and let $1 > \varepsilon > 0$ be a positive real number smaller than one. The functions (f_k) in the following equality

$$(7.7) \quad (\pi^* \omega_\theta + \varepsilon \widehat{\omega})^3 = \sum_{k=0}^3 \binom{3}{k} \varepsilon^k e^{f_k} \widehat{\omega}^3$$

are defined by $e^{f_k} := \frac{\widehat{\omega}^k \wedge \pi^* \omega_\theta^{3-k}}{\widehat{\omega}^3}$. The main property we establish in this step is the following.

Claim 7.8. — There exists $\varepsilon_0 > 0$ such that $e^{(1+\varepsilon_0)f_k} \in L^1(\widehat{X}, \widehat{\omega})$ for each $k = 0, \dots, 3$.

Proof of Claim 7.8. — Recall that ω_θ is quasi-isometric to $\omega_X + (1 - \theta)^2\omega$. In particular, the π -inverse image of the metric ω_θ restricted to the set $\theta = 1$ (i.e. near the non-orbifold singularities) is bounded, and so is each of the functions e^{f_k} .

It remains to analyse what happens in the complement of some open subset containing the set (p_i) ; we argue as follows. We consider a local uniformization $p : B \rightarrow V$ of (X, x_0) , where x_0 belongs to the set $X \setminus \{p_1, \dots, p_k\}$ and $B \subset \mathbb{C}^3$ is open. Define \widehat{B} to be a resolution of singularities of the main component of $B \times_X \widehat{X}$; it comes equipped with two maps

$$\begin{array}{ccc} \widehat{B} & \xrightarrow{\widehat{p}} & \widehat{X} \\ \pi_B \downarrow & & \downarrow \pi \\ B & \xrightarrow{p} & V \hookrightarrow X \end{array}$$

where π_B is a modification and \widehat{p} is generically finite between the manifolds \widehat{B} and \widehat{X} . Let $J_{\widehat{p}}$ be the jacobian of \widehat{p} , viewed as holomorphic section of $K_{\widehat{B}/\widehat{X}}$. Once we fix a Kähler metric $\omega_{\widehat{B}}$ on \widehat{B} one has the tautological formula

$$(7.8) \quad \widehat{p}^* \widehat{\omega}^3 = |J_{\widehat{p}}|^2 \omega_{\widehat{B}}^3.$$

Given any compact subset $K \Subset \widehat{B}$ there are constants $C_1, C_2 > 0$ so that we have

$$(7.9) \quad \widehat{p}^* \widehat{\omega} \leq C_1 \omega_{\widehat{B}}, \quad \widehat{p}^* \pi^* \omega_\theta \leq C_2 \omega_{\widehat{B}}$$

where the second identity follows from $\widehat{p}^* \pi^* \omega_\theta = \pi_B^* p^* \omega_\theta$ while $p^* \omega_\theta$ is a bounded $(1, 1)$ -form on B . Since $\widehat{\omega}^k \wedge \pi^* \omega_\theta^{3-k} = e^{f_k} \widehat{\omega}^3$, (7.8) and (7.9) imply

$$(7.10) \quad e^{f_k \circ \widehat{p}} |J_{\widehat{p}}|^2 \leq C_K$$

on K , for some constant $C_K > 0$ depending only on K . Now, let $V' \Subset V$ be any compact subset, let $L := \pi^{-1}(V')$ and let $K := \widehat{p}^{-1}(L)$. The sets L and K by properness of π and \widehat{p} , respectively. Since \widehat{p} is generically finite, we have

$$\int_L e^{(1+\eta)f_k} \widehat{\omega}^3 \leq \int_K e^{(1+\eta)f_k \circ \widehat{p}} |J_{\widehat{p}}|^2 \omega_{\widehat{B}}^3$$

and this last quantity is smaller than

$$C_K^{1+\eta} \int_K |J_{\widehat{p}}|^{-2\eta} \omega_{\widehat{B}}^3$$

by (7.10). Finally, this integral is finite as soon as $0 \leq \eta \ll 1$, and our claim is established. \square

Step 2. Construction of smooth approximants $\widehat{\omega}_\varepsilon$. We consider next a partition of unity $(\rho_i)_i$ subordinate to a finite covering of \widehat{X} with coordinate sets, say W_i . Consider the smooth function

$$(7.11) \quad F_{k,\varepsilon} := \sum_i \rho_i e^{f_k} \star K_{i,\varepsilon}$$

given by convolution with a family of kernels $K_{i,\varepsilon}$ defined locally on each W_i and then we have

$$(7.12) \quad \lim_{\varepsilon} \|F_{k,\varepsilon} - e^{f_k}\|_{L^{1+\varepsilon_0}(\widehat{X}, \widehat{\omega})} = 0.$$

Next, we solve the Monge-Ampère equation for the class $\{C\pi^*\omega_X + \varepsilon\widehat{\omega}\}$ and volume element $\sum_k \binom{3}{k} \varepsilon^k F_{k,\varepsilon} \widehat{\omega}^3$ – call $\widehat{\omega}_\varepsilon$ the resulting solution so that

$$\widehat{\omega}_\varepsilon^3 = c_\varepsilon \sum_k \binom{3}{k} \varepsilon^k F_{k,\varepsilon} \widehat{\omega}^3$$

where c_ε is a harmless normalizing constant.

Our next claim is that the family of Kähler metrics $(\widehat{\omega}_\varepsilon)_{\varepsilon>0}$ satisfies the properties 1. and 2. stated at the beginning of the paragraph "Proof of (ii)". This is a consequence of the following two remarks, based on [GZ12, Theorem C] and [Sav07, Theorem 1.3], respectively.

- (a) Assume that $\widehat{\omega}_\varepsilon = C\pi^*\omega_X + \varepsilon\widehat{\omega} + dd^c\varphi_\varepsilon$ is normalized by $\max_{\widehat{X}} \varphi_\varepsilon = 0$. Then we have

$$\sup_{\widehat{X}} |\varphi_\varepsilon - (\theta^2\varphi_p + (1-\theta)^2\varphi_{\text{orb}}) \circ \pi| \leq C_\varepsilon$$

where $C_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. We remark that here the parameter ε plays a double role: by adding $\varepsilon\widehat{\omega}$ to $C\pi^*\omega_X$ we transform it into a Kähler metric, and moreover it is used in the convolution. Luckily, the fact that $C\pi^*\omega_X + \varepsilon\widehat{\omega}$ is approaching the boundary of the Kähler cone as $\varepsilon \rightarrow 0$ does not affect the constants in [GZ12]. The main point is that the $L^{1+\varepsilon_0}$ -norms above are finite, combined with (7.12).

- (b) Let $K \subset \widehat{X}$ be any compact subset contained in the inverse image of X_{reg} . Then we have

$$\|\varphi_\varepsilon\|_{C^{2,\alpha}(K)} \leq C_K$$

for some constant independent of the parameter ε . This is a direct application of [Sav07, Theorem 1.3] combined with the point (a) we have just discussed.

The usual arguments in MA theory allow us to conclude. \square

Corollary 7.9. — *Let $x \in X \setminus \{p_1, \dots, p_k\}$. Then, there exist a neighborhood V of x and a number $p_0 > 0$ satisfying the following. Let $f \in C^2(V_{\text{reg}})$ be a non-negative function with support in V such that*

- $f \in L^{p_0}(V, \omega_\theta^3)$.
- $\Delta_{\omega_\theta} f \geq -g$ on V_{reg} for some $g \in L^{p_0}(V, \omega_\theta^3)$.

Then $\sup_{V_{\text{reg}}} f < +\infty$.

Proof. — Thanks to (the proof of) Proposition 3.3 and Proposition 7.6 (ii), all we have left to prove is that $V_{\text{sing}} \subset V$ admits a family of cutoff functions (ρ_δ) such that

$$\limsup_{\delta \rightarrow 0} \int_V (|\nabla \rho_\delta|_{\omega_\theta}^{2+\varepsilon} + |\Delta_{\omega_\theta} \rho_\delta|^{1+\varepsilon}) \omega_\theta^3 = 0$$

for some $\varepsilon > 0$. But this follows from the arguments given in (5.7), (5.8) as well as (5.9) because the metric ω_θ dominates a Kähler form (cf Proposition 4.3 and Proposition 7.6) and $dV_\theta := p^*\omega_\theta^n$ is dominated by a smooth volume form on B . \square

7.3. The induced Hermite-Einstein metric on \mathcal{F} . — We recall that in the article [CGN⁺23], given a coherent, torsion-free sheaf \mathcal{F} a "natural" metric $h_{\mathcal{F},0}$ is introduced by gluing via a partition of unity the quotient metric provided by local resolutions of \mathcal{F} . The resulting curvature form has the remarkable property of being bounded from below in the sense of Griffiths. Moreover it combines very nicely with the desingularization theorem of Rossi, cf. [Ros68].

