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# A characterization of the Taylor expansion of $\lambda$ -terms\*

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## Abstract

The Taylor expansion of  $\lambda$ -terms, as introduced by Ehrhard and Regnier, expresses a  $\lambda$ -term as a series of multi-linear terms, called simple terms, which capture bounded computations. Normal forms of Taylor expansions give a notion of infinitary normal forms, refining the notion of Böhm trees in a quantitative setting.

We give the algebraic conditions over a set of normal simple terms which characterize the property of being the normal form of the Taylor expansion of a  $\lambda$ -term. From this full completeness result, we give further conditions which semantically describe normalizable and total  $\lambda$ -terms.

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## 1 Introduction

Recently various semantics of linear logic and  $\lambda$ -calculus have been proposed (e.g. [5, 6]), where morphisms are infinitely differentiable functions between vector spaces (or more generally, modules). In fact, one can define the Taylor expansion of a function as an infinite sum of terms that are calculated from the values of the function's  $n$ -th derivatives at a given point. These models subsequently provide an intriguing way to describe the regularity of the behavior of a  $\lambda$ -term — the functions interpreting  $\lambda$ -terms are analytic, i.e. they are equal to their Taylor expansion at any point of their domain.

The main interest of Taylor expansion for the  $\lambda$ -calculus lies in the analogy that can be drawn between the usual notion of differentiation and its computational meaning. In fact, applying the derivative at 0 of the  $\lambda$ -term  $M$  on the argument  $N$  corresponds to passing the input  $N$  to  $M$  *exactly once*. This can be formalized in the setting of the *differential  $\lambda$ -calculus* [7] — an extension of the  $\lambda$ -calculus with a syntactic derivative operator, allowing to compute the optimal approximation of a program when applied to depletable arguments. The Taylor formula yields a natural notion of linear approximation of the ordinary application of  $\lambda$ -calculus. Let  $M$  and  $N$  be two  $\lambda$ -terms and assume  $(D^n M \cdot N^n)0$  denotes the  $n$ -th derivative of  $M$  at 0 applied to  $N$  (as in the usual notation  $f^{(n)}(0) \cdot x^n$  with  $M$  as function  $f$ ,  $N$  as argument  $x$ , and 0 as non-linear argument). Then the application of the term  $M$  to

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the term  $N$  becomes:

$$(M)N = \sum_{n=0}^{\infty} \frac{1}{n!} (D^n M \cdot N^n)0. \quad (1)$$

Basically,  $(D^n M \cdot N^n)0$  expresses an evaluation of the application  $(M)N$  using exactly  $n$  linear copies of  $N$ . The coefficient  $\frac{1}{n!}$  is there to take care of the number of permutations on the argument of  $D^n M$ .

More generally, if one fully develops each application occurring in a  $\lambda$ -term  $M$  into its corresponding Taylor expansion, one expresses the term as an infinite sum  $M^*$  of purely “differential programs”, all of which containing only multi-linear applications and applications to 0. Ehrhard and Regnier develop a convenient notation for expressing such “differential programs”, calling them *simple terms* [8, 9]. Also, the authors define a rewriting system inspired by the standard rules for computing derivatives of polynomials which allow to give a normal form  $\text{NF}(M^*)$  to a Taylor series  $M^*$ .

We denote by  $\Delta^{\text{NF}}$  the set of normal simple terms (Equation (10)) and by  $\mathbb{Q}^+[\Delta^{\text{NF}}]$  the formal linear combinations of terms in  $\Delta^{\text{NF}}$  taking coefficients in the semi-ring of positive rational numbers. The combinations of  $\mathbb{Q}^+[\Delta^{\text{NF}}]$  allow to express the normal forms of the Taylor expansion of  $\lambda$ -terms, however not all formal combinations can be associated with a  $\lambda$ -term. We propose here a characterization of the image of the  $\lambda$ -calculus into  $\mathbb{Q}^+[\Delta^{\text{NF}}]$ .

Let us underline that the space  $\mathbb{Q}^+[\Delta^{\text{NF}}]$  is an example of syntax-based semantics of the untyped  $\lambda$ -calculus (as well as of its differential extension) — the interpretation being given by the function  $M \mapsto \text{NF}(M^*)$ , and the composition being assured by a notion of linear substitution (see Equation (5)). Roughly speaking, such a model is a quantitative refinement of the model provided by Böhm trees [1, §10] and it is crucial to the study of vectors based semantics, exactly as Böhm trees are at the core of domain-based denotational semantics. Our result basically amounts to defining a sub-model of  $\mathbb{Q}^+[\Delta^{\text{NF}}]$  which is fully complete for the untyped  $\lambda$ -calculus.

Fortunately, the problem is significantly simplified by Ehrhard and Regnier’s result. In particular, they prove [9, Corollary 35] that the normal form of the Taylor expansion of a  $\lambda$ -term  $M$  can be defined as:

$$\text{NF}(M^*) = \sum_{t \in \text{NF}(\tau(M))} \frac{1}{\mathfrak{m}(t)} t \quad (2)$$

where:  $\mathfrak{m}(t)$  is a natural number (called *multiplicity coefficient*) univocally defined by  $t$ , and  $\text{NF}(\tau(M))$  is the *support* of the normal form of the Taylor expansion of  $M$ , i.e. the set of the simple terms appearing with a non-zero coefficient in  $\text{NF}(M^*)$ . The open issue is then to characterize the sets of simple terms which are of the form  $\text{NF}(\tau(M))$ , for some  $\lambda$ -term  $M$ .

We give three conditions which are necessary and sufficient for a subset  $\mathcal{T}$  of  $\Delta^{\text{NF}}$  to be equal to  $\text{NF}(\tau(M))$ , for some  $\lambda$ -term  $M$  (Theorem 25): (i) the set of free variables occurring in  $\mathcal{T}$  must be finite; (ii)  $\mathcal{T}$  must be recursively enumerable; (iii)  $\mathcal{T}$  must be an ideal with respect to a definedness relation (Definition 9).

Our characterization is based on two previous results: a theorem by Ehrhard and Regnier stating that Taylor expansion and normalization ( $\lambda$ -terms via Böhm trees; and sets of simple terms via the NF operator) commute (here Theorem 8), and Barendregt’s characterization of the Böhm-like trees which are Böhm trees of  $\lambda$ -terms (here Theorem 5). Conditions (i) and (ii) are in fact an adaptation of Barendregt’s characterization, but they are not sufficient to characterize Taylor expansions since the Taylor expansion gives a more atomic decomposition of  $\lambda$ -terms than that obtained from Böhm trees. Condition (iii) is a completeness condition,

assuring that there is no hole in the description of the support of a Taylor expansion. This condition looks like the usual one characterizing the set of finite approximants of a tree as an ideal with respect to the subtree order relation. However, the fine grained notion of approximant given by the simple terms makes our definedness relation a bit subtler than a subtree relation, in fact it is not even a preorder relation.

Our result concerns only the supports of the formal combinations in  $\mathbb{Q}^+[\Delta^{\text{NF}}]$ , i.e. the subsets of  $\Delta^{\text{NF}}$ , the issue about coefficients being completely accomplished by Equation 2. Actually, the powerset of  $\Delta^{\text{NF}}$  has an interest by its own. In fact, the powerset of  $\Delta^{\text{NF}}$  can be formally seen as the space  $\mathbf{Bool}[\Delta^{\text{NF}}]$  of the formal combinations with coefficients in the boolean ring  $\mathbf{Bool} = (\{0, 1\}, \max, \min, 0, 1)$ . The Taylor expansion of a  $\lambda$ -term  $M$  into the space  $\mathbf{Bool}[\Delta^{\text{NF}}]$  is the support  $\tau(M)$  of the Taylor expansion of  $M$  into  $\mathbb{Q}^+[\Delta^{\text{NF}}]$ . Let us remark that  $\mathbf{Bool}[\Delta^{\text{NF}}]$  gives another syntax-based quantitative semantics of the  $\lambda$ -calculus, and our result characterizes the image of the  $\lambda$ -calculus into  $\mathbf{Bool}[\Delta^{\text{NF}}]$ .

**Related works.** The question of characterizing the support of Ehrhard and Regnier’s Taylor expansion has already been addressed by Pagani and Tasson in the setting of the simply typed linear logic proof-nets [12]. In that paper, the authors define a rewriting algorithm taking as input a finite set of cut-free differential nets (corresponding here to the normal simple terms) and either returning a cut-free proof-net or falling in a deadlock. Although related, our approach is different from that of [12] in various points. First, we are considering the untyped  $\lambda$ -calculus, while [12] deals with a simply typed (hence strongly normalizing) framework. Second, the criterion proposed by [12] is the termination of a rewriting relation in a proof-net. In the present setting, this will amount to refer to the termination of an algorithm which starts with a set  $\mathcal{T}$  of simple terms and tries to compute a  $\lambda$ -term  $M$  such that  $\mathcal{T} \subseteq \text{NF}(\tau(M))$ . We are giving more abstract conditions without referring to the termination of a computation. Finally, [12] characterizes the property of being a *finite subset* of the support of the Taylor expansion, while we are capturing here the property of being the whole support ( $\mathcal{T} = \text{NF}(\tau(M))$ ).

**Structure of the paper.** Section 2 recalls the standard notions on  $\lambda$ -calculus and Böhm trees needed in the sequel. In particular, Definition 2 introduces Böhm-like trees and Theorem 5 states Barendregt’s characterization of the set of Böhm-like trees corresponding to  $\lambda$ -terms. Section 3 defines Ehrhard and Regnier’s resource  $\lambda$ -calculus, the normal form operator  $\text{NF}$  and the support  $\tau(M)$  of the Taylor expansion of a  $\lambda$ -term  $M$  (Equation 8). Theorem 8 recalls the commutation of the Taylor expansion with Böhm trees. Finally, Section 4 contains the original results of the paper. Our characterization of the normal forms of the Taylor expansions of  $\lambda$ -terms is given in Theorem 25. Corollary 28 gives an answer to what the maximal cliques of simple terms correspond to, a question addressed in [9]. Finally, Corollary 29 states the conditions characterizing the property of being a normalizable  $\lambda$ -term.

