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WELL-POSEDNESS OF THE GROWTH-COAGULATION EQUATION WITH SINGULAR KERNELS

ANKIK KUMAR GIRI ^{1,*}, PHILIPPE LAURENÇOT ², AND SAROJ SI ¹

ABSTRACT. The well-posedness of the growth-coagulation equation is established for coagulation kernels having singularity near the origin and growing atmost linearly at infinity. The existence of weak solutions is shown by means of the method of the characteristics and a weak L_1 -compactness argument. For the existence result, we also show our gratitude to Banach fixed point theorem and a refined version of the Arzelá-Ascoli theorem. In addition, the continuous dependence of solutions upon the initial data is shown with the help of the DiPerna-Lions theory, Gronwall's inequality and moment estimates. Moreover, the uniqueness of solution follows from the continuous dependence. The results presented in this article extend the contributions made in earlier literature.

1. INTRODUCTION

Coagulation is the process of aggregation where two or more smaller particles or clusters combine to form a larger particle or cluster. In nature, this process is found in phytoplankton aggregation [2], polymer formation [33], powder formation in industry [32], droplet formation in clouds [15] and formation of planets in astrophysics [24, 27]. These physical processes are modelled mathematically by partial integro-differential equations which can be read as

$$\partial_t c(t, v) = Q(c)(t, v) \quad \text{for } (t, v) \in (0, \infty)^2, \quad (1.1)$$

$$c(0, v) = c_0(v) \geq 0, \quad (1.2)$$

where

$$Q(c)(v_1) = \frac{1}{2} \int_0^{v_1} K(v_1 - v_2, v_2) c(v_1 - v_2) c(v_2) dv_2 \\ - \int_0^\infty K(v_1, v_2) c(v_1) c(v_2) dv_2.$$

Here, $K(v_1, v_2)$ denotes the coagulation kernel which gives the rate at which particles of sizes v_1 and v_2 merge to form larger particles of size $v_1 + v_2$ and is a non-negative *symmetric function* on $(0, \infty)^2$. The function $c(t, v_1)$ represents the density of particles with size $v_1 > 0$ at time $t \geq 0$. The first integral in the definition of $Q(c)$ describes the formation of particles of size v_1 due to the interaction

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between the particles of sizes $v_1 - v_2$ and v_2 . On the other hand, the second integral in the definition of $Q(c)$ accounts for the loss of particles of size v_1 due to their merging with particles of arbitrary size. A great amount of progress in this direction has been made over the last hundred years, see for example [4] and the references therein. There is one more kinetic process which can be associated with coagulation, termed as growth process. The growth associated with coagulation is governed by the following mathematical equation

$$\partial_t c(t, v) + \partial_v(gc)(t, v) = Q(c)(t, v) \quad \text{for } (t, v) \in (0, \infty)^2, \quad (1.3)$$

$$c(0, v) = c_0(v) \geq 0, \quad (1.4)$$

where $g(t, v) \geq 0$ represents the rate at which external matter is sticking to the surface of the particles of size v , see [1, 4, 6, 17, 18, 23, 25] and the references therein, so that the second term on the left-hand side of (1.3) represents the growth of the particles of size v_1 . In contrast to the Smoluchowski coagulation equation (1.1), the initial value problem (1.3)–(1.4) has received less attention in the mathematical literature. As far as the existence of a solution to (1.3)–(1.4) is concerned, the initial investigation is conducted by Gajewski and Zacharias [18], where existence is demonstrated in the Hilbert space L^2 for $g \in W^{2,\infty}(0, \infty)$ and the class of coagulation kernels satisfying

$$0 \leq K(v_1, v_2) = K(v_2, v_1) \leq C_0 (1 + (v_1 v_2)^{-\beta_0}), \quad (v_1, v_2) \in (0, \infty)^2, \quad (1.5)$$

for some $\beta_0 \geq 0$ and $C_0 > 0$ and

$$-C_1 \left(1 + \frac{1}{v_1}\right) K(v_1, v_2) \leq \partial_{v_1} K(v_1, v_2) \leq 0, \quad (v_1, v_2) \in (0, \infty)^2,$$

for some $C_1 > 0$. In a subsequent study [17], Gajewski extends the existence result obtained in [18] to the class of coagulation kernels which satisfies only (1.5) using a semigroup theory approach in the Hilbert space L^2 and also includes an additional source term. However, the contributions in [17, 18] do not cover the physically relevant weighted L^1 -space and coagulation kernels which are unbounded for large sizes. Global existence of solutions to (1.3)–(1.4) and uniqueness are studied in [1, 3] for bounded coagulation kernels and in [16] for the class of unbounded coagulation kernels having the following growth condition,

$$K(v_1, v_2) \leq \sigma(v_1 + v_2), \quad (v_1, v_2) \in [1, \infty)^2,$$

where σ is a sublinear function at infinity; that is, $\sigma(v)/v \rightarrow 0$ as $v \rightarrow \infty$. Related models involving growth and multidimensional coagulation have been studied in [10, 19].

The primary focus of this article is to investigate the existence and uniqueness to (1.3)–(1.4) in the weighted L^1 space for the class of coagulation kernels that exhibit singularities for small sizes while being unbounded for large ones and is mainly motivated by [1, 8, 16, 17, 21], with the aim of including a broader class of coagulation kernels in the forthcoming analysis.

The structure of the paper unfolds as follows: In Section 2, we outline the assumptions on coagulation kernels K , growth rate g , and initial data c_0 . This section also introduces the solution concept and highlights the main theorems of the paper. Moving on to Section 3, we delve into characteristics and the mild formulation of our problem (1.3)–(1.4). Dedicated to demonstrating the existence of global weak solutions for (1.3)–(1.4), Section 4 takes center stage. Firstly, we define the truncated problem (4.1)–(4.2). Subsequently, we establish the existence of a unique global mild solution to (4.1)–(4.2) using the Banach fixed-point theorem, following the methodology in [8]. This section also encompasses the derivation of moment estimates for the mild solution, building on the insights

from [21] and relying on the method of weak L_1 -compactness introduced in [29]. Finally, in the last section, we turn our attention to demonstrating the uniqueness and continuous dependence on the initial data of solutions to (1.3)–(1.4).

2. ASSUMPTION AND MAIN RESULT

Assumption 2.1. *Throughout this paper, we impose the following assumptions.*

- (i) K is a non-negative measurable symmetric function on $(0, \infty)^2$. There are $\beta > 0$ and $k, k_1 > 0$ such that

$$K(v_1, v_2) = K(v_2, v_1) \leq \begin{cases} k(v_1 v_2)^{-\beta}, & (v_1, v_2) \in (0, 1)^2, \\ k v_2 v_1^{-\beta}, & (v_1, v_2) \in (0, 1) \times (1, \infty), \\ k(v_1 + v_2), & (v_1, v_2) \in (1, \infty)^2, \end{cases} \quad (2.1)$$

- (ii) g is a non-negative continuous function on $[0, \infty)^2$ and twice differentiable with respect to v_1 with

$$g(t, 0) = 0 \text{ for all } t \geq 0, \quad |\partial_{v_1} g| < A, \quad |\partial_{v_1}^2 g| < B, \quad (2.2)$$

where A and B are positive real numbers.

- (iii) The initial condition c_0 satisfies the following condition

$$0 \leq c_0 \in L^1_{-2\beta, 1}(0, \infty) := L^1(0, \infty; (v + v^{-2\beta}) dv). \quad (2.3)$$

For further use we set $L^1_{m,r}(0, \infty) := L^1(0, \infty; (v^m + v^r) dv)$ for $(m, r) \in \mathbb{R}^2$ and $L^1_m(0, \infty) := L^1_{m,m}(0, \infty)$, and define the moment of order m by

$$M_m(u) := \int_0^\infty v^m u(v) dv, \quad u \in L^1_m(0, \infty).$$

. We also denote the positive cone of $L^1_{m,r}(0, \infty)$ by $L^1_{m,r,+}(0, \infty)$.

Now, let us discuss a class of physically relevant unbounded coagulation kernels that satisfy (2.1). This class includes Smoluchowski's coagulation kernel K_1 [28], the Granulation kernel K_2 [20], and stochastic stirred froths K_3 [7]. These kernels take the following forms

$$K_1(v_1, v_2) = \left(v_1^{\frac{1}{3}} + v_2^{\frac{1}{3}} \right) \left(v_1^{-\frac{1}{3}} + v_2^{-\frac{1}{3}} \right),$$

$$K_2(v_1, v_2) = \frac{(v_1 + v_2)^{\theta_1}}{(v_1 v_2)^{\theta_2}}, \quad \theta_1 \leq 1 \text{ and } \theta_2 \geq 0,$$

$$K_3(v_1, v_2) = (v_1 v_2)^{-\theta}, \quad \theta > 0,$$

for $(v_1, v_2) \in (0, \infty)^2$ and satisfy (2.1) with $\beta = \frac{1}{3}$, $\beta = \theta_2$, and $\beta = \theta$, respectively.

Before presenting our main results, we provide the definition of a weak solution to (1.3)–(1.4) that we will use in the subsequent sections.

Definition 2.2. *Let K and g be two functions satisfying (2.1)–(2.2) and c_0 be an initial condition satisfying (2.3). For $T > 0$, a weak solution to (1.3)–(1.4) on $[0, T]$ is a non-negative function*

$$c \in C([0, T]; w - L^1_{-\beta, 1}(0, \infty)) \quad (2.4)$$

satisfying, for each $t \in [0, T]$ and $\phi \in W^{1,\infty}(0, \infty)$,

$$\begin{aligned} \int_0^\infty c(t, v)\phi(v)dv &= \int_0^\infty c_0(v)\phi(v)dv + \int_0^t \int_0^\infty \phi'(v)g(s, v)c(s, v)dvds \\ &+ \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \tilde{\phi}(v_1, v_2)K(v_1, v_2)c(s, v_2)c(s, v_1)dv_1dv_2ds, \end{aligned} \tag{2.5}$$

where

$$\tilde{\phi}(v_1, v_2) := \phi(v_1 + v_2) - \phi(v_1) - \phi(v_2).$$

We shall say that a weak solution to (1.3)–(1.4) is global if it is a weak solution to (1.3)–(1.4) on $[0, T]$ for all $T > 0$.

