



HAL
open science

Differential graded division algebras, their modules, and dg-simple algebras

Alexander Zimmermann

► **To cite this version:**

Alexander Zimmermann. Differential graded division algebras, their modules, and dg-simple algebras. 2024. hal-04669701v2

HAL Id: hal-04669701

<https://hal.science/hal-04669701v2>

Preprint submitted on 30 Sep 2024 (v2), last revised 21 Oct 2024 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

DIFFERENTIAL GRADED DIVISION ALGEBRAS, THEIR MODULES, AND DG-SIMPLE ALGEBRAS

ALEXANDER ZIMMERMANN

ABSTRACT. We give the definition of a dg-division algebra, that is a concept of a differential graded algebra which may serve as an analogue of a division algebra. Using this concept, we study their dg-modules, give examples when they occur and study their relation to dg-simple algebras.

INTRODUCTION

Differential graded algebras (dg-algebras for short) and their differential graded modules appear in various places, mainly of geometric and topological nature. More precisely, let K be a commutative ring. A dg-algebra over K is a \mathbb{Z} -graded algebra with a graded K -linear endomorphism d of degree 1 with $d^2 = 0$ satisfying the Leibniz equation $d(ab) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)$ for all homogeneous $a, b \in A$. Dg-modules are defined similarly. For more precise definitions we refer to Section 1.1. Though defined by Cartan [5] in 1954 already, the ring theory of dg-algebras remained largely unexplored until quite recently. In 2002 Aldrich and Garcia Rozas characterized the dg-algebras whose dg-module category is semisimple. They obtained that this is the case if and only if the algebra is acyclic and the algebra of cycles $\ker(d)$ is graded semisimple. Orlov studied in [13, 14] finite dimensional dg-algebras over a field K with a geometric motivation. Using this approach Goodbody [8] studied a version of a dg-Jacobson radical. In a sequel of papers we studied more systematically the ring theory of dg-algebras. In [16] we defined dg-Jacobson radicals in a much more general and natural setting, study dg-simple dg-algebras, dg-simple dg-modules, and the relation with dg-simple dg-algebras. We study dg-orders and define locally free dg-class groups. In [18] we define Ore localisation of dg-algebras, proving that under some hypotheses, the localised ring is again dg. Using this, we study and define dg-uniform dimension, and a dg-Goldie theorem for dg-Goldie rings.

In the present paper we first define a dg-division algebra as a dg-ring without non trivial left or right dg-ideals. The question of an appropriate concept for a dg-division algebra was posed recently by Violeta Borges Marques and Julie Symons. We show that under a certain mild technical condition dg-division algebras are precisely those for which the ring of cycles is a \mathbb{Z} -graded-division algebras (cf [12, page 38] for the graded concept). For this result we need a technical assumption, namely that the set of left regular elements of the ring of cycles coincides with the set of right regular elements. This holds true, by a result of Goodearl and Stafford [9] for graded-Noetherian and graded-prime rings of cycles, using a graded version of Goldie's result. We further show that dg-modules over dg-division algebras are free. We then classify completely dg-division rings under these conditions. Namely, they are either acyclic, and hence the structure is completely given by a result from Aldrich and Garcia Rozas [1] and Nastacescu and van Oystaen [12, A.1.4.3 Corollary] or else they have differential 0. Not surprisingly, we prove that endomorphism complexes of dg-simple dg-modules over dg-algebras are dg-division algebras. We provide further examples.

We study the graded centre of a dg-simple algebra, and show that this is a dg-division algebra. In particular, the graded centre of a dg-division algebra is a dg-division algebra. Further, under the above mentioned technical assumption, the homology of a dg-division algebra is a gr-division algebra, as defined by Nastacescu-van Oystaen [12]. Recall that [12] classified completely commutative gr-fields with the group being \mathbb{Z} . Such a gr-field is either a Laurent polynomial ring or a field concentrated in degree 0.

Date: August 8, 2024.

2020 Mathematics Subject Classification. Primary: 16E45; Secondary: 12H05; 13A02; 16N60; 16W50.

Key words and phrases. differential graded algebras.

Further, we study dg-primitive dg-algebras and some of their properties. We finally show that for a dg-Noetherian and dg-artinian dg-algebra being dg-simple is equivalent with being dg-primitive. In a final section we show a dg-version of the Jordan-Chevalley density theorem in case of acyclic dg-algebras.

The paper is organized as follows. In Section 1 we recall the definition of dg-algebras and their modules, the constructions we use in the paper, and we define dg-division algebra, the main structure studied in our paper. In Section 2 we study the main properties of dg-division algebras with respect to the cycles, and some occurrences of dg-division algebras. In Section 3 we show that dg-modules over dg-division algebras are free, and in Section 4 we define graded centres of graded algebras and show that the graded centre of a dg-algebra is a dg-subalgebra. We further consider the graded center of a tensor product of dg-algebras. Section 5 studies dg-primitive algebras, i.e. dg-algebras allowing a dg-simple faithful dg-module, and their relation to dg-simple and dg-prime algebras. Section 6 proves a version of the Jacobson-Chevalley density theorem for dg-algebras.

Acknowledgement: I wish to thank Violeta Borges Marques and Julie Symons for having asked me during the Oberwolfach workshop 'Hochschild (co-)homology and applications' in April 2024 for a possible concept of a dg-field.

1. ELEMENTARY DEFINITIONS

1.1. Differential graded algebra and their modules; definitions and notations. In this subsection we recall notations and basic conventions, as well as the construction of a tensor product of two dg-algebras over a common graded commutative subalgebra of their graded centres. This should be well-known, but we could not find a reference, and in any case the reader may appreciate an explicit verification right in the paper.

Let K be a commutative ring. A differential graded K -algebra (dg- K -algebra for short) (A, d) is a \mathbb{Z} -graded algebra A with a graded K -linear endomorphism of degree 1 with $d^2 = 0$ and satisfying the Leibniz rule $d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)$. By definition a dg-ring is a dg- \mathbb{Z} -algebra.

If (A, d) is a dg- K -algebra, then define A^{op} to be the same graded additive group as A , but multiplication defined as $a \cdot_{op} b := (-1)^{|a||b|} b \cdot a$. Then (A^{op}, d) is again a dg- K -algebra.

A differential graded left module (M, δ) over (A, d) is a \mathbb{Z} -graded A -module M with an endomorphism δ of degree 1 satisfying $\delta(a \cdot m) = d(a) \cdot m + (-1)^{|a|} a \cdot \delta(m)$ for all homogeneous $a \in A$ and $m \in M$. Occasionally we denote a differential graded module by dg-module for short.

A differential graded right module over (A, d) is a differential graded left module over (A^{op}, d) .

For two differential graded left modules (M, δ_M) and (N, δ_N) over (A, d) we put

$$\mathrm{Hom}_A^\bullet((M, \delta_M), (N, \delta_N)) := \{f \in \mathrm{Hom}_{\mathrm{graded}}(M, N) \mid f(am) = (-1)^{|a||f|} a f(m)\}$$

and

$$d_{\mathrm{Hom}}(f) := \delta_N \circ f - (-1)^{|f|} f \circ \delta_M.$$

Further, $\mathrm{End}_A^\bullet(M, \delta_M) := \mathrm{Hom}_A^\bullet((M, \delta_M), (M, \delta_M))$. Then $(\mathrm{End}_A^\bullet(M, \delta_M), d_{\mathrm{Hom}})$ is a dg- K -algebra, and $(\mathrm{Hom}_A^\bullet((M, \delta_M), (N, \delta_N)), d_{\mathrm{Hom}})$ is a left dg-module over $\mathrm{End}_A^\bullet(N, \delta_N)$ and a right dg-module over $\mathrm{End}_A^\bullet(M, \delta_M)$.

Let (A, d) be a dg- K -algebra. Then

$$Z_{gr}(A, d) = \{a \in A \mid a \cdot b = (-1)^{|a||b|} b \cdot a \quad \forall \text{ homogeneous } b \in A\}$$

is the graded centre of (A, d) .

