



**HAL**  
open science

# Looking for Equivalence between Maximum Likelihood and Sparse DOA Estimators

Thomas Aussaguès, Anne Ferréol, Alice Delmer, Pascal Larzabal

► **To cite this version:**

Thomas Aussaguès, Anne Ferréol, Alice Delmer, Pascal Larzabal. Looking for Equivalence between Maximum Likelihood and Sparse DOA Estimators. EUSIPCO 2024 - 32nd European Signal Processing Conference, Aug 2024, Lyon, France. . hal-04669621

**HAL Id: hal-04669621**

**<https://hal.science/hal-04669621v1>**

Submitted on 8 Aug 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License



# Looking for Equivalence between Maximum Likelihood and Sparse DOA Estimators

Thomas Aussaguès<sup>\*,†</sup>, Anne Ferréol<sup>\*,†</sup>, Alice Delmer<sup>\*</sup> & Pascal Larzabal<sup>†</sup>  
<sup>\*</sup>Thales, <sup>†</sup>SATIE, Université Paris-Saclay, UMR CNRS 8029



Flash this QR code to access the paper !

## ABSTRACT

Sparse Direction-of-Arrival (DOA) estimators depend on the regularization parameter  $\lambda$  which is often empirically tuned. In this work, conducted under the vectorized covariance matrix model, we are looking for theoretical equivalence between the Maximum Likelihood (ML) and sparse estimators. We show that under mild conditions,  $\lambda$  can be chosen thanks to the distribution of the minimum of the ML criterion in the case of two impinging sources. We derive this distribution under complex non-circular Gaussian noise. The corresponding  $\lambda$  choice is  $\theta$ -invariant, only requiring an upper bound on the number of sources. Furthermore, it guarantees the global minimum of the sparse  $\ell_0$ -regularized criterion to be the ML solution. Numerical experiments confirm that, for the proposed  $\lambda$ , sparse and ML estimators yield the same statistical performance.

## INTRODUCTION

DOA estimation : a classical signal processing problem with critical applications (radar, telecommunications...) [1]

**Objective** : estimate the sources directions of arrival  $\theta_1, \dots, \theta_M$

Numerous methods have been proposed : Capon's beamformer [2], MUSIC [3], ML [4]:

- ✗ MUSIC can not handle coherent sources
- ✗ ML is intractable for large number of sources
- ✗ Limited number of identifiable sources

