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# ON THE VOLTAGE-CONDUCTANCE KINETIC EQUATION

CLAUDIA FONTE SANCHEZ AND STÉPHANE MISCHLER

ABSTRACT. We consider the nonlinear Voltage-Conductance kinetic equation arising in neuroscience. We establish the existence of solutions in a weighted  $L^\infty$  framework in a weak interaction regime. We also prove the linear asymptotic exponential stability of the steady state making constructive a recent estimate of [18]. Both results are based in a fundamental way on some ultracontractivity property of the flow associated to the linear (possibly time dependent) Voltage-Conductance kinetic equation.

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## 1. INTRODUCTION

The integrate-and-fire (IF) neuron model has proven to be a valuable tool in computational neuroscience for simulating the behavior of individual cortical neurons [31, 42, 5]. However, scaling this approach to model large cortical regions presents significant challenges. Both the computational demands of simulating vast networks and the qualitative analysis of their complex dynamics become increasingly difficult. In this context, the Voltage-Conductance kinetic (VCK) equation emerges as a population-level representation of the neuronal activity [7, 6, 43, 44], analogous to the approach of classical kinetic theory for charged particles [16, 3, 12, 33, 39], and as an alternative to the kinetic Fitzhugh-Nagumo model [20, 37, 34, 30].

Consider a network of IF neurons, where each neuron (indexed by  $j$  out of  $N$  total) is described by two state variables: voltage,  $v^j$ , and conductance,  $y^j$ . The dynamics of the  $j$ th neuron during periods between spikes are governed by the following equations:

$$(1.1) \quad \frac{dv^j}{dt} = -(v^j - v_R) - y^j(v^j - v_E),$$

$$(1.2) \quad \frac{dy^j}{dt} = -y^j + y_* + \frac{\mathfrak{c}}{N} \sum_{l, k \neq j} \delta(t - t_k^\ell).$$

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Here,  $v_R$  represents the resting voltage (set to 0 for convenience),  $v_E$  is the excitatory reversal potential, and  $y_* > 0$  corresponds to the constant external stimulus originating in the lateral geniculate nucleus (LGN) from the visual sensors (e.g., [7]). The term  $\delta(t-t_k^\ell)$  in (1.2) represents a Dirac delta function that models the synaptic input received by neuron  $j$  from the  $\ell$ th spike of neuron  $k$ . The parameter  $\mathfrak{c} \geq 0$  scales the strength of these synaptic interactions.

In the absence of any external or network input, the voltage of the neuron naturally decays towards its resting state ( $v_R$ ) due to the first term in the first equation. The neuron can, however, receive excitatory input from two sources, the external stimulus ( $y_*$ ) and the input that arises from the activity of other neurons in the network, captured by the summation term in the second equation. These combined inputs influence the voltage of the neuron indirectly by affecting its conductance ( $y^j$ ). The conductance acts like a gain control mechanism, modulating how effectively the neuron translates these inputs into changes in its voltage. In complement with this dynamic, when the neuron's voltage reaches a threshold,  $v_F$ , it is assumed to spike. At this point (say, time  $t$ ), the voltage is reset to the resting value,  $v^j(t^+) = v_R$ .

The VCK model is derived from this system by applying a closure in the mean-field limit of a large number of weakly connected neurons [7, 6, 43]. It describe the evolution of the neuronal network through the probability  $F_t(v, y)$  to find neurons at time  $t \geq 0$  with a membrane potential  $v \in (0, v_F)$  and a conductance  $y \in (0, \infty)$ . We refer to [41, 19, 45, 18] for previous mathematical analysis papers, discussions and further references.

**1.1. The VCK equation.** The core mathematical formulation of the VCK model is a partial differential equation (PDE) known as the Voltage-Conductance kinetic equation. This equation governs the time evolution of the probability density function  $F_t(v, y)$  and reads

$$(1.3) \quad \partial_t F + \partial_v(JF) + \partial_y(K_F F) - a_F \partial_{yy}^2 F = 0 \quad \text{in } (0, \infty) \times \mathcal{O}$$

with the shorthand  $(v, y) \in \mathcal{O} := (0, v_F) \times (0, \infty)$  and where the coefficients are given by

$$\begin{aligned} J &:= y(v_E - v) - y_L v, \\ K_F &:= y_* + \mathfrak{c} \mathcal{N}_F - y, \\ a_F &:= a_* + \mathfrak{c}^2 \mathcal{N}_F, \end{aligned}$$

with  $0 < v_F < v_E, y_L, a_*, y_* > 0, \mathfrak{c} \geq 0$ . The definition of  $J$  aligns with equation (1.1), while  $K_F$  and  $a_F$  are derived from equation (1.2) and the closure process. The interaction term in (1.2) becomes a global firing rate  $\mathcal{N}_F$  of the limit system. It influences the evolution of  $F$  through two mechanisms: a current and an internal noise. For  $F_t : \bar{\mathcal{O}} \rightarrow \mathbb{R}$ , the global firing rate is defined as the current generated by all the neurons spiking at time  $t$ , that is

$$(1.4) \quad \mathcal{N}_F(t) := \int_0^\infty J(v_F, y) F_t(v_F, y) dy.$$

It is worth noticing that the equation's nonlinearity, driven by the term  $\mathcal{N}_F$ , is controlled by the connectivity parameter  $\mathfrak{c}$ . In fact, the equation becomes linear when there is no connection between neurons, which corresponds to  $\mathfrak{c} = 0$ . Additionally, similar to classical kinetic models, the noise term in the VCK equation solely affects the conductance variable.

We complement the VCK equation (1.3) with an initial condition

$$(1.5) \quad F(0, \cdot) = F_0 \quad \text{in } \mathcal{O},$$

specifying the probability distribution at time zero. Additionally, boundary conditions are imposed to capture specific behaviors:

$$(1.6) \quad F = 0 \quad \text{on } \Gamma_1 := (0, \infty) \times \Sigma_1,$$

$$(1.7) \quad (JF)(0, y) = (JF)(v_F, y) \quad \text{for any } y > y_F,$$

$$(1.8) \quad K_F F - a_F \partial_y F = 0 \quad \text{on } \Gamma_0 := (0, \infty) \times \Sigma_0,$$

where we define  $y_F := y_L v_F / (v_E - v_F)$ ,

$$\Sigma_0 := (0, v_F) \times \{0\}, \quad \Sigma_1 = (\{0\} \cup \{v_F\}) \times (0, y_F).$$

The first condition ensures that there is no net movement of the probability density across the boundary at low conductance values. Note that  $y_F$  is the solution of the equation  $J(y, v_F) = 0$ , and more precisely, it is the point where the function  $y \mapsto J(y, v_F)$  changes of sign. For values of  $y \leq y_F$ , the vector field  $J$  is negative near  $v_F$  so that the trajectories move away from the boundary at  $v_F$ , i.e., for conductance values smaller than  $y_F$ , the neuron does not receive enough current to spike. On the other hand, as  $y$  pass the value  $y_F$  the voltage of the neuron may reach  $v_F$ , spiking. In that case, it is reset to zero instantaneously, from were the second boundary condition for  $y > y_F$ . The condition in  $\Gamma_0$  follows to assure mass conservation.

We note  $\Sigma_{12} := \Sigma_1 \cup \Sigma_2$ ,  $\Sigma_2 := (\{0\} \cup \{v_F\}) \times (y_F, \infty)$ ,  $\Sigma := \Sigma_{12} \cup \Sigma_0$ ,  $\Gamma = (0, \infty) \times \Sigma$ ,  $\Gamma_i = (0, T) \times \Sigma_i$ ,  $i = 0, 1, 2, 12$ , and  $n_v := +1$  for  $v = v_F$ ,  $n_v := -1$  for  $v = 0$ ,  $n_0 := -1$ . For further references, we also denote

$$\Sigma_i^\pm := \{(v, y) \in \Sigma_i; \pm J n_v > 0\},$$

so that

$$\Sigma_1^+ = \emptyset, \quad \Sigma_1^- = \Sigma_1, \quad \Sigma_2^- = \{0\} \times (y_F, \infty), \quad \Sigma_2^+ = \{v_F\} \times (y_F, \infty),$$

and we next define similarly  $\Sigma_{12}^- = \Sigma_1^- \cup \Sigma_2^-$ ,  $\Gamma_i^\pm := (0, T) \times \Sigma_i^\pm$ . We will sometime summarize (1.8)-(1.6)-(1.7) with the shorthand

$$(1.9) \quad \mathcal{R}_F \gamma F = 0 \quad \text{on} \quad \Gamma := (0, \infty) \times \Sigma,$$

where  $\gamma F$  denotes the trace on  $\Gamma$  of the function  $F$  in order to recall that  $\mathcal{R}_F$  acts on the boundary.

The two important general properties of the model are that (at least formally) any solution is mass conservative, that is

$$(1.10) \quad \langle F(t, \cdot) \rangle = \langle F_0 \rangle, \quad \forall t \geq 0, \quad \langle F \rangle := \int_{\mathcal{O}} F dv dy,$$

and it conserves positivity, that is

$$(1.11) \quad F(t, \cdot) \geq 0, \quad \forall t \geq 0, \quad \text{if} \quad F_0 \geq 0.$$

**1.2. Main results.** We introduce a class of weight functions that will be useful in the sequel. We say that  $\omega : [0, \infty) \rightarrow [0, \infty)$  is an admissible weight function if

$$\omega(y) := \langle y \rangle^k, \quad k > 1, \quad \text{or} \quad \omega(y) := e^{\alpha y}, \quad \alpha > 0,$$

with  $\langle y \rangle^2 := 1 + y^2$ . We then define the weighted Lebesgue space  $L_\omega^p$  associated to the norm

$$\|f\|_{L_\omega^p} := \|f\omega\|_{L^p}.$$

On the one hand, we establish the existence of a solution to the nonlinear problem.

**Theorem 1.1.** *For any admissible weight function  $\omega$ , there exists a constant  $\eta = \eta(\omega) \in (0, \infty)$  such that for any initial datum  $0 \leq F_0 \in L_\omega^\infty$  and any connectivity parameter  $\mathfrak{c} \geq 0$  such that  $(\mathfrak{c} + \mathfrak{c}^2)\|F_0\|_{L_\omega^\infty} < \eta$ , there exists at least one solution  $F \in L^\infty(0, \infty; L_\omega^\infty) \cap C([0, \infty); L_\omega^2)$  to the Voltage-Conductance kinetic (VCK) equation (1.3), (1.4), (1.5), (1.8), (1.6), (1.7) which furthermore satisfies (1.10) and (1.11).*

Inspired by [29, 16, 33, 21, 10], the precise definition of solutions will be given in Section 5. Our result provides in a weak interaction regime the existence of a quite strong solution. Under the normalization hypothesis that  $F_0$  is a probability measure, so that is  $F(t, \cdot)$  for any  $t \geq 0$ , that corresponds to a weak connectivity regime, namely  $\mathfrak{c} > 0$  small enough. Another framework of possible weak solutions has been developed in [41, Sec. 6]. The proof of Theorem 1.1 is based on

the construction of an invariant bounded set for the uniform norm and a compactness argument (Tyknonv fixed point theorem).

It has been established in [41, Theorem 3] that in a weak connectivity regime, and more precisely when  $0 \leq \mathfrak{c} < v_F/v_E$ , there exists a probability measure  $\mathcal{M}$  on  $\mathcal{O}$  such that

$$(1.12) \quad \partial_v(J\mathcal{M}) + \partial_y(K_{\mathcal{M}}\mathcal{M}) - a_{\mathcal{M}}\partial_{yy}^2\mathcal{M} = 0 \quad \text{on } \mathcal{O},$$

$$(1.13) \quad \mathcal{R}_{\mathcal{M}}\gamma\mathcal{M} = 0 \quad \text{on } \Sigma,$$

and that furthermore

$$\mathcal{M} \in L^q, \quad \forall q \in [1, 8/7).$$

This stationary state has been proved to be uniformly bounded and linearly asymptotically stable in [19] and next linearly asymptotic exponentially stable in [18]. We recover and slightly improve these results.

**Theorem 1.2.** *For  $\omega_0 := \langle y \rangle^{k_0}$  and  $\mathfrak{c} < \eta(\omega_0)\|\omega_0^{-1}\|_{L^1}$ , there exists a probability measure  $\mathcal{M}$  which satisfies the stationary equation (1.12), (1.13). Furthermore, for any admissible weight function  $\omega$ , the stationary state  $\mathcal{M}$  satisfies*

$$\mathcal{M} \in L_{\omega}^{\infty}(\mathcal{O}) \cap C(\mathcal{O})$$

and it is linearly asymptotic exponentially stable with constructive constants. More precisely, there exist  $C \geq 1$ ,  $\lambda > 0$  constructive constants such that for any  $f_0 \in L_{\omega}^2 \cap L_{\omega}^p$ ,  $p \in [1, \infty]$ , such that  $\langle f_0 \rangle = 0$ , there exists a unique solution  $f \in C([0, \infty); L_{\omega}^2)$  to the linear equation

$$(1.14) \quad \partial_t f + \partial_v(Jf) + \partial_y(K_{\mathcal{M}}f) - a_{\mathcal{M}}\partial_{yy}^2 f = 0 \quad \text{on } (0, \infty) \times \mathcal{O},$$

$$(1.15) \quad \mathcal{R}_{\mathcal{M}}\gamma f = 0 \quad \text{on } (0, \infty) \times \Sigma,$$

and this one satisfies

$$\|f_t\|_{L_{\omega}^p} \leq C e^{-\lambda t} \|f_0\|_{L_{\omega}^p}, \quad \forall t \geq 0.$$

The proof is in the spirit of the recent works [21, Sec. 11], [11] and [9] where the linear asymptotic exponential stability for the kinetic Fokker-Planck equation set in a domain is established by either taking advantage of hypocoercivity structure (see [46, 2]) or either taking advantage of positivity appearance (through mixing and spreading) and confinement mechanism. We rather follow that second way and more precisely we establish and use a variant of Doblin-Harris theorem as developed recently in [23, 8, 21, 9].

Both above results use in a fundamental way the ultracontractivity property of solutions to the linear (with possibly time dependent coefficients) VCK equation

$$(1.16) \quad \partial_t f = \mathcal{L}_{\mathfrak{a}, K} f := -\partial_v(Jf) - \partial_y(Kf) + \mathfrak{a}\partial_{yy}^2 f \quad \text{in } (0, \infty) \times \mathcal{O},$$

where, for some  $a^* > \max(a_*, y_*)$ ,

$$K := \mathfrak{b} - y, \quad \mathfrak{a}, \mathfrak{b} \in L^{\infty}(0, \infty), \quad y_* \leq \mathfrak{b} \leq a^*, \quad a_* \leq \mathfrak{a} \leq a^*,$$

and the evolution equation is complemented with the boundary conditions (1.6), (1.7) and

$$(1.17) \quad Kf - \mathfrak{a}\partial_y f = 0 \quad \text{on } \Gamma_0.$$

We will sometime summarize (1.6)-(1.7)-(1.17) with the shorthand

$$(1.18) \quad \mathcal{R}_{\mathfrak{a}, K}\gamma f = 0 \quad \text{on } \Gamma.$$

**Theorem 1.3.** *For any exponential weight function  $\omega := e^{\alpha y}$ ,  $\alpha > 0$ , and any initial datum  $f_0 \in L_{\omega}^2$ , the solution  $f$  to the linear Voltage-Conductance kinetic equation (1.16), (1.18) satisfies the ultracontractivity estimate*

$$(1.19) \quad \|f(t, \cdot)\|_{L_{\omega}^{\infty}} \lesssim \frac{e^{\kappa(t-s)}}{(t-s)^{\nu}} \|f(s, \cdot)\|_{L_{\omega}^1},$$

for any  $t > s \geq 0$ , some constants  $\nu > 0$  and  $\kappa$  such that  $0 \leq \kappa \lesssim 1 + \|\mathbf{a}\|_{L^\infty} + \|\mathbf{b}\|_{L^\infty}$ .

Here and below, we use the usual notation  $a \lesssim b$  and  $b \gtrsim a$  when  $a \leq Cb$  for some constant  $C \in (0, \infty)$  and  $a \simeq b$  when  $a \lesssim b$  and  $b \lesssim a$ .

This result is a variant of the ultracontractivity results for the kinetic Fokker-Planck (KFP) equation in a domain established in [11, 9], see also [10]. For general parabolic equations in the whole space, ultracontractivity estimates are a consequence of the De Giorgi-Nash-Moser theory [15, 38, 35, 36], although the notion of ultracontractive semigroup has been introduced later by Davies and Simon [14] (see also [13]). For the classical KFP equation in the whole space, ultracontractivity property is a direct consequence of the representation of the solution thanks to the Kolmogorov kernel [26], of some regularity estimates through Fourier analysis [25, 4] or of some modified energy estimates [24]. Some local uniform estimate of a similar kind for a larger class of KFP equations in the whole space has been established in [40, Theorem 1.2] by using Moser iterative scheme and in [22, Theorem 2.1] by using De Giorgi iterative scheme. We refer to the above mentioned papers, in particular to [11], for more references on the subject as well as to [1] for a general survey about these issues. The proof of Theorem 1.3 is similar to the one of [11, Theorem 1.1] although some steps must be modified. For the gain of integrability  $L^1 \rightarrow L^2$ , the strategy is based on Nash's gain of integrability argument [38] which is performed however on the time integral inequality as in Moser's work [36] what is more convenient in order to use the interior gain of integrability deduced from an adaptation of Hérau regularity result [24]. As in [11], the key argument consists in exhibiting a suitable twisted weight function which makes possible to obtain a priori growth estimate in weighted  $L^p$  space and a nice control (through a penalization term) of the boundary region. Last, the gain of integrability  $L^2 \rightarrow L^\infty$  is obtained by adapting Moser iterative scheme as in [35, 36, 40].

The organization of the paper is as follows. Section 2 is dedicated to the proof of some weighted  $L^p$  a priori growth bounds and the well-posedness of the linear VCK equation in a weighted  $L^2$  framework. Section 3 is devoted to the proof of the ultracontractivity property as stated in Theorem 1.3. In section 4, we establish some useful interior Holder regularity and up to the boundary compactness results. Section 5 is dedicated to the proof of the existence result for the nonlinear problem. In Section 6, we state and prove a constructive version of the Doblin-Harris theorem in a general lattice Banach framework, which is used in Section 7 in order to prove Theorem 1.2.

## 2. GROWTH ESTIMATES AND WELL-POSEDNESS IN $L^2$

**2.1. Growth estimates on the primal problem.** We establish some a priori growth estimates in weighted Lebesgue spaces for a solution  $f$  to the linear VCK equation.

**Lemma 2.1.** *For any admissible weight function  $\omega = \omega(y)$ , there exist a modified weight function  $\tilde{\omega} = \tilde{\omega}(v, y) \simeq \omega$  and a constant  $\kappa \geq 0$  such that any solution  $f$  to the linear VCK equation (1.16), (1.18) satisfies (at least formally)*

$$(2.1) \quad \|f_t\|_{L^p_{\tilde{\omega}}} \leq e^{\kappa t} \|f_0\|_{L^p_{\tilde{\omega}}}, \quad \forall t \geq 0,$$

and  $\kappa \leq C(1 + \|\mathbf{a}\|_{L^\infty} + \|\mathbf{b}\|_{L^\infty})$  for some constant  $C = C_\omega \in (0, \infty)$ .