Let us mention at this point that, if $\phi \in W^{1,\infty}(0, \infty)$, then

$$\begin{aligned} |\phi'(v_1)g(t, v_1)| &\leq g(t, v_1)\|\phi\|_{W^{1,\infty}(0,\infty)} \leq A\|\phi\|_{W^{1,\infty}(0,\infty)}v_1, \\ \tilde{\phi}(v_1, v_2)K(v_1, v_2) &\leq 3\|\phi\|_{L^\infty(0,\infty)} \left(v_1 + v_1^{-\beta}\right) \left(v_2 + v_2^{-\beta}\right), \end{aligned}$$

for $t \in [0, T]$ and $(v_1, v_2) \in (0, \infty)^3$ by (2.1) and (2.2), so that, thanks to (2.4), all the integrals in (2.5) are well-defined.

Let us now state the main results of this article.

Theorem 2.3. *Suppose that K , g , and c_0 satisfy (2.1), (2.2) and (2.3), respectively. Then there exists at least one global weak solution c to (1.3)–(1.4) in the sense of Definition 2.2 satisfying also*

$$c \in L^\infty(0, T; L^1_{-2\beta,1}(0, \infty)) \quad \text{for all } T > 0.$$

The proof of Theorem 2.3 utilizes a method of characteristics introduced in [8] and subsequently employs a weak compactness approach within the space $L^1_{-2\beta,1}(0, \infty)$, a methodology initially introduced in [29] for the coagulation-fragmentation equation and later adapted in [21] for coagulation kernels K growing at most linearly for large sizes; that is, there is $C > 0$ such that $K(v_1, v_2) \leq C(1 + v_1 + v_2)$ for $(v_1, v_2) \in (0, \infty)^2$.

In addition to establishing the existence of weak solutions, a result on uniqueness is proved for a limited class of initial data c_0 .

Theorem 2.4. *Suppose that K and g satisfy (2.1) and (2.2), and consider $c_0 \in L^1_{-2\beta,2,+}(0, \infty)$. Then there exists a unique global weak solution c to (1.3)–(1.4) in the sense of Definition 2.2 such that*

$$c \in L^\infty(0, T; L^1_{-2\beta,2}(0, \infty)) \quad \text{for all } T > 0. \tag{2.6}$$

To establish uniqueness, akin to [14, 30], we demonstrate a more generalized continuous dependence result in Proposition 5.1. The complexity of the proof stems from the low regularity of weak solutions and the task of identifying an appropriate weight function, and in this instance, it is represented by $w(v) := v^{-\beta} + v$, $v > 0$.

Remark 2.5. *When $g \equiv 0$, $c_0 \in L^1_{-2\beta,2}(0, \infty)$ and K satisfies (2.1), we recover from Theorem 2.4 the well-posedness of the continuous coagulation equation (1.1)–(1.2) already established in [5, 11, 12].*

3. CHARACTERISTIC CURVE AND MILD SOLUTION

3.1. Characteristic curves. In this section, first we define the characteristics, and using the characteristic curves, we define the mild solution to (1.3)–(1.4).

To begin with, let us observe that (2.2) implies that

$$0 \leq g(t, v) \leq Av \quad \text{for } (t, v) \in [0, \infty)^2, \quad (3.1)$$

$$|g(t, v_1) - g(t, v_2)| \leq A|v_1 - v_2| \quad \text{for } (t, v_1, v_2) \in [0, \infty)^3. \quad (3.2)$$

Property (3.2) ensures that, for $(t, v) \in [0, \infty)^2$, the following characteristics equation is well-posed and has a unique solution $Y(\cdot; t, v)$ defined for $s \in [0, \infty)$

$$\begin{cases} \partial_s Y(s; t, v) = g(s, Y(s; t, v)), \\ Y(t; t, v) = v. \end{cases} \quad (3.3)$$

Let us now turn to the following fundamental properties of the characteristic curves, which are taken from [8, 9].

Lemma 3.1. *For $(s, t, v) \in [0, \infty)^3$, the characteristics curves Y satisfy*

$$Y(t; s, Y(s; t, v)) = v, \quad (3.4)$$

$$J(s; t, v) := \partial_v Y(s; t, v) = \exp\left(\int_s^t b(\sigma, Y(\sigma; t, v)) d\sigma\right),$$

$$\partial_t Y(s; t, v) = -g(t, v) \exp\left(\int_s^t b(\sigma, Y(\sigma; t, v)) d\sigma\right),$$

$$Y(t; s, 0) = 0,$$

$$Y(t; s, v) > 0 \text{ for } v > 0.$$

where $b := -\partial_v g$.

The proof of this lemma can be seen in [9, Appendix A]. We next discuss some essential properties of solutions to (3.3) which are frequently used in this paper.

Lemma 3.2. *For $(t, v) \in [0, \infty) \times (0, \infty)$ then the characteristic curve $Y(\cdot; t, v)$ solving (3.3) satisfies the following properties:*

$$\text{for } s_1 \leq s_2 \leq t, \quad 0 < Y(s_2; t, v) \leq Y(s_1; t, v)e^{A(s_2-s_1)}, \quad (3.5)$$

$$\lim_{v \rightarrow \infty} Y(0; t, v) = \infty,$$

$$\text{for } s \geq t, \quad Y(s; t, v) \leq ve^{A(s-t)}, \quad (3.6)$$

$$Y(t; 0, v) \leq ve^{At},$$

$$\text{for } s \geq 0, \quad |g(s, Y(t; 0, v))| \leq Ave^{At}, \quad (3.7)$$

$$\text{for } s \geq t, \quad Y(s; t, v) \geq v. \quad (3.8)$$

Proof. The detailed justification of the properties (3.5)–(3.7) can be found in [9, Lemma 1]. Inequality (3.8) follows from (3.3) and the non-negativity of g on $[0, \infty)^2$. \square

3.2. Mild solution. Now, by using the characteristic curves, we define the notion of *mild solution* to (1.3)–(1.4).

Definition 3.3. Let K and g be two functions satisfying (2.1)–(2.2) and c_0 be an initial condition satisfying (2.3). For $T > 0$, a mild solution to (1.3)–(1.4) on $[0, T)$ is a non-negative function $c \in C([0, T]; L^1(0, \infty))$ with $Q(c) \in L^1((0, T) \times (0, \infty))$ which satisfies the mild formulation to (1.3)–(1.4)

$$c(t, v) = c_0(Y(0; t, v))J(0; t, v) + \int_0^t Q(c)(s, Y(s; t, v))J(s; t, v)ds,$$

for all $t \in [0, T]$ and $v \in (0, \infty)$.

We shall say that a mild solution to (1.3)–(1.4) is global if it is a mild solution to (1.3)–(1.4) on $[0, T]$ for all $T > 0$.

In order to prove Theorem 2.3, it is necessary to recover the notion of weak solution from that of mild solution. In this connection, we report a relation between mild and weak solutions, see [9, Proposition 1] and [19, Lemma 2.3].

Proposition 3.4. Let $T > 0$ and assume that (2.1)–(2.3) hold true. The following statements are equivalent:

- (i) $c \in C([0, T]; L^1((0, \infty)))$ is a weak solution in the sense of Definition 2.2.
- (ii) c is a mild solution in the sense of Definition 3.3.

4. GLOBAL EXISTENCE OF WEAK SOLUTIONS

Let K , g and c_0 satisfy (2.1)–(2.3) and consider the following truncated problem

$$\partial_t c_n + \partial_v(gc_n) = Q_n(c_n) \quad \text{for } (t, v) \in (0, \infty)^2, \tag{4.1}$$

$$c_n(0, v) = c_0^n(v) \geq 0, \tag{4.2}$$

where, for $(v_1, v_2) \in (0, \infty)^2$,

$$Q_n(c)(v_1) := \frac{1}{2} \int_0^{v_1} K_n(v_1 - v_2, v_2)c(v_1 - v_2)c(v_2)dv_2 - \int_0^\infty K_n(v_1, v_2)c(v_1)c(t, v_2)dv_2,$$

and

$$K_n(v_1, v_2) := K(v_1, v_2)\chi_{(1/n, n)}(v_1)\chi_{(1/n, n)}(v_2), \quad c_0^n(v_1) := c_0(v_1)\chi_{(0, n)}(v_1).$$

Here, χ_E denotes the characteristic function of the set E ; that is, $\chi_E(v) = 1$ if $v \in E$ and $\chi_E(v) = 0$ if $v \notin E$. Observe that the definition of K_n and (2.1) guarantee that $K_n \in L^\infty((0, \infty)^2)$ and put

$$\beta_n := \max_{(v_1, v_2) \in (0, \infty)^2} K_n(v_1, v_2).$$

Before proving the existence of a mild solution to (4.1)–(4.2), we collect some properties of the coagulation operator Q_n . For $c_1 \in L^1(0, \infty)$ and $c_2 \in L^1(0, \infty)$, we have

$$\|Q_n(c_1)\|_{L^1(0, \infty)} \leq \frac{3}{2}\beta_n \|c_1\|_{L^1(0, \infty)}^2, \tag{4.3}$$

$$\|Q_n(c_1) - Q_n(c_2)\|_{L^1(0, \infty)} \leq 3\beta_n (\|c_1\|_{L^1(0, \infty)} + \|c_2\|_{L^1(0, \infty)}) \|c_1 - c_2\|_{L^1(0, \infty)}. \tag{4.4}$$

Clearly, (4.3) confirms that, for $c_1 \in L^1_+(0, \infty)$, $Q_n(c_1) \in L^1(0, \infty)$ with

$$\int_0^\infty Q_n(c_1)(v)dv \leq 0. \quad (4.5)$$

Additionally, if $c_1 \in L^1_+(0, \infty)$ then

$$\int_0^\infty vQ_n(c_1)(v)dv = 0. \quad (4.6)$$

Theorem 4.1. *Let c_0 be a non-negative function in $L^1_{-2\beta,1}(0, \infty)$. Then the initial value problem (4.1)–(4.2) has a unique global mild solution in the sense of Definition 3.3. Moreover, it satisfies*

$$\begin{aligned} 0 &\leq \int_0^\infty c_n(t, v)dv \leq \int_0^\infty c_0(v)dv, \\ 0 &\leq \int_0^\infty vc_n(t, v)dv \leq e^{At} \int_0^\infty vc_0(v)dv, \end{aligned} \quad (4.7)$$

and

$$\int_0^\infty vc_n(t, v)dv = \int_0^\infty vc_0^n(v)dv + \int_0^t \int_0^\infty g(s, v)c_n(s, v)dvds, \quad (4.8)$$

for all $t \in [0, \infty)$. In addition, c_n is also a global weak solution to (4.1)–(4.2) and thus satisfies

$$\begin{aligned} \int_0^\infty c_n(t, v)\phi(v)dv &= \int_0^\infty c_0^n(v)\phi(v)dv + \int_0^t \int_0^\infty \phi'(v)g(s, v)c_n(s, v)dvds \\ &\quad + \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \tilde{\phi}(v_1, v_2)K_n(v_1, v_2)c_n(s, v_2)c_n(s, v_1)dv_1dv_2ds. \end{aligned} \quad (4.9)$$

for all $t \in [0, \infty)$ and $\phi \in W^{1,\infty}(0, \infty)$.

