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On multi-graded Proj schemes

Arnaud Mayeux and Simon Riche

ABSTRACT

We review the construction (due to Brenner–Schröer) of the Proj scheme associated with a ring graded by a finitely generated abelian group. This construction generalizes the well-known Grothendieck Proj construction for \mathbb{N} -graded rings; we extend some classical results (in particular, regarding quasi-coherent sheaves on such schemes) from the \mathbb{N} -graded setting to this general setting, and prove new results that make sense only in the general setting of Brenner–Schröer. Finally, we show that flag varieties of reductive groups, as well as some vector bundles over such varieties attached to representations of a Borel subgroup, can be naturally interpreted in this formalism.

August 8, 2024

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1. Introduction

1.1 Proj schemes

As part of his refoundation of Algebraic Geometry during the second part of the twentieth century, A. Grothendieck introduced the Proj scheme of an \mathbb{N} -graded ring. This generalization of the notion of projective space is natural in the framework of his theory of schemes. Grothendieck’s Proj construction has since been studied, taught, treated and used many thousands of times. During the first years of the third millennium, O. Brenner and S. Schröer generalized Grothendieck’s Proj construction to rings graded by arbitrary finitely generated abelian groups. Surprisingly, this very nice generalization still seems to be relatively confidential, although it has been recently used and studied in some references, including [Za19, KSU21, MRo24]. In this paper, we provide a slightly different perspective on the Brenner–Schröer Proj construction, and prove a couple

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of new results on this construction. In particular, we explain a generalization in this setting of Serre’s classical description of coherent sheaves on a projective space (or, more generally, a Proj scheme) as a Serre quotient of a category of graded modules over a graded ring, see e.g. [StP, Tag 01YR].

To illustrate this theory, we also explain how to describe flag varieties of reductive algebraic groups over algebraically closed fields, and vector bundles on such flag varieties associated with some representations of a Borel subgroup (including the Springer resolution) as the Proj scheme of a ring graded by characters of a maximal torus. The description of flag varieties as Proj schemes of \mathbb{N} -graded rings is classical, see e.g. [Wa], but it requires a choice of a strictly dominant weight. In the multi-graded setting one does *not* make any such choice, hence gets a more canonical construction, which is sometimes useful. In fact, versions of this construction appear implicitly or explicitly (but without proper references) in various constructions in Geometric Representation Theory, including [AB09, ABG04, ARd16], and it was one of our motivations for this work to make this construction completely explicit and rigorous. Note that [MS05, Exercise 14.16] indicates a special case of this construction.

1.2 The Brenner–Schröer construction

Let us now introduce the ideas of our work more precisely. Recall that given a commutative ring A , its (affine) spectrum $\mathrm{Spec}(A)$ is defined as the set of prime ideals, endowed with a certain topology and a sheaf of rings. Most treatments of Grothendieck’s Proj construction proceed by analogy with this construction, namely by associating to a graded ring the set of its graded prime ideals, and endowing it with a topology and a sheaf of rings after (somewhat tedious) identifications of various sets. However, Brenner–Schröer proceed differently in their generalization in [BS03, § 2]: they start from the fact that, if M is an abelian group, the datum of a grading on a commutative ring A is equivalent to the datum of an action of the associated diagonalizable group scheme $D_{\mathrm{Spec}(\mathbb{Z})}(M)$ on $\mathrm{Spec}(A)$ (see e.g. [SGA3, Exp. I, § 4.7.3]), and then consider a certain open subset in a quotient ringed space constructed in terms of this action. Brenner–Schröer’s Proj scheme is covered by some special affine open subschemes, which generalize similar special affine subschemes appearing in Grothendieck’s Proj construction. In case $M = \mathbb{Z}$ and the negative-degree components in A are 0, this recovers Grothendieck’s construction. (We will refer to this case as the “ \mathbb{N} -graded case.”)

In the present treatment of the Proj construction, we put these “special affine subschemes” at the heart of the Proj construction. In fact, we give them a name: “potions.” Potions glue together to give birth to the Proj scheme thanks to a purely algebraic statement we call “the magic of potions.” To use this statement and glue potion schemes, we have to assume that M is finitely generated. (In the \mathbb{N} -graded case, this result is part, as an auxiliary lemma, of the justification of the tedious identifications mentioned before in most treatments of the Proj construction.) Note that, in general, the underlying set of the Proj scheme is *not* the set of graded prime ideals of the given graded ring; it can be described as a set of graded ideals (see [BS03, Remark 2.3]), but we believe that this point of view is not really useful, and in any case it does not play any role in the present paper. Note that [MS05], already mentioned, provides some specific multi-graded Proj schemes (under the name *spector*) by glueing, cf. [MS05, Definition 10.25]. It seems that [BS03] and [MS05] are independent (each does not cite the other). Some other works related to multi-graded Proj schemes are cited in [BS03, Introduction].

1.3 Contents

We now present the results contained in this document. Let as above M be an abelian group and A be a commutative ring endowed with a grading by M . In Sections 2 and 3 we define the notions of M -relevant families of A (following Brenner–Schröer, see Definition 2.3) and of potions associated with these families, and explain the definition of the Proj scheme $\text{Proj}^M(A)$ of A . The potion associated with a relevant family $(f_i : i \in I)$ is the degree-zero part in the localization of A with respect to the multiplicative subset generated by $(f_i : i \in I)$; see Definition 2.9. The spectra $\text{Spec}(A_{(S)})$ are called potion schemes and are denoted $D_{\dagger}(S)$ (cf. Construction 3.1), and the scheme $\text{Proj}^M(A)$ is defined by glueing the potion schemes $D_{\dagger}(S)$ over all relevant families (cf. Construction 3.1). This glueing is possible by the magic of potions (cf. Proposition 2.10). In other words, Section 2 focuses on commutative algebra around potions while Section 3 applies this material to explain the construction of Proj schemes.

In Sections 2 and 3, we also prove some new basic results. In particular, using the concept of quasi-relevant element (Definition 2.11), we prove that the Proj scheme of a tensor product of graded rings (with the product grading) is the fiber product of the Proj schemes of the rings (Proposition 3.15). We prove some compatibility results regarding the radical of the ideal generated by relevant elements (Propositions 2.5 and 2.7). In Example 2.13, we explain the relation between potions and dilatations of rings (as studied in [MRR23, Ma23b]). We also show that the Proj construction is functorial (Proposition 3.12). In § 3.5 we discuss multi-centered blowups and its relations with dilatations.

In Section 4 we explain how to realize flag varieties of reductive groups, and some vector bundles attached to representations of a Borel subgroup, as the Proj scheme of a natural graded ring (without any choice of strictly dominant character). Finally, in Section 5 we study, given an M -graded commutative ring A , the natural functors relating the categories of M -graded A -modules and quasi-coherent sheaves on $\text{Proj}^M(A)$. In particular, we explain how, under a certain technical assumption, one can describe $\text{QCoh}(\text{Proj}^M(A))$ as a Serre quotient of the category of M -graded A -modules, and therefore obtain a version of Serre’s celebrated theorem.

Many of these results are counterparts of well-known results regarding Grothendieck’s Proj construction, which can be found e.g. in [StP].

1.4 Relations between various constructions for graded rings

Recently, several other basic constructions for \mathbb{N} -graded rings were extended to more general gradings. In particular, in [Ma23a] the theory of \mathbb{G}_m -attractors was extended in the setting of actions of diagonalizable group schemes, and in [Ma23b] mono-centered dilatations (involving \mathbb{N} -gradings) were extended to multi-centered dilatations. The algebraic point of view adopted in the present paper was partly inspired by these parallel generalizations. The diagram in Figure 1 summarizes some relations between these graded constructions. Solid lines materialize immediate connections. If two lines are parallel (not necessarily connected), then the associated connections are also parallel in some appropriate sense.

- (i) Formalisms at the bottom of the diagram involve mono-generated gradings (\mathbb{N} or \mathbb{Z}), while formalisms on higher floors involve more general gradings (finitely generated at the intermediate floor and arbitrary at the top floor).
- (ii) Formalisms in the background are affine in nature (involve rings or affine morphisms of schemes), while formalisms in the foreground involve non-affine morphisms of schemes and use as prerequisites corresponding formalisms in the background.

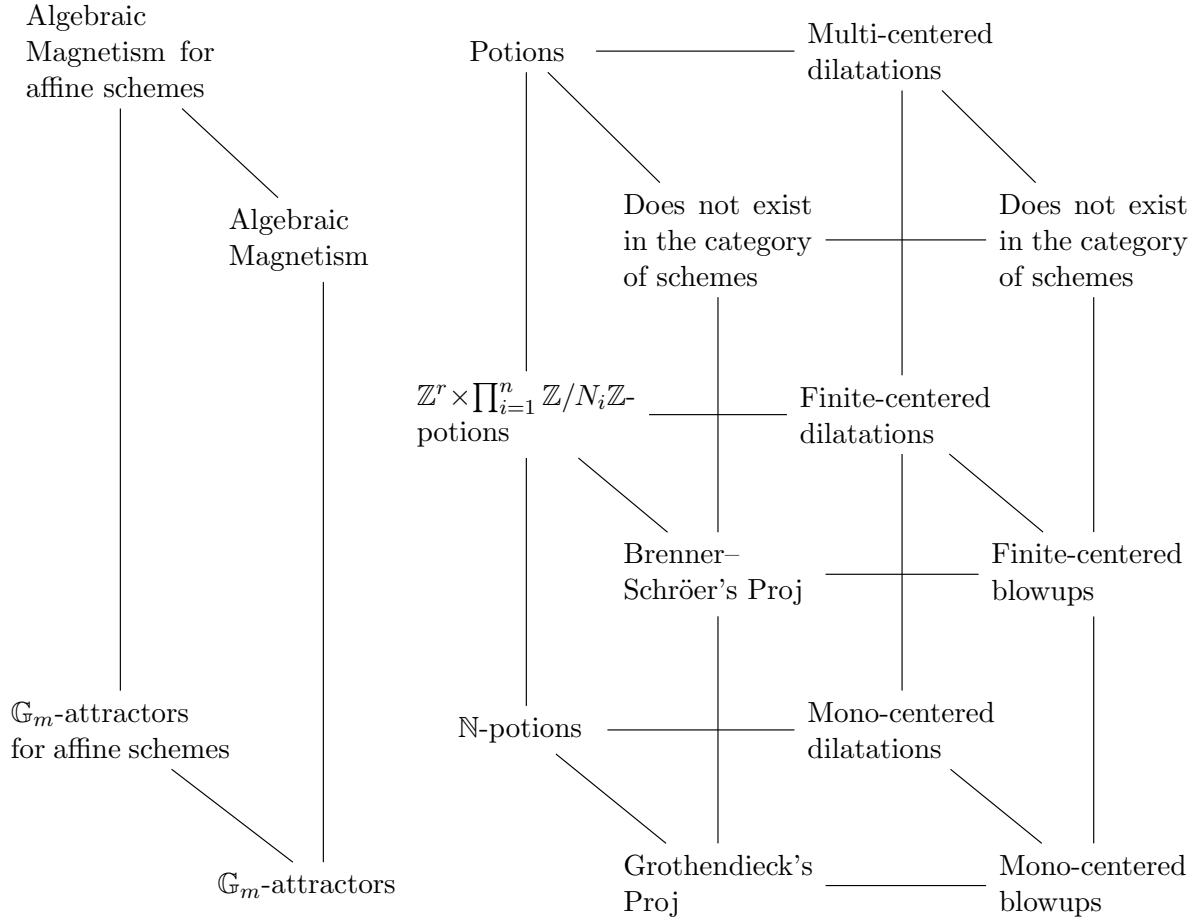


FIGURE 1. Some constructions in algebraic geometry associated with gradings

- (iii) Formalisms on the left study some subschemes of a given scheme endowed with a diagonalizable action (locally, a grading on the background). Formalisms in the middle of the foreground study certain (non-affine) quotients of some given affine schemes with a diagonalizable action, and on the background we have affine local constituents to build them. Formalisms on the right are special cases of formalisms on the middle, but they behave very specifically.
- (iv) On the right and middle, formalisms at the top of the foreground do not exist: in the arbitrary graded case, it is not possible to glue potions as at lower floors. The reason is that if A is a ring and S is an arbitrary multiplicative subset of A , the canonical morphism $\text{Spec}(S^{-1}A) \rightarrow \text{Spec}(A)$ is not an open immersion of schemes in general.

It is possible that other \mathbb{N} -graded constructions in schemes theory can be extended significantly to more general gradings.

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1.6 Some notations and conventions

1.6.1 *Rings* All the rings we will consider in the paper will be tacitly assumed to be unital. (In practice, all the rings we want to consider will be commutative, but we will mention this assumption when it is necessary.)

1.6.2 *Graded rings* Let us make explicit our conventions on graded rings. For this we fix an arbitrary abelian group M .

DEFINITION 1.1. An M -graded ring is a ring A endowed with a direct sum decomposition $A = \bigoplus_{m \in M} A_m$ such that $A_m \cdot A_{m'} \subset A_{m+m'}$ for all $m, m' \in M$.

DEFINITION 1.2. Let A be an M -graded ring. An element $a \in A$ is called *homogeneous* if it belongs to A_m for some $m \in M$. In this case, if a is nonzero, then the element m is unique and denoted $\deg(a)$ and called the *degree* of a .

Remark 1.3. The element 0 is homogeneous, but it has no degree. One can easily check that if the unit 1 is nonzero (i.e. if A is nonzero) then it is homogeneous of degree $0 \in M$.

If R is a ring, an M -graded R -algebra is an M -graded ring A endowed with a ring morphism $R \rightarrow A_0$. (In particular, A is then an R -algebra.)

1.6.3 *Other conventions* If I is a set and \mathbb{M} is a commutative unital semiring we put $\mathbb{M}_I = \bigoplus_{i \in I} \mathbb{M}$. Then \mathbb{M}_I is an \mathbb{M} -semimodule. In this setting we will denote by $(e_i : i \in I)$ the canonical basis of \mathbb{M}_I (i.e. e_i is the I -tuple with 1 in place i and 0 in other places).

If A is a ring and $a \in A$ is an element, we set $a^{\mathbb{N}} := \{a^n : n \in \mathbb{N}\} \subset A$. For any $a \in A$, by convention we have $a^0 = 1$.

If X is a scheme, we will denote by \mathcal{O}_X its structure sheaf, and set $\mathcal{O}(X) = \Gamma(X, \mathcal{O}_X)$. We will also set $X_{\text{aff}} = \text{Spec}(\mathcal{O}(X))$; then we have a canonical morphism of schemes $X \rightarrow X_{\text{aff}}$.

If N is any commutative monoid, there is a universal homomorphism from N to a group N^{gp} (cf. [Og18, § 1.3]). If N is a submonoid of a given abelian group M , then the group N^{gp} naturally identifies with the subgroup $N - N := \{n - n' : n, n' \in N\}$ of M .

Let $(N, +)$ be a commutative monoid. A submonoid F is called a *face* of N if, for all $a, b \in N$, whenever $a + b \in F$ then both a and b belong to F .

2. Potions of graded rings

2.1 Graded rings and localizations

Recall the notion of multiplicative subset in a commutative ring, see [StP, Tag 00CN]. (In particular, a multiplicative subset always contains 1, and it might contain 0.) If a given commutative ring A is M -graded for some abelian group M , we will say that a multiplicative subset is homogeneous if it consists of homogeneous elements.

If A is a commutative M -graded ring and $S \subset A$ is a homogeneous multiplicative subset, the localization A_S of A with respect to S (which is denoted $S^{-1}A$ in many references, including [StP, Tag 00CM]) is canonically M -graded, in such a way that $\deg(\frac{a}{s}) = \deg(a) - \deg(s)$ for $a \in A$

nonzero homogeneous and $s \in S$ such that $\frac{a}{s} \in A_S$ is nonzero. Explicitly, for $m \in M$ we have

$$(A_S)_m = \{0\} \cup \left\{ \frac{a}{s} : a \in A \text{ homogeneous nonzero, } s \in S \text{ such that } \deg(a) - \deg(s) = m \right\}.$$

Given a graded A -module Q , one can also consider the associated localization Q_S (denoted $S^{-1}Q$ in [StP, Tag 07JZ]), which has a natural structure of graded A_S -module.

Given a homogeneous multiplicative subset $S \subset A$, we will denote by \underline{S} the homogeneous multiplicative subset consisting of homogeneous divisors of elements in S . Note that we have a canonical isomorphism of graded rings

$$A_{\underline{S}} \cong A_S. \tag{2.1}$$

If Q is a graded A -module we also have a canonical identification of graded abelian groups

$$Q_{\underline{S}} \cong Q_S \tag{2.2}$$

compatible with the actions of $A_{\underline{S}} = A_S$.

2.2 Relevant families

Consider an abelian group M , and a commutative M -graded ring A .

DEFINITION 2.1. Let S be a homogeneous multiplicative subset of A . We denote by $\deg(S)$ the subset of M consisting of degrees of nonzero homogeneous elements of S (cf. Definition 1.2).

- (i) We will denote by $M[S]$ the submonoid of M generated by $\deg(S)$.
- (ii) We put $M[S] = M[S]^{\text{gp}} = M[S] - M[S]$, the subgroup of M generated by $\deg(S)$.
- (iii) If M is finitely generated, we denote by $M[S]_{\mathbb{R}_{\geq 0}} = M[S] \otimes_{\mathbb{N}} \mathbb{R}_{\geq 0}$ the closed convex cone of $M \otimes_{\mathbb{Z}} \mathbb{R}$ generated by $\deg(S)$.

Remark 2.2. Note that in general $\deg(S)$ is not a monoid, because a product of nonzero elements of A might be 0.

The following definition gives a slight generalization of a terminology used in [BS03].

DEFINITION 2.3. (i) A homogeneous multiplicative subset S of A is called *M -relevant* (or just *relevant* if M is clear from the context) if for any m in M there exists $n \in \mathbb{N}_{>0}$ such that nm belongs to $M[S]$, i.e. if $M/(M[S])$ is a torsion abelian group.

- (ii) A family $(a_i : i \in I)$ of homogeneous elements in A is called *M -relevant* if the multiplicative subset generated by the a_i 's is relevant.
- (iii) A homogeneous element $a \in A$ is called *M -relevant* if the family $\{a\}$ is *M -relevant*.
- (iv) The ideal of A generated by all *M -relevant* elements of A is denoted A_{\dagger} .

Remark 2.4. (i) A necessary condition for an *M -relevant* homogeneous multiplicative subset to exist is that the degrees of nonzero homogeneous elements of A generate a subgroup $M' \subset M$ such that M/M' is a torsion abelian group. If this is the case, $0^{\mathbb{N}}$ is an example of such a subset, since $\underline{0}^{\mathbb{N}}$ consists of all homogeneous elements of A .

- (ii) In case M is finite, all homogeneous multiplicative subsets of A are relevant, and all homogeneous elements are relevant.

The following Proposition generalizes [BS07, Lemma 2.7].¹ Here, we denote by $\text{Rad}(I)$ the radical of an ideal I .

¹We insist that this statement only appears in [BS07], and not in [BS03].

PROPOSITION 2.5. *Let M and M' be two finitely generated abelian groups. Let R be a commutative ring and let A (resp. A') be a commutative M -graded (resp. M' -graded) R -algebra. Then, considering $A \otimes_R A'$ with its natural structure of $M \times M'$ -graded R -algebra:*

- (i) *for any relevant element $x \in A \otimes_R A'$, there exists a positive integer k , a finite set J , and for any $j \in J$ an M -relevant element s_j in A and an M' -relevant element s'_j in A such that $x^k = \sum_{j \in J} s_j \otimes s'_j$;*
- (ii) *Denote by $\langle A_{\dagger} \otimes_R A'_{\dagger} \rangle$ the image of $A_{\dagger} \otimes_R A'_{\dagger}$ in $A \otimes_R A'$. Then $\langle A_{\dagger} \otimes_R A'_{\dagger} \rangle$ is an ideal in $A \otimes_R A'$, and we have $\text{Rad}((A \otimes_R A')_{\dagger}) = \text{Rad}(\langle A_{\dagger} \otimes_R A'_{\dagger} \rangle)$.*

Proof. (i) Let $x \in A \otimes_R A'$ be relevant. Then there exists $k \in \mathbb{Z}_{>0}$ and a factorization $x^k = x_1 \cdots x_l$ into homogeneous factors x_i of degree $d_i \in M \times M'$, such that the degrees d_i generate a subgroup of $M \times M'$ of finite index. For $i \in \{1, \dots, l\}$, write $d_i = (m_i, m'_i)$ with $m_i \in M$ and $m'_i \in M'$. Note that the degrees m_i (resp. m'_i) generate a subgroup of M (resp. M') of finite index. For $i \in \{1, \dots, l\}$, write

$$x_i = \sum_{j \in J_i} s_{ij} \otimes s'_{ij}$$

for an index set J_i , and homogeneous elements $s_{ij} \in A$ and $s'_{ij} \in A'$ of respective degrees m_i and m'_i . Then

$$x^k = \prod_{i=1}^l \left(\sum_{j_i \in J_i} s_{ij_i} \otimes s'_{ij_i} \right) = \sum_{(j_i) \in \prod_{i=1}^l J_i} \left(\prod_{i=1}^l s_{ij_i} \right) \otimes \left(\prod_{i=1}^l s'_{ij_i} \right).$$

For every choice $(j_i) \in \prod_{i=1}^l J_i$, the element $\prod_{i=1}^l s_{ij_i}$, resp. $\prod_{i=1}^l s'_{ij_i}$, is a product of elements of degrees m_1, \dots, m_l , resp. m'_1, \dots, m'_l , and so it is M -relevant in A , resp. M' -relevant in A' . This finishes the proof, taking $J = \prod_{i=1}^l J_i$ and the elements $\prod_{i=1}^l s_{ij_i}$ and $\prod_{i=1}^l s'_{ij_i}$.