*Proof of Lemma 2.1.* For simplicity and because the linear VCK equation is (at least formally) positivity preserving, we may focus on the case  $f = f(t, v, y) \geq 0$ . For  $p \in [1, \infty)$ , we write

$$\frac{1}{p} \frac{d}{dt} \int_{\mathcal{O}} f^p \tilde{\omega}^p = \int_{\mathcal{O}} \partial_v (-Jf) f^{p-1} \tilde{\omega}^p + \int_{\mathcal{O}} \partial_y (\mathbf{a} \partial_y f - Kf) f^{p-1} \tilde{\omega}^p =: A + B,$$

and we then consider each term separately. For the first term, we have

$$\begin{aligned}
A &= - \int_{\mathcal{O}} f^p \tilde{\omega}^p \partial_v J - \frac{1}{p} \int_{\mathcal{O}} J \tilde{\omega}^p \partial_v (f^p) \\
&= \int_{\mathcal{O}} f^p \left( \frac{1}{p} \partial_v (\tilde{\omega}^p J) - \tilde{\omega}^p \partial_v J \right) - \frac{1}{p} \int_{\Sigma} J f^p \tilde{\omega}^p n_v \\
&= \int_{\mathcal{O}} f^p \tilde{\omega}^p \left( J \frac{\partial_v \tilde{\omega}}{\tilde{\omega}} + \left( \frac{1}{p} - 1 \right) \partial_v J \right) - \frac{1}{p} \int_{\Sigma_2} J f^p \tilde{\omega}^p n_v,
\end{aligned}$$

where we have used the Green formula in the second line and the boundary condition on  $\Sigma_1$  in the last line. On the other hand, we compute

$$\begin{aligned}
B &= \int_{\mathcal{O}} (Kf - \mathbf{a} \partial_y f) \partial_y ((f \tilde{\omega})^{p-1}) \tilde{\omega} \\
&= \int_{\mathcal{O}} Kf \tilde{\omega} \partial_y ((f \tilde{\omega})^{p-1}) + \int_{\mathcal{O}} Kf (\partial_y \tilde{\omega}) (f \tilde{\omega})^{p-1} \\
&\quad - \int_{\mathcal{O}} \mathbf{a} \partial_y f \tilde{\omega} \partial_y ((f \tilde{\omega})^{p-1}) - \int_{\mathcal{O}} \mathbf{a} \partial_y f (\partial_y \tilde{\omega}) (f \tilde{\omega})^{p-1} \\
&= \int_{\mathcal{O}} Kf \tilde{\omega} \partial_y ((f \tilde{\omega})^{p-1}) + \int_{\mathcal{O}} Kf (\partial_y \tilde{\omega}) (f \tilde{\omega})^{p-1} \\
&\quad - \int_{\mathcal{O}} \mathbf{a} \partial_y (f \tilde{\omega}) \partial_y ((f \tilde{\omega})^{p-1}) + \int_{\mathcal{O}} \mathbf{a} f (\partial_y \tilde{\omega}) \partial_y ((f \tilde{\omega})^{p-1}) \\
&\quad - \int_{\mathcal{O}} \mathbf{a} \partial_y (f \tilde{\omega}) \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} (f \tilde{\omega})^{p-1} + \int_{\mathcal{O}} \mathbf{a} (f \tilde{\omega})^p \left( \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \right)^2 \\
&= \int_{\mathcal{O}} \left[ \left( 1 - \frac{1}{p} \right) K + \left( 1 - \frac{2}{p} \right) \mathbf{a} \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \right] \partial_y ((f \tilde{\omega})^p) + \int_{\mathcal{O}} K \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} (f \tilde{\omega})^p \\
&\quad - 4 \frac{p-1}{p^2} \int_{\mathcal{O}} \mathbf{a} (\partial_y (f \tilde{\omega})^{p/2})^2 + \int_{\mathcal{O}} \mathbf{a} (f \tilde{\omega})^p \left( \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \right)^2,
\end{aligned}$$

where we have used the Green formula and boundary condition on  $\Sigma_0$  in the first line and we have then just rearrange the terms in order to forcing the dependence on the function  $f \tilde{\omega}$ . Using once more the Green formula in the first integral of the term  $B$  and gathering the two contributions, we obtain

$$\begin{aligned}
\frac{1}{p} \frac{d}{dt} \int_{\mathcal{O}} f^p \tilde{\omega}^p &= - \frac{4(p-1)}{p^2} \int_{\mathcal{O}} \mathbf{a} (\partial_y (f \tilde{\omega})^{p/2})^2 + \int_{\mathcal{O}} f^p \tilde{\omega}^p \varpi \\
&\quad + \int_{\Sigma_0} \left[ \left( 1 - \frac{1}{p} \right) K + \left( 1 - \frac{2}{p} \right) \mathbf{a} \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \right] n_0 (f \tilde{\omega})^p - \frac{1}{p} \int_{\Sigma_2} J f^p \tilde{\omega}^p n_v,
\end{aligned}$$

with

$$(2.2) \quad \varpi := 2 \left( 1 - \frac{1}{p} \right) \mathbf{a} \left( \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \right)^2 + \left( \frac{2}{p} - 1 \right) \mathbf{a} \frac{\partial_{yy}^2 \tilde{\omega}}{\tilde{\omega}} + K \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} + J \frac{\partial_v \tilde{\omega}}{\tilde{\omega}} + \left( \frac{1}{p} - 1 \right) \partial_y K + \left( \frac{1}{p} - 1 \right) \partial_v J.$$

In order to make negative the boundary contribution, we define first

$$(2.3) \quad w := \chi + (1 - \chi) \omega,$$

with  $\chi = \chi(y) \in C^2(\mathbb{R})$ ,  $\mathbf{1}_{[0, y_F/2]} \leq \chi \leq \mathbf{1}_{[0, y_F]}$ , and next

$$(2.4) \quad \tilde{\omega}^p := (\chi + (1 - \chi) J_\xi^{p-1} y^{1-p}) w^p, \quad J_\xi := \xi J(0, y) + (1 - \xi) J(v, y),$$

with  $\xi = \xi(y) \in C^2(\mathbb{R})$ ,  $\mathbf{1}_{[0, y_F]} \leq \chi \leq \mathbf{1}_{[0, 2y_F]}$ .

With this choice, we have  $\tilde{\omega} = 1$  on  $(0, v_F) \times (0, y_F/2)$ , thus

$$(1 - \frac{1}{p})K + (1 - \frac{2}{p})\mathbf{a} \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} = (1 - \frac{1}{p})K \geq (1 - \frac{1}{p})y_* \geq 0 \quad \text{on } \Sigma_0,$$

and the contribution of the boundary term on  $\Sigma_0$  is non positive. On the other hand, the contribution of the boundary term on  $\Gamma_2$  is

$$\begin{aligned} -\frac{1}{p} \int_{\Sigma_2} J f^p \tilde{\omega}^p n_v &= \frac{1}{p} \int_{y_F}^{\infty} (J f^p \tilde{\omega}^p)(0, y) - (J f^p \tilde{\omega}^p)(v_F, y) \\ &= \frac{1}{p} \int_{y_F}^{\infty} ((J^p f^p)(0, y) - (J J_{\xi}^{p-1} f^p)(v_F, y)) w^p y^{1-p} \\ &\leq \frac{1}{p} \int_{y_F}^{\infty} ((J f)^p(0, y) - (J f)^p(v_F, y)) w^2 y^{1-p} = 0, \end{aligned}$$

where we use that  $J_{\xi} \geq J$  in the second line and the boundary condition in the last line. Now, from the very definition of  $\tilde{\omega}$ , we have

$$\varpi \leq \kappa_1 := C(1 + \|\mathbf{a}\|_{L^\infty} + \|\mathbf{b}\|_{L^\infty}) \quad \text{on } (0, T) \times (0, v_F) \times (0, 2y_F]$$

uniformly on  $p \in [1, \infty)$  and for some constant  $C = C(\tilde{\omega}) \geq 0$ . On the other hand, we have  $\tilde{\omega} = Q^{1-1/p} \omega$  on  $(0, v_F) \times (2y_F, \infty)$ ,  $Q := J/y = v_E - v - y_L v/y$ , so that

$$\begin{aligned} \frac{\partial_v \tilde{\omega}}{\tilde{\omega}} &= (1 - \frac{1}{p}) \frac{\partial_v J}{J} \\ \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} &= (1 - \frac{1}{p}) \frac{\partial_y Q}{Q} + \frac{\partial_y \omega}{\omega} = \frac{\partial_y \omega}{\omega} + \mathcal{O}(\langle y \rangle^{-1}) \\ \frac{\partial_{yy}^2 \tilde{\omega}}{\tilde{\omega}} &= -\frac{1}{p} (1 - \frac{1}{p}) \left( \frac{\partial_y Q}{Q} \right)^2 + (1 - \frac{1}{p}) \frac{\partial_{yy}^2 Q}{Q} + 2(1 - \frac{1}{p}) \frac{\partial_y Q}{Q} \frac{\partial_y \omega}{\omega} + \frac{\partial_{yy}^2 \omega}{\omega}. \end{aligned}$$

Observing that

$$Q = v_E - v - y_L v/y, \quad \frac{\partial_y Q}{Q} = \mathcal{O}(y^{-2}), \quad \frac{\partial_{yy}^2 Q}{Q} = \mathcal{O}(y^{-3}), \quad \frac{\partial_v Q}{Q} = -\frac{1}{v_E - v} + \mathcal{O}(y^{-1}),$$

because  $\omega$  is an admissible weight function, we deduce

$$\frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \sim \frac{\partial_y \omega}{\omega}, \quad \frac{\partial_{yy}^2 \tilde{\omega}}{\tilde{\omega}} \sim \frac{\partial_{yy}^2 \omega}{\omega}$$

and next

$$\begin{aligned} \varpi &= 2(1 - \frac{1}{p})\mathbf{a} \left( \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \right)^2 + \left( \frac{2}{p} - 1 \right) \mathbf{a} \frac{\partial_{yy}^2 \tilde{\omega}}{\tilde{\omega}} + K \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} + 1 - \frac{1}{p} \\ &\sim 2(1 - \frac{1}{p})\mathbf{a} \left( \frac{\partial_y \omega}{\omega} \right)^2 + \left( \frac{2}{p} - 1 \right) \mathbf{a} \frac{\partial_{yy}^2 \omega}{\omega} - y \frac{\partial_y \omega}{\omega} + 1 - \frac{1}{p} \end{aligned}$$

as  $y \rightarrow \infty$ . When  $\omega = y^k$ , we have

$$(2.5) \quad \varpi \sim -k + 1 - \frac{1}{p}.$$

When  $\omega = e^{\alpha y}$ ,  $\alpha > 0$ , we have

$$\varpi \sim -\alpha y.$$

More precisely, in both cases, we have

$$(2.6) \quad \varpi \leq \kappa_2 - \varsigma \quad \text{on } (0, T) \times (0, v_F) \times (2y_F, \infty)$$



uniformly on  $p \in [1, \infty)$ , with

$$\begin{aligned}\kappa_2 &:= C(1 + \|\mathbf{a}\|_{L^\infty} + \|\mathbf{b}\|_{L^\infty}) \\ \varsigma &:= k - 1 + \frac{1}{p} \text{ if } \omega = \langle y \rangle^k, \quad \varsigma := \alpha y \text{ if } \omega = e^{\alpha y}.\end{aligned}$$

All together, we have established

$$(2.7) \quad \frac{1}{p} \frac{d}{dt} \int_{\mathcal{O}} f^p \tilde{\omega}^p \leq -\frac{4(p-1)}{p^2} \int_{\mathcal{O}} \mathbf{a}(\partial_y(f\tilde{\omega})^{p/2})^2 + \int_{\mathcal{O}} f^p \tilde{\omega}^p \varpi,$$

and from (2.5) and (2.6), we deduce

$$(2.8) \quad \frac{1}{p} \frac{d}{dt} \int_{\mathcal{O}} f^p \tilde{\omega}^p \leq -\frac{4(p-1)}{p^2} \int_{\mathcal{O}} \mathbf{a}(\partial_y(f\tilde{\omega})^{p/2})^2 + \kappa \int_{\mathcal{O}} f^p \tilde{\omega}^p,$$

with  $\kappa := \max(\kappa_1, \kappa_2)$ . We conclude thanks to the Gronwall lemma in the case  $p \in [1, \infty)$  and passing to the limit  $p \rightarrow \infty$  in the resulting estimate for dealing with the case  $p = \infty$ .  $\square$

**2.2. About the well posedness.** In this section, we build a solution to the Cauchy problem associated to the linear VCK equation in a  $L^2$  framework. We denote  $d\xi_1 := \tilde{\omega}^2 |J| dt dy$  and  $d\xi_2 := \omega^2 J^2 / \langle y \rangle^2 dt dy$  the Borel measures on the boundary  $\Gamma_{12}$  and also  $d\xi_1 := \tilde{\omega}^2 K dt dv$  the Borel measures on the boundary  $\Gamma_0$ .

**Theorem 2.2.** *For any admissible weight function  $\omega$  and any initial datum  $f_0 \in L_\omega^2 \subset L^1$ , there exists a unique solution  $f \in C([0, T]; L_\omega^2) \cap L^2((0, T) \times (0, v_F); H^1(0, \infty))$ ,  $\forall T > 0$ , to the linear VCK equation (1.16)-(1.18) and this one satisfies the growth estimate (2.1) (for  $p = 2$ ). More precisely, there exists a trace function  $\gamma f \in L^2(\Gamma_{12}; d\xi_2)$  such that  $f$  is a renormalized solution, in the sense that*

$$(2.9) \quad \begin{aligned} & \int_{\mathcal{O}} \beta(f_T) \psi + \int_{\Gamma} J \beta(\gamma f) \psi + \int_{\mathcal{U}} \beta(f) [\partial_t \psi + \partial_v (J \psi)] - \int_{\mathcal{U}} (\partial_v J) f \beta'(f) \psi \\ &= - \int_{\mathcal{U}} (K f - \mathbf{a} \partial_y f) \partial_y (\beta'(f) \psi) + \int_{\mathcal{O}} \beta(f_0) \psi, \end{aligned}$$

for any  $\beta \in W^{1, \infty}(\mathbb{R})$  and any  $\psi \in C_c^1(\bar{\mathcal{U}})$ , and where  $\gamma f$  satisfies the boundary conditions (1.6), (1.7) in the a.e. sense. The additional boundary condition (1.17) is encapsulated in the fact that  $\psi$  does not necessarily vanish on the boundary set  $\Gamma_0$ . Furthermore, if  $f_0 \geq 0$ , the solution  $f$  satisfies

$$f(t, \cdot) \geq 0 \quad \text{and} \quad \|f(t, \cdot)\|_{L^1} = \|f_0\|_{L^1}, \quad \forall t \geq 0.$$

*Proof of Theorem 2.2.* The proof is a bit tedious but classical. It is split into 7 steps.

*Step 1.* We define  $\Gamma_2^- := \{(t, v, y) \in \Gamma_2, J n_v < 0\} = (0, T) \times \{0\} \times (y_F, \infty)$ . Given a function  $\mathbf{g} \in L^2(\Gamma_2^-; J \omega^2 dt dy)$ , we solve the inflow problem (1.16), (1.17), (1.6) and

$$(2.10) \quad f = \mathbf{g} \quad \text{on} \quad \Gamma_2^-$$

thanks to Lions' variant of the Lax-Milgram theorem [29, Chap III, §1]. More precisely, we define the Hilbert space  $\mathcal{H}$  associated to the Hilbert norm  $\|\cdot\|_{\mathcal{H}}$  defined by

$$\|f\|_{\mathcal{H}}^2 := \|f\|_{L_{\omega_1}^2(\mathcal{U})}^2 + \|\partial_y f\|_{L_\omega^2(\mathcal{U})}^2,$$

with  $\omega_1 = \omega$  in the polynomial case and  $\omega_1 = \omega \langle y \rangle^{1/2}$  in the exponential case, and we define the bilinear form  $\mathcal{E}_\lambda : \mathcal{H} \times C_c^1([0, T] \times (\mathcal{O} \cup \Sigma_-)) \rightarrow \mathbb{R}$ , by

$$\mathcal{E}_\lambda(f, \varphi) := \int_{\mathcal{U}} f(\lambda - \partial_t - J \partial_v - K \partial_y)(\varphi \tilde{\omega}^2) + \int_{\mathcal{U}} \mathbf{a} \partial_y f \partial_y (\varphi \tilde{\omega}^2),$$

where  $\lambda > 0$ ,  $\Sigma_- := \Sigma_0 \cup \Sigma_1 \cup \Sigma_{2-}$  and  $\tilde{\omega}$  is defined during the proof of Lemma 2.1. Proceeding as during the proof of Lemma 2.1, we have

$$\begin{aligned} \int_{\mathcal{U}} \varphi(-J) \partial_v(\tilde{\omega}^2 \varphi) &= \int_{\mathcal{U}} (\tilde{\omega} \varphi)^2 \left( \frac{1}{2} \partial_v J - J \frac{\partial_v \tilde{\omega}}{\tilde{\omega}} \right) - \frac{1}{2} \partial_v (J(\tilde{\omega} \varphi)^2) \\ &= - \int_{\Gamma_{12}} \frac{1}{2} n_v J(\tilde{\omega} \varphi)^2 + \int_{\mathcal{U}} (\tilde{\omega} \varphi)^2 \left( \frac{1}{2} \partial_v J - J \frac{\partial_v \tilde{\omega}}{\tilde{\omega}} \right) \end{aligned}$$

and similarly

$$\int_{\mathcal{U}} \varphi(-K) \partial_y(\tilde{\omega}^2 \varphi) = - \int_{\Gamma_0} \frac{1}{2} n_y K(\tilde{\omega} \varphi)^2 + \int_{\mathcal{U}} (\tilde{\omega} \varphi)^2 \left( \frac{1}{2} \partial_y K - K \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \right).$$

Observing that

$$\int_{\mathcal{U}} \mathbf{a} \partial_y \varphi \partial_y(\varphi \tilde{\omega}^2) = \int_{\mathcal{U}} \mathbf{a} (\partial_y(\varphi \tilde{\omega}))^2 - \int_{\mathcal{U}} \mathbf{a} (\varphi \tilde{\omega})^2 \frac{(\partial_y \tilde{\omega})^2}{\tilde{\omega}^2},$$

we get

$$\begin{aligned} \mathcal{E}_\lambda(\varphi, \varphi) &= \int_{\mathcal{U}} (\lambda - \varpi)(\tilde{\omega} \varphi)^2 + \int_{\mathcal{U}} \mathbf{a} (\partial_y(\tilde{\omega} \varphi))^2 + \frac{1}{2} \int_{\mathcal{O}} (\tilde{\omega} \varphi)(0, \cdot)^2 \\ &\quad - \frac{1}{2} \int_{\Gamma_{12}} (\tilde{\omega} \varphi)^2 J n_v - \frac{1}{2} \int_{\Gamma_0} (\tilde{\omega} \varphi)^2 K n_y, \end{aligned}$$

for any  $\varphi \in C_c^1(\mathcal{U} \cup \Gamma_-)$ , and where  $\varpi$  is defined (for  $p = 2$ ) in (2.2). We choose  $\lambda > 0$  large enough, in such a way that  $\lambda - \varpi \geq 1$ . Because each contribution is then nonnegative separately, the above quadratic form is coercive in the sense that

$$\mathcal{E}_\lambda(\varphi, \varphi) \geq \min(a_*, 1) \|\varphi\|_{\mathcal{H}}^2, \quad \forall \varphi \in C_c^1(\mathcal{U} \cup \Gamma_-).$$

We define the linear form

$$\ell_\lambda(\varphi) := \int_{\Gamma_{2-}} (-J n_v) \mathbf{g} e^{-\lambda t} \tilde{\omega}^2 \varphi + \int_{\mathcal{O}} f_0 \tilde{\omega}^2 \varphi(0, \cdot), \quad \forall \varphi \in C_c^1(\mathcal{U} \cup \Gamma_-),$$

and we observe that  $|\ell(\varphi)| \lesssim \mathcal{E}_\lambda(\varphi, \varphi)^{1/2}$  for any  $\varphi \in C_c^1(\mathcal{U} \cup \Gamma_-)$ . The above mentioned Lions' theorem implies the existence of a function  $f_\lambda \in \mathcal{H}$  which satisfies the variational equation

$$\mathcal{E}_\lambda(f_\lambda, \psi) = \int_{\Gamma_{2-}} \mathbf{g} e^{-\lambda t} \tilde{\omega}^2 \psi + \int_{\mathcal{O}} f_0 \tilde{\omega}^2 \psi(0, \cdot), \quad \forall \psi \in C_c^1(\mathcal{U} \cup \Gamma_-).$$

Defining  $f := f_\lambda e^{\lambda t}$  and testing this variational equation with  $\psi := \varphi e^{\lambda t}$ , we deduce

$$(2.11) \quad \int_{\mathcal{U}} f(-\partial_t - J \partial_v)(\varphi \tilde{\omega}^2) + \int_{\mathcal{U}} (\mathbf{a} \partial_y f - K) \partial_y(\varphi \tilde{\omega}^2) = \int_{\Gamma_{2-}} (-J n_v) \mathbf{g} \varphi \tilde{\omega}^2 + \int_{\mathcal{O}} f_0 \varphi(0, \cdot) \tilde{\omega}^2,$$

for any  $\varphi \in C_c^1(\mathcal{U} \cup \Gamma_-)$ , and thus for any  $\varphi \in H_0^1(\mathcal{U} \cup \Gamma_-)$ , the closure of  $C_c^1(\mathcal{U} \cup \Gamma_-)$  in  $H^1(\mathcal{U})$ . That last variational equation is classically a weak formulation of the fact that  $f \in \mathcal{H}_T, \forall T > 0$ , is a global solution to the inflow problem (1.16), (1.17), (1.6), (2.10). It is worth emphasizing that here and always below the no flux condition (1.17) has to be understood in the weak sense, namely by the fact that the test functions do not necessarily vanish on  $\Gamma_0$ .

*Step 2.* At least formally, multiplying the equation (1.16) by  $\beta'(f)\psi$  for a renormalizing function  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  and a test function  $\psi : \mathcal{U} \rightarrow \mathbb{R}$ , we have

$$\begin{aligned} \partial_t(\beta(f)\psi) - \beta(f)\partial_t\psi + \partial_v[J\beta(f)\psi] - \beta(f)\partial_v(J\psi) + (\partial_v J)f\beta'(f)\psi \\ + \partial_y[\beta'(f)\psi(Kf - \mathbf{a}\partial_y f)] - [\partial_y(\beta'(f)\psi)](Kf - \mathbf{a}\partial_y f) = 0. \end{aligned}$$

When  $\text{supp}\psi \subset \mathcal{U} \cup \Gamma_0 = (0, T) \times (0, v_F) \times [0, \infty)$  and using the no flux condition (1.17), we get after integration

$$(2.12) \quad - \int_{\mathcal{U}} \beta(f) [\partial_t \psi + \partial_v (J\psi)] + \int_{\mathcal{U}} (\partial_v J) f \beta'(f) \psi - \int_{\mathcal{U}} (Kf - \mathbf{a} \partial_y f) \partial_y (\beta'(f) \psi) = 0.$$

This step is devoted to sketch a rigorous proof of (2.12) by using some classical arguments in the spirit of DiPerna and Lions [17]. For  $\beta \in C^1 \cap W^{1, \infty}$ ,  $\psi \in C_c^1(\mathcal{U} \cup \Gamma_0)$  and a symmetric mollifier  $(\rho_\varepsilon)$  in  $\mathcal{D}(\mathbb{R}^2)$ , we choose  $\varphi_\varepsilon := \tilde{\omega}^{-2} \rho_\varepsilon * (\beta'(f * \rho_\varepsilon) \psi)$  as a test function in (2.11), where  $*$  =  $*_{t,v}$  denotes the convolution operator in the  $t$  and  $v$  variables. Denoting  $f_\varepsilon = f * \rho_\varepsilon$ , for  $\varepsilon > 0$  small enough, we have

$$(2.13) \quad \int_{\mathcal{U}} f (-\partial_t - J \partial_v) (\varphi_\varepsilon \tilde{\omega}^2) = - \int_{\mathcal{U}} f_\varepsilon \partial_t (\beta'(f_\varepsilon) \psi) - \int_{\mathcal{U}} (fJ) * \rho_\varepsilon \partial_v (\beta'(f_\varepsilon) \psi).$$

Observing that  $f_\varepsilon \in H_0^1(\mathcal{U} \cup \Gamma_0)$ , so that we may use the chain rule, we have

$$\begin{aligned} - \int_{\mathcal{U}} f_\varepsilon \partial_t (\beta'(f_\varepsilon) \psi) &= \int_{\mathcal{U}} \partial_t f_\varepsilon \beta'(f_\varepsilon) \psi \\ &= \int_{\mathcal{U}} \partial_t \beta(f_\varepsilon) \psi = - \int_{\mathcal{U}} \beta(f_\varepsilon) \partial_t \psi. \end{aligned}$$

We similarly have

$$- \int_{\mathcal{U}} (fJ) * \rho_\varepsilon \partial_v (\beta'(f_\varepsilon) \psi) = - \int_{\mathcal{U}} \beta(f_\varepsilon) \partial_v (J\psi) + \int_{\mathcal{U}} f_\varepsilon \beta'(f_\varepsilon) \psi \partial_v J + R_\varepsilon,$$

with

$$R_\varepsilon := \int_{\mathcal{U}} r_\varepsilon \beta'(f_\varepsilon) \psi, \quad r_\varepsilon := \partial_v [(fJ) * \rho_\varepsilon - J(f * \rho_\varepsilon)].$$

For the last term, we write  $r_\varepsilon = r_\varepsilon^1 + r_\varepsilon^2$ , with

$$r_\varepsilon^1 := (f \partial_v J) * \rho_\varepsilon - (\partial_v J) f_\varepsilon, \quad r_\varepsilon^2 := (J \partial_v f) * \rho_\varepsilon - J \partial_v f_\varepsilon,$$

where  $r_\varepsilon^1 \rightarrow 0$  in  $L^2$  straightforwardly and  $r_\varepsilon^2 \rightarrow 0$  in  $L^2$  thanks to DiPerna-Lions commutator Lemma [17, Lemma II.1]. We easily pass to the limit in (2.13) and we get

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{U}} f (-\partial_t - J \partial_v) (\varphi_\varepsilon \tilde{\omega}^2) = - \int_{\mathcal{U}} \beta(f) \partial_t \psi - \int_{\mathcal{U}} \beta(f) \partial_v (J\psi) + \int_{\mathcal{U}} f \beta'(f) (\partial_v J) \psi.$$

From the above definition and the fact that  $\beta'(f) \in L_v^2 L_{t, \text{loc}}^2 H_{y, \text{loc}}^1$ , we also have

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{U}} (\mathbf{a} \partial_y f - K) \partial_y (\varphi_\varepsilon \tilde{\omega}^2) = - \int_{\mathcal{U}} (Kf - \mathbf{a} \partial_y f) \partial_y (\beta'(f) \psi).$$

All together, we have thus established (2.12).