If (A, d_A) and (B, d_B) are dg- K -algebras, then $((A \otimes_K B), d_{A \otimes_K B})$ is a dg- K -algebra again with

$$d_{A \otimes B} = d_A \otimes \mathrm{id}_B + \mathrm{id}_A \otimes d_B$$

and

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{|b_1||a_2|} (a_1 a_2 \otimes b_1 b_2).$$

Let (A, d_A) and (B, d_B) be dg- K -algebras, and let $Z = (Z, d)$ be a common graded commutative dg-subalgebra of the graded centre of A and of B . (For a more detailed discussion of the graded center we refer to Section 4 below.) Then there is a differential $d_{A \otimes_Z B}$ on $A \otimes_Z B$ such that $(A \otimes_Z B, d_{A \otimes_Z B})$ is a dg-algebra again. First, we consider the ordinary tensor product $A \otimes_Z B$. This is graded by posing the degree n component as $\bigoplus_{k \in \mathbb{Z}} A_k \otimes_Z B_{n-k}$. This is well-defined since Z is graded. Indeed,

if $a \in A_k$ is in degree $k - \ell$, $b \in B_{n-k}$ is in degree $n - k$, and $z \in Z_\ell$ is in degree ℓ , then $az \otimes b = a \otimes zb$ and the term on both sides is in degree n . Further, put again

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{|b_1||a_2|} (a_1 a_2 \otimes b_1 b_2).$$

This is well-defined again, as is shown by the following computation

$$\begin{aligned} (a_1 z_1 \otimes b_1) \cdot (a_2 \otimes z_2 b_2) &= (-1)^{|b_1||a_2|} (a_1 z_1 a_2 \otimes b_1 z_2 b_2) \\ &= (-1)^{|b_1||a_2|+|z_1||a_2|+|b_1||z_2|} (a_1 a_2 z_1 \otimes z_2 b_1 b_2) \\ &= (-1)^{|b_1||a_2|+|z_1||a_2|+|b_1||z_2|} (a_1 a_2 \otimes z_1 z_2 b_1 b_2) \\ &= (-1)^{|b_1||a_2|+|z_1||a_2|+|b_1||z_2|+|z_1||z_2|} (a_1 a_2 \otimes z_2 z_1 b_1 b_2) \\ &= (-1)^{|b_1||a_2|+|z_1||a_2|+|b_1||z_2|+|z_1||z_2|} (a_1 a_2 z_2 \otimes z_1 b_1 b_2) \\ &= (-1)^{(|b_1|+|z_1|)(|a_2|+|z_2|)} (a_1 a_2 z_2 \otimes z_1 b_1 b_2) \\ &= (a_1 \otimes z_1 b_1) \cdot (a_2 z_2 \otimes b_2) \end{aligned}$$

for $a_1, a_2 \in A$, $b_1, b_2 \in B$, $z_1, z_2 \in Z$ all homogeneous. Then,

$$d_{A \otimes B} = d_A \otimes \text{id}_B + \text{id}_A \otimes d_B$$

is well-defined again. Indeed, since $d_A(z) = d_B(z)$ for all homogeneous $z \in Z$, we get

$$\begin{aligned} d_{A \otimes B}(az \otimes b) &= d_A(az) \otimes b + (-1)^{|az|} az \otimes d_B(b) \\ &= d_A(a)z \otimes b + (-1)^{|a|} ad_A(z) \otimes b + (-1)^{|az|} az \otimes d_B(b) \\ &= d_A(a) \otimes zb + (-1)^{|a|} a \otimes d_A(z)b + (-1)^{|az|} a \otimes z d_B(b) \\ &= d_A(a) \otimes zb + (-1)^{|a|} (a \otimes d_B(z)b + (-1)^{|z|} a \otimes z d_B(b)) \\ &= d_A(a) \otimes zb + (-1)^{|a|} (a \otimes d_B(zb)) \\ &= d_{A \otimes B}(a \otimes zb) \end{aligned}$$

for all homogeneous $a \in A$ and $b \in B$. This shows that $d_{A \otimes B}$ is well-defined. The fact that $d_{A \otimes B}^2 = 0$ is trivial, and actually follows from the classical case, such as the fact that it verifies Leibniz' rule.

We further mention that for a graded algebra A by some group G we denote by a gr-simple A -module an A -module which does not allow a G -graded submodule other than 0 or itself. Similarly, we denote by a gr-simple algebra an algebra which is gr-simple as $A \otimes A^{op}$ -module, and a gr-division algebra an algebra where all homogeneous non zero elements are invertible. Occasionally we use the notion graded-simple for gr-simple, etc.

1.2. Elements of the ring theory of differential graded algebra.

Definition 1.1. A dg- K -algebra (A, d) is *dg-simple* if (A, d) does not contain any twosided dg-ideal other than 0 and A .

Definition 1.2. A dg- K -algebra (A, d) is a *dg-division algebra* if (A, d) does not contain any dg-left ideal nor a dg-right ideal other than 0 and A .

Remark 1.3. In case we want to stress that a dg-division algebra is commutative (or graded commutative) we shall call it a dg-field (or graded commutative dg-field).

Remark 1.4. Note that if $d = 0$, then this was classified by Nastacescu-van Oystaen [12, page 38]. If K is algebraically closed, then $A \simeq K^\alpha G$ for a finite group G and an $\alpha \in H^2(G, K^\times)$.

Example 1.5. Let K be a field. Then by [18] the dg-ring $K[X, X^{-1}]$ with differential $d(X^{2n+1}) = X^{2n}$ and $d(X^{2n}) = 0$ for all $n \in \mathbb{Z}$ is dg-simple. Since it is commutative, it is a dg-field. Note however that the ungraded ring $K[X, X^{-1}]$ is not Artinian (though Noetherian).

2. RELATING DG-DIVISION ALGEBRAS WITH GR-DIVISION ALGEBRA

Lemma 2.1. *Let (A, d) be a dg-ring and suppose that $\ker(d)$ is a \mathbb{Z} -gr division algebra. Then (A, d) is a dg-division algebra.*

Proof. Let (I, d) be a non trivial dg-left ideal of (A, d) . Then $I \cap \ker(d)$ is a \mathbb{Z} -graded ideal of $\ker(d)$.

Since $I \neq 0$, there is $0 \neq x \in I$. If $x \in \ker(d)$, we have shown that $I \cap \ker(d) \neq 0$. If $d(x) \neq 0$, then $d(x) \in I$ since I is a dg-ideal, and $d(x) \in \ker(d)$ since $d^2 = 0$. Hence $I \cap \ker(d) \neq 0$.

Suppose that $I \cap \ker(d) = \ker(d)$. Since $d(1) = 0$, we get $1 \in I \cap \ker(d)$. Hence $1 \in I$ and therefore $A = I$. This is a contradiction to the hypothesis that I is a non trivial dg-ideal. The same holds for dg-right ideals. ■

Lemma 2.2. *Let (A, d) be a dg-ring and suppose that (A, d) does not contain any non trivial dg-left ideal. If the set of homogeneous right regular element of $\ker(d)$ coincides with the set of homogeneous left regular elements of $\ker(d)$, then $\ker(d)$ is a \mathbb{Z} -gr-division algebra.*

Proof. Let $u \neq 0$ be a homogenous element of $\ker(d)$. Then Au is a dg-left ideal of A . Since $u \neq 0$, we get $Au \neq 0$. Suppose that $Au = A$. Then there is $b \in A$ with $1 = bu$. Then

$$0 = d(1) = d(b) \cdot u + (-1)^{|b|} b \cdot d(u) = d(b) \cdot u.$$

Since u has a left inverse in A , it is right regular. Indeed, if $ux = 0$ in A , then $x = bux = 0$. Hence, by hypothesis, u is left regular as well. Therefore, $d(b) = 0$ and hence $b \in \ker(d)$. This shows that the equation $bu = 1$ already holds in $\ker(d)$, and hence u is left invertible in $\ker(d)$. Therefore, $\ker(d)$ does not contain any non trivial graded left ideal, and by [12, page 38; Lemma 1.4.1] u is invertible in $\ker(d)$. ■

Remark 2.3. We may consider situations where the hypothesis that the set of homogeneous right regular element of $\ker(d)$ coincides with the set of homogeneous left regular elements of $\ker(d)$ holds.

- If $\ker(d)$ is finite dimensional over some field, this is true, since multiplication by a left regular element u is realised by a matrix with non zero determinant.
- If $\ker(d)$ is graded commutative, or commutative, then trivially this property holds.
- It is known that if $\ker(d)$ is left gr-Noetherian and gr-prime, this is true. Indeed, by [9] localising at the homogeneous regular elements of $\ker(d)$ yields a gr-simple algebra with all homogeneous regular elements being invertible. Now, the proof of [7, (5.8) Proposition, (5.9) Proposition] applies verbatim to graded rings and homogeneous left invertible elements.