Furthermore, the following result was established (notations as in *loc. cit.*).

Theorem 7.10. — [CGN⁺23] *Let X be a 3-dimensional compact complex Kähler space with at most klt singularities. The restriction $\mathcal{F}|_{X_0}$ admits a metric $h_{\mathcal{F}} = h_{\mathcal{F},0} \exp(s)$ with the following properties:*

- (i) *It is smooth and it satisfies the Hermite-Einstein equation on $(X_0, \omega_\theta|_{X_0})$, i.e.*

$$\Theta(\mathcal{F}, h_{\mathcal{F}}) \wedge \omega_\theta^2 = \lambda \text{Id}_{\mathcal{F}} \otimes \omega_\theta^3,$$

where $\lambda = \frac{c_1(\mathcal{F}) \cdot [\omega_\theta]^2}{r[\omega_\theta]^3}$. Moreover, $h_{\mathcal{F}}$ is a smooth metric in the orbifold sense when restricted to $X \setminus U$.

- (ii) *There exists a constant $C > 0$ such that the inequalities*

$$\text{Tr} \exp(s) \leq C, \quad \exp(s) \geq C e^{N\phi_Z} \text{Id}_{\mathcal{F}}$$

hold pointwise on X_0 , where $N > 0$ is a positive, large enough constant.

- (iii) *We have*

$$\int_{X_0} \frac{1}{-\phi_Z} |D's|^2 \omega_\theta^3 \leq C, \quad \int_{X_0} |\Theta(\mathcal{F}, h_{\mathcal{F}})|^2 \omega_\theta^3 < \infty$$

where the norm $|\cdot|$ is taken with respect to $h_{\mathcal{F},0}$ and ω_θ .

The notations in the statement above are as follows:

- $Z \subset X$ is the proper analytic subset of X given by the union of X_{sing} with the non locally-free locus of \mathcal{F} ;
- ϕ_Z is a function on X with log poles along Z ;
- $X_0 := X \setminus Z$ is the open subset of X_{reg} such that $\mathcal{F}|_{X_0}$ admits a vector bundle structure;
- $U \subset X$ is the support of the cut-off function θ in (7.5), i.e. a small open subset containing the non-quotient singularities of X .

Note that the result in [CGN⁺23] holds true in arbitrary dimension (provided that one modifies the definition of ω_θ accordingly), but this is the version we will need here.

Remark 7.11. — An important consequence of the property (iii) is that if η is smooth $(1,1)$ -form on X_0 such that $\sup_{X_0} |\eta|_{\omega_\theta} < +\infty$, then the improper integral

$$\int_{X_0} \Delta(\mathcal{F}, h_{\mathcal{F}}) \wedge \eta < \infty$$

is convergent. Indeed $\Delta(\mathcal{F}, h_{\mathcal{F}})$ involves quadratic terms in $\Theta(\mathcal{F}, h_{\mathcal{F}})$ hence it is enough to show that $|\Theta(\mathcal{F}, h_{\mathcal{F}})|_{h_{\mathcal{F}}, \omega_{\theta}}^2$ is in L^1 . But this follows from the first (resp. second) statement in item (ii) (resp. item (iii)) in Theorem 7.10 above.

In particular, since ω_{θ} dominates a multiple of ω_X , $\Delta(\mathcal{F}, h_{\mathcal{F}})$ induces a $(1, 1)$ -current on the whole space X .

7.4. C^0 estimates on the orbifold locus. — The aim of this section is to show that $h_{\mathcal{F}}$ is quasi-isometric to a smooth orbifold metric on $X \setminus \{p_1, \dots, p_k\}$, therefore (partially) generalizing the second part of item (i) in Theorem 7.10. In order to state the result, it is convenient to introduce the following

Setup 7.12. — Let $x \in X \setminus \{p_1, \dots, p_k\}$, and let V be a small open neighborhood of x such that $V \simeq B/G$ where $B \subset \mathbb{C}^n$ the unit ball of the Euclidean space, and G is a finite group acting on B holomorphically. We can assume that the action is free in the complement of a set W of codimension at least two. Let $p : B \rightarrow V$ be the quotient map, which is étale over V_{reg} .

Let $\mathcal{E} := (p^* \mathcal{F})^{**}$ which is a vector bundle on B and which we endow with a smooth hermitian metric h_0 . Since \mathcal{F} is reflexive, $\mathcal{F}|_{X_{\text{reg}}}$ is locally free in codimension two hence up to adding more points (p_i) , one can assume that \mathcal{F} is locally free on $X_{\text{reg}} \setminus \{p_1, \dots, p_k\}$, hence on V_{reg} , too. In particular, one has

$$(7.13) \quad \mathcal{E}|_{B \setminus W} = p^*(\mathcal{F}|_{V_{\text{reg}}})$$

and this allows us to consider the smooth hermitian metric $h_{\mathcal{E}} := p^* h_{\mathcal{F}}$ defined over $B \setminus W$.

Finally, we denote by

$$(7.14) \quad \omega_B := p^* \omega_{\theta}$$

the inverse image of the metric $\omega_{\theta}|_V$.

Theorem 7.13. — In Setup 7.12, there exists a constant $C > 0$ such that

$$(7.15) \quad C^{-1} h_0 \leq h_{\mathcal{E}} \leq C h_0$$

holds pointwise on $B \setminus W$.

Remark 7.14. — In other words, the HE metric $h_{\mathcal{F}}$ is quasi-isometric to an orbifold metric on $\mathcal{F}|_V$. Notice as well that one can obtain (7.15) along the lines sketched in [CGN⁺23], since the Sobolev inequality for (X, ω_{θ}) holds true.

Proof. — We proceed in three steps.

Step 1. A weaker C^0 estimate.

We first claim that the metric $h_{\mathcal{E}}$ compares with a fixed, arbitrary Hermitian metric h_0 on \mathcal{E} as follows

$$(7.16) \quad C^{-1} \exp(N\phi_Z \circ p) h_0 \leq h_{\mathcal{E}} \leq C \exp(-N\phi_Z \circ p) h_0.$$

To see this, we first observe that such a two-sided bound is valid for the pull-back of $h_{\mathcal{F},0}$, as it follows by using the definition of this metric. Indeed, we can assume that V is sufficiently small, so that the restriction $\mathcal{F}|_V$ is given by the exact sequence

$$\mathcal{O}_V^p \rightarrow \mathcal{O}_V^q \rightarrow \mathcal{F}|_V \rightarrow 0.$$

Given the equality (7.13), the pull-back of this sequence to U gives

$$(7.17) \quad \mathcal{O}_B^p \rightarrow \mathcal{O}_B^q \rightarrow \mathcal{E}|_B \rightarrow 0$$

which is moreover exact when restricted to the complement $B \setminus W$ of W in B . The pull-back of $h_{\mathcal{F},0}$ on $\mathcal{E}|_{B \setminus W}$ is the quotient metric induced by (7.17) above by construction. The important point is that $\text{codim}_B(W) \geq 2$ is at least two, and therefore the holomorphic maps defining the morphisms in (7.17) locally near a point of W extend. Then e.g the RHS of (7.16) with the pull-back of $h_{\mathcal{F},0}$ instead of $h_{\mathcal{E}}$ is consequence of a direct evaluation of the quotient metric: say that $(e_i)_{i=1,\dots,q}$ is the canonical basis of the trivial bundle of rank r , and $(s_j)_{j=1,\dots,r}$ is a trivialising frame of \mathcal{E} . If we denote by π the second arrow (from the left) in (7.17), then we have

$$\pi(e_i) = \sum f_i^j s_j$$

for some holomorphic functions (f_i^j) defined in the complement of W . By Hartogs, these functions extend holomorphically on B . Moreover, the rank of the matrix (f_i^j) is maximal, equal to r except maybe on W . Therefore, the sum of absolute value square of its $r \times r$ minors is greater than $\exp(N\phi_Z \circ p)$ for some $N \gg 0$. It follows that the norm of each s_i with respect to the quotient metric is smaller than $\exp(-N\phi_Z \circ p)$, which establish our claim. Similar arguments show that the first inequality in (7.17) equally holds true. Then the assertion (7.17) follows from the estimates in Theorem 7.10.