*Step 3.* We assume furthermore that  $f$  is smooth up to the boundary and we establish three additional estimates. Taking  $\beta(s) := s^2$  and  $\psi = \chi_1(t) \chi_2(v)$ ,  $\chi_1 \in \mathcal{D}((0, T))$ ,  $\chi_2 \in \mathcal{D}((0, v_F))$ ,  $\chi_i \geq 0$ , in (2.12), we get

$$\frac{d}{dt} \int_{\mathcal{O}} f^2 \chi_2 = \int_{\mathcal{O}} I_f, \quad I_f := f^2 [J \partial_v \chi_2 - \chi_2 \partial_v J] + 2(Kf - \mathbf{a} \partial_y f) (\partial_y f) \chi_2,$$

and integrating twice in the time variable, we obtain

$$(2.14) \quad \sup_{[0, T]} \int_{\mathcal{O}} f^2 \chi_2 \leq T \int_{\mathcal{U}} I_f \lesssim T \|\chi_2\|_{W^{1, \infty}} \|f\|_{\mathcal{H}}^2.$$

Taking now  $\psi := \chi_1(t)\chi_2(v)$ ,  $\chi_2 := (v_F/2 - v)_+$ , in the equation preceding (2.12) or equivalently taking  $\psi := \chi_1(t)\chi_2(v)\chi_{2\varepsilon}(v)$  in (2.12) with  $\chi_{2\varepsilon} \rightarrow 1$ ,  $\chi'_{2\varepsilon} \rightarrow -\delta_0$  and passing to the limit  $\varepsilon \rightarrow 0$ , we get

$$\int_{\Gamma_{12}^0} f^2 J \chi_1 \frac{v_F}{2} = \int_{\mathcal{U}} f^2 [\partial_t \psi + J \partial_v \psi - \psi \partial_v J] + 2 \int_{\mathcal{U}} \psi (Kf - \mathbf{a} \partial_y f) \partial_y f,$$

with  $\Gamma_{12}^0 := (0, T) \times \Sigma_{12}^0$ ,  $\Sigma_{12}^0 := \{0\} \times (0, \infty)$ . We deduce

$$(2.15) \quad \int_{\Gamma_{12}^0} f^2 J \chi_1 \lesssim \|\chi_1\|_{W^{1,\infty}} \|f\|_{\mathcal{H}}^2.$$

We finally take  $\psi := \chi_1(t)\chi_2(v)$ ,  $\chi_2 := J(v - v_F/2)_+$ , in the equation preceding (2.12) and we get

$$\int_{\Gamma_{12}^{v_F}} f^2 J^2 \chi_1 \frac{v_F}{2} = \int_{\mathcal{U}} f^2 [-\partial_t \psi - J \partial_v \psi + \psi \partial_v J] + 2 \int_{\mathcal{U}} \psi (\mathbf{a} \partial_y f - Kf) \partial_y f,$$

with  $\Gamma_{12}^{v_F} := (0, T) \times \Sigma_{12}^{v_F}$ ,  $\Sigma_{12}^{v_F} := \{v_F\} \times (0, \infty)$ . We deduce

$$(2.16) \quad \int_{\Gamma_{12}^{v_F}} f^2 J^2 \chi_1 \lesssim \|\chi_1\|_{W^{1,\infty}} \|f\|_{\mathcal{H}}^2.$$

*Step 4.* Using the same suitably modified convolution trick as in the trace theory developed in [32, 33, 21, 10] for the Vlasov equation and the kinetic Fokker-Planck equation, we can take up again the arguments used in Step 2 and we easily establish the existence of a sequence  $(f_\varepsilon)$  of  $H^1(\mathcal{U})$  such that  $f_\varepsilon \rightarrow f$  strongly in  $\mathcal{H}$  and a.e. on  $\mathcal{U}$ , and such that

$$- \int_{\mathcal{U}} \beta(f_\varepsilon) [\partial_t \psi + \partial_v (J\psi)] + \int_{\mathcal{U}} (\partial_v J) f_\varepsilon \beta'(f_\varepsilon) \psi - \int_{\mathcal{U}} (Kf - \mathbf{a} \partial_y f) \partial_y \varphi_\varepsilon = \int_{\mathcal{U}} r_\varepsilon \beta'(f_\varepsilon) \psi,$$

for any  $\varepsilon > 0$ , any  $\psi \in C_c^1(\mathcal{U})$  and any  $\beta \in C^1 \cap W^{1,\infty}$ , with  $r_\varepsilon \rightarrow 0$  in  $L^1_{\text{loc}}(\bar{\mathcal{U}})$  and  $\varphi_\varepsilon \rightarrow \beta'(f)\psi$  in  $\mathcal{H}$ . The above relation is the same as the one leading to (2.12) except on the fact that we do not impose any smallness condition on  $\varepsilon > 0$  which should depend on  $\psi$ . As a consequence, for  $\psi \in C_c^1(\bar{\mathcal{U}})$ , we may deduce from it the Green formula which tells us that

$$(2.17) \quad \left[ \int_{\mathcal{O}} \beta(f_{\varepsilon t}) \psi \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Sigma_{12}} J h_\varepsilon \psi - \int_{t_1}^{t_2} \int_{\mathcal{O}} [\beta(f_\varepsilon) (\partial_t \psi + \partial_v (J\psi)) - (\partial_v J) f_\varepsilon \beta'(f_\varepsilon) \psi] \\ - \int_{t_1}^{t_2} \int_{\mathcal{O}} (Kf - \mathbf{a} \partial_y f) \partial_y \varphi_\varepsilon = \int_{t_1}^{t_2} \int_{\mathcal{O}} r_\varepsilon \beta'(f_\varepsilon) \psi.$$

The same identity holds with  $f_\varepsilon$  replaced by  $f_{\varepsilon\varepsilon'}$  :=  $f_\varepsilon - f_{\varepsilon'}$  because this identity has been established starting from the weak formulation (2.11) which depends linearly on  $f$ . Using (2.14), (2.15) and (2.16) in Step 3, we deduce that  $(f_{\varepsilon\varepsilon'})$  tends to 0 in  $C([0, T]; L^2_{\text{loc}}(\mathcal{O}))$ , in  $L^2((\tau, T - \tau) \times \Sigma_{12}^0; J dy dt)$ ,  $\forall \tau \in (0, T)$ , and in  $L^2((\tau, T - \tau) \times \Sigma_{12}^{v_F}; J^2 dy dt)$ ,  $\forall \tau \in (0, T)$ , so that  $(f_\varepsilon)$  is a Cauchy sequence in the same spaces. In other words, there exist a function  $t \mapsto f_t \in C([0, T]; L^2_{\text{loc}}(\mathcal{O}))$ , and a function  $\gamma f$  defined on  $\Gamma_{12}$  satisfying  $\gamma f \in L^2((\tau, T - \tau) \times \Sigma_{12}^0; J dy dt)$ ,  $\forall \tau \in (0, T)$ , and  $\gamma f \in L^2((\tau, T - \tau) \times \Sigma_{12}^{v_F}; J^2 dy dt)$ ,  $\forall \tau \in (0, T)$ , such that

$$f_\varepsilon(t, \cdot) \rightarrow f_t \text{ in } C([0, T]; L^2_{\text{loc}}(\mathcal{O})) \text{ and a.e. on } \mathcal{O}, \forall t \in [0, T]; \\ f_\varepsilon \rightarrow \gamma f \text{ in } L^2((\tau, T - \tau) \times \Sigma_{12}^0; J dy dt), \forall \tau \in (0, T), \text{ and a.e. on } \Gamma_{12}^0; \\ f_\varepsilon \rightarrow \gamma f \text{ in } L^2((\tau, T - \tau) \times \Sigma_{12}^{v_F}; J^2 dy dt), \forall \tau \in (0, T), \text{ and a.e. on } \Gamma_{12}^{v_F}.$$

Passing to the limit in (2.17), we get

$$(2.18) \quad - \int_{t_1}^{t_2} \int_{\mathcal{O}} [\beta(f) (\partial_t \psi + \partial_v (J\psi)) - (\partial_v J) f \beta'(f) \psi] - \int_{t_1}^{t_2} \int_{\mathcal{O}} (Kf - \mathbf{a} \partial_y f) \partial_y (\beta'(f) \psi) \\ + \left[ \int_{\mathcal{O}} \beta(f(t, \cdot)) \psi \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Sigma_{12}} J n_v \beta(\gamma f) \psi = 0,$$

for any  $\psi \in C_c^1(\bar{\mathcal{U}})$  and any  $\beta \in C^1 \cap W^{1,\infty}$ , and we use indifferently the notations  $f_t = f(t, \cdot)$ . Because  $f \in \mathcal{H}$ , the function  $f$  admits a trace on  $\Gamma_0$ , also denoted by  $\gamma f \in L^2(\Gamma_0; dt dv)$ , and integrating twice by part, we have

$$\begin{aligned} - \int_{t_1}^{t_2} \int_{\mathcal{O}} K f \partial_y (\beta'(f) \psi) &= - \int_{t_1}^{t_2} \int_{\Sigma_0} n_y K \gamma f \beta'(\gamma f) \psi + \int_{t_1}^{t_2} \int_{\mathcal{O}} \beta'(f) \psi \partial_y (K f) \\ &= \int_{t_1}^{t_2} \int_{\Sigma_0} K \gamma f \beta'(\gamma f) \psi + \int_{t_1}^{t_2} \int_{\mathcal{O}} [\beta'(f) f \psi \partial_y K + \psi K \partial_y \beta(f)] \\ &= \int_{t_1}^{t_2} \int_{\Sigma_0} K (\gamma f \beta'(\gamma f) - \beta(\gamma f)) \psi + \int_{t_1}^{t_2} \int_{\mathcal{O}} [\beta'(f) f \psi \partial_y K - \beta(f) \partial_y (\psi K)]. \end{aligned}$$

Together with (2.18), we also have

$$(2.19) \quad \begin{aligned} & \left[ \int_{\mathcal{O}} \beta(f(t, \cdot)) \psi \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Sigma_{12}^-} J n_v \beta(\gamma f) \psi + \int_{t_1}^{t_2} \int_{\Sigma_0} K [\gamma f \beta'(\gamma f) - \beta(\gamma f)] \psi \\ & - \int_{t_1}^{t_2} \int_{\mathcal{O}} [\beta(f) (\partial_t \psi + \partial_v (J \psi) + \partial_y (\psi K)) - (\partial_v J + \partial_y K) f \beta'(f) \psi] \\ & + \int_{t_1}^{t_2} \int_{\mathcal{O}} \mathbf{a} \partial_y f \partial_y (\beta'(f) \psi) = 0, \end{aligned}$$

for any  $\psi \in C_c^1(\bar{\mathcal{U}})$  and any  $\beta \in C^1 \cap W^{1,\infty}$ .

*Step 5.* Because of the previous estimates on  $f$ ,  $f_t$  and  $\gamma f$ , we may take  $\beta(s) = s$  in (2.18) by using an approximation procedure in the three following cases (and more precisely, we apply (2.18) to a renormalized function  $\beta_n \in C^1 \cap W^{1,\infty}$ , we assume that  $\beta_n(s) \rightarrow s$ ,  $\beta'_n(s) \rightarrow 1$ , with  $|\beta_n(s)| \leq |s|$ ,  $|\beta'_n(s)| \leq 1$ , and we pass to the limit as  $n \rightarrow \infty$ ). Taking  $\psi = \chi_1(t) \chi_2$ ,  $\chi_1 \in C_c^1([0, T])$ ,  $\chi_2 \in C_c^1(\mathcal{O})$ , we have

$$- \int_{\mathcal{O}} f(0, \cdot) \psi(0, \cdot) - \int_{\mathcal{U}} f (\partial_t \psi + J \partial_v \psi) - \int_{\mathcal{U}} (K f - \mathbf{a} \partial_y f) \partial_y \psi = 0.$$

Together with (2.11), we deduce that  $f(0, \cdot) = f_0$ . Taking now  $\psi \in C_c^1((0, T) \times (\mathcal{O} \cup \Sigma_{12}^-))$ , we also have

$$\int_{\Gamma_{12}^-} J n_v \gamma f \psi - \int_{\mathcal{U}} f (\partial_t \psi + J \partial_v \psi) - \int_{\mathcal{U}} (K f - \mathbf{a} \partial_y f) \partial_y \psi = 0,$$

Together with (2.11), we deduce that  $\gamma f = 0$  on  $\Gamma_1$  and  $\gamma f = \mathbf{g}$  on  $\Gamma_2^-$ .

Because of the above identification of the trace functions, the renormalized equation (2.18) writes now

$$(2.20) \quad \begin{aligned} & \int_{\mathcal{O}} \beta(f_T) \psi + \int_{\Gamma_2^+} J \beta(\gamma f) \psi = \int_{\mathcal{U}} \beta(f) [\partial_t \psi + \partial_v (J \psi)] - \int_{\mathcal{U}} (\partial_v J) f \beta'(f) \psi \\ & + \int_{\mathcal{U}} (K f - \mathbf{a} \partial_y f) \partial_y (\beta'(f) \psi) + \int_{\Gamma_2^-} J \beta(\mathbf{g}) \psi + \int_{\mathcal{O}} \beta(f_0) \psi, \end{aligned}$$

for any  $\beta \in C^1 \cap W^{1,\infty}$  and  $\psi \in C_c^1(\bar{\mathcal{U}})$ . Using that  $f \in \mathcal{H}$ ,  $f_0 \in L_\omega^2$  and  $J^{1/2} \mathbf{g} \in L_\omega^2(\Gamma_2^-)$  at the RHS, we may take  $\psi = \tilde{\omega}^2$  and a sequence  $(\beta_n)$  of  $C^1 \cap W^{1,\infty}$  such that  $0 \leq \beta_n(s) \leq s^2$ ,  $|\beta'_n(s)| \leq 2|s|$ ,  $\beta_n \rightarrow s^2$ ,  $\beta'_n(s) \rightarrow 2s$ , and passing to the limit we deduce

$$\begin{aligned} & \int_{\mathcal{O}} f_T^2 \tilde{\omega}^2 + \int_{\Gamma_2^+} J \gamma f^2 \tilde{\omega}^2 + \int_{\Gamma_0} K \gamma f^2 \tilde{\omega}^2 + \int_{\mathcal{U}} \mathbf{a} (\partial_y (\omega f))^2 + \int_{\mathcal{U}} f^2 \varpi_- \tilde{\omega}^2 \\ & \leq \int_{\mathcal{U}} f^2 \varpi_+ \tilde{\omega}^2 + \int_{\mathcal{O}} f_0^2 \tilde{\omega}^2 + \int_{\Gamma_2^+} J \mathbf{g}^2 \tilde{\omega}^2, \quad \forall T > 0. \end{aligned}$$

Applying the Gronwall lemma to the function  $t \mapsto \|f_t\|_{L_{\tilde{\omega}}^2}^2$  which belongs to  $L^1(0, T)$  (because  $f \in \mathcal{H}$ ) and is lsc (because of the regularity of  $f$  obtained in Step 4), we deduce

$$(2.21) \quad \begin{aligned} & \|f_t\|_{L_{\tilde{\omega}}^2}^2 + \int_0^t \left( \|\gamma f_s\|_{L_{\tilde{\omega}}^2(\Sigma_2^+ \cup \Sigma_0; d\xi_1)}^2 + \|f\|_{H_{\tilde{\omega}}^{1,+}}^2 \right) e^{\lambda_0(t-s)} ds \\ & \leq \|f_0\|_{L_{\tilde{\omega}}^2}^2 e^{\lambda_0 t} + \int_0^t \|\mathfrak{g}_s\|_{L_{\tilde{\omega}}^2(\Sigma_2^-; d\xi_1)}^2 e^{\lambda_0(t-s)} ds, \end{aligned}$$

for any  $t > 0$ , with  $\lambda_0 := \sup \varpi_+ < \infty$  and

$$\|f\|_{H_{\tilde{\omega}}^{1,+}}^2 := \int_{\mathcal{O}} a_*(\partial_y(\omega f))^2 + \int_{\mathcal{O}} f^2 \varpi_- \tilde{\omega}^2.$$

*Step 6.* With that last estimate at end, we may pass to the limit in (2.18) written with  $\beta_n \rightarrow s^2$  and  $\psi_n \rightarrow \tilde{\omega}^2$  and we deduce the identity

$$\begin{aligned} & \|f_{t_2}\|_{L_{\tilde{\omega}}^2}^2 + \int_{t_1}^{t_2} \int_{\Sigma_2^+} J \gamma f^2 \tilde{\omega}^2 + \int_{t_1}^{t_2} \int_{\Sigma_0} K \gamma f^2 \tilde{\omega}^2 + \int_{t_1}^{t_2} \int_{\mathcal{O}} \mathfrak{a}(\partial_y(\omega f))^2 \\ & = \int_{t_1}^{t_2} \int_{\mathcal{O}} f^2 \varpi \tilde{\omega}^2 + \|f_{t_1}\|_{L_{\tilde{\omega}}^2}^2 + \int_{t_1}^{t_2} \int_{\Sigma_2^-} J \mathfrak{g}^2 \tilde{\omega}^2, \end{aligned}$$

for any  $0 \leq t_1 \leq t_2 < \infty$ , in particular  $t \mapsto \|f_t\|_{L_{\tilde{\omega}}^2}^2$  is continuous. Together with the already known weak continuity property, we classically deduce that  $f \in C(\mathbb{R}_+; L_{\tilde{\omega}}^2)$ . Similarly, taking  $\beta_n \rightarrow s_-^2$  and  $\psi_n \rightarrow \tilde{\omega}^2$  in (2.18), we deduce

$$\|f_-(t)\|_{L_{\tilde{\omega}}^2}^2 \leq \int_0^t \int_{\mathcal{O}} f_-^2 \varpi \tilde{\omega}^2 + \|f_{0-}\|_{L_{\tilde{\omega}}^2}^2 + \int_{t_1}^{t_2} \int_{\Sigma_2^+} J \mathfrak{g}_-^2 \tilde{\omega}^2, \quad \forall t > 0.$$

Thanks to the Gronwall lemma, we deduce the positivity property:  $f(t) \geq 0$  if  $\mathfrak{g} \geq 0$  and  $f_0 \geq 0$ . Similarly, for two solutions  $f_1$  and  $f_2$  to the inflow problem (1.16), (1.17), (1.6), (2.10) in the sense of the variational formulation (2.11), the difference  $f := f_2 - f_1$  is also a solution to the variational problem (2.11) but associated to  $f_0 = \mathfrak{g} = 0$ . Applying the conclusion (2.21) to that solution, we get  $f = 0$ , and we have thus proved the uniqueness of the solution to the inflow problem (1.16), (1.17), (1.6), (2.10).