(ii) It is clear that $\langle A_{\dagger} \otimes_R A'_{\dagger} \rangle$ is an ideal contained in $(A \otimes_R A')_{\dagger}$, hence $\text{Rad}(\langle A_{\dagger} \otimes_R A'_{\dagger} \rangle) \subset \text{Rad}((A \otimes_R A')_{\dagger})$. On the other hand, by (i) we have $(A \otimes_R A')_{\dagger} \subset \text{Rad}(\langle A_{\dagger} \otimes_R A'_{\dagger} \rangle)$, hence $\text{Rad}((A \otimes_R A')_{\dagger}) \subset \text{Rad}(\langle A_{\dagger} \otimes_R A'_{\dagger} \rangle)$, which finishes the proof. \square

Remark 2.6. Assume that $M' = \{0\}$, so that A' is simply an R -algebra, which we will denote R' for clarity, and we consider the M -graded algebra $A \otimes_R R'$. In this case we have $R'_{\dagger} = R'$ (see Remark 2.4(ii)), hence the proposition says that $\text{Rad}((A \otimes_R R')_{\dagger}) = \text{Rad}(\langle A_{\dagger} \otimes_R R' \rangle)$. This implies that the preimage of the closed subset $V(A_{\dagger}) \subset \text{Spec}(A)$ under the projection

$$\text{Spec}(A \otimes_R R') = \text{Spec}(A) \times_{\text{Spec}(R)} \text{Spec}(R') \rightarrow \text{Spec}(A)$$

is

$$V(\langle A_{\dagger} \otimes_R R' \rangle) = V(\text{Rad}(\langle A_{\dagger} \otimes_R R' \rangle)) = V(\text{Rad}((A \otimes_R R')_{\dagger})) = V((A \otimes_R R')_{\dagger}).$$

PROPOSITION 2.7. *Let M be a finitely generated abelian group, let A, B be M -graded rings, and let $\Psi : A \rightarrow B$ be a morphism of M -graded rings.*

- (i) *We have $\Psi(A_{\dagger}) \subset B_{\dagger}$.*
- (ii) *If Ψ is surjective, then $\text{Rad}(\Psi(A_{\dagger})) = \text{Rad}(B_{\dagger})$.*

Proof. It is easily seen that the image under Ψ of any relevant element of A is a relevant element of B , which proves (i).

Now, assume that Ψ is surjective. Then $\Psi(A_{\dagger})$ is an ideal, and by (i) we have $\text{Rad}(\Psi(A_{\dagger})) \subset \text{Rad}(B_{\dagger})$. Let $f \in B$ be a relevant element. Then there exists $N \in \mathbb{Z}_{\geq 1}$, nonzero homogeneous

elements $f_1, \dots, f_r \in B$ of respective degrees m_1, \dots, m_r , and a homogeneous element $g \in B$ such that $f^N = f_1 \cdots f_r \cdot g$ and the subgroup of M generated by m_1, \dots, m_r has finite index. Choose homogeneous elements $\tilde{f}_1, \dots, \tilde{f}_r, \tilde{g} \in A$ whose respective images are f_1, \dots, f_r, g . Then $\tilde{f} = \tilde{f}_1 \cdots \tilde{f}_r \cdot \tilde{g}$ is relevant and satisfies $\Psi(\tilde{f}) = f^N$, hence $f \in \text{Rad}(\Psi(A_{\dagger}))$. This implies that $B_{\dagger} \subset \text{Rad}(\Psi(A_{\dagger}))$, hence $\text{Rad}(B_{\dagger}) \subset \text{Rad}(\Psi(A_{\dagger}))$, which finishes the proof of (ii). \square

Remark 2.8. In the setting of Proposition 2.7(ii), the preimage of the closed subset $V(A_{\dagger})$ under the closed immersion $\text{Spec}(B) \subset \text{Spec}(A)$ is

$$V(\Psi(A_{\dagger})) = V(\text{Rad}(\Psi(A_{\dagger}))) = V(\text{Rad}(B_{\dagger})) = V(B_{\dagger}).$$

2.3 Potions

Let M be an abelian group, and A a commutative M -graded ring.

DEFINITION 2.9 (Potions). Let S be a homogeneous multiplicative subset of A .

- (i) The degree-0 part $(A_S)_0$ of the localization A_S is denoted $A_{(S)}$ and is called the *portion* of A with respect to S .
- (ii) If Q is a graded A -module, we will also denote by $Q_{(S)}$ the degree-0 part $(Q_S)_0$ in Q_S ; it admits a canonical structure of $A_{(S)}$ -module.

Note that in the setting of Definition 2.9, by (2.1)–(2.2) we have canonical identifications

$$A_{(\underline{S})} \cong A_{(S)}, \quad Q_{(\underline{S})} \cong Q_{(S)}.$$

If $\{a_i : i \in I\}$ is a family of homogeneous elements of A , we will denote by

$$A_{\{a_i : i \in I\}}$$

the potion associated with the multiplicative subset of A generated by $\{a_i : i \in I\}$; in case $\#I = 1$ we will write $A_{(a)}$ for $A_{\{a\}}$.

If S and T are multiplicative subsets of A , we will denote by ST the multiplicative subset of A generated by $S \cup T$, i.e. $ST = \{st : s \in S, t \in T\}$. Of course, ST is homogeneous if S and T are. The following result generalizes [Gr61, Lemma 2.2.2] and [Za19, Lemma 3.7], and is the key result that makes the Proj construction work.

PROPOSITION 2.10 (Magic of potions). *Let S and T be homogeneous multiplicative subsets of A .*

- (i) *We have a canonical morphism of potion rings $A_{(S)} \rightarrow A_{(ST)}$.*
- (ii) *Assume that S is relevant, and that T does not contain 0. Fix a subset $T' \subset T$ which generates T as a submonoid of (A, \times) and, for any t in T' , fix $n_t \in \mathbb{N}_{>0}$ and $s_t, s'_t \in \underline{S}$ nonzero such that $\deg(t^{n_t}) = \deg(s_t) - \deg(s'_t)$. Then $\frac{t^{n_t} s'_t}{s_t}$ belongs to the potion $A_{(\underline{S})} = A_{(S)}$. Moreover we have a canonical isomorphism of $A_{(S)}$ -algebras between $A_{(ST)}$ and the localization of $A_{(S)}$ with respect to the multiplicative subset of $A_{(S)}$ generated by the elements $\{\frac{t^{n_t} s'_t}{s_t} : t \in T'\}$.*
- (iii) *Assume that S is relevant and that T is finitely generated as submonoid of (A, \times) . The morphism of schemes*

$$\text{Spec}(A_{(ST)}) \rightarrow \text{Spec}(A_{(S)})$$

induced by the ring morphism in (i) is an open immersion of schemes.

(iv) Let $f_1, \dots, f_n \in A$ be relevant homogeneous elements of the same degree. Then we have a canonical open immersion

$$\mathrm{Spec}(A_{(f_1+\dots+f_n)}) \rightarrow \mathrm{Spec}(A_{(f_1)}) \cup \dots \cup \mathrm{Spec}(A_{(f_n)})$$

where the right-hand side is defined as the glueing (in the sense of [StP, Tag 01JA]) of the affine schemes $\mathrm{Spec}(A_{(f_i)})$ along the open subschemes $\mathrm{Spec}(A_{(f_i \cdot f_j)}) \subset \mathrm{Spec}(A_{(f_i)})$ (see (iii)).

Proof. (i) The desired morphism is obtained as the degree-zero part of the canonical morphism of graded rings $A_S \rightarrow A_{ST}$.

(ii) Without loss of generality, we can (and will) assume that $S = \underline{S}$. Consider the localization $(A_{(S)})_{\{\frac{t^{n_t} s'_t}{s_t} : t \in T'\}}$ of $A_{(S)}$ with respect to the homogeneous multiplicative subset generated by $\{\frac{t^{n_t} s'_t}{s_t} : t \in T'\}$. Since the image of $\frac{t^{n_t} s'_t}{s_t}$ in $A_{(ST)}$ is invertible for any $t \in T'$, the universal property of localizations gives us a canonical morphism of $A_{(S)}$ -algebras

$$\phi : (A_{(S)})_{\{\frac{t^{n_t} s'_t}{s_t} : t \in T'\}} \rightarrow A_{(ST)}$$

sending $\frac{\frac{a}{s}}{\prod_t (\frac{t^{n_t} s'_t}{s_t})^{k_t}}$ to $\frac{a}{s} \cdot \prod_t (\frac{s_t}{t^{n_t} s'_t})^{k_t}$. (Here and below we consider products indexed by T' ; we tacitly assume that only finitely many of the exponents are nonzero, i.e. that $(k_t : t \in T')$ belongs to $\mathbb{N}_{T'}$.)

Let us prove that ϕ is an isomorphism. If an element $\frac{\frac{a}{s}}{\prod_t (\frac{t^{n_t} s'_t}{s_t})^{k_t}}$ belongs to $\ker(\phi)$, then we have $\frac{a(\prod_t s_t^{k_t})}{s(\prod_t t^{n_t k_t} (s'_t)^{k_t})} = 0$ in $A_{(ST)}$. So there exist $\mathfrak{s} \in S$ and $\mathfrak{t} \in T$ such that $a \cdot (\prod_t s_t^{k_t}) \cdot \mathfrak{s}\mathfrak{t} = 0$ in A . Now the element $\frac{\frac{a}{s}}{\prod_t (\frac{t^{n_t} s'_t}{s_t})^{k_t}}$ equals zero in $(A_{(S)})_{\{\frac{t^{n_t} s'_t}{s_t} : t \in T'\}}$ if and only if there exists $(d_t) \in \mathbb{N}_{T'}$ such that $\frac{a}{s} \prod_t (\frac{t^{n_t} s'_t}{s_t})^{d_t}$ equals zero in $A_{(S)}$; that is if and only if there exists $\mathfrak{s} \in S$ such that $a \cdot \mathfrak{s} \cdot \prod_t t^{n_t d_t} = 0$ in A . This shows that ϕ is injective. Let now $\frac{a}{\mathfrak{s}\mathfrak{t}}$ be an element in $A_{(ST)}$, with $\mathfrak{s} \in S$, $\mathfrak{t} \in T$, and $a \in A$ homogeneous of degree $\deg(\mathfrak{s}\mathfrak{t})$. Write $\mathfrak{t} = \prod_t t^{d_t}$ for some $(d_t) \in \mathbb{N}_{T'}$. The equality

$$\frac{a}{\mathfrak{s}\mathfrak{t}} = \frac{a \cdot \prod_t t^{d_t(n_t-1)} (s'_t)^{d_t}}{\mathfrak{s} \prod_t s_t^{d_t}} \cdot \prod_t \left(\frac{s_t}{t^{n_t} s'_t} \right)^{d_t}$$

holds in $A_{(ST)}$, and implies that $\frac{a}{\mathfrak{s}\mathfrak{t}}$ belongs to the image of ϕ . This proves that ϕ is surjective, and concludes the proof.

(iii) This follows immediately from (ii), since finite localizations of rings induce open immersions.

(iv) For $i \in \{1, \dots, n\}$, put $f'_i = \frac{f_i}{f_1+\dots+f_n} \in A_{(f_1+\dots+f_n)}$ and

$$D(f'_i) = \mathrm{Spec} \left((A_{(f_1+\dots+f_n)})_{f'_i} \right) \subset \mathrm{Spec}(A_{(f_1+\dots+f_n)}),$$

where $(A_{(f_1+\dots+f_n)})_{f'_i}$ is the localization of $A_{(f_1+\dots+f_n)}$ with respect to the multiplicative subset generated by f'_i . We have $f'_1 + \dots + f'_n = 1$ in $A_{(f_1+\dots+f_n)}$, so by [StP, Tag 01HS] we obtain that

$$D(f'_1) \cup \dots \cup D(f'_n) = \mathrm{Spec}(A_{(f_1+\dots+f_n)}).$$

Now by (ii) we have $(A_{(f_1+\dots+f_n)})_{f'_i} = A_{((f_1+\dots+f_n)_{f_i})}$ and therefore $D(f'_i) \subset \mathrm{Spec}(A_{(f_i)})$, which finishes the proof. \square

Proposition 2.10(iv) leads us to introduce the following terminology.

DEFINITION 2.11. A nonzero homogeneous element $a \in A$ is called *quasi- M -relevant* (or just quasi-relevant) if it is a sum of M -relevant elements of the same degree.

Remark 2.12. Note that if $M \neq \mathbb{Z}$ there might exist quasi-relevant elements which are not relevant. For example, let $A = \mathbb{Z}[X, Y, Z, T]$ be \mathbb{Z}^2 -graded with $\deg(X) = \deg(Y) = e_1$ and $\deg(Z) = \deg(T) = e_2$. Then XZ and YT are relevant. So $XZ + YT$ is quasi-relevant, but it is not relevant.

2.4 Examples and basic properties

Example 2.13 (Multi-centered dilatations). Let A be a commutative ring and let $\{[M_i, a_i] : i \in I\}$ be a multi-center in A in the sense of [Ma23b], i.e. M_i is an ideal of A and a_i is an element of A for any $i \in I$. For $i \in I$, let L_i be the ideal $M_i + (a_i)$ of A . Let $\text{Bl}_{\{L_i : i \in I\}} A = \bigoplus_{\nu \in \mathbb{N}_I} L^\nu$ be the multi-Rees A -algebra associated with A and $\{L_i : i \in I\}$. (Here, if $\nu = (\nu_i : i \in I)$, then L^ν is the product of the $(L_i)^{\nu_i}$'s.) The ring $\text{Bl}_{\{L_i : i \in I\}} A$ is naturally \mathbb{Z}_I -graded, and each a_i defines an element in its degree- e_i part, also denoted a_i . Then the family $\{a_i : i \in I\}$ is relevant, and [Ma23b, Fact 2.29] shows that the potion ring $(\text{Bl}_{\{L_i : i \in I\}} A)_{(\{a_i : i \in I\})}$ identifies with the dilatation $A[\{\frac{M_i}{a_i} : i \in I\}]$.

Example 2.14. Any commutative ring A can be considered as graded by the abelian group $M = 0$. In this case any element a in A is relevant and $A_{(a)} = A_a$ is the localization of A with respect to a as in [StP, Tag 02C5].

3. Brenner–Schröer Proj

3.1 Proj scheme of a graded ring

From now on we assume that M is a *finitely generated* abelian group, and fix a commutative M -graded ring A . Consider the affine scheme $\text{Spec}(A_0)$, and the diagonalizable group scheme

$$D_{\text{Spec}(A_0)}(M) = \text{Spec}(A_0[M])$$

over $\text{Spec}(A_0)$. The M -grading on A defines an action of $D_{\text{Spec}(A_0)}(M)$ on $\text{Spec}(A)$; we will denote by

$$a, p : D_{\text{Spec}(A_0)}(M) \times_{\text{Spec}(A_0)} \text{Spec}(A) \rightarrow \text{Spec}(A)$$

the action and projection morphisms, respectively.

In [BS03, Definition 2.2], Brenner and Schröer define the Proj scheme associated with A as a certain open subset in the ringed space $\text{Quot}(A)$ obtained as the cokernel (in the sense of [SGA3, Exp. V, § 1.b]) of the maps a and p . In Construction 3.1 below we explain an equivalent description of this scheme, obtained by gluing spectra of certain potion rings. (See the comments following the construction for a justification that our description is indeed equivalent with that in [BS03].)

We will denote by \mathcal{F}_A the set of all relevant homogeneous multiplicative subsets of A which are finitely generated as submonoids of (A, \times) .

CONSTRUCTION 3.1 (Proj as glueing potions). Let $\mathcal{F} \subset \mathcal{F}_A$ be a subset. For each $S \in \mathcal{F}$, let $D_{\dagger}(S)$ be the spectrum of the potion $A_{(S)}$. By Proposition 2.10(iii), if $S, T \in \mathcal{F}$, the affine scheme $D_{\dagger}(ST)$ identifies canonically with an open subscheme of $D_{\dagger}(S)$. For each $S, T \in \mathcal{F}$, we have equalities

$$D_{\dagger}(SS) = D_{\dagger}(S) \quad \text{and} \quad D_{\dagger}(ST) = D_{\dagger}(TS).$$

Moreover, for each triple $S, T, U \in \mathcal{F}$, we have

$$D_{\dagger}(ST) \cap D_{\dagger}(SU) = D_{\dagger}(TS) \cap D_{\dagger}(TU)$$

(intersections in $D_{\dagger}(S)$ and $D_{\dagger}(T)$ respectively; equalities in $D_{\dagger}(S)$, $D_{\dagger}(T)$ or $D_{\dagger}(U)$). Indeed, using Proposition 2.10(ii), these intersections identify with $\text{Spec}(A_{(STU)})$. Now, by glueing [StP, Tag 01JA], from these data we obtain a scheme $\text{Proj}_{\mathcal{F}}^M(A)$ and, for each $S \in \mathcal{F}$, an open immersion $\varphi_S : D_{\dagger}(S) \rightarrow \text{Proj}_{\mathcal{F}}^M(A)$, such that

$$\text{Proj}_{\mathcal{F}}^M(A) = \bigcup_{S \in \mathcal{F}} \varphi_S(D_{\dagger}(S))$$

and that for $S, T \in \mathcal{F}$ we have

$$\varphi_S(D_{\dagger}(ST)) = \varphi_T(D_{\dagger}(TS)) = \varphi_S(D_{\dagger}(S)) \cap \varphi_T(D_{\dagger}(T)).$$

In practice, we will often identify $D_{\dagger}(S)$ and $\varphi_S(D_{\dagger}(S))$.

In the case when $\mathcal{F} = \mathcal{F}_A$, the scheme $\text{Proj}_{\mathcal{F}_A}^M(A)$ will be denoted $\text{Proj}^M(A)$, or just $\text{Proj}(A)$ when M is clear from the context.

Remark 3.2. (i) As explained in Remark 2.4(i), if the subgroup of M generated by degrees of nonzero homogeneous elements in A is of finite index, $\{0\} \in \mathcal{F}_A$. However, the localization $A_{\{0\}}$ is the zero ring, so that $D_{\dagger}(\{0\})$ is empty.

(ii) More generally, if $0 \in S$ then $A_S = \{0\}$, so that $D_{\dagger}(S) = \emptyset$. The scheme $\text{Proj}^M(A)$ is therefore covered by the open subschemes $D_{\dagger}(S)$ where S does not contain 0.

(iii) Let $S, T \in \mathcal{F}_A$, and assume that $S \subset T \subset \underline{S}$. Then $T = ST$, so that we have an embedding $D_{\dagger}(T) \subset D_{\dagger}(S)$. Since $A_S = A_T$, this embedding is an equality.

LEMMA 3.3. *The scheme $\text{Proj}^M(A)$ is quasi-separated.*

Proof. This follows from [StP, Tag 01KO], since by construction the intersections in $\text{Proj}^M(A)$ of affine open subschemes of the form $D_{\dagger}(S)$ are affine. \square

For any $S \in \mathcal{F}_A$ we have a canonical morphism $A_0 \rightarrow A_{(S)}$; these morphisms glue to a morphism of schemes

$$\text{Proj}^M(A) \rightarrow \text{Spec}(A_0). \quad (3.1)$$

Using the notation introduced in Definition 2.3(iii), in case $S = a^{\mathbb{N}}$ where $a \in A$ is M -relevant, we will write $D_{\dagger}(a)$ for $D_{\dagger}(a^{\mathbb{N}})$.

If $f \in A$ is relevant, for any homogeneous $g \in A$ the product fg is relevant.

LEMMA 3.4. *Let $f \in A$ be relevant, and let $g \in A$ be homogeneous.*

(i) *We have $D_{\dagger}(fg) \subset D_{\dagger}(f)$.*

(ii) *If g is also relevant, then we have*

$$D_{\dagger}(fg) = D_{\dagger}(f) \cap D_{\dagger}(g).$$

(iii) *If $\deg(g) = 0$, then $D_{\dagger}(fg)$ is the preimage of $D(g) \subset \text{Spec}(A_0)$ under the composition*

$$D_{\dagger}(f) \hookrightarrow \text{Proj}^M(A) \xrightarrow{(3.1)} \text{Spec}(A_0).$$

Proof. (i) We have $(fg)^{\mathbb{N}} \subset f^{\mathbb{N}} \cdot (fg)^{\mathbb{N}} \subset (fg)^{\mathbb{N}}$, hence by Remark 3.2(iii) we have $D_{\dagger}(f) \cap D_{\dagger}(fg) = D_{\dagger}(f^{\mathbb{N}} \cdot (fg)^{\mathbb{N}}) = D_{\dagger}(fg)$, so that $D_{\dagger}(fg) \subset D_{\dagger}(f)$.

(ii) This follows similarly from Remark 3.2(iii).

(iii) By (i) we have $D_{\dagger}(fg) \subset D_{\dagger}(f)$. Now it is clear that $A_{(fg)}$ is the localization of $A_{(f)}$ with respect to g , which gives the claim. \square

In particular, Lemma 3.4(ii) shows that if $a \in A$ is relevant, for any $k \in \mathbb{Z}_{\geq 1}$ we have

$$D_{\dagger}(a) = D_{\dagger}(a^k). \quad (3.2)$$

Note that if $S \in \mathcal{F}_A$, and if f_1, \dots, f_n are multiplicative generators of S , then $f := f_1 \cdots f_n$ is a relevant element of A , and by Remark 3.2(iii) we have $D_{\dagger}(S) = D_{\dagger}(f)$. It follows that

$$\text{Proj}^M(A) = \bigcup_{f \in A \text{ relevant}} D_{\dagger}(f), \quad (3.3)$$

which justifies that the scheme considered in Construction 3.1 coincides with that defined in [BS03, Definition 2.2]. In particular the scheme $\text{Proj}^M(A)$ can be constructed by gluing open subschemes associated with relevant elements along their intersections, which by Lemma 3.4(ii) are again open subschemes associated with relevant elements.

Remark 3.5. (i) In case $M = \mathbb{Z}$, any nonzero homogeneous element of nonzero degree is relevant. If we furthermore have $A_n = 0$ for any $n \in \mathbb{Z}_{<0}$ one recovers the usual Proj scheme associated with a (nonnegatively) graded ring as in [StP, Tag 01M3]; we will refer to this setting as the \mathbb{N} -graded setting.

(ii) In the \mathbb{N} -graded setting, the scheme $\text{Proj}^{\mathbb{Z}}(A)$ is always separated; see [StP, Tag 01MC]. For general M and A this is not true; for separatedness criteria, see [BS03, § 3] and Proposition 3.6 below.

The following result is an immediate corollary of [BS03, Proposition 3.3]. Given a finite homogeneous family $T = (a_i : i \in I)$, generating a multiplicative subset S of A , we will denote by $C_T \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ the closed convex cone $M[\underline{S}]_{\mathbb{R}_{\geq 0}}$ (cf. Definition 2.1). Note that T is relevant if and only if C_T has nonempty interior.