Proof. For $n \geq 2$, we set

$$t_n := \frac{1}{2[\kappa_n + 6\beta_n M_0(c_0)]}, \quad \kappa_n := \beta_n M_0(c_0), \quad (4.10)$$

and define the set

$$P_n := \left\{ u \in C([0, t_n]; L^1_+(0, \infty)) : \sup_{t \in [0, t_n]} \{M_0(u(t))\} \leq M_0(c_0), M_1(u(t)) \leq M_1(c_0)e^{At} \right\},$$

recalling that A is the Lipschitz constant of g , see (2.2), and the moments M_m are defined right after the statement of Assumption 2.1. Clearly, P_n is a complete metric space for the distance

$$d_{P_n}(u_1, u_2) := \sup_{t \in [0, t_n]} \{ \|(u_1 - u_2)(t)\|_{L^1(0, \infty)} \}, \quad (u_1, u_2) \in P_n^2.$$

induced by the norm of $C([0, t_n]; L^1(0, \infty))$.

Now, for $u \in P_n$, we define the map $\mathcal{T}_n(u)$ by

$$\begin{aligned} \mathcal{T}_n(u)(t, v) &:= c_0^n(Y(0; t, v))J(0; t, v)e^{-\kappa_n t} \\ &\quad + \int_0^t (Q_n(u)(s, Y(s; t, v)) + \kappa_n u(s, Y(s; t, v))) J(s; t, v)e^{-\kappa_n(t-s)} ds \end{aligned}$$

for $(t, v) \in [0, t_n] \times (0, \infty)$ and aim at showing the existence of a fixed point of \mathcal{T}_n in P_n . Since $\mathcal{T}_n(u)$ is a mild solution to

$$\begin{cases} \partial_t U + \partial_{v_1}(gU) + \kappa_n U = Q_n(u) + \kappa_n u & \text{for } (t, v) \in (0, t_n) \times (0, \infty), \\ U(0, v) = c_0^n(v), \end{cases}$$

a fixed point of \mathcal{T}_n is clearly a mild solution to (4.1)–(4.2) on $[0, t_n]$.

Let us now check that \mathcal{T}_n is a strict contraction on P_n . Let $u \in P_n$ and $t \in [0, t_n]$. We first note that, since $u(t) \geq 0$ and $M_0(u(t)) \leq M_0(c_0)$, we deduce from (4.10) that

$$\begin{aligned} Q_n(u)(t, v) + \kappa_n u(t, v) &\geq \kappa_n u(t, v) - \int_0^\infty K_n(v, v_1) u(t, v) u(t, v_1) dv_1 \\ &\geq [\beta_n M_0(c_0) - \beta_n M_0(u(t))] u(t, v) \geq 0. \end{aligned} \quad (4.11)$$

Combining this lower bound with the non-negativity of c_0^n and J gives

$$\mathcal{T}_n(u) \geq 0 \quad \text{for } u \in P_n. \quad (4.12)$$

Then we compute the norm of $\mathcal{T}_n(u)(t)$ in $L^1(0, \infty)$ for all $t \in [0, t_n]$.

$$\begin{aligned} \int_0^\infty \mathcal{T}_n(u)(t, v) dv &= e^{-\kappa_n t} \int_0^\infty c_0^n(v) dv \\ &\quad + \int_0^t \int_0^\infty (Q_n(u)(s, v) + \kappa_n u(s, v)) e^{-\kappa_n(t-s)} dv ds. \end{aligned}$$

By using (4.5), we end up with

$$M_0(\mathcal{T}_n(u)(t)) \leq e^{-\kappa_n t} M_0(c_0) + \kappa_n M_0(c_0) \left(\frac{1 - e^{-\kappa_n t}}{\kappa_n} \right) = M_0(c_0)$$

when $\kappa_n > 0$ and $M_0(\mathcal{T}_n(u)(t)) \leq M_0(c_0)$ when $\kappa_n = 0$. Therefore, we have established

$$M_0(\mathcal{T}_n(u)(t)) \leq M_0(c_0). \quad (4.13)$$

We next compute the first moment of $\mathcal{T}_n(u)$ for $u \in P_n$ and find, for $t \in [0, t_n]$,

$$\begin{aligned} \int_0^\infty v \mathcal{T}_n(u)(t, v) dv &= e^{-\kappa_n t} \int_0^\infty Y(t, 0, v) c_0^n(v) dv \\ &\quad + \int_0^t \int_0^\infty [Y(t; s, v) (Q_n(u)(s, v) + \kappa_n u(s, v)) e^{-\kappa_n(t-s)}] dv ds. \end{aligned}$$

Owing to (3.6), (4.6), and (4.11), we deduce that

$$e^{-\kappa_n t} \int_0^\infty Y(t, 0, v) c_0^n(v) dv \leq M_1(c_0) e^{(A-\kappa_n)t}$$

and

$$\begin{aligned} &\int_0^\infty [Y(t; s, v) (Q_n(u)(s, v) + \kappa_n u(s, v)) e^{-\kappa_n(t-s)}] dv \\ &\leq \int_0^\infty [v (Q_n(u)(s, v) + \kappa_n u(s, v)) e^{(A-\kappa_n)(t-s)}] dv \\ &= \kappa_n M_1(u(s)) e^{(A-\kappa_n)(t-s)} \leq \kappa_n M_1(c_0) e^{At - \kappa_n(t-s)}. \end{aligned}$$

Consequently,

$$\int_0^\infty v \mathcal{T}_n(u)(t, v) dv \leq M_1(c_0) e^{(A-\kappa_n)t} + \kappa_n M_1(c_0) \int_0^t e^{At-\kappa_n(t-s)} ds = M_1(c_0) e^{At}. \quad (4.14)$$

Therefore, (4.12), (4.13), and (4.14) imply that P_n is an invariant set for the map \mathcal{T}_n .

Now we prove that \mathcal{T}_n is a contraction on P_n with respect to the distance d_{P_n} . Consider $(u_1, u_2) \in P_n^2$ and $t \in [0, t_n]$. From (4.4), we obtain

$$\begin{aligned} & \|(\mathcal{T}_n(u_1) - \mathcal{T}_n(u_2))(t)\|_{L^1(0, \infty)} \\ & \leq \int_0^t [\|(Q_n(u_1) - Q_n(u_2))(s)\|_{L^1(0, \infty)} + \kappa_n \|(u_1 - u_2)(s)\|_{L^1(0, \infty)}] e^{-\kappa_n(t-s)} ds, \\ & \leq \int_0^t [3\beta_n (M_0(u_1(s)) + M_0(u_2(s))) + \kappa_n] ds d_{P_n}(u_1, u_2) \\ & \leq t_n (\kappa_n + 6\beta_n M_0(c_0)) d_{P_n}(u_1, u_2). \end{aligned}$$

Hence, the choice (4.10) of t_n leads us to

$$d_{P_n}(\mathcal{T}_n(u_1), \mathcal{T}_n(u_2)) \leq \frac{1}{2} d_{P_n}(u_1, u_2),$$

which holds for all $(u_1, u_2) \in P_n^2$. Therefore, \mathcal{T}_n is a strict contraction on P_n . Hence, by Banach contraction theorem, there is a unique fixed point $c_n \in P_n$ of \mathcal{T}_n and c_n is the sought-for mild solution to (4.1)–(4.2) on $[0, t_n]$, as already pointed out. Moreover, from the definition of P_n , we have

$$0 \leq M_0(c_n(t)) \leq M_0(c_0), \quad M_1(c_n(t)) \leq M_1(c_0) e^{At}, \quad (4.15)$$

for $0 \leq t \leq t_n$.

Now, since $M_0(c_n(t_n)) \leq M_0(c_0)$ by (4.15), we can repeat the same procedure as above with $c_n(t_n)$ as initial condition and thereby extend c_n uniquely on the time interval $[0, 2t_n]$, showing as well that the extension of c_n to $[t_n, 2t_n]$ also satisfies (4.15) for $[t_n, 2t_n]$. Iterating this argument provides the existence and uniqueness of a global mild solution c_n to (4.1)–(4.2) in the sense of Definition 3.3, which satisfies (4.15) for all $t \geq 0$. In particular, c_n satisfies (4.7) and we are left with checking (4.8).

Let $t \geq 0$ and $\eta_R \in C_c^1((0, \infty))$ with

$$\eta_R(v) = v \quad \text{for } 0 \leq v \leq R, \quad \eta_R(v) = 0 \quad \text{for } v \geq 2R$$

and

$$0 \leq \eta_R(v) \leq 2v, \quad |\eta_R'(v)| \leq 2.$$

Inserting η_R in (4.9), we obtain

$$\begin{aligned} \int_0^\infty c_n(t, v) \eta_R(v) dv &= \int_0^\infty \eta_R(v) c_0^n(v) dv + \int_0^t \int_0^\infty g(s, v) c_n(s, v) \eta_R'(v) dv ds \\ &\quad + \int_0^t \int_0^\infty \int_0^\infty \widetilde{\eta}_R(v_1, v_2) K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_2 dv_1 ds. \end{aligned}$$

Since $c_n(t)$ and c_0^n belong to $L_1^1(0, \infty)$ and

$$\widetilde{\eta}_R(v_1, v_2) = 0, \quad v_1 + v_2 \in (0, R), \quad |\widetilde{\eta}_R(v_1, v_2)| \leq 4(v_1 + v_2), \quad (v_1, v_2) \in (0, \infty)^2,$$

it follows from the boundedness of K_n and (2.2) that we may take the limit $R \rightarrow \infty$ and deduce from Lebesgue's dominated convergence theorem that

$$\int_0^\infty v c_n(t, v) dv = \int_0^\infty v c_0^n(v) dv + \int_0^t \int_0^\infty g(s, v) c_n(s, v) dv ds,$$

hence (4.8). Finally, c_n is also a global weak solution to (4.1)–(4.2) by Proposition 3.4 and the proof of Theorem 4.1 is complete. \square

Next, we investigate the behavior of the mild solution c_n to (4.1)–(4.2) for small sizes in the upcoming Lemma 4.2.