Theorem 2.4. *Let (A, d) be a differential graded ring. Suppose that in $\ker(d)$ the set of the left regular homogeneous elements and the set of the right regular homogeneous elements coincide. Then (A, d) is a dg-division algebra if and only if $\ker(d)$ is a gr-division algebra.*

Proof. This is a direct consequence of Lemma 2.1 and Lemma 2.2. Note that the lemmas give a more general statement for the only if direction. ■

Corollary 2.5. *Let (A, d) be a differential graded ring. Suppose that in $\ker(d)$ the set of the left regular homogeneous elements and the set of the right regular homogeneous elements coincide. Then the following statements are equivalent.*

- (A, d) admits only trivial dg-left ideals.
- (A, d) admits only trivial dg-right ideals.
- each non zero homogeneous element of $\ker(d)$ is invertible (i.e. $\ker(d)$ is a gr-division algebra).

Proof. The first statement implies the third statement by Lemma 2.2. Similarly, the second statement implies the third statement by Lemma 2.2. The third statement implies each of the first and the second statement by Lemma 2.1. ■

Remark 2.6. Recall that in a graded commutative ring, the square of homogeneous elements of odd degree is 0, unless the characteristic of the base ring is 2. Indeed, let x be a homogeneous element of odd degree. Then, swapping the two factors yields

$$x \cdot x = (-1)^{|x||x|} x \cdot x = -x \cdot x.$$

Hence $2 \cdot x^2 = 0$.

Corollary 2.7. *Let (D, ∂) be a dg-division algebra. Suppose that in $\ker(\partial)$ the set of the left regular homogeneous elements and the set of the right regular homogeneous elements coincide. Then $(Z_{gr}(D), \partial)$ is a dg-division algebra as well. In particular, if the characteristic of D is different from 2, then $Z_{gr}(D) \cap \ker(\partial)$ is concentrated in even degrees, and in any case $Z_{gr}(D) \cap \ker(\partial)$ is commutative and either isomorphic to a Laurent polynomial ring or concentrated in degree 0.*

Proof. By Lemma 4.1 we see that $Z_{gr}(D)$ is a dg-subalgebra of (D, ∂) . Further, all elements in $Z_{gr}(D) \cap \ker(\partial)$ are invertible in $\ker(d)$. However, if $z \in Z_{gr}(D)$ and $a \in \ker(\partial)$ are homogeneous elements with $az = za = 1$ (using that in a group the left and the right inverse of a fixed element coincide), then

$$b = baz = (-1)^{|a||b|} abz$$

and since z is right regular, $ba = (-1)^{|a||b|} ab$ for all homogeneous $b \in D$. Hence $a \in Z_{gr}(D)$ as well. Using Corollary 2.5 and Remark 2.6 we proved the statement. The classification of commutative \mathbb{Z} -graded gr-division algebras is given in [12, Section A.1.4; Section B.1.1], and it is shown that $Z_{gr}(D) \cap \ker(\partial)$ is isomorphic to a Laurent polynomial ring or concentrated in degree 0. ■

Theorem 2.8. *Let (A, d) be a dg-division algebra and suppose that the set of homogeneous left regular elements of $\ker(d)$ coincides with the set of homogeneous right regular elements of $\ker(d)$. Then either (A, d) is acyclic or $d = 0$. Further, $\ker(d) =: R$ is either concentrated in degree 0, or else $\ker(d)$ is a twisted Laurent polynomial ring $\ker(d) = R_0[T, T^{-1}; \sigma]$ for some automorphism σ of R_0 with $Xr = \sigma(r)X$ for all $r \in R_0$.*

Proof. Indeed, since $H(A, d)$ is a dg-algebra and since $\ker(d)/\text{im}(d) = H(A, d)$ as an algebra, $\text{im}(d)$ is a two-sided ideal of $\ker(d)$. However, as $\ker(d)$ is a gr-division algebra, every homogeneous non zero element of $\ker(d)$ is invertible in $\ker(d)$. Therefore, either $\ker(d) = \text{im}(d)$, or $\text{im}(d)$ does not contain any non zero homogeneous element of $\ker(d)$. Hence, either $H(A, d) = 0$ in the first case, or in the second case $d = 0$. The structure of $\ker(d)$ follows from Nastacescu-van Oystaen [12, A.1.4.3 Corollary]. ■

Example 2.9. Both cases mentioned in Theorem 2.8 do occur. Let $A = K[X, X^{-1}]$ the Laurent polynomial ring over a field K . Then the differential 0 yields a gr-field, whence also a dg-field. However, also $d(X^{2n+1}) := X^{2n}$ and $d(X^{2n}) = 0$ for all $n \in \mathbb{N}$ yields a dg-algebra, which is acyclic, and a dg-field, since it is already a gr-field. But also the cycles is a Laurent polynomial algebra over K in all even degrees, which is again a gr-field.

Corollary 2.10. *Let (A, d) be a dg-division algebra, and suppose that the set of homogeneous left regular elements of $\ker(d)$ coincides with the set of homogeneous right regular elements of $\ker(d)$. Then, either (A, d) is acyclic, or $\ker(d) = H(A, d)$ is a gr-division algebra.*

Proof. By Theorem 2.4 we see that $\ker(d)$ is a gr-division algebra, whence any homogeneous non zero element is invertible. Therefore, also in $H(A, d) = \ker(d)/\text{im}(d)$ every homogeneous non zero element is invertible. This shows the statement. ■

Further, we recall

Lemma 2.11. [1, Lemma 4.1, Lemma 4.2] *Let (A, d) be a dg-algebra. Then (A, d) is acyclic if and only if $1 \in \text{im}(d)$.*

Let (A, d) be an acyclic dg-algebra over a commutative ring R . Then, as an R -module $A = \ker(d) \oplus \ker(d) \cdot y = \ker(d) \oplus y \cdot \ker(d)$ for any homogeneous element $y \in A$ with $d(y) = 1$.

Remark 2.12. Note that using Lemma 2.11, if (A, d) is acyclic, then [1, Proposition 4.3] gives more explicitly $A \simeq \ker(d)[X; D]/(X^2 - y^2)$, with $d(y) = 1$ and $D(a) = -(-1)^{|a|} d(yay)$ for any homogeneous $a \in \ker(d)$. Furthermore, $A = \ker(d) \oplus \ker(d)y$, and the isomorphism is $\Phi : \ker(d)[X; D] \rightarrow A$ by $\Phi(X) = y$ and $\Phi(a) = a$ for any $a \in \ker(d)$. For any homogeneous $a, b \in \ker(d)$ we get $d(b + ay) = a$. Moreover, $D(a) = Xa - (-1)^{|a|} aX$ for any homogeneous $a \in \ker(d)$. Recall that a dg-division algebra has either differential 0 or (A, d) is acyclic. The cycles are still given by $\ker(d)$, which need to be a gr-division ring.

Proposition 2.13. *Let K be a field, let (A, d) be a differential graded algebra and let (S, δ) be a dg-simple left dg-module over (A, d) . Then $(\text{End}_A^\bullet(S, \delta), d_{\text{Hom}})$ is a dg-division algebra. Moreover, the set of homogeneous left regular elements of $\ker(d_{\text{Hom}})$ coincides with the set of homogeneous right regular elements of $\ker(d_{\text{Hom}})$.*

Proof. We shall use Corollary 2.5 and we shall see that the hypotheses are satisfied..
Now, let $f \neq 0$ be homogeneous with $f \in \ker(d_{\text{Hom}})$. Then this is equivalent with

$$0 = d_{\text{Hom}}(f) = \delta \circ f - (-1)^{|f|} f \circ \delta$$

and hence

$$f \circ \delta = (-1)^{|f|} \delta \circ f.$$

We claim that $\ker(f)$ is a dg-submodule of (S, δ) . Let $x \in \ker(f)$ be homogeneous.

$$0 = \delta(f(x)) = (-1)^{|f|} f(\delta(x))$$

and hence $\delta(x) \in \ker(f)$. Moreover, for any homogeneous $a \in A$

$$f(ax) = (-1)^{|a||f|} a \cdot f(x) = 0$$

and hence $ax \in \ker(f)$ again.

Since (S, δ) is dg-simple, either $f = 0$ or $\ker(f) = 0$.

