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A holographic uniqueness theorem for the two-dimensional Helmholtz equation

A. V. Nair, R. G. Novikov

Abstract

We consider a plane wave, a radiation solution, and the sum of these solutions (total solution) for the Helmholtz equation in an exterior region in \mathbb{R}^2 . We consider a straight line in this region, such that the direction of propagation of the plane wave is not parallel to this line. We show that the radiation solution in the exterior region is uniquely determined by the intensity of the total solution on an interval of this line. In particular, this result solves one of the old mathematical questions of holography in its two-dimensional setting. Our proofs also contribute to the theory of the Karp expansion of radiation solutions in two dimensions.

Keywords: Helmholtz equation, Karp expansion, phase recovering, holography
AMS Subject classification: 35J05, 35P25, 35R30

1 Introduction

We consider the two-dimensional Helmholtz equation

$$\Delta\psi(x) + \kappa^2\psi(x) = 0, \quad x \in \mathcal{U}, \quad \kappa > 0, \quad (1)$$

where Δ is the Laplacian in x , and \mathcal{U} is a region (open connected set) in \mathbb{R}^2 consisting of all points outside a closed bounded regular curve S (as in [8]). This equation particularly arises in electrodynamics, acoustics, and quantum mechanics. For equation (1) we consider solutions ψ_0 and ψ_1 such that:

$$\psi_0 = e^{ikx}, \quad k \in \mathbb{R}^2, \quad |k| = \kappa, \quad (2)$$

ψ_1 is of class C^2 and satisfies the Sommerfeld's radiation condition

$$\sqrt{|x|} \left(\frac{\partial}{\partial |x|} - i\kappa \right) \psi_1(x) \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty, \quad (3)$$

uniformly in $x/|x|$. We say that ψ_0 is the plane wave solution and ψ_1 is a radiation solution.

Let

$$\begin{aligned}
L &= L_{x_0, \theta} = \{x \in \mathbb{R}^2 : x = x(s) = x_1 + s\theta, \quad -\infty < s < +\infty\}, \\
L^+ &= L_{x_1, \theta}^+ = \{x \in \mathbb{R}^2 : x = x(s) = x_1 + s\theta, \quad 0 < s < +\infty\}, \\
L^- &= L_{x_2, \theta}^- = \{x \in \mathbb{R}^2 : x = x(s) = x_2 - s\theta, \quad 0 < s < +\infty\}, \tag{4}
\end{aligned}$$

where $x_0, x_1, x_2 \in \mathbb{R}^2$, $\theta \in \mathbb{S}^1$ and $x_1, x_2 \in L$.

Thus, $L = L_{x_0, \theta}$ is the oriented straight line in \mathbb{R}^2 that passes through x_0 and has direction θ . $L^+ = L_{x_1, \theta}^+$ is the ray in \mathbb{R}^2 that starts at $x_0 \in \mathbb{R}^2$ and has direction θ , and $L^- = L_{x_2, \theta}^-$ a ray that lies on the extension of L^+ in the $(-\theta)$ direction.

In the present work we show that, for a fixed plane wave solution ψ_0 , any complex-valued radiation solution ψ_1 on $L^+ \cup L^-$ is uniquely determined by the intensity $|\psi|^2$ of the total solution $\psi = \psi_0 + \psi_1$ on arbitrary intervals $\Lambda^+ \cup \Lambda^-$ of L^+ and L^- respectively, under the assumptions that $L^+ = L_{x_1, \theta}^+ \subset \mathcal{U}$, $L^- = L_{x_2, \theta}^- \subset \mathcal{U}$ and $\pm\theta \neq k/|k|$. Here, \mathcal{U} is the region in (1) and k is the vector in (2). This result is given as Theorem 1 in Section 2.

As a corollary, we also obtain that,

for any straight line $L \subset \mathcal{U}$ such that $\pm\theta \neq k/|k|$, any complex-valued ψ_1 on L is uniquely determined by the intensity $|\psi_0 + \psi_1|^2$ on an arbitrary interval Λ of L , for fixed ψ_0 ; see Theorem 2 in Section 2,

Our studies are motivated by problems of holography and phaseless inverse scattering. These research areas go back, in particular, to [5], [23], [24]. In connection with recent or relatively recent mathematical results obtained in these directions, see, for example, [6], [7], [9] - [21], [23], [24], and references therein.