Lemma 4.2. *The mild solution c_n to (4.1)–(4.2) belongs to $L^\infty(0, \infty; L^1_{-2\beta}(0, \infty))$ and satisfies*

$$M_{-2\beta}(c_n(t)) = \int_0^\infty v^{-2\beta} c_n(t, v) dv \leq M_{-2\beta}(c_0), \quad t \geq 0. \quad (4.16)$$

Proof. Let us define

$$\phi_\epsilon(v) := (v + \epsilon)^{-2\beta} \text{ for } (\epsilon, v) \in (0, 1) \times (0, \infty).$$

Then, it is clear that $\phi_\epsilon \in W^{1,\infty}(0, \infty)$. By Theorem 4.1, we know that the mild solution is also a weak solution. Hence, applying ϕ_ϵ in the weak formulation (4.9), we have

$$\begin{aligned} \int_0^\infty c_n(t, v) \phi_\epsilon(v) dv &= \int_0^\infty c_0^n(v) \phi_\epsilon(v) dv + \int_0^t \int_0^\infty \phi'_\epsilon(v) g(s, v) c_n(s, v) dv ds \\ &\quad + \frac{1}{2} \int_0^t \int_0^\infty \tilde{\phi}_\epsilon(v_1, v_2) K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_2 dv_1 ds, \end{aligned} \quad (4.17)$$

where

$$\tilde{\phi}_\epsilon(v_1, v_2) = (v_1 + v_2 + \epsilon)^{-2\beta} - (v_1 + \epsilon)^{-2\beta} - (v_2 + \epsilon)^{-2\beta} \leq 0. \quad (4.18)$$

Let us first compute the first term in the right-hand side of (4.17)

$$\int_0^\infty c_0^n(v) \phi_\epsilon(v) dv = \int_0^\infty (v + \epsilon)^{-2\beta} c_0^n(v) dv \leq \int_0^\infty v^{-2\beta} c_0(v) dv. \quad (4.19)$$

Then, we consider the second term in the right-hand side of (4.17)

$$\begin{aligned} &\int_0^t \int_0^\infty \phi'_\epsilon(v) g(s, v) c_n(s, v) dv ds \\ &= -2\beta \int_0^t \int_0^\infty (v + \epsilon)^{-2\beta-1} g(s, v) c_n(s, v) dv ds. \end{aligned}$$

The non-negativity of g and c_n implies that

$$\int_0^t \int_0^\infty \phi'_\epsilon(v) g(s, v) c_n(s, v) dv ds \leq 0. \quad (4.20)$$

Finally, we evaluate the last term in the right-hand side of (4.17) and infer from (4.18) and the non-negativity of K_n that

$$\frac{1}{2} \int_0^t \int_0^\infty \tilde{\phi}_\epsilon(v_1, v_2) K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_2 dv_1 ds \leq 0. \quad (4.21)$$

Now, using (4.19), (4.20), and (4.21) in (4.17), we have

$$\int_0^\infty (v + \epsilon)^{-2\beta} c_n(t, v) dv \leq \int_0^\infty v^{-2\beta} c_0(v) dv.$$

Taking the limit $\epsilon \rightarrow 0$ and using Fatou's lemma, we obtain the desired result (4.16). \square

The following results are an immediate consequence of Theorem 4.1 and Lemma 4.2.

Proposition 4.3. *Let $n \geq 1$. There is a unique mild solution c_n to (4.1)–(4.2) (which is also a weak solution according to Proposition 3.4) that satisfies, for $t > 0$,*

$$\begin{aligned} \int_0^\infty c_n(t, v) dv &\leq \int_0^\infty c_0(v) dv, \\ \int_0^\infty v c_n(t, v) dv &= \int_0^\infty v c_0^n(v) dv + \int_0^t \int_0^\infty g(s, v) c_n(s, v) dv ds, \\ \int_0^\infty v^{-2\beta} c_n(t, v) dv &\leq \int_0^\infty v^{-2\beta} c_0(v) dv. \end{aligned}$$

Moreover, for all $T > 0$, there is a constant $C(T) > 0$ such that

$$\int_0^\infty v c_n(t, v) dv \leq C(T) \quad \text{for } t \in [0, T].$$

We now focus on estimating superlinear moments, exploiting the condition (2.1) that the growth of K is at most linear for large volumes.

Lemma 4.4. *Let $\Gamma[c_0]$ be the set of non-negative convex functions $j \in C^1([0, \infty))$ such that j' is a concave function on $[0, \infty)$ with $j(0) = 0$, $j'(0) \geq 0$ and*

$$N(j, c_0) := \int_0^\infty j(v) c_0(v) dv < \infty. \quad (4.22)$$

Let c_n be the global mild solution to (4.1)–(4.2) and $T \in (0, \infty)$. Then there is a constant $C(T)$ depending only on A , k , c_0 , j , and T such that

$$\int_0^\infty j(v) c_n(t, v) dv \leq C(T) \quad \text{for } t \in (0, T). \quad (4.23)$$

Proof. We set $C(T)$ as any positive constant which depends only on A , K , j , and T throughout the proof. For $R \geq 1$, set $j_R(v) := \min(j(v), j(R))$, $v \in (0, \infty)$. Clearly, $j_R \in W^{1, \infty}(0, \infty)$ and, from [21, Lemmas A.1–A.2], we have, for $(v_1, v_2) \in (0, \infty)^2$,

$$(v_1 + v_2) (j_R(v_1 + v_2) - j_R(v_1) - j_R(v_2)) \leq 2(v_2 j_R(v_1) + v_1 j_R(v_2)), \quad (4.24)$$

$$0 \leq v_1 j_R'(v_1) \leq 2j_R(v_1). \quad (4.25)$$

Let $t \in [0, T]$. We infer from (4.9) with $\phi = j_R$ that

$$\begin{aligned} \int_0^\infty j_R(v) c_n(t, v) dv &= \int_0^\infty j_R(v) c_0^n(v) dv + \int_0^t \int_0^\infty j_R'(v) g(s, v) c_n(s, v) dv ds \\ &\quad + \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \tilde{j}_R(v_1, v_2) K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_2 dv_1 ds, \end{aligned}$$

where $\tilde{j}_R(v_1v_2) := j_R(v_1 + v_2) - j_R(v_1) - j_R(v_2)$ for $(v_1, v_2) \in (0, \infty)^2$. Using (4.22), we obtain

$$\begin{aligned} \int_0^\infty j_R(v)c_n(t, v)dv &\leq N(j, c_0) + \int_0^t \int_0^\infty j'_R(v)g(s, v)c(s, v)dvd s \\ &\quad + \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \tilde{j}_R(v_1, v_2)K_n(v_1, v_2)c_n(s, v_1)c_n(s, v_2)dv_2dv_1ds. \end{aligned} \quad (4.26)$$

Using (3.1) and (4.25) in the second term of the right-hand side of (4.26), we get

$$\begin{aligned} \int_0^t \int_0^\infty j'_R(v)g(s, v)c_n(s, v)dvd s &\leq A \int_0^t \int_0^\infty j'_R(v)vc_n(s, v)dvd s \\ &\leq 2A \int_0^t \int_0^\infty j_R(v)c_n(s, v)dvd s. \end{aligned} \quad (4.27)$$

Let us now estimate the last term in the right-hand side of (4.26). For that purpose, we divide the above integral in the following manner:

$$\int_0^\infty \int_0^\infty \tilde{j}_R(v_1, v_2)K_n(v_1, v_2)c_n(s, v_1)c_n(s, v_2)dv_1dv_2 = \sum_{i=1}^4 Z_i, \quad (4.28)$$

where

$$\begin{aligned} Z_1 &:= \int_0^1 \int_0^1 \tilde{j}_R(v_1, v_2)K_n(v_1, v_2)c_n(s, v_1)c_n(s, v_2)dv_1dv_2, \\ Z_2 &:= \int_0^1 \int_1^\infty \tilde{j}_R(v_1, v_2)K_n(v_1, v_2)c_n(s, v_1)c_n(s, v_2)dv_1dv_2, \\ Z_3 &:= \int_1^\infty \int_0^1 \tilde{j}_R(v_1, v_2)K_n(v_1, v_2)c_n(s, v_1)c_n(s, v_2)dv_1dv_2 = Z_2, \\ Z_4 &:= \int_1^\infty \int_1^\infty \tilde{j}_R(v_1, v_2)K_n(v_1, v_2)c_n(s, v_1)c_n(s, v_2)dv_1dv_2. \end{aligned}$$

Applying (2.1), (4.24) and using the definition of K_n and Proposition 4.3, the term Z_1 can be evaluated as

$$\begin{aligned} Z_1 &\leq 2 \int_0^1 \int_0^1 \left(\frac{v_2j_R(v_1) + v_1j_R(v_2)}{v_1 + v_2} \right) K_n(v_1, v_2)c_n(s, v_1)c_n(s, v_2)dv_1dv_2 \\ &\leq 2k \int_0^1 \int_0^1 (j_R(v_1) + j_R(v_2))(v_1v_2)^{-\beta}c_n(s, v_1)c_n(s, v_2)dv_1dv_2 \\ &\leq 4kj(1) \int_0^1 \int_0^1 v_1^{-\beta}v_2^{-\beta}c_n(s, v_1)c_n(s, v_2)dv_1dv_2 \\ &\leq 4kj(1)M_{-2\beta}^2(c_0). \end{aligned}$$

Analogously, one can estimate

$$\begin{aligned} Z_2 = Z_3 &\leq 2k \int_0^1 \int_1^\infty \left(\frac{v_1 v_2^{1-\beta}}{v_1 + v_2} j_R(v_1) + \frac{v_1^2 v_2^{-\beta}}{v_1 + v_2} j_R(v_2) \right) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2 \\ &\leq 2k \int_0^1 \int_1^\infty \left(v_2^{1-\beta} j_R(v_1) + v_1 v_2^{-\beta} j(1) \right) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2 \\ &\leq 2k \left[M_{-2\beta}(c_0) \int_0^\infty j_R(v_1) c_n(s, v_1) dv_1 + j(1) M_{-2\beta}(c_0) M_1(c_0) e^{AT} \right], \end{aligned}$$

and

$$\begin{aligned} Z_4 &\leq 2k \int_1^\infty \int_1^\infty (v_1 j_R(v_2) + v_2 j_R(v_1)) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2 \\ &\leq 4k M_1(c_0) e^{AT} \int_0^\infty j_R(v_1) c_n(s, v_1) dv_1 \leq C(T) \int_0^\infty j_R(v_1) c_n(s, v_1) dv_1. \end{aligned}$$

Gathering the estimates on Z_1 , Z_2 , Z_3 and Z_4 and inserting them in (4.28), we obtain

$$\begin{aligned} &\frac{1}{2} \int_0^\infty \int_0^\infty \tilde{j}_R(v_1, v_2) K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2 \\ &\leq C(T) + C(T) \int_0^\infty j_R(v_1) c_n(s, v_1) dv_1. \end{aligned} \tag{4.29}$$

Now, using (4.27) and (4.29) in (4.26), we obtain

$$\int_0^\infty j_R(v) c_n(t, v) dv \leq C(T) + C(T) \int_0^t \int_0^\infty j_R(v) c_n(s, v) dv ds.$$

Hence, by Gronwall's inequality, we have

$$\int_0^\infty j_R(v) c_n(t, v) dv \leq C(T),$$

and (4.23) follows from the above inequality by Fatou's Lemma after letting $R \rightarrow \infty$. \square

Corollary 4.5. *Suppose that c_0 belongs to $L_2^1(0, \infty)$. Then, for every $T > 0$, there exists a positive constant $C_{sm}(T)$ such that*

$$M_2(c_n(t)) \leq C_{sm}(T), \quad t \in [0, T], \quad n \geq 1.$$

Proof. The function $J_2 : v \mapsto v^2$ is convex on $[0, \infty)$ with concave derivative, and satisfies $J_2(0) = J_2'(0) = 0$. Hence, $J_2 \in \Gamma[c_0]$ and Corollary 4.5 straightforwardly follows from Lemma 4.4 with $j = J_2$. \square

We are left with the tail behavior of c_n , which we analyze in the next lemma.