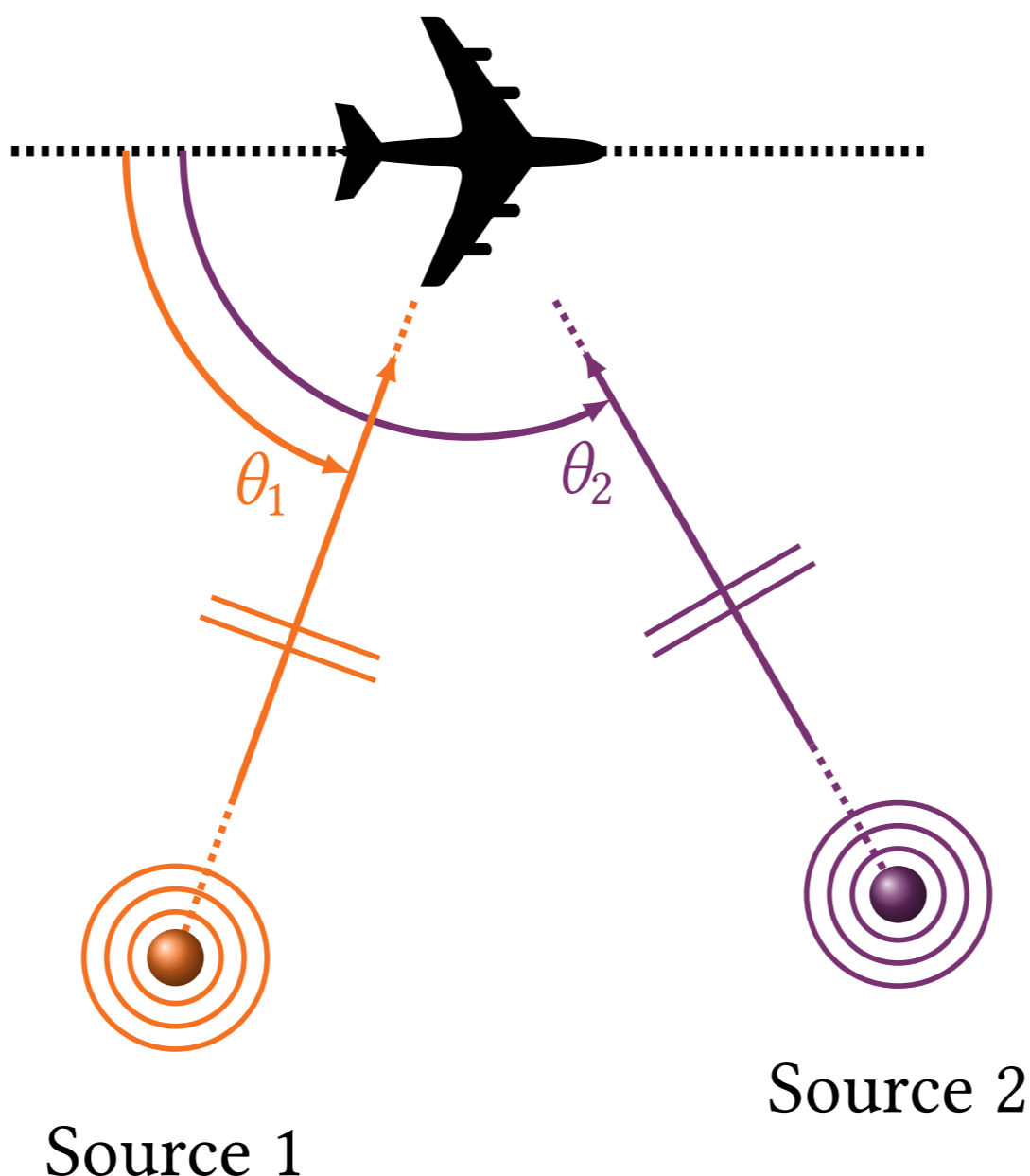
The vectorized covariance matrix model

- ✓ High number of identifiable sources
- ✓ Enhanced performances in severe scenarios

DOA estimation with the vectorized covariance matrix model : **sparse methods**

Sparse methods rely on the minimization of a regularized criterion parametrized by  $\lambda$  an hyperparameter which is **often empirically tuned**

**Contribution**: we propose a novel  $\theta$ -invariant regularization parameter choice that ensures equivalence between ML and sparse DOA estimators



## THE VECTORIZED COVARIANCE MATRIX MODEL

Let  $M$  sources of directions  $\Theta = \{\theta_1, \dots, \theta_M\}$  impinge on an array of  $N$  antennas

**Classical model**:

Steering matrix  $\leftarrow$  Emitted signals

$$\mathbf{x}(t) = \mathbf{A}(\Theta)\mathbf{s}(t) + \mathbf{n}(t)$$

with covariance matrix  $\mathbf{R}_x = \mathbf{A}(\Theta)\mathbf{R}_s\mathbf{A}(\Theta) + \sigma^2\mathbf{I}_N$

Circular Gaussian noise  $\mathcal{C}\mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}_N)$

✗ Number of identifiable sources limited to  $N - 1$  sources

**The vectorized covariance matrix model [5] (for uncorrelated sources)**:

Virtual array steering matrix  $\leftarrow$  Sources powers vector :

$$\mathbf{r} = \text{vec}(\mathbf{R}_x - \sigma^2\mathbf{I}) = \mathbf{B}(\Theta)\boldsymbol{\gamma}_\Theta + \boldsymbol{\delta}$$