The present work continues [17] where holography uniqueness results were proved for the Helmholtz equation in the three-dimensional case.

Theorems 1, 2 mentioned above and presented in detail in Section 2 solve one of the old mathematical questions arising in holography in its two-dimensional setting.

In the present work we use the Karp expansion (7) (recalled in Section 3) instead of the Atkinson-Wilcox expansion used in [17] for radiation solutions of the Helmholtz equation. Although the Karp expansion can be considered as a two-dimensional analogue of the three-dimensional Atkinson-Wilcox expansion used in [17], its properties are somewhat different, more complicated, and less studied. In this connection the present work also contributes to the theory of Karp expansion of the radiation solutions ψ_1 of the two-dimensional Helmholtz equation (1); see Lemmas 2, 3, and Theorem 3 in Section 4, and proof of Lemma 1 in Section 7.

The main results of this work are presented in more detail and proved in Sections 2, 4, 5, 6, 7. In our proofs, we proceed from the results recalled in Section 3.

2 Main results

Our key result is as follows.

Theorem 1. *Let $\psi_0 = e^{ikx}$ and ψ_1 be solutions of equation (1) as in formulas (2) and (3). Let L , L^+ and L^- be as given in (4) such that $L^+ = L_{x_1, \theta}^+ \subset \mathcal{U}$ and $L^- = L_{x_2, \theta}^- \subset \mathcal{U}$, $x_1, x_2 \in L$, $\theta \neq k/|k|$, and $-\theta \neq k/|k|$, where \mathcal{U} is the region in (1). Then ψ_1 on $L^+ \cup L^-$ is uniquely determined by the intensity $|\psi_0 + \psi_1|^2$ on $\Lambda^+ \cup \Lambda^-$, for fixed k , where Λ^+ and Λ^- are arbitrary non-empty intervals of L^+ and L^- respectively.*

As a corollary, we also get the following result.

Theorem 2. *Let $\psi_0 = e^{ikx}$ and ψ_1 be solutions of equation (1) as in formulas (2) and (3). Let L be a straight line in \mathbb{R}^2 such that $L \subset \mathcal{U}$ as described in (4). If $\theta \neq k/|k|$ and $-\theta \neq k/|k|$, then ψ_1 on L is uniquely determined by the intensity $|\psi_0 + \psi_1|^2$ on Λ , for fixed k , where Λ is an arbitrary non-empty interval of L .*

Theorem 1 is proved in Sections 4, 5, and 6. This proof uses techniques developed in [8] and [17]. Note that ψ_0, ψ_1 are real-analytic on \mathcal{U} , and, therefore, on L^+, L^- in Theorem 1 and on L in Theorem 2. Therefore, the function $|\psi|^2 = |\psi_0 + \psi_1|^2 = (\psi_0 + \psi_1)(\bar{\psi}_0 + \bar{\psi}_1)$ is real-analytic on L^+, L^- in Theorem 1 and on L in Theorem 2. Given this analyticity, Theorem 1 reduces to the case when $\Lambda^+ = L^+, \Lambda^- = L^-$ and Theorem 2 reduces to the case when $\Lambda = L$.

Theorem 2 is proved as follows (for example). We assume that $\Lambda = L$. Then we simply consider $L^+ \subset L$ and $L^- \subset L$ such that $L^+ \cap L^- \neq \emptyset$. Then we may apply theorem 1.

Corollary 1. *Under the assumptions of Theorem 2, the intensity $|\psi_0 + \psi_1|^2$ on L , for fixed k , uniquely determines ψ_1 in the entire region \mathcal{U} .*

Corollary 1 follows from Theorem 2, formula (14), and analyticity of ψ_1 in \mathcal{U} .