PROPOSITION 3.6. *Let \mathcal{F} be a collection of relevant finite homogeneous families such that $C_T \cap C_{T'} \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ has nonempty interior for all $T, T' \in \mathcal{F}$. Then the scheme $\text{Proj}_{\mathcal{F}}^M(A)$ is separated.*

Proof. This is an immediate corollary of [BS03, Proposition 3.3] and the comment preceding (3.3). \square

The natural morphism of ringed spaces $\text{Spec}(A) \rightarrow \text{Quot}(A)$ restricts to a canonical morphism

$$\text{Spec}(A) \setminus V(A_{\dagger}) \rightarrow \text{Proj}^M(A). \quad (3.4)$$

In the description given in (3.3), this morphism can be understood as follows. For any $f \in A$ relevant, we have a canonical embedding $A_{(f)} \subset A_f$, which provides a morphism $\text{Spec}(A_f) \rightarrow \text{Spec}(A_{(f)})$. The latter morphisms glue to a morphism from

$$\text{Spec}(A) \setminus V(A_{\dagger}) = \bigcup_{f \in A \text{ relevant}} \text{Spec}(A_f)$$

to

$$\text{Proj}^M(A) = \bigcup_{f \in A \text{ relevant}} \text{Spec}(A_{(f)}).$$

This morphism is discussed in [BS03, Comments after Definition 2.2]. The closed subscheme $V(A_{\dagger}) \subset \text{Spec}(A)$ associated with A_{\dagger} is stable under the action of $D_{\text{Spec}(A_0)}(M)$; there is hence also an action on the open complement $\text{Spec}(A) \setminus V(A_{\dagger})$. This action stabilizes each open subscheme $\text{Spec}(A_f)$, and by [BS03, Lemma 2.1] the morphism $\text{Spec}(A_S) \rightarrow \text{Spec}(A_{(f)})$ is a geometric quotient for the action of $D_{\text{Spec}(A_0)}(M)$ in the sense of [MFK93, Definition 0.6]. It follows that (3.4) is also a geometric quotient. As explained in [BS03], this morphism is also affine.

Remark 3.7. Assume we are given a flat affine group scheme H over $\text{Spec}(A_0)$ and an action of H on A which preserves degrees. This induces an action on $\text{Spec}(A)$, which stabilizes $V(A_{\dagger})$, hence also an action on $\text{Spec}(A) \setminus V(A_{\dagger})$. Since a geometric quotient is a categorical quotient (see [MFK93, Proposition 0.1]), the composition

$$H \times_{\text{Spec}(A_0)} (\text{Spec}(A) \setminus V(A_{\dagger})) \rightarrow \text{Spec}(A) \setminus V(A_{\dagger}) \rightarrow \text{Proj}^M(A)$$

factors through a morphism

$$H \times_{\text{Spec}(A_0)} \text{Proj}^M(A) \rightarrow \text{Proj}^M(A).$$

It is easily seen that this morphism defines an action of H on $\text{Proj}^M(A)$.

Example 3.8. In view of Example 2.14, when $M = \{0\}$, for any ring A we have $\text{Proj}^{\{0\}}(A) = \text{Spec}(A)$.

3.2 Open subschemes defined by quasi-relevant elements

Recall the notion of quasi- M -relevant element of A from Definition 2.11. If $a \in A$ is M -quasi-relevant, then we can write $a = \sum_{i=1}^n f_i$ where each f_i is M -relevant of degree $\deg(a)$, and by Proposition 2.10(iv) we have an open immersion

$$\text{Spec}(A_{(a)}) \rightarrow \bigcup_{i=1}^n D_{\dagger}(f_i).$$

By construction the right-hand side is an open subscheme in $\text{Proj}^M(A)$, so we deduce an open immersion

$$\text{Spec}(A_{(a)}) \rightarrow \text{Proj}^M(A). \quad (3.5)$$

LEMMA 3.9. *The morphism (3.5) is canonical, i.e. it does not depend on the way a is written as a sum of relevant elements.*

Proof. Assume that $a = \sum_{i=1}^n f_i = \sum_{j=1}^m g_j$ where the f_i 's and the g_j 's are relevant, all of the same degree. What we have to prove is that the compositions

$$\text{Spec}(A_{(a)}) \rightarrow \bigcup_{i=1}^n \text{Spec}(A_{(f_i)}) \subset \text{Proj}^M(A) \quad \text{and} \quad \text{Spec}(A_{(a)}) \rightarrow \bigcup_{j=1}^m \text{Spec}(A_{(g_j)}) \subset \text{Proj}^M(A)$$

coincide. Here the first, resp. second, immersion is obtained using the open covering

$$\text{Spec}(A_{(a)}) = \bigcup_{i=1}^n \text{Spec}(A_{(af_i)}), \quad \text{resp.} \quad \text{Spec}(A_{(a)}) = \bigcup_{j=1}^m \text{Spec}(A_{(ag_j)})$$

and the natural open immersions $\text{Spec}(A_{(af_i)}) \rightarrow \text{Spec}(A_{(f_i)})$, resp. $\text{Spec}(A_{(ag_j)}) \rightarrow \text{Spec}(A_{(g_j)})$. However, in $A_{(a)}$ we have $\sum_{i,j} \frac{f_i g_j}{a^2} = 1$, so that using similar considerations we can refine both

of these decompositions into a decomposition

$$\mathrm{Spec}(A_{(a)}) = \bigcup_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \mathrm{Spec}(A_{(af_i g_j)}),$$

and it is clear that both immersions can be obtained from this decomposition using the canonical open immersions $\mathrm{Spec}(A_{(af_i g_j)}) \rightarrow \mathrm{Proj}^M(A)$. \square

Lemma 3.9 justifies that the image of (3.5) can be denoted $D_{\dagger}(a)$; this open scheme identifies canonically with $\mathrm{Spec}(A_{(a)})$. In case a is relevant, this coincides with the open subscheme denoted similarly above.

COROLLARY 3.10. *Let $(a_i : i \in I)$ be relevant elements in A . If*

$$A_{\dagger} \subset \mathrm{Rad} \left(\sum_{i \in I} A \cdot a_i \right),$$

then we have

$$\mathrm{Proj}^M(A) = \bigcup_{i \in I} D_{\dagger}(a_i).$$

Proof. By (3.3), $\mathrm{Proj}^M(A)$ is covered by the open subschemes $D_{\dagger}(f)$ where f is relevant. By assumption, for such f there exists $k \in \mathbb{Z}_{>0}$ such that $f^k = \sum_{j=1}^r b_j \cdot a_{i_j}$ for some elements $b_j \in A$ and some indices $i_j \in I$. Since f is homogeneous, here we can assume that for any j the element b_j is homogeneous, with $\deg(b_j) + \deg(a_{i_j}) = \deg(f)$. Then we have

$$D_{\dagger}(f) \stackrel{(3.2)}{=} D_{\dagger}(f^k) \subset \bigcup_{j=1}^r D_{\dagger}(b_j a_{i_j}),$$

and by Lemma 3.4(i) we have $D_{\dagger}(b_j a_{i_j}) \subset D_{\dagger}(a_{i_j})$ for any j . The claim follows. \square

Remark 3.11. In particular, in case I can be chosen finite, this corollary implies that $\mathrm{Proj}^M(A)$ is quasi-compact.

3.3 Some basic properties

We start with a functoriality property of the Proj scheme construction, which generalizes [StP, Tag 01MY]. For this we note that if A, B are commutative M -graded rings and $S \subset A$ is a homogeneous multiplicative subset, then so is $\Psi(S)$. Clearly, $\Psi(S)$ is relevant if S is, and finitely generated as a submonoid of (B, \times) if S is finitely generated as a submonoid of (A, \times) . Hence Ψ induces a map $\mathcal{F}_A \rightarrow \mathcal{F}_B$, which will also be denoted Ψ .

PROPOSITION 3.12 (Functoriality of Proj). *Let $\Psi : A \rightarrow B$ be a morphism of M -graded rings. For any $\mathcal{F} \subset \mathcal{F}_A$ we have a canonical morphism of schemes $\mathrm{Proj}_{\Psi(\mathcal{F})}^M(B) \rightarrow \mathrm{Proj}_{\mathcal{F}}^M(A)$. Moreover, for any $S \in \mathcal{F}$ we have*

$$\mathrm{Proj}_{\Psi(\mathcal{F})}^M(B) \times_{\mathrm{Proj}_{\mathcal{F}}^M(A)} D_{\dagger}(S) = D_{\dagger}(\Psi(S)); \quad (3.6)$$

in particular, the morphism is affine.

Proof. The morphism is obtained by glueing from the morphisms induced by the canonical morphisms $A_{(S)} \rightarrow B_{(\Psi(S))}$ for $S \in \mathcal{F}$.

Now fix $S \in \mathcal{F}$. From the definition of our morphism we see that we have an inclusion

$$D_{\dagger}(\Psi(S)) \subset \text{Proj}_{\Psi(\mathcal{F})}^M(B) \times_{\text{Proj}_{\mathcal{F}}^M(A)} D_{\dagger}(S)$$

as open subschemes of $\text{Proj}_{\Psi(\mathcal{F})}^M(B)$. On the other hand, by definition $\text{Proj}_{\Psi(\mathcal{F})}^M(B)$ is covered by the open subschemes $D_{\dagger}(\Psi(T))$ when T runs over \mathcal{F} , so that $\text{Proj}_{\Psi(\mathcal{F})}^M(B) \times_{\text{Proj}_{\mathcal{F}}^M(A)} D_{\dagger}(S)$ is covered by the open subschemes

$$D_{\dagger}(\Psi(T)) \times_{\text{Proj}_{\mathcal{F}}^M(A)} D_{\dagger}(S).$$

Now the morphism $D_{\dagger}(\Psi(T)) \rightarrow \text{Proj}_{\mathcal{F}}^M(A)$ factors through $D_{\dagger}(T)$, so that

$$D_{\dagger}(\Psi(T)) \times_{\text{Proj}_{\mathcal{F}}^M(A)} D_{\dagger}(S) = D_{\dagger}(\Psi(T)) \times_{D_{\dagger}(T)} (D_{\dagger}(T) \cap D_{\dagger}(S)).$$

Using Proposition 2.10(ii) one sees that the right-hand side coincides with $D_{\dagger}(\Psi(ST))$; in particular it is contained in $D_{\dagger}(\Psi(S))$, which finishes the proof of (3.6).

Using [StP, Tag 01S8], the equalities (3.6) imply that our morphism is affine. \square

LEMMA 3.13. *In the setting of Proposition 3.12, assume that Ψ is surjective. Then we have $\text{Proj}_{\Psi(\mathcal{F}_A)}^M(B) = \text{Proj}^M(B)$, and the morphism*

$$\text{Proj}^M(B) \rightarrow \text{Proj}^M(A)$$

is a closed immersion.

Proof. Let $S \in \mathcal{F}_B$. By definition, there exist $s_1, \dots, s_r \in \underline{S}$ whose degrees generate a subgroup of M of finite index. Let $S' \subset B$ be the homogenous multiplicative subset generated by s_1, \dots, s_r , and let $S'' = S \cdot S'$. Then S' and S'' are relevant homogeneous multiplicative subsets of B , and by Remark 3.2(iii) we have $D_{\dagger}(S) = D_{\dagger}(S'')$. We therefore have an embedding $D_{\dagger}(S) \rightarrow D_{\dagger}(S')$. Now, choosing for each $i \in \{1, \dots, r\}$ a homogeneous preimage t_i of s_i in A and denoting by $\tilde{S} \subset A$ the homogeneous multiplicative subset generated by t_1, \dots, t_r , it is clear that \tilde{S} is relevant and that $\Psi(\tilde{S}) = S'$. Hence $D_{\dagger}(S') \subset \text{Proj}_{\Psi(\mathcal{F}_A)}^M(B)$, which implies that $D_{\dagger}(S)$ is also contained in $\text{Proj}_{\Psi(\mathcal{F}_A)}^M(B)$. Since S was arbitrary, this proves the equality $\text{Proj}_{\Psi(\mathcal{F}_A)}^M(B) = \text{Proj}^M(B)$.

Since the property of being a closed immersion is local on the target (see [StP, Tag 02L6]), to prove the second assertion it suffices to prove that for any $S \in \mathcal{F}_A$ the induced morphism

$$\text{Proj}^M(B) \times_{\text{Proj}^M(A)} D_{\dagger}(S) \rightarrow D_{\dagger}(S)$$

is a closed immersion. This follows from (3.6), since the morphism $A_{(f)} \rightarrow B_{(\Psi(f))}$ is surjective. \square

Remark 3.14. The conclusion of Lemma 3.13 can often be obtained under weaker assumptions using more specific information on relevant elements in A . See [StP, Tag 01N0] for the case of \mathbb{N} -graded rings, and Proposition 4.7 below for an example with a more general M .

PROPOSITION 3.15. *Let M and M' be two finitely generated abelian groups. Let R be a commutative ring and let A (resp. A') be a commutative M -graded (resp. M' -graded) R -algebra. Then for the natural $(M \times M')$ -grading on $A \otimes_R A'$, we have a canonical isomorphism*

$$\text{Proj}^{M \times M'}(A \otimes_R A') \cong \text{Proj}^M(A) \times_{\text{Spec}(R)} \text{Proj}^{M'}(A').$$

Proof. The identification is provided by the following equalities:

$$\begin{aligned}
 \mathrm{Proj}^M(A) \times_{\mathrm{Spec}(R)} \mathrm{Proj}^{M'}(A') &= \left(\bigcup_{f \in A \text{ relevant}} \mathrm{Spec}(A_{(f)}) \right) \times_{\mathrm{Spec}(R)} \left(\bigcup_{f' \in A' \text{ relevant}} \mathrm{Spec}(A_{(f')}) \right) \\
 &= \bigcup_{f \in A, f' \in A' \text{ relevant}} \mathrm{Spec} \left(A_{(f)} \otimes_R A'_{(f')} \right) \\
 &= \bigcup_{f \in A, f' \in A' \text{ relevant}} \mathrm{Spec} \left((A \otimes_R A')_{(f \otimes f')} \right) \\
 &= \mathrm{Proj}^{M \times M'}(A \otimes_R A').
 \end{aligned}$$

Here the first equality follows from (3.3). The second equality follows from a basic property of fiber products of schemes, see [StP, Tag 01JS]. The third equality follows from the obvious isomorphism of $(M \times M')$ -graded rings

$$A_f \otimes_R A'_{f'} \cong (A \otimes_R A')_{f \otimes f'}$$

(for $f \in A$ and $f' \in A'$ homogeneous) by restriction to the components of degree $(0, 0)$.

To conclude we have to explain the fourth equality. Let $\mathcal{F}_{A \otimes_R A'}^\otimes \subset \mathcal{F}_{A \otimes_R A'}$ be the subset consisting of homogeneous multiplicative subsets of the form $(f \otimes f')^\mathbb{N}$ with $f \in A$ M -relevant and $f' \in A'$ M' -relevant. (Here we use once again the notation introduced in Definition 2.3(iii).) The desired equality will follow from the equality

$$\mathrm{Proj}_{\mathcal{F}_\otimes}^{M \times M'}(A \otimes_R A') = \mathrm{Proj}^{M \times M'}(A \otimes_R A'),$$

which can be justified as follows. By definition, the left-hand side is an open subset of the right-hand side. To prove that this open subset is all of $\mathrm{Proj}^{M \times M'}(A \otimes_R A')$, by (3.3) it suffices to prove that it contains the open subscheme attached to any relevant element x in $A \otimes_R A'$. Now if $x \in A \otimes_R A'$ is relevant, by Proposition 2.5(i) there exist a positive integer k , a finite set J and for any $j \in J$ an M -relevant element s_j in A and an M' -relevant element s'_j in A such that $x^k = \sum_{j \in J} s_j \otimes s'_j$. Then we have

$$D_{\dagger}(x) = D_{\dagger}(x^k) \subset \bigcup_{j \in J} D_{\dagger}(s_j \otimes s'_j),$$

which finishes the proof. \square

Remark 3.16. Another proof of Proposition 3.15 can be obtained using Proposition 2.5(ii) and the fact that (3.4) is a geometric quotient.

The following corollary appears in the proof of [BS07, Proposition 2.5].

COROLLARY 3.17 (Base change). *Let M be a finitely generated abelian group, let R be a commutative ring, and let A be a commutative M -graded R -algebra. If R' is a commutative R -algebra, then for the natural M -grading on $A \otimes_R R'$ we have*

$$\mathrm{Proj}(A \otimes_R R') \cong \mathrm{Proj}(A) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R').$$

Proof. In view of Example 3.8, the statement is the special case of Proposition 3.15 corresponding to $M' = \{0\}$. \square

Remark 3.18. Let $S \in \mathcal{F}_A$, and let $S_{R'}$ be its image in $A \otimes_R R'$. Then it is clear that under the isomorphism of Corollary 3.17 the open subscheme $D_{\dagger}(S_{R'}) \subset \mathrm{Proj}(A \otimes_R R')$ identifies with $D_{\dagger}(S) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$.

The following example generalizes [StP, Tag 01MI].

Example 3.19. Let R be a commutative ring and N be a commutative cancellative monoid such that the group $M = N - N$ is finitely generated. Let $R[N] := \bigoplus_{m \in N} R \cdot X^m$ be the R -algebra of the monoid N , with its natural M -grading. The homogeneous elements in this ring are those of the form rX^m with $r \in R$ and $m \in N$; such an element is relevant if and only if $N \cap (m - N)$ generates M . For each such element we have $(R[N])_{(rX^m)} = R_r$; therefore we have $\text{Proj}(R[N]) = \text{Spec}(R)$.

PROPOSITION 3.20. *Let M be a finitely generated abelian group, and let A be a commutative M -graded ring. Let $M' \subset M$ be a subgroup of finite index. Consider the subring $A' = \bigoplus_{m \in M'} A_m$; it can be considered naturally as an M' -graded ring, or as an M -graded ring. We have canonical isomorphisms of schemes*

$$\text{Proj}^M(A) \cong \text{Proj}^M(A') \cong \text{Proj}^{M'}(A').$$

Proof. We use the description of these schemes using (3.3). If f is an M -relevant element in A , then by (3.2) we have $A_{(f)} = A_{(f^d)}$ for all $d \in \mathbb{Z}_{>0}$. If n is the index of M' in M , then f^n belongs to A' and is M' -relevant in A' ; moreover we have $A_{(f)} = A_{(f^n)} = A'_{(f^n)}$. This shows that $\text{Proj}^M(A)$ identifies with an open subscheme of $\text{Proj}^{M'}(A')$. Reciprocally, if $f \in A'$ is M' -relevant, then f is an M -relevant element of A ; it follows that this open subscheme is the whole of $\text{Proj}^{M'}(A')$.

The other isomorphism $\text{Proj}^M(A') \cong \text{Proj}^{M'}(A')$ is clear since an element in A' is M' -relevant if and only if it is M -relevant. \square

Remark 3.21. Note that Proposition 3.20 is not true if M/M' is infinite. For example, let R be a ring, which we endow with the \mathbb{Z} -grading where R is in degree 0. One can also consider R as a $\{0\}$ -graded. There is no \mathbb{Z} -relevant element in R so $\text{Proj}^{\mathbb{Z}}(R) = \emptyset$; however by Example 3.8 we have $\text{Proj}^{\{0\}}(R) = \text{Spec}(R)$.

We now state a few immediate (but useful) consequences of Proposition 3.20. Here, intuitively, (i) says that “the Proj scheme only depends on components in any subgroup of finite index,” and (ii) allows to reduce the theory to the case M is a free \mathbb{Z} -module. Finally, (iii) generalizes Example 3.8.

COROLLARY 3.22. *Let M be a finitely generated abelian group.*

- (i) *Let A and B be commutative M -graded rings, and assume that there exists a subgroup $M' \subset M$ such that M/M' is finite and the subrings of A and B consisting of their homogeneous components associated with elements in M' are isomorphic (as M' -graded rings). Then we have*

$$\text{Proj}^M(A) \cong \text{Proj}^M(B).$$

- (ii) *Let A be a commutative M -graded ring. Let $M' \subset M$ be a complement to the torsion subgroup $M_{\text{tor}} \subset M$, and consider the M' -graded ring $A' = \bigoplus_{m \in M'} A_m$. Then we have $\text{Proj}^M(A) \cong \text{Proj}^{M'}(A')$.*
- (iii) *Assume that M is finite. For any commutative M -graded ring, we have $\text{Proj}(A) = \text{Spec}(A_0)$.*

3.4 Relative Proj schemes

Proposition 3.17 allows us to extend the construction of Proj schemes to a relative setting, as follows. Let X be a scheme, M be a finitely generated abelian group, and \mathcal{A} be an M -graded

quasi-coherent \mathcal{O}_X -algebra.

CONSTRUCTION 3.23 (Relative Proj). For any affine open subscheme $U \subset X$ we consider the Proj-scheme $\text{Proj}^M(\Gamma(U, \mathcal{A}))$ of the M -graded ring $\Gamma(U, \mathcal{A})$; it admits a canonical morphism to $\text{Spec}(\Gamma(U, \mathcal{A}_0))$, hence to U . If $U' \subset U \subset X$ are open affine subschemes, by Corollary 3.17 we have a canonical isomorphism of U' -schemes.

$$U' \times_U \text{Proj}^M(\Gamma(U, \mathcal{A})) \cong \text{Proj}^M(\Gamma(U', \mathcal{A})).$$

By relative glueing (see [StP, Tag 01LH]), there exists an X -scheme $\text{Proj}_X^M(\mathcal{A})$ such that

$$\text{Proj}_X^M(\mathcal{A}) \times_X U = \text{Proj}^M(\mathcal{A}(U))$$

for any affine open subscheme $U \subset X$.

Most of the results of this section have obvious analogues in this setting; we leave it to the reader to formulate these variants and adapt the proofs.

3.5 Proj, blowups and dilatations

Let A be a ring and $\{J_i : i \in I\}$ be ideals of A . Recall the ring $\text{Bl}_{\{J_i : i \in I\}} A = \bigoplus_{\nu \in \mathbb{N}_I} J_i^\nu$ (cf. Example 2.13 and [Ma23b, Definition 2.28]), which is \mathbb{Z}_I -graded. We assume now that I is finite. Any family $(a_i : i \in I)$ of elements of A such that $a_i \in J_i$ for each $i \in I$ can be considered as a relevant family in the \mathbb{Z}_I -graded ring $\text{Bl}_{\{J_i : i \in I\}} A$, so we have an associated affine open subscheme

$$D_{\dagger}(a_i : i \in I) \subset \text{Proj}^{\mathbb{Z}_I}(\text{Bl}_{\{J_i : i \in I\}} A)$$

and this open subscheme identifies with the dilatation $\text{Spec}(A[\{\frac{J_i}{a_i} : i \in I\}])$ (cf. Example 2.13 and [Ma23b, Fact 2.29]).