Step 2. *A differential inequality.*

Let τ be any (non identically zero) holomorphic section of \mathcal{E}^* over B , and let $\psi := |\tau|_{h_{\mathcal{E}}}^2$. One has

$$(7.18) \quad \Delta''_{\omega_B} \log \psi \geq -C$$

pointwise on $B \setminus W$ for some constant $C_0 \geq 0$ independent of τ . ⁽¹⁾ We will not provide an argument for (7.18), since it is a standard calculation combined with the fact that the dual metric $h_{\mathcal{E}}^*$ is HE with respect to $\omega_B = p^*\omega_\theta$. Note that the inequality (7.18) is equivalent to

$$(7.19) \quad \psi \Delta''_{\omega_B} \psi \geq |\partial\psi|_{\omega_B}^2 - C_0\psi^2.$$

Next, let $\varepsilon > 0$ be a positive constant, and let χ be a cutoff function on B , as in Step 1 of Theorem 5.2; recall that χ is obtained by pull-back from V . We claim that there exist a positive constant C_ε and another cut-off function $\tilde{\chi}$ so that we have

$$(7.20) \quad \Delta''_{\omega_B} (\chi\psi^\varepsilon) \geq -C_\varepsilon \tilde{\chi}\psi^\varepsilon$$

pointwise on $B \setminus W$.

⁽¹⁾One can show more generally that this holds in the sense of distributions on B .

The rest of this step is devoted to the proof of (7.20). We will do the same type of calculations as in Step 2 of Theorem 5.2; here are the details. The identity

$$(7.21) \quad \Delta''_{\omega_B}(\chi\psi^\varepsilon) = \psi^\varepsilon \Delta''_{\omega_B}(\chi) + \chi \Delta''_{\omega_B}(\psi^\varepsilon) + 2\Re\left(\mathrm{Tr}_{\omega_B} \sqrt{-1} \partial\chi \wedge \bar{\partial}\psi^\varepsilon\right)$$

holds for every $\varepsilon > 0$, and so does the following one

$$(7.22) \quad \Delta''_{\omega_B}(\psi^\varepsilon) = \varepsilon\psi^{\varepsilon-1} \Delta''_{\omega_B}(\psi) - \varepsilon(1-\varepsilon)\psi^{\varepsilon-2} |\partial\psi|_{\omega_B}^2.$$

By combining (7.22) with (7.19) we infer that the inequality

$$(7.23) \quad \Delta''_{\omega_B}(\psi^\varepsilon) \geq \varepsilon\psi^{\varepsilon-2} |\partial\psi|_{\omega_B}^2 - C_0\varepsilon\psi^\varepsilon$$

holds.

We still have

$$(7.24) \quad |\Delta''_{\omega_B}\chi| \leq C(n), \quad |d\chi|_{\omega_B}^2 \leq C(n)\chi$$

given that χ is the pull-back of a function on V by hypothesis and that ω_θ dominates a Kähler form by Proposition 4.3 and Proposition 7.6. This will take care of the first term in (7.21). For the rest of them, remark that we have

$$\chi \Delta''_{\omega_B}(\psi^\varepsilon) + 2\Re\left(\mathrm{Tr}_{\omega_B} \sqrt{-1} \partial\chi \wedge \bar{\partial}\psi^\varepsilon\right) \geq \varepsilon\chi\psi^{\varepsilon-2} |\partial\psi|_{\omega_B}^2 - C\varepsilon\chi^{1/2}\psi^{\varepsilon-1} |\partial\psi|_{\omega_B} - C_0\varepsilon\chi\psi^\varepsilon$$

but the mean inequality shows for any $\delta := C^{-2}$, one has

$$\begin{aligned} C\varepsilon\chi^{1/2}\psi^{\varepsilon-1} |\partial\psi|_{\omega_B} &\leq \delta C^2\varepsilon\chi\psi^{\varepsilon-2} |\partial\psi|_{\omega_B}^2 + \delta^{-1}\varepsilon\psi^\varepsilon \\ &\leq \varepsilon\chi\psi^{\varepsilon-2} |\partial\psi|_{\omega_B}^2 + C'\tilde{\chi}\psi^\varepsilon. \end{aligned}$$

The inequality (7.20) now follows.

Step 3. End of the proof.

In order to show that the LHS inequality in (7.15) holds true, we apply Corollary 7.9 and the function $f := \chi\psi^\varepsilon$ (or better say, the G -invariant version of it). Notice that f belongs to the space $L^{p_0}(B, \omega_B)$ as soon as $\varepsilon \ll 1$ is sufficiently small: this is a consequence of the weaker estimate (7.16) obtained in the first step.

The proof of the RHS of (7.15) is completely similar but instead one considers a local holomorphic section $\tilde{\tau}$ of \mathcal{E} over B and apply the above computations to $\tilde{\psi} := |\tilde{\tau}|_{h_\varepsilon}^2$. \square

7.5. $\bar{\partial}$ -closedness of the current induced by $\Delta(\mathcal{F}, h_{\mathcal{F}})$ on the orbifold locus. — The main result we are after in this section is the following.

Theorem 7.15. — *We assume that (X, ω_X) and \mathcal{F} are satisfying the hypothesis of Theorem 7.10. Let $h_{\mathcal{F}}$ be the metric on \mathcal{F} which verifies the HE equation with respect to the metric ω_θ . Let α be a $(1, 0)$ -form with support in the open subset $X \setminus \{p_1, \dots, p_k\}$ such that it is smooth on X_{reg} and such that*

$$\sup_{X_{\mathrm{reg}}} (|\alpha|_{\omega_\theta} + |\bar{\partial}\alpha|_{\omega_\theta}) < \infty.$$

Then the equality

$$(7.25) \quad \int_{X_0} \Delta(\mathcal{F}, h_{\mathcal{F}}) \wedge \bar{\partial}\alpha = 0$$

holds.

Note that the integral in the above statement is convergent thanks to Remark 7.11. It is, however, not immediate that we can perform the aforementioned integration by parts a priori, since we ignore the regularity properties of $h_{\mathcal{F}}$ on the closure of the annulus $0 < \theta < 1$, this is the main source of difficulties. Even though Theorem 7.13 plays an important role in the proof which will follow, Theorem 7.15 does not follow immediately from it, since it only gives C^0 estimates, whereas (7.25) involves the curvature of $h_{\mathcal{F}}$.

Proof. — Again, we proceed in several steps.

Step 1. *Reduction to some integrability properties of cut-off functions.*

First of all, the result is local. Using a partition of unity, one can assume that α has support in a set V as in Setup 7.12. Without loss of generality, \mathcal{E} is trivial on B with frame (e_1, \dots, e_r) . We let A be the connection 1-form on $B \setminus W$ associated to $h_{\mathcal{E}}$ and the chosen basis. We have $\Theta(\mathcal{E}, h_{\mathcal{E}}) = \bar{\partial}A$ on $B \setminus W$.

Next, let (χ_{ε}) be a family of cut-off functions for V_{sing} which will be specified later. The statement that needs to be proved is equivalent to

$$(7.26) \quad \lim_{\varepsilon \rightarrow 0} \int_V \Delta(\mathcal{F}, h_{\mathcal{F}}) \wedge \bar{\partial}\chi_{\varepsilon} \wedge \alpha = 0.$$

By pulling back to B via the map p , the main assignment in (7.26) is the following: show that the equality

$$\lim_{\varepsilon \rightarrow 0} \int_B \text{Tr}(\Theta(\mathcal{E}, h_{\mathcal{E}}) \wedge \Theta(\mathcal{E}, h_{\mathcal{E}})) \wedge p^*(\bar{\partial}\chi_{\varepsilon} \wedge \alpha) = 0$$

holds, which in turn is equivalent to

$$\lim_{\varepsilon \rightarrow 0} \int_B \text{Tr}(A \wedge \Theta(\mathcal{E}, h_{\mathcal{E}})) \wedge p^*(\bar{\partial}\chi_{\varepsilon} \wedge \bar{\partial}\alpha) = 0.$$

By using Cauchy-Schwarz inequality, it would be sufficient to prove that

$$(7.27) \quad \lim_{\varepsilon \rightarrow 0} \int_B |A|_{h_{\mathcal{E}}, \omega_B}^2 \cdot |p^*(\bar{\partial}\chi_{\varepsilon})|_{\omega_B}^2 \omega_B^3 < \infty.$$

since the L^2 norm of the curvature tensor $\Theta(\mathcal{E}, h_{\mathcal{E}})$ with respect to the inverse image metric ω_B is finite. From now on, set

$$\psi_{\varepsilon} := |p^*(\bar{\partial}\chi_{\varepsilon})|_{\omega_B}^2,$$

keeping in mind that it could be replaced by any function dominating it.