For the three-dimensional Helmholtz equation, prototypes of Theorems 1,2 and Corollary 1 were obtained in [17]. In the present work, we proceed from Karp's results on two-dimensional radiation solutions (in [8]) for equation (1) and the approach of [17].

Note that the problem in two dimensions is more difficult than in three dimensions. The reason is that the Karp expansion (7) for the radiation solution in two dimensions is considerably more complicated than the Atkinson-Wilson expansion (in [2], [22]) for the three-dimensional case.

In particular in Theorem 1 we need to consider two rays L^+ and L^- in place of a single ray L^+ as was in [17]. Both these rays need to have different directions

than the propagation direction of the plane wave $\psi_0 = e^{ikx}$. In Theorem 2 as well we need both the directions of the line L to be different from the propagation direction of the plane wave $\psi_0 = e^{ikx}$.

In the present work, we strongly use the uniqueness of Karp's expansion (7), which is formulated as Lemma 1 in the next section. For completeness of presentation, we prove this lemma, as we did not find a proof in the literature.

In the present work we also contribute to the theory of the Karp expansion for radiation solutions of the two-dimensional Helmholtz equation (1) by Lemmas 2, 3, and Theorem 3 given in Section 4. In particular, in Lemma 2 we write explicitly recursive relations for coefficients of Karp expansion, in Lemma 3 we establish important symmetries of these coefficients and in Theorem 3, we present a recursive method for finding all coefficients in Karp expansion from coefficients arising in standard scattering theory.

Note also that Theorems 1, 2, and Corollary 1 admit straightforward applications to phaseless inverse scattering in two dimensions.

3 Preliminaries

Let (r, ϕ) denote the polar coordinates of a point x and let $\theta = \frac{x}{|x|}$ be its direction. So $-\theta$ will be the direction of $-x$ whose polar coordinates will be $(r, \phi + \pi)$.

We use the Laplacian in polar coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{r^2 \partial \phi^2}. \quad (5)$$

Let

$$B_\rho = \{x \in \mathbb{R}^2 \mid |x| < \rho\}, \quad \rho > 0. \quad (6)$$

Suppose that ψ_1 is a radiation solution of equation (1), and $\mathbb{R}^2 \setminus B_\rho \subset \mathcal{U}$.

Due to Theorems I, II of [8], we have that:

$$\psi_1(x) = H_0(\kappa|x|) \sum_{j=0}^{\infty} \frac{F_j(\phi)}{|x|^j} + H_1(\kappa|x|) \sum_{j=0}^{\infty} \frac{G_j(\phi)}{|x|^j}, \quad (7)$$

$$\text{for } x \in \mathbb{R}^2 \setminus B_\rho,$$

where H_0 and H_1 are the Hankel functions of the first kind of order zero and one respectively. The series converges absolutely and uniformly for $|x| \geq \rho_1 > \rho$ (for any such ρ_1).

Lemma 1. *Suppose that:*

$$H_0(r) \sum_{j=0}^{\infty} \frac{A_j}{r^j} + H_1(r) \sum_{j=0}^{\infty} \frac{B_j}{r^j} = 0, \quad (8)$$

for $r \geq r_1 > 0$, $A_j, B_j \in \mathbb{C}$, and both the series converges absolutely and uniformly. Then $A_j = B_j = 0$, $\forall i \in \mathbb{N}$.

In this article \mathbb{N} denotes natural numbers including 0.

We did not find a proof for this result in the available literature. Therefore for completeness of presentation this lemma is proved in Section 7. Lemma 1 implies uniqueness of the Karp expansion (7) for a radiation solution.

In addition to expansion (7), we also have the following asymptotic expansion:

$$\psi_1(x) \sim \sqrt{\frac{2}{\pi\kappa|x|}} e^{i(\kappa|x| - \frac{\pi}{4})} \sum_{j=0}^{\infty} \frac{f_j(\phi)}{|x|^j} \quad \text{for } x \in \mathbb{R}^2, \quad |x| \rightarrow \infty. \quad (9)$$

However, the series in (9) diverges in general and in particular for $\psi_1(x) = H_0(\kappa|x|)$ (see [8] for more details).