The goal of the present subsection is to study related connections and compatibilities between dilatations and these specific Proj schemes, and to show that this allows to easily provide a proof of the existence of (projective) blowups, possibly with multiple centers. We start with a multi-centered version of [StP, Tag 0804].

LEMMA 3.24. *In the setting above, the scheme $\text{Proj}^{\mathbb{Z}_I}(\text{Bl}_{\{J_i : i \in I\}} A)$ is covered by the open subschemes $\text{Spec}(A[\{\frac{J_i}{a_i} : i \in I\}])$ where $(a_i : i \in I)$ runs over families such $a_i \in J_i$ for all $i \in I$.*

Proof. By Corollary 3.10, the lemma will follow if we prove that any relevant element in the \mathbb{Z}_I -graded ring $\text{Bl}_{\{J_i : i \in I\}} A$ belongs to the sum of the ideals $(\text{Bl}_{\{J_i : i \in I\}} A) \cdot \prod_i a_i$ where $(a_i : i \in I)$ runs over the families as above. Now it is clear that this sum is the sum of the graded components in $\text{Bl}_{\{J_i : i \in I\}} A$ associated with elements in \mathbb{Z}_I all of whose components are positive. The claim therefore follows from the observation that the degree of any relevant element in $\text{Bl}_{\{J_i : i \in I\}} A$ must have positive components, because if the i -component of the degree of a homogeneous element is 0, then the same will hold for all divisors of all of its powers, so that this element cannot be relevant. \square

We now work in a relative setting; we therefore fix a base X , and let $\{Y_i : i \in I\}$ be a finite family of closed subschemes.

DEFINITION 3.25 (Multi-centered blowups). A (projective) blowup of X with center $\{Y_i : i \in I\}$ is a pair consisting of a scheme \tilde{X} and a morphism $\pi : \tilde{X} \rightarrow X$ such that $\pi^{-1}(Y_i)$ is an effective Cartier divisor for all $i \in I$, and which is universal with respect to this property in the following sense. If $\pi' : \tilde{X}' \rightarrow X$ is any morphism such that $(\pi')^{-1}(Y_i)$ is an effective Cartier divisor for all i , then there exists a unique morphism $m : \tilde{X}' \rightarrow \tilde{X}$ such that $\pi' = \pi \circ m$.

Clearly, a blowup of X with center $\{Y_i : i \in I\}$ is unique up to unique isomorphism if it exists. For each i , let $\mathcal{J}_i \subset \mathcal{O}_X$ be the quasi-coherent ideal of Y_i , so that $Y_i = V(\mathcal{J}_i)$. Let

$$\mathrm{Bl}_{\{\mathcal{J}_i : i \in I\}} \mathcal{O}_X = \bigoplus_{\nu \in \mathbb{N}_I} \mathcal{J}^\nu$$

be the associated Rees algebra (cf. e.g. [Ma23b, § 3.1]). This is canonically a \mathbb{Z}_I -graded quasi-coherent \mathcal{O}_X -algebra. Using the construction of § 3.4 we now define an X -scheme

$$\underline{\mathrm{Bl}}_{\{Y_i : i \in I\}} X := \mathrm{Proj}_X^{\mathbb{Z}_I}(\mathrm{Bl}_{\{\mathcal{J}_i : i \in I\}} \mathcal{O}_X).$$

We will prove in Proposition 3.30 below that this scheme is a blowup of X with center $\{Y_i : i \in I\}$ in the sense of Definition 3.25. (The case $|I| = 1$ of this statement is classical, see e.g. [StP, Tag 01OF].)

LEMMA 3.26. *The canonical morphism $\underline{\mathrm{Bl}}_{\{Y_i : i \in I\}} X \rightarrow X$ is separated.*

Proof. Since the property of being separated is local on the target (see e.g. [StP, Tag 02KU]), we can assume $X = \mathrm{Spec}(A)$ is affine, and by [StP, Tag 01KV] the claim will follow if in this case we prove that $\underline{\mathrm{Bl}}_{\{Y_i : i \in I\}} X$ is a separated scheme. We proceed with usual local notations from the beginning of the subsection. Let \mathcal{F} be the collection of relevant homogeneous families $(a_i : i \in I)$ with $a_i \in J_i$ for all $i \in I$. By Lemma 3.24, $\mathrm{Proj}^M(\mathrm{Bl}_{\{J_i : i \in I\}} A) = \mathrm{Proj}_{\mathcal{F}}^M(\mathrm{Bl}_{\{J_i : i \in I\}} A)$. For any family $T = (a_i : i \in I)$ in \mathcal{F} , we have $C_T = (\mathbb{R}_{\geq 0})_I \subset \mathbb{R}_I$. So if T, T' belong to \mathcal{F} , $C_T \cap C_{T'}$ has nonempty interior. Now Proposition 3.6 finishes the proof. \square

PROPOSITION 3.27 (Definiteness). *If C is an effective Cartier divisor on X , then $(\underline{\mathrm{Bl}}_{\{Y_i : i \in I\}} X) \times_X C$ is an effective Cartier divisor on $\underline{\mathrm{Bl}}_{\{Y_i : i \in I\}} X$.*

Proof. This statement is local on X , so we can assume $X = \mathrm{Spec}(A)$ for some ring A , and write J_i for the global sections of \mathcal{J}_i . In view of Lemma 3.24, in this case it suffices to prove that for any family $(a_i : i \in I)$ with $a_i \in J_i$ for any i , if C is an effective Cartier divisor on X then $\mathrm{Spec}(A[\{\frac{J_i}{a_i} : i \in I\}]) \times_X C$ is an effective Cartier divisor on $\mathrm{Spec}(A[\{\frac{J_i}{a_i} : i \in I\}])$. This statement is a special case of [Ma23b, Proposition 3.18]. \square

In the following statement we use the notation of dilatations of schemes from [Ma23b] (see also [MRR23, DMdS23]). For simplicity of notation we set

$$X' := \underline{\mathrm{Bl}}_{\{Y_i : i \in I\}} X,$$

and for $i \in I$ we set

$$Y'_i := Y_i \times_X X'.$$

PROPOSITION 3.28 (Compatibility). *Let $\{D_i : i \in I\}$ be a family of locally principal closed subschemes of X , such that $Y_i \subset D_i$ for all i , and set $D'_i := D_i \times_X X'$. There exists a canonical isomorphism of X -schemes*

$$\mathrm{Bl}_{\{Y'_i : i \in I\}}^{D'_i : i \in I} X' \cong \mathrm{Bl}_{\{Y_i : i \in I\}}^{D_i : i \in I} X.$$

Proof. To simplify notation we set $X'' = \mathrm{Bl}_{\{Y_i : i \in I\}}^{D_i : i \in I} X$. We have a canonical morphism of X -schemes $X'' \rightarrow X'$ which is an open immersion. (In fact this question is local on the target and explained at the beginning of the subsection in the local case). Setting $Y''_i = Y_i \times_X X''$ and $D''_i = D_i \times_X X''$ (for $i \in I$), by [Ma23b, § 3.5] we therefore obtain canonical morphisms

$$\mathrm{Bl}_{\{Y''_i : i \in I\}}^{D''_i : i \in I} X'' \rightarrow \mathrm{Bl}_{\{Y'_i : i \in I\}}^{D'_i : i \in I} X' \rightarrow \mathrm{Bl}_{\{Y_i : i \in I\}}^{D_i : i \in I} X.$$

By [Ma23b, Proposition 3.28], we have $\text{Bl}_{\{Y_i'' : i \in I\}}^{\{D_i'' : i \in I\}} X'' = X''$. So we get X -morphisms $a : X'' \rightarrow \text{Bl}_{\{Y_i' : i \in I\}}^{\{D_i' : i \in I\}} X'$ and $b : \text{Bl}_{\{Y_i' : i \in I\}}^{\{D_i' : i \in I\}} X' \rightarrow X''$. It is easily checked that $b \circ a = \text{id}$ and $a \circ b = \text{id}$ (e.g. using the universal property of dilatations), so these morphisms provide the desired identification. \square

PROPOSITION 3.29 (Complementary). *For all $i \in I$, Y_i' is a Cartier divisor on X' . Moreover, the composition $\text{Bl}_{\{\emptyset : i \in I\}}^{\{Y_i' : i \in I\}} X' \rightarrow X' \rightarrow X$ is an open immersion; in fact, this is just the inclusion $X \setminus (\cup_{i \in I} Y_i) \rightarrow X$.*

Proof. First, assume that $X = \text{Spec}(A)$ for some ring A , and write J_i for the global sections of \mathcal{J}_i . In this case, we have explained in the course of the proof of Proposition 3.27 that $\text{Bl}_{\{Y_i : i \in I\}} X$ is covered by affine open subschemes $\text{Spec}((\bigoplus_{\nu \in \mathbb{N}_I} J^\nu)_{(\{a_i : i \in I\})})$ where $(a_i : i \in I)$ is a family of elements of A such that $a_i \in J_i$ for any i . As explained above, $(\bigoplus_{\nu \in \mathbb{N}_I} J^\nu)_{(\{a_i : i \in I\})}$ is the dilatation ring $A[\{\frac{J_i}{a_i} : i \in I\}]$. Now the first assertion follows from [Ma23b, Facts 2.10 & 2.25]. The second assertion follows from the formula

$$(A[\{\frac{J_i}{a_i} : i \in I\}]) \left[\left\{ \frac{A[\{\frac{J_i}{a_i} : i \in I\}]}{a_i} : i \in I \right\} \right] = A[\{\frac{A}{a_i} : i \in I\}]$$

(see e.g. the computation in [Ma23b, Proof of Corollary 2.23]) and the fact the the right-hand side is the localization of A with respect to the a_i 's, see [Ma23b, Fact 2.11].

To treat the general case, one takes an affine open covering of X and use the affine case. \square

PROPOSITION 3.30 (Existence of blowups). *Let $\pi : T \rightarrow X$ be a scheme such that, for all $i \in I$, $T \times_X Y_i$ is a Cartier divisor on T . Then there exists a unique morphism of X -schemes $T \rightarrow X'$. As a consequence, the pair consisting of X' and the canonical morphism $X' \rightarrow X$ is a blowup of X with center $\{Y_i : i \in I\}$.*

Proof. It suffices to prove the statement in the affine case; namely we assume that $X = \text{Spec}(A)$ and $T = \text{Spec}(R)$ for some rings A and R . Then we denote by J_i the global sections of \mathcal{J}_i , so that $Y_i = \text{Spec}(A/J_i)$. Then $\pi : T \rightarrow X$ corresponds to a morphism of rings $f : A \rightarrow R$. By assumption, for any $i \in I$, we have $f(J_i)R = \gamma_i R$ for some element $\gamma_i \in R$ which is not a zero divisor.

Let us first show the existence of a morphism of X -schemes $T \rightarrow X'$. We have a canonical morphism of \mathbb{N}_I -graded rings

$$\bigoplus_{\nu \in \mathbb{N}_I} J^\nu \rightarrow \bigoplus_{\nu \in \mathbb{N}_I} (f(J)R)^\nu$$

where $(f(J)R)^\nu$ denotes the ideal $\prod_{i \in I} (f(J_i)R)^{\nu_i} \subset R$. Now since $f(J_i)R = \gamma_i R$ with γ_i non-zero divisor for any i , we have a canonical isomorphism of R -algebras $\bigoplus_{\nu \in \mathbb{N}_I} (f(J)R)^\nu \cong R[\mathbb{N}_I]$. So we get a canonical morphism of graded A -algebras

$$\bigoplus_{\nu \in \mathbb{N}_I} J^\nu \rightarrow R[\mathbb{N}_I].$$

By Example 3.19 we have $\text{Proj}^{\mathbb{Z}_I}(R[\mathbb{N}_I]) = \text{Spec}(R)$, so that by Proposition 3.12 this morphism provides a morphism of X -schemes $\phi : \text{Spec}(R) \rightarrow \text{Proj}(\bigoplus_{\nu \in \mathbb{N}_I} J^\nu)$ as required.

To show uniqueness we have to show that if $\phi' : T \rightarrow X'$ is an arbitrary morphism of X -schemes, we have $\phi = \phi'$. By lemma 3.26, the morphism $X' \rightarrow X$ is separated; as a consequence, the difference kernel $\ker(\phi, \phi')$ (see [GW20, Definition 9.1]) is a closed subscheme of T by [GW20, Definition/Proposition 9.7]. Let $U \subset T$ be the complement of the closed subscheme defined by the

ideal generated by the γ_i 's; then the scheme theoretic image of the open immersion $\iota_U : U \rightarrow T$ is T by [StP, Tag 056A]. Since $U \times_X Y_i = \emptyset$ for all i , the universal property of dilatations (see [Ma23b, Proposition 3.17]) implies that both $\phi|_U$ and $\phi'|_U$ both factor through $\text{Bl}_{\{\emptyset:i \in I\}}^{\{Y'_i:i \in I\}} X'$. By Proposition 3.29, $\text{Bl}_{\{\emptyset:i \in I\}}^{\{Y'_i:i \in I\}} X'$ is an open subscheme of X , so that this implies that $\phi|_U = \phi'|_U$. We deduce that $U \subset \ker(\phi, \phi')$, and then that $\ker(\phi, \phi') = T$, so that $\phi = \phi'$.

What we have now proved and Proposition 3.29 imply that X' satisfy the defining property of blowups (see Definition 3.25). \square

Remark 3.31. In other words, Proposition 3.30 says that $\underline{\text{Bl}}_{\{Y_i:i \in I\}} X$ is a final object in the category $\text{Sch}_X^{\{Y_i:i \in I\}\text{-reg}}$ of [Ma23b, Definition 3.4].

Recall the “+” operation on closed subschemes of X , see [Ma23b, Notation 3.1].

COROLLARY 3.32. *We have a canonical isomorphism $\underline{\text{Bl}}_{\{Y_i:i \in I\}} X \cong \underline{\text{Bl}}_{\sum_{i \in I} Y_i} X$.*

Proof. It is enough to prove that $\underline{\text{Bl}}_{\sum_{i \in I} Y_i} X$ is also a blowup of X with center $\{Y_i : i \in I\}$. This is immediate using that $\underline{\text{Bl}}_{\sum_{i \in I} Y_i} X$ is a blowup of X with center $\{\sum_{i \in I} Y_i\}$ and the fact that Cartier divisors form a face of the monoid of closed subschemes (see [Ma23b, Notation 3.1]). \square

4. Examples arising from reductive algebraic groups

In this section we show that a number of varieties of interest in Geometric Representation Theory can be described as Proj schemes associated with certain graded rings in a canonical way (which, in particular, do not require a specific choice of dominant character). We see these examples as a motivation to study the general theory considered in the rest of the paper, but none of the considerations of the present sections will be used in proofs of results of other sections. (In Section 5, these results will only be used as illustrations of certain general statements.)

4.1 Flag varieties

In this section we fix an algebraically closed field \mathbb{k} ,² and a connected reductive algebraic group \mathbf{G} over \mathbb{k} . We fix a Borel subgroup \mathbf{B} and a maximal torus $\mathbf{T} \subset \mathbf{B}$, and denote by \mathbf{U} the unipotent radical of \mathbf{B} . We denote by \mathbb{X} the character lattice of \mathbf{T} (a free \mathbb{Z} -module of finite rank), and by $\mathfrak{R} \subset \mathbb{X}$ the root system of (\mathbf{G}, \mathbf{T}) , i.e. the subset of nonzero \mathbf{T} -weights in the Lie algebra of \mathbf{G} . We will denote by $\mathfrak{R}_+ \subset \mathfrak{R}$ the subset consisting of the opposites of the \mathbf{T} -weights in the Lie algebra of \mathbf{U} ; this forms a positive system. We will also consider the cocharacter lattice \mathbb{X}^\vee of \mathbf{T} , which identifies with the dual of \mathbb{X} , and the system of coroots $\mathfrak{R}^\vee \subset \mathbb{X}^\vee$. There exists a canonical bijection $\mathfrak{R} \xrightarrow{\sim} \mathfrak{R}^\vee$, which we will as usual denote $\alpha \mapsto \alpha^\vee$. Our choice of positive system \mathfrak{R}_+ lets us define a submonoid $\mathbb{X}_+ \subset \mathbb{X}$ of dominant weights as

$$\mathbb{X}_+ = \{\lambda \in \mathbb{X} \mid \forall \alpha \in \mathfrak{R}_+, \langle \lambda, \alpha^\vee \rangle \geq 0\}.$$

We will also consider the submonoid of strictly dominant weights, defined by

$$\mathbb{X}_{++} = \{\lambda \in \mathbb{X} \mid \forall \alpha \in \mathfrak{R}_+, \langle \lambda, \alpha^\vee \rangle > 0\}.$$

Consider the flag variety \mathbf{G}/\mathbf{B} , a smooth projective scheme over \mathbb{k} . To any $\lambda \in \mathbb{X}$ is associated in a natural way a line bundle $\mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)$ on \mathbf{G}/\mathbf{B} ; see e.g. [Ja03, § II.4.1]. (Our $\mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)$ corre-

²It is likely that most of the results in this section can be generalized to reductive group schemes over more general base schemes, but we will not go in this direction.

sponds to $\mathbf{G}/\mathbf{B}(\mathbb{k}_B(\lambda))$ in the notation of [Ja03], where $\mathbb{k}_B(\lambda)$ is the 1-dimensional B -module determined by λ .) It is a standard fact that the space

$$\Gamma(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda))$$

vanishes unless $\lambda \in \mathbb{X}^+$, and is finite-dimensional in this case; see e.g. [Ja03, § II.2.1 and Proposition II.2.6]. For $\lambda, \mu \in \mathbb{X}$ there exists a canonical isomorphism

$$\mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda) \otimes_{\mathcal{O}_{\mathbf{G}/\mathbf{B}}} \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\mu) \cong \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda + \mu), \quad (4.1)$$

giving rise to a canonical morphism

$$\Gamma(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)) \otimes_{\mathbb{k}} \Gamma(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\mu)) \rightarrow \Gamma(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda + \mu)), \quad (4.2)$$

which is known to be surjective, see [BK05, Theorem 3.1.2].

We consider the \mathbb{X} -graded ring

$$A := \bigoplus_{\lambda \in \mathbb{X}_+} \Gamma(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)).$$

Its spectrum $\mathbf{X} := \text{Spec}(A)$ is the ‘‘affine completion’’ of the space \mathbf{G}/\mathbf{U} ; it is a classical object, studied in particular in [AR, § 6.2.1]³ (where it is denoted \mathcal{X}).

The main result of this subsection is the following statement.

PROPOSITION 4.1. *There exists a canonical isomorphism*

$$\mathbf{G}/\mathbf{B} \xrightarrow{\sim} \text{Proj}^{\mathbb{X}}(A).$$

Remark 4.2. (i) A similar description of \mathbf{G}/\mathbf{B} as the Proj-scheme associated with an \mathbb{N} -graded ring, which requires to choose a strictly dominant weight, is classical, although we were not able to find an explicit mention in the published literature. For an exposition of the proof, see the notes [Wa].

(ii) The algebra A has a canonical action of the group \mathbf{G} . As explained in Remark 3.7, this induces a canonical action of \mathbf{G} on $\text{Proj}^{\mathbb{X}}(A)$. It will be clear from the proofs below that this action corresponds to the obvious action on \mathbf{G}/\mathbf{B} via the identification of Proposition 4.1.

We will give two independent proofs of Proposition 4.1. The first one uses the description of the Proj-scheme as a geometric quotient under the action of a diagonalizable group scheme explained in § 3.1. The second one uses results of [BS03] on ample families of line bundles.

The first proof will require a preliminary lemma. Here we consider the ideal A_{\dagger} with respect to the \mathbb{X} -grading, as defined in Definition 2.3.

LEMMA 4.3. *We have*

$$A_{\dagger} \subset \bigoplus_{\lambda \in \mathbb{X}_{++}} \Gamma(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)).$$

Moreover, if \mathbf{G} has a simply connected derived subgroup this inclusion is an equality.

Proof. It is clear that $\bigoplus_{\lambda \in \mathbb{X}_{++}} \Gamma(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda))$ is an ideal in A . If $f \in A$ is a homogenous relevant element, then $\deg(f)$ must be strictly dominant; in fact, if there exists $\alpha \in \mathfrak{A}_+$ such that

³In this section of [AR] it is assumed that the base field has characteristic 0. This assumption is however not used for the statements we use below.

$\langle \deg(f), \alpha^\vee \rangle = 0$, then any homogeneous divisor g of a power of f will also satisfy $\langle \deg(g), \alpha^\vee \rangle = 0$, contradicting the fact that f is relevant. As a consequence, we have

$$A_\dagger \subset \bigoplus_{\lambda \in \mathbb{X}_{++}} \Gamma(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)).$$

Now, assume that \mathbf{G} has simply connected derived subgroup, and let \mathfrak{R}_s be the set of simple roots. Our assumption ensures that for any $\alpha \in \mathfrak{R}_s$ there exists a weight $\varpi_\alpha \in \mathbb{X}$ such that

$$\langle \varpi_\alpha, \beta^\vee \rangle = \delta_{\alpha, \beta}$$

for $\beta \in \mathfrak{R}_s$. (In case \mathbf{G} is semisimple these weights are uniquely determined, and called the fundamental weights. If \mathbf{G} is not semisimple, they are not unique.) Setting

$$\mathbb{X}_0 = \{\lambda \in \mathbb{X} \mid \forall \alpha \in \mathfrak{R}_s, \langle \lambda, \alpha^\vee \rangle = 0\},$$

then we have

$$\mathbb{X} = \mathbb{X}_0 \oplus \left(\bigoplus_{\alpha \in \mathfrak{R}_s} \mathbb{Z} \varpi_\alpha \right), \quad \mathbb{X}_+ = \left(\bigoplus_{\alpha \in \mathfrak{R}_s} \mathbb{Z}_{\geq 0} \cdot \varpi_\alpha \right) \oplus \mathbb{X}_0. \quad (4.3)$$

Any $\lambda \in \mathbb{X}_0$ extends to a character of \mathbf{G} , so that the space $\Gamma(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda))$ is 1-dimensional (by the tensor identity [Ja03, Proposition I.3.6]), and its nonzero elements are invertible.

By the surjectivity of (4.2) when λ, μ are dominant, the ideal

$$\bigoplus_{\lambda \in \mathbb{X}_{++}} \Gamma(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)) \subset A$$

is generated by elements which are products $\prod_{\alpha \in \mathfrak{R}_s} f_\alpha$ with $\deg(f_\alpha) = \varpi_\alpha$. Such products are relevant since they admit a divisor of degree ϖ_α for each $\alpha \in \mathfrak{R}_s$, and also of degree λ for any $\lambda \in \mathbb{X}_0$. This proves the inclusion

$$\bigoplus_{\lambda \in \mathbb{X}_{++}} \Gamma(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)) \subset A_\dagger$$

and finishes the proof. \square

Now we can give the first proof of Proposition 4.1.