Non-Circular Gaussian noise  $\mathcal{C}\mathcal{N}(\mathbf{0}, \Gamma, \mathbf{C})$

✓ Allows the processing of at most  $N^2 - 1$  sources with *increased* resolution

**Maximum Likelihood estimation on the vectorized covariance matrix model**:

Pre-whitening  $\mathbf{y} = \mathbf{W}\mathbf{r} = \mathbf{B}_w(\Theta)\boldsymbol{\gamma}_\Theta + \boldsymbol{\delta}_w$  with  $\mathbf{W} = \Gamma^{-1/2}$ ,  $\mathbf{B}_w(\Theta) = \mathbf{W}\mathbf{B}(\Theta)$ ,  $\boldsymbol{\delta}_w = \mathbf{W}\boldsymbol{\delta}$

ML estimation  $\hat{\Theta} = \arg \min J_{ML}(\Theta)$  with  $J_{ML}(\Theta) = \mathbf{y}^H \boldsymbol{\Pi}^\perp(\Theta) \mathbf{y}$

✗ Minimizing  $J_{ML}$  requires  $M$ -dimensional non-convex optimization with numerous local minimas

**Objective** : solve the ML through sparse DOA estimators

## SPARSE MODELING AND SPARSE ESTIMATION

### Sparse modeling

Let  $\Phi = \{\varphi_1, \dots, \varphi_G\}$  a grid of  $G$  pre-defined directions such that  $\Theta \subset \Phi$

$\mathbf{B}_w(\Phi) = \mathbf{W}[\mathbf{b}(\varphi_1), \dots, \mathbf{b}(\varphi_G)]$  :  $N^2 \times G$  matrix containing the virtual array response  $\mathbf{b}(\varphi_g)$  in each of the  $G$  grid directions

**Sparse** vectorized covariance matrix model

$$\mathbf{y} = \mathbf{B}_w(\Theta)\boldsymbol{\gamma}_\Theta + \boldsymbol{\delta}_w = \mathbf{B}_w(\Phi)\boldsymbol{\gamma}_0 + \boldsymbol{\delta}_w$$

Sparse  $G \times 1$  vector with only  $M$  non null coefficients

DOAs corresponds to the directions  $\varphi_G$  of  $\boldsymbol{\gamma}_0$  non zeros components