Our first observation is that $|A|_{h_{\mathcal{E}}, \omega_B}^2 \leq C|A|_{h_0, \omega_B}^2$ by Theorem 7.13. We can assume that the holomorphic frame $(e_i)_{i=1, \dots, r}$ is almost orthogonal with respect to h_0 , and then we have

$$|A|_{h_0, \omega_B}^2 \leq C \sum_i |D'e_i|_{h_0, \omega_B}^2$$

for some constant $C > 0$ which is uniform on $B \setminus W$ (since it only depends on the metric h_0 , and not on A). Another application of Theorem 7.13 shows that it is enough

to control the integral $\int_B |D'\tau|_{h_\varepsilon, \omega_B}^2 \cdot \psi_\varepsilon \omega_B^3$ for $\tau = e_1, \dots, e_r$. We have the elementary formula

$$(7.28) \quad \Delta''_{\omega_B} |\tau|_{h_\varepsilon}^2 = |D'\tau|_{h_\varepsilon}^2 - C_1 |\tau|_{h_\varepsilon}^2$$

which holds pointwise on $B \setminus W$ for some constant C_1 . After multiplying with $\chi\psi_\varepsilon$ where χ is a cut-off function for B as in Step 2 of Theorem 7.13 whose gradient and Hessian are bounded with respect to ω_B , one can perform an integration by parts. Since $|\tau|_{h_\varepsilon}$ is bounded (cf (7.16)) (7.27) reduces to showing that

$$(7.29) \quad \limsup_{\varepsilon \rightarrow 0} \int_B (\psi_\varepsilon + |\partial\psi_\varepsilon| + |\Delta''_{\omega_B} \psi_\varepsilon|) \omega_B^3 < \infty.$$

Step 2. Estimating the integral (7.29).

Recall that $(V, x) \simeq (\mathbb{C}, 0) \times (S, s)$ where $i : (S, s) \hookrightarrow (\mathbb{C}^3, 0)$ is a germ of surface with log terminal singularities. Let $q : (\mathbb{C}^2, 0) \rightarrow (S, s)$ be a uniformizing chart and let $f := i \circ q : \mathbb{C}^2 \rightarrow \mathbb{C}^3$ be the composition. Therefore, up to shrinking V , the ramified cover $p : B \rightarrow V$ can be written as follows

$$(7.30) \quad (z, u, v) \rightarrow (z, f_1(u, v), f_2(u, v), f_3(u, v))$$

Consider next the truncation function

$$(7.31) \quad \chi_\varepsilon := \Xi_\varepsilon(\log \log(1/\rho)),$$

where the function ρ is defined by the formula

$$\rho(z, u, v) := (1 - \theta)^2(|u|^2 + |v|^2) + \sum_{i=1}^3 |f_i(u, v)|^2$$

and $\Xi_\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is equal to 1 on the interval $[0, \varepsilon^{-1}[$ and with zero on $[1 + \varepsilon^{-1}, \infty[$. We will also assume that the absolute value of the first and second derivative of Ξ_ε is uniformly bounded.

The reason for the choice of ρ as above is that we obviously have

$$(7.32) \quad \omega_B \simeq dd^c \rho + \sqrt{-1} dz \wedge d\bar{z},$$

given Proposition 7.6 (since the functions f_i above are in fact restrictions of the coordinates Z_i in \mathbb{C}^4 to X).

A rough estimate gives the inequality

$$(7.33) \quad |\bar{\partial}\chi_\varepsilon|_{\omega_B} \leq \frac{-\Xi'_\varepsilon(\log \log(1/\rho))}{\rho^{1/2} \log \frac{1}{\rho}} =: \psi_\varepsilon^{1/2},$$

where we use the fact that

$$\sup_B |\partial\rho|_{\omega_B} < \infty.$$

According to Step 1 (more precisely, (7.29)) we have to show that the following

$$(7.34) \quad \limsup_{\varepsilon \rightarrow 0} \int_{\text{supp } \Xi'_\varepsilon} (\psi_\varepsilon + |\partial\psi_\varepsilon|_{\omega_B} + |\Delta_{\omega_B}(\psi_\varepsilon)|) \omega_B^3 < \infty,$$

holds true. This basically boils down to showing that the following inequality

$$(7.35) \quad \lim_{\varepsilon \rightarrow 0} \int_{\text{supp } \Xi'_\varepsilon} \frac{((1-\theta)^2 \omega_{\text{orb}} + p^* \omega_X)^2 \wedge \sqrt{-1} dz \wedge d\bar{z}}{\rho^2 \log^2(|u|^2 + |v|^2)} < \infty$$

is true, where we use the notation

$$\omega_{\text{orb}} := dd^c(|u|^2 + |v|^2).$$

We will show at the beginning of Step 3 that (7.35) implies (7.34), cf Lemma 7.16. For the moment let us focus on proving that (7.35) holds true.

To this end, we remark that $\frac{1}{\rho^2} ((1-\theta)^2 \omega_{\text{orb}} + p^* \omega_X)^2$ is dominated by

$$\frac{\omega_{\text{orb}}^2}{(|u|^2 + |v|^2)^2} + \frac{2\omega_{\text{orb}} \wedge p^* \omega_X}{(|u|^2 + |v|^2) \cdot (\sum_{i=1}^3 |f_i(u, v)|^2)} + \frac{p^* \omega_X^2}{(\sum_{i=1}^3 |f_i(u, v)|^2)^2}$$

so that we basically to bound expressions of the form

$$(7.36) \quad \int_{\mathfrak{A}_\varepsilon} \frac{1}{\log^2 \varphi_1} \frac{dd^c \varphi_1 \wedge dd^c \varphi_2}{\varphi_1 \varphi_2}$$

independently of ε , where we use the following notations.

1. We integrate over the domain

$$\mathfrak{A}_\varepsilon := a + \frac{1}{\varepsilon} \leq \log \log \frac{1}{|u|^2 + |v|^2} \leq b + \frac{1}{\varepsilon}$$

for some constants $a < b$.

2. The functions φ_ℓ for $\ell = 1, 2$ are given by

$$(7.37) \quad \varphi_\ell := |u|^2 + |v|^2, \quad \text{or} \quad \varphi_\ell := \sum_i |f_i|^2.$$

This type of integral is being taken care of in the next and final step, cf Lemma 7.17 below.

Step 3. The final lemmas.

In this paragraph, we prove the auxiliary results, Lemma 7.16 and Lemma 7.17 that were used in Step 2 above.

First, we prove the following.

Lemma 7.16. — (7.35) implies (7.34).

Proof of Lemma 7.16. — In order to simplify the writing, we introduce the notation $\tau := (\Xi'_\varepsilon)^2$, so that we have

$$\psi_\varepsilon = \frac{\tau(\log \log(1/\rho))}{\rho \log^2 \rho}.$$

The first derivative of ψ_ε is equal to

$$\bar{\partial}\psi_\varepsilon = \left(\frac{\tau'(\log \log(1/\rho))}{\rho^2 \log^3 \rho} - \frac{\tau(\log \log(1/\rho))}{\rho^2 \log^2 \rho} - 2 \frac{\tau(\log \log(1/\rho))}{\rho^2 \log^3 \rho} \right) \bar{\partial}\rho.$$

We will not compute explicitly the $\partial\bar{\partial}\psi_\varepsilon$ since it would be too painful, but simply remark that the most singular terms are

$$\frac{1}{\rho^2 \log^2 \rho} \partial\bar{\partial}\rho, \quad \frac{1}{\rho^3 \log^2 \rho} \partial\rho \wedge \bar{\partial}\rho.$$

Now, as already remarked, we have

$$|\partial\bar{\partial}\rho|_{\omega_\theta} \leq C, \quad |\partial\rho \wedge \bar{\partial}\rho|_{\omega_\theta} \leq C\rho$$

from which it follows that

$$|\partial\psi_\varepsilon| \leq C \frac{1}{\rho^{\frac{3}{2}} \log^2 \rho}, \quad |\Delta''_{\omega_\theta} \psi_\varepsilon| \leq C \frac{1}{\rho^2 \log^2 \rho}.$$

□

Next, we show that the family of real numbers (7.36) is bounded.