First proof of Proposition 4.1. By [BS03, Proposition 4.2], there exists a canonical rational map from \mathbf{G}/\mathbf{B} to $\text{Proj}^{\mathbb{X}}(A)$; what we will show is that this rational morphism is an isomorphism.

It is a standard fact that there exists a connected reductive algebraic group $\tilde{\mathbf{G}}$ over \mathbb{k} with simply connected derived subgroup and a finite central isogeny $\tilde{\mathbf{G}} \rightarrow \mathbf{G}$. (This follows e.g. from the considerations in [Ja03, § I.1.18].) The preimages $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{T}}$ of \mathbf{B} and \mathbf{T} are a Borel subgroup and a maximal torus in $\tilde{\mathbf{G}}$ respectively. Let also $\tilde{\mathbb{X}}$ and \tilde{A} be the counterparts of \mathbb{X} and A for $\tilde{\mathbf{G}}$, $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{T}}$. We have an isomorphism

$$\tilde{\mathbf{G}}/\tilde{\mathbf{B}} \xrightarrow{\sim} \mathbf{G}/\mathbf{B}, \quad (4.4)$$

and composition with the surjection $\tilde{\mathbf{T}} \rightarrow \mathbf{T}$ induces an embedding $\mathbb{X} \hookrightarrow \tilde{\mathbb{X}}$ whose cokernel is finite. For $\lambda \in \mathbb{X}$, under the identification (4.4) the line bundle on \mathbf{G}/\mathbf{B} associated with λ identifies with the line bundle on $\tilde{\mathbf{G}}/\tilde{\mathbf{B}}$ associated with its image in $\tilde{\mathbb{X}}$; hence A identifies with the subring of \tilde{A} given by the sum of the components whose label is in \mathbb{X} . In view of Proposition 3.20, we deduce an isomorphism

$$\text{Proj}^{\mathbb{X}}(A) \cong \text{Proj}^{\tilde{\mathbb{X}}}(\tilde{A}).$$

Comparing with (4.4), this reduces the proof to the case \mathbf{G} has simply connected derived subgroup.

In this case, we have described the ideal $A_{\dagger} \subset A$ in Lemma 4.3. This ideal coincides with that considered in [AR, Remark 6.2.3]; comparing these description we obtain an identification

$$\mathrm{Spec}(A) \setminus V(A_{\dagger}) = \mathbf{G}/\mathbf{U}. \quad (4.5)$$

Using the comments preceding Remark 3.7, we see that $\mathrm{Proj}^{\mathbb{X}}(A)$ identifies with the geometric quotient of the left-hand side by the action of \mathbf{T} . (This quotient is unique by [MFK93, Proposition 0.1].) It is clear that the projection $\mathbf{G}/\mathbf{U} \rightarrow \mathbf{G}/\mathbf{B}$ identifies \mathbf{G}/\mathbf{B} with the geometric quotient of \mathbf{G}/\mathbf{U} by the action of \mathbf{T} , which implies our identification. \square

Remark 4.4. Decomposing the space $\mathcal{O}(\mathbf{G}/\mathbf{U})$ according to the weight spaces of the natural action of \mathbf{T} and looking at the definitions we obtain an identification

$$\mathcal{O}(\mathbf{G}/\mathbf{U}) = \bigoplus_{\lambda \in \mathbb{X}} \Gamma(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)) = A.$$

Hence we have $\mathrm{Spec}(A) = (\mathbf{G}/\mathbf{U})_{\mathrm{aff}}$. The above proof amounts to noticing that, in case \mathbf{G} has simply connected derived subgroup, the natural morphism

$$\mathbf{G}/\mathbf{U} \rightarrow (\mathbf{G}/\mathbf{U})_{\mathrm{aff}}$$

is an open immersion, with image $\mathrm{Spec}(A) \setminus V(A_{\dagger})$.

Second proof of Proposition 4.1. As in the first proof, one can assume that \mathbf{G} has simply connected derived subgroup. In fact, one can even reduce the proof to the case \mathbf{G} is *semisimple* and simply connected. Indeed, \mathbf{G} and its derived subgroup have the same flag variety. On the other hand, let \mathbb{X}_0 , \mathfrak{R}_s and $(\varpi_{\alpha} : \alpha \in \mathfrak{R}_s)$ are as in the proof of Lemma 4.3, and recall the descriptions (4.3) of \mathbb{X} and \mathbb{X}_+ . Denoting by A_1 , resp. A_2 , the sum of the components of A whose degrees belong to $\bigoplus_{\alpha \in \mathfrak{R}_s} \mathbb{Z}_{\geq 0} \cdot \varpi_{\alpha}$, resp. to \mathbb{X}_0 , multiplication induces an isomorphism

$$A_1 \otimes A_2 \xrightarrow{\sim} A.$$

By Proposition 3.15 we deduce an isomorphism

$$\mathrm{Proj}^{\mathbb{X}}(A) \cong \mathrm{Proj}^{\bigoplus_{\alpha \in \mathfrak{R}_s} \mathbb{Z} \cdot \varpi_{\alpha}}(A_1) \times \mathrm{Proj}^{\mathbb{X}_0}(A_2).$$

Now A_2 identifies with $\mathbb{k}[\mathbb{X}_0]$, so that by Example 3.19 we have $\mathrm{Proj}^{\mathbb{X}_0}(A_2) = \mathrm{Spec}(\mathbb{k})$. Since A_1 identifies with the version of A associated with the derived subgroup of \mathbf{G} , this finishes the reduction of the proof to the semisimple case.

From now on, we therefore assume that \mathbf{G} is semisimple and simply connected. In this case, we will show that the claim follows from [BS03, Corollary 4.6]. First, let us prove that $\mathrm{Proj}^{\mathbb{X}}(A)$ is separated. As explained above we have $\mathbb{X}_+ = \bigoplus_{\alpha \in \mathfrak{R}_s} \mathbb{Z}_{\geq 0} \cdot \varpi_{\alpha}$, and by the surjectivity of (4.2) the natural morphism of \mathbb{X} -graded rings

$$\left(\bigotimes_{\alpha \in \mathfrak{R}_s} S(A_{\varpi_{\alpha}}) \right) \rightarrow A \quad (4.6)$$

is surjective, where we denote by $S(V)$ the symmetric algebra of a vector space V . By Lemma 3.13 we deduce a closed immersion

$$\mathrm{Proj}^{\mathbb{X}}(A) \hookrightarrow \mathrm{Proj}^{\mathbb{X}} \left(\bigotimes_{\alpha \in \mathfrak{R}_s} S(A_{\varpi_{\alpha}}) \right)$$

and by Proposition 3.15 the right-hand side identifies with

$$\prod_{\alpha \in \mathfrak{R}_s} \text{Proj}^{\mathbb{Z} \cdot \varpi_\alpha}(\mathbb{S}(A_{\varpi_\alpha})).$$

By the theory of \mathbb{N} -graded Proj schemes, for $\alpha \in \mathfrak{R}_s$ the scheme $\text{Proj}^{\mathbb{Z} \cdot \varpi_\alpha}(\mathbb{S}(A_{\varpi_\alpha}))$ identifies with the projective space of the dual vector space $(A_{\varpi_\alpha})^*$ (which is the Weyl module of highest weight $-w_\circ(\varpi_\alpha)$, where w_\circ is the longest element in the Weyl group). Hence we obtain a closed immersion

$$\text{Proj}^{\mathbb{X}}(A) \hookrightarrow \prod_{\alpha \in \mathfrak{R}_s} \mathbb{P}((A_{\varpi_\alpha})^*). \quad (4.7)$$

Since projective spaces (and, more generally, Proj schemes associated with \mathbb{N} -graded rings, see Remark 3.5) are separated and closed immersions are separated (see [StP, Tag 01QU]), this indeed shows that $\text{Proj}^{\mathbb{X}}(A)$ is separated.

Let us note also that we have $(\mathbf{G}/\mathbf{B})_{\text{aff}} = \text{Spec}(\mathbb{k})$, so that the morphism $\mathbf{G}/\mathbf{B} \rightarrow (\mathbf{G}/\mathbf{B})_{\text{aff}}$ is projective, hence proper.

Finally, choose a numbering $\alpha_1, \dots, \alpha_r$ of the elements in \mathfrak{R}_s . It is a standard fact (see e.g. [Ja03, Proposition II.4.4]) that $\mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)$ is ample for any $\lambda \in \mathbb{X}_{++}$; in particular, $\mathcal{O}_{\mathbf{G}/\mathbf{B}}(\varpi_{\alpha_1} + \dots + \varpi_{\alpha_r})$ is ample. Comparing the characterization of ample line bundles in [StP, Tag 01PS] with [BS03, Proposition 1.1] we deduce that the collection

$$\mathcal{O}_{\mathbf{G}/\mathbf{B}}(\varpi_{\alpha_1}), \dots, \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\varpi_{\alpha_r})$$

is an ample family in the sense of [BS03].

We have now checked that the conditions in [BS03, Corollary 4.6] are satisfied, and this statement allows us to conclude. \square

Remark 4.5. In the case when \mathbf{G} is semisimple and simply connected, the closed immersion (4.7) recovers the standard closed immersion of the flag variety in a product of projective spaces of fundamental Weyl modules.

Remark 4.6. One can construct an explicit open covering of $\text{Proj}^{\mathbb{X}}(A)$ as follows. Choose a finite central isogeny $\tilde{\mathbf{G}} \rightarrow \mathbf{G}$ where $\tilde{\mathbf{G}}$ has simply connected derived subgroup, see the first proof of Proposition 4.1. Let $\tilde{\mathbb{X}}$ and \tilde{A} be as in this proof, and recall that A identifies with the sum of the graded components of A labelled by elements in $\mathbb{X} \subset \tilde{\mathbb{X}}$. Choose elements $(\varpi_\alpha : \alpha \in \mathfrak{R}_s)$ as above for the group $\tilde{\mathbf{G}}$ and, for any $\alpha \in \mathfrak{R}_s$, choose a basis $(f_i^\alpha : i \in I_\alpha)$ of $\tilde{A}_{\varpi_\alpha}$. Let E be the set of sections of the obvious map

$$\bigsqcup_{\alpha \in \mathfrak{R}_s} I_\alpha \rightarrow \mathfrak{R}_s;$$

for $\sigma \in E$ we set $f_\sigma = \prod_{\alpha \in \mathfrak{R}_s} f_{\sigma(\alpha)}^\alpha$. Since $\mathbb{X} \subset \tilde{\mathbb{X}}$ has finite index, for any $\sigma \in E$ there exists $n_\sigma \in \mathbb{Z}_{\geq 1}$ such that $g_\sigma := (f_\sigma)^{n_\sigma} \in A$. Then we claim that

$$A_\dagger \subset \text{Rad} \left(\sum_{\sigma \in E} A \cdot g_\sigma \right),$$

so that by Corollary 3.10 we have

$$\text{Proj}^{\mathbb{X}}(A) = \bigcup_{\sigma \in E} D_\dagger(g_\sigma).$$

In fact, by Lemma 4.3, to prove this inclusion it suffices to prove that

$$A_\lambda \subset \text{Rad} \left(\sum_{\sigma \in E} A \cdot g_\sigma \right) \quad \text{for any } \lambda \in \mathbb{X}_{++}. \quad (4.8)$$

Fix such λ . If $a \in A_\lambda$, there exist elements $(b_\sigma : \sigma \in E)$ in $\tilde{A}_{\lambda - \sum_\alpha \varpi_\alpha}$ such that $a = \sum_\sigma b_\sigma \cdot f_\sigma$. Then if $N > |E| \cdot \max_\sigma n_\sigma$, we have $a^N = \sum_\sigma c_\sigma g_\sigma$ for some homogeneous elements $c_\sigma \in \tilde{A}$, where $\deg(c_\sigma) + \deg(g_\sigma) = N\lambda$ for any σ . Here we have $\deg(c_\sigma) \in \mathbb{X}$, hence $c_\sigma \in A$, and the desired claim follows.

4.2 Some vector bundles over the flag variety

We continue with the setting of §4.1, and consider a finite-dimensional \mathbf{G} -module \tilde{V} and a \mathbf{B} -stable subspace $V \subset \tilde{V}$. We can then consider the induced scheme

$$\mathbf{G} \times^{\mathbf{B}} V,$$

i.e. the quotient of the product $\mathbf{G} \times V$ by the (free) action of \mathbf{B} defined by $b \cdot (g, x) = (gb^{-1}, b \cdot x)$. (This construction is a special case of that considered in [Ja03, §II.5.14].) It is a vector bundle over \mathbf{G}/\mathbf{B} (in particular, a smooth variety). For $\lambda \in \mathbb{X}$ we will denote by $\mathcal{O}_{\mathbf{G} \times^{\mathbf{B}} V}(\lambda)$ the pullback to $\mathbf{G} \times^{\mathbf{B}} V$ of the line bundle $\mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)$. By (4.1), for $\lambda, \mu \in \mathbb{X}$ we have a canonical isomorphism

$$\mathcal{O}_{\mathbf{G} \times^{\mathbf{B}} V}(\lambda) \otimes_{\mathcal{O}_{\mathbf{G} \times^{\mathbf{B}} V}} \mathcal{O}_{\mathbf{G} \times^{\mathbf{B}} V}(\mu) \cong \mathcal{O}_{\mathbf{G} \times^{\mathbf{B}} V}(\lambda + \mu),$$

which provides a structure of \mathbb{X} -graded ring on

$$A_V := \bigoplus_{\lambda \in \mathbb{X}_+} \Gamma(\mathbf{G} \times^{\mathbf{B}} V, \mathcal{O}_{\mathbf{G} \times^{\mathbf{B}} V}(\lambda)).$$

The \mathbf{B} -equivariant embedding $V \subset \tilde{V}$ induces a closed immersion

$$\mathbf{G} \times^{\mathbf{B}} V \hookrightarrow \mathbf{G} \times^{\mathbf{B}} \tilde{V}, \quad (4.9)$$

and the right-hand side identifies with $\mathbf{G}/\mathbf{B} \times \tilde{V}$ (via the map $[g, x] \mapsto (g\mathbf{B}, g \cdot x)$ for $g \in G$ and $x \in \tilde{V}$). For any $\lambda \in \mathbb{X}_+$ we deduce a canonical morphism

$$\mathcal{O}(\tilde{V}) \otimes A_\lambda \rightarrow (A_V)_\lambda. \quad (4.10)$$

Since \mathbf{G}/\mathbf{B} is projective, these considerations also show that there exists a canonical projective morphism

$$\mathbf{G} \times^{\mathbf{B}} V \rightarrow \tilde{V}. \quad (4.11)$$

Our goal in this subsection is to prove the following statement.

PROPOSITION 4.7. *There exists a canonical isomorphism*

$$\mathbf{G} \times^{\mathbf{B}} V \cong \text{Proj}^{\mathbb{X}}(A_V).$$

Remark 4.8. In case $V = \tilde{V} = \{0\}$, Proposition 4.7 recovers Proposition 4.1 (but our proof will use the latter statement). Another interesting case is when \tilde{V} is the Lie algebra \mathfrak{g} of \mathbf{G} and V is the Lie algebra \mathfrak{u} of the unipotent radical of \mathbf{B} . In this case, $\mathbf{G} \times^{\mathbf{B}} V$ is the so-called *Springer resolution*.

The proof of Proposition 4.7 will rely on the following preliminary result.

LEMMA 4.9. *There exists $N \in \mathbb{Z}_{>0}$ such that for any $\lambda \in \mathbb{X}^+$ which satisfies $\langle \lambda, \alpha^\vee \rangle \geq N$ for any simple root α , the morphism (4.10) is surjective.*

Proof. Consider the natural (affine) morphism $\pi : \mathbf{G} \times^{\mathbf{B}} V \rightarrow \mathbf{G}/\mathbf{B}$. If we denote by \mathcal{V} the vector bundle on \mathbf{G}/\mathbf{B} associated with the \mathbf{B} -module V (this vector bundle is denoted $\mathcal{L}_{\mathbf{G}/\mathbf{B}}(V)$ in [Ja03, § I.5.8]), and by \mathcal{V}^\vee the dual vector bundle, then $\pi_* \mathcal{O}_{\mathbf{G} \times^{\mathbf{B}} V}$ identifies with the symmetric algebra $S_{\mathcal{O}_{\mathbf{G}/\mathbf{B}}}(\mathcal{V}^\vee)$. Similarly, the pushforward to \mathbf{G}/\mathbf{B} of $\mathcal{O}_{\mathbf{G} \times^{\mathbf{B}} \tilde{V}}$ identifies with $\mathcal{O}(\tilde{V}^*) \otimes \mathcal{O}_{\mathbf{G}/\mathbf{B}}$, and the closed immersion (4.9) corresponds to a surjection of sheaves

$$\mathcal{O}(\tilde{V}) \otimes \mathcal{O}_{\mathbf{G}/\mathbf{B}} \twoheadrightarrow S_{\mathcal{O}_{\mathbf{G}/\mathbf{B}}}(\mathcal{V}^\vee).$$

Denoting by $\mathcal{V}^\perp \subset \tilde{V}^* \otimes \mathcal{O}_{\mathbf{G}/\mathbf{B}}$ the orthogonal of $\mathcal{V} \subset \tilde{V} \otimes \mathcal{O}_{\mathbf{G}/\mathbf{B}}$ (a sub-vector bundle), this surjection can be “extended” to the Koszul resolution

$$\bigwedge^{-\bullet}(\mathcal{V}^\perp) \otimes_{\mathcal{O}_{\mathbf{G}/\mathbf{B}}} \left(\mathcal{O}(\tilde{V}) \otimes \mathcal{O}_{\mathbf{G}/\mathbf{B}} \right) \rightarrow S_{\mathcal{O}_{\mathbf{G}/\mathbf{B}}}(\mathcal{V}^\vee),$$

a quasi-isomorphism of complexes of (quasi-coherent) $\mathcal{O}_{\mathbf{G}/\mathbf{B}}$ -modules. For any $\lambda \in \mathbb{X}$, tensoring with $\mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)$ we deduce a quasi-isomorphism

$$\bigwedge^{-\bullet}(\mathcal{V}^\perp) \otimes_{\mathcal{O}_{\mathbf{G}/\mathbf{B}}} \left(\mathcal{O}(\tilde{V}) \otimes \mathcal{O}_{\mathbf{G}/\mathbf{B}} \right) \otimes_{\mathcal{O}_{\mathbf{G}/\mathbf{B}}} \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda) \rightarrow \pi_* \mathcal{O}_{\mathbf{G} \times^{\mathbf{B}} V}(\lambda). \quad (4.12)$$

Now \mathcal{V}^\perp is the vector bundle on \mathbf{G}/\mathbf{B} associated with the orthogonal $V^\perp \subset \tilde{V}^*$ of $V \subset \tilde{V}$. Hence, if we denote by $\Lambda \subset \mathbb{X}$ the (finite) subset consisting of the T -weights appearing in the various exterior powers of V^\perp , then each $\bigwedge^i \mathcal{V}^\perp$ is an iterated extension of line bundles of the form $\mathcal{O}_{\mathbf{G}/\mathbf{B}}(\mu)$ with $\mu \in \Lambda$. If we choose N such that $\langle \mu, \alpha^\vee \rangle \geq -N$ for any $\mu \in \Lambda$ and any simple root α , then if $\lambda \in \mathbb{X}_+$ satisfies $\langle \lambda, \alpha^\vee \rangle \geq N$ for any simple root α , each weight $\lambda + \mu$ with $\mu \in \Lambda$ is dominant. Since line bundles on \mathbf{G}/\mathbf{B} associated with dominant weights have no higher cohomology (this is Kempf’s vanishing theorem, see [Ja03, Proposition II.4.5]), for such λ we have

$$H^n(\bigwedge^i \mathcal{V}^\perp \otimes_{\mathcal{O}_{\mathbf{G}/\mathbf{B}}} \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)) = 0 \quad \text{for any } i \text{ and any } n > 0.$$

Breaking the resolution (4.12) into short exact sequences and using the long exact sequences obtained by applying the global sections functor and its derived functors, we deduce in this case a quasi-isomorphism

$$\mathcal{O}(\tilde{V}) \otimes \Gamma\left(\mathbf{G}/\mathbf{B}, \bigwedge^{-\bullet}(\mathcal{V}^\perp) \otimes_{\mathcal{O}_{\mathbf{G}/\mathbf{B}}} \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)\right) \rightarrow \Gamma(\mathbf{G}/\mathbf{B}, \pi_* \mathcal{O}_{\mathbf{G} \times^{\mathbf{B}} V}(\lambda)) = (A_V)_\lambda,$$

proving in particular that the morphism of the lemma is surjective. \square

Proof of Proposition 4.7. As in the second proof of Proposition 4.1, one can (and will) assume that \mathbf{G} is semisimple and simply connected. Here again, to prove the proposition we will show that the conditions in [BS03, Corollary 4.6] are satisfied.

Recall the notation of Remark 4.6 and, for any σ , denote by \tilde{f}_σ the image of f_σ in A_V . We claim that

$$\text{Proj}^{\mathbb{X}}(A_V) = \bigcup_{\sigma \in E} D_{\dagger}(\tilde{f}_\sigma). \quad (4.13)$$

In fact, as in the proof of Lemma 4.3 one sees that the degree of any relevant element f in A_V belongs to \mathbb{X}_{++} . If N satisfies the condition in Lemma 4.9, then f^N belongs to the image of (4.10) for $\lambda = N \deg(f)$, hence to the ideal generated by the elements \tilde{f}_σ . This implies the claim in view of Corollary 3.10.