*Step 7.* We briefly explain how we may deduce the existence and uniqueness of a solution to the linear VCk equation (1.16), (1.18). We define  $g_0 = 0$  and next recursively the sequence  $(g_k)$  defined for  $k \geq 1$  as the solution to the equation

$$\begin{aligned} & \partial_t g_k + \partial_v(Jg_k) + \partial_y(Kg_k - \mathfrak{a}\partial_y g_k) = 0 \text{ in } \mathcal{U}, \\ & g_k(0) = f_0 \text{ in } \mathcal{O}, \quad Kg_k - \mathfrak{a}\partial_y g_k = 0 \text{ on } \Gamma_0, \\ & \gamma g_k = 0 \text{ on } \Gamma_1, \quad J(0, \cdot)\gamma_- g_k = J(v_F, \cdot)\gamma_+ g_{k-1} \text{ on } \Gamma_{2-} \end{aligned}$$

provided by the previous steps. The Cesaro means

$$f_k := \frac{1}{k}(g_1 + \dots + g_k)$$

is then a solution to the equation

$$\begin{aligned} & \partial_t f_k + \partial_v(Jf_k) + \partial_y(Kf_k - \mathfrak{a}\partial_y f_k) = 0 \text{ in } \mathcal{U}, \\ & f_k(0) = f_0 \text{ in } \mathcal{O}, \quad Kf_k - \mathfrak{a}\partial_y f_k = 0 \text{ on } \Gamma_0, \\ & \gamma f_k = 0 \text{ on } \Gamma_1, \quad J(0, \cdot)\gamma_- f_k = (1 - 1/k)J(v_F, \cdot)\gamma_+ f_{k-1} \text{ on } \Gamma_{2-}. \end{aligned}$$

Using repeatedly the estimates (2.21) corresponding to the equations on  $(g_k)$ , we deduce that  $(g_k)$  is bounded in  $\mathcal{H}_T$ ,  $\forall T > 0$ . Similarly, summing up the estimates (2.21) corresponding to

the equations on  $g_1, \dots, g_k$  and using the elementary inequality

$$\left(\frac{1}{k}(g_1 + \dots + g_k)\right)^2 \leq \frac{1}{k}(g_1^2 + \dots + g_k^2),$$

we deduce

$$(2.22) \quad \|f_k(t)\|_{L_\omega^2}^2 + \int_0^t \|f_k(s)\|_{H_\omega^{1,\dagger}}^2 e^{\lambda_0(t-s)} ds \leq \|f_0\|_{L_\omega^2}^2 e^{\lambda_0 t}, \quad \forall t > 0.$$

We deduce that there exist  $f \in \mathcal{H}_T \cap L^\infty(0, T; L_\omega^2)$ ,  $\forall T > 0$ , and a subsequence  $(f_{k_n})$  such that  $f_{k_n} \rightharpoonup f$  in  $\mathcal{H}_T \cap L^\infty(0, T; L_\omega^2)$ . Passing first to the limit in the variational formulation of the equation on  $(f_{k_n})$  with test functions in  $C_c^1(\mathcal{U} \cup \Gamma_0)$ , we deduce

$$\int_{\mathcal{U}} f(-\partial_t - J\partial_v)\psi + \int_{\mathcal{U}} (\mathfrak{a}\partial_y f - K)\partial_y \psi = 0,$$

for any  $\psi \in C_c^1(\mathcal{U} \cup \Gamma_0)$ . Repeating the arguments of sections 2, 3 and 4, we know that there exists a trace function  $\gamma f$  on  $\Gamma$  connected to  $f$  through the renormalized formulations (2.18) and (2.19). Proceeding as in section 3, we also get that  $(\gamma g_{k_n})$  and  $(\gamma f_{k_n})$  are bounded in  $L^2(\Gamma_{12}^0, d\xi_1)$  and in  $L^2(\Gamma_{12}^{v_F}, d\xi_2)$ . Passing to the limit in the variational formulation of the equation on  $(f_{k_n})$  with test functions in  $C_c^1(\mathcal{U} \cup \Gamma_{12}^0)$  and in  $C_c^1(\mathcal{U} \cup \Gamma_{12}^{v_F})$ , we then deduce first

$$\gamma f_{k_n} \rightharpoonup \gamma f \quad \text{in } L^2(\Gamma_{12}^0, d\xi_1) \cap L^2(\Gamma_{12}^{v_F}, d\xi_2) \quad \text{as } k \rightarrow \infty$$

and next

$$\gamma f_{k_n-1} = \frac{k_n}{k_n-1} \gamma f_{k_n} - \frac{1}{k_n-1} \gamma g_{k_n} \rightharpoonup \gamma f \quad \text{in } L^2(\Gamma_{12}^{v_F}, d\xi_2) \quad \text{as } k \rightarrow \infty.$$

We may then pass to the limit in the variational formulation associated to the equation on  $f_{k_n}$  with test functions in  $C_c^1(\mathcal{U} \cup \Gamma_-)$  and we deduce that  $f$  is a variational solution to the linear VCK equation (1.16), (1.18), and more precisely

$$(2.23) \quad \int_{\mathcal{U}} f(-\partial_t - J\partial_v)\psi + \int_{\mathcal{U}} (\mathfrak{a}\partial_y f - K)\partial_y \psi = \int_0^T \int_{y_F}^\infty (J\gamma f)(t, v_F, y)\psi(t, 0, y) + \int_{\mathcal{O}} f_0 \psi(0, \cdot)$$

for any  $\psi \in C_c^1(\mathcal{U} \cup \Gamma_-)$ . Repeating the arguments of Step 6, we similarly establish the further properties of the solution  $f$ .  $\square$

### 3. ULTRA CONTRACTIVITY

This section is dedicated to the proof of the ultracontractivity Theorem 1.3, in weighted Lebesgue spaces  $L_\omega^p$  for a strongly confining weight function  $\omega := e^{\alpha y}$ ,  $\alpha > 0$ .

**3.1. A boundary penalization  $L^2$  estimate.** We take back the  $L^2$  estimate established in Section 2.1 and improve it by introducing a penalization of the neighborhood of the boundary  $\Sigma_{12}$  thanks to a suitable power of the distance  $\delta(v) := \min(v, v_F - v)$ .

**Lemma 3.1.** *Assume  $\omega := e^{\alpha y}$ ,  $\alpha > 0$ . There exist a weight function  $\tilde{\omega} = \tilde{\omega}(v, y) \simeq \omega$  and a constant  $\tilde{\kappa} \geq 0$  such that any solution  $f$  to the linear VCK equation (1.16)-(1.18) satisfies*

$$(3.1) \quad \int_0^T \varphi^2 \int_{\mathcal{O}} \left\{ f^2 \omega^2 \left[ \frac{J^2}{\delta^{1/2} \langle y \rangle^2} + \alpha \langle y \rangle \right] + (\partial_y(f\tilde{\omega}))^2 \right\} \leq \tilde{\kappa} \int_0^T (\varphi \varphi'_+ + \varphi^2) \int_{\mathcal{O}} f^2 \omega^2,$$

for any  $\varphi \in C_c^1(0, T)$  and where  $\tilde{\kappa}$  only depends on  $a^*, a_*, y_*$ .

*Proof of Lemma 3.1.* We come back to the proof of Lemma 2.1. We define

$$(3.2) \quad \tilde{\omega}^2 := \tilde{\omega}^2 \mathfrak{P}, \quad \mathfrak{P} := 1 - \frac{1}{2} \frac{\delta^{1/2}}{\delta_*^{1/2}} \frac{\delta' J}{\langle J \rangle^2},$$

with  $\delta_* := \sup[\delta(\delta')^2]$ ,  $\tilde{\omega}^2$  defining during the proof of Lemma 2.1 and where we abuse notation by rather denoting  $\delta \in C^2(\mathbb{R})$  such that  $\delta > 0$  on  $(0, v_F)$  and  $\delta(v) = \min(v, v_F - v)$  for any  $v \in (0, v_F/3) \cup (2v_F/3, v_F)$  so that it is equivalent to establish (3.14) for the previously defined  $\delta$  or this smooth variant. Proceeding exactly as during the proof of Lemma 2.1, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathcal{O}} f^2 \tilde{\omega}^2 &= - \int_{\mathcal{O}} \mathfrak{a}(\partial_y(f\tilde{\omega}))^2 + \int_{\mathcal{O}} f^2 \tilde{\omega}^2 \varpi \\ &\quad + \int_{\Sigma_0} \frac{1}{2} K n_0 (f\tilde{\omega})^2 - \frac{1}{p} \int_{\Sigma_2} J f^2 \tilde{\omega}^2 n_v, \end{aligned}$$

with

$$\varpi := \mathfrak{a} \left( \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \right)^2 + K \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} + J \frac{\partial_v \tilde{\omega}}{\tilde{\omega}} - \frac{1}{2} \partial_y K - \frac{1}{2} \partial_v J,$$

and thus again

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{O}} f^2 \tilde{\omega}^2 + \int_{\mathcal{O}} a_* (\partial_y(f\tilde{\omega}))^2 \leq \int_{\mathcal{O}} f^2 \tilde{\omega}^2 \tilde{\omega}.$$

We observe that

$$\frac{\partial_z \tilde{\omega}}{\tilde{\omega}} = \frac{\partial_z \tilde{\omega}}{\tilde{\omega}} + \frac{1}{2} \frac{\partial_z \mathfrak{P}}{\mathfrak{P}}$$

for  $z = y, v$ , and thus

$$\tilde{\omega} = \tilde{\omega} + \mathfrak{a} \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \frac{\partial_y \mathfrak{P}}{\mathfrak{P}} + \frac{K}{2} \frac{\partial_y \mathfrak{P}}{\mathfrak{P}} + \frac{J}{2} \frac{\partial_v \mathfrak{P}}{\mathfrak{P}}$$

with  $\tilde{\omega}$  defined by (2.2),

$$\partial_v \mathfrak{P} = -\frac{1}{4} \frac{1}{\delta^{1/2}} \frac{(\delta')^2}{\delta_*^{1/2}} \frac{J}{\langle J \rangle^2} + \frac{\phi_1}{\langle J \rangle}, \quad \phi_1 \in L^\infty(\mathcal{O}),$$

and

$$\partial_y \mathfrak{P} = -\frac{1}{2} \frac{\delta^{1/2}}{\delta_*^{1/2}} \delta' \partial_y \frac{J}{\langle J \rangle^2} = \frac{\phi_2}{\langle J \rangle^2}, \quad \phi_2 \in L^\infty(\mathcal{O}).$$

We deduce that

$$(3.3) \quad \tilde{\omega} \leq \tilde{\kappa} - \frac{1}{8\delta_*^{1/2}} \frac{1}{\delta^{1/2}} \frac{J^2}{\langle y \rangle^2} - \alpha \langle y \rangle$$

with  $\tilde{\kappa} \leq C(1 + \|\mathfrak{a}\|_{L^\infty} + \|\mathfrak{b}\|_{L^\infty})$  and where we observe that  $\langle J \rangle \simeq \langle y \rangle$ . The previous estimate gives

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{O}} f^2 \tilde{\omega}^2 + \int_{\mathcal{O}} f^2 \tilde{\omega}^2 \left( \frac{1}{8\delta_*^{1/2}} \frac{1}{\delta^{1/2}} \frac{J^2}{\langle y \rangle^2} + \alpha \langle y \rangle \right) + \int_{\mathcal{O}} a_* (\partial_y(f\tilde{\omega}))^2 \leq \tilde{\kappa} \int_{\mathcal{O}} f^2 \omega^2.$$

We conclude by multiplying the above equation by  $\varphi^2$  and integrating in the time variable.  $\square$

We reformulate that last result in a more tractable one by using several times the following estimate

$$(3.4) \quad \int_a^c \frac{1}{|y-b|^\mu} h^2 dy \lesssim \int_a^c ((\partial_y h)^2 + h^2) dy,$$

for  $\mu \in (0, 1)$  and any  $h \in H^1(a, c)$ ,  $a, b \in \mathbb{R}$ ,  $c \in \mathbb{R} \cup \{+\infty\}$ ,  $a \leq b \leq c$ . The estimate (3.4) is an immediate consequence of the classical embedding  $H^1(a, c) \subset L^\infty(a, c)$  used in the region where  $|y-b| \leq 1$  in the LHS integral of (3.4).



**Lemma 3.2.** *Assume  $\omega := e^{\alpha y}$ ,  $\alpha > 0$ . With the notations of Lemma 3.1, any solution  $f$  to the linear VCK equation (1.16)-(1.18) satisfies*

$$\int_0^T \varphi^2 \int_{\mathcal{O}} f^2 \frac{1}{\delta^{1/9}} \frac{\omega^2}{\langle y \rangle^2} \lesssim \int_0^T (\varphi \varphi'_+ + \varphi^2) \int_{\mathcal{O}} f^2 \omega^2$$

for any  $\varphi \in C_c^1(0, T)$

*Proof of Lemma 3.2.* We define  $\mathcal{O} = \mathfrak{D}_1 \cup \mathfrak{D}_2 \cup \mathfrak{D}_3 \cup \mathfrak{D}_4$  with

$$\begin{aligned} \mathfrak{D}_1 &:= \{(v, y); 0 < v < \frac{1}{2}v_F, \frac{1}{2}v_E \leq v + y_L v^{29/36}, y > v^{7/36}\}, \\ \mathfrak{D}_2 &:= \{(v, y); 0 < v < \frac{1}{2}v_F, \frac{1}{2}v_E > v + y_L v^{29/36}, y > v^{7/36}\}, \\ \mathfrak{D}_3 &:= \{(v, y); 0 < v < \frac{1}{2}v_F, y \leq v^{7/36}\}, \\ \mathfrak{D}_4 &:= \{(v, y); \frac{1}{2}v_F \leq v < v_F, 0 \leq y < \frac{1}{2}y_F\}, \\ \mathfrak{D}_5 &:= \{(v, y); \frac{1}{2}v_F \leq v < v_F, y > \frac{1}{2}y_F\}, \end{aligned}$$

and we estimate separately each of the term

$$\int_0^T \int_{\mathfrak{D}_i} \varphi^2 f^2 \frac{\omega_0^2}{\delta^{1/9}},$$

for  $i = 1, \dots, 5$  and where we use the shorthand  $\omega_0 := \omega / \langle y \rangle$ .

• On  $\mathfrak{D}_1$ , we have  $\delta \geq \delta_1 > 0$ , so that

$$\int_0^T \int_{\mathfrak{D}_1} \varphi^2 f^2 \frac{\omega_0^2}{\delta^{1/9}} \lesssim \int_0^T \int_{\mathfrak{D}_1} \varphi^2 f^2 \omega_0^2.$$

• On  $\mathfrak{D}_2$ , we observe that

$$J \geq y(v_E - v - y_L v^{29/36}) \geq \frac{1}{2}v_E y,$$

so that

$$\frac{J^2}{\delta^{1/2}} \gtrsim \frac{y^2}{v^{1/2}} \gtrsim \frac{1}{v^{1/9}} \gtrsim \frac{1}{\delta^{1/9}},$$

where we have used  $y > v^{7/36}$  and next  $v = \delta$  in the two last inequalities. We deduce

$$\int_0^T \int_{\mathfrak{D}_2} \varphi^2 f^2 \frac{\omega_0^2}{\delta^{1/9}} \lesssim \int_0^T \int_{\mathfrak{D}_2} \varphi^2 f^2 \omega_0^2 \frac{J^2}{\delta^{1/2}}.$$

• On  $\mathfrak{D}_3$ , we have

$$\begin{aligned} \int_0^T \int_{\mathfrak{D}_3} \varphi^2 f^2 \frac{\omega_0^2}{\delta^{1/9}} &\lesssim \int_0^T \int_0^{v_F/2} \int_0^{v_F^{7/36}} \varphi^2 (\omega f)^2 \frac{1}{y^{4/7}} \\ &\lesssim \int_0^T \int_0^{v_F/2} \int_0^\infty \varphi^2 (\partial_y(\omega f))^2 \end{aligned}$$

where we have used that  $y^{4/7} \leq \delta^{1/9}$  on the first line and the inequality (3.4) in the last line.

• On  $\mathfrak{D}_4$ , we observe that there exists  $v_F^* \in [\frac{1}{2}v_F, v_F)$  and  $J_* > 0$  such that

$$J^2 \geq J_*^2 > 0 \text{ if } (v, y) \in (v_F^*, v_F) \times (0, y_F/2),$$

so that

$$\frac{1}{\delta^{1/9}} \leq \frac{1}{\delta(v_F^*)^{1/9}} \mathbf{1}_{v < v_F^*} + \frac{J_*^2}{J_*^2} \frac{1}{\delta^{1/9}} \mathbf{1}_{v > v_F^*} \lesssim 1 + \frac{J_*^2}{\delta^{1/2}}$$

on  $\mathfrak{D}_4$ . We deduce

$$\int_0^T \int_{\mathfrak{D}_4} \varphi^2 f^2 \frac{\omega_0^2}{\delta^{1/9}} \lesssim \int_0^T \int_{\mathfrak{D}_4} \varphi^2 f^2 \omega_0^2 \left(1 + \frac{J^2}{\delta^{1/2}}\right).$$

• On  $\mathfrak{D}_5$ , we first observe that

$$\frac{1}{\delta^{1/9}} \leq \frac{J^2}{\delta^{1/2}} + \frac{1}{|J|^{4/7}},$$

thanks to the Young inequality, and that

$$\int_{y_{F/2}}^{\infty} \frac{f^2}{|J|^{4/7}} \omega_0^2 dy \lesssim \int_0^{\infty} (\partial_y(f\omega))^2 dy$$

from the inequality (3.4) again and the fact that we may write  $|J(v, y)| = \zeta|y - c|$ , with  $\zeta = \zeta(v) \in [\zeta_*, \zeta^*]$ ,  $c = c(v) \in [c_*, c^*]$ ,  $\zeta_*, \zeta^*, c_*, c^* \in (0, \infty)$ . We deduce

$$\int_0^T \int_{\mathfrak{D}_5} \varphi^2 f^2 \frac{\omega_0^2}{\delta^{1/9}} \lesssim \int_0^T \int_{v_{F/2}}^{v_F} \int_0^{\infty} \varphi^2 \left( (\partial_y(f\omega))^2 + f^2 \omega_0^2 \frac{J^2}{\delta^{1/2}} \right).$$

We conclude by gathering the above five contributions and by using the previous estimate (3.14).  $\square$

Interpolating the two previous estimates, we conclude with the following formulation of penalization of the boundary (in the  $v$  variable) and the infinity (in the  $y$  variable).

**Proposition 3.3.** *Assume  $\omega := e^{\alpha\langle y \rangle}$ , with  $\alpha > 0$ . Any solution  $f$  to the linear VCk equation (1.16)-(1.18) satisfies*

$$\int_0^T \varphi^2 \int_{\mathcal{O}} f^2 \omega^2 \left[ \frac{1}{\delta^{1/27}} + \langle y \rangle \right] \lesssim \int_0^T (\varphi \varphi'_+ + \varphi^2) \int_{\mathcal{O}} f^2 \omega^2$$

for any  $\varphi \in C_c^1(0, T)$ .

*Proof of Proposition 3.3.* The estimate immediately follows from the Young inequality

$$\frac{1}{\delta^{1/27}} = \frac{1}{\delta^{1/27} \langle y \rangle^{2/3}} \langle y \rangle^{2/3} \leq \frac{1}{\delta^{1/9} \langle y \rangle^2} + \langle y \rangle,$$

together with Lemma 3.1 and Lemma 3.2.  $\square$

**3.2. Gain of integrability estimate.** We introduce the function

$$(3.5) \quad \bar{f} := f\psi, \quad \psi := \varphi(t)\chi(v)\omega_0(y),$$

with  $\varphi \in C_c^1((0, T))$ ,  $\chi \in C_c^1((0, v_F))$ ,  $0 \leq \varphi, \chi \leq 1$ ,  $\omega_0 := e^{\alpha y}$  for  $y \geq y_0 > 0$ ,  $\alpha > 0$ ,  $0 \leq \omega_0 \in C_c^1((0, \infty))$ , which is a solution to

$$(3.6) \quad \partial_t \bar{f} + \hat{J} \partial_v \bar{f} + \check{K} \partial_y \bar{f} - a \partial_{yy}^2 \bar{f} + \alpha \langle y \rangle \bar{f} = F$$

on  $\mathbb{R}_+ \times \mathbb{R}^2$ , with

$$\begin{aligned} \hat{J} &:= J(v, y), \text{ if } y > 0, \quad \hat{J} := J(v, -y), \text{ if } y < 0, \\ \check{K} &:= K(v, y), \text{ if } y > 0, \quad \check{K} := -K(v, -y), \text{ if } y < 0, \\ F &:= (\partial_t \varphi) \chi \omega_0 f + \hat{J} \varphi (\partial_v \chi) \omega_0 f - (\partial_v \hat{J}) \varphi \chi \omega_0 f + \check{K} \varphi \chi (\partial_y \omega_0) f - (\partial_y \check{K}) \varphi \chi \omega_0 f \\ &\quad - a \varphi \chi (\partial_{yy}^2 \omega_0) f - 2a \varphi \chi (\partial_y \omega_0) (\partial_y f) + \alpha \langle y \rangle \varphi \chi \omega_0 f \end{aligned}$$

We first focus on the homogeneous equation

$$(3.7) \quad \partial_t g + \bar{J} \partial_v g + \bar{K} \partial_y g - a \partial_{yy}^2 g + \alpha \langle y \rangle g = 0, \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^2,$$

complemented with the initial condition

$$g(0) = g_0, \quad \text{on } \mathbb{R}^2,$$

where

$$\bar{K}, \bar{J} \in C_c^1((0, v_F) \times \mathbb{R}), \quad \bar{K} = \check{K}, \bar{J} = \check{J} \quad \text{on } \text{supp } \chi \times \mathbb{R}.$$

More precisely, we take  $\bar{K} = \bar{\chi}\check{K}$ ,  $\bar{J} = \bar{\chi}\check{J}$ ,  $\bar{\chi} \in C_c^1((0, v_F))$ ,  $\mathbf{1}_{\text{supp } \chi} \leq \bar{\chi} \leq 1$ , and  $\alpha > \|\bar{\chi}'\|_{(v_E + y_L v_F + 1)}$ . We establish the following series of growth estimates and gain of integrability result.

**Lemma 3.4.** *For any  $p \in [1, \infty]$ , any solution  $g$  to the homogeneous equation (3.7) satisfies*

$$(3.8) \quad \|g_t\|_{L^p} \leq \|g_0\|_{L^p}, \quad \forall t \geq 0.$$

*Proof of Lemma 3.4.* For  $p \in (1, \infty)$ , any nonnegative solution  $g$  to (3.7) satisfies

$$\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^2} g^p = \frac{1}{p} \int_{\mathbb{R}^2} g^p (\partial_v \bar{J} + \partial_y \bar{K} - p\alpha \langle y \rangle) - (p-1)a \int_{\mathbb{R}^2} (\partial_y g)^2 g^{p-2} \leq 0,$$

because  $\partial_v \bar{J} + \partial_y \bar{K} \leq \bar{\chi}' \check{J} \leq \alpha \langle y \rangle$ . We conclude to (3.8) thanks to the Gronwall lemma when  $p \in (1, \infty)$  and by passing to the limit  $p \rightarrow 1$  and  $p \rightarrow \infty$  in (3.8) for the two limit cases.  $\square$

**Proposition 3.5.** *Any solution  $g$  to the homogeneous equation (3.7) satisfies*

$$(3.9) \quad \|g_t\|_{H_v^1 L_y^2 \cap L_v^2 H_y^2} \lesssim \frac{1}{t^2} \|g_0\|_{L^2}, \quad \forall t \in (0, 1).$$

*Proof of Proposition 3.5.* We use a regularization argument in the spirit of Hérau [24] (see also Villani [46, Appendix]). More precisely, we define

$$\mathcal{F}(t, g) := \varepsilon^{-3} \|g\|_{L^2}^2 + \varepsilon^4 t^4 \|\partial_v g\|_{L^2}^2 + \varepsilon^3 t^3 (\partial_v g, \partial_y g)_{L^2} + \varepsilon t^2 \|\partial_y g\|_{L^2}^2 + \varepsilon^4 t^4 \|\partial_{yy}^2 g\|_{L^2}^2,$$

for  $t, \varepsilon \in (0, 1)$ , and for a solution  $g$  to the homogeneous equation (3.7), we estimate the derivative quantity  $\frac{d}{dt} \mathcal{F}(t, g_t)$ . For that purpose, we start computing each square of norm or scalar product on  $g$  involved separately.