Theorem III in [8] gives us the following relations:

$$F_0(\phi) = \frac{1}{2}[f_0(\phi) + f_0(\phi + \pi)], \quad -iG_0(\phi) = \frac{1}{2}[f_0(\phi) - f_0(\phi + \pi)], \quad (10)$$

where F_0, G_0 , and f_0 are the leading coefficients in (7) and (9) respectively.

In the present work we also use the following properties of H_0 and H_1 :

$$H_\nu(r) \sim \sqrt{\frac{2}{\pi r}} e^{i(r - \frac{1}{2}\nu\pi - \frac{\pi}{4})} \sum_{j=0}^{\infty} \frac{C_{\nu j}}{r^j}, \quad \text{as } |x| \rightarrow \infty, \quad (11)$$

$$C_{\nu j} = \frac{(\frac{1}{2} - \nu)_j \Gamma(\nu + j + \frac{1}{2})}{j! \Gamma(\nu + \frac{1}{2}) (2i)^j},$$

$$H'_0 = -H_1, \quad H''_0 = \frac{1}{r} H_1 - H_0, \quad (12)$$

$$\frac{H'_1}{r} = \frac{-2H''_0}{r}, \quad H''_1 = -(\frac{1}{r} H_1 - H_0)', \quad (13)$$

where $'$ denotes derivative with respect to r .

Recall that,

$$\psi_1(x) = 2 \int_L \frac{\partial G^+(x-y, \kappa)}{\partial \nu_y} \psi_1(y) dy, \quad x \in V_L, \quad (14)$$

$$G^+(x, \kappa) = \frac{i}{4} H_0(\kappa|x|), \quad x \in \mathbb{R}^2,$$

where ψ_1 is a radiation solution of equation (1), L and V_L are line and open half-plane in \mathcal{U} , where L is the boundary of V_L , ν is the outward normal to L relative to V_L ; see, for example, formula 5.84 in [3].

4 Determining F_n and G_n from f_i

In this section, we give a method to determine $F_n(\phi)$ and $G_n(\phi)$ in (7), from $f_0(\phi), f_0(\phi + \pi), \dots, f_n(\phi), f_n(\phi + \pi)$ values in (9), for ϕ and $(\phi + \pi)$.

4.1 Recursion relations for F_n, G_n

Lemma 2. *The coefficients $F_n = F_n(\phi), G_n = G_n(\phi)$ in (7) satisfy the following recursive relations:*

$$F_{n+1} = \frac{-((n+1)^2 G_n + G_n'')}{2\kappa(n+1)}, \quad G_{n+1} = \frac{n^2 F_n + F_n''}{2\kappa(n+1)}, \quad n \in \mathbb{N}, \quad (15)$$

where F_n'', G_n'' denote double derivatives with respect to ϕ .

The fact that F_n, G_n can be obtained recursively in terms of F_0, G_0 was already mentioned in [8], where formula (15) was given for $n = 0$.

The derivation of relations (15) is as follows.

We substitute expansion (7) into the Helmholtz equation (1) (using (8)) for $|x| > \rho$ and express the derivatives of H_0 and H_1 with respect to r using the formulas (12) and (13).

As a result, we have:

$$\begin{aligned} (\Delta + \kappa^2)\psi_1(x) = \\ H_0(\kappa r) \sum_{n=0}^{\infty} (F_n'' - 2\kappa(n+1)G_{n+1} + n(n+1)F_n - nF_n) \frac{1}{r^{n+2}} \\ + H_1(\kappa r) \sum_{n=0}^{\infty} (G_n'' + 2\kappa(n+1)F_{n+1} + n(n+1)G_n + (n+1)G_n) \frac{1}{r^{n+2}} = 0, \end{aligned} \quad (16)$$

for $r > \rho$.

Due to [8], both series in (16) converge absolutely and uniformly for $r \geq \rho_1 > \rho$. Therefore, on applying Lemma 1, we get the relations in (15).