## 2 $\Lambda$ -Calculus and Böhm trees

We denote by  $\Lambda$  the set of  $\lambda$ -terms, written using Krivine’s convention [11]:

$$\Lambda : M, N ::= x \mid (M)N \mid \lambda x.M,$$

where  $x$  ranges over a countable set  $\text{Var}$  of variables. As usual, we suppose that application associates to the left and  $\lambda$ -abstraction to the right. The  $\alpha$ -conversion and the set  $\text{FV}(M)$  of *free variables of*  $M$  are defined following [11]. A term  $M$  is *closed* whenever  $\text{FV}(M) = \emptyset$ .

Given two terms  $M, N$ , we denote by  $M\{N/x\}$  the term obtained by simultaneously substituting  $N$  to all free occurrences of  $x$  in  $M$ , subject to the usual proviso about renaming bound variables in  $M$  to avoid capture of free variables in  $N$ .

Hereafter terms are considered up to  $\alpha$ -conversion. We define the following terms:

$$\begin{aligned} \mathbf{I} &:= \lambda x.x, & \mathbf{S} &:= \lambda xyz.((x)z)(y)z, & \mathbf{\Omega} &:= (\lambda x.(x)x)\lambda x.(x)x, \\ \mathbf{\Theta}_f &:= \lambda x.(f)(x)x, & \mathbf{\Theta} &:= \lambda f.(\mathbf{\Theta}_f)\mathbf{\Theta}_f. \end{aligned}$$

The  $\beta$ -reduction  $\xrightarrow{\beta}$  is the smallest relation over  $\Lambda$  containing the  $\beta$ -step  $(\lambda x.M)N \xrightarrow{\beta} M\{N/x\}$  and closed under the following context rules (supposing  $M \xrightarrow{\beta} M'$ ):

$$(\text{abs}) : \lambda x.M \xrightarrow{\beta} \lambda x.M', \quad (\text{fun}) : (M)N \xrightarrow{\beta} (M')N, \quad (\text{arg}) : (N)M \xrightarrow{\beta} (N)M'. \quad (3)$$

We denote by  $\xrightarrow{\beta^*}$  the reflexive and transitive closure of  $\xrightarrow{\beta}$ . A  $\beta$ -normal form is a normal form for  $\xrightarrow{\beta}$ . The reduction  $\xrightarrow{\beta}$  is confluent which implies that the  $\beta$ -normal form of a  $\lambda$ -term is unique, whenever it exists. However,  $\xrightarrow{\beta}$  is not normalizing, i.e. there are  $\lambda$ -terms  $M$  without a  $\beta$ -normal form, e.g.  $\mathbf{\Omega}$  and  $\mathbf{\Theta}$ . Indeed,  $\mathbf{\Omega}$  reduces only to itself, i.e.  $\mathbf{\Omega} \xrightarrow{\beta} \mathbf{\Omega}$ , while  $\mathbf{\Theta}$  yields the infinite reduction sequence  $\mathbf{\Theta} \xrightarrow{\beta} \lambda f.(f)(\mathbf{\Theta}_f)\mathbf{\Theta}_f \xrightarrow{\beta} \lambda f.(f)(f)(\mathbf{\Theta}_f)\mathbf{\Theta}_f \xrightarrow{\beta} \dots$ .

Notice however that  $\mathbf{\Omega}$  and  $\mathbf{\Theta}$  are quite different. Any application  $(\mathbf{\Omega})M_0 \dots M_{n-1}$  gives a non-normalizing  $\lambda$ -term, independently from  $M_0, \dots, M_{n-1}$ . In that sense  $\mathbf{\Omega}$  represents the everywhere undefined function. On the contrary,  $\mathbf{\Theta}$  is a fundamental term producing a potentially infinite iteration of its first argument, i.e.  $(\mathbf{\Theta})M_0 \dots M_{n-1} \xrightarrow{\beta^*} (M_0)(\mathbf{\Theta})M_0M_1 \dots M_{n-1} \xrightarrow{\beta^*} (M_0)(M_0)(\mathbf{\Theta})M_0M_1 \dots M_{n-1} \xrightarrow{\beta^*} \dots$ , and so it converges for certain  $M_0, \dots, M_{n-1}$  for appropriate reduction strategies. Böhm trees are introduced in [1, §10] and provide a notion of infinitary normal form allowing to distinguish between totally undefined terms, like  $\mathbf{\Omega}$ , and terms like  $\mathbf{\Theta}$ , which are not normalizing but can interact with their arguments giving convergent computations.

The Böhm tree of a term is defined by using the *head reduction*  $\xrightarrow{h}$ , a deterministic restriction of  $\xrightarrow{\beta}$  obtained by forbidding (arg) in the set of rules (3) and by restricting the (fun) rule to the case where  $M$  is not an abstraction. A *head normal form* is a normal form for the reduction  $\xrightarrow{h}$ , and it always has the shape:  $\lambda x_0 \dots x_{m-1}.(y)M_0 \dots M_{n-1}$ , for  $n, m \geq 0$ ,  $x_0, \dots, x_{m-1}, y$  variables, and  $M_0, \dots, M_{n-1}$   $\lambda$ -terms. The variable  $y$  is called the *head variable*, and  $M_0, \dots, M_{n-1}$  its arguments.

Notice that  $\mathbf{\Omega}$  has no head normal form, while  $\lambda f.(f)(\mathbf{\Theta}_f)\mathbf{\Theta}_f$  is a head normal form of  $\mathbf{\Theta}$ . Basically, the Böhm tree  $\text{BT}(M)$  of a  $\lambda$ -term  $M$  is a (possibly infinite) nesting of head normal forms: if  $M$  is  $\beta$ -normalizable, then  $\text{BT}(M)$  is just the applicative tree of its  $\beta$ -normal form (e.g. Figure 1a, 1c), otherwise it can be either infinite in depth (e.g. Figure 1d) or with “holes” denoting totally undefined sub-terms (e.g. Figure 1b) or both (e.g. Figure 1e).

We recall some definitions of [1, §10]. We introduce the set of Böhm-like trees (Definition 2), which is the codomain of the function  $\text{BT}(\cdot)$ . In general, a labelled tree can be represented as a partial function from the set  $\mathbb{N}^*$  of finite sequences of integers to the set of possible labels. In the case of Böhm-like trees, a label is a pair  $(\lambda x_0 \dots x_{m-1}.y, n)$ , where  $\lambda x_0 \dots x_{m-1}.y$  represents the  $\lambda$ -prefix and the head variable of a head normal form, and  $n$  the number of its arguments, which is also a bound to the number of children of the node labelled by  $(\lambda x_0 \dots x_{m-1}.y, n)$ . Definition 4 associates with any  $\lambda$ -term  $M$  a Böhm-like tree  $\text{BT}(M)$ . Not every Böhm-like tree is the Böhm tree of a  $\lambda$ -term, and we recall in Theorem 5 Barendregt’s characterization of the image set of  $\text{BT}(\cdot)$ .

► **Notation 1.** Greek letters  $\alpha, \beta, \gamma$  will vary over  $\mathbb{N}^*$ , the set of finite sequences of natural numbers. We denote by  $@$  the concatenation operator, by  $<$  the strict prefix order, by  $\text{length}(\alpha)$  the length of a sequence  $\alpha \in \mathbb{N}^*$ , and by  $\langle n_1, \dots, n_k \rangle$  the sequence of  $n_1, \dots, n_k$ . In particular,  $\langle \rangle$  is the empty sequence. For example  $\langle 1, 2 \rangle < \langle 1, 2, 3 \rangle = \langle 1 \rangle @ \langle 2, 3 \rangle$ .

We recall that, when  $f$  is a partial function,  $f(x) \downarrow$  means that  $f(x)$  is defined and that  $f(x) \uparrow$  means that  $f(x)$  is undefined.

► **Definition 2.** Let  $\Sigma \triangleq \{\lambda x_0 \dots x_{m-1}.y ; m \in \mathbb{N}, x_i, y \in \text{Var}\}$ . A *Böhm-like tree*<sup>1</sup> is a partial function  $\mathcal{B}$  from  $\mathbb{N}^*$  to  $\Sigma \times \mathbb{N}$  such that:

- if  $\mathcal{B}(\alpha) \downarrow$  and  $\beta < \alpha$ , then  $\mathcal{B}(\beta) \downarrow$ ,
- if  $\mathcal{B}(\alpha) = (a, n)$ , then  $\forall k \geq n, \mathcal{B}(\alpha @ \langle k \rangle) \uparrow$ .

Actually, a sequence  $\alpha$  in the domain of  $\mathcal{B}$  describes a path from the root to a node labelled by  $\mathcal{B}(\alpha)$ . Notice that the enumeration of the children of a node allows some holes, representing the presence of totally undefined arguments in the head normal form associated with such a node. Figure 1 gives examples of a convenient graphical representation of Böhm-like trees, where the number  $n$  of a label  $(a, n)$  is encoded by using pending edges. More precisely,

Figure 1a is the function  $\langle \rangle \mapsto (\lambda x.x, 0)$ ,

Figure 1b is the function  $\langle \rangle \mapsto (\lambda x.x, 1)$ ,

Figure 1c is the function  $\langle \rangle \mapsto (\lambda xyz.z, 2), \langle 0 \rangle \mapsto (z, 0), \langle 1 \rangle \mapsto (y, 1), \langle 10 \rangle \mapsto (z, 0)$ ,

Figure 1d is the function  $\langle \rangle \mapsto (\lambda f.f, 1), \langle 0 \rangle \mapsto (f, 1), \langle 00 \rangle \mapsto (f, 1), \langle 000 \rangle \mapsto (f, 1) \dots$ ,

Figure 1e is the function  $\langle \rangle \mapsto (f, 2), \langle 1 \rangle \mapsto (f, 2), \langle 11 \rangle \mapsto (f, 2), \langle 111 \rangle \mapsto (f, 2) \dots$ .