Also $\text{im}(f)$ is a dg-submodule of (S, δ) . If $x = f(y)$, then

$$\delta(x) = \delta(f(y)) = (-1)^{|f|} f(\delta(y)) \in \text{im}(f)$$

Further,

$$a \cdot f(x) = (-1)^{|a||f|} f(a \cdot x) \in \text{im}(f)$$

again. Since (S, δ) is dg-simple, and since $\text{im}(f)$ is a dg-submodule, either $\text{im}(f) = 0$ (which is equivalent with $f = 0$ and this was excluded) or $\text{im}(f) = S$. Therefore f is surjective as well.

Hence f is an isomorphism between (S, δ) and a shifted copy, whence invertible, and this shows that $\ker(d_{\text{Hom}})$ is a gr-division algebra.

We want to apply Corollary 2.5. If f is a left invertible non invertible dg-endomorphism of (S, δ) , then (S, δ) has a non trivial direct factor, which is contradictory to (S, δ) being simple. Likewise a right invertible non invertible dg-endomorphism leads to a contradiction.

Hence $(\text{End}_A^\bullet(S, \delta), d_{\text{Hom}})$ is a dg-division algebra by Corollary 2.5. ■

Recall the following statement from Aldrich and Garcia Rozas [1].

Theorem 2.14. [1, Theorem 5.3]. *Let (A, d) be a dg-algebra. Then the following statements are equivalent.*

- (1) *The category of left dg-modules over (A, d) is semisimple.*
- (2) *The left module (A, d) over (A, d) is a direct sum of a family of dg-simple dg-modules.*
- (3) *(A, d) is acyclic and the category of \mathbb{Z} -graded modules over $\ker(d)$ is semisimple.*

Note that this shows that for a dg-simple dg-algebra (A, d) with non zero homology the regular (A, d) -left module (A, d) cannot be the direct sum of dg-simple dg-modules. However, in case the category of left dg-modules is semisimple, we have the following statement.

Proposition 2.15. *Let (A, d) be a dg-simple dg-algebra. If (A, d) is left dg-artinian. Then, up to isomorphism of dg-modules and shift in degrees, there is a unique simple dg-module (S, δ) over (A, d) .*

Suppose in addition that one (and hence all) of the conditions of Theorem 2.14 hold. Then, for all finitely generated dg-module (M, δ_M) over (A, d) , such as (A, d) , there is a positive integer n and we get

$$\text{End}_A^\bullet(M, \delta_M) \simeq \text{Mat}_n((D, \partial))$$

for some dg-division algebra

$$(D, \partial) \simeq (\text{End}_A^\bullet(S, \delta), d_{\text{Hom}}).$$

Proof. As (A, d) is dg-Artinian, there is a minimal element in the set of non zero dg-left submodules of (A, d) . This minimal element is necessarily a dg-simple dg-module (S, δ) . By [16, Lemma 4.22] we get that all dg-simple dg-modules over (A, d) are isomorphic to (S, δ) .

Suppose now that in addition the category of left dg-modules over (A, d) is semisimple. Then the regular left dg-module (A, d) over (A, d) is a direct sum of dg-simple dg-modules. As (S, δ) is the only dg-simple dg-module over (A, d) up to isomorphism,

$$(A, d) = \bigoplus_{j \in J} (S_j, \delta_j)$$

for $(S_j, \delta_j) \simeq (S, \delta)$ for all $j \in J$. We claim that the index set J for (A, d) is finite. Else, there is a countable subset $\{j_i \mid i \in \mathbb{N}\} \subseteq J$. Then

$$U_i := \sum_{j \in J \setminus \{j_1, \dots, j_i\}} (S_j, \delta_j)$$

defines an infinite decreasing sequence of dg-submodules. This contradicts (A, d) being left dg-artinian, and hence J is finite.

But now, for any finitely generated dg-modules (M, δ_M) over (A, d) we get that

$$M = \bigoplus_{j=1}^n S_j$$

for $S_j \simeq S$. Hence,

$$\text{End}_A^\bullet(M, \delta_M) \simeq \text{Mat}_{n \times n}(D, \partial).$$

for $(D, \partial) = (\text{End}_A^\bullet(S, \delta), d_{\text{Hom}})$. ■

3. DG-MODULES OVER DG-DIVISION ALGEBRAS

We shall prove that a dg-module over dg-division algebra contains a basis.

Proposition 3.1. *Let (A, d) be a dg-division algebra and let (M, δ) be a dg-module. Suppose that the set of left invertible elements of $\ker(d)$ coincides with the set of right invertible elements of $\ker(d)$. Then there is a set $\Lambda \subset \ker(\delta)$ such that*

$$M = \bigoplus_{m \in \Lambda} A \cdot m$$

and $A \cdot m \simeq {}_A A[-|m|]$, i.e. each direct factor is isomorphic to the regular module.

Proof. Let $0 \neq m \in \ker(\delta)$. Then the map

$$\begin{array}{ccc} A[-|m|] & \xrightarrow{\rho_m} & A \cdot m \\ a & \mapsto & am \end{array}$$

is a homomorphism of dg left modules. Indeed, it is obviously a left module homomorphism. Further,

$$\rho_m(d(a)) = d(a) \cdot m = \delta(a \cdot m) - (-1)^{|a|} a \cdot \delta(m) = \delta(\rho_m(a)),$$

where the first equation is Leibniz formula and the second equation comes from $m \in \ker \delta$.

Further, (A, d) is a dg-division algebra, and hence does not contain any non trivial dg left ideal. Hence ρ_m is injective, and surjective by definition.

Now, let \mathfrak{L} be the set of subsets $\Lambda \subseteq \ker \delta$ with

$$\sum_{m \in \Lambda} Am = \bigoplus_{m \in \Lambda} Am.$$

This set is non empty since each non zero singleton in $\ker \delta$ is an element in \mathfrak{L} . The set \mathfrak{L} is partially ordered by inclusion. Let \mathfrak{D} be a totally ordered subset on \mathfrak{L} . Then put

$$\Gamma := \bigcup_{\Lambda \in \mathfrak{D}} \Lambda.$$

We claim that

$$D := \sum_{m \in \Gamma} Am = \bigoplus_{m \in \Gamma} Am.$$

Indeed, if $x \in \sum_{m \in \Gamma} Am$, then $x = \sum_{i=1}^n x_i m_i \in \sum_{i=1}^n Am_i$ for some $m_i \in \Gamma$ and $x_i \in A$. Hence there is $\Lambda \in \mathfrak{D}$ such that $m_1, \dots, m_n \in \Lambda$. But, by hypothesis, $\sum_{m \in \Lambda} Am = \bigoplus_{m \in \Lambda} Am$. Hence $x = 0 \Leftrightarrow x_i = 0 \ \forall i \in \{1, \dots, n\}$.

By Zorn's lemma there is a maximal element Δ in \mathfrak{L} and set $F := \bigoplus_{m \in \Delta} Am$. Let $x \in M \setminus F$. Suppose $x \in \ker(\delta)$. Then the map $A \rightarrow Ax$ defined by $a \mapsto ax$ is injective, since the kernel is a dg-left ideal of A , whence 0 or A . The latter is impossible, and the injectivity follows. The map is surjective by definition, whence an isomorphism. Now, we have an inclusion of dg-left modules $F \cap Ax \leq Ax$. Since Ax is dg-simple, being isomorphic to A , the intersection is 0 or $Ax \subseteq F$, which is absurd by the choice of x . Hence if $x \in \ker(\delta)$ we have that $Ax + F = Ax \oplus F \leq M$ and $\Delta \cup \{x\} \in \mathfrak{L}$ is strictly larger than Δ , contrary to the maximality of Δ .