*Step 1.* From Lemma 3.4, we already know that

$$\frac{1}{2} \frac{d}{dt} \|g\|_{L^2}^2 \leq -\|g \langle y \rangle^{1/2}\|_{L^2}^2 - a \|\partial_y g\|_{L^2}^2.$$

Similarly, we compute

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (\partial_v g)^2 = \int_{\mathbb{R}^2} (\partial_v g)^2 \left( \frac{1}{2} \partial_y \bar{K} - \frac{1}{2} \partial_v \bar{J} - \alpha \langle y \rangle \right) - a \int_{\mathbb{R}^2} (\partial_{vy}^2 g)^2,$$

so that

$$\frac{1}{2} \frac{d}{dt} \|\partial_v g\|_{L^2}^2 \leq C_1 \|\partial_v g\|_{L^2}^2 - \|\langle y \rangle^{1/2} \partial_v g\|_{L^2}^2 - a \|\partial_{vy}^2 g\|_{L^2}^2.$$

We next compute

$$\frac{d}{dt} \int_{\mathbb{R}^2} \partial_v g \partial_y g = \int_{\mathbb{R}^2} \left( -\frac{1}{2} \partial_v \bar{J} (\partial_y g)^2 - \frac{1}{2} \partial_y \bar{J} (\partial_v g)^2 - 2a \partial_{vy}^2 g \partial_{yy}^2 g - 2\alpha \langle y \rangle \partial_v g \partial_y g \right),$$

so that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} \partial_v g \partial_y g &\leq -C_{21} \|\partial_v g\|_{L^2}^2 + C_{22} \|\langle y \rangle^{1/2} \partial_y g\|_{L^2}^2 \\ &\quad + C_{23} \|\partial_{vy} g\|_{L^2} \|\partial_{yy} g\|_{L^2} + C_{24} \|\langle y \rangle^{1/2} \partial_v g\|_{L^2} \|\langle y \rangle^{1/2} \partial_y g\|_{L^2}. \end{aligned}$$

We also compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (\partial_y g)^2 &= \int_{\mathbb{R}^2} (\partial_y g)^2 \left( \frac{1}{2} \partial_v \bar{J} - \frac{1}{2} \partial_y \bar{K} - \alpha \langle y \rangle \right) \\ &\quad + \int_{\mathbb{R}^2} g^2 \frac{\alpha}{2} \partial_{yy}^2 \langle y \rangle - \int_{\mathbb{R}^2} \partial_y \bar{J} \partial_y g \partial_v g - a \int_{\mathbb{R}^2} (\partial_{yy}^2 g)^2, \end{aligned}$$

so that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_y g\|_{L^2}^2 &\leq -\|\langle y \rangle^{1/2} \partial_y g\|_{L^2}^2 - a \|\partial_{yy}^2 g\|_{L^2}^2 \\ &\quad + C_{31} \|\partial_y g\|_{L^2}^2 + C_{32} \|g\|_{L^2}^2 + C_{33} \|\partial_v g\|_{L^2} \|\partial_y g\|_{L^2}. \end{aligned}$$

We finally compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (\partial_{yy} g)^2 &= \int_{\mathbb{R}^2} (\partial_{yy} g)^2 \left( \frac{1}{2} \partial_v \bar{J} - \frac{3}{2} \partial_y \bar{K} - \alpha \langle y \rangle \right) + \int_{\mathbb{R}^2} (\partial_y g)^2 2\alpha \partial_{yy}^2 \langle y \rangle \\ &\quad - \int_{\mathbb{R}^2} g^2 \frac{\alpha}{2} \partial_{yyy}^4 \langle y \rangle - a \int_{\mathbb{R}^2} (\partial_{yyy}^2 g)^2 - 2 \int_{\mathbb{R}^2} \partial_y \bar{J} \partial_{vy} g \partial_{yy} g, \end{aligned}$$

so that

$$\frac{1}{2} \frac{d}{dt} \|\partial_{yy} g\|_{L^2}^2 \leq +C_{41} \|\partial_{yy}^2 g\|_{L^2}^2 + C_{42} \|\partial_y g\|_{L^2}^2 + C_{43} \|g\|_{L^2}^2 + C_{44} \|\partial_{vy} g\|_{L^2} \|\partial_{yy} g\|_{L^2}.$$

Step 2. The derivative of  $\mathcal{F}$  takes the form

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(t, g_t) &= \varepsilon^{-3} \frac{d}{dt} \|g\|_{L^2}^2 + 4\varepsilon^4 t^3 \|\partial_v g\|_{L^2}^2 + \varepsilon^4 t^4 \frac{d}{dt} \|\partial_v g\|_{L^2}^2 \\ &\quad + 3\varepsilon^3 t^2 (\partial_v g, \partial_y g)_{L^2} + \varepsilon^3 t^3 \frac{d}{dt} (\partial_v g, \partial_y g)_{L^2} \\ &\quad + 2\varepsilon t \|\partial_y g\|_{L^2}^2 + \varepsilon t^2 \frac{d}{dt} \|\partial_y g\|_{L^2}^2 \\ &\quad + 4\varepsilon^4 t^3 \|\partial_{yy} g\|_{L^2}^2 + \varepsilon^4 t^4 \frac{d}{dt} \|\partial_{yy} g\|_{L^2}^2, \end{aligned}$$

for any  $\varepsilon, t \in (0, 1)$ . Using the estimates established in the first step and the Young inequalities

$$\begin{aligned} &(\varepsilon^3 t^3 C_{23} + 2\varepsilon^4 t^4 C_{44}) \|\partial_{vy} g\|_{L^2} \|\partial_{yy} g\|_{L^2} \\ &\leq (\varepsilon^{3/2} t^2 C_{23} + \varepsilon^3 t^4 C_{44}) \|\partial_{yy} g\|_{L^2}^2 + (\varepsilon^{9/2} t^4 C_{23} + \varepsilon^5 t^4 C_{44}) \|\partial_{vy} g\|_{L^2}^2, \\ &(2\varepsilon t^2 C_{33} + 3\varepsilon^3 t^2) \|\partial_v g\|_{L^2} \|\partial_y g\|_{L^2} \\ &\leq (\varepsilon^4 t^4 C_{33} + 2\varepsilon^4 t^4) \|\partial_v g\|_{L^2}^2 + (\varepsilon^{-2} C_{33} + 2\varepsilon^2) \|\partial_y g\|_{L^2}^2 \\ &\varepsilon^3 t^3 C_{24} \|\langle y \rangle^{1/2} \partial_v g\|_{L^2} \|\langle y \rangle^{1/2} \partial_y g\|_{L^2} \\ &\leq \varepsilon^{3/2} t^2 C_{24} \|\langle y \rangle^{1/2} \partial_v g\|_{L^2}^2 + \varepsilon^{9/2} t^4 C_{24} \|\langle y \rangle^{1/2} \partial_y g\|_{L^2}^2, \end{aligned}$$

we immediately deduce

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(t, g_t) &\leq \|g \langle y \rangle^{1/2}\|_{L^2}^2 (-2\varepsilon^{-3} + 2\varepsilon C_{32} t^2 + 2\varepsilon^4 C_{43} t^4) \\ &\quad + \|\partial_y g\|_{L^2}^2 (-2a\varepsilon^{-3} + 2\varepsilon C_{31} t^2 + 2\varepsilon^4 C_{42} t^4 + 2\varepsilon t + \varepsilon^{-2} C_{33} + 2\varepsilon^2) \\ &\quad + \|\langle y \rangle^{1/2} \partial_y g\|_{L^2}^2 (\varepsilon^{3/2} t^2 C_{24} + \varepsilon^3 t^3 C_{22} - 2\varepsilon t^2) \\ &\quad + \|\partial_v g\|_{L^2}^2 (4\varepsilon^4 t^3 + 2\varepsilon^4 t^4 C_1 + \varepsilon^4 t^4 C_{33} + 2\varepsilon^4 t^4 - C_{21} \varepsilon^3 t^3) \\ &\quad + \|\partial_{yy} g\|_{L^2}^2 (-2\varepsilon a t^2 + 4\varepsilon^4 t^3 + 2\varepsilon^4 t^4 C_{41} + \varepsilon^{3/2} t^2 C_{23} + \varepsilon^3 t^4 C_{44}) \\ &\quad + (\varepsilon^{9/2} t^4 C_{23} + \varepsilon^5 t^4 C_{44} - 2\varepsilon^4 t^4 a) \|\partial_{vy} g\|_{L^2}^2 \\ &\quad + (\varepsilon^{9/2} t^4 C_{24} - 2\varepsilon^4 t^4) \|\langle y \rangle^{1/2} \partial_v g\|_{L^2}^2. \end{aligned}$$

Choosing  $\varepsilon \in (0, 1)$  small enough, all the terms are nonnegative separately, and we thus obtain

$$\mathcal{F}(t, g_t) \leq \mathcal{F}(0, g_0) = \varepsilon^{-3} \|g_0\|_{L^2}^2, \quad \forall t \in (0, 1).$$

Observing that

$$\mathcal{F}(t, g) \geq \frac{1}{2} \varepsilon^4 t^4 (\|g\|_{L^2}^2 + \|\partial_v g\|_{L^2}^2 + \|\partial_y g\|_{L^2}^2 + \|\partial_{yy}^2 g\|_{L^2}^2),$$

we then conclude to (3.9) with  $C_1 := (2\varepsilon^{-7})^{1/2}$ .  $\square$

**Corollary 3.6.** *Any solution  $g$  to the homogeneous equation (3.7) satisfies*

$$\|g_t\|_{L^\infty} \lesssim \frac{1}{t^2} \|g_0\|_{L^2}, \quad \forall t \in (0, 1).$$

*Proof of Corollary 3.6.* We first observe that the following Sobolev type embedding

$$H_v^1 L_y^2 \cap L_v^2 H_y^2 \subset L^\infty(\mathbb{R}^2)$$

holds true. We may indeed write

$$\|f\|_{L^\infty} \lesssim \|\hat{f}\|_{L^1} \lesssim AB$$

using the Cauchy-Schwarz inequality in the second inequality, with

$$\begin{aligned} A^2 &:= \int_{\mathbb{R}^2} (1 + |\xi|^2 + |\eta|^4) |\hat{f}|^2 d\xi d\eta \\ &= \int_{\mathbb{R}^2} (|\hat{f}|^2 + |\widehat{\partial_v f}|^2 + |\widehat{\partial_{yy}^2 f}|^2) d\xi d\eta \leq \|f\|_{H_v^1 L_y^2 \cap L_v^2 H_y^2}^2, \end{aligned}$$

where we have used the Plancherel identity in the last inequality, and

$$B^2 := \int_{\mathbb{R}^2} \frac{1}{1 + |\xi|^2 + |\eta|^4} d\xi d\eta = \int_{\mathbb{R}} \frac{d\eta}{(1 + |\eta|^4)^{1/2}} \int_{\mathbb{R}} \frac{dz}{1 + z^2} < \infty,$$

by performing the change of variables  $z := \xi/(1 + |\eta|^4)^{1/2}$ . We then conclude the proof thanks to (3.9).  $\square$

We come back to the analysis of the linear VCk equation (1.16)-(1.18).

**Proposition 3.7.** *Assume  $\omega := e^{\alpha y}$ ,  $\alpha > 0$ . For any solution  $f$  to the linear VCk equation (1.16)-(1.18), the function  $\bar{f}$  defined by (3.5) satisfies*

$$(3.10) \quad \int_0^T \|\bar{f}\|_{L^{5/2}}^2 dt \lesssim \|\chi\|_{W^{1,\infty}}^2 \int_0^T \|f\omega\|_{L^2}^2 (\varphi^2 + (\varphi')^2) dt,$$

for any  $T \in (0, 1)$ ,  $\varphi \in C_c^1(0, T)$  and  $\chi \in C_c^1(0, v_F)$ .

*Proof of Proposition 3.8.* We denote by  $S_t^0$  the semigroup associated to homogeneous equation (3.7) which is well defined by arguing as in Theorem 2.2 taking advantage of the estimates established in Lemma 3.4. From Lemma 3.4 and Corollary 3.6, we have

$$\|S_t^0\|_{L^2 \rightarrow L^2} \lesssim 1, \quad \|S_t^0\|_{L^2 \rightarrow L^\infty} \lesssim \frac{1}{t^2},$$

for any  $t \in (0, 1)$ . By interpolation (Holder inequality), we deduce

$$\|S_t^0\|_{L^2 \rightarrow L^{5/2}} \lesssim \frac{1}{t^{2/5}},$$

for any  $t \in (0, 1)$ . Because  $\bar{f}$  satisfies (3.6), we may use the Duhamel formula together with the fact that  $\bar{f}(0, \cdot) = 0$  on  $\mathbb{R}^2$ , and we get

$$\bar{f}_t = \int_0^t S_{t-s}^0 F_s ds.$$

As a consequence, we have

$$\begin{aligned}
\|\bar{f}_t\|_{L^{5/2}(\mathbb{R}^2)} &\leq \int_0^t \|S_{t-s}^0\|_{L^2 \rightarrow L^{5/2}} \|F_s\|_{L^2} ds \\
&\lesssim \int_0^t \frac{1}{(t-s)^{2/5}} \|F_s\|_{L^2} ds \\
&\lesssim \left( \int_0^t \frac{ds}{(t-s)^{4/5}} \right)^{1/2} \left( \int_0^t \|F_s\|_{L^2}^2 ds \right)^{1/2} \\
&\leq t^{1/10} \|F\|_{L^2(\mathcal{U})},
\end{aligned}$$

for any  $t \in (0, 1)$ , where we have used the previous estimate in the second line, the Cauchy-Schwarz inequality in the third line and the very definition of  $F$  in the last line. We deduce (3.10) by observing that

$$|F| \lesssim |\partial_t \varphi f \omega| + |\varphi \partial_v \chi f \omega| + |\varphi f \omega| + |\varphi \partial_y (f \tilde{\omega})|,$$

with  $\tilde{\omega}$  defined by (2.4), next by integrating the above estimate in the time variable and by using Lemma 3.1.  $\square$

We then come back on the analysis of a solution  $f$  to the linear VCK equation (1.16)-(1.18).

**Proposition 3.8.** *For any  $\beta > 5/2$ , there exists  $C$  such that*

$$\int_0^T \|f \delta^{2\beta/5} \frac{\omega}{\langle y \rangle}\|_{L^{5/2}}^2 \varphi^2 dt \leq C \int_0^T \|f\|_{L^2}^2 (\varphi^2 + (\varphi')^2) dt$$

*Proof of Proposition 3.8.* We take  $\chi = \chi_k$  with  $\|\nabla \chi_k\| 2^{-k} \lesssim 1$  uniformly in  $k$ , and we repeat the proof of Step 2 in [11, Proposition 3.5].  $\square$

We are now able to establish the cornerstone estimate of gain of integrability in  $(v, y)$  variables for a solution  $f$  to the linear VCK equation (1.16)-(1.18) by gathering the last result and Proposition 3.3.

**Proposition 3.9.** *There exists  $r > 2$  such that*

$$\int_0^T \varphi^2 \|f \omega\|_{L^r(\mathcal{O})}^2 dt \lesssim \|(\varphi + |\varphi'|) f \omega\|_{L^2(\mathcal{U})}^2,$$

for any  $\varphi \in C^1(0, T) \cap C_0(0, T)$ ,  $T \in (0, 1)$ .

*Proof of Proposition 3.9.* We set  $\Delta := \delta^{2\beta/5} / \langle y \rangle$ , so that Proposition 3.8 writes

$$(3.11) \quad \int_0^T \|f \omega \Delta\|_{L^{5/2}(\mathcal{O})}^2 \varphi^2 dt \lesssim \int_0^T \|f \omega\|_{L^2(\mathcal{O})}^2 (\varphi^2 + (\varphi')^2) dt.$$

From Proposition 3.3 and the Holder inequality, we have next

$$\begin{aligned}
\|f \varphi \omega \frac{\langle y \rangle^{\theta/2}}{\delta^{(1-\theta)/12}}\|_{L^2} &\leq \|f \varphi \omega \langle y \rangle^{1/2}\|_{L^2}^\theta \|f \varphi \omega \frac{1}{\delta^{1/12}}\|_{L^2}^{1-\theta} \\
&\lesssim \|(\varphi + |\varphi'|) f \omega\|_{L^2},
\end{aligned}$$

for any  $\theta \in (0, 1)$ . Choosing  $\theta = \theta_0$  such that

$$(1/\theta_0 - 1)/6 = 2\beta/5$$

and setting  $\mu := \theta_0/2$ , we thus deduce

$$(3.12) \quad \int_0^T \|f \omega \Delta^{-\mu}\|_{L^2(\mathcal{O})}^2 \varphi^2 dt \lesssim \int_0^T \|f \omega\|_{L^2(\mathcal{O})}^2 (\varphi^2 + (\varphi')^2) dt.$$

We conclude by interpolating (3.11) and (3.12) as in [11, Proposition 3.7].  $\square$

**3.3. Half ultracontractivity estimate using Nash's argument.** In the spirit of Nash work [38], we deduce a time pointwise gain of integrability estimate.

**Corollary 3.10.** *There exists  $\nu_1 > 0$  such that*

$$(3.13) \quad \|f(t, \cdot)\|_{L_\omega^2(\mathcal{O})} \lesssim \frac{1}{t^{\nu_1}} \|f_0\|_{L_\omega^1(\mathcal{O})}, \quad \forall t \in (0, 1).$$

*Proof of Corollary 3.10.* For  $\varphi \in C^1(0, T) \cap C_0(0, T)$ ,  $T \in (0, 1)$ , using first an interpolation inequality and next an Holder inequality, we have

$$\begin{aligned} \int_0^T (\varphi^2 + (\varphi')^2) \|f\|_{L_\omega^2}^2 dt &\leq \int_0^T (1 + (\varphi'/\varphi)^2) \varphi^{2\theta} \|f\|_{L_\omega^1}^{2\theta} \varphi^{2(1-\theta)} \|f\|_{L_\omega^2}^{2(1-\theta)} dt \\ &\leq \left( \int_0^T (1 + (\varphi'/\varphi)^2)^{1/\theta} \varphi^2 \|f\|_{L_\omega^1}^2 dt \right)^\theta \left( \int_0^T \varphi^2 \|f\|_{L_\omega^2}^2 dt \right)^{1-\theta}, \end{aligned}$$

with  $r > 2$  given by Proposition 3.9 and  $\theta \in (0, 1)$  such that  $1/2 = \theta + (1 - \theta)/r$ . Gathering this estimate and the estimate of Proposition 3.9, we obtain after simplification

$$\int_0^T (\varphi^2 + (\varphi')^2) \|f\|_{L_\omega^2}^2 dt \lesssim \int_0^T (1 + (\varphi'/\varphi)^2)^{1/\theta} \varphi^2 \|f\|_{L_\omega^1}^2 dt.$$

Using the estimates

$$\|f(T, \cdot)\|_{L_\omega^2} \lesssim \|f(t, \cdot)\|_{L_\omega^2}, \quad \|f(t, \cdot)\|_{L_\omega^1} \lesssim \|f_0\|_{L_\omega^1},$$

for any  $0 < t < T < 1$  established in Lemma 2.1, we deduce that

$$C_2(\varphi, T) \|f(T, \cdot)\|_{L_\omega^2} \lesssim C_1(\varphi, T) \|f\|_{L_\omega^1},$$

with

$$C_2(\varphi, T)^2 := \int_0^T (\varphi^2 + (\varphi')^2) dt, \quad C_1(\varphi, T)^2 := \int_0^T (1 + (\varphi'/\varphi)^2)^{1/\theta} \varphi^2 dt.$$

We finally take  $\varphi(t) := \varphi_0(t/T)$ ,  $\varphi_0(s) = s^{1/\theta}(1-s)^{1/\theta}$ , and we observe that

$$C_2(\varphi, T)^2 \geq C_2(\varphi_0, 1)^2 T^{-1}, \quad C_1(\varphi, T)^2 \leq C_1(\varphi_0, 1)^2 T^{-1-2/\theta}$$

and  $(1 + (\varphi'_0/\varphi_0)^2)^{1/\theta} \varphi_0^2 \in L^\infty(0, 1)$ , from what we immediately conclude that (3.13) holds with  $\nu := 1/\theta$ .  $\square$

**3.4. The other half ultracontractivity estimate using Moser's argument.** With Corollary 3.10, we have made half of the proof of Theorem 1.3. In order to cover the other half of the way, we may proceed using one of De Giorgi, Nash or Moser approaches. We briefly present the Moser approach based on a iterated scheme.

**Lemma 3.11.** *Assume  $\omega := e^{\alpha y}$ ,  $\alpha > 0$ . There exists a constant  $\tilde{\kappa} \geq 0$  such that any (nonnegative) solution  $f$  to the linear VCK equation (1.16)-(1.18) satisfies*

$$(3.14) \quad \int_0^T \varphi^2 \int_{\mathcal{O}} \left\{ (f\tilde{\omega})^p \left[ \frac{J^2}{\delta^{1/2} \langle y \rangle^2} + \alpha \langle y \rangle \right] + (\partial_y (f\tilde{\omega})^{p/2})^2 \right\} \leq p\tilde{\kappa} \int_0^T (\varphi\varphi'_+ + \varphi^2) \int_{\mathcal{O}} (f\tilde{\omega})^p,$$

for any  $p \in [2, \infty)$  and  $\varphi \in C_c^1(0, T)$ , where  $\tilde{\kappa} = \tilde{\kappa}(a_*, y_*, a^*)$  and  $\tilde{\omega}$  is defined in (2.4).