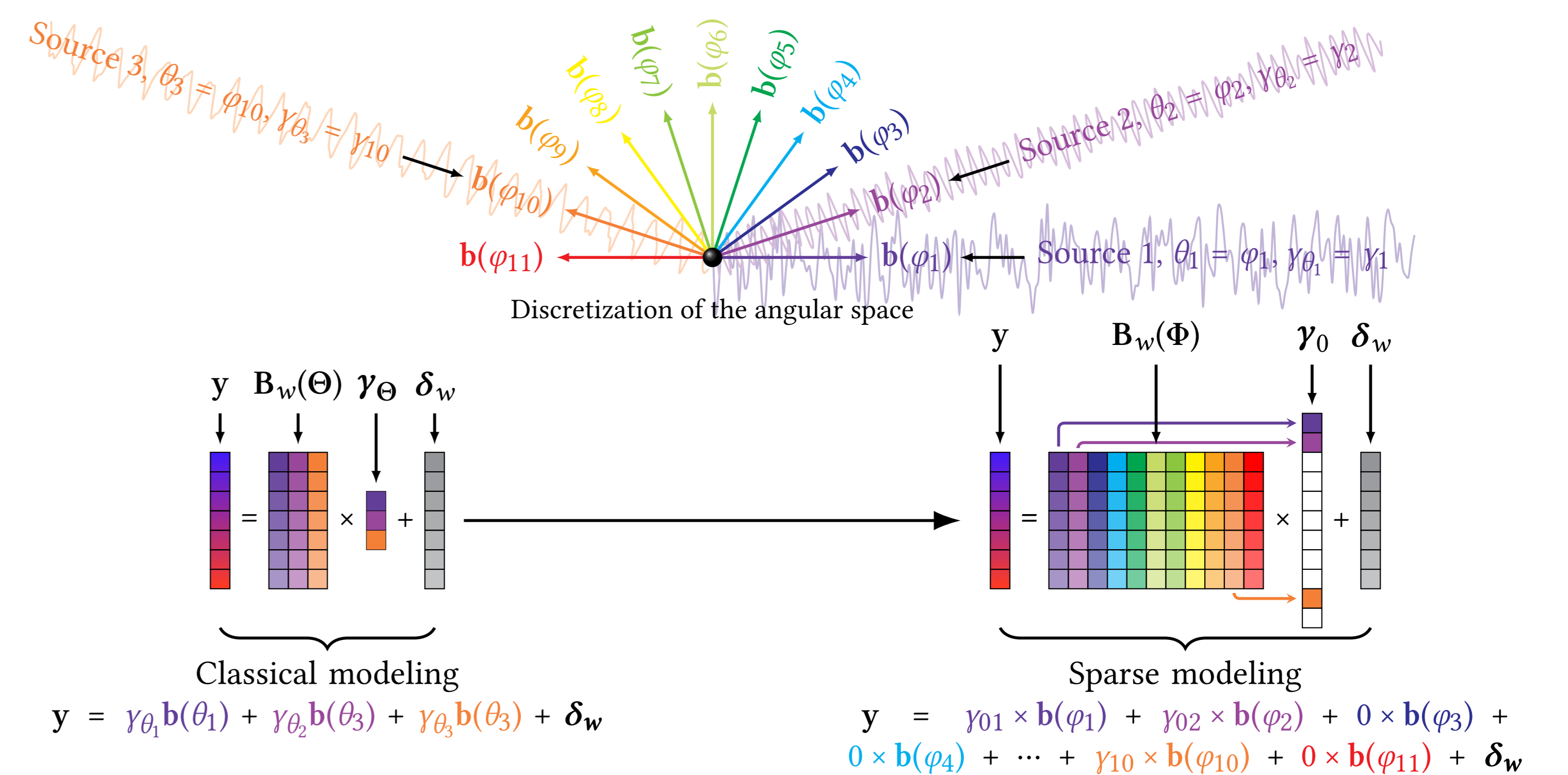
### Sparse estimation

✗  $G \gg N^2$  : the problem is ill-posed

Use of the sparsity prior : regularized problem formulation

Penalty function  $\Psi$  :  $\ell_0$  norm, Continuous Exact  $\ell_0$  (CEL0) [6]

$$\hat{\boldsymbol{\gamma}} = \arg \min_{\boldsymbol{\gamma}} J_\Psi(\lambda, \boldsymbol{\gamma}) \quad \text{with} \quad J_\Psi(\lambda, \boldsymbol{\gamma}) = \frac{1}{2} \underbrace{\|\mathbf{y} - \mathbf{B}_w(\Phi)\boldsymbol{\gamma}\|_2^2}_{\text{Data fidelity}} + \underbrace{\Psi(\lambda, \boldsymbol{\gamma})}_{\text{Penalty}}$$