Lemma 4.6. *For $T > 0$,*

$$\limsup_{R \rightarrow \infty} \sup_{n \geq 1} \sup_{t \in [0, T]} \int_R^\infty v c_n(t, v) dv = 0. \tag{4.30}$$

Proof. A refined form of the de la Vallée-Poussin theorem, originally derived in [22, Proposition I.1.1] and also reported in [4, Theorem 7.1.6], establishes the existence of a function $j_0 \in \Gamma[c_0]$ satisfying also

$$\lim_{v \rightarrow \infty} j_0'(v) = \lim_{v \rightarrow \infty} \frac{j_0(v)}{v} = \infty. \quad (4.31)$$

For $R \geq 1$, we have

$$\begin{aligned} & \sup_{n \geq 1} \sup_{t \in [0, T]} \int_R^\infty v c_n(t, v) dv \\ & \leq \sup_{n \geq 1} \sup_{t \in [0, T]} \sup_{v \in [R, \infty)} \left(\frac{v}{j_0(v)} \right) \int_R^\infty j_0(v) c_n(t, v) dv. \end{aligned}$$

Since $j_0 \in \Gamma[c_0]$, we can utilize Lemma 4.4 to obtain

$$\sup_{n \geq 1} \sup_{t \in [0, T]} \int_R^\infty v c_n(t, v) dv \leq C(T) \sup_{v \in [R, \infty)} \left(\frac{v}{j_0(v)} \right).$$

This inequality implies (4.30) by (4.31). \square

We now examine the behavior of c_n on small measurable subsets of $(0, \infty)$. Accordingly, we present two lemmas; detailed proofs can be found in [21, Lemmas 4.3–4.4].

Lemma 4.7. *Let $Z \in \mathcal{M}$ with $|Z| < \infty$ and m be a positive integer. Then there exist 2^m subsets $(Z_l) \in \mathcal{M}$ such that*

$$\begin{aligned} |Z_l| & \leq |Z|, \quad l \in \{1, \dots, 2^m\}, \\ 2^m Z & := \{2^m v, v \in Z\} \subset \bigcup_{l=1}^{2^m} Z_l, \end{aligned}$$

where \mathcal{M} is the set of all measurable subsets of $(0, \infty)$ and $|Z|$ denotes the Lebesgue measure of Z .

Lemma 4.8. *Let $T > 0$ and $Z \in \mathcal{M}$ with $|Z| < \infty$. Then there exists an integer m_0 depending only on A and T such that*

$$|Y(s; t, Z)| \leq 2^{m_0} |Z|, \quad (s, t) \in [0, T]^2.$$

Having completed this preparation, we are poised to articulate the main result of this section.

Proposition 4.9. *Given $T > 0$ and $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that for all $n \geq 1$ and $t \in [0, T]$*

$$\int_Z v^{-\beta} c_n(t, v) dv \leq \epsilon \quad \text{whenever} \quad |Z| \leq \delta_\epsilon.$$

Proof. We take $C_p = 2^{m_0}$ where m_0 is given in Lemma 4.8.

Let $t \in [0, T]$, $n \geq 1$, $\delta \in (0, 1)$ and define

$$\mathcal{E}_{n, \delta}(t) := \sup \left\{ \int_0^\infty \chi_Z(v) v^{-\beta} c_n(t, v) dv, Z \in \mathcal{M} \text{ with } |Z| \leq \delta \right\}.$$

Consider $Z \in \mathcal{M}$ having $|Z| \leq \delta$. According to Lemma 4.7 and Lemma 4.8, for each $t \in [0, T]$ and $s \in [0, t]$, we have C_p subsets $(Z_l^{s, t}) \in \mathcal{M}$ such that $|Z_l^{s, t}| \leq \delta$ and

$$Y(s; t, Z) \subset \bigcup_{l=1}^{C_p} Z_l^{s, t}. \quad (4.32)$$

Next, since c_n is a mild solution to (4.1)–(4.2), we obtain

$$\begin{aligned} \int_Z v^{-\beta} c_n(t, v) dv &= \int_0^t \int_Z v_1^{-\beta} Q_n(c_n)(s, Y(s; t, v_1)) J(s; t, v_1) dv_1 ds \\ &\quad + \int_Z v^{-\beta} c_0(Y(0; t, v)) J(0; t, v) dv, \end{aligned} \quad (4.33)$$

Now let

$$I_1 := \int_0^t \int_Z v_1^{-\beta} Q_n(c_n)(s, Y(s; t, v_1)) J(s; t, v_1) dv_1 ds.$$

Therefore, by (3.4), and the non-negativity of c_n and K_n , we have

$$I_1 \leq \frac{1}{2} \int_0^t \int_{Y(s; t, Z)} Y^{-\beta}(t; s, v_1) \int_0^{v_1} K_n(v_1 - v_2, v_2) c_n(s, v_1 - v_2) c_n(s, v_2) dv_2 dv_1 ds.$$

Owing to (3.8), we have

$$I_1 \leq \frac{1}{2} \int_0^t \int_{Y(s; t, Z)} v_1^{-\beta} \int_0^{v_1} K_n(v_1 - v_2, v_2) c_n(s, v_1 - v_2) c_n(s, v_2) dv_2 dv_1 ds.$$

Hence, by Fubini's theorem,

$$I_1 \leq \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty (v_2 + v_1)^{-\beta} \chi_{Y(s; t, Z)}(v_2 + v_1) K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2 ds = \sum_{i=1}^4 W_i,$$

where

$$W_1 := \int_0^t \int_0^1 \int_0^1 (v_1 + v_2)^{-\beta} \chi_{Y(s; t, Z)}(v_1 + v_2) K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_2 dv_1 ds,$$

$$W_2 := \int_0^t \int_0^1 \int_1^\infty (v_1 + v_2)^{-\beta} \chi_{Y(s; t, Z)}(v_1 + v_2) K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_2 dv_1 ds,$$

$$W_3 := \int_0^t \int_1^\infty \int_0^1 (v_1 + v_2)^{-\beta} \chi_{Y(s; t, Z)}(v_1 + v_2) K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_2 dv_1 ds = W_2,$$

and

$$W_4 := \int_0^t \int_1^\infty \int_1^\infty (v_1 + v_2)^{-\beta} \chi_{Y(s; t, Z)}(v_1 + v_2) K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_2 dv_1 ds.$$

By the definition of K_n and (2.1),

$$\begin{aligned} W_1 &\leq k \int_0^t \int_0^1 \int_0^1 v_1^{-\beta} \chi_{Y(s; t, Z)}(v_1 + v_2) v_1^{-\beta} v_2^{-\beta} c_n(s, v_1) c_n(s, v_2) dv_2 dv_1 ds \\ &\leq k \int_0^t \int_0^1 v_1^{-2\beta} \int_0^1 \chi_{-v_1+Y(s; t, Z)}(v_2) v_2^{-\beta} c_n(s, v_1) c_n(s, v_2) dv_2 dv_1 ds. \end{aligned}$$

Next, Proposition 4.3, (4.32), and the translation invariance of Lebesgue measure give

$$\begin{aligned} W_1 &\leq k \sum_{l=1}^{C_p} \int_0^t \int_0^1 v_1^{-2\beta} c_n(s, v_1) \left(\int_0^1 \chi_{-v_1+Z_l^{s,t}}(v_2) v_2^{-\beta} c_n(s, v_2) dv_2 \right) dv_1 ds \\ &\leq k \sum_{l=1}^{C_p} \int_0^t \mathcal{E}_{n,\delta}(s) \int_0^1 v_1^{-2\beta} c_n(s, v_1) dv_1 ds \leq C(T) \int_0^t \mathcal{E}_{n,\delta}(s) ds. \end{aligned}$$

Similarly,

$$\begin{aligned} W_2 = W_3 &\leq k \int_0^t \int_0^1 \int_1^\infty \chi_{Y(s;t,Z)}(v_1 + v_2) v_1^{-\beta} v_2 c_n(s, v_1) c_n(s, v_2) dv_2 dv_1 ds \\ &\leq k \int_0^t \int_1^\infty v_2 c_n(s, v_2) \int_0^1 v_1^{-\beta} \chi_{-v_2+Y(s;t,Z)}(v_1) c_n(s, v_1) dv_1 dv_2 ds \\ &\leq C(T) \int_0^t \mathcal{E}_{n,\delta}(s) ds, \end{aligned}$$

and

$$\begin{aligned} W_4 &\leq k \int_0^t \int_1^\infty \int_1^\infty (v_1 + v_2)^{1-\beta} \chi_{Y(s;t,Z)}(v_1 + v_2) c_n(s, v_1) c_n(s, v_2) dv_2 dv_1 ds \\ &\leq k \int_0^t \int_1^\infty \int_1^\infty (v_1 v_2^{-\beta} + v_1^{-\beta} v_2) \chi_{Y(s;t,Z)}(v_1 + v_2) c_n(s, v_1) c_n(s, v_2) dv_2 dv_1 ds \\ &\leq 2k \int_0^t \int_1^\infty v_1 c_n(s, v_1) \int_1^\infty v_2^{-\beta} \chi_{-v_1+Y(s;t,Z)}(v_2) c_n(s, v_1) dv_2 dv_1 ds \\ &\leq C(T) \int_0^t \mathcal{E}_{n,\delta}(s) ds. \end{aligned}$$