*Proof of Lemma 3.11.* We adapt the proofs of Lemma 2.1 and Lemma 3.1. We set

$$\tilde{\omega}^p := \tilde{\omega}^p \mathfrak{P},$$

with now  $\mathfrak{P}$  defined by

$$\mathfrak{P} := 1 - \varphi \delta^{1/2} \frac{\delta' J}{\langle J_\xi \rangle^2}, \quad \varphi := \frac{1}{2} \min\left(\frac{1}{\delta^*}, \frac{y_*}{a^* \delta^\#}\right),$$

where we define  $\delta_{\sharp}^2 := \sup[\delta(\delta')^2(v_E - v)^2]$  and we recall that  $\delta_*^2 := \sup[\delta(\delta')^2]$ . Proceeding exactly as during the proof of Lemma 2.1, we have

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\mathcal{O}} f^p \tilde{\omega}^p &= -\frac{4(p-1)}{p^2} \int_{\mathcal{O}} \mathbf{a}(\partial_y(f\tilde{\omega})^{p/2})^2 + \int_{\mathcal{O}} f^p \tilde{\omega}^p \tilde{\omega} \\ &\quad + \int_{\Sigma_0} \left[ \left(1 - \frac{1}{p}\right)K + \left(1 - \frac{2}{p}\right)\mathbf{a} \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \right] n_0 (f\tilde{\omega})^p - \frac{1}{p} \int_{\Sigma_2} J f^p \tilde{\omega}^p n_v, \end{aligned}$$

with

$$\tilde{\omega} := 2\left(1 - \frac{1}{p}\right)\mathbf{a} \left(\frac{\partial_y \tilde{\omega}}{\tilde{\omega}}\right)^2 + \left(\frac{2}{p} - 1\right)\mathbf{a} \frac{\partial_{yy}^2 \tilde{\omega}}{\tilde{\omega}} + K \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} + J \frac{\partial_v \tilde{\omega}}{\tilde{\omega}} + \left(\frac{1}{p} - 1\right)\partial_y K + \left(\frac{1}{p} - 1\right)\partial_v J.$$

It is worth emphasizing that because of the definition of  $\wp$ , we have  $\mathfrak{P} \geq 1/2$  on  $\mathcal{O}$ , we have  $|\partial_y \mathfrak{P}| = \wp \delta^{1/2} |\delta'| (v_E - v) \leq \wp / \delta_{\sharp}$  on  $\Sigma_0$ , and thus

$$\begin{aligned} \left(1 - \frac{1}{p}\right)K + \left(1 - \frac{2}{p}\right)\mathbf{a} \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} &= \left(1 - \frac{1}{p}\right)\mathbf{b} + \left(1 - \frac{2}{p}\right)\mathbf{a} \frac{\partial_y \mathfrak{P}}{\mathfrak{P}} \\ &\geq \left(1 - \frac{1}{p}\right)y_* + \left(\frac{2}{p} - 1\right)a^* 2 \frac{\wp}{\delta_{\sharp}} \geq 0 \end{aligned}$$

on  $\Sigma_0$ . As a consequence, the arguments in Lemma 2.1 remain valid for the two terms involving the boundary sets  $\Sigma_0$  and  $\Sigma_2$  and these ones are thus nonpositive. Also observing that

$$-\int_{\mathcal{O}} \mathbf{a}(\partial_y(f\tilde{\omega})^{p/2})^2 \leq -\frac{1}{2} \int_{\mathcal{O}} \mathbf{a}(\partial_y(f\tilde{\omega})^{p/2})^2 \mathfrak{P} + \int_{\mathcal{O}} \mathbf{a}(f\tilde{\omega})^p (\partial_y \mathfrak{P}^{1/2})^2,$$

we deduce that

$$\frac{1}{p} \frac{d}{dt} \int_{\mathcal{O}} f^p \tilde{\omega}^p \leq -\frac{(p-1)}{p^2} \int_{\mathcal{O}} \mathbf{a}(\partial_y(f\tilde{\omega})^{p/2})^2 + \int_{\mathcal{O}} f^p \tilde{\omega}^p \tilde{\omega},$$

with

$$\tilde{\omega} := \mathfrak{P} \tilde{\omega} + \mathbf{a}(\partial_y \mathfrak{P}^{1/2})^2.$$

Observing that

$$\frac{\partial \tilde{\omega}}{\tilde{\omega}} = \frac{\partial \tilde{\omega}}{\tilde{\omega}} + \frac{1}{p} \frac{\partial \mathfrak{P}}{\mathfrak{P}}$$

and that

$$\frac{\partial_{yy}^2 \tilde{\omega}}{\tilde{\omega}} = \frac{\partial_{yy}^2 \tilde{\omega}}{\tilde{\omega}} + \frac{2}{p} \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \frac{\partial_y \mathfrak{P}}{\mathfrak{P}} + \frac{1-p}{p^2} \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \frac{\partial_y \mathfrak{P}}{\mathfrak{P}} + \frac{1}{p} \frac{\partial_{yy}^2 \mathfrak{P}}{\mathfrak{P}},$$

we get

$$\tilde{\omega} = \mathfrak{P} \varpi + \mathbf{a} \left\{ \frac{2}{p} \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \partial_y \mathfrak{P} + \left(\frac{1}{2} + \frac{1}{p} - \frac{1}{p^2}\right) \frac{(\partial_y \mathfrak{P})^2}{\mathfrak{P}} + \frac{2-p}{p^2} \partial_{yy}^2 \mathfrak{P} \right\} + \frac{1}{p} K \partial_y \mathfrak{P} + \frac{1}{p} J \partial_v \mathfrak{P},$$

where  $\varpi$  is defined by (2.2) and thus uniformly bounded in  $p$ . Moreover, as

$$\left(\frac{\partial_y \mathfrak{P}}{\mathfrak{P}}\right)^2 \sim \frac{1}{y^4}, \quad \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \frac{\partial_y \mathfrak{P}}{\mathfrak{P}} \sim \frac{1}{y^2}, \quad \frac{\partial_{yy} \mathfrak{P}^{1/2}}{\mathfrak{P}^{1/2}} \sim \frac{1}{y^2}, \quad K \left(\frac{1}{2} \frac{\partial_y \mathfrak{P}}{\mathfrak{P}}\right) \sim \frac{1}{y^2}$$

and

$$\partial_v \mathfrak{P} = -\frac{\wp}{2} \frac{(\delta')^2}{\delta^{1/2}} \frac{J}{\langle J \rangle^2} + \frac{\phi}{\langle J \rangle}, \quad \phi \in L^\infty(\mathcal{O}),$$

we conclude that

$$\tilde{\omega} - \frac{1}{p} J \partial_v \mathfrak{P} \sim \varpi \mathfrak{P} \sim -\mathfrak{P} \alpha y$$



when  $y \rightarrow \infty$  and any value of  $\alpha$  uniformly in  $p$ . Because  $\langle J \rangle \simeq \langle y \rangle$ , the previous estimate gives

$$\frac{d}{dt} \int_{\mathcal{O}} f^p \tilde{\omega}^p + \int_{\mathcal{O}} f^p \tilde{\omega}^p \left( \frac{1}{2} \frac{1}{\delta^{1/2}} \frac{J^2}{\langle y \rangle^2} + \frac{\alpha}{2} \langle y \rangle \right) + \int_{\mathcal{O}} a_* (\partial_y (f \tilde{\omega})^{p/2})^2 \lesssim p \int_{\mathcal{O}} f^p \tilde{\omega}^p.$$

We conclude by multiplying the above equation by  $\varphi^2$  and integrating in the time variable.  $\square$

**Lemma 3.12.** *Assume  $\omega := e^{\alpha y}$ ,  $\alpha > 0$ . There exists a constant  $C = C(a_*, a^*) \in (0, \infty)$  such that any solution  $f$  to the linear VCk equation (1.16), (1.18) satisfies*

$$\int_0^T \varphi^2 \int_{\mathcal{O}} (f \tilde{\omega})^p \left[ \frac{1}{\delta^{1/27}} + \langle y \rangle \right] \leq pC \int_0^T (\varphi \varphi'_+ + \varphi^2) \int_{\mathcal{O}} (f \tilde{\omega})^p,$$

for any  $\varphi \in C_c^1(0, T)$  and any  $p \in [2, \infty)$ .

*Proof of Lemma 3.12.* The proof is straightforward by using Lemma 3.11 and by following step by step the proof of Lemma 3.2 and Proposition 3.3.  $\square$

Similarly as in Section 3.2, for a weight function  $\omega := e^{\alpha y}$ ,  $\alpha > 0$ , a solution  $f$  to the linear VCk equation (1.16)-(1.18) and an exponent  $p \geq 2$ , we define

$$(3.15) \quad q := p/2, \quad h := (f \tilde{\omega})^q, \quad \bar{h} := h \varphi(t) \chi(v) \omega_0(y),$$

with  $\varphi \in C_c^1((0, T))$ ,  $\chi \in C_c^1((0, v_F))$ ,  $0 \leq \varphi, \chi \leq 1$ ,  $\omega_0 = \frac{1}{\langle y \rangle}$ . The function  $\bar{h}$  is a solution to the modified VCk equation

$$(3.16) \quad \partial_t \bar{h} + \hat{J} \partial_v \bar{h} + \check{K} \partial_y \bar{h} - \mathbf{a} \partial_{yy}^2 \bar{h} + \alpha \langle y \rangle \bar{h} = H - \mathbf{a} \chi \varphi \omega_0 q(q-1) f^{q-2} (\partial_y f)^2 \tilde{\omega}^q$$

with coefficients

$$\begin{aligned} \hat{J} &:= J(v, y), \text{ if } y > 0, & \hat{J} &:= J(v, -y), \text{ if } y < 0, \\ \check{K} &:= K(v, y), \text{ if } y > 0, & \check{K} &:= -K(v, -y), \text{ if } y < 0, \end{aligned}$$

and source term

$$\begin{aligned} H &:= (\partial_t \varphi) \chi \omega_0 h + \hat{J} \varphi (\partial_v \chi) \omega_0 h - q (\partial_v \hat{J}) \varphi \chi \omega_0 h + \check{K} \varphi \chi (\partial_y \omega_0) h - q (\partial_y \check{K}) \varphi \chi \omega_0 h \\ &\quad - \mathbf{a} \varphi \chi (\partial_{yy}^2 \omega_0) h - 2 \mathbf{a} \varphi \chi (\partial_y \omega_0 + q \omega_0 \frac{\partial_y \tilde{\omega}}{\tilde{\omega}}) (\partial_y h) + \alpha \langle y \rangle \varphi \chi \omega_0 h \\ &\quad + q \left( \frac{\partial_v \tilde{\omega} J}{\tilde{\omega}} + \frac{\partial_y \tilde{\omega} K}{\tilde{\omega}} \right) \varphi \chi \omega_0 h + \mathbf{a} (q+1) q h \left( \frac{\partial_y \tilde{\omega}}{\tilde{\omega}} \right)^2 \varphi \chi \omega_0 - \mathbf{a} q h \frac{\partial_{yy}^2 \tilde{\omega}}{\tilde{\omega}} \varphi \chi \omega_0. \end{aligned}$$

We note that the associated homogeneous equation is again (3.7). Moreover, since

$$(3.17) \quad |H| \lesssim |\partial_t \varphi h| + |\varphi \partial_v \chi h| + q^2 |\varphi h| + q |\varphi \partial_y (h)|,$$

and the term  $-\mathbf{a} q (q-1) f^{q-2} (\partial f)^2 \omega^p$  is negative, a result analogous to Proposition 3.8 holds for the function  $\bar{h}$ .

**Proposition 3.13.** *The function  $\bar{h}$  defined in (3.15) satisfies*

$$(3.18) \quad \|\bar{h}\|_{L^{5/2}(U)}^2 \lesssim q^3 \|\chi\|_{W^{1,\infty}}^2 \int_0^T \|h\|_{L^2}^2 (\varphi^2 + (\varphi')^2) dt,$$

for any  $T \in (0, 1)$ ,  $\varphi \in C_c^1(0, T)$  and  $\chi \in C_c^1(0, v_F)$ .

*Proof of Proposition 3.13.* As in Proposition 3.8, we may use Duhamel formula to obtain

$$\bar{h} = \int_0^t S_{t-s}^0 H_s ds - q(q-1) \int_0^t S_{t-s}^0 (\mathbf{a} f^{q-2} (\partial_y f)^2 \omega^q)(s) ds,$$

for  $t \in (0, 1)$ . We can ignore the second term since it is negative, which allows us to proceed as before to obtain

$$\|\bar{h}\|_{L^{5/2}(\mathbb{R}^2)} \leq t^{1/10} \|H\|_{L^2(U)}.$$

We use then Proposition 3.12 and observation (3.17), to get

$$\|\bar{h}\|_{L^{5/2}(\mathbb{R}^2)} \leq t^{1/10} q^3 \|\chi\|_{W^{1,\infty}}^2 \int_0^T \|h\|_{L^2}^2 (\varphi^2 + (\varphi')^2) dt.$$

From where, we deduce

$$\|\bar{h}\|_{L^{5/2}(\mathcal{U})} \leq q^3 \|\chi\|_{W^{1,\infty}}^2 \int_0^T \|h\|_{L^2}^2 (\varphi^2 + (\varphi')^2) dt \left( \int_0^T t^{1/4} \right)^{2/5} dt,$$

and then immediately conclude.  $\square$

**Proposition 3.14.** *There exists  $r > 2$  such that the function  $h$  defined in (3.15) satisfies*

$$\|\varphi h\|_{L^r(\mathcal{U})}^2 \lesssim q^3 \|(\varphi + |\varphi'|)h\|_{L^2(\mathcal{U})}^2,$$

for any  $\varphi \in C_c^1((0, T))$ .

*Proof of Proposition 3.14.* On the one hand, from Proposition 3.13 and arguing as during the proof of Proposition 3.8, for any  $\beta > 5/2$  and setting  $\Delta := \delta^{2\beta/5} \langle y \rangle$ , we have

$$\|\varphi \Delta h\|_{L^{5/2}(\mathcal{U})}^2 \lesssim q^3 \|(\varphi + |\varphi'|)h\|_{L^2(\mathcal{U})}^2.$$

On the other hand, arguing as during proof of Proposition 3.9, we may rewrite the conclusion of Lemma 3.12 as

$$\|\varphi \Delta^{-\mu} h\|_{L^2(\mathcal{U})}^2 \lesssim q \|(\varphi + |\varphi'|)h\|_{L^2(\mathcal{U})}^2,$$

for some suitable  $\mu > 0$ . We conclude thanks to an interpolation argument.  $\square$

**Proposition 3.15.** *Assume  $\omega := e^{\alpha y}$ ,  $\alpha > 0$ . Any solution  $f$  to the linear VCK equation (1.16)-(1.18) satisfies*

$$\|f_T \omega\|_{L^\infty} \lesssim T^{-\frac{r}{r-2}} \|f_0 \omega\|_{L^2}, \quad \forall T \in (0, 1),$$

where  $r > 2$  is defined in Proposition 3.14.

*Proof of Proposition 3.15.* We follow Moser's iterative method and thus introduce the following sequences

$$t_k := \frac{T}{2} - \frac{T}{2^k}, \quad \mathcal{U}_k := (t_k, T] \times \mathcal{O}, \quad p_k := (r/2)^k 2,$$

where  $r$  is defined in Proposition 3.14 and  $T \in (0, 1]$ .

*Step 1.* In Proposition 3.14, we take  $q = p_k/2$  and  $\varphi \in C_c^1(0, 1)$  such that  $\mathbf{1}_{(t_{k+1}, T)} \leq \varphi \leq \mathbf{1}_{(t_k, T)}$  and  $|\varphi'| \leq 2 \frac{1}{t_{k+1} - t_k}$ . We obtain then

$$\|(f\tilde{\omega})^{p_k/2}\|_{L^r(\mathcal{U}_{k+1})}^2 \lesssim p_k^3 \frac{1}{(t_{k+1} - t_k)^2} \|(f\tilde{\omega})^{p_k/2}\|_{L^2(\mathcal{U}_k)}^2.$$

Noticing that  $p_{k+1} = p_k \frac{r}{2}$ , we may rewrite equivalently the above estimate as

$$(3.19) \quad \|f\tilde{\omega}\|_{L^{p_{k+1}}(\mathcal{U}_{k+1})} \leq \left( C p_k^3 \frac{2^{2k}}{T^2} \right)^{\frac{1}{p_k}} \|f\tilde{\omega}\|_{L^{p_k}(\mathcal{U}_k)}.$$

*Step 2.* Observing that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{p_k} &= \frac{1}{2} \sum_{j=0}^{\infty} \left( \frac{2}{r} \right)^j = \frac{r}{r-2}, \\ \sum_{k=1}^{\infty} \frac{k}{p_k} &= \frac{1}{2} \sum_{j=0}^{\infty} j \left( \frac{2}{r} \right)^j = c_1 < \infty, \end{aligned}$$

and

$$\prod_{k=1}^{\infty} p_j^{3/p_j} = e^{3 \sum_{k=1}^{\infty} \frac{\log p_j}{p_j}} = e^{3 \log(\frac{r}{2}) \sum_{k=1}^{\infty} j \left(\frac{2}{r}\right)^j} = e^{3 \log(\frac{r}{2}) c_1},$$

we deduce that

$$\prod_{j=1}^k \left( Cp_j^3 \frac{2^{2j}}{T^2} \right)^{1/p_j} \lesssim T^{-2\frac{r}{r-2}}.$$

As a consequence, we have

$$\begin{aligned} \|f\tilde{\omega}\|_{L^\infty(\mathcal{U}_\infty)} &\leq \liminf_{k \rightarrow \infty} \|f\tilde{\omega}\|_{L^{p_k}(\mathcal{U}_k)} \\ &\leq \liminf_{k \rightarrow \infty} \prod_{j=1}^k \left( Cp_j^3 \frac{2^{2j}}{T} \right)^{1/p_j} \|f\tilde{\omega}\|_{L^{p_1}(\mathcal{U}_1)} \\ &\lesssim T^{-2\frac{r}{r-2}} \|f\tilde{\omega}\|_{L^2(\mathcal{U}_1)}. \end{aligned}$$

We conclude by using Lemma 2.1 with  $p = 2$  in order to bound the RHS term.  $\square$

We finally come to the

*Proof of Theorem 1.3.* We may reduce the proof to the case  $s = 0$  by using a mere time translation. From (3.13) and Proposition 3.15, we deduce that

$$(3.20) \quad \|f(t, \cdot)\|_{L^\infty(\mathcal{O})} \lesssim \frac{1}{t^\nu} \|f_0\|_{L^1_\omega(\mathcal{O})}, \quad \forall t \in (0, 1),$$

with  $\nu := \nu_1 + 2r/(r-2)$ . Estimate (1.19) is then just a combination of (3.20) used on the time interval  $(0, 1)$  and the estimate (2.1) for  $p = \infty$  used on the time interval  $[1, \infty)$ .  $\square$

#### 4. INTERIOR HOLDER, HARNACK AND COMPACTNESS ESTIMATES

We briefly explain how we can establish that any solution  $f$  to the linear kinetic equation (1.16) enjoys some further regularity estimates.

**Theorem 4.1.** *For any  $0 < t_0 < t_1 < \infty$  and any relatively compact region  $O \subset\subset \mathcal{O}$ , there exist two constants  $C, \beta > 0$  such that any solution  $f$  to the linear Voltage-Conductance kinetic equation (1.16), (1.18) satisfies*

$$\|f\|_{C^\beta((t_0, t_1) \times O)} \leq C \|f_0\|_{L^1_{\omega_1}}.$$

*Proof of Theorem 4.1.* Introducing the localization  $\bar{f}$  as in (3.6), we write

$$(4.1) \quad \partial_t \bar{f} + J \partial_v \bar{f} + K \partial_y \bar{f} - \mathbf{a} \partial_{yy}^2 \bar{f} = F - \alpha \langle y \rangle \bar{f} =: \mathfrak{F} \quad \text{in } (0, \infty) \times \mathcal{O}.$$

We introduce the change of variables

$$\bar{g}(t, w, z) = \bar{f}(t, v, y), \quad \mathfrak{G}(t, w, z) = \mathfrak{F}(t, v, y),$$

with  $w = w(v, y)$  and  $z = z(v, y)$ . Assuming

$$(4.2) \quad \partial_y z = 1, \quad \partial_y w = 0,$$

we straightforwardly obtain

$$\begin{aligned} \mathfrak{G} &= \partial_t \bar{f} + J \partial_v \bar{f} + K \partial_y \bar{f} - \mathbf{a} \partial_{yy}^2 \bar{f} \\ &= \partial_t \bar{g} + J \partial_v w \partial_w \bar{g} + J \partial_v z \partial_z \bar{g} + K \partial_z \bar{g} - \mathbf{a} \partial_{zz}^2 \bar{g}. \end{aligned}$$

The previous conditions on  $w$  and  $z$  are equivalent to

$$(4.3) \quad w := \phi(v), \quad z := y + \psi(v),$$

and we look for  $\phi$  and  $\psi$  such that

$$J \partial_v w = z,$$

or in other words

$$(y(v_E - v) - y_L v) \phi'(v) = y + \psi(v).$$

A convenient choice is

$$(4.4) \quad \phi(v) := -\log(v_E - v), \quad \psi(v) := -\frac{yLv}{v_E - v}.$$

As a conclusion, with this change of variables, the function  $\bar{g}$  is a solution to

$$(4.5) \quad \partial_t \bar{g} + z \partial_w \bar{g} + Q \partial_z \bar{g} - \mathfrak{a} \partial_{zz}^2 \bar{g} = \mathfrak{G},$$

with  $Q := J\psi' + K$ . Because of the compact support condition on  $\bar{g}$ , we may for instance understand the equation on  $\mathbb{R}_+ \times \mathbb{R}^2$  by defining  $Q \in W^{1,\infty}(\mathbb{R}^2)$  arbitrarily outside of the support of  $\bar{g}$ . By definition and using Theorem 1.3, we have  $\bar{g}, \mathfrak{G} \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$ .