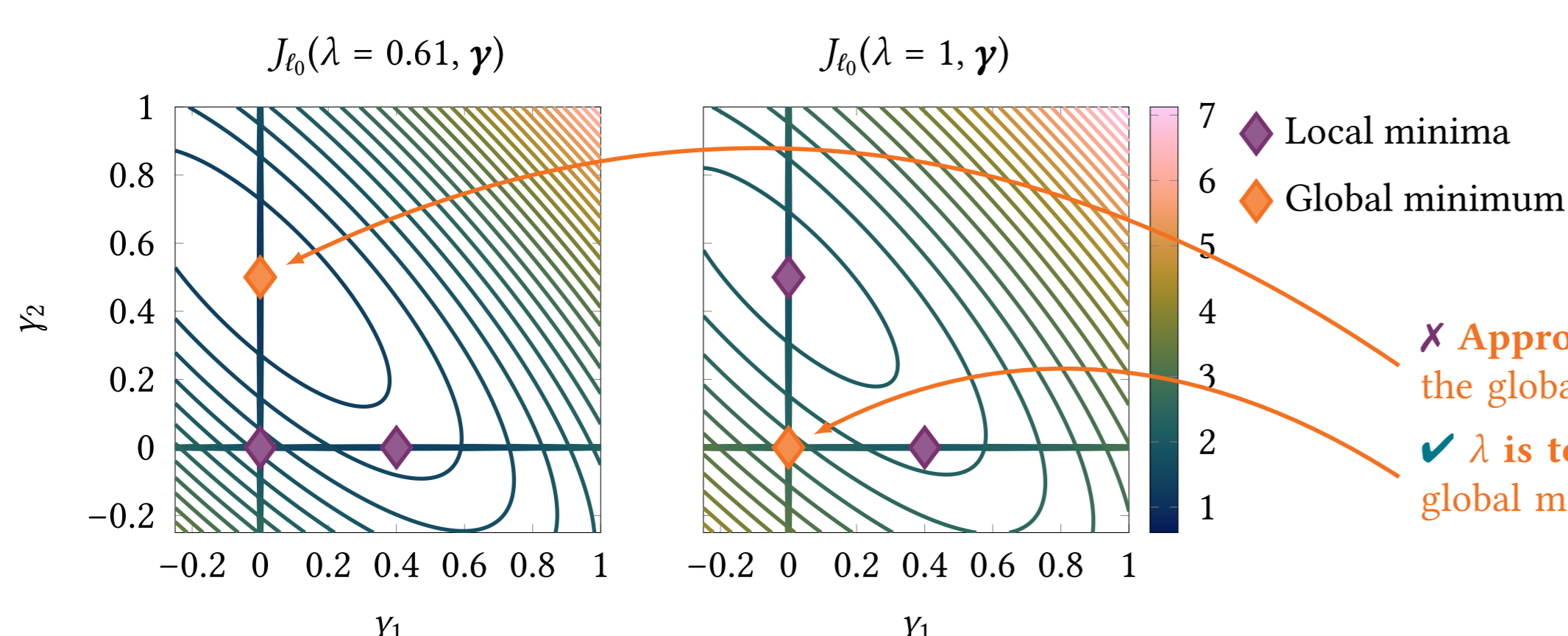


## REGULARIZATION PARAMETER $\lambda$ CHOICE

$\lambda$  balances data fidelity towards the solution sparsity

Example : influence of the regularization parameter on the criterion minima

$$\text{Contour lines of } J_0: \mathbf{y} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } \boldsymbol{\gamma}_0 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$



**Equivalence**: both ML and sparse criterion **have the same global minimizer**

Delmer's [7] interval  $[\lambda^-, \lambda^+]$  ensures the equivalence **but** the statistics of  $\lambda^-, \lambda^+$ :