From the estimates of W_1 , W_2 , W_3 , and W_4 , we infer that

$$I_1 \leq C(T) \int_0^t \mathcal{E}_{n,\delta}(s) ds. \quad (4.34)$$

Next, by (3.4) and (3.8),

$$\begin{aligned} \int_Z v^{-\beta} c_0(Y(0;t,v)) J(0;t,v) dv &= \int_{Y(0;t,Z)} Y^{-\beta}(t;0,v) c_0(v) dv \\ &\leq \int_{Y(0;t,Z)} v^{-\beta} c_0(v) dv, \end{aligned}$$

and it follows from (4.32) that

$$\int_Z v^{-\beta} c_0(Y(0;t,v)) J(0;t,v) dv \leq \sum_{l=1}^{C_p} \int_{Z_l^{0,t}} v^{-\beta} c_0(v) dv \leq C_p \mathcal{E}_\delta(0), \quad (4.35)$$

where

$$\mathcal{E}_\delta(0) := \sup \left\{ \int_0^\infty \chi_Z(v) v^{-\beta} c_0(v) dv, Z \in \mathcal{M} \text{ with } |Z| \leq \delta \right\}.$$

Using (4.34) and (4.35) in (4.33), we get

$$\int_Z v^{-\beta} c^n(t, v) dv \leq C_p \mathcal{E}_\delta(0) + C(T) \int_0^t \mathcal{E}_{n,\delta}(s) ds.$$

Taking the supremum on both sides over the set $Z \in \mathcal{M}$ with $|Z| \leq \delta$, we have

$$\mathcal{E}_{n,\delta}(t) \leq C_p \mathcal{E}_\delta(0) + C(T) \int_0^t \mathcal{E}_{n,\delta}(s) ds, \quad t \in [0, T].$$

By Gronwall's inequality, we get

$$\mathcal{E}_{n,\delta}(t) \leq C(T) \mathcal{E}_\delta(0), \quad t \in [0, T]. \quad (4.36)$$

Now, by the absolute continuity of the Lebesgue integral, for a given $\epsilon > 0$, we can find $\delta_\epsilon > 0$ such that

$$\mathcal{E}_{\delta_\epsilon}(0) \leq \frac{\epsilon}{C(T)}. \quad (4.37)$$

From (4.36) and (4.37), we get

$$\mathcal{E}_{n,\delta_\epsilon}(t) \leq \epsilon, \quad t \in [0, T],$$

for all $n \geq 1$, and thus complete the proof of Proposition 4.9. \square

From Proposition 4.9, Lemma 4.6, and the Dunford-Pettis theorem, we infer that

$$\left\{ \begin{array}{l} (c_n(t))_{n \geq 1} \text{ is weakly compact in} \\ L^1(0, \infty) \text{ for each } t \in [0, T]. \end{array} \right. \quad (4.38)$$

We now proceed to prove that $(c_n)_{n \geq 1}$ is weakly equicontinuous with respect to t in $L^1(0, \infty)$.

Lemma 4.10. *For every $T > 0$ and $\phi \in L^\infty(0, \infty)$, the following result is true.*

$$\lim_{h \rightarrow 0} \sup_{t \in [0, T-h]} \sup_{n \geq 1} \left| \int_0^\infty (c_n(t+h, v) - c_n(t, v)) \phi(v) dv \right| = 0. \quad (4.39)$$

Proof. Let $T > 0$, $\phi \in C_c^1(0, \infty)$, $n \geq 1$, $h \in (0, T)$, and $t \in (0, T-h)$. By (4.9), we have

$$\begin{aligned} & \left| \int_0^\infty (c_n(t+h, v) - c_n(t, v)) \phi(v) dv \right| \\ & \leq \|\phi'\|_{L^\infty(0, \infty)} \int_t^{t+h} \int_0^\infty g(t, v) c_n(s, v) dv ds \\ & \quad + \frac{1}{2} \int_t^{t+h} \int_0^\infty \int_0^\infty |\tilde{\phi}(v_1, v_2)| K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2 ds, \end{aligned} \quad (4.40)$$

with $\tilde{\phi}(v_1, v_2) = \phi(v_1 + v_2) - \phi(v_1) - \phi(v_2)$.

First, by (3.1) and Proposition 4.3, we get

$$\int_t^{t+h} \int_0^\infty g(s, v) c_n(s, v) dv ds \leq C(T)h. \quad (4.41)$$

Let us now estimate the second term on the right-hand side of (4.40):

$$\begin{aligned} & \frac{1}{2} \int_t^{t+h} \int_0^\infty \int_0^\infty \left| \tilde{\phi}(v_1, v_2) \right| K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2 ds \\ & \leq \frac{3}{2} \|\phi\|_{L^\infty(0, \infty)} \int_t^{t+h} \int_0^\infty \int_0^\infty K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2 ds. \end{aligned} \quad (4.42)$$

In order to further estimate the inequality (4.42), let us consider

$$\int_0^\infty \int_0^\infty K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2 ds = \sum_{i=1}^4 \widehat{W}_i,$$

where

$$\begin{aligned} \widehat{W}_1 &:= \int_0^1 \int_0^1 K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2, \\ \widehat{W}_2 &:= \int_0^1 \int_1^\infty K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2, \\ \widehat{W}_3 &:= \int_1^\infty \int_0^1 K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2 = \widehat{W}_2, \end{aligned}$$

and

$$\widehat{W}_4 := \int_1^\infty \int_1^\infty K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2.$$

With the help of definition of K_n , (2.1) and Proposition 4.3, \widehat{W}_1 can be estimated as

$$\begin{aligned} \widehat{W}_1 &\leq k \int_0^1 \int_0^1 v_1^{-\beta} v_2^{-\beta} c_n(s, v_1) c_n(s, v_2) dv_1 dv_2 \\ &\leq k \left(\int_0^\infty v_1^{-2\beta} c_n(s, v_1) dv_1 \right) \left(\int_0^\infty v_2^{-2\beta} c_n(s, v_2) dv_2 \right) \leq k M_{-2\beta}^2(c_0). \end{aligned}$$

Similarly

$$\widehat{W}_2 = \widehat{W}_3 \leq k \left(\int_0^\infty v_2^{-2\beta} c_n(s, v_2) dv_2 \right) \left(\int_0^\infty v_1 c_n(s, v_1) dv_1 \right) \leq k e^{AT} M_{-2\beta}(c_0) M_1(c_0),$$

and

$$\begin{aligned} \widehat{W}_4 &= \int_1^\infty \int_1^\infty K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2 \\ &\leq 2k \left(\int_0^\infty v_1 c_n(s, v_1) dv_1 \right) \left(\int_0^\infty c_n(s, v_2) dv_2 \right) \leq 2k e^{AT} M_1(c_0) M_0(c_0). \end{aligned}$$

From the estimates on $\{\widehat{W}_i\}_{1 \leq i \leq 4}$, we infer that

$$\int_0^\infty \int_0^\infty K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2 \leq C(T), \quad s \in [0, T]. \quad (4.43)$$

Inserting (4.43) into (4.42), we obtain

$$\frac{1}{2} \int_t^{t+h} \int_0^\infty \int_0^\infty \left| \tilde{\phi}(v_1, v_2) \right| K_n(v_1, v_2) c_n(s, v_1) c_n(s, v_2) dv_1 dv_2 ds \leq C(T) \|\phi\|_{L^\infty(0, \infty)} h. \quad (4.44)$$

Gathering (4.40), (4.41) and (4.44), we end up with

$$\left| \int_0^\infty (c_n(t+h, v) - c_n(t, v)) \phi(v) dv \right| \leq C(T) \|\phi\|_{W^{1, \infty}(0, \infty)} h$$

for all $h \in (0, T)$, $t \in [0, T-h]$, and $n \geq 1$. Therefore, (4.39) is true for every $\phi \in C_c^1(0, \infty)$. We next use a density argument as in the proof of [21, Lemma 4.5] to extend the validity of (4.39) to arbitrary functions in $L^\infty(0, \infty)$. \square

Thus, from Lemma 4.10, we infer that

$$\begin{cases} (c_n)_{n \geq 1} \text{ is weakly equicontinuous in } L^1(0, \infty) \\ \text{for each } t \in [0, T], \end{cases} \quad (4.45)$$

see [31, Definition 1.3.1]. Due to (4.38) and (4.45), we may apply a variant of the Arzelá-Ascoli theorem [31, Theorem 1.3.2] to deduce that the sequence (c_n) is relatively compact in $C([0, T]; w - L^1(0, \infty))$ for any $T > 0$. Therefore, by a diagonal argument, we conclude that there exists a subsequence (c_n) (not relabeled) which converges to some limit function c in $C([0, \infty); w - L^1(0, \infty))$ in the following sense:

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^\infty (c_n(t, v) - c(t, v)) \phi(v) dv \right| = 0 \quad (4.46)$$

for each $T \in (0, \infty)$ and $\phi \in L^\infty(0, \infty)$.