The *height* of a Böhm-like tree  $\mathcal{B}$  is defined by:  $\text{height}(\mathcal{B}) \triangleq \sup\{1 + \text{length}(\alpha) \mid \mathcal{B}(\alpha) \downarrow\}$ . Remark that the domain of  $\mathcal{B}$  can be empty and in that case it has a zero height, otherwise the height is positive or infinite. Given  $h \in \mathbb{N}$ , the *restriction of  $\mathcal{B}$  to  $h$*  is the Böhm-like tree  $\mathcal{B}|_h$  defined by cutting off subtrees at height  $h$ :  $\mathcal{B}|_h(\alpha) = \mathcal{B}(\alpha)$  if  $\text{length}(\alpha) < h$ , otherwise  $\mathcal{B}|_h(\alpha) \uparrow$ . Moreover, if  $\mathcal{B}(\langle i \rangle) = (a, n)$  and  $i < n$ , we define the  *$i$ -th subtree  $\mathcal{B}_i$  of  $\mathcal{B}$*  as:  $\mathcal{B}_i(\alpha) = \mathcal{B}(\langle i \rangle @ \alpha)$ .

Trees can be seen as ordered by the set-theoretical inclusion on the graph of their functions. With respect to this order  $\{\mathcal{B}|_h\}_{h \in \mathbb{N}}$  is a chain whose limit is  $\mathcal{B}$ , i.e.  $\mathcal{B} = \bigcup_{h \in \mathbb{N}} \mathcal{B}|_h$ . Such a remark is useful for proofs and definitions on infinite Böhm-like trees. For example, the set  $\text{FV}(\mathcal{B})$  of *free variables* of a Böhm-like tree is defined as follows: if  $\text{height}(\mathcal{B}) = 0$ , then  $\text{FV}(\mathcal{B}) \triangleq \emptyset$ ; if  $\text{height}(\mathcal{B}) = h + 1$ , let  $\mathcal{B}(\langle \rangle) = (\lambda x_0 \dots x_{m-1}.y, n)$ , then  $\text{FV}(\mathcal{B}) \triangleq (\{y\} \cup \bigcup_{i=0}^{n-1} \text{FV}(\mathcal{B}_i)) \setminus \{x_0, \dots, x_{m-1}\}$ ; finally, if  $\text{height}(\mathcal{B}) = \infty$ , then  $\text{FV}(\mathcal{B}) \triangleq \bigcup_{h \in \mathbb{N}} \text{FV}(\mathcal{B}|_h)$ .

► **Definition 3.** A Böhm-like tree  $\mathcal{B}$  is *recursively enumerable, r.e.* for short, if  $\mathcal{B}$  is a partial recursive function (after some coding of  $\Sigma$ ).

► **Definition 4.** The *Böhm tree*  $\text{BT}(M)$  of a  $\lambda$ -term  $M$  is defined as follows:

- if  $M \xrightarrow{h^*} \lambda x_0 \dots x_{m-1}.(y)M_0 \dots M_{n-1}$ , then,

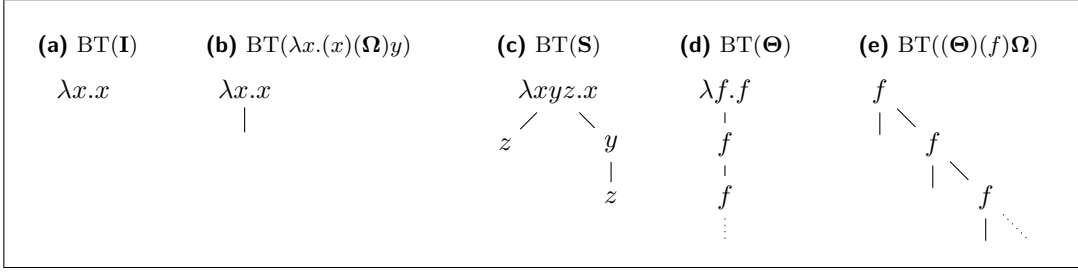
$$\text{BT}(M)(\langle \rangle) \triangleq (\lambda x_0 \dots x_{m-1}.y, n)$$

$$\text{BT}(M)(\langle i \rangle @ \alpha) \triangleq \text{BT}(M_i)(\alpha) \quad \text{if } i < n,$$

$$\text{BT}(M)(\langle i \rangle @ \alpha) \uparrow \quad \text{otherwise.}$$

- Otherwise,  $\text{BT}(M)$  is the totally undefined function.

<sup>1</sup> This is called an *effective Böhm-like tree* in [1, Def. 10.1.9].



■ **Figure 1** Some examples of Böhm trees.

Figure 1 gives examples of  $\text{BT}()$  values. A Böhm-like tree which is not in the image of a  $\lambda$ -term is the infinite branch

$$\epsilon \mapsto (x_0, 1), \quad \langle 0 \rangle \mapsto (x_1, 1), \quad \langle 00 \rangle \mapsto (x_2, 1), \quad \langle 000 \rangle \mapsto (x_3, 1) \dots$$

for  $x_0, x_1, x_2, x_3, \dots$  pairwise distinct variables.

► **Theorem 5** ([1, Theorem 10.1.23]). *Let  $\mathcal{B}$  be a Böhm-like tree. There is a  $\lambda$ -term  $M$  such that  $\text{BT}(M) = \mathcal{B}$  if, and only if,  $\text{FV}(\mathcal{B})$  is finite and  $\mathcal{B}$  is r.e..*

The left-to-right direction is a straight consequence of the remark that  $\text{FV}(\text{BT}(M)) \subseteq \text{FV}(M)$  and of the fact that the definition of  $\text{BT}(M)$  is effective (hence by Church's Thesis it is r.e.). The proof of the right-to-left direction is more involved. Essentially, it uses a theorem stating that the set  $\Lambda^0$  of closed  $\lambda$ -terms can be enumerated in such a way that there are two  $\lambda$ -terms  $\mathbf{E}$  and  $\mathbf{F}$  such that  $\forall M \in \Lambda^0$ , denoting by  $\underline{M}$  the Church numeral associated with  $M$  by the enumeration, we have  $(\mathbf{E})M =_\beta \underline{M}$  and, vice versa,  $(\mathbf{F})\underline{M} =_\beta M$ . We refer to [1, 10] for more details.

### 3 Taylor expansion

We recall the resource calculus as presented in [8, 9]. Let us warn the reader that the name “resource calculus” also refers in the literature to slightly different calculi. In particular, we have the calculus by Boudol et al.'s in [3], which is a resource-sensitive extension of the lazy call-by-value  $\lambda$ -calculus, and Tranquilli's resource  $\lambda$ -calculus [13] which is basically a different notation for Ehrhard and Regnier's differential  $\lambda$ -calculus. The resource calculus of [8, 9], which we briefly recall here, is a fragment of Tranquilli's calculus.

We define the set  $\Delta$  of the simple terms (Grammar (4)), a rewriting relation  $\xrightarrow{r}$  over the finite powerset of  $\Delta$  (rules (5) and (6)), and a normal form operator  $\text{NF}$  over the (finite and infinite) powerset of  $\Delta$  (Equation 7). Concerning the discussion in the Introduction, the simple terms in  $\Delta$  are a notation that express the terms in a Taylor series (Equation 2) and the subsets of  $\Delta$  are the supports of such series.

Equation (8) defines the Taylor expansion of the  $\lambda$ -calculus and Equation (9) extends it to Böhm-like trees. Theorem 8 states Ehrhard and Regnier's correspondence between the resource reduction on the Taylor expansion of  $\lambda$ -terms and their Böhm trees.

**Resource calculus.** The set  $\Delta$  of *simple terms* is defined by:

$$\Delta : t ::= x \mid \lambda x.t \mid \langle t \rangle \mu, \tag{4}$$

where  $x$  ranges over the set  $\text{Var}$  of variables, and  $\mu$  is a finite multiset of simple terms, called *bag*. Small Latin letters like  $s, t, u$  will vary on simple terms, and small Greek letters  $\mu, \nu, \rho$

on bags. We recall that we adopt Krivine's notation [11], so, in particular,  $\langle t \rangle \mu_1 \dots \mu_n$  is a shortcut for  $\langle \dots \langle \langle t \rangle \mu_1 \rangle \dots \rangle \mu_n$ .

► **Notation 6.** We recall that for any set  $E$ , a *multiset* on  $E$  is a function  $\mu : E \rightarrow \mathbb{N}$ . The *support*  $|\mu|$  of  $\mu$  is the set of all elements  $e$  of  $E$  such that  $\mu(e) \neq 0$ , and  $\mu$  is called a finite multiset when  $|\mu|$  is a finite set. We denote by  $\mathcal{M}(E)$  the set of multisets on  $E$  and by  $\mathcal{M}_f(E)$  the set of finite multisets on  $E$ . A multiplicative notation is used for bags,  $1$  is the empty bag and  $\mu \cdot \nu$  is the disjoint union of  $\mu$  and  $\nu$ . The bag  $[t]$  is the singleton containing exactly one occurrence of the simple term  $t$ . More occurrences of  $t$  can be written as a power:  $[t^3] = [t, t, t] = [t] \cdot [t] \cdot [t]$ .

Free and bound variables and the  $\alpha$ -equivalence  $\equiv_\alpha$  are defined as in the  $\lambda$ -calculus.