4.2 Symmetries for F_n, G_n

Lemma 3. *The coefficients F_n, G_n in (7) have the following important symmetries:*

$$F_n(\phi) = (-1)^n F_n(\phi + \pi), \quad G_n(\phi) = (-1)^{n+1} G_n(\phi + \pi), \quad (17)$$

for $n \in \mathbb{N}$.

The proof of Lemma 3 is as follows.

Due to formulas (10), symmetries (17) are true for $n = 0$. In addition, using relations (15) one can see that if symmetries (17) hold for n then they are true for $(n + 1)$. Hence, by induction (17) is true $\forall n \in \mathbb{N}$.

Remark 1. *Suppose ψ_1 satisfies (1) and (3). Let L, L^+, L^- be as in (4), $L^+, L^- \subset \mathcal{U}$ and $0 \subset L$. Then from analyticity of ψ_1 in \mathcal{U} , symmetries (17) and expansion (7) it follows that ψ_1 on L^+ uniquely determines ψ_1 on L^- .*

4.3 Comparison of expansions (7) and (9)

Now we equate asymptotically the series (9) with the convergent series (7) as $r \rightarrow \infty$.

The coefficients of r^{-n} must be equal for these asymptotic expansions $\forall n \in \mathbb{N}$.

As a result, we get the following relations:

$$f_n(\phi) = F_n(\phi) - iG_n(\phi) + \sum_{j=1}^n (C_{0j}F_{n-j}(\phi) - iC_{1j}G_{n-j}(\phi)), \quad (18)$$

$$f_n(\phi + \pi) = F_n(\phi + \pi) - iG_n(\phi + \pi) + \sum_{j=1}^n (C_{0j}F_{n-j}(\phi + \pi) - iC_{1j}G_{n-j}(\phi + \pi)). \quad (19)$$

Proceeding from formulas (17), (18), (19) we obtain the following result.

Theorem 3. *The coefficients F_n, G_n in (7) and f_n in (9) are related by the formulas:*

$$F_n(\phi) - iG_n(\phi) = f_n(\phi) - \sum_{j=1}^n (C_{0j}F_{n-j}(\phi) - iC_{1j}G_{n-j}(\phi)), \quad (20)$$

$$F_n(\phi) + iG_n(\phi) = (-1)^n (f_n(\phi + \pi) - \sum_{j=1}^n (-1)^{(n-j)} (C_{0j}F_{n-j}(\phi) + iC_{1j}G_{n-j}(\phi))). \quad (21)$$

By these formulas, the coefficients f_0, \dots, f_n uniquely determine the coefficients $F_0, G_0, \dots, F_n, G_n$ in a recursive way.

Theorem 3 for $n = 0$ reduces to formulas (10).

In particular, we consider relations (20), (21) as a linear system for finding $F_n(\phi), G_n(\phi)$ from $f_n(\phi), f_n(\phi + \pi)$ and $F_0(\phi), G_0(\phi), \dots, F_{n-1}(\phi), G_{n-1}(\phi)$, that are found from $f_0(\phi), \dots, f_{n-1}(\phi)$.

5 Determining f_i

Recall that already work [16] gives formulas for finding all f_i in (9) from $|\psi|^2 = |\psi_0 + \psi_1|^2$ on $L_{0,\theta}^+$. However, these formulas in [16] are not very simple. Therefore, in a similar way to [17] (where three-dimensional case is considered) we give a very simple proof that $|\psi|^2$ on $L^+ \subseteq L_{0,\theta}^+$ defined as in (4) where 0 denotes the origin in \mathbb{R}^2 , uniquely determines $f_j(\phi)$ in (9) $\forall j \in \mathbb{N}$, under the assumption that $\theta \neq k/|k|$, where $\psi = \psi_0 + \psi_1$, ψ_0 and ψ_1 are solutions of equation (1) as in (2) and (3).