So, suppose that $x \notin \ker(\delta)$. Then consider the inclusion of dg left A -modules

$$Ax + A\delta(x) \geq A\delta(x).$$

We claim that there is $y \in A\delta(x)$ with $\delta(y) = \delta(x)$. Indeed, let $z \in A$ be homogeneous with $d(z) \neq 0$, then

$$A\delta(x) \xrightarrow{\varphi} A$$

with $\varphi(a \cdot \delta(x)) := a \cdot d(z)$ for all $a \in A$ defines a dg-left module isomorphism. Now, $y := \varphi^{-1}(z)$ is the required element. But then $y - x \in \ker(\delta)$ and $A(y - x)$ is again isomorphic to A as dg left module over (A, d) . Hence

$$\begin{aligned} Ax + A\delta(x) &= Ax + (Ay + A\delta(x)) \\ &= Ay + A\delta(x) + A(y - x) \\ &= (Ay + A\delta(x)) + A(y - x) \\ &= A\delta(x) + A(y - x) \\ &= A\delta(x) \oplus A(y - x) \end{aligned}$$

since there is no proper non zero submodule of $A\delta(x) \simeq A \simeq A(y - x)$, the dg-algebra (A, d) being a dg-division algebra. This shows that $F \oplus A\delta(x) \subseteq M$ and $\Delta \cup \{\delta(x)\}$ is strictly larger than Δ . This contradiction proves the statement. ■

Lemma 3.2. *Let (D, ∂) be a dg-division algebra and let (M, δ) be a dg-module over (D, ∂) . Suppose that (N, δ) is a dg-submodule of M . If (M, δ) is of finite dimension n over D , i.e. $M = \bigoplus_{i=1}^n D \cdot x_i$ for $x_i \in \ker(\delta)$. Then if $N = \bigoplus_{i=1}^n D y_i$ for $y_i \in \ker(\delta)$. Then $y_j = \sum_{i=1}^m d_{j,i} x_i$ for $d_{j,i} \in \ker(\partial)$.*

If D is graded commutative, then $n \leq m$ and $m = n$ is equivalent with $M = N$.

Proof. By definition we get

$$y_j = \sum_{i=1}^m d_{j,i} x_i$$

for uniquely determined homogeneous $d_{i,j} \in D$. But then,

$$0 = \delta(y_j) = \sum_{i=1}^m \partial(d_{j,i}) x_i$$

since $x_i \in \ker(\delta)$. By the direct sum decomposition $M = \bigoplus_{i=1}^m D \cdot x_i$ we get $\partial(d_{j,i}) = 0$.

Suppose now that D is graded commutative. Then the set of left and the set of right regular homogeneous elements of D coincide, and hence any homogeneous element of $\ker(\partial)$ is invertible in $\ker(\partial)$. Further, this implies, using that D is graded commutative, that $\ker(\partial)$ is concentrated in even degrees (or D is of characteristic 2). In any case, $\ker(\partial)$ is commutative. By the classification of \mathbb{Z} -graded gr-division algebras (cf Corollary 2.7 or [12, Section A.1.4; Section B.1.1]), either $\ker(\partial)$ is a field concentrated in degree 0, or $\ker(\partial) \simeq K[X, X^{-1}]$ for X an element in degree $2k$ for some k , and K being the degree 0 component.

Consider the matrix Δ formed by the elements $d_{i,j}$. The elementary divisors of Δ are unique, and give the embedding of N into M . Since N is a graded submodule of M , the elementary divisors have to be homogeneous in X as well, whence all invertible. Whence $n \leq m$ and if $n < m$, then $N < M$. If $n = m$, then Δ is invertible. This shows the statement. ■

4. THE GRADED CENTER

Lemma 4.1. *Let (A, d) be a differential graded ring. Then $(Z_{gr}(A), d)$ is a differential graded subalgebra. For all even degrees $2k$, the homogeneous components of the centre and the graded centre coincide. More precisely,*

$$(Z_{gr}(A), d)^{2k} = Z(A)^{2k} \quad \forall k \in \mathbb{Z}.$$

Proof. By construction, $Z_{gr}(A)$ is a subalgebra. Further, for any $a \in Z_{gr}(A)$ we get

$$\begin{aligned} d(a)b &= d(ab) - (-1)^{|a|} a \cdot d(b) \\ &= (-1)^{|a||b|} d(ba) - (-1)^{|a|+(|a|(|b|+1))} d(b)a \\ &= (-1)^{|a||b|} (d(ba) - d(b)a) \\ &= (-1)^{|a||b|+|b|} b \cdot d(a) \\ &= (-1)^{|d(a)||b|} b \cdot d(a) \end{aligned}$$

Hence $(Z_{gr}(A), d|_{Z_{gr}(A)})$ is a dg-algebra again. Further, by definition,

$$\begin{aligned} (Z_{gr}(A))^{2k} &:= \{a \in A_{2k} \mid ab = (-1)^{|2a||b|} ba \quad \forall b \text{ homogeneous in } A.\} \\ &= \{a \in A_{2k} \mid ab = ba \quad \forall b \text{ homogeneous in } A.\} \\ &= Z(A)^{2k}. \end{aligned}$$

Hence, the even degree elements of $Z_{gr}(A)$ coincide with those of $Z(A)$. ■

Lemma 4.2. *Let (A, d) be a dg-simple dg-algebra. Then $Z_{gr}(A, d)$ is a dg-division algebra.*

Proof. Let (I, d) be a dg-ideal of $Z_{gr}(A, d)$. Then (IA, d) is a twosided dg-ideal of (A, d) . Hence IA contains 1, whence also I has to contain 1. This shows the lemma. ■

Lemma 4.3. *Let (A, d_A) and (B, d_B) be two dg-simple dg-algebras with $Z_A = Z_{gr}(A, d_A)$ and $Z_B = Z_{gr}(B, d_B)$. Let Z be a common dg-subalgebra of Z_A and Z_B . Then $Z_A \otimes_Z Z_B$ is in the graded centre of $(A \otimes_Z B, d_{A \otimes_Z B})$.*

Proof. Indeed, $Z_A \otimes_Z Z_B$ is in the graded center, as

$$\begin{aligned} (z_1 \otimes z_2) \cdot (a \otimes b) &= (-1)^{|a||z_2|} (z_1 a \otimes z_2 b) \\ &= (-1)^{|a||z_2|+|z_1||a|+|z_2||b|} (az_1 \otimes bz_2) \\ &= (-1)^{|a||z_2|+|z_1||a|+|z_2||b|+|b||z_1|} (a \otimes b)(z_1 \otimes z_2) \\ &= (-1)^{|z_1 \otimes z_2||a \otimes b|} (a \otimes b)(z_1 \otimes z_2) \end{aligned}$$

This shows the lemma. ■

Lemma 4.4. *Let (A, d_A) and (B, d_B) be two dg-simple dg-algebras with $Z_{gr}(A, d_A) = Z = Z_{gr}(B, d_B)$. Then $Z \otimes_Z Z$ is the graded centre of $(A \otimes_Z B, d_{A \otimes_Z B})$.*

Proof. By Lemma 4.3, we get that $Z \otimes_Z Z$ is in the graded center. We shall adapt the usual proof from the classical case (cf e.g. [15, Theorem 7.6]) to this situation. By Proposition 3.1 and Lemma 4.2 we get $B = \bigoplus_{i \in I} Z \cdot b_i$ for $b_i \in \ker(d_B)$ and hence

$$A \otimes_Z B = \bigoplus_{i \in I} A \otimes_Z b_i$$

for homogeneous $b_i \in B$. If $x = \sum_{i \in I} a_i \otimes b_i \in Z_{gr}(A \otimes_Z B)$ is homogeneous of degree $|x|$, then for all homogeneous $a \in A$ we get

$$\begin{aligned} 0 &= (a \otimes 1) \cdot x - (-1)^{|a||x|} x \cdot (a \otimes 1) \\ &= \sum_{i \in I} (a \otimes 1) \cdot (a_i \otimes b_i) - (-1)^{|a||x|} (a_i \otimes b_i) \cdot (a \otimes 1) \\ &= \sum_{i \in I} (aa_i - (-1)^{|a|(|x|+|b_i|)} a_i a) \otimes b_i \end{aligned}$$

and hence $(aa_i - (-1)^{|a|(|x|+|b_i|)} a_i a) = 0$ for all i . However, $(-1)^{|x|+|b_i|} = (-1)^{|a_i|}$, which shows that $a_i \in Z_{gr}(A)$ for all $i \in I$. But this shows that $x \in Z_{gr}(Z \otimes_Z Z) = Z \otimes_Z Z$. ■

Theorem 4.5. *Let (D_A, ∂_A) and (D_B, ∂_B) be dg-division algebras over a dg-division algebra (Z, ∂) . Suppose that $Z = Z_{gr}(A) = Z_{gr}(B)$ and suppose that the set of left regular elements of $\ker \partial_A$ equals the set of right regular elements of $\ker \partial_A$, and likewise the set of left regular elements of $\ker \partial_B$ equals the set of right regular elements of $\ker \partial_B$. Then $(D_A \otimes_Z D_B, \partial_{A \otimes_Z B})$ is a dg-simple dg-algebra.*