The symbols  $\mathcal{S}, \mathcal{T}, \mathcal{U}$  will vary over the powerset  $\mathcal{P}(\Delta^{\text{NF}})$ . By notational convention, we extend all the constructs of the grammar of  $\Delta$  as point-wise operations on (possibly infinite) sets of simple terms, like for example

$$\lambda x.t \triangleq \{\lambda x.t \mid t \in \mathcal{T}\}, \quad \langle \mathcal{S} \rangle [\mathcal{T}^2, \mathcal{U}] \triangleq \{\langle s \rangle [t_1, t_2, u] \mid s \in \mathcal{S}, t_1, t_2 \in \mathcal{T}, u \in \mathcal{U}\}.$$

The number of free occurrences of  $x$  in  $t$ , called *degree* of  $x$  in  $t$ , is written  $\text{deg}_x(t)$ . A *redex* is a simple term of the shape  $\langle \lambda x.t \rangle [s_1, \dots, s_n]$ . Its reduction gives a finite set of simple terms, which is empty whenever  $\text{deg}_x(t) \neq n$ , otherwise it is the set of all possible simple terms obtained by linearly replacing each free occurrence of  $x$  with exactly one  $s_i$ , for  $i = 1, \dots, n$ . Formally,

$$\langle \lambda x.t \rangle [s_1, \dots, s_n] \xrightarrow{r} \begin{cases} \{t\{s_{\sigma(1)}/x_1, \dots, s_{\sigma(n)}/x_n\} \mid \sigma \in \mathcal{S}_n\} & \text{if } \text{deg}_x(t) = n, \\ \emptyset & \text{otherwise.} \end{cases} \quad (5)$$

where  $\mathcal{S}_n$  is the group of permutations over  $n = \{1, \dots, n\}$  and  $x_1, \dots, x_n$  is any enumeration of the free occurrences of  $x$  in  $t$ , so that  $t\{s_{\sigma(i)}/x_i\}$  denotes the term obtained from  $t$  by replacing the  $i$ -th free occurrence of  $x$  with the term  $s_{\sigma(i)}$ .

The relation  $\xrightarrow{r} \subseteq \Delta \times \mathcal{P}_f(\Delta)$  is extended to  $\mathcal{P}_f(\Delta) \times \mathcal{P}_f(\Delta)$  by context closure, i.e.  $\xrightarrow{r}$  is the smallest relation on  $\mathcal{P}_f(\Delta) \times \mathcal{P}_f(\Delta)$  satisfying Equation 5 and such that, whenever  $t \xrightarrow{r} \mathcal{T}$  and  $\mathcal{S} \in \mathcal{P}_f(\Delta)$ , we have:

$$\lambda x.t \xrightarrow{r} \lambda x.\mathcal{T}, \quad \langle t \rangle \mu \xrightarrow{r} \langle \mathcal{T} \rangle \mu, \quad \langle s \rangle [t] \cdot \mu \xrightarrow{r} \langle s \rangle [\mathcal{T}] \cdot \mu, \quad \{t\} \cup \mathcal{S} \xrightarrow{r} \mathcal{T} \cup \mathcal{S}. \quad (6)$$

Notice that the size (*i.e.* the number of symbols) of each simple term in  $\mathcal{T}$  is strictly less than the size of  $t$ , whenever  $t \xrightarrow{r} \mathcal{T}$ . Therefore,  $\xrightarrow{r}$  is strongly normalizing on  $\mathcal{P}_f(\Delta)$ . Moreover, one can easily check that  $\xrightarrow{r}$  is weakly confluent, thus confluent by Newman's lemma.

► **Proposition 7.** *The relation  $\xrightarrow{r}$  is strongly normalizing and confluent on  $\mathcal{P}_f(\Delta)$ .*

In particular, given any simple term  $t$ , its unique *normal form*  $\text{NF}(t)$  is always well-defined. Such an operator is extended over  $\mathcal{P}(\Delta^{\text{NF}})$  by:

$$\text{NF}(\mathcal{T}) \triangleq \bigcup_{t \in \mathcal{T}} \text{NF}(t). \quad (7)$$

We also extend over the notation for free variables by setting  $\text{FV}(\mathcal{T}) \triangleq \bigcup_{t \in \mathcal{T}} \{\text{FV}(t)\}$ .

**Taylor expansion.** We define the map  $\tau$  from  $\Lambda$  to  $\mathcal{P}_f(\Delta)$  by structural induction on the  $\lambda$ -terms. As said in the Introduction,  $\tau$  is the support of the Taylor expansion described in [7, 9], from the  $\lambda$ -calculus to the infinite formal linear combinations in  $\mathbb{Q}^+[\![\Delta]\!]$ , as well as the Taylor expansion into  $\mathbf{Bool}[\![\Delta]\!]$ .

$$\tau(x) \triangleq x, \quad \tau(\lambda x.M) \triangleq \lambda x.\tau(M), \quad \tau((M)N) \triangleq \bigcup_{n=0}^{\infty} \langle \tau(M) \rangle [\tau(N)^n]. \quad (8)$$

For example, we have

$$\begin{aligned} \tau(\mathbf{I}) &\triangleq \{\lambda x.x\}, \\ \tau(\mathbf{S}) &\triangleq \{\lambda xyz.\langle x \rangle [z^{n_0}] [\langle y \rangle [z^{n_1}], \dots, \langle y \rangle [z^{n_k}]] ; k, n_0, \dots, n_k \in \mathbb{N}\}, \\ \tau(\mathbf{\Omega}) &\triangleq \{\langle \lambda x.\langle x \rangle [x^{n_0}] [\lambda x.\langle x \rangle [x^{n_1}], \dots, \lambda x.\langle x \rangle [x^{n_k}]] ; k, n_0, \dots, n_k \in \mathbb{N}\}, \\ \tau(\mathbf{\Theta}) &\triangleq \{\lambda f.\langle \lambda x.\langle f \rangle [\langle x \rangle [x^{n_1}], \dots, \langle x \rangle [x^{n_k}]] [\lambda x.\langle f \rangle [\langle x \rangle [x^{n_{1,1}}], \dots, \langle x \rangle [x^{n_{1,k_1}}]], \dots, \\ &\quad \lambda x.\langle f \rangle [\langle x \rangle [x^{n_{h,1}}], \dots, \langle x \rangle [x^{n_{h,k_h}}]] ; k, n_i, h, n_{i,j} \in \mathbb{N}\} \end{aligned}$$

The Taylor expansion of a  $\beta$ -redex contains resource redexes. If one looks for an invariant under  $\beta$ -reduction, one should consider the normal forms of the Taylor expansions. By induction on the size of the simple terms, one can for example prove that  $\text{NF}(\tau(\mathbf{\Omega})) = \emptyset$ , while

$$\text{NF}(\tau(\mathbf{\Theta})) = \{\lambda f.\langle f \rangle 1, \lambda f.\langle f \rangle [(\langle f \rangle 1)^n], \lambda f.\langle f \rangle [(\langle f \rangle [(\langle f \rangle 1)^{n_1}], \dots, \langle f \rangle [(\langle f \rangle 1)^{n_k}]], \dots\}.$$

Indeed, notice that these examples can be seen as a *thick* version [2] of the finite approximants of the Böhm tree of the corresponding  $\lambda$ -terms, where each sub-tree has been recursively replaced by a finite multiset of copies of it. We recall the following Theorem 8, stating a commutation between the resource reduction and the Taylor expansion through the Böhm tree operator. To do this, we need to extend the notion of Taylor expansion to Böhm-like trees:

$$\tau(\mathcal{B}) \triangleq \begin{cases} \emptyset & \text{if } \text{height}(\mathcal{B}) = 0, \\ \{\lambda x_0 \dots x_{m-1}.\langle y \rangle \mu_0 \dots \mu_{n-1} ; \mu_i \in \mathcal{M}_F(\tau(\mathcal{B}_i))\} & \text{if } \text{height}(\mathcal{B}) = h + 1 \text{ and} \\ \bigcup_h \tau(\mathcal{B}|_h) & \text{if } \text{height}(\mathcal{B}) = \infty. \end{cases} \quad \mathcal{B}(\langle \rangle) = (\lambda x_0 \dots x_{m-1}.y, n), \quad (9)$$

► **Theorem 8** ([8, Theorem 2]). *For every  $\lambda$ -term  $M$ ,  $\text{NF}(\tau(M)) = \tau(\text{BT}(M))$ .*

## 4 Characterizing the Taylor expansion

In this section, we prove our main result (Theorem 25), an algebraic characterization of the normal forms of the Taylor expansion of  $\Lambda$ . Definitions 9 and 10 give the two crucial notions (a notion of ideal and a notion of effective element) for such a characterization.

Let  $\Delta^{\text{NF}}$  be the set of the simple terms which are normal forms with respect to  $\xrightarrow{\tau}$ . They have the following shape:

$$\Delta^{\text{NF}} : t ::= \lambda x_0 \dots x_{m-1}.\langle y \rangle \mu_0 \dots \mu_{n-1} \quad (10)$$

where  $m, n \geq 0$  and each  $\mu_i$  is a bag of simple terms in normal form.

► **Definition 9** (Definedness). The *definedness relation*  $\preceq$  on  $\Delta^{\text{NF}}$  is given as follows:

$$\lambda x_0 \dots x_{m-1} \langle y \rangle \mu_0 \dots \mu_{n-1} \preceq t \text{ iff } \begin{cases} t = \lambda x_0 \dots x_{m-1} \langle y \rangle \nu_0 \dots \nu_{n-1} \text{ and} \\ \forall i < n, |\mu_i| \neq \emptyset \implies \exists v \in |\nu_i|, \forall u \in |\mu_i|, u \preceq v. \end{cases}$$

Given  $\mathcal{T} \in \mathcal{P}(\Delta^{\text{NF}})$  we say that:

- $\mathcal{T}$  is *downward closed* whenever  $\forall t$  s.t.  $\exists t' \in \mathcal{T}, t \preceq t'$ , we have  $t \in \mathcal{T}$ ;
- $\mathcal{T}$  is *directed*, whenever  $\forall t, t' \in \mathcal{T}$ , we have  $\exists t'' \in \mathcal{T}$ , s.t.  $t, t' \preceq t''$ ;
- $\mathcal{T}$  is an *ideal* whenever it is downward closed and directed.