Let

$$a(x, k) = \sqrt{|x|}(|\psi(x)|^2 - 1), \quad x \in \mathcal{U}, \quad (22)$$

where $\psi = \psi_0 + \psi_1$, ψ_0 and ψ_1 are solutions of equation (1) as in (2) and (3), k is the wave vector in (2). Then as $|x| \rightarrow +\infty$ (see [14], [16]):

$$a(x, k) = \sqrt{\frac{2}{\pi\kappa}} (e^{i(\kappa|x| - kx - \frac{\pi}{4})} f_0(\phi) + e^{-i(\kappa|x| - kx - \frac{\pi}{4})} \overline{f_0(\phi)}) + O\left(\frac{1}{\sqrt{|x|}}\right), \quad (23)$$

uniformly in $\theta = x/|x|$;

$$f_0(\phi) = \frac{i\sqrt{\frac{\pi\kappa}{2}}}{D} (e^{i(ky - \kappa|y| + \frac{\pi}{4})} a(x, k) - e^{i(kx - \kappa|x| + \frac{\pi}{4})} a(y, k)) + O\left(\frac{1}{\sqrt{|x|}}\right), \quad (24)$$

$$D = 2\sin(\tau(k\theta - \kappa)), \quad \theta \in \mathbb{S}^1, \quad \tau > 0,$$

$$x, y \in L_{x_1, \theta}^+, \quad x_1 = 0, \quad y = x + \tau\theta,$$

for $D \neq 0$ for fixed θ and τ , where $L_{x_1, \theta}^+$ is defined in (4).

Remark 2. *If an arbitrary function a on $L_{0,\theta}^+$ satisfies (23), for fixed $\theta \in \mathbb{S}^1$, $k \in \mathbb{R}^2$, $\kappa = |k| > 0$, then formula (24) holds.*

The determination of f_0 follows from (24).

Suppose that f_0, \dots, f_n are determined, then the determination of f_{n+1} is as follows.

Let

$$\psi_{1,n}(x) = \sqrt{\frac{2}{\pi\kappa|x|}} e^{i(\kappa|x| - \frac{\pi}{4})} \sum_{j=0}^n \frac{f_j(\phi)}{|x|^j}, \quad \text{where } \theta = \frac{x}{|x|}, \quad (25)$$

$$a_n(x, k) = \sqrt{|x|}(|e^{ikx} + \psi_{1,n}(x)|^2 - 1), \quad (26)$$

$$b_n(x, k) = |x|^{n+1}(a(x, k) - a_n(x, k)), \quad (27)$$

where x is as in (24), $a(x, k)$ is defined by (22).

We have that:

$$a(x, k) = \sqrt{|x|}((e^{ikx} + \psi_{1,n}(x) + \sqrt{\frac{2}{\pi\kappa|x|}}e^{i(\kappa|x|-\frac{\pi}{4})}\frac{f_{n+1}(\phi)}{|x|^{n+1}} + O(|x|^{-n-2-\frac{1}{2}})) \quad (28)$$

$$\begin{aligned} & \times (e^{-ikx} + \overline{\psi_{1,n}(x)} + \sqrt{\frac{2}{\pi\kappa|x|}}e^{-i(\kappa|x|-\frac{\pi}{4})}\frac{\overline{f_{n+1}(\phi)}}{|x|^{n+1}} + O(|x|^{-n-2-\frac{1}{2}})) - 1) \\ & = a_n(x, k) + \sqrt{\frac{2}{\pi\kappa}}(e^{i(\kappa|x|-\frac{\pi}{4})}\frac{f_{n+1}(\phi)}{|x|^{n+1}} + e^{-i(\kappa|x|-\frac{\pi}{4})}\frac{\overline{f_{n+1}(\phi)}}{|x|^{n+1}}) + O(|x|^{-n-1-\frac{1}{2}}) \end{aligned}$$

$$b_n(x, k) = \sqrt{\frac{2}{\pi\kappa}}(e^{i(\kappa|x|-\frac{\pi}{4}-kx)}f_{n+1}(\phi) + e^{-i(\kappa|x|-\frac{\pi}{4}-kx)}\overline{f_{n+1}(\phi)}) + O\left(\frac{1}{\sqrt{|x|}}\right), \quad (29)$$

as $|x| \rightarrow +\infty$, uniformly in $\theta = x/|x|$, where a is defined by (22).