Proof. Recall that Z is of characteristic different from 2 implies that the cycles of Z are supported in even degrees. As (D_B, ∂_B) is a dg-module over the dg-division algebra $(Z_A, d_A) =: (Z, d)$, by Proposition 3.1 there is a set $\{a_i \mid i \in I_A\}$ of homogeneous elements in $\ker(\partial_A)$ and a set $\{b_i \mid i \in I_B\}$ of homogeneous elements in $\ker(\partial_B)$ such that

$$D_A = \bigoplus_{i \in I_A} a_i \cdot Z \text{ and } D_B = \bigoplus_{i \in I_B} Z \cdot b_i.$$

Put $I := I_A \times I_B$, which is a Z -basis for $D_A \otimes_Z D_B$. Then, any z in $D_A \otimes_Z D_B$ can be written as

$$D_A \otimes_Z D_B \ni z = \sum_{(i_A, i_B) \in I} (a_{i_A} \otimes b_{i_B}) \lambda_{(i_A, i_B)}$$

for elements $\lambda_{(i_A, i_B)} \in Z$. All elements are homogeneous, and hence

$$|z| = |a_{i_A}| + |b_{i_B}| + |\lambda_{(i_A, i_B)}|.$$

Since $a_i \in \ker \partial_A$, we get

$$\begin{aligned} \partial_{D_A \otimes D_B}(z) &= \sum_{(i_A, i_B) \in I} (\partial_A \otimes 1 + 1 \otimes \partial_B)((a_{i_A} \otimes b_{i_B}) \lambda_{(i_A, i_B)}) \\ &= \sum_{(i_A, i_B) \in I} (-1)^{|a_{i_A}| + |b_{i_B}|} (a_{i_A} \otimes b_{i_B}) \partial(\lambda_{(i_A, i_B)}) \end{aligned}$$

and hence $z \in \ker(\partial_{D_A \otimes D_B})$ if and only if $\lambda_{(i_A, i_B)} \in \ker(\partial)$ for all $(i_A, i_B) \in I$. Suppose that J is a twosided dg-ideal in $D_A \otimes D_B$. As J is a dg-ideal, a non zero element x in J is either in $\ker(\partial_{D_A \otimes D_B})$, or $\partial_{D_A \otimes D_B}(x) \neq 0$ is in $J \cap \ker(\partial_{D_A \otimes D_B})$. Hence, $\ker(d) \cap J \neq 0$. Let

$$x = \sum_{(i_A, i_B) \in I} (a_{i_A} \otimes b_{i_B}) \lambda_{(i_A, i_B)} = (a_{i_1, A} \otimes b_{i_1, B}) \lambda_{(i_1, A; i_1, B)} + \dots + (a_{i_n, A} \otimes b_{i_n, B}) \lambda_{(i_n, A; i_n, B)}$$

be a non zero homogeneous element of minimal length n within the non zero elements in $J \cap \ker(\partial_{D_A \otimes D_B})$. Note that this is equivalent with $\lambda_{(i_j, A; i_j, B)} \neq 0$ for all $j \in \{1, \dots, n\}$.

In case D_B is dg-simple only, and not a dg-division algebra, then we may like to proceed as in [10] as follows. Since D_B is dg-simple, and since $0 \neq b_{i_1, B} \in \ker(\partial_B)$, we see that $D_B \cdot b_{i_1, B} \cdot D_B$ is a twosided dg-ideal of D_B . Since D_B is dg-simple, we may multiply from the left with an element $b_\ell \in D_B$ and an element $b_r \in D_B$ from the right such that $b_\ell b_{i_1, A} b_r = 1$. Moreover, the length of z and of $b_\ell z b_r$ coincide. However, $b_\ell z b_r$ may no longer be in $\ker(\partial_{A \otimes_Z B})$. This would hold, if one had that $\ker \partial_B$ is gr-simple as well.

Since we assumed that D_B is a dg-division algebra, by Corollary 2.5 we get that each $b_{i_j, B}$ is invertible for all j , and multiplying by $1 \otimes b_{i_1, B}^{-1}$ from the right, we may assume that $b_{i_1, B} = 1$. For any homogeneous non zero $y \in \ker(\partial_B)$, we consider

$$\begin{aligned} u_y := (1 \otimes y)x(1 \otimes y^{-1}) &= \sum_{j=1}^n (1 \otimes y)(a_{i_j} \otimes b_{i_j}) \lambda_{(i_j, A; i_j, B)} (1 \otimes y^{-1}) \\ &= (-1)^{(|a_{i_1, A}| + |\lambda_{(i_j, A; i_j, B)}|)|y|} (a_{i_1, A} \otimes 1) \lambda_{(i_1, A; i_1, B)} \\ &\quad + \sum_{j=2}^n (-1)^{|y||a_{i_j, A}|} (a_{i_j, A} \otimes y b_{i_j, B}) \lambda_{(i_j, A; i_j, B)} y^{-1} \end{aligned}$$

But

$$x - (-1)^{(|a_{i_1, A}| + |\lambda_{(i_1, A; i_1, B)}|)|y|} u_y \in J \cap \ker(\partial_{D_A \otimes D_B})$$

has length at most $n-1$, which is strictly smaller than the length of x , and still is in $J \cap \ker \partial_{D_A \otimes D_B}$. Since the length of x was assumed to be minimal we get $x - (-1)^{(|a_{i_1, A}| + |\lambda_{(i_1, A; i_1, B)}|)|y|} u_y = 0$. Therefore,

$$b_{i_j, B} \lambda_{(i_j, A; i_j, B)} = (-1)^{|y||a_{i_1, A}| + |y||\lambda_{(i_1, A; i_1, B)}| + |y||a_{i_j, A}|} y b_{i_j, B} \lambda_{(i_j, A; i_j, B)} y^{-1}$$

for all j . Since x is homogeneous,

$$|x| = |a_{i_j, A}| + |b_{i_j, B}| + |\lambda_{(i_j, A; i_j, B)}| = |a_{i_1, A}| + |1| + |\lambda_{(i_1, A; i_1, B)}| = |a_{i_1, A}| + |\lambda_{(i_1, A; i_1, B)}|$$

and therefore,

$$\begin{aligned}
 b_{i_j, B} \lambda_{(i_j, A; i_j, B)} y &= (-1)^{|y||a_{i_1, A}| + |y||\lambda_{(i_1, A; i_1, B)}| + |y||a_{i_j, A}|} y b_{i_j, B} \lambda_{(i_j, A; i_j, B)} y^{-1} y \\
 &= (-1)^{|y|(|a_{i_j, A}| + |a_{i_1, A}| + |\lambda_{(i_1, A; i_1, B)}|)} y b_{i_j, B} \lambda_{(i_j, A; i_j, B)} \\
 &= (-1)^{|y|(|a_{i_j, A}| + |x|)} y b_{i_j, B} \lambda_{(i_j, A; i_j, B)} \\
 &= (-1)^{|y|(|b_{i_j, B}| + |\lambda_{(i_j, A; i_j, B)}|)} y b_{i_j, B} \lambda_{(i_j, A; i_j, B)}
 \end{aligned}$$

for all j . This means that $b_{i_j, B} \lambda_{(i_j, A; i_j, B)} \in Z_{gr}(B)$, and also in $\ker(\partial_B)$, which shows that x is homogeneous in $\ker(\partial_A) \otimes 1$. This element is invertible. Therefore $J = D_A \otimes_Z D_B$. This shows the statement. ■

5. DG-PRIMITIVE DG-ALGEBRAS

Definition 5.1. Let K be a commutative ring and let (A, d) be a dg- K -algebra.

- A dg-module (M, δ) over a dg-algebra (A, d) is *dg-faithful* if for each homogeneous element $a \in A$ we get $aM = 0 \Rightarrow a = 0$.
- A dg- K -algebra (A, d) is called *left (resp. right) dg-primitive* if there is a dg-simple dg-faithful left (resp. right) dg-module over (A, d) .
- A twosided dg-ideal (I, d) of (A, d) is called left (resp. right) dg-primitive if $(A/I, \bar{d})$ is a left (resp. right) dg-primitive algebra.