Notice that the relation  $\preceq$  is transitive but not reflexive (i.e. it is not a preorder). For instance,  $\langle y \rangle [\langle y \rangle 1[y], \langle y \rangle [y]1] \not\preceq \langle y \rangle [\langle y \rangle 1[y], \langle y \rangle [y]1]$ , as  $\langle y \rangle 1[y] \not\preceq \langle y \rangle [y]1$  as well as  $\langle y \rangle [y]1 \not\preceq \langle y \rangle 1[y]$ . It is not antireflexive either, for example  $\langle y \rangle [x] \preceq \langle y \rangle [x]$ . Moreover, it is neither symmetric nor antisymmetric. We have  $\langle y \rangle [x]1 \preceq \langle y \rangle [x][x]$ , but  $\langle y \rangle [x][x] \not\preceq \langle y \rangle [x]1$ , and  $\langle y \rangle [t] \preceq \langle y \rangle [t, t]$ ,  $\langle y \rangle [t, t] \preceq \langle y \rangle [t]$  but  $\langle y \rangle [t] \neq \langle y \rangle [t, t]$ . However, on the simple terms having bags of at most one element,  $\preceq$  is an order relation.

We need to formalize what is an “effective element” of  $\mathcal{P}(\Delta^{\text{NF}})$ . In order to do that, we use the notion of a recursively enumerable subset of  $\mathbb{N}$  and an encoding of  $\Delta^{\text{NF}}$  into  $\mathbb{N}$ . Let  $G$  be any effective bijection between  $\Delta^{\text{NF}} / \equiv_\alpha$  and  $\mathbb{N}$ . One way of defining  $G$  is by using the De Bruijn notation [4], which gives a system of canonical representatives for the  $\alpha$ -equivalence. We will omit such details, so we fix once and for all a recursive Gödel numbering  $G : \Delta^{\text{NF}} \mapsto \mathbb{N}$ .

► **Definition 10** (Effectiveness). An element  $\mathcal{T}$  of  $\mathcal{P}(\Delta^{\text{NF}})$  is *recursively enumerable*, r.e. for short, whenever the set  $G(\mathcal{T}) \triangleq \{G(t) ; t \in \mathcal{T}\} \subseteq \mathbb{N}$  is recursively enumerable, i.e. either  $\mathcal{T} = \emptyset$ , or there exists a total recursive function  $\phi : \mathbb{N} \mapsto \mathbb{N}$  such that  $G(\mathcal{T}) = \{\phi(n) ; n \in \mathbb{N}\}$ .

Notice that the notion of a r.e. element of  $\mathcal{P}(\Delta^{\text{NF}})$  does not depend on the chosen Gödel enumeration of  $\Delta^{\text{NF}}$ .

► **Definition 11.** We say that a set  $\mathcal{T} \in \mathcal{P}(\Delta^{\text{NF}})$  is *single-headed with width  $n$* , whenever there exist  $m \in \mathbb{N}$ ,  $x_0, \dots, x_{m-1}, y \in \text{Var}$ ,  $J \subseteq \mathbb{N}$  and, for each  $i < n$ , a family  $(\mu_i^j)_{j \in J}$  of bags such that :

$$\mathcal{T} = \{\lambda x_0 \dots x_{m-1} \langle y \rangle \mu_0^j \dots \mu_{n-1}^j \mid j \in J\}. \quad (11)$$

For each  $i$ , we denote by  $\widehat{\mathcal{T}}_i$  the set  $\{\mu_i^j \mid j \in J\}$  and simply by  $\mathcal{T}_i$  the set  $\bigcup_{j \in J} |\mu_i^j|$ . Notice that  $\widehat{\mathcal{T}}_i \subseteq \mathcal{M}_F(\mathcal{T}_i)$ .

► **Lemma 12.** *If  $\mathcal{T}$  is directed then  $\mathcal{T}$  is single-headed and  $\mathcal{T}_i$  is directed (for every  $i$  less than the width of  $\mathcal{T}$ ). Let  $\mathcal{T} \in \mathcal{P}(\Delta^{\text{NF}})$  be non-empty. If  $\mathcal{T}$  is single-headed with width  $n \in \mathbb{N}$  and downward closed then  $\mathcal{T}_i$  is downward closed (for every  $i < n$ ).*

**Proof.** It is straightforward to check that  $\mathcal{T}$  directed implies  $\mathcal{T}$  single-headed.

From now on assume  $\mathcal{T}$  is single-headed of width  $n$  and let us prove that  $\mathcal{T}_i$  is downward closed (resp. directed) whenever  $\mathcal{T}$  is downward closed (resp. directed).

Let  $u' \preceq u \in \mathcal{T}_i$ . This means that there is  $t = \lambda x_0 \dots x_{m-1} \langle y \rangle \mu_0 \dots \mu_{n-1} \in \mathcal{T}$ , such that  $u \in |\mu_i|$ . Define  $t' \triangleq \lambda x_0 \dots x_{m-1} \langle y \rangle 1 \dots 1 [u'] 1 \dots 1$  and notice that  $t' \preceq t$ . By the downward closedness of  $\mathcal{T}$  we conclude  $t' \in \mathcal{T}$ , so  $u' \in \mathcal{T}_i$  and hence  $\mathcal{T}_i$  is downward closed.

Let  $u_1, u_2 \in \mathcal{T}_i$ . We have in  $\mathcal{T}$  two elements  $t_1 = \lambda x_0 \dots x_{m-1} \langle y \rangle \mu_0^1 \dots \mu_{n-1}^1$  and  $t_2 = \lambda x_0 \dots x_{m-1} \langle y \rangle \mu_0^2 \dots \mu_{n-1}^2$ , such that  $u_1 \in |\mu_i^1|$  and  $u_2 \in |\mu_i^2|$ . By directedness of  $\mathcal{T}$ ,

there is an element  $t_3 = \lambda x_0 \dots x_{m-1} \langle y \rangle \mu_0^3 \dots \mu_{n-1}^3 \in \mathcal{T}$  such that  $t_1, t_2 \preceq t_3$ . So, there is  $v_1, v_2 \in |\mu_i^3|$ , such that  $u_1 \preceq v_1$  and  $u_2 \preceq v_2$ . Again by directedness of  $\mathcal{T}$ , there exists  $t_4 = \lambda x_0 \dots x_{m-1} \langle y \rangle \mu_0^4 \dots \mu_{n-1}^4 \in \mathcal{T}$  such that  $t_3 \preceq t_4$ , so there exists  $w \in |\mu_i^4|$ , such that  $v_1, v_2 \preceq w$ . We have  $w \in \mathcal{T}_i$  and by transitivity,  $u_1, u_2 \preceq w$ . Hence  $\mathcal{T}_i$  is directed.  $\blacktriangleleft$

We now define a coherence relation on normal simple terms, originally introduced in [9], which will mainly be used to approximate the notion of ideal. Indeed, we will show that every ideal is a clique (Proposition 16) and every clique is included in an ideal (Proposition 20).

► **Definition 13** (Coherence, [9, §3]). The coherence relation  $\circ$  on  $\Delta^{\text{NF}}$  is defined by:

$$\lambda x_0 \dots x_{m-1} \langle y \rangle \mu_0 \dots \mu_{n-1} \circ t \text{ iff } \begin{cases} t = \lambda x_0 \dots x_{m-1} \langle y \rangle \nu_0 \dots \nu_{n-1} \text{ and} \\ \forall i < n, \forall u, u' \in |\mu_i \cdot \nu_i|, u \circ u'. \end{cases}$$

We call  $\mathcal{T} \in \mathcal{P}(\Delta^{\text{NF}})$  a *clique* whenever  $\forall t, t' \in \mathcal{T}, t \circ t'$ .

Notice that  $\circ$  is symmetric, but not reflexive. For example,  $\langle y \rangle [x, z] \not\circ \langle y \rangle [x, z]$ . Let us stress, however, that the previous example  $\langle y \rangle [\langle y \rangle 1 [y], \langle y \rangle [y] 1]$  showing the non-reflexivity of  $\preceq$ , is indeed coherent with itself. Furthermore,  $\circ$  is not transitive. We have  $\langle y \rangle [x] [z] \circ \langle y \rangle [x] 1$  and  $\langle y \rangle [x] [y] \circ \langle y \rangle [x] 1$  but  $\langle y \rangle [x] [z] \not\circ \langle y \rangle [x] [y]$ .

As a direct consequence of the definition of coherence we get the following lemma.

► **Lemma 14** (Hereditary characterization of cliques). *Let  $\mathcal{C} \in \mathcal{P}(\Delta^{\text{NF}})$ ,  $\mathcal{C}$  is a clique iff  $\mathcal{C}$  is single-headed of a certain width  $n$  and for each  $i < n$ ,  $\mathcal{C}_i$  is a clique.*

Hereafter, we will often use the induction of the following notion of height, analogous to the definition of height of a Böhm-like tree.

► **Definition 15** (Height). Given a simple normal form  $t = \lambda x_0 \dots x_{m-1} \langle y \rangle \mu_0 \dots \mu_{n-1}$ , we define  $\text{height}(t) = 1 + \max_i (\max_{t_i \in |\mu_i|} \text{height}(t_i))$ . The height of a set  $\mathcal{T} \in \mathcal{P}(\Delta^{\text{NF}})$  is  $\text{height}(\mathcal{T}) = \sup_{t \in \mathcal{T}} (\text{height}(t))$ . We define  $\mathcal{T}|_h = \{t \in \mathcal{T} \mid \text{height}(t) < h\}$ .

Notice that  $\text{height}(t) = 1$  if and only if  $t$  has only empty bags. Of course, the height of a set of simple normal forms  $\mathcal{T}$  can be infinite. For example,  $\text{height}(\text{NF}(\tau(\Theta))) = \infty$ , while  $\text{height}(\text{NF}(\tau(\mathbf{S}))) = 3$ .