Due to (29) and Remark 2, as $|x| \rightarrow +\infty$ we get:

$$f_{n+1}(\phi) = \frac{i\sqrt{\frac{\pi\kappa}{2}}}{D}(e^{i(ky-\kappa|y|+\frac{\pi}{4})}b_n(x, k) - e^{i(kx-\kappa|x|+\frac{\pi}{4})}b_n(y, k)) + O\left(\frac{1}{\sqrt{|x|}}\right), \quad (30)$$

$$D = 2\sin(\tau(k\theta - \kappa)), \quad x, y \in L^+ \subseteq L_{0,\theta}^+, \quad y = x + \tau\theta, \quad \theta \in \mathbb{S}^1, \quad \tau > 0,$$

assuming that $D \neq 0$ for fixed θ and τ (where the parameter τ can be always fixed in such a way that $D \neq 0$, under our assumption that $\theta \neq k/|k|$).

Formulas (22), (25)-(27) and (30) determine f_{n+1} , give the step of induction for finding all f_j .

6 Proof of Theorem 1

6.1 Case $L^+ \subseteq L_{0,\theta}^+$, $L^- \subseteq L_{0,\theta}^-$

First, we give the proof for the case when $L^+ \subseteq L_{0,\theta}^+$, $L^- \subseteq L_{0,\theta}^-$, where L^+ , $L_{0,\theta}^+$, L^- , $L_{0,\theta}^-$ are defined in (4) and 0 denotes the origin in \mathbb{R}^2 .

In this case, $|\psi|^2$ on L^+ and L^- uniquely determines $f_j(\phi)$ and $f_j(\phi+\pi)$ respectively in (9), $\forall j \in \mathbb{N}$ via formulas (29), (30), where $\psi = \psi_0 + \psi_1$; see Section 5.

In turn, using Theorem 3, in Section 4 we uniquely determine $F_j(\phi), F_j(\phi + \pi)$ and $G_j(\phi), G_j(\phi + \pi)$, $\forall j \in \mathbb{N}$.
 Finally, the convergence of the series in (7) and analyticity of ψ_1 and $|\psi|^2$ on L^+ and L^- gives us ψ_1 on $L^+ \cup L^-$.

6.2 General case

In fact, in a similar way with [17], the general case reduces to the case of Subsection 6.1 by the change of variables

$$\begin{aligned} x' &= x - q & (31) \\ \text{for some fixed } q &\in \mathbb{R}^2 \text{ such that } L^+ \subseteq L_{q,\theta}^+, \quad L^- \subseteq L_{q,\theta}^-. \end{aligned}$$

In the new variables x' , we have that:

$$\psi_0 = e^{ikq} e^{ikx'}, \quad (32)$$

$$\psi_1 \text{ satisfies (3) and admits presentation (7)} \quad (33)$$

with x' in place of x , with some new r', F'_j, G'_j and f'_j ,

$$|\psi|^2 = |e^{ikx'} + e^{-ikq} \psi_1(x')|^2, \quad (34)$$

where $x' \in \mathcal{U}' = \mathcal{U} - q$, and r' is such that $\mathbb{R}^2 \setminus B_{r'} \subset \mathcal{U}'$;

$$L^+ \subseteq L_{q,\theta}^+ = L_{0,\theta}^+ \quad L^- \subseteq L_{q,\theta}^- = L_{0,\theta}^-. \quad (35)$$

In addition,

$$e^{-ikq} \psi_1(x') = \sqrt{\frac{2}{\pi \kappa |x'|}} e^{i(\kappa|x'| - \frac{\pi}{4})} \sum_{j=0}^{\infty} \frac{f_j''(\phi)}{|x'|^j} \text{ for } x' \in \mathbb{R}^2 \setminus B_{r'}, \quad (36)$$

where $f_j'' = e^{-ikq} f'_j$.