✗ Are difficult to compute

✗ Depend on the directions

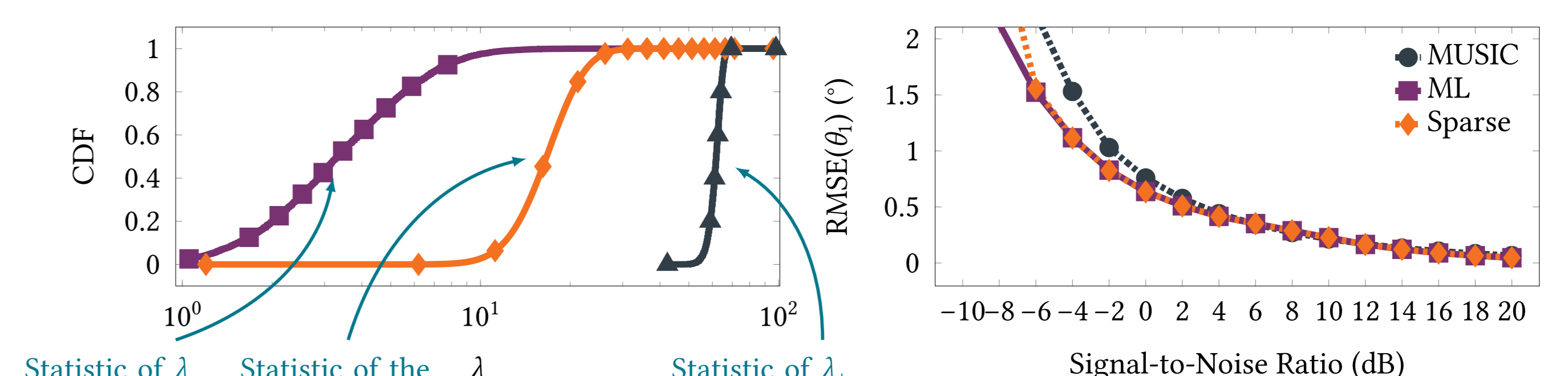
**Contributions**[8]:

- ✓ The min. of the ML,  $\epsilon$ , criterion belongs to the interval  $[\lambda^-, \lambda^+]$
- ✓  $\epsilon$  follows, asymptotically, a  $\chi^2$  distribution with  $N^2 - M$  degrees of freedom
- ✓  $\theta$ -invariant distribution

## SIMULATIONS

The proposed  $\lambda$  choice belongs to the interval  $[\lambda^-, \lambda^+]$  : it **ensures** equivalence with the ML

Numerical simulations confirm that both ML and sparse estimators have the **same statistical performance**



## CONCLUSION

- ✓ Novel  $\theta$ -invariant regularization parameter choice
- ✓ Ensuring equivalence between sparse and ML DOA estimators
- ✓ Efficient implementation of the ML

[1] Hamid Krim and Mats Viberg, "Two Decades of Array Signal Processing Research: The Parametric Approach". In: *Signal Processing Magazine, IEEE* 13 (Aug. 1996), pp. 67–94.  
 [2] J. Capon, "High-resolution frequency-wavenumber spectrum analysis". In: *Proceedings of the IEEE* 57.8 (1969), pp. 1408–1418.  
 [3] G. Bienvenu and L. Kopp, "Optimality of high resolution array processing using the eigensystem approach". In: *IEEE Transactions on Acoustics, Speech, and Signal Processing* 31.5 (1983), pp. 1235–1248.  
 [4] B. Ottersten et al. "Exact and Large Sample Maximum Likelihood Techniques for Parameter Estimation and Detection in Array Processing". In: *Radar Array Processing*, Ed. by Simon Haykin, John Litva, and Terence J. Shepherd. Berlin, Heidelberg: Springer Berlin Heidelberg, 1993, pp. 99–151.  
 [5] Joseph S. Picard and Anthony J. Weiss, "Direction finding of multiple emitters by spatial sparsity and linear programming". In: *2009 9th International Symposium on Communications and Information Technology*, 2009, pp. 1258–1262.  
 [6] Emmanuel Soubies, Laure Blanc-Féraud, and Gilles Aubert, "A Continuous Exact  $\ell_0$  penalty (CEL0) for least squares regularized problem". In: *SIAM Journal on Imaging Sciences* 8.3 (July 2015), pp. 1607–1639 (33 pages).  
 [7] Alice Delmer, Anne Ferréol, and Pascal Larzabal, "On regularization Parameter for  $\ell_0$ -Sparse Covariance Fitting Based DOA Estimation". In: *ICASSP 2020 - 2020 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2020.  
 [8] Thomas Aussaguès et al. "Looking for Equivalence between Maximum Likelihood and Sparse DOA Estimators". In: *2024 32th European Signal Processing Conference (EUSIPCO)*, 2024.