We shall improve this convergence in the next lemma but first combine Proposition 4.3 and the convergence (4.46) to derive additional properties on c . First, c is clearly non-negative due to (4.46) and the non-negativity of c_n for all $n \geq 1$. Next, for $m \in [-2\beta, 1]$, we infer from Proposition 4.3 and (4.46) that, for $T > 0$, $t \in [0, T]$, and $R > 1$,

$$\int_{\frac{1}{R}}^R v^m c(t, v) dv = \lim_{n \rightarrow \infty} \int_{\frac{1}{R}}^R v^m c_n(t, v) dv \leq C(T).$$

We let $R \rightarrow \infty$ and use Fatou's lemma to conclude that

$$\sup_{t \in [0, T]} M_m(c(t)) \leq C(T), \quad m \in [-2\beta, 1]. \quad (4.47)$$

Similarly, we infer from Lemma 4.6 and (4.46) that

$$\lim_{R \rightarrow \infty} \sup_{t \in [0, T]} \int_R^\infty v c(t, v) dv = 0. \quad (4.48)$$

Lemma 4.11. *For $-2\beta < m \leq 1$ and $T > 0$, $(c_n)_{n \geq 1}$ converges to c in $C([0, T]; w - L_m^1(0, \infty))$.*

Proof. Let $T > 0$, $t \in [0, T]$, and $\phi \in L^\infty(0, \infty)$. For $R \geq 1$, let us consider the following term

$$\begin{aligned} & \left| \int_0^\infty v^m \phi(v) (c_n(t, v) - c(t, v)) dv \right| \\ & \leq \int_0^{\frac{1}{R}} v^{m+2\beta} v^{-2\beta} |\phi(v)| (c_n(t, v) + c(t, v)) dv \\ & \quad + \left| \int_{\frac{1}{R}}^R v^m \phi(v) (c_n(t, v) - c(t, v)) dv \right| \\ & \quad + \int_R^\infty v |\phi(v)| (c_n(t, v) + c(t, v)) dv. \end{aligned}$$

By Proposition 4.3 and (4.47), we have

$$\begin{aligned} & \left| \int_0^\infty v^m \phi(v) (c_n(t, v) - c(t, v)) dv \right| \\ & \leq \left| \int_{\frac{1}{R}}^R v^m \phi(v) (c_n(t, v) - c(t, v)) dv \right| + C(T) \left(\frac{1}{R} \right)^{m+2\beta} \|\phi\|_{L^\infty(0, \infty)} \\ & \quad + \left[\int_R^\infty v c(t, v) dv + \sup_{n \geq 1} \int_R^\infty v c_n(t, v) dv \right] \|\phi\|_{L^\infty(0, \infty)}. \end{aligned}$$

Taking the supremum with respect to t on both sides over $[0, T]$, we get

$$\begin{aligned} & \sup_{t \in [0, T]} \left| \int_0^\infty v^m \phi(v) (c_n(t, v) - c(t, v)) dv \right| \\ & \leq \sup_{t \in [0, T]} \left| \int_{\frac{1}{R}}^R v^m \phi(v) (c_n(t, v) - c(t, v)) dv \right| + C(T) \left(\frac{1}{R} \right)^{m+2\beta} \|\phi\|_{L^\infty(0, \infty)} \\ & \quad + \left[\sup_{t \in [0, T]} \int_R^\infty v c(t, v) dv + \sup_{n \geq 1} \sup_{t \in [0, T]} \int_R^\infty v c_n(t, v) dv \right] \|\phi\|_{L^\infty(0, \infty)}. \end{aligned}$$

Now taking the limit $n \rightarrow \infty$ and applying (4.46) yield

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^\infty v^m \phi(v) (c_n(t, v) - c(t, v)) dv \right| \\ & \leq C(T) \left(\frac{1}{R} \right)^{m+2\beta} \|\phi\|_{L^\infty(0, \infty)} \\ & \quad + \left[\sup_{t \in [0, T]} \int_R^\infty v c(t, v) dv + \sup_{n \geq 1} \sup_{t \in [0, T]} \int_R^\infty v c_n(t, v) dv \right] \|\phi\|_{L^\infty(0, \infty)}. \end{aligned}$$

Next, letting $R \rightarrow \infty$ and using Lemma 4.6 and (4.48), we obtain

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^\infty v^m \phi(v) (c_n(t, v) - c(t, v)) dv \right| = 0,$$

and thus complete the proof. \square

At this stage, we are prepared to conclude the proof of Theorem 2.3.

Proof of Theorem 2.3. Let $T > 0$. We have already shown in (4.47) that $c \in L^\infty(0, T; L_m^1(0, \infty))$ for all $m \in [-2\beta, 1]$, while Lemma 4.11 ensures that $c \in C([0, T]; L_m^1(0, \infty))$ for all $m \in (-2\beta, 1]$. Finally, the assumptions (2.1) and (2.2), along with the convergences stated in Lemma 4.11, allows us to use classical arguments, see [4, 29] for instance, to pass to the limit as $n \rightarrow \infty$ in (4.9) and obtain that c satisfies (2.5). \square

Corollary 4.12. *If all the assumptions made in Theorem 2.4 are satisfied, then there exists at least one global weak solution c to (1.3)–(1.4) in the sense of Definition 2.2 such that (2.6) holds true.*

Proof. Existence of at least one weak solution c satisfying (2.6) follows directly from Theorem 2.3 and Corollary 4.5. \square

5. UNIQUENESS

This section is devoted to the proof of Theorem 2.4. Since the existence of at least one weak solution to (1.3)–(1.4) as stated in Theorem 2.4 follows from Corollary 4.12, we are left with the uniqueness issue. In the subsequent study, we use *solution to (1.3)–(1.4)* to mean the *solution in the sense of Theorem 2.4*. We now focus on the key element of the uniqueness result, as presented in Theorem 2.4. The uniqueness result of Theorem 2.4 is a consequence of the following continuous dependence result.

Proposition 5.1. *Assume that K and g satisfy (2.1) and (2.2). Let $c_{1,0}$ and $c_{2,0}$ be two non-negative functions in $L^1_{-2\beta,2}(0, \infty)$. If $T > 0$ and c_1 and c_2 are two weak solutions to (1.3)–(1.4) on $[0, T]$ with respective initial conditions $c_{1,0}$ and $c_{2,0}$ such that*

$$c_i \in L^\infty(0, T; L^1_{-2\beta,2}(0, \infty)), \quad i = 1, 2, \quad (5.1)$$

then there exists $C(T) > 0$ depending only on k , A , and the $L^\infty(0, T; L^1_{-2\beta,2}(0, \infty))$ -norms of c_1 and c_2 such that

$$\|c_1(t) - c_2(t)\|_{L^1_{-\beta,1}(0, \infty)} \leq C(T) \|c_{1,0} - c_{2,0}\|_{L^1_{-\beta,1}(0, \infty)}, \quad t \in [0, T].$$

We split the proof of Proposition 5.1 into several steps and first establish a differential inequality for a weight L^1 -norm of $c_1 - c_2$ in Lemma 5.2. More precisely, introducing

$$\begin{aligned} E &:= c_1 - c_2, & E_0 &:= E(0, \cdot) = c_{1,0} - c_{2,0}, \\ S &:= Q(c_1) - Q(c_2) & \text{and } \nu(t) &:= \int_0^\infty w(v) |E(t, v)| dv, \end{aligned}$$

for $(t, v) \in [0, T] \times (0, v)$, along with the weight

$$w(v) := v^{-\beta} + v, \quad v \in (0, \infty),$$

we shall establish the following inequality.

Lemma 5.2. *There is $L_0 > 0$ depending only on A such that*

$$\nu(t) \leq \nu(0) + L_0 \int_0^t \nu(s) ds + \int_0^t \int_0^\infty w(v) S(s, v) \text{sign}(E(s, v)) dv ds, \quad t \in [0, T]. \quad (5.2)$$

Observe that the integrability properties of c_1 and c_2 ensure that $\nu(t)$ is well-defined for all $t \in [0, T]$, while the finiteness of the last term on the right-hand side of (5.2) is provided by the next result.

Lemma 5.3. $E \in L^\infty(0, T; L^1_{-2\beta, 2}(0, \infty))$ and $S \in L^\infty(0, T; L^1_{-\beta, 1}(0, \infty))$.

Proof. The stated integrability properties of E readily follow from (5.1).

Next, let $i \in \{1, 2\}$ and $t \in [0, T]$. It follows from (2.1) and Fubini's theorem that

$$\begin{aligned}
\int_0^\infty w(v)|Q(c_i)(t, v)|dv &\leq \frac{1}{2} \int_0^\infty \int_0^\infty w(v_1 + v_2)K(v_1, v_2)c_i(t, v_1)c_i(t, v_2)dv_2dv_1 \\
&\quad + \int_0^\infty \int_0^\infty w(v_1)K(v_1, v_2)c_i(t, v_1)c_i(t, v_2)dv_2dv_1 \\
&\leq 2 \int_0^\infty \int_0^\infty [w(v_1) + w(v_2)]K(v_1, v_2)c_i(t, v_1)c_i(t, v_2)dv_2dv_1 \\
&\leq 4k \int_0^1 \int_0^1 (v_1^{-\beta} + v_2^{-\beta}) (v_1v_2)^{-\beta} c_i(t, v_1)c_i(t, v_2)dv_2dv_1 \\
&\quad + 8k \int_0^1 \int_1^\infty (v_1^{-\beta} + v_2) v_1^{-\beta} v_2 c_i(t, v_1)c_i(t, v_2)dv_2dv_1 \\
&\quad + 4k \int_1^\infty \int_1^\infty (v_1 + v_2)^2 c_i(t, v_1)c_i(t, v_2)dv_2dv_1 \\
&\leq 8kM_{-2\beta}(c_i(t))M_{-\beta}(c_i(t)) + 8kM_{-2\beta}(c_i(t))M_1(c_i(t)) \\
&\quad + 8kM_{-\beta}(c_i(t))M_2(c_i(t)) + 16kM_2(c_i(t))M_0(c_i(t)),
\end{aligned}$$

and we infer from the integrability properties (5.1) of c_i that the right-hand side of the above inequality belongs to $L^\infty(0, T)$. Therefore, $Q(c_i)$ belongs to $L^\infty(0, T; L^1_{-\beta, 1}(0, \infty))$ for $i \in \{1, 2\}$, and so does $S = Q(c_1) - Q(c_2)$. \square

Formally, Lemma 5.2 follows from the multiplication of the equation solved by E (derived from (1.3) for c_1 and c_2) by $w \operatorname{sign}(E)$ and integration with respect to both volume and time. However, since $c_1 - c_2$ need not be differentiable, a suitable regularization in the spirit of DiPerna & Lions theory for transport equations [13] is used, see also [26, Appendix 6.1 & 6.2]. Specifically, we first extend E , S , and g to $[0, T] \times \mathbb{R}$ by zero and define their extensions \tilde{E} , \tilde{S} , and \tilde{g} as follows: for $t \in [0, T]$,

$$\begin{aligned}
\tilde{E}(t, v) &:= \begin{cases} E(t, v) & \text{for } v \in (0, \infty), \\ 0 & \text{for } v \in (-\infty, 0), \end{cases} & \tilde{E}_0(v) &:= \begin{cases} E(0, v) & \text{for } v \in (0, \infty), \\ 0 & \text{for } v \in (-\infty, 0), \end{cases} \\
\tilde{S}(t, v) &:= \begin{cases} S(t, v) & \text{for } v \in (0, \infty), \\ 0 & \text{for } v \in (-\infty, 0), \end{cases} & \tilde{g}(t, v) &:= \begin{cases} g(t, v) & \text{for } v \in (0, \infty), \\ 0 & \text{for } v \in (-\infty, 0). \end{cases}
\end{aligned}$$

It readily follows from (2.2) that \tilde{g} is weakly differentiable with respect to v on \mathbb{R} and, for $t \in [0, T]$, $\partial_v \tilde{v}(t)$ is given by

$$\partial_v \tilde{g}(t, v) = \begin{cases} 0 & \text{for } v \in (-\infty, 0), \\ \partial_v g(t, v) & \text{for } v \in (0, \infty). \end{cases}$$