Next, we will prove that the scheme $\text{Proj}^{\mathbb{X}}(A_V)$ is separated. In fact, combining the morphisms (4.10) we obtain a canonical morphism of \mathbb{X} -graded rings

$$\mathcal{O}(\tilde{V}) \otimes A \rightarrow A_V. \quad (4.14)$$

(Here, the grading on the left-hand side is such that the degree- λ component is $\mathcal{O}(\tilde{V}) \otimes A_\lambda$.) Consider the associated rational morphism

$$\mathrm{Proj}^{\mathbb{X}}(A_V) \dashrightarrow \mathrm{Proj}^{\mathbb{X}}(\mathcal{O}(\tilde{V}) \otimes A) = \tilde{V} \times \mathrm{Proj}^{\mathbb{X}}(A) \quad (4.15)$$

provided by Proposition 3.12, where the identification is provided by Corollary 3.17. By (4.13) this morphism is defined everywhere. We claim that it is a closed immersion. In fact, by Remarks 3.18 and 4.6 we have

$$\mathrm{Proj}^{\mathbb{X}}(\mathcal{O}(\tilde{V}) \otimes A) = \bigcup_{\sigma \in E} D_{\dagger}(1 \otimes f_{\sigma}).$$

Using the fact that the property of being a closed immersion is local on the target (see [StP, Tag 02L6]), we deduce that to prove the claim it suffices to prove that for any $\sigma \in E$ the projection

$$\mathrm{Proj}^{\mathbb{X}}(A_V) \times_{\mathrm{Proj}^{\mathbb{X}}(\mathcal{O}(\tilde{V}) \otimes A)} D_{\dagger}(1 \otimes f_{\sigma}) \rightarrow D_{\dagger}(1 \otimes f_{\sigma})$$

is a closed immersion. Fix such a σ . By (3.6), we have

$$\mathrm{Proj}^{\mathbb{X}}(A_V) \times_{\mathrm{Proj}^{\mathbb{X}}(\mathcal{O}(\tilde{V}) \otimes A)} D_{\dagger}(1 \otimes f_{\sigma}) = D_{\dagger}(\tilde{f}_{\sigma}).$$

Therefore, to conclude it suffices to prove that the morphism $\mathcal{O}(\tilde{V}) \otimes A_{(f_{\sigma})} \rightarrow (A_V)_{(\tilde{f}_{\sigma})}$ is surjective. But, if N is as above any element in $(A_V)_{(\tilde{f}_{\sigma})}$ can be written in the form $\frac{a}{(\tilde{f}_{\sigma})^{kN}}$ for some $k > 0$, where $a \in A$ is homogeneous of degree $kN \deg(\tilde{f}_{\sigma})$. Then a belongs to the image of $\mathcal{O}(\tilde{V}) \otimes A$, which implies the desired surjectivity.

By Proposition 4.1 the right-hand side in (4.15) is separated, hence as in the second proof of Proposition 4.1 we deduce that $\mathrm{Proj}^{\mathbb{X}}(A_V)$ is separated.

Next, we show that the natural morphism

$$\mathbf{G} \times^{\mathbf{B}} V \rightarrow (\mathbf{G} \times^{\mathbf{B}} V)_{\mathrm{aff}}$$

is proper. In fact this map fits in the commutative diagram

$$\begin{array}{ccc} \mathbf{G} \times^{\mathbf{B}} V & \longrightarrow & \mathrm{Spec}(\mathcal{O}(\mathbf{G} \times^{\mathbf{B}} V)) \\ & \searrow & \swarrow \\ & \tilde{V} & \end{array} \quad (4.11)$$

Here the left diagonal arrow is proper, and the right diagonal arrow is induced by the corresponding morphism on function algebras. The latter morphism is separated (as is any morphism whose domain is affine, see [StP, Tag 01KN]), hence the horizontal morphism is proper by [StP, Tag 01W6].

Finally, we observe that if $\lambda \in \mathbb{X}_{++}$ the line bundle $\mathcal{O}_{\mathbf{G} \times^{\mathbf{B}} V}(\lambda)$ is ample by [StP, Tag 0892] and the similar claim for \mathbf{G}/\mathbf{B} . As in the second proof of Proposition 4.1, these arguments show that the conditions in [BS03, Corollary 4.6] are satisfied, and this statement provides the desired identification of $\mathbf{G} \times^{\mathbf{B}} V$ with $\mathrm{Proj}^{\mathbb{X}}(A_V)$. \square

Remark 4.10. Recall the notation from Remark 4.6 and, for $\sigma \in E$, denote by \tilde{g}_{σ} the image of g_{σ} in A_V . We claim that

$$(A_V)_{\dagger} \subset \mathrm{Rad} \left(\sum_{\sigma \in E} (A_V) \cdot \tilde{g}_{\sigma} \right),$$

so that as in this remark we have

$$\mathrm{Proj}^{\mathbb{X}}(A_V) = \bigcup_{\sigma \in E} D_{\dagger}(\tilde{g}_{\sigma}).$$

(This generalizes (4.13), proved under the assumption that \mathbf{G} is semisimple and simply connected.) In fact, as in the proof of Lemma 4.3 the degree of any relevant element f in A_V belongs to \mathbb{X}_{++} ; to conclude, it therefore suffices to prove that we have

$$(A_V)_{\lambda} \subset \mathrm{Rad} \left(\sum_{\sigma \in E} (A_V) \cdot \tilde{g}_{\sigma} \right) \quad \text{for any } \lambda \in \mathbb{X}_{++}.$$

Fix such a λ , and let $a \in (A_V)_{\lambda}$. If N is as in Lemma 4.9, then a^N belongs to the image of $\mathcal{O}(\tilde{V}) \otimes A_{N\lambda}$; i.e. there exists a finite set I and elements $(b_i : i \in I)$ in $\mathcal{O}(\tilde{V})$ and $(c_i : i \in I)$ in $A_{N\lambda}$ such that a^N is the image of $\sum_{i \in I} b_i \otimes c_i$. By (4.8), for any $i \in I$ there exists $n_i \in \mathbb{Z}_{>0}$ such that $(c_i)^{n_i} \in \sum_{\sigma} A \cdot g_{\sigma}$. Then if $M \in \mathbb{Z}$ satisfies $M > |I| \cdot (\max_i n_i - 1)$ we have $a^{MN} \in \sum_{\sigma} (A_V) \cdot \tilde{g}_{\sigma}$, which concludes the proof.

4.3 The Springer resolution

We continue with the setting of § 4.2, in the special case when \tilde{V} is the Lie algebra \mathfrak{g} of \mathbf{G} and V is the Lie algebra \mathfrak{u} of the unipotent radical \mathbf{U} of \mathbf{B} . In this case the vector bundle $\mathbf{G} \times^{\mathbf{B}} \mathfrak{u}$ is called the *Springer resolution*, and usually denoted $\tilde{\mathcal{N}}$. (The name is justified by the fact that, under appropriate technical assumptions, $\tilde{\mathcal{N}}$ is a resolution of singularities of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ of \mathbf{G} , first introduced by Springer.) This variety appears in numerous works in Geometric Representation Theory, among which [AB09] (in the case when \mathbb{k} is an algebraic closure of the field \mathbb{Q}_{ℓ}). In that reference the authors introduce an \mathbb{X} -graded ring, the spectrum of which is denoted $\hat{\mathcal{N}}_{af}$ in [AB09, § 3.1]. This construction is reproduced in [AR, § 6.2.2], where the spectrum is denoted $\hat{\mathcal{N}}_{\mathcal{X}}$. The definition makes sense over an arbitrary algebraically closed field; here, we will denote this ring by $A'_{\mathfrak{u}}$. By construction, we have morphisms of \mathbb{X} -graded rings

$$\mathcal{O}(\mathfrak{g}) \otimes A \rightarrow A'_{\mathfrak{u}} \rightarrow A_{\mathfrak{u}},$$

the first of which is surjective. (The main point of this construction is that $A'_{\mathfrak{u}}$ is a somewhat explicit quotient of $\mathcal{O}(\mathfrak{g}) \otimes A$, whereas $A_{\mathfrak{u}}$ does not have a very explicit description.)

LEMMA 4.11. *There exists a canonical isomorphism*

$$\mathrm{Proj}^{\mathbb{X}}(A_{\mathfrak{u}}) \xrightarrow{\sim} \mathrm{Proj}^{\mathbb{X}}(A'_{\mathfrak{u}}).$$

Proof. The morphism $A'_{\mathfrak{u}} \rightarrow A_{\mathfrak{u}}$ provides a rational morphism $\mathrm{Proj}^{\mathbb{X}}(A_{\mathfrak{u}}) \dashrightarrow \mathrm{Proj}^{\mathbb{X}}(A'_{\mathfrak{u}})$, see Proposition 3.12. As in the proof of Proposition 4.7, it follows from (4.13) that this morphism is defined everywhere. To show that it is an isomorphism, we will construct an isomorphism $\mathrm{Proj}^{\mathbb{X}}(A'_{\mathfrak{u}}) \cong \tilde{\mathcal{N}}$, under which this morphism is the identity.

Consider the morphisms

$$\mathrm{Spec}(A'_{\mathfrak{u}}) \hookrightarrow \mathrm{Spec}(\mathcal{O}(\mathfrak{g}) \otimes_{\mathbb{k}} A) = \mathfrak{g} \times_{\mathrm{Spec}(\mathbb{k})} \mathrm{Spec}(A) \rightarrow \mathrm{Spec}(A),$$

where the left arrow is the closed immersion induced by the surjection $\mathcal{O}(\mathfrak{g}) \otimes A \rightarrow A'_{\mathfrak{u}}$, and the right one is the obvious projection. By Remark 2.6, $V((\mathcal{O}(\mathfrak{g}) \otimes_{\mathbb{k}} A)_{\dagger})$ is the preimage of $V(A_{\dagger})$ under the second morphism, and by Remark 2.8 $V((A'_{\mathfrak{u}})_{\dagger})$ is the preimage of $V((\mathcal{O}(\mathfrak{g}) \otimes_{\mathbb{k}} A)_{\dagger})$ under the first morphism. Combining these informations, we obtain that $\mathrm{Spec}(A'_{\mathfrak{u}}) \setminus V((A'_{\mathfrak{u}})_{\dagger})$ is the preimage of $\mathrm{Spec}(A) \setminus V(A_{\dagger})$, which we have identified with \mathbf{G}/\mathbf{U} in (4.5). By [AR,

Equation (6.2.10)] this preimage is the variety denoted $\widehat{\mathcal{N}}$ in [AR, § 6.2], whose quotient by the action of \mathbf{T} is $\widetilde{\mathcal{N}}$ (see the discussion preceding [AR, Equation 6.2.7]). This concludes the proof. \square

Remark 4.12. (i) The second part of the proof can alternatively be replaced by an argument based on the fact that the morphism $(A'_{\mathbf{u}})_{\lambda} \rightarrow (A_{\mathbf{u}})_{\lambda}$ is an isomorphism if λ is sufficiently dominant, see [AR, Lemma 6.2.4].

(ii) The same comments as in Remark 4.10 apply in this case.

5. Quasi-coherent sheaves on Proj schemes

In this section we study quasi-coherent sheaves on Proj schemes, generalizing classical results for Proj schemes of \mathbb{N} -graded rings. Our presentation follows and extends [StP, Tag 01MJ] and [StP, Tag 01MM]. Other pioneering works on quasi-coherent sheaves on Brenner-Schröer Proj may be found in [MRo24].

Most of the results of this section have obvious analogues in the relative setting of § 3.4. We leave it to the reader to formulate these variants and adapt the proofs.

5.1 Sheaves associated with graded modules

We proceed with the notation of § 3.1; in particular we fix a finitely generated abelian group M and a commutative M -graded ring A . Recall also the notation \mathcal{F}_A .

Let Q be an M -graded A -module. For any homogeneous multiplicative subset $S \subset A$, we have considered in Definition 2.9 the $A_{(S)}$ -module $Q_{(S)}$.

FACT 5.1. *There exists a unique quasi-coherent $\mathcal{O}_{\mathrm{Proj}^M(A)}$ -module \widetilde{Q} such that*

$$\Gamma(D_{\dagger}(S), \widetilde{Q}) = Q_{(S)}$$

for every $S \in \mathcal{F}_A$.

Proof. This is immediate by glueing. \square

It is clear that the assignment $Q \mapsto \widetilde{Q}$ is functorial. More specifically, denoting by $\mathrm{Mod}^M(A)$ the abelian category of M -graded A -modules, and by $\mathrm{QCoh}(\mathrm{Proj}^M(A))$ the abelian category of quasi-coherent sheaves on the scheme $\mathrm{Proj}^M(A)$, this assignment defines a functor

$$\mathrm{Mod}^M(A) \rightarrow \mathrm{QCoh}(\mathrm{Proj}^M(A)),$$

which is exact by exactness of localization. Note that this functor commutes with all colimits. (This follows from the facts that restriction to open subscheme and localization commute with colimits.)

An M -graded A -module Q will be called *negligible* if $\widetilde{Q} = 0$. We will denote by $\mathrm{Mod}^M(A)_{\mathrm{neg}}$ the full subcategory of $\mathrm{Mod}^M(A)$ whose objects are the negligible modules. Since the functor $Q \mapsto \widetilde{Q}$ is exact, this is a Serre subcategory, see [StP, Tag 02MQ], and our functor factors through an exact functor

$$L : \mathrm{Mod}^M(A) / \mathrm{Mod}^M(A)_{\mathrm{neg}} \rightarrow \mathrm{QCoh}(\mathrm{Proj}^M(A)).$$

see [StP, Tag 02MS].

FACT 5.2. *There is a canonical morphism of A_0 -modules*

$$Q_0 \rightarrow \Gamma(\mathrm{Proj}^M(A), \tilde{Q})$$

such that for any $S \in \mathcal{F}_A$ the composition $Q_0 \rightarrow \Gamma(\mathrm{Proj}^M(A), \tilde{Q}) \rightarrow \Gamma(D_{\dagger}(S), \tilde{Q}) = Q_{(S)}$ coincides with the map $Q_0 \rightarrow Q_{(S)}$ given by $x \mapsto \frac{x}{1}$.

Proof. The morphisms $Q_0 \rightarrow Q_{(S)}$ given by $x \mapsto \frac{x}{1}$, where S runs through \mathcal{F}_A , are compatible by restriction, hence glue to the desired morphism. \square

The next proposition studies the relation between tensor products of graded modules and of quasi-coherent sheaves.

PROPOSITION 5.3. *Let P, Q be graded A -modules. There is a canonical map of $\mathcal{O}_{\mathrm{Proj}^M(A)}$ -modules*

$$\tilde{P} \otimes_{\mathcal{O}_X} \tilde{Q} \longrightarrow \widetilde{P \otimes_A Q}$$

which induces, for any $S \in \mathcal{F}_A$, the canonical map

$$P_{(S)} \otimes_{A_{(S)}} Q_{(S)} \rightarrow (P \otimes_A Q)_{(S)}$$

on sections over $D_{\dagger}(S)$. Moreover, the diagram

$$\begin{array}{ccc} P_0 \otimes_{A_0} Q_0 & \longrightarrow & (P \otimes_A Q)_0 \\ \downarrow & & \downarrow \\ \Gamma(\mathrm{Proj}^M(A), \tilde{P} \otimes_{\mathcal{O}_X} \tilde{Q}) & \longrightarrow & \Gamma(\mathrm{Proj}^M(A), \widetilde{P \otimes_A Q}) \end{array}$$

commutes, where the upper horizontal arrow is the natural map, and the vertical ones are induced by the morphisms from Fact 5.2.

Proof. Constructing a morphism as displayed is equivalent to constructing an $\mathcal{O}_{\mathrm{Proj}^M(A)}$ -bilinear map

$$\tilde{P} \times \tilde{Q} \longrightarrow \widetilde{P \otimes_A Q},$$

see [StP, Tag 01CA]. It suffices to define this map on sections over the opens $(D_{\dagger}(S) : S \in \mathcal{F}_A)$ compatible with restriction maps. On $D_{\dagger}(S)$, with $S = \{a_i : i \in I\}$, we use the $A_{(S)}$ -bilinear map $P_{(S)} \times Q_{(S)} \rightarrow (P \otimes_A Q)_{(S)}$ given by $(x/a^{\nu}, y/a^{\nu'}) \mapsto (x \otimes y)/a^{\nu+\nu'}$. The commutation of the diagram follows from definitions. \square

In general, the morphism of Proposition 5.3 is not an isomorphism, as seen already in the \mathbb{N} -graded setting, see [StP, Tag 01ML].

Remark 5.4. Consider the setting of Remark 3.7, and denote by $\mathrm{Mod}^{M,H}(A)$ the category of H -equivariant M -graded A -modules. Let also $\mathrm{Mod}^{M,H}(A)_{\mathrm{neg}}$ be the Serre subcategory consisting of objects which are negligible as M -graded A -modules. On the other hand, consider the category $\mathrm{QCoh}^H(\mathrm{Proj}^M(A))$ of H -equivariant quasi-coherent sheaves on $\mathrm{Proj}^M(A)$. (For a review of equivariant quasi-coherent sheaves, see e.g. [MRi16, Appendix A].) Then very similar considerations to those about allow to construct an exact functor

$$\mathrm{Mod}^{M,H}(A) \rightarrow \mathrm{QCoh}^H(\mathrm{Proj}^M(A))$$

which factors through the quotient $\mathrm{Mod}^{M,H}(A)/\mathrm{Mod}^{M,H}(A)_{\mathrm{neg}}$.

5.2 Twisting sheaves

We now define the versions in our setting of the twisting sheaves from [StP, Tag 01MN]. If Q is a graded A -module and $\alpha \in M$, we will denote by $Q(\alpha)$ the M -graded A -module which coincides with Q as an A -module, but with the M -grading defined by $(Q(\alpha))_\beta = Q_{\alpha+\beta}$ for $\beta \in M$.

DEFINITION 5.5 (Twisting sheaves). Let $\alpha \in M$.

- (i) The quasi-coherent sheaf $\widetilde{A(\alpha)}$ is called the α -th-twist of the structure sheaf of $\text{Proj}^M(A)$, and is denoted $\mathcal{O}_{\text{Proj}^M(A)}(\alpha)$.
- (ii) If \mathcal{Q} is a sheaf of $\mathcal{O}_{\text{Proj}^M(A)}$ -modules, we set $\mathcal{Q}(\alpha) = \mathcal{O}_{\text{Proj}^M(A)}(\alpha) \otimes_{\mathcal{O}_{\text{Proj}^M(A)}} \mathcal{Q}$.

Recall that if A is noetherian as a non-graded ring, then by [BS03, Lemma 2.4] the ring A_0 is noetherian, and by [BS03, Proposition 2.5] the canonical morphism $\text{Proj}^M(A) \rightarrow \text{Spec}(A_0)$ is of finite type; as a consequence, $\text{Proj}^M(A)$ is a noetherian scheme.

LEMMA 5.6. Assume that A is a noetherian ring.

- (i) For any $\alpha \in M$, the quasi-coherent sheaf $\mathcal{O}_{\text{Proj}^M(A)}(\alpha)$ is coherent.
- (ii) If Q is a finitely generated M -graded A -module, then \widetilde{Q} is coherent.

Proof. (i) We need to prove that for any $S \in \mathcal{F}_A$ the $A_{(S)}$ -module $(A(\alpha))_{(S)}$ is finitely generated. Here $A_{(S)}$, resp. $(A(\alpha))_{(S)}$, is the degree-0, resp. degree- α , component in the localization A_S . By [BS03, Lemma 2.4], A_S is noetherian. By a simple argument (see e.g. [GY83, Lemma 2.2]), this implies that each of its graded components is finitely generated over its degree-0 part, which concludes the proof.

(ii) Any finitely generated M -graded A -module is a quotient of a finite direct sum of modules $A(\alpha)$ with $\alpha \in M$. The claim therefore follows from (i) and exactness of the functor $Q \mapsto \widetilde{Q}$, since finite direct sums and quotients of coherent sheaves by quasi-coherent subsheaves are coherent. \square

We now drop the assumption that A is noetherian. Note that $\mathcal{O}_{\text{Proj}^M(A)}(\alpha)$ is not a line bundle in general, even in the \mathbb{N} -graded setting. If $\alpha, \alpha' \in M$, since $A(\alpha) \otimes_A A(\alpha') = A(\alpha + \alpha')$, Proposition 5.3 implies that there is a canonical map

$$\mathcal{O}_{\text{Proj}^M(A)}(\alpha) \otimes_{\mathcal{O}_{\text{Proj}^M(A)}} \mathcal{O}_{\text{Proj}^M(A)}(\alpha') \longrightarrow \mathcal{O}_{\text{Proj}^M(A)}(\alpha + \alpha'). \quad (5.1)$$

These maps define on

$$\bigoplus_{\alpha \in M} \mathcal{O}_{\text{Proj}^M(A)}(\alpha) \quad (5.2)$$

a structure of M -graded sheaf of $\mathcal{O}_{\text{Proj}^M(A)}$ -algebras, and then an M -graded ring structure on

$$\bigoplus_{\alpha \in M} \Gamma(\text{Proj}^M(A), \mathcal{O}_{\text{Proj}^M(A)}(\alpha)). \quad (5.3)$$

Note that the morphism (5.1) is not an isomorphism in general. (Again, this can already be false in the \mathbb{N} -graded setting.)

More generally, if Q is an M -graded A -module, and if $\alpha, \alpha' \in M$, we also have $A(\alpha) \otimes_A Q(\alpha') = Q(\alpha + \alpha')$; Proposition 5.3 therefore provides a canonical morphism

$$\mathcal{O}_{\text{Proj}^M(A)}(\alpha) \otimes_{\mathcal{O}_{\text{Proj}^M(A)}} \widetilde{Q(\alpha')} \rightarrow \widetilde{Q(\alpha + \alpha')}. \quad (5.4)$$

These maps define on

$$\bigoplus_{\alpha \in M} \widetilde{Q}(\alpha)$$

a structure of M -graded sheaf of modules over (5.2), and on

$$\bigoplus_{\alpha \in M} \Gamma(\text{Proj}^M(A), \widetilde{Q}(\alpha))$$

a structure of M -graded module over (5.3).

FACT 5.7. (i) *There is a canonical morphism of graded rings*

$$A \rightarrow \bigoplus_{\alpha \in M} \Gamma(\text{Proj}^M(A), \mathcal{O}_{\text{Proj}^M(A)}(\alpha)).$$

(ii) *For any graded A -module Q we have a canonical morphism of M -graded abelian groups*

$$Q \rightarrow \bigoplus_{\alpha \in M} \Gamma(X, \widetilde{Q}(\alpha)).$$

which is a morphism of M -graded A -modules with respect to the structure on the right-hand side provided by (i).