Lemma 7.17. — *Let $n \geq 2$ and let $I = \{1, \dots, n\}$. Let u_1, \dots, u_n be psh functions with analytic singularities on a ball $B \subset \mathbb{C}^n$ centered at the origin such that $(u_i = -\infty) = \{0\}$ for all $i \in I$. Assume that $u_i \leq -1$ for all $i \in I$, pick real numbers $a < b$ and consider the domain*

$$A_\varepsilon = e^{a+\frac{1}{\varepsilon}} \leq -\log |z|^2 \leq e^{b+\frac{1}{\varepsilon}}.$$

With the notation above, we have

$$\limsup_{\varepsilon \rightarrow 0} \int_{A_\varepsilon} \frac{(-1)^n}{u_1 \cdots u_n} \frac{dd^c e^{u_1}}{e^{u_1}} \wedge \dots \wedge \frac{dd^c e^{u_n}}{e^{u_n}} < +\infty.$$

Proof of Lemma 7.17. — First, since u_i have analytic singularities concentrated at the origin, there exists constant $\alpha, \beta, C > 0$ such that up to shrinking the ball slightly

$$(7.38) \quad \forall i \in I, \quad \alpha u_1 - C \leq u_i \leq \beta u_1 + C.$$

There exists a finite set Γ such that for each index $i \in I$, there exists holomorphic functions $(f_{i,\gamma})_{\gamma \in \Gamma}$ on B such that

$$u_i = \log \left(\sum_{\gamma \in \Gamma} |f_{i,\gamma}|^2 \right) + \text{smooth},$$

where we allowed $f_{i,\gamma} \equiv 0$ so that Γ can be chosen independently of i . Now, let $\mu : X \rightarrow B$ be a log resolution of the ideal sheaf $(f_{i,\gamma})_{i \in I, \gamma \in \Gamma} \subset \mathcal{O}_B$ and let $E = \mu^{-1}(0)$ be the exceptional divisor. We need to show that

$$\limsup_{\varepsilon \rightarrow 0} \int_{\mu^{-1}(A_\varepsilon)} \frac{(-1)^n}{\mu^* u_1 \cdots \mu^* u_n} \mu^* \left(\frac{dd^c e^{u_1}}{e^{u_1}} \wedge \dots \wedge \frac{dd^c e^{u_n}}{e^{u_n}} \right) < +\infty.$$

Let $x \in E$. We can find a neighborhood U_x of $x \in X$, an integer $1 \leq k \leq n$, a system of holomorphic coordinates z_1, \dots, z_n on U_x centered at x and positive numbers a_{i1}, \dots, a_{ik} such that

$$(7.39) \quad \forall i \in I, \quad \mu^* u_i = \sum_{j=1}^k a_{ij} \log |z_j|^2 + \text{smooth} \quad \text{on } U_x.$$

The fact that all coefficients a_{ij} are positive follows from (7.38). Moreover, there exist constants a_x, b_x such that

$$\mu^{-1}(A_\varepsilon) \cap U_x \subset \left(e^{a_x + \frac{1}{\varepsilon}} \leq -\log |z_1 \cdots z_k|^2 \leq e^{b_x + \frac{1}{\varepsilon}} \right) =: A_{x,\varepsilon}.$$

Thanks to (7.39) and the formula

$$e^{-u} dd^c e^u = dd^c u + du \wedge d^c u,$$

it is sufficient to bound uniformly the integral

$$\int_{A_{x,\varepsilon}} \frac{(-1)^{n^2/2}}{(-\log |z_1 \cdots z_k|^2)^k} \bigwedge_{j=1}^k \frac{dz_j \wedge d\bar{z}_j}{|z_j|^2} \wedge \bigwedge_{j=k+1}^n dz_j \wedge d\bar{z}_j.$$

Set $r_i = |z_i|$ for $1 \leq i \leq k$, which are subject to the conditions $r_i \leq 1$ and $e^{-e^{b_x + \frac{1}{\varepsilon}}} \leq r_1^2 \cdots r_k^2 \leq e^{-e^{a_x + \frac{1}{\varepsilon}}}$. In particular, we have $r_i^2 \geq e^{-e^{b_x + \frac{1}{\varepsilon}}}$ for all i . Therefore, the integral above is dominated by

$$e^{-ka_x - \frac{k}{\varepsilon}} \prod_{i=1}^k \int_{e^{-e^{b_x + \frac{1}{\varepsilon}}} \leq r_i^2 \leq 1} \frac{dr_i}{r_i} = 2^{-k} e^{k(b_x - a_x)}.$$

Hence our integral is under control over $\mu^{-1}(A_\varepsilon) \cap U_x$, and we can conclude by compactness of E . \square

This concludes the proof of Theorem 7.15. \square

7.6. Proof of Theorem 7.2. — Consider the metric $h_{\mathcal{F}}$ on \mathcal{F} constructed at the previous step. Since it is an orbifold hermitian metric on $X \setminus U$ and ϕ satisfies (7.1), we have for ε small enough

$$(7.40) \quad \begin{aligned} C\Delta(\mathcal{F}) \cdot [\omega_X] &= C \int_X \Delta(\mathcal{F}, h_{\mathcal{F}}) \wedge (\omega_X + dd^c \phi) \\ &= \int_X \chi_\varepsilon \Delta(\mathcal{F}, h_{\mathcal{F}}) \wedge \omega_\theta - \int_X \chi_\varepsilon \Delta(\mathcal{F}, h_{\mathcal{F}}) \wedge dd^c \left((C\theta^2 + \theta^2) \varphi_p + (1 - \theta)^2 \varphi_{\text{orb}} \right) \end{aligned}$$

where $(\chi_\varepsilon)_{\varepsilon > 0}$ is converging to the characteristic function of $X \setminus \{p_1, \dots, p_k\}$. Note that the integrals above are convergent thanks to Remark 7.11.

The first integral on the RHS of the above equality is semi-positive since its integrand is pointwise semi-positive by the HE condition, cf e.g. the proof of [Kob87, Theorem 4.4.7].

Concerning the second one, we show next that one can integrate by parts thanks to Theorem 7.15. Indeed, if we define

$$\alpha := \chi_\varepsilon \partial \left((C\theta^2 + \theta^2) \varphi_p + (1 - \theta)^2 \varphi_{\text{orb}} \right),$$

then we have to make sure that α and its $\bar{\partial}$ are bounded with respect to ω_θ –note that the bound in question is not required to be independent of ε . But this is clear, since φ_p is a potential for ω_X , and moreover the derivative of θ is bounded.

Thus, the quantity we have to analyse becomes the following expression

$$(7.41) \quad \int_X \Delta(\mathcal{F}, h_{\mathcal{F}}) \wedge \partial \chi_\varepsilon \wedge \bar{\partial} ((C\theta^2 + \theta^2)\varphi_p + (1 - \theta)^2\varphi_{\text{orb}}).$$

For $\varepsilon \ll 1$, the integral (7.41) coincides with

$$(7.42) \quad (C + 1) \int_U \Delta(\mathcal{F}, h_{\mathcal{F}}) \wedge \partial \chi_\varepsilon \wedge \bar{\partial} \varphi_p,$$

and this last expression converges to zero provided that we choose carefully the truncation family $(\chi_\varepsilon)_{\varepsilon > 0}$ as follows. For each point p_i we take

$$\Xi_\varepsilon(\log \log(1/|Z|^2))$$

where Z above are coordinates centred at p_i . The function φ_p is supposed to be a potential for ω_X locally near each of p_i and we certainly can construct one such that $|\partial \varphi_p| = \mathcal{O}(|Z|)$, i.e. its gradient is vanishing at p . It follows that

$$(7.43) \quad \sup_{U_{\text{reg}}} |\partial \chi_\varepsilon \wedge \bar{\partial} \varphi_p|_{\omega_X} \leq C$$

for some constant C independent of ε . Then, since the curvature form belongs to L^2 we can conclude that (7.42) tends to zero as $\varepsilon \rightarrow 0$.

Remark 7.18. — The property (7.43) is precisely what forces us to consider the metric ω_θ : if we simply work with ω , then we have no trouble with the integration by parts, but it is not clear how to evaluate

$$\sup_{U_{\text{reg}}} |\partial \chi_\varepsilon \wedge \bar{\partial} \varphi_{\text{orb}}|_\omega$$

as $\varepsilon \rightarrow 0$. Of course, the L^1 -norm of the function $|\partial \chi_\varepsilon \wedge \bar{\partial} \varphi_{\text{orb}}|_\omega$ is converging to zero, but then in order to obtain the desired conclusion, we would need the curvature of $(\mathcal{F}, h_{\mathcal{F}})$ to be bounded, instead of L^2 .