Lemma 5.2. Let (A, d) be a dg-algebra and let (M, δ) be a left dg- (A, d) -module. Then (M, δ) is faithful if and only if the action of A on M yields an embedding of dg-algebras

$$(A, d) \hookrightarrow (End_A^\bullet(M, \delta), d_{\text{Hom}}).$$

If (M, δ) is a right dg-module, then the action is faithful if and only if the action of A on M yields an embedding of dg-algebras

$$(A, d) \hookrightarrow (End_A^\bullet(M, \delta), d_{\text{Hom}})^{op}.$$

Proof. Suppose that M is dg-faithful. Let $0 \neq a \in A$ be homogeneous. Then

$$\begin{array}{ccc}
 M & \xrightarrow{\mu_a} & M \\
 m & \mapsto & am
 \end{array}$$

is a K -linear endomorphism of M of degree $|a|$. Further, M is faithful if and only if μ_a is injective for all non zero homogeneous $a \in A$. Observe that $\mu_{a+b} = \mu_a + \mu_b$ and

$$\mu_{ab}(m) = abm = \mu_a(\mu_b(m)) = (\mu_a \circ \mu_b)(m).$$

If M is a right module, then $\mu_a(m) = ma$ and

$$\mu_{ab}(m) = mab = \mu_b(\mu_a(m)) = (\mu_b \circ \mu_a)(m).$$

Now, denote by $\mu : A \rightarrow End_K^\bullet(M)$ the morphism given by the action. We need to show $\mu \circ d = d_{\text{Hom}} \circ \mu$. For all homogeneous $a \in A$ and all $m \in M$ we get

$$\begin{aligned}
 (d_{\text{Hom}}(\mu(a)))(m) &= (\delta \circ \mu_a - (-1)^{|a|} \mu_a \circ \delta)(m) \\
 &= \delta(a \cdot m) - (-1)^{|a|} a \cdot \delta(m) \\
 &= d(a) \cdot m \\
 &= ((\mu \circ d)(a))(m)
 \end{aligned}$$

by the Leibniz formula. This shows the lemma. ■

Corollary 5.3. Let (A, d) be a differential graded K -algebra over a field K . If (A, d) admits a faithful finite dimensional (over K) dg-module, then (A, d) is isomorphic to a subalgebra of a differential graded matrix algebra over K .

Proof. Indeed, forgetting the grading and the differential, $(End_A^\bullet(M, \delta), d_{\text{Hom}})$ is a matrix algebra over K . The rest follows from Lemma 5.2. ■

Lemma 5.4. *Let (A, d) be a dg- K -algebra. Then a twosided dg-ideal (I, d) of (A, d) is left (resp. right) dg-primitive if and only if there is a left (resp. right) dg-simple dg-module (S, δ) over (A, d) such that $I = \text{ann}_\ell(S)$ (resp. $I = \text{ann}_r(S)$), i.e. I is the left (resp. right) annihilator of S .*

Proof. By symmetry, it is sufficient to consider the left case.

The left annihilator of a differential graded left module is a twosided dg-ideal (cf [18]). Since (S, δ) is a simple dg-module over (A, d) , it is also a dg-simple $(A/\text{ann}_\ell(S, \delta), \bar{d})$ -module. Since we take the quotient modulo the annihilator, (S, δ) is a faithful module over $(A/\text{ann}_\ell(S, \delta), \bar{d})$. Hence, $(A/\text{ann}_\ell(S, \delta), \bar{d})$ is left dg-primitive.

If $(A/I, \bar{d})$ is left dg-primitive, then there is a faithful dg-simple left dg-module (S, δ) over $(A/I, \bar{d})$. Then, (S, δ) is a dg-module over (A, d) as well. Hence, the twosided ideal $J := \text{ann}_\ell(S, \delta)$ of (A, d) contains I , by definition. Further, $(J/I, \bar{d})$ is the annihilator of (S, δ) in $(A/I, \bar{d})$. Hence, (S, δ) is a faithful $(A/I, \bar{d})$ -module if and only if $I = J$. ■

Corollary 5.5. *Let (A, d) be a dg-algebra over K . Then $\text{dgrad}_2(A, d)$ is the intersection of all left dg-primitive dg-ideals of (A, d) .*

Proof. This is a direct consequence of Lemma 5.4. ■

Lemma 5.6. *A dg-simple dg-ring admitting a dg-simple module is left dg-primitive.*

A left dg-primitive dg-ring (A, d) is dg-prime.

Proof. Let (A, d) be dg-simple. Since (A, d) admits a dg-simple module, its annihilator is a twosided dg-ideal, whence 0 by the simplicity of (A, d) . Hence (A, d) is primitive.

Let (I, d) and (J, d) be non zero twosided dg-ideals with $IJ = 0$, and let (S, δ) be a faithful dg-simple dg-module. Then JS is a dg-submodule of S , and hence $JS = 0$ or $JS = S$. Since S is faithful, the first case implies $J = 0$, which was excluded. Hence $JS = S$. Similarly, $IS = S$. Therefore

$$0 = (IJ)S = I(JS) = IS = S$$

and we obtain a contradiction. This shows that (A, d) is dg-prime. ■

6. A DIFFERENTIAL GRADED JACOBSON-CHEVALLEY DENSITY THEOREM

Let (A, d) be a differential graded dg-simple algebra and suppose that (A, d) is left dg-Noetherian and left dg-artinian. By Lemma 5.6 (A, d) is dg-primitive. Hence there is a faithful dg-simple dg-module $(S_{(A, d)}, \delta)$ over (A, d) . Since (A, d) is dg-simple, [16, Proposition 4.28] shows that any other dg-simple dg-module over (A, d) is isomorphic to $(S_{(A, d)}, \delta)$. By Proposition 2.13 we get that $\text{End}_A^\bullet(S_{(A, d)}, \delta)$ is a dg-division algebra.

A very basic result in ring theory is the Chevalley-Jacobson density theorem. A graded version of the Chevalley-Jacobson density theorem can be found in Chen et al. [6]. The result actually follows from earlier work of Liu, Beattie and Fang [11].

Theorem 6.1. [6, 11] *Let A be a group graded algebra. Let M be a gr-simple graded A -module and let $D = \text{End}_{A\text{-graded}}(M)$. Let x_1, \dots, x_k be homogeneous D -independent elements of M , and let y_1, \dots, y_k be any elements of M . Then there is a $a \in A$ with $ax_i = y_i$ for any $i \in \{1, \dots, k\}$.*

We shall need to show that if (A, d) is a dg-algebra and (M, δ) is a dg-simple dg-module over (A, d) , then $\ker(\delta)$ is a \mathbb{Z} -gr-simple $\ker(d)$ -module. In the special case of an acyclic (A, d) -algebra, this follows from a result due to Aldrich and Garcia-Rozas [1].

Theorem 6.2. [1, Theorem 4.7] *Let (A, d) be a differential graded K -algebra. Then the following statements are equivalent.*

- (1) *the regular dg-module is a projective object in the category of (A, d) -dg-modules.*
- (2) *(A, d) is acyclic.*
- (3) *$1 \in \text{im}(d)$.*
- (4) *Any left dg-module over (A, d) is acyclic.*
- (5) *Taking the kernel of the differential is a right exact functor $Z(-)$ from the category of left dg-modules over (A, d) to the category of \mathbb{Z} -graded modules over $\ker(d)$.*

- (6) *The pair of adjoint functors $(A \otimes_{\ker(d)} -, Z(-))$ are inverse equivalences between the category of left dg-modules over (A, d) to the category of \mathbb{Z} -graded modules over $\ker(d)$.*

As a consequence, if (A, d) is acyclic, then a dg-module (M, δ) over (A, d) is dg-simple if and only if $\ker(\delta)$ is a graded-simple $\ker(d)$ -module. We hence may apply Theorem 6.1 to dg-simple dg-modules in this case.

We shall need to use Lemma 2.11 to determine the cycles of a tensor product of acyclic algebras.