Proof. The morphisms are given by Fact 5.2, after noticing that $A_\alpha = (A(\alpha))_0$ and $Q_\alpha = (Q(\alpha))_0$. The fact that the morphism in (i) is a ring morphism follows from Proposition 5.3. \square

5.3 Graded modules associated with sheaves

For any $\mathcal{O}_{\text{Proj}^M(A)}$ -module \mathcal{Q} , the morphism (5.1) induces a morphism

$$\mathcal{O}_{\text{Proj}^M(A)}(\alpha) \otimes_{\mathcal{O}_{\text{Proj}^M(A)}} \mathcal{Q}(\alpha') \longrightarrow \mathcal{Q}(\alpha + \alpha'), \quad (5.5)$$

and these morphisms define on the direct sum

$$\bigoplus_{\alpha \in M} \mathcal{Q}(\alpha)$$

the structure of an M -graded sheaf of modules over (5.2), and on

$$\bigoplus_{\alpha \in M} \Gamma(\text{Proj}^M(A), \mathcal{Q}(\alpha))$$

a structure of M -graded module over the ring (5.3), hence over the ring A by Fact 5.7(i). Denoting by $\mathbf{Mod}(\mathcal{O}_{\text{Proj}^M(A)})$ the abelian category of sheaves of $\mathcal{O}_{\text{Proj}^M(A)}$ -modules, this construction provides a functor

$$\Gamma_\bullet : \mathbf{Mod}(\mathcal{O}_{\text{Proj}^M(A)}) \rightarrow \mathbf{Mod}^M(A).$$

In the following proposition we use the notation $M[\underline{S}]$ introduced in § 2.2.

PROPOSITION 5.8. *Let $S \in \mathcal{F}_A$, and let $\alpha \in M[\underline{S}]$.*

- (i) *The sheaf $\mathcal{O}_{\text{Proj}^M(A)}(\alpha)|_{D_\dagger(S)}$ is invertible, and in fact isomorphic to $(\mathcal{O}_{\text{Proj}^M(A)})|_{D_\dagger(S)}$.*
- (ii) *For any M -graded A -module Q , the morphism*

$$\widetilde{Q}(\alpha)|_{D_\dagger(S)} \rightarrow \widetilde{Q}(\alpha)|_{D_\dagger(S)}$$

obtained by restriction from (5.4) is an isomorphism.

(iii) For any $\alpha' \in M$, the morphism

$$\mathcal{O}_{\text{Proj}^M(A)}(\alpha)|_{D_{\dagger}(S)} \otimes_{\mathcal{O}_{D_{\dagger}(S)}} \mathcal{O}_{\text{Proj}^M(A)}(\alpha')|_{D_{\dagger}(S)} \rightarrow \mathcal{O}_{\text{Proj}^M(A)}(\alpha + \alpha')|_{D_{\dagger}(S)}$$

obtained by restriction from (5.1) is an isomorphism.

(iv) For any $\mathcal{O}_{\text{Proj}^M(A)}$ -module \mathcal{Q} and any $\alpha' \in M$, the morphism

$$\mathcal{O}_{\text{Proj}^M(A)}(\alpha)|_{D_{\dagger}(S)} \otimes_{\mathcal{O}_{D_{\dagger}(S)}} \mathcal{Q}(\alpha')|_{D_{\dagger}(S)} \rightarrow \mathcal{Q}(\alpha + \alpha')|_{D_{\dagger}(S)}$$

obtained by restriction from (5.5) is an isomorphism.

Proof. Since $\alpha \in M[\underline{S}]$, there exists an invertible element $a \in A_S$ of degree α . Then the map $x \mapsto a \cdot x$ induces an isomorphism of $A_{(S)}$ -modules $A_{(S)} \cong (A(\alpha))_{(S)}$. We deduce (i).

Similarly, given a graded A -module Q , the map $q \mapsto a \cdot q$ induces an isomorphism of $A_{(S)}$ -modules $Q_{(S)} \cong (Q(\alpha))_{(S)}$, which implies (ii). The statement in (iii) is the special case of (ii) where $Q = A(\alpha')$. Finally, (iv) is a direct consequence of (iii). \square

LEMMA 5.9. For any $\mathcal{Q} \in \text{Mod}(\mathcal{O}_{\text{Proj}^M(A)})$, there exists a canonical (in particular, functorial) morphism

$$\widetilde{\Gamma_{\bullet}(\mathcal{Q})} \rightarrow \mathcal{Q}$$

in $\text{Mod}(\mathcal{O}_{\text{Proj}^M(A)})$.

Proof. Since $\text{Proj}^M(A)$ is covered (by definition) by the open subschemes $D_{\dagger}(S)$, to prove the lemma it suffices to construct morphisms

$$\widetilde{\Gamma_{\bullet}(\mathcal{Q})}|_{D_{\dagger}(S)} \rightarrow \mathcal{Q}|_{D_{\dagger}(S)} \tag{5.6}$$

for $S \in \mathcal{F}_A$, which coincide on intersections of such open subschemes. And given $S \in \mathcal{F}_A$, by basic properties of quasi-coherent sheaves on affine schemes (see [StP, Tag 01I7]), to define such a morphism it suffices to define a morphism of $A_{(S)}$ -modules

$$\Gamma_{\bullet}(\mathcal{Q})_{(S)} \rightarrow \Gamma(D_{\dagger}(S), \mathcal{Q}). \tag{5.7}$$

Now an element of $\Gamma_{\bullet}(\mathcal{Q})_{(S)}$ can be represented by a fraction $\frac{m}{s}$ where $s \in S$ and m is a homogeneous element of $\Gamma_{\bullet}(\mathcal{Q})$ of degree $\deg(s)$, i.e. a global section of $\mathcal{Q}(\deg(s))$. Consider

$$m|_{D_{\dagger}(S)} \in \Gamma(D_{\dagger}(S), \mathcal{Q}(\deg(s))).$$

The element $\frac{1}{s} \in A_S$ has degree $-\deg(s)$, hence defines a section of $\mathcal{O}_{\text{Proj}^M(S)}(-\deg(s))$ on $D_{\dagger}(S)$, which we denote s^{-1} . Then the product $m|_{D_{\dagger}(S)} \otimes s^{-1}$ defines a section of

$$\mathcal{Q}(\deg(s)) \otimes_{\mathcal{O}_{\text{Proj}^M(A)}} \mathcal{O}_{\text{Proj}^M(A)}(-\deg(s)) \cong \mathcal{Q}$$

over $D_{\dagger}(S)$. (Here the isomorphism is provided by Proposition 5.8(iv).)

One easily checks that this section is independent of the representation of the element of $\Gamma_{\bullet}(\mathcal{Q})_{(S)}$ as a fraction, hence that this process defines a map (5.7), and then that this map is a morphism of $A_{(S)}$ -modules. One also easily sees that the associated morphisms (5.6) glue on the intersections of open subschemes $D_{\dagger}(S)$, hence provide the desired morphism of sheaves. \square

5.4 Maximally relevant families

A family $S \in \mathcal{F}_A$ will be called *maximally relevant* if $M[\underline{S}] = M$. A homogeneous element $f \in A$ will be called maximally relevant if $f^{\mathbb{N}}$ is maximally relevant. We will denote by $\mathcal{F}_A^{\text{m}} \subset \mathcal{F}_A$

the subset consisting of maximally relevant families. In this subsection we will explore various consequences of the following condition (which may or may not hold, depending on A):

$$\mathrm{Proj}^M(A) = \bigcup_{S \in \mathcal{F}_A^m} D_{\dagger}(S). \quad (5.8)$$

Remark 5.10. If in the setting of Corollary 3.10 one can choose the a_i 's to be maximally relevant, then condition (5.8) holds. This setting covers at least some cases of interest, as follows.

- (i) Assume that $M = \mathbb{Z}$, and that A is generated by A_1 as an A_0 -algebra. (In particular, we are in the \mathbb{N} -graded case.) If $(a_i : i \in I)$ is a family of generators of A_1 as an A_0 -module, since any relevant element must have positive degree the condition of Corollary 3.10 is satisfied. Clearly, each a_i is maximally relevant.
- (ii) Consider the setting of § 4.1. We have explained in Remark 4.6 how to construct a family $(g_{\sigma} : \sigma \in E)$ for which Corollary 3.10 applies. We claim that any g_{σ} in this family is maximally relevant. In fact, fix σ , and recall the notation introduced in this remark. Since \mathbb{X} is generated by \mathbb{X}_{+} as a group, it suffices to justify that $\mathbb{X}[(g_{\sigma})^{\mathbb{N}}]$ contains \mathbb{X}^{+} . Let $\lambda \in \mathbb{X}$, and write $\lambda = \sum_{\alpha} m_{\alpha} \varpi_{\alpha} + \lambda_0$ with $m_{\alpha} \in \mathbb{N}$ and $\lambda_0 \in \tilde{\mathbb{X}}_0$ in the decomposition (4.3) (for the group $\tilde{\mathbf{G}}$). If a is a nonzero vector in the 1-dimensional vector space \tilde{A}_{λ_0} , then the element $a_0 \cdot \prod_{\alpha} (f_{\sigma(\alpha)}^{\alpha})^{m_{\alpha}}$ is of weight λ hence belongs to A , and divides a power of g_{σ} (in \tilde{A} , hence in A for weight reasons), which justifies that $\lambda \in \mathbb{X}[(g_{\sigma})^{\mathbb{N}}]$.
- (iii) Consider the setting of § 4.2. In Remark 4.10, we have constructed a family $(\tilde{g}_{\sigma} : \sigma \in E)$ for which Corollary 3.10 applies. It follows from the case treated above using the morphism (4.14) that each \tilde{g}_{σ} is maximally relevant. Similar comments apply in the setting of § 4.3.

First, the following statement is a direct consequence of Proposition 5.8.

COROLLARY 5.11. *Assume that (5.8) is satisfied. Then for any $\alpha \in M$, the quasi-coherent $\mathcal{O}_{\mathrm{Proj}^M(A)}$ -module $\mathcal{O}_{\mathrm{Proj}^M(A)}(\alpha)$ is a line bundle. Moreover:*

- (i) *the morphism (5.1) is an isomorphism for any $\alpha, \alpha' \in M$;*
- (ii) *the morphism (5.4) is an isomorphism for any $\alpha, \alpha' \in M$ and any M -graded A -module Q ;*
- (iii) *the morphism (5.5) is an isomorphism for any $\alpha, \alpha' \in M$ and any $\mathcal{O}_{\mathrm{Proj}^M(A)}$ -module Q .*

Remark 5.12. In the setting of § 4.1, for $\lambda \in \mathbb{X}$ the line bundle $\mathcal{O}_{\mathrm{Proj}^{\mathbb{X}}(A)}(\lambda)$ corresponds to the line bundle $\mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)$. A similar comment applies in the setting considered in § 4.2.

PROPOSITION 5.13. *Assume that (5.8) is satisfied, and that M is a free abelian group. Then the natural morphism*

$$\mathrm{Spec}(A) \setminus V(A_{\dagger}) \rightarrow \mathrm{Proj}^M(A)$$

(see (3.4)) is a Zariski trivial principal bundle for the group scheme $D_{\mathrm{Spec}(A_0)}(M)$.

Proof. By assumption $\mathrm{Proj}^M(A)$ is covered by the open subschemes $D_{\dagger}(S)$ where S is maximally relevant. Now for such S , the ring A_S contains invertible elements in each degree. Choose an isomorphism $M = \mathbb{Z}^n$ and, for any $i \in \{1, \dots, n\}$, choose an invertible element a_i in A_S of degree the i -th vector in the canonical basis of \mathbb{Z}^n . Then we obtain an isomorphism of M -graded rings

$$A_{(S)}[x_i^{\pm 1} : 1 \leq i \leq n] \xrightarrow{\sim} A_S$$

sending $\prod_i (x_i)^{n_i}$ to $\prod_i (a_i)^{n_i}$. In other words we have

$$\mathrm{Spec}(A_S) \cong \mathrm{Spec}(A_{(S)}) \times_{\mathrm{Spec}(A_0)} \mathrm{D}_{\mathrm{Spec}(A_0)}(M),$$

so that the restriction of our map to the preimage of $D_{\dagger}(S)$ is a *trivial* $\mathrm{D}_{\mathrm{Spec}(A_0)}(M)$ -bundle. \square

Remark 5.14. Assume that the conditions in Proposition 5.13 are satisfied. Then this proposition implies that pullback induces an equivalence of categories between $\mathrm{QCoh}(\mathrm{Proj}^M(A))$ and the category of $\mathrm{D}_{\mathrm{Spec}(A_0)}(M)$ -equivariant quasi-coherent sheaves on $\mathrm{Spec}(A) \setminus V(A_{\dagger})$. On the other hand, the category of M -graded A -modules identifies with the category of $\mathrm{D}_{\mathrm{Spec}(A_0)}(M)$ -equivariant quasi-coherent sheaves on $\mathrm{Spec}(A)$. Under these identifications, the functor $Q \mapsto \tilde{Q}$ corresponds to restriction along the open immersion $\mathrm{Spec}(A) \setminus V(A_{\dagger}) \rightarrow \mathrm{Spec}(A)$.

For the next lemma, recall the notion of negligible M -graded A -module from § 5.1.

LEMMA 5.15. *Assume that (5.8) is satisfied, and fix a subset $\mathcal{F} \subset \mathcal{F}_A^m$ such that*

$$\mathrm{Proj}^M(A) = \bigcup_{S \in \mathcal{F}} D_{\dagger}(S).$$

Then if Q is an M -graded A -module the following conditions are equivalent:

- (i) Q is negligible;
- (ii) for any $S \in \mathcal{F}$ and any $q \in Q$, there exists $s \in S$ such that $s \cdot q = 0$.

Proof. Since $\mathrm{Proj}^M(A)$ is covered by the open subschemes $(D_{\dagger}(S) : S \in \mathcal{F})$, we have $\tilde{Q} = 0$ iff $Q_{(S)} = 0$ for any $S \in \mathcal{F}$. Now since $\mathcal{F} \subset \mathcal{F}^{\max}$, for any $S \in \mathcal{F}$ the ring A_S has invertible elements of all degrees, so that $Q_{(S)} = 0$ iff $Q_S = 0$. Finally, it is clear from definitions that $Q_S = 0$ iff for any $q \in Q$ there exists $s \in S$ such that $s \cdot q = 0$. \square

Remark 5.16. Let us make the condition in Lemma 5.15 more explicit in some cases considered in Remark 5.10.

- (i) First, assume that A_0 is noetherian, that A is generated by A_1 as an A_0 -algebra, and moreover that A_1 is finite as an A_0 -module. (This is the setting considered in [StP, Tag 01YR].) For $N \in \mathbb{Z}$ we set $A_{\geq N} = \bigoplus_{m \geq N} A_m$. Then Q is negligible iff for any $q \in Q$ there exists $N > 0$ such that $A_{\geq N} \cdot q = 0$. (Since $A_{\geq N} = (A_{\geq 1})^N$, this condition is equivalent to Q being $A_{\geq 1}$ -power torsion in the sense of [StP, Tag 05E6].) In fact, choose $(a_i : i \in I)$ as in Remark 5.10, with I finite; then we can choose $\mathcal{F} = \{(a_i)^{\mathbb{N}} : i \in I\}$. If Q satisfies our condition, then it is negligible by Lemma 5.15. On the other hand, assume that Q is negligible, and let $q \in Q$. By the lemma, for any $i \in I$ there exists $n_i \in \mathbb{Z}_{>0}$ such that $(a_i)^{n_i} \cdot q = 0$. Then if $N = 1 + \sum_{i \in I} (n_i - 1)$, we have

$$A_{\geq N} \subset \sum_{i \in I} A \cdot (a_i)^{n_i},$$

so that $A_{\geq N} \cdot q = 0$.

- (ii) Now, consider the setting of § 4.1, and recall the notation of Remarks 4.6 and 5.10. For $N \in \mathbb{Z}$ we denote by $\mathbb{X}_{\geq N} \subset \mathbb{X}$ the submonoid consisting of elements λ which satisfy $\langle \lambda, \alpha^{\vee} \rangle \geq N$ for any simple root α . (In particular, we have $\mathbb{X}_{\geq 0} = \mathbb{X}_+$, and $\mathbb{X}_{\geq 1} = \mathbb{X}_{++}$.) We also denote by $A_{\geq N}$ the sum of the components in A whose degrees belong to $\mathbb{X}_{\geq N}$. We will use similar notation for $\tilde{\mathbb{X}}$ and \tilde{A} . Then Q is negligible iff for any $q \in Q$ there exists $N > 0$ such that $A_{\geq N} \cdot q = 0$. In fact, in Lemma 5.15 we can take $\mathcal{F} = \{(g_{\sigma})^{\mathbb{N}} : \sigma \in E\}$.

Since each g_σ belongs to $A_{\geq 1}$, it is clear from this lemma that if Q satisfies our condition, then it is negligible. On the other hand assume that Q is negligible, and fix $q \in Q$. By the lemma, for any $\sigma \in E$ there exists N_σ such that $(g_\sigma)^{N_\sigma} \cdot q = 0$. If $N = 1 + \sum_\sigma (N_\sigma n_\sigma - 1)$, we have

$$A_{\geq N} \subset \sum_{\sigma \in E} A \cdot (g_\sigma)^{N_\sigma}, \quad (5.9)$$

which will imply the claim. In fact, by the surjectivity of the maps (4.2) we have

$$\tilde{A}_{\geq 1} \subset \sum_{\sigma \in E} \tilde{A} \cdot f_\sigma,$$

and then

$$\tilde{A}_{\geq N} = (\tilde{A}_{\geq 1})^N \subset \sum_{\sigma \in E} \tilde{A} \cdot (g_\sigma)^{N_\sigma},$$

which implies (5.9).

- (iii) Consider the setting of § 4.2. Defining $\mathbb{X}_{\geq N}$ as above, and then $(A_V)_{\geq N}$ as the sum of the graded components of A_V whose label belongs to $\mathbb{X}_{\geq N}$, one checks using Lemma 4.9 and (5.9) that a graded A_V -module Q is negligible iff for any $q \in Q$ there exists $N > 0$ such that $(A_V)_{\geq N} \cdot q = 0$. Similar comments apply in the setting of § 4.3.

PROPOSITION 5.17. *Assume that (5.8) is satisfied. Then the composition*

$$\mathrm{Mod}^M(A) \xrightarrow{Q \mapsto \tilde{Q}} \mathrm{QCoh}(\mathrm{Proj}^M(A)) \rightarrow \mathrm{Mod}(\mathcal{O}_{\mathrm{Proj}^M(A)})$$

(where the second functor is the obvious forgetful functor) is left adjoint to the functor Γ_\bullet .

Proof. To prove the proposition we need to define functorial morphisms

$$\varepsilon_Q : \widetilde{\Gamma_\bullet(Q)} \rightarrow Q$$

for $Q \in \mathrm{Mod}(\mathcal{O}_{\mathrm{Proj}^M(A)})$ and

$$\eta_Q : Q \rightarrow \Gamma_\bullet(\tilde{Q})$$

for $Q \in \mathrm{Mod}^M(A)$, which satisfy the usual zigzag relations. Here ε is provided by Lemma 5.9. (This does not require any assumption.) To define η , we observe that for $Q \in \mathrm{Mod}^M(A)$ we have

$$\Gamma_\bullet(\tilde{Q}) = \bigoplus_{\alpha \in M} \Gamma(\mathrm{Proj}^M(A), \tilde{Q}(\alpha)) \cong \bigoplus_{\alpha \in M} \Gamma(\mathrm{Proj}^M(A), \widetilde{Q(\alpha)}),$$

where the isomorphism follows from Item (ii) in Corollary 5.11. The desired morphism is therefore provided by Fact 5.7(ii).

We leave it to the reader to check that these morphisms indeed satisfy the zigzag relations. \square

Recall that if \mathcal{L} is a line bundle on a scheme X and $s \in \Gamma(X, \mathcal{L})$ is a global section, we have an open subscheme $X_s \subset X$ defined by the nonvanishing of s , see [StP, Tag 01CY].

LEMMA 5.18. *Assume that (5.8) is satisfied. Let $f \in A$ be homogeneous and relevant, and let \tilde{f} be the image of f under the morphism of Fact 5.7(i). We have*

$$\mathrm{Proj}^M(A)_{\tilde{f}} = D_\dagger(f).$$

Proof. Our assumptions imply that $\mathcal{O}_{\mathrm{Proj}^M(A)}(\mathrm{deg}(f))$ is a line bundle (see Corollary 5.11), so that $\mathrm{Proj}^M(A)_{\tilde{f}}$ is well defined. In view of our assumption, to prove the statement it suffices to

prove that for any $S \in \mathcal{F}_A^m$ we have

$$D_{\dagger}(S)_{\tilde{f}_S} = D_{\dagger}(S) \cap D_{\dagger}(f),$$

where \tilde{f}_S is the restriction of \tilde{f} to $D_{\dagger}(S)$. Fix such an S . As in the proof of Proposition 5.8(i), there exist $a, s \in S$ such that $\deg(a) - \deg(s) = \deg(f)$, and then multiplication by $\frac{a}{s}$ induces an isomorphism of sheaves

$$\mathcal{O}_{D_{\dagger}(S)} \xrightarrow{\sim} \mathcal{O}_{\text{Proj}^M(A)}(\deg(f))|_{D_{\dagger}(S)}.$$

The inverse image of \tilde{f}_S under this isomorphism is $\frac{fs}{a}$, which implies that

$$D_{\dagger}(S)_{\tilde{f}_S} = \text{Spec} \left((A_{(S)})_{\frac{fs}{a}} \right).$$

By Proposition 2.10(ii), the right-hand side identifies with $\text{Spec}(A_{(S \cdot f^{\mathbb{N}})})$, i.e. with $D_{\dagger}(S) \cap D_{\dagger}(f)$, which finishes the proof. \square

PROPOSITION 5.19. *Assume that (5.8) is satisfied, and moreover that $\text{Proj}^M(A)$ is quasi-compact. Then for any \mathcal{Q} in $\text{QCoh}(\text{Proj}^M(A))$, the morphism*

$$\Gamma_{\bullet}(\widetilde{\mathcal{Q}}) \rightarrow \mathcal{Q}$$

of Lemma 5.9 is an isomorphism.

