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Abstract

The linear sampling method has been applied to the detection of defects in an elastic plate by the use of Lamb modes in a series of papers: [1], Inverse Problems, 29:025017, 34:075006. However these papers are based on the $X-Y$ formalism and on a conjecture which states that some families of eigenfunctions form a Riesz basis. We propose to give a mathematical justification of the method without any conjecture and without the $X-Y$ formalism and based on the Kondratiev theory, [2].

Keywords: Lamb modes, Green matrix, linear sampling method

1 Introduction

The diffraction of elastic waves by a defect in a plate can be modelled using Lamb modes. Lamb modes are obtained by searching solutions to the homogeneous time-harmonic elasticity equations in a plate assuming an invariance with respect to the direction in the plate orthogonal to the direction of wave propagation, and then the problem in the plate is reduced to a problem in a strip.

2 Elementary solutions and direct problem

Let $\Omega = \{x = (x_1, x_3),\, x_1 \in S,\, x_3 \in \mathbb{R}\}$ ($S = (-h, h)$, $h > 0$) be the strip (without defect). Consider the homogeneous time-harmonic elasticity equations in Ω:

$$\begin{cases} \sigma_{ij,j}(u) + \omega^2 \rho u_i = 0 \text{ in } \Omega, \\ \sigma_{i3}(u) = 0 \text{ on } \partial\Omega. \end{cases} \tag{1}$$

The complex numbers $\nu \in \mathbb{C}$ such that (1) has an elementary solution of the form $u(x_1, x_3) = \varphi(x_1) e^{\nu x_3}$, $\varphi \neq 0$ give Lamb modes if $\varphi_2 = 0$ and SH modes if $\varphi_1 = \varphi_3 = 0$ and are eigenvalues of a polynomial operator pencil $\mathscr{L}(\nu)$: $\boldsymbol{H}^1(S) \to (\boldsymbol{H}^1(S))^*$. In the sequel it is assumed that all the eigenvalues of $\mathscr{L}(\nu)$ are algebraic simple and that 0 is not an eigenvalue of $\mathscr{L}(\nu)$.

The spectral problem relative to the Lamb modes is non-selfadjoint with compact resolvent, so that there is no Hilbert basis of eigenfunctions. A solution to the problem of non-orthogonality is given by the $X-Y$ formalism. Define the mixed variables $X = \begin{pmatrix} \sigma_{31} \\ u_3 \end{pmatrix}$, $Y = \begin{pmatrix} u_1 \\ -\sigma_{33} \end{pmatrix}$, and set $\begin{pmatrix} X(x_1, x_3) \\ Y(x_1, x_3) \end{pmatrix} = \begin{pmatrix} \mathcal{X}(x_1) \\ \mathcal{Y}(x_1) \end{pmatrix} e^{\nu x_3}$. There exists a sequence of guided modes $\begin{pmatrix} \mathcal{X}_n \\ \mathcal{Y}_n \end{pmatrix}$, $n \in \mathbb{N}$ satisfying a biorthogonality relation: $(\mathcal{X}_n | \mathcal{Y}_m)_S = \delta_{nm} J_n$ where $(\mathcal{X}|\mathcal{Y})_S = \int_{-h}^{h} (\mathcal{X}^1 \mathcal{Y}^1 + \mathcal{X}^2 \mathcal{Y}^2) ds$. The study of [1] is based on the conjecture that the families of $\mathcal{X}_n$ and $\mathcal{Y}_n$ form a Riesz basis in their respective spaces. Roughly speaking the main difference between the method of [1] and the method proposed here is that instead of having a development of the outgoing Green matrix as an infinite series due to the aforementioned conjecture, we get a rigorous development of the outgoing Green matrix as the sum of a linear combination of terms related to outgoing elementary solutions and a rest which is exponentialy decreasing at infinity. It is sufficient to justify the linear sampling method.

Set $\Sigma_R = S \times \{R\}$ ($R \in \mathbb{R}$). Then if u and v are solutions of (1), $q_\Omega(u, v) = \int_{\Sigma_R} \sigma_{i3}(\overline{v}) u_i - \sigma_{i3}(u) \overline{v}_i$ does not depend on $R \in \mathbb{R}$. Let u be an elementary solution of (1). Then u is outgoing (resp. incoming) iff $i q_\Omega(u, u) > 0$ (resp. < 0) for an imaginary eigenvalue or corresponds to an eigenvalue with negative (resp. positive) real part. If $\delta > 0$, there exists a basis $(Z_i^+)_{i=1,\dots,p}$, $(Z_i^-)_{i=1,\dots,p}$ of elementary solutions corresponding to eigenvalues in the strip $-\delta < Re\nu < \delta$ such that $(Z_i^+)_{i=1,\dots,p}$ are outgoing and $(Z_i^-)_{i=1,\dots,p}$ are incoming.

Let D such that $\overline{D} \subset \Omega$ be the defect, and set $\Omega_D = \Omega \setminus \overline{D}$. Consider the direct problem:

$$\begin{cases} \sigma_{ij,j}(u) + \omega^2 \rho u_i = 0 \text{ in } \Omega_D, \\ \sigma_{i3}(u) = 0 \text{ on } \partial\Omega, \\ u|_{\partial D} = g \in \boldsymbol{H}^{1/2}(\partial D). \end{cases} \tag{2}$$

Let $\delta > 0$ be such that $\mathscr{L}(\nu)$ has no eigenvalue on the lines Re $\nu = \pm\delta$ and assume that there is

no solution $\in \boldsymbol{H}^1(\Omega_D)$ to (2) with $g = 0$. Define $\boldsymbol{W}^1_\delta(\Omega)$ as the completion of $\boldsymbol{\mathcal{C}}^\infty_0(\overline{\Omega})$ for $\|u\|_{\delta,1,\Omega} = \|e^{\delta|x_3|}(\sum_{\alpha,\gamma\in\mathbb{N},\alpha+\gamma\leq 1}\partial^\alpha_{x_1}\partial^\gamma_{x_3}u)\|_{0,\Omega}$. Then for all $g \in \boldsymbol{H}^{1/2}(\partial D)$ there is a unique weak solution to (2) which coincides for $x_3 \gtrless 0$ large with the sum of a linear combination of outgoing (resp. incoming) elementary solutions and an element of $\boldsymbol{W}^1_\delta(\Omega)$.