► **Proposition 16** (Ideals are cliques). *If  $\mathcal{D} \in \mathcal{P}(\Delta^{\text{NF}})$  is an ideal then  $\mathcal{D}$  is a clique.*

**Proof.** It is enough to check that, for every two terms  $t_1, t_2 \in \Delta^{\text{NF}}$ , if there exists an ideal  $\mathcal{D}'$  such that  $t_1, t_2 \in \mathcal{D}'$  then  $t_1 \circ t_2$ . Let us prove this claim by induction on the maximal height among  $t_1$  and  $t_2$ . Since  $\mathcal{D}'$  is an ideal there exists  $t_3 \in \mathcal{D}'$  such that  $t_1, t_2 \preceq t_3$  and, by Lemma 12,  $\mathcal{D}'$  is single-headed of a certain width  $n$ , so there exist bags  $\mu_i^j$  such that

$$t_j = \lambda x_0 \dots x_{m-1} \langle y \rangle \mu_0^j \dots \mu_{n-1}^j \text{ (for each } j = 1, 2, 3),$$

moreover each  $\mathcal{D}'_i$  is directed.

If  $\max(\text{height}(t_1), \text{height}(t_2)) = 1$  then  $t_1 = t_2 = \lambda x_0 \dots x_{m-1} \langle y \rangle 1 \dots 1$  and this term is coherent with itself. Now let  $h = \max(\text{height}(t_1), \text{height}(t_2))$ . In order to prove  $t_1 \circ t_2$  we have to prove that for each  $i < n$ , for every  $u_1, u_2 \in |\mu_i^1 \cdot \mu_i^2|$ , we have  $u_1 \circ u_2$ . But  $\max(\text{height}(u_1), \text{height}(u_2)) \leq h - 1$  and  $u_1, u_2$  belong to  $\mathcal{D}'_i$  which is directed. We conclude by induction hypothesis.  $\blacktriangleleft$

In fact, every subset of an ideal is a clique (just check that the proof above does not use the downward closure).

► **Definition 17** (Linearization). Let  $\mathcal{C}$  be a non-empty clique of finite height. The *linearization* of  $\mathcal{C}$ , defined by induction on  $\text{height}(\mathcal{C})$ , is the simple term  $\mathcal{L}(\mathcal{C}) \triangleq \lambda x_0 \dots x_{m-1} \langle y \rangle \xi_0 \dots \xi_{n-1}$  where, for every  $i < n$ ,  $\xi_i = [\mathcal{L}(\mathcal{C}_i)]$  if  $\mathcal{C}_i$  non-empty, otherwise  $\xi_i = 1$ .

► **Lemma 18.** *Let  $\mathcal{C} \subseteq \mathcal{D}$  be two non-empty cliques of finite height. We have:  $\text{height}(\mathcal{L}(\mathcal{C})) = \text{height}(\mathcal{C})$ ,  $\mathcal{L}(\mathcal{C}) \preceq \mathcal{L}(\mathcal{D})$  and, finally,  $\forall t \in \mathcal{C}, t \preceq \mathcal{L}(\mathcal{C})$ .*

**Proof.** Let  $\mathcal{C}$  and  $\mathcal{D}$  satisfying the hypothesis. By Lemma 14,  $\mathcal{C}$  and  $\mathcal{D}$  are single headed with same width  $n$ . For any  $i < n$ , notice that  $\mathcal{C}_i \subseteq \mathcal{D}_i$ . We proceed by induction on  $\text{height}(\mathcal{C})$ . If  $\text{height}(\mathcal{C}) = 1$ , then  $\mathcal{C} = \{\mathcal{L}(\mathcal{C}) = \lambda x_0 \dots x_{m-1} \langle y \rangle 1 \dots 1\}$ . Clearly, one gets the statement. Otherwise, let  $\text{height}(\mathcal{C}) > 1$ . Then  $\mathcal{L}(\mathcal{C})$  is defined as in the equation of Definition 17. By Lemma 14,  $\mathcal{D}_i$  and  $\mathcal{C}_i$  are cliques. Hence, whenever  $\mathcal{C}_i$  is non-empty, we have by induction hypothesis that  $\mathcal{L}(\mathcal{C}_i)$  satisfies the statement of the lemma for  $\mathcal{C}_i$  and  $\mathcal{D}_i$ . One can then easily conclude that  $\text{height}(\mathcal{L}(\mathcal{C})) = \text{height}(\mathcal{C})$ , and  $\mathcal{L}(\mathcal{C}) \preceq \mathcal{L}(\mathcal{D})$  and, finally,  $\forall t \in \mathcal{C}, t \preceq \mathcal{L}(\mathcal{C})$ . ◀

► **Lemma 19.** *Let  $\mathcal{D}$  be an ideal and let  $\mathcal{C} \subseteq \mathcal{D}$  be a non-empty clique of finite height. Then,  $\mathcal{L}(\mathcal{C}) \in \mathcal{D}$ .*

**Proof.** Let  $\mathcal{D}$  and  $\mathcal{C}$  satisfying the hypothesis. By Lemma 12,  $\mathcal{D}$  (and hence  $\mathcal{C}$ ) are single headed and of same width  $n$ . The proof of  $\mathcal{L}(\mathcal{C}) \in \mathcal{D}$  is by induction on  $\text{height}(\mathcal{C})$ .

Let  $I$  be the set of indices  $i < n$  such that  $\mathcal{C}_i$  is non-empty. If  $I$  is empty, then  $\mathcal{L}(\mathcal{C}) = \lambda x_0 \dots x_{m-1} \langle y \rangle 1 \dots 1$ , and one can deduce that  $\mathcal{L}(\mathcal{C}) \in \mathcal{D}$  since  $\mathcal{D}$  is downward closed and  $\mathcal{L}(\mathcal{C})$  is  $\preceq$  to any element of  $\mathcal{D}$ . If otherwise  $I$  is non-empty, then for every  $i \in I$ , we have that  $\mathcal{C}_i \subseteq \mathcal{D}_i$ , as well as that  $\mathcal{C}_i$  is a clique (Lemma 14). By induction hypothesis we can suppose  $\mathcal{L}(\mathcal{C}_i) \in \mathcal{D}_i$ . This means that there is  $w_i \in \mathcal{D}, w_i = \lambda x_0 \dots x_{m-1} \langle y \rangle \rho_0 \dots \rho_{n-1}$ , and  $\mathcal{L}(\mathcal{C}_i) \in |\rho_i|$ . As  $\mathcal{D}$  is directed and  $\preceq$  is transitive, we can construct  $w \in \mathcal{D}$  such that  $w_i \preceq w$ , for every  $i \in I$ . We remark that  $\mathcal{L}(\mathcal{C}) \preceq w \in \mathcal{D}$ , therefore  $\mathcal{L}(\mathcal{C}) \in \mathcal{D}$ . ◀

► **Proposition 20** (Cliques can be ideals). *Let  $\mathcal{C}$  be a clique. Then:*

1. *the set  $\mathcal{I}(\mathcal{C}) \triangleq \{t \mid \exists h > 0, t \preceq \mathcal{L}(\mathcal{C}|_h)\}$  is an ideal;*
2. *This set  $\mathcal{I}(\mathcal{C})$  is the smallest ideal containing  $\mathcal{C}$ .*

**Proof.** Obviously  $\mathcal{I}(\mathcal{C})$  is downward closed. Moreover, by Lemma 18,  $\{\mathcal{L}(\mathcal{C}|_h)\}_{h \in \mathbb{N}}$  is a chain (in fact  $\mathcal{C}|_h \subseteq \mathcal{C}|_{h+1}$ ), hence directed. We conclude that  $\mathcal{I}(\mathcal{C})$  is directed because the downward closure of a directed set is directed ( $\preceq$  being a transitive relation).

As for item 2, first, we prove that  $\mathcal{C} \subseteq \mathcal{I}(\mathcal{C})$ . By Lemma 18,  $\mathcal{C}|_h \subseteq \mathcal{I}(\mathcal{C})$ , for every  $h$ . We conclude because  $\mathcal{C} = \bigcup_h \mathcal{C}|_h$ . Second, let  $\mathcal{D}$  be an ideal containing  $\mathcal{C}$ , it is enough to prove  $\mathcal{L}(\mathcal{C}|_h) \in \mathcal{D}$  for every  $h$  for concluding  $\mathcal{C} \subseteq \mathcal{D}$ . This is a consequence of Lemma 19. ◀

► **Corollary 21.** *Let  $\mathcal{C}$  be a maximal clique. Then,  $\mathcal{C}$  is an ideal.*

**Proof.** Let  $\mathcal{C}$  be a maximal clique. By Proposition 20,  $\mathcal{I}(\mathcal{C})$  is an ideal and  $\mathcal{C} \subseteq \mathcal{I}(\mathcal{C})$ . And by Proposition 16,  $\mathcal{I}(\mathcal{C})$  is a clique. So we conclude by maximality of  $\mathcal{C}$ , that  $\mathcal{C}$  is equal to the ideal  $\mathcal{I}(\mathcal{C})$ . ◀

Recall from Section 2 that we consider Böhm-like trees ordered by the set-theoretical inclusion on the graph of their functions. Indeed, such an order is reflected in their Taylor expansion.

► **Lemma 22.** *If  $\mathcal{B}, \mathcal{B}'$  are Böhm-like trees,  $\tau(\mathcal{B}) \subseteq \tau(\mathcal{B}')$  iff  $\mathcal{B} \subseteq \mathcal{B}'$ . Moreover,  $\text{height}(\mathcal{B}) = \text{height}(\tau(\mathcal{B}))$ .*

**Proof.**  $\implies$  We suppose that  $\tau(\mathcal{B}) \subseteq \tau(\mathcal{B}')$ . By induction on  $\text{length}(\alpha)$ , for any  $\alpha \in \mathbb{N}^*$ , we show that  $\mathcal{B}(\alpha) = \mathcal{B}'(\alpha)$ , whenever  $\mathcal{B}(\alpha)$  is defined.