Proof. Since the sheaves under consideration are quasi-coherent, and since (5.8) holds, to prove the statement it suffices to prove that for any maximally relevant $f \in A$ the morphism

$$\Gamma(D_{\dagger}(f), \widetilde{\Gamma_{\bullet}(\mathcal{Q})}) \rightarrow \Gamma(D_{\dagger}(f), \mathcal{Q})$$

induced by the morphism of Lemma 5.9 is an isomorphism. Now by definition the left-hand side identifies with $\Gamma_{\bullet}(\mathcal{Q})_{(f)}$. Considering the \mathbb{Z} -graded module

$$Q := \bigoplus_{n \in \mathbb{Z}} \Gamma(\text{Proj}^M(A), \mathcal{Q}(n \cdot \deg(f)))$$

over the \mathbb{N} -graded ring

$$\bigoplus_{n \in \mathbb{N}} \Gamma(\text{Proj}^M(A), \mathcal{O}_{\text{Proj}^M(A)}(n \cdot \deg(f))),$$

and denoting by \tilde{f} the image of f in $\Gamma(\text{Proj}^M(A), \mathcal{O}_{\text{Proj}^M(A)}(\deg(f)))$, then we have

$$\Gamma_{\bullet}(\mathcal{Q})_{(f)} = Q_{(\tilde{f})}.$$

Since $\text{Proj}^M(A)$ is quasi-compact (by assumption) and quasi-separated (see Lemma 3.3), by [StP, Tag 01PW] the right-hand side identifies with

$$\Gamma(\text{Proj}^M(A)_{\tilde{f}}, \mathcal{Q}),$$

i.e. with $\Gamma(D_{\dagger}(s), \mathcal{Q})$ by Lemma 5.18, which finishes the proof. \square

Remark 5.20. Under the additional assumption that M is a free abelian group, one can give an alternative proof of Proposition 5.19 as follows. As explained in the comments following (3.4), the natural morphism $\text{Spec}(A) \setminus V(A_{\dagger}) \rightarrow \text{Proj}^M(A)$ is affine, hence quasi-compact, so that the scheme $A \setminus V(A_{\dagger})$ is quasi-compact. Hence the open immersion $j : \text{Spec}(A) \setminus V(A_{\dagger}) \rightarrow \text{Spec}(A)$ is quasi-compact and separated, so that the pushforward functor j_* preserves quasi-coherent sheaves, see [StP, Tag 01LC]. Under the identifications considered in Remark 5.14 the functor $Q \mapsto \widetilde{Q}$ corresponds to j^* , while the functor Γ_{\bullet} corresponds to j_* , and the statement of the proposition becomes the familiar fact that the adjunction morphism $j^*j_* \rightarrow \text{id}$ is an isomorphism.

COROLLARY 5.21. *Assume that (5.8) is satisfied, and moreover that $\text{Proj}^M(A)$ is quasi-compact. Then the functor*

$$L : \text{Mod}^M(A)/\text{Mod}^M(A)_{\text{neg}} \rightarrow \text{QCoh}(\text{Proj}^M(A))$$

and the composition

$$\text{QCoh}(\text{Proj}^M(A)) \rightarrow \text{Mod}(\mathcal{O}_{\text{Proj}^M(A)}) \xrightarrow{\Gamma_\bullet} \text{Mod}^M(A) \rightarrow \text{Mod}^M(A)/\text{Mod}^M(A)_{\text{neg}}$$

(where the first arrow is the obvious embedding and the third one is the quotient functor) are mutually inverse equivalences of categories.

This corollary follows immediately from Propositions 5.17 and 5.19, in view of the following general fact. (For the notion of kernel of an exact functor between abelian categories, see [StP, Tag 02MR].)

LEMMA 5.22. *Let \mathbf{A}, \mathbf{B} be abelian categories, and let*

$$L : \mathbf{A} \rightarrow \mathbf{B}, \quad R : \mathbf{B} \rightarrow \mathbf{A}$$

be functors. Assume that

- (i) L is left adjoint to R ;
- (ii) L is exact;
- (iii) the adjunction morphism $LR \rightarrow \text{id}$ is an isomorphism.

Then L factors through an equivalence of categories

$$\bar{L} : \mathbf{A}/\ker(L) \xrightarrow{\sim} \mathbf{B},$$

whose quasi-inverse is the composition of R with the quotient functor $\mathbf{A} \rightarrow \mathbf{A}/\ker(L)$.

Proof. Let us denote by $\pi : \mathbf{A} \rightarrow \mathbf{A}/\ker(L)$ the quotient functor. By [StP, Tag 02MS], L factors through a functor $\bar{L} : \mathbf{A}/\ker(L) \rightarrow \mathbf{B}$. Then our assumption (iii) shows that $\bar{L} \circ (\pi R) \cong \text{id}$. On the other hand, using adjunction we have a canonical morphism

$$\pi \rightarrow \pi RL = \pi R \bar{L} \pi.$$

We claim that this morphism is an isomorphism, which will conclude the proof in view of the fact that if $F, G : \mathbf{A}/\ker(L) \rightarrow \mathbf{B}$ are two functors, each morphism of functors $F\pi \rightarrow G\pi$ is induced by a unique morphism of functors $F \rightarrow G$.

To prove the claim it suffices to prove that for $X \in \mathbf{B}$, the kernel and cokernel of the adjunction morphism $X \rightarrow RLX$ belong to $\ker(L)$. Since L is exact, this is equivalent to showing that the image under L of this morphism is an isomorphism, which follows from the zigzag relation and our assumption (iii). \square

Remark 5.23. As in Remark 5.20, under the additional assumption that M is a free abelian group, one can give an alternative proof of Corollary 5.21 by noticing that the functor j^* induces an equivalence between the category of $D_{\text{Spec}(A_0)}(M)$ -equivariant quasi-coherent sheaves on $\text{Spec}(A) \setminus V(A_+)$ and the Serre quotient of the category of $D_{\text{Spec}(A_0)}(M)$ -equivariant quasi-coherent sheaves on $\text{Spec}(A)$ by the Serre subcategory of sheaves supported set-theoretically on $V(A_+)$, i.e. whose restriction to $\text{Spec}(A) \setminus V(A_+)$ vanishes.

We now consider the case when A is noetherian. Recall that in this case the scheme $\text{Proj}^M(A)$ is noetherian (in particular, quasi-compact), see § 5.2. In this case we consider the category

$\text{Mod}_{\text{fg}}^M(A)$ of finitely generated M -graded A -module, its Serre subcategory $\text{Mod}_{\text{fg}}^M(A)_{\text{neg}}$ of objects which are negligible modules, and the category $\text{Coh}(\text{Proj}^M(A))$ of coherent sheaves on $\text{Proj}^M(A)$. Recall (see Lemma 5.6) that in this case the functor $Q \mapsto \tilde{Q}$ restricts to a functor $\text{Mod}_{\text{fg}}^M(A) \rightarrow \text{Coh}(\text{Proj}^M(A))$, which must factor through a functor

$$\mathbf{L}_{\text{Coh}} : \text{Mod}_{\text{fg}}^M(A) / \text{Mod}_{\text{fg}}^M(A)_{\text{neg}} \rightarrow \text{Coh}(\text{Proj}^M(A)).$$

PROPOSITION 5.24. *If A is noetherian, the functor \mathbf{L}_{Coh} is an equivalence of categories.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} \text{Mod}_{\text{fg}}^M(A) / \text{Mod}_{\text{fg}}^M(A)_{\text{neg}} & \xrightarrow{\mathbf{L}_{\text{Coh}}} & \text{Coh}(\text{Proj}^M(A)) \\ \downarrow & & \downarrow \\ \text{Mod}^M(A) / \text{Mod}^M(A)_{\text{neg}} & \xrightarrow{\mathbf{L}} & \text{QCoh}(\text{Proj}^M(A)) \end{array}$$

where the lower horizontal arrow is known to be an equivalence (see Corollary 5.21) and the vertical arrows are fully faithful. It follows that \mathbf{L}_{Coh} is fully faithful.

To prove essential surjectivity, we consider $\mathcal{F} \in \text{Coh}(\text{Proj}^M(A))$, and the M -graded A -module $Q = \Gamma_{\bullet}(\mathcal{F})$. Since A is noetherian, Q is the filtered colimit of its finitely generated M -graded A -submodules; in other words there exists a filtered set I and finitely generated M -graded A -submodules $Q_i \subset Q$ such that $Q = \text{colim}_i Q_i$. By exactness of the functor $P \mapsto \tilde{P}$, and since this functor commutes with colimits (see § 5.1), each \tilde{Q}_i is a coherent subsheaf of $\mathcal{F} = \tilde{Q}$, and we have $\mathcal{F} = \text{colim}_i \tilde{Q}_i$. As in [StP, Tag 01Y8], this implies that $\mathcal{F} = \tilde{Q}_i$ for some i , hence that \mathcal{F} belongs to the essential image of \mathbf{L}_{Coh} . \square

Remark 5.25. In the setting of Remark 5.4 we similarly obtain an equivalence of abelian categories

$$\text{Mod}^{M,H}(A) / \text{Mod}^{M,H}(A)_{\text{neg}} \rightarrow \text{QCoh}^H(\text{Proj}^M(A))$$

and, in case A is noetherian, an equivalence of abelian categories

$$\text{Mod}_{\text{fg}}^{M,H}(A) / \text{Mod}_{\text{fg}}^{M,H}(A)_{\text{neg}} \rightarrow \text{Coh}^H(\text{Proj}^M(A))$$

where we use obvious notation in the left-hand side.

5.5 Derived categories

We come back to the general setting of § 5.4, and consider for $? \in \{+, -, \text{b}\}$ the derived categories $D^? \text{QCoh}(\text{Proj}^M(A))$ and $D^? \text{Mod}^M(A)$. We will denote by $D_{\text{neg}}^? \text{Mod}^M(A)$ the full triangulated subcategory of the latter category consisting of complexes all of whose cohomology objects are negligible. In the following statement we use the Verdier quotient of a triangulated category by a full triangulated subcategory; for this notion, we refer to [StP, Tag 05RA].

PROPOSITION 5.26. *Assume that (5.8) is satisfied, and moreover that $\text{Proj}^M(A)$ is quasi-compact. Then if $? \in \{+, -, \text{b}\}$ the functor $Q \mapsto \tilde{Q}$ induces an equivalence of triangulated categories*

$$D^? \text{Mod}^M(A) / D_{\text{neg}}^? \text{Mod}^M(A) \xrightarrow{\sim} D^? \text{QCoh}(\text{Proj}^M(A)).$$

Proof. Since the functor $Q \mapsto \tilde{Q}$ is exact, it induces a triangulated functor

$$D^? \text{Mod}^M(A) \rightarrow D^? \text{QCoh}(\text{Proj}^M(A))$$

on derived categories. This functor sends objects in $D_{\text{neg}}^? \text{Mod}^M(A)$ to complexes all of whose cohomology objects are trivial, i.e. to the zero object, so that our functor factors through a

triangulated functor

$$D^? \text{Mod}^M(A)/D_{\text{neg}}^? \text{Mod}^M(A) \rightarrow D^? \text{QCoh}(\text{Proj}^M(A)).$$

To check that this functor is an equivalence, we note that by [Mi91, Theorem 3.2] there exists a canonical equivalence of triangulated categories

$$D^? \text{Mod}^M(A)/D_{\text{neg}}^? \text{Mod}^M(A) \xrightarrow{\sim} D^?(\text{Mod}^M(A)/\text{Mod}^M(A)_{\text{neg}}).$$

This reduces the statement to the abelian case, which was proved in Corollary 5.21. \square

The following statement is the version of Proposition 5.26 for coherent sheaves, in the case of noetherian rings. Here we denote by $D_{\text{neg}}^? \text{Mod}_{\text{fg}}^M(A)$ the full subcategory of $D^? \text{Mod}_{\text{fg}}^M(A)$ consisting of complexes all of whose cohomology objects are negligible. The proof is the same, simply replacing the reference to Corollary 5.21 by a reference to Proposition 5.24.

PROPOSITION 5.27. *Assume that A is noetherian, and that (5.8) is satisfied. Then if $? \in \{+, -, b\}$ the functor $Q \mapsto \tilde{Q}$ induces an equivalence of triangulated categories*

$$D^? \text{Mod}_{\text{fg}}^M(A)/D_{\text{neg}}^? \text{Mod}_{\text{fg}}^M(A) \xrightarrow{\sim} D^? \text{Coh}(\text{Proj}^M(A)).$$

Remark 5.28. As in Remark 5.25, in the setting of Remark 5.4 one obtains similar equivalences for derived categories of *equivariant* modules and quasi-coherent sheaves.

5.6 On a lemma by Arkhipov–Bezrukavnikov

Continue with the setting of Proposition 5.27, and denote by $\text{Mod}_{\text{fg,fr}}^M(A)$ the additive category of *free* M -graded A -modules. Since any finitely generated M -graded A -module is a quotient of an object of $\text{Mod}_{\text{fg,fr}}^M(A)$, we have a canonical equivalence of triangulated categories

$$K^- \text{Mod}_{\text{fg,fr}}^M(A) \xrightarrow{\sim} D^- \text{Mod}_{\text{fg}}^M(A). \quad (5.10)$$

In view of Proposition 5.27, denoting by $K_{\text{neg}}^- \text{Mod}_{\text{fg,fr}}^M(A)$ the full subcategory of $K^- \text{Mod}_{\text{fg,fr}}^M(A)$ consisting of complexes all of whose cohomology objects are negligible, we deduce an equivalence of categories

$$K^- \text{Mod}_{\text{fg,fr}}^M(A)/K_{\text{neg}}^- \text{Mod}_{\text{fg,fr}}^M(A) \xrightarrow{\sim} D^- \text{Coh}(\text{Proj}^M(A)).$$

In [AB09, Sublemma 1], the authors state a similar claim for *bounded* homotopy categories.⁴ The proof is very sketchy, and it is not clear to us if it really applies in the stated generality. Here we provide a complete argument for this claim (under additional assumptions), kindly explained to one of us by R. Bezrukavnikov.

PROPOSITION 5.29. *Assume that A is a finitely generated \mathbb{k} -algebra for some field \mathbb{k} , that M is a free abelian group, and that (5.8) is satisfied. Consider the bounded homotopy category $K^b \text{Mod}_{\text{fg,fr}}^M(A)$, and the full subcategory $K_{\text{neg}}^b \text{Mod}_{\text{fg,fr}}^M(A)$ of complexes all of whose cohomology objects are negligible. The functor $Q \mapsto \tilde{Q}$ induces a fully faithful functor*

$$K^b \text{Mod}_{\text{fg,fr}}^M(A)/K_{\text{neg}}^b \text{Mod}_{\text{fg,fr}}^M(A) \rightarrow D^b \text{Coh}(\text{Proj}^M(A)).$$

Proof. As in Remark 5.14, under the present assumptions, the functor $Q \mapsto \tilde{Q}$ identifies with the functor

$$\text{Mod}_{\text{fg}}^M(A) = \text{QCoh}^T(\text{Spec}(A)) \xrightarrow{j^*} \text{Coh}^T(\text{Spec}(A) \setminus V(A_{\dagger})) \cong \text{Coh}(\text{Proj}^M(A)),$$

⁴Note that for a triangulated category \mathbf{A} and full subcategories \mathbf{B}, \mathbf{C} , it is not true in general that the natural functor $\mathbf{B}/(\mathbf{B} \cap \mathbf{C}) \rightarrow \mathbf{A}/\mathbf{C}$ is fully faithful; the bounded case therefore does not immediately follow from the unbounded case.

where T is the torus $D_{\text{Spec}(\mathbb{k})}(M)$ and $j : \text{Spec}(A) \setminus V(A_{\dagger}) \rightarrow \text{Spec}(A)$ is the embedding. What we have to show is therefore that for any bounded complexes $\mathcal{F}, \mathcal{F}'$ of objects in $\text{Mod}_{\text{fg,fr}}^M(A)$ the morphism

$$\phi : \text{Hom}_{K^b \text{Mod}_{\text{fg,fr}}^M(A)/K_{\text{neg}}^b \text{Mod}_{\text{fg,fr}}^M(A)}(\mathcal{F}, \mathcal{F}') \xrightarrow{\sim} \text{Hom}_{D^b \text{Coh}^T(\text{Spec}(A) \setminus V(A_{\dagger}))}(j^* \mathcal{F}, j^* \mathcal{F}').$$

induced by the functor j^* an isomorphism. For this we will construct a morphism ψ in the reverse direction, and check that ϕ and ψ are inverse to each other. In the course of the proof we will use the obvious fact that the natural functor

$$K^b \text{Mod}_{\text{fg,fr}}^M(A) \rightarrow D^b \text{Mod}_{\text{fg}}^M(A) \quad (5.11)$$

is fully faithful.

Fix $\mathcal{F}, \mathcal{F}'$ as above, and consider a morphism $f : j^* \mathcal{F} \rightarrow j^* \mathcal{F}'$ in $D^b \text{Coh}^T(\text{Spec}(A) \setminus V(A_{\dagger}))$. By Proposition 5.27, this morphism can be represented by a diagram

$$\mathcal{F} \xrightarrow{g} \mathcal{F}'' \xleftarrow{h} \mathcal{F}' \quad (5.12)$$

where $\mathcal{F}'' \in D^b \text{Mod}_{\text{fg}}^M(A)$ and h is a morphism whose cone \mathcal{G} has all of its cohomology objects supported set-theoretically on $V(A_{\dagger})$. Then, for some $n \gg 0$, \mathcal{G} can be represented by a bounded complex of M -graded A -modules all of whose components are annihilated by $(A_{\dagger})^n$ (see e.g. [BR24, Proposition A.1]). Consider a finite-dimensional graded subspace $E \subset (A_{\dagger})^n$ that generates this ideal, and the Koszul complex \mathcal{K} of the multiplication morphism $E \otimes_{\mathbb{k}} A \rightarrow A$, see [StP, Tag 0621]. By definition, this is a bounded complex of free M -graded A -modules, concentrated in nonnegative degrees, and whose degree-0 component is A . Moreover, by [StP, Tag 0663] the restriction of this complex to $\text{Spec}(A) \setminus V(A_{\dagger})$ is acyclic. Denote by \mathcal{C} the cokernel of the natural embedding of complexes $A \rightarrow \mathcal{K}$; then we have a morphism $\mathcal{C}[-1] \rightarrow A$ whose cone (namely, \mathcal{K}) is supported on $V(A_{\dagger})$, hence a morphism

$$\mathcal{F} \otimes_A \mathcal{C}[-1] \rightarrow \mathcal{F}$$

with the same property. We claim that the composition

$$\mathcal{F} \otimes_A \mathcal{C}[-1] \rightarrow \mathcal{F} \xrightarrow{g} \mathcal{F}'' \rightarrow \mathcal{G}$$

vanishes. This will imply that the composition of the first two morphisms factors through a morphism

$$\mathcal{F} \otimes_A \mathcal{C}[-1] \rightarrow \mathcal{F}'$$

(in $D^b \text{Mod}_{\text{fg}}^M(A)$, or equivalently in $K^b \text{Mod}_{\text{fg,fr}}^M(A)$); then the diagram

$$\mathcal{F} \leftarrow \mathcal{F} \otimes_A \mathcal{C}[-1] \rightarrow \mathcal{F}'$$

will define the desired morphism $\psi(f)$ in $K^b \text{Mod}_{\text{fg,fr}}^M(A)/K_{\text{neg}}^b \text{Mod}_{\text{fg,fr}}^M(A)$.

To prove the claim, it suffices to notice that the morphism

$$\text{Hom}_{D^b \text{Mod}_{\text{fg}}^M(A)}(\mathcal{F} \otimes_A \mathcal{K}, \mathcal{G}) \rightarrow \text{Hom}_{D^b \text{Mod}_{\text{fg}}^M(A)}(\mathcal{F}, \mathcal{G})$$

is surjective. This follows from the isomorphism

$$\text{Hom}_{D^b \text{Mod}_{\text{fg}}^M(A)}(\mathcal{F} \otimes_A \mathcal{K}, \mathcal{G}) \cong \text{Hom}_{D^b \text{Mod}_{\text{fg}}^M(A)}(\mathcal{F}, \mathcal{G} \otimes_A \mathcal{K}^{\vee})$$

where \mathcal{K}^{\vee} is the dual complex of A -modules, and the fact that \mathcal{G} is a direct summand in $\mathcal{G} \otimes_A \mathcal{K}^{\vee}$ by our choice of n .

It is clear from construction that $\phi \circ \psi = \text{id}$. On the other hand, any morphism $f' : \mathcal{F} \rightarrow \mathcal{F}'$ in the quotient category can be represented by a diagram (5.12) where now g, h are morphisms

in $K^b\mathrm{Mod}_{\mathrm{fg},\mathrm{fr}}^M(A)$. For the construction of $\psi(\phi(f'))$ we can take the images of these morphisms in $D^b\mathrm{Mod}_{\mathrm{fg}}^M(A)$; we deduce a commutative diagram

$$\begin{array}{ccc}
 & \mathcal{F} \otimes_A \mathcal{C}[-1] & \\
 k \swarrow & & \searrow l \\
 \mathcal{F} & & \mathcal{F}' \\
 g \searrow & & \swarrow h \\
 & \mathcal{F}'' &
 \end{array}$$

in $D^b\mathrm{Mod}_{\mathrm{fg}}^M(A)$, in which k and h have their cones supported on $V(A_+)$. Since the functor (5.11) is fully faithful, we can regard this diagram as a diagram in $K^b\mathrm{Mod}_{\mathrm{fg},\mathrm{fr}}^M(A)$, and its commutativity shows that $\psi(\phi(f')) = f'$. \square

Remark 5.30. (i) Consider the setting of Remark 5.4, with H a linearly reductive algebraic group over \mathbb{k} . Then one can consider the full subcategory $\mathrm{Mod}_{\mathrm{fg},\mathrm{fr}}^{M,H}(A)$ of the category $\mathrm{Mod}_{\mathrm{fg}}^{M,H}(A)$ of objects which sums of objects of the form $V \otimes A(\alpha)$ where V is a finite-dimensional H -module and $\alpha \in M$. Let $K_{\mathrm{neg}}^b\mathrm{Mod}_{\mathrm{fg},\mathrm{fr}}^{M,H}(A)$ be the full triangulated subcategory of $K^b\mathrm{Mod}_{\mathrm{fg},\mathrm{fr}}^{M,H}(A)$ whose objects are the complexes all of whose cohomology objects are supported on $V(A_+)$. Then the same proof as for Proposition 5.29 shows that the natural functor

$$K^b\mathrm{Mod}_{\mathrm{fg},\mathrm{fr}}^{M,H}(A)/K_{\mathrm{neg}}^b\mathrm{Mod}_{\mathrm{fg},\mathrm{fr}}^{M,H}(A) \rightarrow D^b\mathrm{Coh}^H(\mathrm{Proj}^M(A))$$

is fully faithful

- (ii) Another possible approach to this question, also suggested by R. Bezrukavnikov, would be to use the standard fact that the perfect derived category of an open subscheme $U \subset X$ is the Verdier quotient of the perfect derived category of X by the subcategory of complexes supported on $X \setminus U$, suitably generated to quotient stacks. (For a result of this form, see e.g. [TT90].) In the presence of a contracting \mathbb{G}_m -action, any equivariant vector bundle is free, which relates this statement with Proposition 5.29.

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