3 Linear sampling method

It is possible to construct an outgoing Green matrix Gr^{out} such that for all fixed $x \in \overline{\Omega}$, for all $\varphi \in \boldsymbol{\mathcal{C}}^\infty_0(\overline{\Omega})$,

$$\int_\Omega (-\partial_j\sigma_{ij}(\varphi) - \omega^2\varphi_i)Gr^{out}(x,.)_{ki} + \int_{\partial\Omega}\sigma_{ij}(\varphi)n_j Gr^{out}(x,.)_{ki} = \varphi_k(x),$$

and such that for y fixed, $Gr^{out}(x,y)$ coincides for $x_3 \gtrless y_3$ large with the sum of $\sum_{k=1}^p Z^\pm_k(x)(Z^\mp_k(y))^T$ and an element of $\boldsymbol{W}^1_\delta(\Omega)$.

For all $y \notin \partial D$ let $u^s(.,y)$ be the solution of the direct problem with $g = -Gr^{out}(.,y)|_{\partial D}$. Let $0 < t_0 < R$ and define $F : \boldsymbol{L}^2(\Sigma_R) \to \boldsymbol{L}^2(\Sigma_R)$ by: $\forall h \in \boldsymbol{L}^2(\Sigma_R)$, $(Fh)(x) = \int_{\Sigma_R} u^s(x,y)h(y)ds(y)$, $x \in \Sigma_R$. Let $z \in \Omega$ between Σ_{-t_0} and Σ_{t_0} be a sampling point, $p \in \mathbb{R}^3$, $p \neq 0$ and consider the problem: find $h(.,z,p) \in \boldsymbol{L}^2(\Sigma_R)$ such that

$$Fh(.,z,p) = Gr^{out}(.,z)p \text{ on } \Sigma_R.$$

Then $z \in \Omega\backslash\overline{D} \Leftrightarrow ||h(.,z,p)||_{\boldsymbol{L}^2(\Sigma_R)}$ "explodes". This is the "linear sampling method".

4 Properties of the operators

Let $\mathcal{H}$, S, $\mathcal{F}$ be the operators defined by:

$\mathcal{H} : \boldsymbol{L}^2(\Sigma_R) \to \boldsymbol{H}^{1/2}(\partial D)$, $\forall h \in \boldsymbol{L}^2(\Sigma_R)$,

$$(\mathcal{H}h)(x) = \int_{\Sigma_R} Gr^{out}(x,y)h(y)ds(y),\ x \in \partial D,$$

S: $\boldsymbol{H}^{-1/2}(\partial D) \to \boldsymbol{H}^{1/2}(\partial D)$, $\forall\varphi \in \boldsymbol{H}^{-1/2}(\partial D)$,

$$(S\varphi)(x) = \int_{\partial D} Gr^{out}(x,y)\varphi(y)ds(y),\ x \in \partial D.$$

$\mathcal{F} : \boldsymbol{H}^{-1/2}(\partial D) \to \boldsymbol{L}^2(\Sigma_R)$, $\forall\varphi \in \boldsymbol{H}^{-1/2}(\partial D)$,

$$(\mathcal{F}\varphi)(x) = \int_{\partial D} Gr^{out}(x,y)\varphi(y)ds(y),\ x \in \Sigma_R.$$

Assume that ω^2 is not an eigenvalue of the Dirichlet problem in D (Hypothesis (H)). In that case S is an isomorphism. We have the following properties: $\mathcal{H}$, $\mathcal{F}$, F are compact operators, $\mathcal{H}$, $\mathcal{F}$, F are one-to-one and with dense range and $F = -\mathcal{F}S^{-1}\mathcal{H}$. Then Theorem 3.6 of [1], which justifies the linear sampling method can be proved without any conjecture. However from the numerical point of view, there is a priori no significant progress since the error made by truncating Gr^{out} is not estimated precisely.

5 An example of proof

Let us prove that $\mathcal{H}$ is one-to-one. Assume for example that D is on the left of Σ_R. Let $h \in \boldsymbol{L}^2(\Sigma_R)$ and for $x_3 \neq R$ set $v_h(x) = \int_{\Sigma_R} Gr^{out}(x,y)h(y)ds(y)$, so that $v_h = 0$ on ∂D. v_h is solution to the homogeneous equations with radiation conditions for $x_3 < R$ and $x_3 > R$. The hypothesis (H) implies that $v_h = 0$ on D so that $v_h(x) = 0$ if $x_3 < R$ (analyticity) and $v_h(x) = 0$ if $x_3 = R$ (single layer potential). The uniqueness of an outgoing solution in an elastic half-strip [3] implies that $v_h(x) = 0$ if $x_3 > R$ and $v_h \equiv 0$. The form of Gr^{out} implies that for all $k = 1,\ldots,p$,

$$\int_{\Sigma_R}(Z^\pm_k(y))^T h(y)ds(y) = 0$$

this being true for all $\delta > 0$ so that h is orthogonal to all the Lamb and SH modes. The completeness of Lamb modes [4] implies $h = 0$.

References

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