Remark 7.19. — Let ω be the metric obtained in Theorem 5.2. In the context of Theorem 7.10, let $h_{\mathcal{F}}$ be the metric on \mathcal{F} , which satisfies the HE equation with respect to ω on X_{reg} . Then we have

$$\int_{X_{\text{reg}}} \Delta(\mathcal{F}, h_{\mathcal{F}}) \wedge \bar{\partial} \tau = 0$$

for any $(n - 2, n - 3)$ form τ whose support lies in $X \setminus Z$, and such that it has at most orbifold singularities. It would be very interesting to analyze further the properties of $\Delta(\mathcal{F}, h_{\mathcal{F}})$.

7.7. Proof of Corollary 7.4. — We have seen in (7.40) that the first term of the RHS is non-negative while the second one goes to zero when $\varepsilon \rightarrow 0$. Therefore, in the equality case we must have

$$\int_X \Delta(\mathcal{F}, h_{\mathcal{F}}) \wedge \omega_{\theta} = 0.$$

Recall from Setup 7.12 that \mathcal{F} is locally free on $X_{\text{reg}}^{\circ} := X_{\text{reg}} \setminus \{p_1, \dots, p_k\}$, hence the integrand is smooth on that locus. Since $h_{\mathcal{F}}$ is HE with respect to ω_{θ} , the integrand coincides up to a dimensional constant with the squared norm of $\Theta(\mathcal{F}, h_{\mathcal{F}}) - \frac{1}{r} \text{tr}_{\text{End}}(\Theta(\mathcal{F}, h_{\mathcal{F}})) \cdot \text{Id}_{\mathcal{F}}$, cf the proof of [Kob87, Theorem 4.4.7]. Therefore $\Theta(\mathcal{F}, h_{\mathcal{F}})$ is diagonal on X_{reg}° , i.e. $\mathcal{F}|_{X_{\text{reg}}^{\circ}}$ is projectively hermitian flat there. In other words, there is a representation $\rho : \pi_1(X_{\text{reg}}^{\circ}) \rightarrow \text{PU}(r, \mathbb{C})$ such that $\mathbb{P}(\mathcal{F}|_{X_{\text{reg}}^{\circ}}) \simeq (\widetilde{X_{\text{reg}}^{\circ}} \times \mathbb{P}^{r-1})/\rho$ where ρ acts diagonally in the natural way, cf e.g. [Kob87, Proposition 1.4.22].

Now, let $p \in X_{\text{reg}} \setminus X_{\text{reg}}^{\circ}$ and let U be a (small) contractible Stein neighborhood of p . Set $U^{\circ} := U \setminus \{p\}$, which is still simply connected. We have $\mathbb{P}(\mathcal{F}|_{U^{\circ}}) \simeq \mathbb{P}(\mathcal{O}_{U^{\circ}}^{\oplus r})$, hence there exists a line bundle L° on U° such that $\mathcal{F}|_{U^{\circ}} \simeq (L^{\circ})^{\oplus r}$. Let $i : U^{\circ} \hookrightarrow U$ be the inclusion. We claim that $L := i_*(L^{\circ})$ is coherent. Indeed since \mathcal{F} is coherent and reflexive, we have $\mathcal{F}|_U = i_*(\mathcal{F}|_{U^{\circ}}) = L^{\oplus r}$ hence L is coherent (and reflexive, too). Since L has rank one and U is smooth, L is a line bundle. Moreover, L is trivial since U is Stein and contractible. In particular, \mathcal{F} is locally free near p . Moreover, the isomorphism

$$\pi_1(X_{\text{reg}}^{\circ}) \rightarrow \pi_1(X_{\text{reg}})$$

induced by the inclusion $X_{\text{reg}}^{\circ} \hookrightarrow X_{\text{reg}}$ shows that $\mathcal{F}|_{X_{\text{reg}}}$ is projectively hermitian flat, which shows the first item in the corollary.

Next, we claim that X admits a maximally quasi-étale cover $p : Y \rightarrow X$ in the sense of e.g. [CGGN22, Definition 5.3]. This is a consequence of the proof of the existence of such a cover when X is quasi-projective given by [GKP16] and the validity of the results of [BCHM10] for projective morphisms between complex spaces as proved by Fujino [Fuj22], as explained in [CGGN22, Remark 6.10]. From the previous step, $p^*(\mathcal{F}|_{X_{\text{reg}}})$ is locally free and projectively hermitian flat on $p^{-1}(X_{\text{reg}})$, and for the same reasons as above, it extends to a projectively hermitian flat bundle on Y_{reg} which is nothing but $p^{[*]}\mathcal{F}$ by reflexivity of both sheaves. The second item of the corollary is now straightforward by the definition of maximally quasi-étale cover, cf e.g. [GKP22, Proposition 3.10] for the relevant details.

Appendix by F. Campana⁽¹⁾, A. Höring⁽²⁾ and T. Peternell⁽³⁾

Abundance on Kähler threefolds. — In the paper [CHP16], the following abundance type theorem (Theorem 8.2) is stated

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Theorem A.1. — Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities such that K_X is nef. If the numerical dimension $\nu(X) \geq 2$, then the Kodaira dimension $\kappa(X) \geq 1$.

This is an important step in the proof the Abundance Conjecture for Kähler threefolds [CHP16, Thm.1.1]:

Theorem A.2. — Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities such that K_X is nef. Then K_X is semi-ample.

Note that Theorem A.1 was already shown, [Pet01], for all compact Kähler threefolds X unless there is no positive-dimensional compact subvariety through the very general point of X and X is not bimeromorphic to an étale quotient of a torus, a case which needs to be ruled out.

The proof of Theorem A.1 however contains a gap, noticed and communicated to us by Das and Ou. We explain the gap and show how to fix it using the main result of the companion paper, that is Theorem A above. For the reader's convenience, we recall below Theorem A which is the "orbifold version" of [CHP16, Theorem 7.1]:

Theorem A.3. — Let X be a normal compact Kähler threefold with klt singularities, and let α be a Kähler class on X . Let \mathcal{F} be a reflexive coherent \mathbb{Q} -sheaf on X of rank $r > 0$ that is α -stable. Then we have

$$\alpha \cdot \hat{c}_2(\mathcal{F}) \geq \left(\frac{r-1}{2r}\right) \alpha \cdot c_1^2(\mathcal{F}).$$

If X is projective, Theorem A.3 is well-known, see [Kwc92, chap.10] for the surface case and [GKPT19, Thm.6.1]. The technique is to reduce to the surface case by taking hyperplane sections. This is of course not possible in the Kähler case.

Proof of Theorem A.1. — The issue in the proof of [CHP16, Thm.8.2] is concerned with the proof of Equality (33). To be precise, we are in the following situation. Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with klt singularities. Further, X carries a divisor $D \in |mK_X|$ with the following properties.

1. Set $B := \text{Supp}D$. The pair (X, B) is lc and $X \setminus B$ has terminal singularities.
2. $K_X + B$ is nef with $\nu(K_X + B) = 2$; further, $\kappa(X) = \kappa(K_X + B)$.
3. For all irreducible components $T \subset B$, we have $(K_X + B)_T \neq 0$.
4. $(K_X + B) \cdot K_X^2 \geq 0$.

Then we claim that $\kappa(X) \geq 1$.

To prove this, we consider a terminal modification $\mu : X' \rightarrow X$, set $L := m(K_X + B)$; $L' = \mu^*(L)$ and things come down to prove Equality (33) in [CHP16], stating

$$(7.44) \quad L' \cdot (K_{X'}^2 + c_2(X')) \geq 0.$$

To prove Equation (7.44), we claimed that we may assume that X has canonical singularities in codimension two. This however is not true in general. Once Equation (7.44) is settled (which holds when X has canonical singularities), the proof given in [CHP16] is complete. Thus we have to treat the case that X has (possibly) non-canonical singularities.