Lemma 6.3. *Let (A, d_A) and (B, d_B) be acyclic dg-algebras over some graded commutative dg-ring K . Then for any homogeneous $z \in A$ with $d_A(z) = 1$ and any homogeneous $w \in B$ with $d_B(w) = 1$ we get that*

$$\ker(d_{A \otimes_K B}) = (\ker(d_A) \otimes_K \ker(d_B)) \oplus (1 \otimes w - z \otimes 1) \cdot (\ker(d_A) \otimes_K \ker(d_B))$$

Proof. By Lemma 2.11 we get that

$$A = \ker(d_A) \oplus z \cdot \ker(d_A) \text{ and } B = \ker(d_B) \oplus w \cdot \ker(d_B).$$

Then

$$A \otimes B = (\ker(d_A) \otimes \ker(d_B)) \oplus (\ker(d_A) \otimes w \ker(d_B)) \oplus (z \ker(d_A) \otimes \ker(d_B)) \oplus (z \ker(d_A) \otimes w \ker(d_B))$$

Now,

$$d_{A \otimes B} = d_A \otimes \text{id}_B + \text{id}_A \otimes d_B.$$

Hence

$$\begin{aligned} d_{A \otimes B}(\ker(d_A) \otimes \ker(d_B)) &= 0, \\ d_{A \otimes B}(\ker(d_A) \otimes w \ker(d_B) \oplus z \ker(d_A) \otimes \ker(d_B)) &\subseteq \ker(d_A) \otimes \ker(d_B) \end{aligned}$$

and

$$d_{A \otimes B}(z \ker(d_A) \otimes w \ker(d_B)) \subseteq \ker(d_A) \otimes w \ker(d_B) \oplus z \ker(d_A) \otimes \ker(d_B).$$

Let

$$u \in \ker(d_A) \otimes w \ker(d_B) \oplus z \ker(d_A) \otimes \ker(d_B)$$

be homogeneous with $d_{A \otimes B}(u) = 0$. Then

$$u = (z \otimes 1)u_1 + (1 \otimes w)u_2$$

and

$$d_{A \otimes B}(u) = u_1 + u_2.$$

Hence

$$\ker(d_{A \otimes B}) = \ker(d_A) \otimes \ker(d_B) \oplus (1 \otimes w - z \otimes 1) \cdot (\ker(d_A) \otimes_K \ker(d_B)).$$

This shows the lemma. ■

Remark 6.4. • An unpublished result [4] due to G.M.Bergman shows that for \mathbb{Z} -graded rings, the Jacobson radical (ungraded version) is homogeneous. This means that any simple module is automatically graded.

- Recall that Bahturin, Zaicev and Sehgal classified in [2] finite-dimensional simple G -graded K -algebras A for a group G and an algebraically closed field K , subject to some hypotheses with respect to G and to the base field. In particular, if either K is of characteristic 0 or the order of any finite subgroup of G is coprime to the characteristic of K , then A is a matrix algebra over a graded skew-field $K^\alpha H$ for some finite subgroup H of G and α a 2-cocycle with values in K^\times .

If $G = \mathbb{Z}$, then there is no non trivial finite subgroup. Hence, considering finite dimensional graded simple algebras we are left with gradings on full matrix algebras over F .

- Lemma 6.3 indicates that if $\ker(d_A)$ and $\ker(d_B)$ are graded semisimple, then this is might not be true for $\ker(d_{A \otimes_K B})$. Compare with the classification of graded-simple algebras by Bahturin et al. as mentioned in the item above in case of algebraically closed base fields.

Theorem 6.5. *Let (A, d) be an acyclic dg-algebra, i.e. a dg-algebra with $H(A, d) = 0$. Let (M, δ) be a dg-simple dg-module over (A, d) and let*

$$(D, \partial) := \text{End}_A^\bullet((M, \delta), d_{\text{Hom}}).$$

Then (D, ∂) is a dg-division algebra, and also a \mathbb{Z} -gr-division algebra. Moreover, for each family x_1, \dots, x_k of D -independent elements of $\ker(\delta)$ and each family y_1, \dots, y_k of elements of $\ker(\delta)$, there is an element $a \in \ker(d)$ with $ax_i = y_i$ for all $i \in \{1, \dots, k\}$.

Proof. Let $N := \ker(\delta)$ and $B := \ker(d)$ to shorten the notation. As (M, δ) is dg-simple over (A, d) , by the equivalence of categories in Theorem 6.2.(6) we get that N is \mathbb{Z} -graded simple as graded B -module. Further, $\text{End}_{B\text{-graded}}(N) \simeq D = \text{End}_A^\bullet((M, \delta), d_{\text{Hom}})$ again by the equivalence of categories in Theorem 6.2.(6). Hence, D is also a \mathbb{Z} -gr-division algebra. The statement now follows directly from Theorem 6.1. ■

Lemma 6.6. *Let (A, d_A) and (B, d_B) be dg-algebras with $Z_{\text{gr}}(A, d_A) = Z_{\text{gr}}(B, d_B) = Z$ and suppose that (A, d_A) or (B, d_B) is acyclic. Then $(A \otimes_Z B, d_{A \otimes_Z B})$ is acyclic as well.*

Proof. Indeed, by Lemma 2.11 we need to show that $1_{A \otimes_Z B} \in \text{im}(d_{A \otimes_Z B})$. Since (A, d_A) or (B, d_B) is acyclic, there is a homogeneous element $z_A \in A$ or $z_B \in B$ with $d_A(z_A) = 1_A$ or $d_B(z_B) = 1_B$. However, $d_{A \otimes_Z B} = d_A \otimes_Z \text{id}_B + \text{id}_A \otimes_Z d_B$. Suppose that (A, d_A) is acyclic. But then

$$d_{A \otimes_Z B}(z_A \otimes 1_B) = d_A(z_A) \otimes 1_B = 1_A \otimes_Z 1_B$$

This shows the statement. ■

REFERENCES

- [1] S. Tempest Aldrich and J. R. Garcia Rozas, *Exact and Semisimple Differential Graded Algebras*, Communications in Algebra **30** (no 3) (2002) 1053-1075.
- [2] Yu. A. Bahturin, M. V. Zaicev, Sudarshan K. Sehgal, *Finite-dimensional simple graded algebras*, (English version) Sbornik Mathematics **199:7** 965-983.
- [3] George M. Bergman, *Tensor products of faithful modules*, arxiv: 1610.05178v1
- [4] George M. Bergman, *On Jacobson radicals of graded rings*, preprint (1975)
<https://math.berkeley.edu/~gbergman/papers/unpub/J.G.pdf>
- [5] Henri Cartan, *DGA-algèbres et DGA-modules*, Séminaire Henri Cartan, tome 7, no 1 (1954-1955), exp. no 2, p. 1-9.
- [6] Tung-Shyan Chen, Chin-Fang Huang, and Jing-Whei Liang, *Extended Jacobson Density Theorem for Graded Rings with Derivations and Automorphisms*, Taiwanese Journal of Mathematics **14** no 5 (2010) 1993-2014.
- [7] Charles W. Curtis and Irving Reiner, *METHODS OF REPRESENTATION THEORY*, Vol 1, John Wiley Interscience 1981.
- [8] Isambard Goodbody, *Reflecting perfection for finite dimensional differential graded algebras*. preprint october 5, 2023; arxiv:2310.02833.
- [9] Ken Goodearl and Tobi Stafford, *The graded version of Goldie's theorem*, Contemporary Math. **259**, (2000) 237-240.
- [10] Israel Nathan Herstein, *NONCOMMUTATIVE RINGS*, The Mathematical Association of America, fifth printing 2005
- [11] S. X. Liu, Margaret Beattie, and H. J. Fang, *Graded division rings and the Jacobson density theorem*, Journal of the Beijing Normal University (Natural Science) **27** (2) (1991) 129-134.
- [12] Constantin Nastasescu and Fred van Oystaen, *GRADED RING THEORY*. North Holland 1982; Amsterdam
- [13] Dimitri Orlov, *Finite dimensional differential graded algebras and their geometric realisations*, Advances in Mathematics **366** (2020) 107096, 33 pp.
- [14] Dimitri Orlov, *Smooth DG algebras and twisted tensor product*. arxiv 2305.19799
- [15] Irving Reiner, *MAXIMAL ORDERS*, Academic Press London, New-York, San-Francisco 1975.
- [16] Alexander Zimmermann, *Differential graded orders, their class groups and idèles*, preprint December 30, 2022; 32 pages
- [17] Alexander Zimmermann, *Differential graded Brauer groups*, preprint March 31, 2023; final version 11 pages; to appear in Revista de la Union Matematica Argentina.
- [18] Alexander Zimmermann, *Ore Localisation for differential graded rings; Towards Goldie's theorem for differential graded algebras*, Journal of Algebra **663** (2025) 48-80.

UNIVERSITÉ DE PICARDIE,
DÉPARTEMENT DE MATHÉMATIQUES ET LAMFA (UMR 7352 DU CNRS),
33 RUE ST LEU,
F-80039 AMIENS CEDEX 1,
FRANCE
Email address: `alexander.zimmermann@u-picardie.fr`