We also have

$$e^{-ikq} \psi_1(x') = H_0(\kappa|x'|) \sum_{j=0}^{\infty} \frac{\mathcal{F}_j(\phi)}{|x'|^j} + H_1(\kappa|x'|) \sum_{j=0}^{\infty} \frac{\mathcal{G}_j(\phi)}{|x'|^j}, \quad (37)$$

where $\mathcal{F}_j = e^{-ikq} F'_j$ and $\mathcal{G}_j = e^{-ikq} G'_j$, and the series converges absolutely and uniformly in $|x| > r'$.

The coefficients F'_j, G'_j and f'_j are mentioned in (33).

In view of (34)-(37), similarly to the case of Subsection 6.1, it is sufficient to prove that $|\psi|^2$ on L^+ and L^- uniquely determines $\mathcal{F}_j(\phi), \mathcal{G}_j(\phi)$ (subsequently for $(\phi + \pi)$) in (37) $\forall j \in \mathbb{N}$. This determination of $\mathcal{F}_j, \mathcal{G}_j$ is completely similar to the determination of F_j and G_j in Subsection 6.1.

This completes the proof of Theorem 1.

7 Proof of Lemma 1

First, we recall some properties of H_0 and H_1 .

Both H_0 and H_1 admit holomorphic continuation from \mathbb{R}_+ to $\mathbb{C}_- = \{z \in \mathbb{C} \mid \text{Im}(z) < 0\}$; see [1].

In view of (12) we also have that:

$$\frac{dH_0(z)}{dz} = -H_1(z), \quad z \in \mathbb{C}_-. \quad (38)$$

In addition, H_0 has infinitely many zeroes $z_j \in \mathbb{C}_-$ such that $|z_j| \rightarrow \infty$, as $j \rightarrow \infty$; see [1] and [4].

Using formula (38) we get:

$$\lim_{z \rightarrow z_j} \frac{H_0(z)}{H_1(z)} = 0, \quad \forall j \in \mathbb{N}. \quad (39)$$

Next, we consider:

$$\Sigma_0(r) = \sum_{j=0}^{\infty} \frac{A_j}{r^j}, \quad \Sigma_1(r) = \sum_{j=0}^{\infty} \frac{B_j}{r^j}, \quad r \geq r_1. \quad (40)$$

Due to the assumptions of Lemma 1, the functions $\Sigma_0(r)$ and $\Sigma_1(r)$ admit holomorphic continuation from the interval $[r_1, +\infty)$ to the domain \mathcal{D}_{r_1} where,

$$\mathcal{D}_\rho = \{z \in \mathbb{C} \mid |z| \geq \rho\} \cup \{\infty\}. \quad (41)$$

The rest of the proof of Lemma 1 is as follows.

Suppose that Σ_1 is not identically 0. Then we consider the smallest j such that $B_j \neq 0$, and in particular:

$$z^j \Sigma_1(z) \rightarrow B_j \text{ as } z \rightarrow \infty. \quad (42)$$

Using formula (42) we get that there exists a neighbourhood of ∞ in \mathcal{D}_{r_1} where ∞ is the only zero of Σ_1 .

Then proceeding from (8) we get:

$$H_0(z)\Sigma_0(z) = -H_1(z)\Sigma_1(z) \quad \text{in } \mathbb{C}_- \cap \mathcal{D}_{r_1}. \quad (43)$$

Using also (39) we get that:

$$-\lim_{z \rightarrow z_j} \frac{H_0(z)\Sigma_0(z)}{H_1(z)} = \lim_{z \rightarrow z_j} \Sigma_1(z) = 0, \quad \forall j \in \mathbb{N}, \quad \text{such that } z_j \in \mathbb{C}_- \cap \mathcal{D}_{r_1}. \quad (44)$$

However, formula (44) is in contradiction with the aforementioned conclusion that ∞ is the only zero of Σ_1 in \mathcal{D}_{r_1} .
Hence,

$$B_j = 0 \quad \forall j \in \mathbb{N}, \quad \Sigma_1 \equiv 0, \quad (45)$$

under the assumptions of Lemma 1.

Using (8), (40), and (45) we get, $A_j = 0 \quad \forall j \in \mathbb{N}$.

This completes the proof of Lemma 1.

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