Recall that a klt space has quotient singularities in codimension two, so we can use the orbifold Chern classes defined in an analytic context in [GK20]. Let $S \subset X$ be the finite locus where X does not have quotient singularities. We define the intersection $\omega \cdot \hat{c}_2(\mathcal{F})$ for a Kähler form ω on X and a reflexive sheaf \mathcal{F} on X which is a \mathbb{Q} -vector bundle on $X \setminus S$ as follows: the orbifold Chern class $\hat{c}_2(\mathcal{F}) \in H^4(X \setminus S, \mathbb{R})$ extends uniquely to a class γ on X , and we simply define

$$\omega \cdot \hat{c}_2(\mathcal{F}) := [\omega] \cdot \gamma,$$

where $[\omega] \in H^2(X, \mathbb{R})$ is the class of ω .

For the proof of Theorem A.1, instead of taking a terminalization, we let $\mu : X' \rightarrow X$ be a desingularization. We proceed following the lines of [Kwc92, chapter 14]. As in [Kwc92, Lemma 14.3.1], things come down to prove

$$(7.45) \quad L \cdot (K_X^2 + \hat{c}_2(X)) \geq 0.$$

Since

$$L \cdot (K_X + B) \cdot B = 0,$$

it suffices by [Kwc92, Lemma 14.3.2] to show that

$$(7.46) \quad L \cdot \hat{c}_2(\Omega_X(\log B)) \geq 0.$$

To prove Equation 7.46, we proceed as in [CHP16] in an orbifold context: as a first step note that Theorem A.3 yields the following orbifold version of [CHP16, Theorem 7.2].

Theorem A.4. — Let (X, ω) be a compact Kähler threefold with klt singularities, and let $\mathcal{F} \rightarrow X$ be a non-zero reflexive coherent \mathbb{Q} -sheaf such that $\det \mathcal{F}$ is \mathbb{Q} -Cartier. Suppose that there exists a pseudoeffective class $P \in N^1(X)$ such that

$$L := c_1(\mathcal{F}) + P$$

is a nef class. Suppose furthermore that for all $0 < \varepsilon \ll 1$ the sheaf \mathcal{F} is $(L + \varepsilon\omega)$ -generically nef. Then we have

$$L \cdot \hat{c}_2(\mathcal{F}) \geq \frac{1}{2}(L \cdot c_1^2(\mathcal{F}) - L^3).$$

In particular, if $L \cdot c_1^2(\mathcal{F}) \geq 0$ and $L^3 = 0$, then

$$(7.47) \quad L \cdot \hat{c}_2(\mathcal{F}) \geq 0.$$

Now we aim to apply Theorem A.4 for the reflexive sheaf $\mathcal{F} = \Omega_X(\log B)$ to prove Equation 7.46. Therefore, we need to check generic nefness of \mathcal{F} which follows from the generic nefness of Ω_X . This is done by the following adaption of [CHP16, Proposition 8.2], thereby finalizing the proof of Theorem A.1.

Proposition A.5. — Let X be a normal non-algebraic compact Kähler space of dimension three with klt singularities such that $\kappa(X) \geq 0$. Then Ω_X is generically nef with respect to any nef class α , i.e. for every torsion-free quotient sheaf

$$\Omega_X \rightarrow \mathcal{Q} \rightarrow 0,$$

we have $\alpha^2 \cdot c_1(\mathcal{Q}) \geq 0$.

As a preparation we show a technical lemma:

Lemma A.6. — Let X be a normal non-algebraic compact Kähler space of dimension three with klt singularities such that $\kappa(X) \geq 0$. Let $\mu : \hat{X} \rightarrow X$ be a resolution of singularities by a compact Kähler threefold. Then we have $\kappa(\hat{X}) \geq 0$.

It is well-known that the statement is not true for projective threefolds: there are projective threefolds with numerically trivial canonical class that are klt and uniruled, in particular every resolution has Kodaira dimension $-\infty$.

Proof. — We can suppose without loss of generality that μ is a composition of blowups. Then the μ -exceptional locus E is a union of projective varieties: indeed the morphism is projective since it is a composition of blowups. Moreover $\mu(E)$ is a finite union of varieties of dimension at most one, so $\mu(E)$ is projective.

Arguing by contradiction we assume that $\kappa(\hat{X}) = -\infty$. Since \hat{X} is a smooth compact Kähler threefold, this is equivalent to \hat{X} being uniruled [HP16, Cor.1.4].

Since \hat{X} is not algebraic, \hat{X} cannot be rationally connected. In fact the MRC fibration is an almost holomorphic fibration

$$\hat{f} : \hat{X} \dashrightarrow \hat{S}$$

onto a non-projective surface [HP15, Intro]. Since E is a union of projective surfaces, it does not surject onto \hat{S} . Therefore \hat{f} descends to an almost holomorphic map $f : X \dashrightarrow \hat{S}$ whose general fiber is a smooth rational curve. This contradicts $\kappa(X) \geq 0$. \square

Proof of Proposition A.5. — Let $\pi : \hat{X} \rightarrow X$ be a desingularisation by a compact Kähler manifold, and let \mathcal{K} be the kernel of the induced epimorphism

$$(\pi^* \Omega_X) / \text{torsion} \rightarrow (\pi^* \mathcal{Q}) / \text{torsion} \rightarrow 0.$$

Using the injective map $(\pi^* \Omega_X) / \text{torsion} \hookrightarrow \Omega_{\hat{X}}$ we may view \mathcal{K} as a subsheaf of $\Omega_{\hat{X}}$, and we denote by $\hat{\mathcal{K}}$ its saturation in $\Omega_{\hat{X}}$. Set

$$\hat{\mathcal{Q}} := \Omega_{\hat{X}} / \hat{\mathcal{K}},$$

then $\hat{\mathcal{Q}}$ is a torsion-free quotient of $\Omega_{\hat{X}}$ coinciding with $(\pi^* \mathcal{Q}) / \text{torsion}$ in the complement of the exceptional locus. In particular we have

$$\alpha^2 \cdot c_1(\mathcal{Q}) = (\pi^* \alpha)^2 \cdot c_1((\pi^* \mathcal{Q}) / \text{torsion}) = (\pi^* \alpha)^2 \cdot c_1(\hat{\mathcal{Q}}).$$

Thus we are left to show that $\pi^*(\alpha)^2 \cdot c_1(\hat{\mathcal{Q}}) \geq 0$. To do this, we apply Enoki's theorem [Eno88, Theorem 1.4], and need to write as \mathbb{Q} -divisors,

$$K_{\hat{X}} = L + D$$

with L nef and D effective. By Lemma A.6 we have $\kappa(\hat{X}) \geq 0$. Thus we can just set $L = 0$ and D an effective \mathbb{Q} -divisor such that $D \sim_{\mathbb{Q}} K_X$. \square

Remark A.7. — In their recent preprints [DO23a, DO23b] Das and Ou have extended Theorem A.2 to log-canonical pairs, in particular they give an independent proof of Theorem A.1.

