Looking for Equivalence between Maximum Likelihood and Sparse DOA Estimators (poster)

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Sparse Direction-of-Arrival (DOA) estimators depend on the regularization parameter $\lambda$ which is often empirically tuned. In this work, conducted under the vectorized covariance matrix model, we are looking for theoretical equivalence between the Maximum Likelihood (ML) and sparse estimators. We show that under mild conditions, $\lambda$ can be chosen thanks to the distribution of the minimum of the ML criterion in the case of two impinging sources. We derive this distribution under complex non-circular Gaussian noise. The corresponding $\lambda$ choice is $\theta$-invariant, only requiring an upper bound on the number of sources. Furthermore, it guarantees the global minimum of the sparse $\ell_1$-regularized criterion to be the ML solution. Numerical experiments confirm that, for the proposed $\lambda$, sparse and ML estimators yield the same statistical performance.

**Introduction**

- **Objective**: estimate the sources directions of arrival $\theta_1, \ldots, \theta_N$
- **DOA estimation**: a classical signal processing problem with critical applications (radar, telecommunications...)
- **Numerous methods have been proposed**: Capon’s beamformer [2], MUSIC [3], ML [4]:
  - **MUSIC**: can handle coherent sources
  - **ML**: is tractable for large number of sources
  - **Limited number of identifiable sources**
- **The vectorized covariance matrix model**
  - **High number of identifiable sources**
  - **Enhanced performances in severe scenarios**
- **DOA estimation with the vectorized covariance matrix model**: sparse methods
- **Sparse methods rely on the minimization of a regularized criterion parametrized by $\lambda$** which is often empirically tuned

**Contribution**: we propose a novel $\theta$-invariant regularization parameter choice that ensures equivalence between ML and sparse DOA estimators

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**Sparse modeling and Sparse Estimation**

- **Sparse modeling**: let $\Phi = \{\varphi_1, \ldots, \varphi_N\}$ a grid of $G$ pre-defined directions such that $\Theta = \Phi$
- $\mathbf{B}(\Theta) = \mathbf{W}[\mathbf{b}(\varphi_1), \ldots, \mathbf{b}(\varphi_N)]$: $\mathbb{C}^N \times G$ matrix containing the virtual array response $\mathbf{b}(\varphi_p)$ in each of the $G$ grid directions
- **Sparse vectorized covariance matrix model**
  - $\mathbf{y} = \mathbf{B}(\Theta)\mathbf{y}_0 + \mathbf{\delta}$: $\mathbb{C}^G \times 1$ vector with only $M$ non null coefficients

**DOAs corresponds to the directions $\varphi_{\hat{\Theta}}$ of $\mathbf{y}_0$ non zeros components**

**Sparse estimation**

- $\lambda \gg \mathbb{N}^2$: the problem is ill-posed
- **Use of the sparsity prior**: regularized problem formulation
- **Penalty function** $\Psi$: $f$ norm, Continuous Exact $l_0$ (CEL) [6]:
  $$y = \arg\min_{\mathbf{y}} J_{\lambda}(\mathbf{y}, \mathbf{y}_0) = \frac{1}{2} \| \mathbf{B}(\Theta)\mathbf{y} - \mathbf{y}_0 \|^2_2 + \Psi(\mathbf{y}, \mathbf{y}_0)$$
- **Data fidelity**: $\ell_2$
- **Pareto**: $\Psi(\mathbf{y}, \mathbf{y}_0)$

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**Regularization parameter $\lambda$ choice**

- $\lambda$ balances data fidelity towards the solution sparsity
- **Example**: influence of the regularization parameter on the criterion minima
  - **Contour lines of $J_{\lambda}$**
    - $J_{\lambda}(\mathbf{y}) = \frac{1}{2} \| \mathbf{B}(\Theta)\mathbf{y} - \mathbf{y}_0 \|^2_2 + \frac{\lambda}{2} \| \mathbf{y} \|^2_2$
    - $\lambda = 0.5$
    - $\lambda = 2$
    - $\lambda = 12$
    - **Local minima**: $J_{\lambda}(\mathbf{y}) = 0$
    - **Global minima**: $J_{\lambda}(\mathbf{y}) = 0$
    - **Appropriate $\lambda$ choice**:
      - the global min is $\mathbf{y}_0$
      - in too large: the global min is $0$

**Equivalence**: both ML and sparse criterion have the same global minimizer

- **Delmer’s criterion** [7] $[\lambda; \lambda^*]$ ensures the equivalence but the statistics of $\lambda; \lambda^*$: $\mathbb{N}^2$ are difficult to compute
- **Depend on the directions**

**Contributions** [8]:

- **Min. of the ML, $\epsilon$, criterion belongs to the interval $[\lambda; \lambda^*]$**
- $\epsilon$ follows, asymptotically, a $\chi^2$ distribution with $N^2 - M$ degrees of freedom
- **$\theta$-invariant distribution**

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**The vectorized Covariance Matrix Model**

- **Let $M$ sources of directions $\Theta = \{\theta_1, \ldots, \theta_M\}$ impinging on an array of $N$ antennas**
  - **Classical model**:
    - **Steering matrix**: $\mathbf{a}(\Theta)$
    - **Emitting signals**: $\mathbf{\xi}(\Theta)$
    - **Circular Gaussian noise**: $\mathcal{CN}(0, \sigma^2)$
    - **Number of identifiable sources limited to $N - 1$ sources**

- **The vectorized covariance matrix model** [5] (for uncorrelated sources):
  - **Source powers vector**: $\mathbf{\rho} = \text{diag}(\mathbf{R}_0)$
  - **Virtual array steering matrix**: $\mathbf{r} = \text{vec}(\mathbf{R}_0 - \sigma^2) = \mathbf{B}(\Theta)\mathbf{\rho} + \mathbf{\delta}$
  - **Non-Circular Gaussian noise**: $\mathcal{CN}(0, \Gamma, \mathbf{C})$

**Objective**: solving the $\mathbb{C}^N$ ML problem
- **Ensuring equivalence between sparse and ML DOA estimators**
- **Minimizing $J_{\lambda}$ requires $M$-dimensional non-convex optimization with numerous local minima**

**Conclusion**

- **Novel $\theta$-invariant regularization parameter choice**
- **Ensuring equivalence between sparse and ML DOA estimators**
- **Efficient implementation of the ML**

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**References**