We introduce a mollifier  $(\rho_\varepsilon)$  in  $\mathcal{D}(\mathbb{R}^2)$  and the regularization sequence  $\bar{g}_\varepsilon := \bar{g} *_{wz} \rho_\varepsilon$  which satisfies

$$\partial_t \bar{g}_\varepsilon + z \partial_w \bar{g}_\varepsilon + Q \partial_z \bar{g}_\varepsilon - \mathfrak{a} \partial_{zz}^2 \bar{g}_\varepsilon = \mathfrak{G}_\varepsilon,$$

with  $\mathfrak{G}_\varepsilon := \mathfrak{G} * \rho_\varepsilon + [\rho_\varepsilon, z \partial_w + Q \partial_z] \bar{g}$ . Using [17, Lemma II.1], we have  $\mathfrak{G}_\varepsilon \rightarrow \mathfrak{G}$  in any  $L^p(\mathbb{R}^2)$  space,  $1 \leq p < \infty$ . We may multiply the above equation by  $\bar{g}_\varepsilon$  and integrate in all variables. Using the Green formula and the fact that the boundary term vanishes because of the compact support of  $\bar{g}_\varepsilon$ , we get

$$\int_{\mathcal{U}} (\partial_z \bar{g}_\varepsilon)^2 = \int_{\mathcal{U}} \left[ \frac{1}{2} \bar{g}_\varepsilon^2 \partial_z Q + \bar{g}_\varepsilon \mathfrak{G}_\varepsilon \right] \lesssim \|\bar{g}\|_{L^2(\mathcal{U})}^2 + \|\mathfrak{G}\|_{L^2(\mathcal{U})}^2.$$

Passing to the limit, we deduce  $\partial_z \bar{g} \in L^2(\mathcal{U})$ . We may thus apply [22, Theorem 1.3], from which we immediately conclude that  $\bar{g} \in C^\beta$ , for some  $\beta \in (0, 1)$ , so that  $\bar{f}$  and  $f$  are also  $C^\beta$  functions.  $\square$

Using the same change of variables as above and the Harnack inequality for the standard kinetic Fokker-Planck equation as developed in [22, 27, 28, 9], we deduce the Harnack inequality for the VCK equation (1.16).

**Theorem 4.2.** *For  $\varepsilon \in (0, 1)$ , we define  $\mathcal{O}_\varepsilon := \{(v, y) \in \mathcal{O}; \delta(v) > \varepsilon, y < 1/\varepsilon\}$ . For every  $T > T_0 > 0$ , and  $\varepsilon > 0$ , there exists  $C$  such that any solution  $0 \leq f \in L^\infty(\mathcal{U})$  to the linear VCK equation (1.16) satisfies*

$$(4.6) \quad \sup_{\mathcal{O}_\varepsilon} f_{T_0} \leq C \inf_{\mathcal{O}_\varepsilon} f_T.$$

*Proof of Theorem 4.2.* We define  $g(t, w, z) = f(t, v, y)$  with the previous change of variables (4.3) and (4.4). The function  $g$  thus satisfies

$$\partial_t g + z \partial_w g - \mathfrak{a} \partial_{zz}^2 g + Q \partial_z g + Ag = 0$$

on  $U := (0, \infty) \times O$ , with  $Q$  defined during the proof of Theorem 4.1,  $A := \partial_v J + \partial_y K$  and  $O := \Xi(\mathcal{O})$ ,  $\Xi(v, y) := (\phi(v), y + \psi(v))$ . Because  $\Xi$  is a homeomorphism, the set  $O_\varepsilon := \Xi(\mathcal{O}_\varepsilon)$  is relatively compact in  $O$ . We may thus use the Harnack inequality established in [9, Theorem 6.1] which tells us that there exists a constant  $C = C(\varepsilon, T, T_0)$  such that

$$\sup_{O_\varepsilon} g_{T_0} \leq C \inf_{O_\varepsilon} g_T,$$

from what (4.6) immediately follows.  $\square$

We conclude this section by formulating the following compactness result.

**Theorem 4.3.** *Consider a sequence  $(f_n)$  bounded in  $L^\infty_{\omega}(\mathcal{U}) \cap \mathcal{H}_\omega$  such that*

$$\begin{aligned} \partial_t f_n + \partial_v (J f_n) + \partial_y (K_n f_n) - \mathfrak{a}_n \partial_{yy}^2 f_n &= 0 \quad \text{in } \mathcal{U}, \\ \mathcal{R}_{\mathfrak{a}_n, K_n} \gamma f_n &= 0 \quad \text{on } \Gamma, \quad f_n(0) = f_{0n} \quad \text{in } \mathcal{O}, \end{aligned}$$

with  $K_n := \mathbf{b}_n - y$ ,  $(\mathbf{a}_n)$  and  $(\mathbf{b}_n)$  bounded in  $L^\infty(0, T)$ ,  $\mathbf{a}_n \geq a_*$ ,  $\mathbf{b}_n \geq y_*$  and  $f_{0n} \rightarrow f_0$  in  $L^1(\mathcal{O})$ . There exist  $f \in L^\infty_\omega(\mathcal{U}) \cap \mathcal{H}_\omega$  and  $\mathbf{a}, \mathbf{b} \in L^\infty(0, T)$  such that, up to the extraction of a subsequence,  $f_n \rightarrow f$  strongly in  $L^1(\mathcal{U})$ ,  $\mathbf{a}_n \rightarrow \mathbf{a}$  and  $\mathbf{b}_n \rightarrow \mathbf{b}$  weakly in  $L^\infty(0, T)$  and

$$\begin{aligned} \partial_t f + \partial_y(Jf) + \partial_y(Kf) - \mathbf{a} \partial_{yy}^2 f &= 0 \quad \text{in } \mathcal{U}, \\ \mathcal{R}_{\mathbf{a}, K} \gamma f &= 0 \quad \text{on } \Gamma, \quad f(0) = f_0 \quad \text{in } \mathcal{O}, \end{aligned}$$

in the variational sense with  $K := \mathbf{b} - y$ .

*Proof of Theorem 4.3. Step 1.* We define the sequence  $(\bar{f}_n)$  as in (3.6) for a localization function  $\psi \in C_c^1(\mathcal{U})$ . From Theorem 4.1, we have

$$\|\bar{f}_n\|_{C^\beta(U)} \leq C$$

for some  $\beta \in (0, 1)$  and any  $U \subset\subset \mathcal{U}$ . From the Ascoli theorem and a Cantor diagonal process, the sequence  $(\bar{f}_n)$  and next the sequence  $(f_n)$  belong to a strong compact set of  $L^1(\mathcal{U})$ . Up to the extraction of a subsequence, there exists thus  $f \in L^\infty_\omega(\mathcal{U}) \cap \mathcal{H}_\omega$  such that  $f_n \rightarrow f$  strongly in  $L^1(\mathcal{U})$  and  $\partial_y f_n \rightarrow \partial_y f$  weakly in  $L^2(\mathcal{U})$ . We may also assume that, up to the extraction of the same subsequence,  $\mathbf{a}_n \rightarrow \mathbf{a}$  and  $\mathbf{b}_n \rightarrow \mathbf{b}$  weakly in  $L^\infty(0, T)$ .

*Step 2.* We claim that for any  $\psi \in C_c^1(\mathcal{U} \cup \Gamma_-)$ , we have

$$(4.7) \quad \int_{\mathcal{O}} \partial_y f_n \partial_y \psi \rightarrow \int_{\mathcal{O}} \partial_y f \partial_y \psi \quad \text{strongly } L^1(0, T).$$

We first assume  $\psi \in C_c^4(\mathcal{U})$  and we denote

$$\Psi_n := \int_{\mathcal{O}} \partial_y f_n \partial_y \psi.$$

We clearly have  $(\Psi_n)$  is bounded in  $L^2(0, T)$  from the fact that  $(f_n)$  is bounded in  $\mathcal{H}_\omega$ . On the other hand, we compute

$$\begin{aligned} \frac{d}{dt} \Psi_n &= \int_{\mathcal{O}} \partial_y f_n \partial_t \partial_y \psi - \int_{\mathcal{O}} \partial_t f_n \partial_{yy}^2 \psi \\ &= \int_{\mathcal{O}} f_n [J \partial_{vyy}^3 \psi - K_n \partial_{yyy}^2 \psi - \mathbf{a}_n \partial_{yyy}^4 \psi - \partial_{tyy}^3 \psi], \end{aligned}$$

where the RHS is bounded in  $L^\infty(0, T)$ . From the Rellich theorem, we deduce that  $(\Psi_n)$  is relatively compact in  $L^1(0, T)$  and thus (4.7) holds. For a general function  $\psi \in C_c^1(\mathcal{U} \cup \Gamma_-)$ , we may introduce an approximation family  $(\psi_\varepsilon)$  of  $C^4(\mathcal{U})$  such that  $\|\psi - \psi_\varepsilon\|_{W^{1, \infty}} \leq \varepsilon$  for any  $\varepsilon > 0$ . We obvious notations, we have already established that the associated sequence  $(\Psi_{n, \varepsilon})_{n \geq 1}$  is relatively compact in  $L^1(0, T)$  for any  $\varepsilon > 0$ . On the other hand, we compute

$$\|\Psi_{n, \varepsilon} - \Psi_n\|_{L^1(0, T)} \leq \int_{\mathcal{U}} |\partial_y f_n (\partial_y \psi - \partial_y \psi_\varepsilon)| \leq \|f_n\|_{\mathcal{H}} \|\partial_y \psi - \partial_y \psi_\varepsilon\| \lesssim \varepsilon$$

for any  $n \geq 1$ . That precisely mean that the sequence  $(\Psi_n)$  is relatively pre-compact in  $L^1(0, T)$ , from what we classically deduce (4.7).

*Step 3.* Proceeding as in Step 7 of the proof of Theorem 2.2, we also know that  $\gamma f_n \rightarrow \gamma f$  weakly in  $L^\infty(\Gamma)$ . From the variational formulation (2.23) of  $f_n$ , we have

$$(4.8) \quad \int_{\mathcal{U}} f_n (-\partial_t - J \partial_v) \psi + \int_{\mathcal{U}} (\mathbf{a}_n \partial_y f_n - K_n) \partial_y \psi = \int_0^T \int_{y_F}^\infty (J \gamma f_n)(t, v_F, y) \psi(t, 0, y) + \int_{\mathcal{O}} f_{0n} \psi(0, \cdot)$$

for any  $\psi \in C_c^1(\mathcal{U} \cup \Gamma_-)$ . Because of the strong convergence properties of  $(f_n)$  and  $(\Psi_n)$  as well as the weak convergence properties of  $(\mathbf{a}_n)$ ,  $(K_n)$  and  $(\gamma f_n)$ , we may pass to the limit in the above formulation, and we get that  $f$  satisfies the variational formulation (2.23).  $\square$

## 5. EXISTENCE RESULT FOR THE NONLINEAR PROBLEM

In this section, we present the proof of Theorem 1.1 that we split into four steps

*Step 1.* We fix an admissible weight function  $\omega$ , a constant  $a^* > 2(a_* + y_*)$  and  $\mathfrak{c}, N^* > 0$  such that  $N^*(2\mathfrak{c} + 2\mathfrak{c}^2) \leq a^*$ . For any function  $0 \leq N \in L^\infty(\mathbb{R}_+)$  such that  $\|N\|_{L^\infty} \leq N^*$ , we define the three functions  $\mathfrak{a} := a_* + \mathfrak{c}^2 N \in [a_*, a^*]$ ,  $\mathfrak{b} := y_* + \mathfrak{c}N \in [y_*, a^*]$  and  $K := \mathfrak{b} - y$ . We introduce the splitting

$$(5.1) \quad \mathcal{L} := \mathcal{L}_{\mathfrak{a}, K} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A}f := M\chi_R f,$$

From the proof of Lemma 2.1, we see that we may choose  $M = M(a^*) > 0$  and  $R = R(a^*) > 0$  large enough in such a way that

$$\varpi_\omega^{\mathcal{B}} = \varpi_\omega^{\mathcal{L}} - M\chi_R \leq -1$$

and for  $\omega := e^{\alpha y}$ ,  $\alpha > 0$ ,

$$\varpi_{\omega_\alpha}^{\mathcal{B}} = \varpi_{\omega_\alpha}^{\mathcal{L}} - M\chi_R \leq -1 - \alpha'y, \quad \alpha' \in (0, \alpha).$$

We denote by  $S_{\mathcal{B}}(t, s)$  and  $S_{\mathcal{L}}(t, s)$  the associated semigroups and we use the shorthand  $S_{\mathcal{B}}(t) := S_{\mathcal{B}}(t, 0)$ ,  $S_{\mathcal{L}}(t) := S_{\mathcal{L}}(t, 0)$ . Repeating the proof of Lemma 2.1, we know that

$$(5.2) \quad \|S_{\mathcal{B}}(t, s)f_s\|_{L_\omega^p} \leq e^{-(t-s)}\|f_s\|_{L_\omega^p}, \quad \forall t \geq s \geq 0.$$

Let us introduce  $\omega_\alpha = e^{\alpha y}$ ,  $\alpha > 0$ , such that  $\omega_\alpha \geq \omega$ . Repeating the proof of Theorem 1.3, we also have

$$(5.3) \quad \|S_{\mathcal{B}}(t, s)f_s\|_{L_{\omega_\alpha}^\infty} \leq C_1 \frac{e^{-(t-s)}}{(t-s)^\nu} \|f_s\|_{L_{\omega_\alpha}^1}, \quad \forall t \geq s \geq 0,$$

for a constant  $C_1 \in (0, \infty)$  only dependent of  $a^*$ . Introducing the notations

$$(U \star V)(t, s) := \int_s^t U(t, \tau)V(\tau, s)d\tau,$$

$U^{\star 1} = U$ ,  $U^{\star k} = U^{\star(k-1)} \star U$ , the Duhamel identity writes

$$(5.4) \quad S_{\mathcal{L}} = V + W \star S_{\mathcal{L}}, \quad V := S_{\mathcal{B}} + \dots + (S_{\mathcal{B}}\mathcal{A})^{\star(N-1)} \star S_{\mathcal{B}}, \quad W := (S_{\mathcal{B}}\mathcal{A})^{\star N}.$$

As a consequence of (5.2), (5.3) and the fact that  $\mathcal{A} : L^1 \rightarrow L_{\omega_\alpha}^1$ , we have

$$(5.5) \quad \|V(t)\|_{\mathcal{B}(L_\omega^\infty)} \leq C_V e^{-t}, \quad \|W(t, s)\|_{\mathcal{B}(L^1, L_\omega^\infty)} \leq C_W e^{-(t-s)}.$$

For any  $f_0 \in L_\omega^\infty$ , we deduce

$$\begin{aligned} \|S_{\mathcal{L}}(t)f_0\|_{L_\omega^\infty} &\leq C_V e^{-t}\|f_0\|_{L_\omega^\infty} + \int_0^t C_W e^{-(t-s)}\|S_{\mathcal{L}}(s)f_0\|_{L^1} ds \\ &\leq C_V e^{-t}\|f_0\|_{L_\omega^\infty} + \int_0^t C_W e^{-(t-s)} ds \|f_0\|_{L^1}, \end{aligned}$$

and thus

$$(5.6) \quad \|S_{\mathcal{L}}(t)f_0\|_{L_\omega^\infty} \leq C_2 \|f_0\|_{L_\omega^\infty},$$

for a constant  $C_2 \in (1, \infty)$  only dependent of  $a^*$ .

*Step 2.* We set

$$C_3 := \int_{y_F} J(v_F, y)\omega^{-1} dy, \quad \eta^* := a^*/(2C_2 C_3),$$

and

$$\mathfrak{N} := \{N \in L^\infty(0, T); 0 \leq N \leq N^*\}.$$

For  $0 \leq F_0 \in L^\infty_\omega$ , we define  $f = f_N(t) := S_{\mathcal{L}}(t)F_0$  which is then the solution to the linear VCK equation

$$(5.7) \quad \partial_t f = \mathcal{L}_{\mathfrak{a},K} f \text{ in } (0, \infty) \times \mathcal{O}, \quad \mathcal{B}_{\mathfrak{a},K} f = 0 \text{ in } (0, \infty) \times \Sigma, \quad f(0) = F_0 \text{ in } \mathcal{O}.$$

Observing that

$$\begin{aligned} 0 \leq \mathcal{N}(f_N) &\leq C_3 \|\gamma f_N\|_{L^\infty_\omega} \leq C_3 \|f_N\|_{L^\infty_\omega} \\ &\leq C_3 C_2 \|F_0\|_{L^\infty_\omega}, \end{aligned}$$

we have thus

$$\Lambda : \mathfrak{N} \rightarrow \mathfrak{N}, \quad N \mapsto \Lambda(N) := \mathcal{N}(f_N),$$

under the smallness condition  $\|F_0\|_{L^\infty_\omega}(\mathfrak{c} + \mathfrak{c}^2) \leq \eta^*$ .

*Step 3.* We endow  $\mathfrak{N}$  with the weak  $L^2$  convergence. We claim that  $\Lambda$  is continuous. Take indeed  $(N_\ell)$  a sequence of  $\mathfrak{N}$  such that  $N_\ell \rightarrow N$  in  $L^2$ , and denote by  $f_\ell$  the corresponding solution of (5.7) so that  $(f_\ell)$  is bounded in  $\mathfrak{F} \cap \mathcal{H}$  with

$$\mathfrak{F} := \{f \in L^\infty_\omega; f \geq 0, \|f\|_{L^1} = \|F_0\|_{L^1}, \|f\|_{L^\infty} \leq C_2 \|F_0\|_{L^\infty}\}.$$

Thanks to Theorem 4.3, there exists a subsequence  $(f_{\ell_n})$  such that  $f_{\ell_n} \rightarrow f$ , where  $f \in \mathfrak{F} \cap \mathcal{H}$  is a solution to (5.7) associated to  $N$ . By uniqueness of  $f$  in Theorem 2.2, it is the full sequence  $(f_\ell)$  which converges to  $f$ . On the other hand, because  $(\gamma f_\ell)$  is bounded in  $L^\infty_\omega(\Gamma)$ , there exists a subsequence  $(\gamma f_{\ell_n})$  such that  $\gamma f_{\ell_n} \rightarrow \mathfrak{g}$  weakly in  $L^\infty_\omega(\Gamma)$ . Passing to the limit in the Green formula which defines the trace function, we get that  $\mathfrak{g} = \gamma f$ , and thus it is the full sequence  $(\gamma f_\ell)$  which converges to  $\gamma f$ . We thus deduce that

$$\Lambda(N_\ell) = \mathcal{N}(\gamma f_\ell) \rightarrow \mathcal{N}(\gamma f) = \Lambda(N),$$

so that  $\Lambda$  is indeed continuous.

*Step 4.* The set  $\mathfrak{N}$  being obviously convex and compact for the weak  $L^2$  convergence, we may use the Schauder-Tykonov theorem which claims in that situation that there exists  $\bar{N}$  such that  $\Lambda(\bar{N}) = \bar{N}$ . The function  $F := f_{\bar{N}}$  is thus a solution to the nonlinear problem (1.5), (1.8), (1.6), (1.7).  $\square$

## 6. DOBLIN-HARRIS THEOREM IN A BANACH LATTICE

We formulate a general abstract constructive Doblin-Harris theorem in the spirit of the ones presented in the recent works [21, Section 6] and [9, Theorem 7.1], see also [23, 8] for similar results and approaches in the classical probability measures framework.

The proof is a consequence of the previous estimates and of some Doblin-Harris techniques developed in [8, 21] (see also [23] and the references therein).

We consider a Banach lattice  $X$ , which means that  $X$  is a Banach space endowed with a closed positive cone  $X_+$  (we write  $f \geq 0$  if  $f \in X_+$  and we recall that  $f = f_+ - f_-$  with  $f_\pm \in X_+$  for any  $f \in X$ . We also denote  $|f| := f_+ + f_-$ ). We assume that  $X$  is in duality with another Banach lattice  $Y$ , with closed positive cone  $Y_+$ , so that the bracket  $\langle \phi, f \rangle$  is well defined for any  $f \in X$ ,  $\phi \in Y$ , and that  $f \in X_+$  (resp.  $\phi \geq 0$ ) iff  $\langle \psi, f \rangle \geq 0$  for any  $\psi \in Y_+$  (resp. iff  $\langle \phi, g \rangle \geq 0$  for any  $g \in X_+$ ), typically  $X = Y'$  or  $Y = X'$ . We write  $\psi \in Y_{++}$  if  $\psi \in Y$  satisfies  $\langle \psi, f \rangle > 0$  for any  $f \in X_+ \setminus \{0\}$ .

We consider a positive and conservative (or stochastic) semigroup  $S = (S_t) = (S(t))$  on  $X$ , that means that  $S_t$  is a bounded linear mapping on  $X$  such that

- $S_t : X_+ \rightarrow X_+$  for any  $t \geq 0$ ,

- there exist  $\phi_1 \in Y_{++}$ ,  $\|\phi_1\| = 1$ , and a dual semigroup  $S^* = S_t^* = S^*(t)$  on  $Y$  such that  $S_t^* \phi_1 = \phi_1$  for any  $t \geq 0$ . More precisely, we assume that  $S_t^*$  is a bounded linear mapping on  $Y$  such that  $\langle S(t)f, \phi \rangle = \langle f, S^*(t)\phi \rangle$ , for any  $f \in X$ ,  $\phi \in Y$  and  $t \geq 0$ , and thus in particular  $S_t^* : Y_+ \rightarrow Y_+$  for any  $t \geq 0$ .

We denote by  $\mathcal{L}$  the generator of  $S$  with domain  $D(\mathcal{L})$ . For  $\psi \in Y_+$ , we define the seminorm

$$[f]_\psi := \langle |f|, \psi \rangle, \quad \forall f \in X.$$

In order to obtain a very accurate and constructive description of the longtime asymptotic behaviour of the semigroup  $S$ , we introduce additional assumptions.