If  $\mathcal{B}(\langle \rangle) = (\lambda x_0 \dots x_{m-1}.y, n)$ , then  $\lambda x_0 \dots x_{m-1}.y 1 \dots 1 \in \tau(\mathcal{B}) \subseteq \tau(\mathcal{B}')$ , therefore by definition  $\mathcal{B}'(\langle \rangle) = (\lambda x_0 \dots x_{m-1}.y, n)$ . The induction case follows because  $\tau(\mathcal{B}) \subseteq \tau(\mathcal{B}')$  implies  $\tau(\mathcal{B}_i) \subseteq \tau(\mathcal{B}'_i)$  for every  $i < n$ . Then,  $\mathcal{B}(\langle i \rangle @ \alpha) = \mathcal{B}_i(\alpha)$  which, by induction hypothesis, is equal to  $\mathcal{B}'_i(\alpha) = \mathcal{B}'(\langle i \rangle @ \alpha)$ .

$\Leftarrow$  We suppose that  $\mathcal{B} \subseteq \mathcal{B}'$ . Let  $t \in \tau(\mathcal{B})$ , by induction on  $\text{height}(t)$ , one shows that  $t \in \tau(\mathcal{B}')$ . The reasoning is similar to the left-to-right implication.

The last statement regarding the height is trivial.  $\blacktriangleleft$

**► Lemma 23.** *Let  $\mathcal{T} \in \mathcal{P}(\Delta^{\text{NF}})$ . There is a Böhm-like tree  $\mathcal{B}$  such that  $\tau(\mathcal{B}) = \mathcal{T}$  iff  $\mathcal{T}$  is an ideal.*

**Proof.**  $\implies$  The proof depends whether  $\text{height}(\mathcal{B})$  is finite or infinite.

If  $\text{height}(\mathcal{B})$  is finite, we proceed by induction on  $\text{height}(\mathcal{B})$ . If  $\mathcal{B}(\langle \rangle) = (\lambda x_0 \dots x_{m-1}.y, n)$ , then  $\tau(\mathcal{B}) = \{\lambda x_0 \dots x_{m-1}.y \mu_0 \dots \mu_{n-1} \mid i < n, \mu_i \in \mathcal{M}_F(\tau(\mathcal{B}_i))\}$ .

**Downward closure.** Let  $t \preceq t' \in \tau(\mathcal{B})$ . Since  $t' \in \tau(\mathcal{B})$  we have  $t'$  of the shape

$$t' = \lambda x_0 \dots x_{m-1}.y \mu'_0 \dots \mu'_{n-1}. \quad (12)$$

Then, since  $t \preceq t'$ ,  $t$  is of the form

$$t = \lambda x_0 \dots x_{m-1}.y \mu_0 \dots \mu_{n-1}, \quad (13)$$

such that, for every  $i < n$ , either  $\mu_i = 1 \in \mathcal{M}_F(\tau(\mathcal{B}_i))$ , or there exists  $u' \in \mu'_i$  such that for any  $u \in \mu_i$ ,  $u \preceq u'$ . Notice that  $u' \in \tau(\mathcal{B}_i)$ , so by induction hypothesis  $u' \in \tau(\mathcal{B}_i)$ . Since this is the case for every  $u \in \mu_i$ , we get  $\mu_i \in \mathcal{M}_F(\tau(\mathcal{B}_i))$ . We conclude  $t \in \tau(\mathcal{B})$ .

**Directedness.** Let  $t, t' \in \tau(\mathcal{B})$ , and let us find  $t''$  such that  $t, t' \preceq t''$ . The terms  $t$  and  $t'$  are as in Equation (13) and (12), respectively.

For  $i < n$ ,  $\mu_i \cdot \mu'_i \in \mathcal{M}_F(\tau(\mathcal{B}_i))$ . By induction hypothesis,  $\tau(\mathcal{B}_i)$  is directed. Since  $\mu_i \cdot \mu'_i$  is finite (and  $\preceq$  is transitive), if  $\mu_i \cdot \mu'_i$  is non-empty, then there exists  $t_i \in \tau(\mathcal{B}_i)$ , such that for any  $u \in \mu_i \cdot \mu'_i$ ,  $u \preceq t_i$ . Then, define  $\xi_i = [t_i]$  if  $\mu_i \cdot \mu'_i$  non-empty, otherwise  $\xi_i = 1$ . We set  $t'' = \lambda x_0 \dots x_{m-1}.y \xi_0 \dots \xi_{n-1}$ . Notice that  $t'' \in \tau(\mathcal{B})$ , since  $\xi_i \in \mathcal{M}_F(\tau(\mathcal{B}_i))$ , and  $t, t' \preceq t''$ .

Now let us consider the case  $\text{height}(\mathcal{B}) = \infty$ . By definition  $\tau(\mathcal{B}) = \bigcup_{h \in \mathbb{N}} \tau(\mathcal{B}|_h)$ . Notice that  $\tau(\mathcal{B}|_h)$  is a Böhm-like tree of finite height, so, by the previous reasoning, we can conclude that  $\tau(\mathcal{B}|_h)$  is an ideal. Since  $\{\mathcal{B}|_h\}_{h \in \mathbb{N}}$  is a chain, we can easily conclude that the whole  $\tau(\mathcal{B})$  is an ideal.

$\Leftarrow$  In this case, we also split in two subcases, depending whether  $\text{height}(\mathcal{T})$  is finite or infinite. In the first case our induction is on  $\text{height}(\mathcal{T})$ .

If  $\text{height}(\mathcal{T}) = 0$ , then  $\mathcal{T} = \emptyset$  and we choose the Böhm-like tree undefined everywhere. Otherwise,  $\mathcal{T}$  is non-empty and we can write it as in Equation (11), by Lemma 12. Moreover, for any  $i < n$ ,  $\mathcal{T}_i$  is an ideal. By induction hypothesis, we can assume that there exists a Böhm-like tree  $\mathcal{B}^i$  such that  $\tau(\mathcal{B}^i) = \mathcal{T}_i$  (notice that  $\mathcal{B}^i$  is the everywhere undefined function whenever  $\mathcal{T}_i = \emptyset$ ). Define then  $\mathcal{B}(\langle \rangle) \triangleq (\lambda x_0 \dots x_{m-1}.y, n)$  and  $\mathcal{B}(\langle i \rangle @ \alpha) = \mathcal{B}^i(\alpha)$ . Note that in particular  $\mathcal{B}_i = \mathcal{B}^i$ .

In order to prove  $\tau(\mathcal{B}) = \mathcal{T}$ , we must show that  $\widehat{\mathcal{T}}_i = \mathcal{M}_F(\tau(\mathcal{B}_i))$ , for every  $i < n$ . By Lemma 12, we only know that  $\widehat{\mathcal{T}}_i \subseteq \mathcal{M}_F(\mathcal{T}_i) = \mathcal{M}_F(\tau(\mathcal{B}^i)) = \mathcal{M}_F(\tau(\mathcal{B}_i))$ . Let  $\mu_i \in \mathcal{M}_F(\mathcal{T}_i)$ , we prove that  $\mu_i \in \widehat{\mathcal{T}}_i$ . Actually, we prove that the term  $t_{\mu_i} \triangleq \lambda x_0 \dots x_{m-1}.y 1 \dots 1 \mu_i 1 \dots 1 \in \mathcal{T}$ , which is enough to conclude.

If  $\mu_i = 1$  then the bags in  $t_{\mu_i}$  are all empty and so one concludes by the downward closure of  $\mathcal{T}$ , remarking that  $t_{\mu_i}$  is  $\preceq$  to any element in  $\mathcal{T}$ . Otherwise, we have that  $|\mu_i|$  is a non-empty sub-set of  $\mathcal{T}_i$  of finite height (since it has finite cardinality). Moreover, since  $\mathcal{T}_i$  is an ideal,  $\mathcal{T}_i$ , and hence  $|\mu_i|$ , are cliques (Proposition 16). We first apply Lemma 19 to  $\mathcal{C} = |\mu_i| \subseteq \mathcal{T}_i$ . We then obtain  $\mathcal{L}(|\mu_i|) \in \mathcal{T}_i$ , so there exists  $t \in \mathcal{T}$  whose  $i$ -th bag contains  $\mathcal{L}(|\mu_i|)$ . We further apply Lemma 18, arguing that  $\forall u \in |\mu_i|, u \preceq \mathcal{L}(|\mu_i|)$ . This induces that  $t_{\mu_i} \preceq t$ , so finally,  $t_{\mu_i} \in \mathcal{T}$ .

We now suppose that  $\text{height}(\mathcal{T}) = \infty$ . Recall Definition 15, and notice that  $\mathcal{T}|_h \subseteq \mathcal{T}$  is downward closed because  $t \preceq t'$  implies  $\text{height}(t) \leq \text{height}(t')$ . We now show that  $\mathcal{T}|_h$  is directed. In fact, take any two  $t_1, t_2 \in \mathcal{T}|_h$ . Set  $\mathcal{C} = \{t_1, t_2\} \subseteq \mathcal{T}$ . By Proposition 16,  $\mathcal{T}$  and  $\mathcal{C}$  are cliques, so by Lemma 18 and 19,  $\mathcal{L}(\{t_1, t_2\}) \in \mathcal{T}$ ,  $t_1, t_2 \preceq \mathcal{L}(\{t_1, t_2\})$  and  $\text{height}(\mathcal{L}(\{t_1, t_2\})) \leq h$ . We conclude  $\mathcal{L}(\{t_1, t_2\}) \in \mathcal{T}|_h$ .