References

- [Agm60] S. AGMON – “The L_p approach to the Dirichlet problem”, *Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser.* **13** (1960), p. 405–448 (English).
- [Art69] M. ARTIN – “Algebraic approximation of structures over complete local rings”, *Inst. Hautes Études Sci. Publ. Math.* (1969), no. 36, p. 23–58.
- [BBE⁺19] R. J. BERMAN, S. BOUCKSOM, P. EYSSIDIEUX, V. GUEDJ & A. ZERIAHI – “Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties”, *J. Reine Angew. Math.* **751** (2019), p. 27–89.
- [BCHM10] C. BIRKAR, P. CASCINI, C. HACON & J. MCKERNAN – “Existence of minimal models for varieties of log general type”, *J. Amer. Math. Soc.* **23** (2010), p. 405–468.
- [BS94] S. BANDO & Y.-T. SIU – “Stable sheaves and Einstein-Hermitian metrics”, in *Geometry and analysis on complex manifolds. Festschrift for Professor S. Kobayashi’s 60th birthday*, Singapore: World Scientific, 1994, p. 39–59 (English).
- [CC21] X. CHEN & J. CHENG – “On the constant scalar curvature Kähler metrics. I: A priori estimates”, *J. Am. Math. Soc.* **34** (2021), no. 4, p. 909–936 (English).
- [CGG22] B. CLAUDON, P. GRAF & H. GUENANCIA – “Numerical characterization of complex torus quotients”, *Comment. Math. Helv.* (2022), no. 4, p. pp. 769–799.
- [CGGN22] B. CLAUDON, P. GRAF, H. GUENANCIA & P. NAUMANN – “Kähler spaces with zero first Chern class: Bochner principle, Albanese map and fundamental groups”, *J. Reine Angew. Math.* **786** (2022), p. 245–275.
- [CGN⁺23] J. CAO, P. GRAF, P. NAUMANN, M. PÄUN, T. PETERNELL & X. WU – “Hermite-Einstein metrics in singular settings”, Preprint [arXiv:2303.08773](https://arxiv.org/abs/2303.08773), 2023.
- [Che22] X. CHEN – “Admissible Hermitian-Yang-Mills connections over normal varieties”, Preprint [arXiv:2205.12305](https://arxiv.org/abs/2205.12305), 2022.
- [CHP16] F. CAMPANA, A. HÖRING & T. PETERNELL – “Abundance for Kähler threefolds”, *Ann. Sci. Éc. Norm. Supér. (4)* **49** (2016), no. 4, p. 971–1025 (English).
- [CHP23] F. CAMPANA, A. HÖRING & T. PETERNELL – “Erratum and addendum to the paper: Abundance for Kähler threefolds”, Preprint [arXiv:2304.10161](https://arxiv.org/abs/2304.10161), 2023.
- [CW24] X. CHEN & R. A. WENTWORTH – “A Donaldson-Uhlenbeck-Yau theorem for normal varieties and semistable bundles on degenerating families”, *Math. Ann.* **388** (2024), no. 2, p. 1903–1935 (English).
- [DO23a] O. DAS & W. OU – “On the log abundance for compact Kähler 3-folds”, *arXiv preprint, to appear in manuscripta mathematica* **2201.01202** (2023).
- [DO23b] ———, “On the log abundance for compact Kähler threefolds II”, *arXiv preprint* **2306.00671** (2023).
- [Don85] S. DONALDSON – “Anti self-dual Yang Mills connections over complex algebraic surfaces and stable vector bundles”, *Proc. Lond. Math. Soc., III. Ser.* **50** (1985), p. 1–26.
- [EGZ09] P. EYSSIDIEUX, V. GUEDJ & A. ZERIAHI – “Singular Kähler-Einstein metrics”, *J. Amer. Math. Soc.* **22** (2009), p. 607–639.
- [Eno88] I. ENOKI – “Stability and negativity for tangent sheaves of minimal Kähler spaces”, in *Geometry and analysis on manifolds (Katata/Kyoto, 1987)*, Lecture Notes in Math., vol. 1339, Springer, Berlin, 1988, p. 118–126.
- [Fuj22] O. FUJINO – “Minimal model program for projective morphisms between complex analytic spaces”, Preprint [arXiv:2201.11315](https://arxiv.org/abs/2201.11315), 2022.
- [GGZ23] V. GUEDJ, H. GUENANCIA & A. ZERIAHI – “Strict positivity of Kähler-Einstein currents”, Preprint [arXiv:2305.12422](https://arxiv.org/abs/2305.12422), 2023.

- [GK20] P. GRAF & T. KIRSCHNER – “Finite quotients of three-dimensional complex tori”, *Ann. Inst. Fourier (Grenoble)* **70** (2020), no. 2, p. 881–914.
- [GKKP11] D. GREB, S. KEBEKUS, S. J. KOVÁCS & T. PETERNELL – “Differential forms on log canonical spaces”, *Publ. Math. Inst. Hautes Études Sci.* (2011), no. 114, p. 87–169.
- [GKP16] D. GREB, S. KEBEKUS & T. PETERNELL – “Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of abelian varieties”, *Duke Math. J.* **165** (2016), no. 10, p. 1965–2004.
- [GKP22] ———, “Projective flatness over klt spaces and uniformisation of varieties with nef anti-canonical divisor”, *J. Algebr. Geom.* **31** (2022), no. 3, p. 467–496 (English).
- [GKPT19] D. GREB, S. KEBEKUS, T. PETERNELL & B. TAJI – “The Miyaoka-Yau inequality and uniformisation of canonical models”, *Ann. Scient. Éc. Norm. Sup. (4)* **52** (2019), no. 6, p. 1487–1535.
- [GPSS22] B. GUO, D. PHONG, J. SONG & J. STURM – “Diameter estimates in Kähler geometry”, Preprint [arXiv:2209.09428](https://arxiv.org/abs/2209.09428), to appear in CPAM, 2022.
- [GPSS23] ———, “Sobolev inequalities on Kähler spaces”, Preprint [arXiv:2311.00221](https://arxiv.org/abs/2311.00221), 2023.
- [GPSS24] ———, “Diameter estimates in Kähler geometry II: removing the small degeneracy assumption”, Preprint [arXiv:2405.18280](https://arxiv.org/abs/2405.18280), 2024.
- [GPT23] B. GUO, D. PHONG & F. TONG – “On L^∞ estimates for complex Monge-Ampère equations”, *Ann. Math. (2)* **198** (2023), no. 1, p. 393–418 (English).
- [GT77] D. GILBARG & N. TRUDINGER – *Elliptic partial differential equations of second order*, Springer-Verlag, 1977.
- [GT24] V. GUEDJ & T. D. TÔ – “Kähler families of Green’s functions”, Preprint [arXiv:2405.17232](https://arxiv.org/abs/2405.17232), 2024.
- [Gue16] H. GUENANCIA – “Semistability of the tangent sheaf of singular varieties”, *Algebraic Geometry* **3** (2016), no. 5, p. 508–542.
- [GZ12] V. GUEDJ & A. ZERIAHI – “Stability of solutions to complex Monge-Ampère equations in big cohomology classes”, *Math. Res. Lett.* **19** (2012), no. 5, p. 1025–1042.
- [HP15] A. HÖRING & T. PETERNELL – “Mori fibre spaces for Kähler threefolds”, *J. Math. Sci. Univ. Tokyo* **22** (2015), no. 1, p. 219–246.
- [HP16] A. HÖRING & T. PETERNELL – “Minimal models for Kähler threefolds”, *Invent. Math.* **203** (2016), no. 1, p. 217–264.
- [Kob87] S. KOBAYASHI – *Differential geometry of complex vector bundles.*, Princeton, NJ: Princeton University Press; Tokyo: Iwanami Shoten Publishers, 1987 (English).
- [Kwc92] J. KOLLÁR (WITH 14 COAUTHORS) – *Flips and abundance for algebraic threefolds*, Société Mathématique de France, Paris, 1992, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).
- [LT19] C. LI & G. TIAN – “Orbifold regularity of weak Kähler-Einstein metrics”, in *Advances in complex geometry*, Contemp. Math., vol. 735, Amer. Math. Soc., [Providence], RI, [2019] ©2019, p. 169–178.
- [MP97] Y. MIYAOKA & T. PETERNELL – *Geometry of higher dimensional algebraic varieties*, DMV Semin., vol. 26, Basel: Birkhäuser, 1997 (English).
- [Mum83] D. MUMFORD – “Towards an enumerative geometry of the moduli space of curves”, in *Arithmetic and geometry, Vol. II*, Progr. Math., vol. 36, Birkhäuser Boston, Boston, MA, 1983, p. 271–328.
- [Ou24] W. OU – “Orbifold modifications of complex analytic varieties”, Preprint [arXiv:2401.07273](https://arxiv.org/abs/2401.07273), 2024.

- [Pet01] T. PETERNELL – “Towards a Mori theory on compact Kähler threefolds. III”, *Bull. Soc. Math. France* **129** (2001), no. 3, p. 339–356.
- [Ros68] H. ROSSI – “Picard variety of an isolated singular point”, *Rice Univ. Stud.* **54** (1968), no. 4, p. 63–73 (English).
- [Rub14] Y. A. RUBINSTEIN – “Smooth and singular Kähler-Einstein metrics”, in *Geometric and spectral analysis*, Contemp. Math., vol. 630, Amer. Math. Soc., Providence, RI, 2014, p. 45–138.
- [Sav07] O. SAVIN – “Small perturbation solutions for elliptic equations”, *Commun. Partial Differ. Equations* **32** (2007), no. 4, p. 557–578 (English).
- [UY86] K. UHLENBECK & S. YAU – “On the existence of Hermitian-Yang-Mills connections in stable vector bundles.”, *Commun. Pure Appl. Math.* **39** (1986), p. S257–S293.
- [Vu24] D.-V. VU – “Uniform diameter estimates for Kähler metrics”, Preprint [arXiv:2405.14680](https://arxiv.org/abs/2405.14680), 2024.
- [Yau75] S.-T. YAU – “Harmonic functions on complete Riemannian manifolds.”, *Commun. Pure Appl. Math.* **28** (1975), p. 201–228.

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