- We first make the strong dissipativity assumption

$$(6.1) \quad \|S(t)f\| \leq C_0 e^{\lambda t} \|f\| + C_1 \int_0^t e^{\lambda(t-s)} [S(s)f]_{\phi_1} ds,$$

for any  $f \in X$  and  $t \geq 0$ , where  $\lambda < 0$  and  $C_i \in (0, \infty)$ .

- Next, we make the slightly relaxed Doblin-Harris positivity assumption

$$(6.2) \quad S_T f \geq \eta_{\varepsilon, T} g_\varepsilon [S_{T_0} f]_{\psi_\varepsilon}, \quad \forall f \in X_+,$$

for any  $T \geq T_1 > T_0 \geq 0$  and  $\varepsilon > 0$ , where  $\eta_{\varepsilon, T} > 0$ ,  $g \in X_+ \setminus \{0\}$  and  $(\psi_\varepsilon)$  is a bounded and decreasing family of  $Y_+ \setminus \{0\}$ . We say that the above condition is relaxed because we possibly have  $T_0 > 0$  while the condition (6.2) holds with  $T_0 = 0$  in the usual Doblin-Harris.

- We finally assume the following compatibility interpolation like condition

$$(6.3) \quad [f]_{\phi_1} \leq \xi_\varepsilon \|f\| + \Xi_\varepsilon [f]_{\psi_\varepsilon}, \quad \forall f \in X, \quad \varepsilon \in (0, 1],$$

for two positive real numbers families  $(\xi_\varepsilon)$  and  $(\Xi_\varepsilon)$  such that  $\xi_\varepsilon \searrow 0$  as  $\varepsilon \searrow 0$ .

We refer to [9] for a detailed discussion about these assumptions.

**Theorem 6.1.** *Consider a semigroup  $S$  on a Banach lattice  $X$  which satisfies the above conditions. Then, there exists a unique normalized positive stationary state  $f_1 \in D(\mathcal{L})$ , that is*

$$\mathcal{L}f_1 = 0, \quad f_1 \geq 0, \quad \langle \phi_1, f_1 \rangle = 1.$$

Furthermore, there exist some constructive constants  $C \geq 1$  and  $\lambda_2 < 0$  such that

$$(6.4) \quad \|S(t)f - \langle f, \phi_1 \rangle f_1\| \leq C e^{\lambda_2 t} \|f - \langle f, \phi_1 \rangle f_1\|$$

for any  $f \in X$  and  $t \geq 0$ .

*Sketch of the proof of Theorem 6.1.* We just allude the proof which is very similar to the proof of [9, Theorem 7.1]

*Step 1. The Lyapunov condition.* On the one hand, we classically have

$$(6.5) \quad [S(t)f]_{\phi_1} \leq [f]_{\phi_1}, \quad \forall t \geq 0, \quad \forall f \in X.$$

For  $f \in X$ , we may indeed write  $f = f_+ - f_-$ ,  $f_\pm \in X_+$ , and then compute

$$\begin{aligned} |S_t f| &\leq |S_t f_+| + |S_t f_-| \\ &= S_t f_+ + S_t f_- = S_t |f|, \end{aligned}$$

where we have used the positivity property of  $S_t$  in the second line. We deduce

$$[S_t f]_{\phi_1} \leq \langle S_t |f|, \phi_1 \rangle = \langle |f|, S_t^* \phi_1 \rangle$$

and thus (6.5), because of the stationarity property of  $\phi_1$ .

On the other hand, from (6.1) and (6.5), we have

$$\|S_t f\| \leq C_0 e^{\lambda t} \|f\| + C_1 \int_0^t e^{\lambda(t-s)} [f]_{\phi_1} ds,$$



and we may thus choose  $T \geq T_1$  large enough in such a way that

$$(6.6) \quad \|S_T f\| \leq \gamma_L \|f\| + K[f]_{\phi_1},$$

with

$$\gamma_L := C_0 e^{\lambda T} \in (0, 1), \quad K := C_1/\lambda.$$

*Step 2. The Doblin-Harris condition.* Take  $f \geq 0$  such that  $\|f\| \leq A[f]_{\phi_1}$  with  $A > K/(1 - \gamma_L)$ . We have

$$\begin{aligned} \|S_{T_0} f\| &\leq (\gamma_L + K) \|f\| \\ &\leq (\gamma_L + K) A[f]_{\phi_1} \\ &= (\gamma_L + K) A[S_{T_0} f]_{\phi_1} \\ &\leq (\gamma_L + K) A(\xi_\varepsilon \|S_{T_0} f\| + \Xi_\varepsilon [S_{T_0} f]_{\psi_\varepsilon}) \end{aligned}$$

for any  $\varepsilon > 0$ , where we have used successively the growth estimate (6.6) in the first line, the condition on  $f$  in the second line, the stationarity property of  $\phi_1$  in the third line and the interpolation inequality (6.3) in the last line. Choosing  $\varepsilon > 0$  small enough, we immediately obtain

$$\|S_{T_0} f\| \leq 2(\gamma_L + K) A \Xi_\varepsilon [S_{T_0} f]_{\psi_\varepsilon}.$$

Together with

$$[f]_{\phi_1} = [S_{T_0} f]_{\phi_1} \leq \|S_{T_0} f\|$$

and the relaxed Doblin-Harris positivity condition (6.2), we conclude to the conditional Doblin-Harris positivity estimate

$$S_T f \geq c g_\varepsilon [f]_{\phi_1}$$

for all  $T \geq T_1$ , with  $c^{-1} = c_A^{-1} := 2(\gamma_L + K) A \Xi_\varepsilon \eta_{\varepsilon, T}^{-1}$ . We may now classically improve the non-expensive estimate (6.5) on the set  $\mathcal{N} := \{f \in X; \langle \phi_1, f \rangle = 0\}$ . Take indeed  $f \in \mathcal{N}$  such that  $\|f\| \leq A[f]_{\phi_1}$ . Observing that  $[f_\pm]_{\phi_1} = [f]_{\phi_1}/2$  and thus  $\|f_\pm\| \leq \|f\| \leq 2A[f_\pm]_{\phi_1}$ , the previous estimate tells us that

$$S_T f_\pm \geq \varrho g_\varepsilon \quad \varrho := c_{2A} [f]_{\phi_1}.$$

Slightly modifying the arguments of Step 1, we compute now

$$\begin{aligned} |S_t f| &\leq |S_t f_+ - \varrho g_\varepsilon| + |S_t f_- - \varrho g_\varepsilon| \\ &= S_t |f| - 2\varrho g_\varepsilon. \end{aligned}$$

We deduce

$$[S_t f]_{\phi_1} \leq \langle |f|, \phi_1 \rangle - 2\varrho \langle \phi_1, g_\varepsilon \rangle,$$

and thus conclude to the *conditional coupling estimate*

$$(6.7) \quad [S_t f]_{\phi_1} \leq \gamma_H [f]_{\phi_1},$$

with  $\gamma_H := 1 - 2c_{2A} \langle \phi_1, g_\varepsilon \rangle \in (0, 1)$ .

*Step 3.* We introduce a new equivalent norm  $\| \cdot \|$  on  $X$  defined by

$$(6.8) \quad \| \| f \| \| := [f]_{\phi_1} + \beta \| f \|.$$

Using the three properties (6.5), (6.6) and (6.7), we may prove that there exist  $\beta > 0$  small enough and  $\gamma \in (0, 1)$  such that

$$(6.9) \quad \| \| S_T f \| \| \leq \gamma \| \| f \| \|, \quad \text{for any } f \in \mathcal{N}.$$

We refer to [9, Proof of Theorem 7.1] where this claim is established, and to [23, 8] for previous variants. We then classically conclude, see again for instance [9, Proof of Theorem 7.1].  $\square$

## 7. CONVERGENCE IN THE LARGE TIME ASYMPTOTIC

In this section, we prove Theorem 1.2 as a consequence of the previous analysis of the linear VCk equation, in particular Theorem 1.3, and of the constructive Doblin-Harris stability result stated in Theorem 6.1.

*Proof of Theorem 1.2.* We split the proof into four steps, checking first that the conditions of Theorem 6.1 are met.

*Step 1. The stochastic semigroup and the strong dissipativity condition.* We proceed similarly as during the proof of Theorem 1.1. We fix an admissible weight function  $\omega$  and we consider the Banach lattice  $X := L_\omega^2$ . We fix a constant  $a^* > 2(a_* + y_*)$  and for any two constants  $N, \mathfrak{c} \geq 0$  such that  $(\mathfrak{c} + \mathfrak{c}^2)N \leq a^*/2$ , we set  $\mathfrak{a} := a_* + \mathfrak{c}^2 N \in [a_*, a^*]$ ,  $\mathfrak{b} := y_* + \mathfrak{c}N \in [y_*, a^*]$ ,  $K := \mathfrak{b} - y$  and  $\mathcal{L} := \mathcal{L}_{\mathfrak{a}, \mathfrak{b}}$ . Because of Theorem 2.9, we know that  $\mathcal{L}$  generates a semigroup on  $L_\omega^2$  which is associated to the linear equation (1.16) with boundary conditions (1.18) and which is mass and positivity conservative, so that  $S_{\mathcal{L}}$  is stochastic and  $\phi_1 = 1$ . Introducing next the same splitting (5.1) and using (5.2), the  $L^2$  variant of (5.3), (5.4), the  $L^2$  variant of (5.5), we deduce that

$$\|S_{\mathcal{L}}(t)f_0\|_{L_\omega^2} \leq C_V e^{-t} \|f_0\|_{L_\omega^2} + \int_0^t C_W e^{-(t-s)} \|S_{\mathcal{L}}(s)f_0\|_{L^1} ds$$

which is nothing but the strong dissipativity condition (6.1).

*Step 2. The Doblin-Harris condition and the Doblin-Harris Theorem.* For  $0 \leq f_0 \in L_\omega^2$ , let us denote  $f_t := S_{\mathcal{L}}(t)f_0$ . From the Harnack inequality (4.6), for any  $T > T_0 > 0$  and  $\varepsilon > 0$ , there exists a constant  $C \in (0, \infty)$ , independent of  $f_0$ , so that

$$\begin{aligned} f_T &\geq \inf_{\mathcal{O}_\varepsilon} f_T \mathbf{1}_{\mathcal{O}_\varepsilon} \\ &\geq \frac{1}{C} \frac{1}{|\mathcal{O}_\varepsilon|} \mathbf{1}_{\mathcal{O}_\varepsilon} \int_{\mathcal{O}_\varepsilon} f_{T_0}, \end{aligned}$$

what is nothing but (6.2) with  $g_\varepsilon = \psi_\varepsilon := \mathbf{1}_{\mathcal{O}_\varepsilon}$ . On the other hand, for any  $f \in L_\omega^2$ , we have

$$\begin{aligned} \int_{\mathcal{O}} |f| &= \int_{\mathcal{O}_\varepsilon^c} |f| + \int_{\mathcal{O}_\varepsilon} |f| \\ &\leq \left( \int_{\mathcal{O}_\varepsilon^c} \omega^{-2} \right)^{1/2} \|f\|_{L_\omega^2} + \int |f| \mathbf{1}_{\mathcal{O}_\varepsilon}, \end{aligned}$$

so that (6.3) holds true with  $\Xi_\varepsilon := 1$ ,  $\xi_\varepsilon := \|\mathbf{1}_{\mathcal{O}_\varepsilon}\|_{L_\omega^2} \rightarrow 0$  because  $\omega^{-1} \in L^2(\mathcal{O})$ . As a consequence, of Theorem 6.1, there exists a nonnegative and normalized steady state  $\mathcal{M}_N$  which is furthermore asymptotically exponential stable.

*Step 3. The fixed point argument.* Take  $0 \leq F_* \in L_\omega^\infty$  such that  $\|F_*\|_{L^1} = 1$ . Choosing  $\mathfrak{c} > 0$  small enough in such a way that  $(\mathfrak{c} + \mathfrak{c}^2)\|F_*\|_{L_\omega^\infty} \leq \eta^*$ , we see that

$$\mathfrak{F}_* := \{f \in L_\omega^\infty; f \geq 0, \|f\|_{L^1} = 1, \|f\|_{L_\omega^\infty} \leq C_2 \|F_*\|_{L_\omega^\infty}\}$$

is not empty because  $F_* \in \mathfrak{F}_*$ . From Steps 1 & 2, we may apply Theorem 6.1 and we deduce that  $S_{\mathcal{L}}(t)F_* \rightarrow \mathcal{M}_N$  as  $t \rightarrow \infty$ . On the other hand, from (5.6), we have  $S_{\mathcal{L}}(t)F_* \in \mathfrak{F}_*$  for any  $t \geq 0$ . We deduce that  $\mathcal{M}_N \in \mathfrak{F}_*$ , and in particular

$$0 \leq \mathcal{N}(\mathcal{M}_N) \leq C_3 \|\gamma \mathcal{M}_N\|_{L_\omega^\infty} \leq C_3 \|\mathcal{M}_N\|_{L_\omega^\infty} \leq C_3 C_2 \|F_*\|_{L_\omega^\infty} \leq \frac{C_3 C_3}{\mathfrak{c} + \mathfrak{c}^2} \eta^* = N^*.$$

We have thus built a mapping

$$\Lambda_* : [0, N^*] \rightarrow [0, N^*], \quad N \mapsto \Lambda_*(N) := \mathcal{N}(\mathcal{M}_N)$$

From Step 3 in the proof of Theorem 1.1, the mapping  $\Lambda$  is continuous. As a consequence, there exists at least a fix point  $N^\sharp \in [0, N^*]$ , so that  $\Lambda_*(N^\sharp) = N^\sharp$ . The function  $\mathcal{M} := \mathcal{M}_{N^\sharp}$  is then

a nonnegative normalized stationary solution to the nonlinear VCK equation (1.3)-(1.9) and it is linearly asymptotically exponential stable by construction.  $\square$

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#### REFERENCES

- [1] Francesca Anceschi and Sergio Polidoro. A survey on the classical theory for Kolmogorov equation. *Matematiche (Catania)*, 75(1):221–258, 2020.
- [2] Armand Bernou, Kleber Carrapatoso, Stéphane Mischler, and Isabelle Tristani. Hypocoercivity for kinetic linear equations in bounded domains with general Maxwell boundary condition. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 40(2):287–338, 2023.
- [3] François Bouchut. Existence and uniqueness of a global smooth solution for the Vlasov-Poisson-Fokker-Planck system in three dimensions. *J. Funct. Anal.*, 111(1):239–258, 1993.
- [4] François Bouchut. Hypocoercivity in kinetic equations. *J. Math. Pures Appl. (9)*, 81(11):1135–1159, 2002.
- [5] Romain Brette and Wolfram Gerstner. Adaptive exponential integrate-and-fire model as an effective description of neuronal activity. *Journal of neurophysiology*, 94(5):3637–3642, 2005.
- [6] David Cai, Louis Tao, Aaditya V. Rangan, and David W. McLaughlin. Kinetic theory for neuronal network dynamics. *Commun. Math. Sci.*, 4(1):97–127, 2006.
- [7] David Cai, Louis Tao, Michael Shelley, and David W. McLaughlin. An effective kinetic representation of fluctuation-driven neuronal networks with application to simple and complex cells in visual cortex. *Proceedings of the National Academy of Sciences*, 101:7757–7762, 2004.
- [8] José A. Cañizo and Stéphane Mischler. Harris-type results on geometric and subgeometric convergence to equilibrium for stochastic semigroups. *J. Funct. Anal.*, 284(7):46, 2023. Id/No 109830.
- [9] Kleber Carrapatoso, Pierre Gabriel, Richard Medina, and Stéphane Mischler. Constructive Krein-Rutman result for the Kinetic Fokker-Planck equation in a domain. arXiv 2407.10530 (2024).
- [10] Kleber Carrapatoso and Stéphane Mischler. The Landau equation in a bounded domain. arXiv 2407.09031 (2024).
- [11] Kleber Carrapatoso and Stéphane Mischler. The Kinetic Fokker-Planck equation in a domain: ultracontractivity, hypocoercivity and long-time asymptotic behavior, arXiv 2406.10112 (2024).
- [12] José A. Carrillo. Global weak solutions for the initial-boundary-value problems to the Vlasov-Poisson-Fokker-Planck system. *Math. Methods Appl. Sci.*, 21(10):907–938, 1998.
- [13] Edward B. Davies. Spectral properties of metastable Markov semigroups. *J. Functional Analysis*, 52(3):315–329, 1983.
- [14] Edward B. Davies and Barry Simon. Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. *J. Funct. Anal.*, 59(2):335–395, 1984.
- [15] Ennio De Giorgi. Sulla differenziabilità e l’analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)*, pages 25–43, 1957.
- [16] Pierre Degond. Global existence of smooth solutions for the Vlasov-Fokker-Planck equation in 1 and 2 space dimensions. *Ann. Sci. École Norm. Sup. (4)*, 19(4):519–542, 1986.
- [17] Ronald J. DiPerna and Pierre-Louis Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.
- [18] Xu’an Dou, Fanhao Kong, Weijun Xu, and Zhennan Zhou. A voltage-conductance kinetic system from neuroscience: probabilistic reformulation and exponential ergodicity, arXiv 2305.03992 (2023).
- [19] Xu’an Dou, Benoît Perthame, Delphine Salort, and Zhennan Zhou. Bounds and long term convergence for the voltage-conductance kinetic system arising in neuroscience. *Discrete Contin. Dyn. Syst.*, 43(3-4):1366–1382, 2023.
- [20] Richard FitzHugh. Mathematical models of threshold phenomena in the nerve membrane. *Bull. Math. Biol.*, 17(4):257–278, 1955.
- [21] Claudia Fonte Sanchez, Pierre Gabriel, and Stéphane Mischler. On the Krein-Rutman theorem and beyond, arxiv 2305.06652 (2023).
- [22] François Golse, Cyril Imbert, Clément Mouhot, and Alexis Vasseur. Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation. *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)*, 19(1):253–295, 2019.

- [23] Martin Hairer and Jonathan C. Mattingly. Yet another look at Harris' ergodic theorem for Markov chains. In *Seminar on Stochastic Analysis, Random Fields and Applications VI*, volume 63 of *Progr. Probab.*, pages 109–117. Birkhäuser/Springer Basel AG, Basel, 2011.
- [24] Frédéric Hérau. Short and long time behavior of the Fokker-Planck equation in a confining potential and applications. *J. Funct. Anal.*, 244(1):95–118, 2007.
- [25] Lars Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.
- [26] Andreï Kolmogoroff. Zufällige Bewegungen (zur Theorie der Brownschen Bewegung). *Ann. of Math. (2)*, 35(1):116–117, 1934.
- [27] Alberto Lanconelli, Andrea Pascucci, and Sergio Polidoro. Gaussian lower bounds for non-homogeneous Kolmogorov equations with measurable coefficients. *J. Evol. Equ.*, 20(4):1399–1417, 2020.
- [28] Tony Lelièvre, Mouad Ramil, and Julien Reygner. A probabilistic study of the kinetic Fokker-Planck equation in cylindrical domains. *J. Evol. Equ.*, 22(2):Paper No. 38, 74, 2022.
- [29] Jacques-Louis Lions. *Équations différentielles opérationnelles et problèmes aux limites*, volume Band 111 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1961.
- [30] Eric Luçon and Christophe Poquet. Emergence of oscillatory behaviors for excitable systems with noise and mean-field interaction: a slow-fast dynamics approach. *Comm. Math. Phys.*, 373(3):907–969, 2020.
- [31] David McLaughlin, Robert Shapley, Michael Shelley, and Dingeman J Wielaard. A neuronal network model of macaque primary visual cortex (v1): Orientation selectivity and dynamics in the input layer 4ca. *Proceedings of the National Academy of Sciences*, 97(14):8087–8092, 2000.
- [32] Stéphane Mischler. On the trace problem for solutions of the Vlasov equation. *Comm. Partial Differential Equations*, 25(7-8):1415–1443, 2000.
- [33] Stéphane Mischler. Kinetic equations with Maxwell boundary conditions. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(5):719–760, 2010.
- [34] Stéphane Mischler, Cristóbal Quiñinao, and Jonathan Touboul. On a kinetic Fitzhugh-Nagumo model of neuronal network. *Comm. Math. Phys.*, 342(3):1001–1042, 2016.
- [35] Jürgen Moser. A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations. *Comm. Pure Appl. Math.*, 13:457–468, 1960.
- [36] Jürgen Moser. A Harnack inequality for parabolic differential equations. *Comm. Pure Appl. Math.*, 17:101–134, 1964.
- [37] J. Nagumo, S. Arimoto, and S. Yoshizawa. An active pulse transmission line simulating nerve axon. *Proc. IRE*, 50(10):2061–2070, 1962.
- [38] John Nash. Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.*, 80:931–954, 1958.
- [39] Francis Nier. Boundary conditions and subelliptic estimates for geometric Kramers-Fokker-Planck operators on manifolds with boundaries. *Mem. Amer. Math. Soc.*, 252(1200):v+144, 2018.
- [40] Andrea Pascucci and Sergio Polidoro. The Moser's iterative method for a class of ultraparabolic equations. *Commun. Contemp. Math.*, 6(3):395–417, 2004.
- [41] Benoît Perthame and Delphine Salort. On a voltage-conductance kinetic system for integrate & fire neural networks. *Kinet. Relat. Models*, 6(4):841–864, 2013.
- [42] Aaditya V Rangan, David Cai, and David W McLaughlin. Modeling the spatiotemporal cortical activity associated with the line-motion illusion in primary visual cortex. *Proceedings of the National Academy of Sciences*, 102(52):18793–18800, 2005.
- [43] Aaditya V. Rangan, David Cai, and Louis Tao. Numerical methods for solving moment equations in kinetic theory of neuronal network dynamics. *J. Comput. Phys.*, 221(2):781–798, 2007.
- [44] Aaditya V. Rangan, Gregor Kovačič, and David Cai. Kinetic theory for neuronal networks with fast and slow excitatory conductances driven by the same spike train. *Phys. Rev. E (3)*, 77(4):041915, 13, 2008.
- [45] Delphine Salort and Didier Smets. Convergence towards equilibrium for a model with partial diffusion. working paper or preprint, hal-03845918 (Nov. 2022).
- [46] Cédric Villani. Hypocoercivity. *Mem. Amer. Math. Soc.*, 202(950):iv+141, 2009.

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