We can then apply the reasoning for sets of finite heights and conclude that there exists  $\mathcal{B}^h$  such that  $\tau(\mathcal{B}^h) = \mathcal{T}|_h$ . As  $\{\mathcal{T}|_h\}_{h \in \mathbb{N}}$  is by definition a chain, we conclude with Lemma 22 that  $\{\mathcal{B}^h\}_{h \in \mathbb{N}}$  is a chain. Let  $\mathcal{B} = \bigcup_{h \in \mathbb{N}} \mathcal{B}^h$ , we have that  $\mathcal{B}$  is a Böhm-like tree. Remark that  $\mathcal{B}|_h = \mathcal{B}^h$ , as for any  $h \in \mathbb{N}$ , the difference between  $\mathcal{B}^h$  and  $\mathcal{B}^{h+1}$  lies only on the sequences of length  $h$ , which are undefined for  $\mathcal{B}^h$ . Finally  $\tau(\mathcal{B}) \stackrel{\Delta}{=} \bigcup_{h \in \mathbb{N}} \tau(\mathcal{B}|_h) = \bigcup_{h \in \mathbb{N}} \tau(\mathcal{B}^h) = \bigcup_{h \in \mathbb{N}} \mathcal{T}|_h = \mathcal{T}$ . ◀

► **Lemma 24.** *Let  $\mathcal{B}$  be a Böhm-like tree, then*

- $\text{FV}(\mathcal{B}) = \text{FV}(\tau(\mathcal{B}))$ ,
- $\mathcal{B}$  is r.e. iff  $\tau(\mathcal{B})$  is r.e.

**Proof.** The only difficulty is to prove that  $\tau(\mathcal{B})$  r.e. implies  $\mathcal{B}$  r.e. Observe that an element  $t$  of  $\tau(\mathcal{B})$  defines a kind of subtree of  $\mathcal{B}$  (in fact, a rooted *thick* subtree [2]) which can be used to define a partial function  $f_t : \mathbb{N}^* \rightarrow \Sigma \times \mathbb{N}$  such that, whenever  $f_t(\alpha)$  is defined,  $f_t(\alpha)$  equals  $\mathcal{B}(\alpha)$ . Moreover for every  $\alpha \in \mathbb{N}^*$ , such that  $\mathcal{B}(\alpha)$  is defined, there exists  $t \in \tau(\mathcal{B})$  such that  $f_t(\alpha)$  is defined. Hence, given an effective enumeration of  $\tau(\mathcal{B})$ , one can compute  $\mathcal{B}(\alpha)$  by iterating over  $\tau(\mathcal{B})$  with  $t$  until  $f_t(\alpha)$  is defined (if  $\mathcal{B}(\alpha)$  is undefined this will never end). ◀

► **Theorem 25.** *Let  $\mathcal{T} \in \mathcal{P}(\Delta^{\text{NF}})$ . There is a  $\lambda$ -term  $M$  such that  $\text{NF}(\tau(M)) = \mathcal{T}$  iff the following conditions hold:*

1.  $\text{FV}(\mathcal{T})$  is finite,
2.  $\mathcal{T}$  is r.e.,
3.  $\mathcal{T}$  is an ideal wrt  $\preceq$ .

**Proof.** By Theorem 8, the equality  $\text{NF}(\tau(M)) = \mathcal{T}$  can be replaced by  $\tau(\text{BT}(M)) = \mathcal{T}$ . Then, by Theorem 5 the condition at the left-hand side of the iff can be replaced by “there is a Böhm-like tree  $\mathcal{B}$  such that (i)  $\text{FV}(\mathcal{B})$  is finite, (ii)  $\mathcal{B}$  is r.e., (iii)  $\tau(\mathcal{B}) = \mathcal{T}$ ”. The equivalence is then achieved by Lemma 23 and 24. ◀

In [9], Ehrhard and Regnier noticed that the Taylor expansion of normal forms are maximal cliques with respect to the set-theoretical inclusion. However, not every maximal clique  $\mathcal{T}$  represents a normalizable  $\lambda$ -term, even if  $\mathcal{T}$  enjoys the conditions of Theorem 25. For example,  $\text{NF}(\tau(\Theta))$  is a maximal clique. We define the total  $\lambda$ -terms (called  $\perp$ -free in [1, Definition 10.1.12]) and prove that they correspond to the property of being a maximal clique in  $\mathcal{P}(\Delta^{\text{NF}})$ .

► **Definition 26** (Totality). A Böhm-like tree  $\mathcal{B}$  is *total* whenever  $\mathcal{B}(\langle \rangle) \downarrow$  and  $\mathcal{B}(\alpha) = (a, n)$  implies that for all  $i < n$ ,  $\mathcal{B}(\alpha @ \langle i \rangle) \downarrow$ . A  $\lambda$ -term  $M$  is *total* whenever  $M \xrightarrow{h^*} \lambda x_0 \dots x_{m-1}.(y)M_0 \dots M_{n-1}$  and  $M_0, \dots, M_{n-1}$  are total, i.e.  $\text{BT}(M)$  is total.

► **Lemma 27.** *Let  $\mathcal{B}$  be a Böhm-like tree. The set  $\tau(\mathcal{B})$  is a maximal clique (with respect to the set-theoretical inclusion) iff  $\mathcal{B}$  is total.*

**Proof.** One proves the equivalent statement: there exists  $t \in \Delta^{\text{NF}}$ ,  $t \notin \tau(\mathcal{B})$ , coherent with any element of  $\tau(\mathcal{B})$ , iff  $\mathcal{B}$  is not total. The left-to-right direction is by induction on  $\text{height}(t)$ , noticing that if a subtree  $\mathcal{B}_i$  is not total, then  $\mathcal{B}$  is not total. The converse direction is by induction on  $\text{length}(\alpha)$  for a sequence  $\alpha$  such that  $\mathcal{B}(\alpha) = (a, n)$  but there is  $i < n$ ,  $\mathcal{B}(\alpha @ \langle i \rangle) \uparrow$ . ◀

► **Corollary 28.** *Let  $\mathcal{T} \in \mathcal{P}(\Delta^{\text{NF}})$ . There is a total  $\lambda$ -term  $M$  such that  $\text{NF}(\tau(M)) = \mathcal{T}$  iff*

1.  $\text{FV}(\mathcal{T})$  is finite,
2.  $\mathcal{T}$  is r.e.,
3.  $\mathcal{T}$  is a maximal clique.

**Proof.**  $\implies$  By Theorem 25, we get **1** and **2**. Condition **3** is a consequence of Theorem 8 and Lemma 27.

$\impliedby$  By Lemma 21 and Theorem 25 we get a  $\lambda$ -term  $M$  such that  $\text{NF}(\tau(M)) = \mathcal{T}$ . By Theorem 8 and Lemma 27 we conclude that  $M$  is total. ◀

Of course, one would like to characterize the property of having a  $\beta$ -normal form.

► **Corollary 29.** *Let  $\mathcal{T} \in \mathcal{P}(\Delta^{\text{NF}})$ . There is a normalizable  $\lambda$ -term  $M$  such that  $\text{NF}(\tau(M)) = \mathcal{T}$  iff*

1.  $\text{height}(\mathcal{T})$  is finite,
2.  $\mathcal{T}$  is a maximal clique.

**Proof.**  $\implies$  Remark that if  $M$  is normalizable, then  $\text{BT}(M)$  is total and of finite height. In fact,  $\text{BT}(M)$  is the applicative tree of the normal form of  $M$ . By Corollary 28,  $\text{NF}(\tau(M))$  is a maximal clique. By Theorem 8 and Lemma 22 we get that  $\text{height}(\text{NF}(\tau(M)))$  is finite.

$\impliedby$  Consider the simple term  $\mathcal{L}(\mathcal{T})$ . One can easily prove that  $\mathcal{L}(\mathcal{T})$  does not contain empty bags, otherwise  $\mathcal{T}$  would not be maximal. Moreover, by induction of  $\text{height}(\mathcal{L}(\mathcal{T}))$ , one also proves that the  $\lambda$ -term  $M$  obtained from  $\mathcal{L}(\mathcal{T})$  by replacing any resource application  $\langle t \rangle[u]$  with a  $\lambda$ -calculus application  $(t)u$  is such that  $\mathcal{L}(\mathcal{T}) \in \tau(M)$ . Let us prove that  $\mathcal{T} = \tau(M)$ . By Proposition 20,  $\mathcal{I}(\{\mathcal{L}(\mathcal{T})\})$  is the smallest ideal containing  $\mathcal{L}(\mathcal{T})$ , so  $\mathcal{I}(\{\mathcal{L}(\mathcal{T})\}) \subseteq \tau(M)$ . By Lemma 18, for every  $t \in \mathcal{T}$ ,  $t \preceq \mathcal{L}(\mathcal{T})$ , hence  $\mathcal{T} \subseteq \mathcal{I}(\{\mathcal{L}(\mathcal{T})\}) \subseteq \tau(M)$ . By the maximality of  $\mathcal{T}$  we conclude  $\mathcal{T} = \tau(M)$ . ◀

Let us notice that the simple term  $\mathcal{L}(\mathcal{T})$  used in the proof of Corollary 29 for proving the existence of a  $\beta$ -normal form  $M$  s.t.  $\tau(M) = \mathcal{T}$  is called the *linearization* of  $M$  in [9].

## 5 Conclusion

The main Theorem 25, combined with Ehrhard and Regnier's Equation 2, gives a characterization of the support  $\text{NF}(\tau(M))$  of  $\text{NF}(M^*)$  that can be extended to a characterization of  $\text{NF}(M^*)$  itself (with positive rational coefficients instead of booleans). This refinement can be done by requiring in addition that the coefficient of each non-zero element  $t$  of the formal combination should exactly be the so-called *multiplicity coefficient*  $\text{m}(t)$  of  $t$ , a number defined by induction on  $t$  through the use of a generalization of binomial coefficients to

multisets (see [9]). Roughly speaking the multiplicity coefficient of a simple term  $t$  expresses the various ways  $t$  can be recombined into itself by varying enumeration of multisets.

Such a strict correspondence between supports and coefficients is lost in a non-deterministic setting. Ehrhard and Regnier's Equation 2 is due to the fact that the supports of the Taylor expansions of the  $\lambda$ -terms are cliques, and this is false as soon as one allows to superpose programs (e.g. by adding an `or` constructor, or a random operator). An open issue is then to capture the convergence of formal combinations of simple terms to non-deterministic, or probabilistic  $\lambda$ -terms.

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