In particular, $\tilde{g}(t, \cdot) \in W^{1,\infty}(I)$ for any bounded interval $I \subset \mathbb{R}$ for every $t \in [0, T]$. This implies that $g(t, \cdot)$ is absolutely continuous on I for every $t \in [0, T]$ and satisfies

$$\tilde{g}(t, v_2) - \tilde{g}(t, v_1) = \int_{v_1}^{v_2} \partial_v \tilde{g}(t, v) dv, \quad (v_1, v_2) \in \mathbb{R}^2.$$

Moreover, from (2.2), we obtain

$$|\partial_v \tilde{g}(t, v)| \leq A \quad \text{and} \quad |\tilde{g}(t, v)| \leq A|v|, \quad (t, v) \in [0, T] \times \mathbb{R}. \quad (5.3)$$

Next, given a regularizing sequence of mollifiers $\rho_\delta(v) := \frac{1}{\delta} \rho(\frac{v}{\delta})$, $v \in \mathbb{R}$, with $\delta \in (0, 1)$, $\rho \in \mathcal{D}(\mathbb{R})$, $\text{supp}(\rho) \subset (-1, 1)$, $\rho \geq 0$ and $\|\rho\|_{L^1} = 1$, we put

$$\tilde{E}^\delta := \tilde{E} \star \rho_\delta, \quad \tilde{E}_0^\delta := \tilde{E}_0 \star \rho_\delta \quad \text{and} \quad \tilde{S}^\delta := \tilde{S} \star \rho_\delta.$$

For further use, we report the following properties of the convolution in weighted L^1 -spaces.

Lemma 5.4. *Let $m \geq 0$ and $\varphi \in L^1(\mathbb{R}, (1 + |v|^m)dv)$. For every $\delta \in (0, 1)$, the C^∞ -smooth function $\rho_\delta \star \varphi$ belongs to $L^1(\mathbb{R}, (1 + |v|^m)dv)$ with*

$$\int_{\mathbb{R}} |(\rho_\delta \star \varphi)(v)|(1 + |v|^m)dv \leq 2^{m+1} \int_{\mathbb{R}} |\varphi(v)|(1 + |v|^m)dv.$$

Moreover,

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}} |(\rho_\delta \star \varphi - \varphi)(v)|(1 + |v|^m)dv = 0.$$

Lemma 5.4 is a classical property of the convolution in \mathbb{R} for $m = 0$ and the proof for $m > 0$ relies on the splitting of the contribution of the interval $(-R, R)$ on which the classical result for $m = 0$ can be applied and a control of the tails on $(-\infty, R) \cup (R, \infty)$ which is due to the integrability properties of φ and the elementary inequality $(|v_1| + |v_2|)^m \leq 2^m(|v_1|^m + |v_2|^m)$ for $(v_1, v_2) \in \mathbb{R}^2$. We omit the proof.

We next report an immediate consequence of Lemma 5.3 and Lemma 5.4.

Corollary 5.5. *For each $\delta \in (0, 1)$, $\tilde{E}^\delta \in L^\infty(0, T; L^1(\mathbb{R}, (1 + v^2)dv))$ and $\tilde{S}^\delta \in L^\infty(0, T; L^1(\mathbb{R}, (1 + |v|)dv))$.*

Next, according to the definition of \tilde{E} , \tilde{S} , and \tilde{g} , it readily follows from Definition 2.2 that, for $t \in [0, T]$ and $\phi \in W^{1,\infty}(\mathbb{R})$,

$$\begin{aligned} \int_{\mathbb{R}} \tilde{E}(t, v_1) \phi(v_1) dv_1 &= \int_{\mathbb{R}} \tilde{E}_0(v_1) \phi(v_1) dv_1 + \int_0^t \int_{\mathbb{R}} \phi'(v_1) \tilde{g}(s, v_1) \tilde{E}(s, v_1) dv_1 ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \phi(v_1) \tilde{S}(s, v_1) \phi(v_1) dv_1 ds. \end{aligned}$$

For $(t, v) \in [0, T] \times \mathbb{R}$, we take $\phi(v_1) = \rho_\delta(v - v_1)$ in the above identity to deduce that \tilde{E}^δ is a smooth solution to

$$\partial_t \tilde{E}^\delta + \partial_v \left(\rho_\delta \star (\tilde{g} \tilde{E}) \right) = \tilde{S}^\delta, \quad (t, v) \in (0, T) \times \mathbb{R}.$$

Equivalently,

$$\partial_t \tilde{E}^\delta + \partial_v \left(\tilde{g} \tilde{E}^\delta \right) = D^\delta + \tilde{S}^\delta, \quad (t, v) \in (0, T) \times \mathbb{R}, \quad (5.4)$$

where

$$D^\delta := \partial_v \left(\tilde{g} \tilde{E}^\delta - \rho_\delta \star (\tilde{g} \tilde{E}) \right).$$

Next, for $\epsilon \in (0, 1)$, we define

$$\Sigma_\epsilon(z) := \frac{z^2}{\sqrt{z^2 + \epsilon}}, \quad z \in \mathbb{R}.$$

Clearly, for every $\epsilon \in (0, 1)$, Σ_ϵ is a continuously differentiable function satisfying the following properties:

$$|\Sigma_\epsilon(z) - |z|| \leq \min\{\sqrt{\epsilon}, |z|\}, \quad |\Sigma'_\epsilon(z) - \text{sign}(z)| \leq \frac{2\epsilon}{z^2 + \epsilon} \leq 2 \quad \text{and} \quad |\Sigma'_\epsilon(z)| \leq 2, \quad z \in \mathbb{R}. \quad (5.5)$$

$$|z \Sigma'_\epsilon(z) - \Sigma_\epsilon(z)| \leq \min\{\sqrt{\epsilon}, |z|\}, \quad z \in \mathbb{R}. \quad (5.6)$$

Using the chain rule, we obtain the following equation from (5.4)

$$\partial_t \Sigma_\epsilon(\tilde{E}^\delta) + \partial_v \left(\tilde{g} \Sigma_\epsilon(\tilde{E}^\delta) \right) = \Sigma'_\epsilon(\tilde{E}^\delta) D^\delta + \Sigma'_\epsilon(\tilde{E}^\delta) \tilde{S}^\delta - \partial_v \tilde{g} \left[\tilde{E}^\delta \Sigma'_\epsilon(\tilde{E}^\delta) - \Sigma_\epsilon(\tilde{E}^\delta) \right] \quad (5.7)$$

for $(t, v) \in (0, T) \times \mathbb{R}$. Now, setting $w_\epsilon(v) := \sqrt{v^2 + \epsilon} + (v^2 + \epsilon)^{-\beta/2}$ for $v \in \mathbb{R}$, we note that

$$0 \leq w_\epsilon(v) \leq |v| + \sqrt{\epsilon} + |v|^{-\beta} \leq (1 + \sqrt{\epsilon})w(|v|), \quad v \in \mathbb{R} \setminus \{0\}, \quad (5.8)$$

and

$$\begin{aligned} |w_\epsilon(v_1) - w_\epsilon(v_2)| &\leq \left| \int_{v_1}^{v_2} \left[\frac{v}{\sqrt{v^2 + \epsilon}} - \beta \frac{v}{(v^2 + \epsilon)^{(\beta+2)/2}} \right] dv \right| \\ &\leq (1 + \beta \epsilon^{-(\beta+1)/2}) |v_1 - v_2| \end{aligned} \quad (5.9)$$

for $(v_1, v_2) \in \mathbb{R}^2$. In addition,

$$w_\epsilon(v) \leq 2\epsilon^{-\beta/2}(1 + |v|), \quad v \in \mathbb{R}. \quad (5.10)$$

Before going on, let us check that all terms in (5.7) belong to the appropriate space.

Lemma 5.6. *There is $C(T) > 0$ depending on K, g, c_1 and c_2 such that, for $t \in [0, T]$,*

$$\int_{\mathbb{R}} (1 + |v|) \left| \partial_v \left(\tilde{g}(t, v) \Sigma_\epsilon(\tilde{E}^\delta(t, v)) \right) \right| dv \leq \frac{C(T)}{\delta}, \quad (5.11a)$$

$$\int_{\mathbb{R}} (1 + |v|) \left| \Sigma'_\epsilon(\tilde{E}^\delta(t, v)) D^\delta(t, v) \right| dv \leq \frac{C(T)}{\delta}, \quad (5.11b)$$

$$\int_{\mathbb{R}} (1 + |v|) \left| \Sigma'_\epsilon(\tilde{E}^\delta(t, v)) \tilde{S}^\delta(t, v) \right| dv \leq C(T), \quad (5.11c)$$

$$\int_{\mathbb{R}} (1 + |v|) \left| \partial_v \tilde{g}(t, v) \left[\tilde{E}^\delta \Sigma'_\epsilon(\tilde{E}^\delta) - \Sigma_\epsilon(\tilde{E}^\delta) \right] (t, v) \right| dv \leq C(T), \quad (5.11d)$$

$$\int_{\mathbb{R}} (1 + |v|) \left| \partial_t \Sigma_\epsilon(\tilde{E}^\delta(t, v)) \right| dv \leq \frac{C(T)}{\delta}. \quad (5.11e)$$

Proof. We begin with the proof of (5.11a) and (5.11b). Since

$$\partial_v \left(\tilde{g} \Sigma_\epsilon(\tilde{E}^\delta) \right) = \Sigma_\epsilon(\tilde{E}^\delta) \partial_v \tilde{g} + \tilde{g} \Sigma'_\epsilon(\tilde{E}^\delta) (\rho'_\delta \star \tilde{E})$$

and

$$D^\delta = \tilde{g}(\rho'_\delta \star \tilde{E}) + \tilde{E}^\delta \partial_v \tilde{g} - \rho'_\delta \star (\tilde{g} \tilde{E}), \quad (5.12)$$

it suffices to study the integrability of each term separately. First, owing to (5.3), (5.5), and Lemma 5.4 (with $m = 1$), we obtain

$$\begin{aligned} \int_{\mathbb{R}} (1 + |v|) \left| \Sigma_\epsilon(\tilde{E}^\delta(v)) \partial_v \tilde{g}(v) \right| dv &\leq 2 \int_{\mathbb{R}} (1 + |v|) \left| \tilde{E}^\delta(v) \partial_v \tilde{g}(v) \right| dv \leq 2A \int_{\mathbb{R}} (1 + |v|) \left| \tilde{E}^\delta(v) \right| dv \\ &\leq 8A \int_{\mathbb{R}} (1 + |v|) \left| \tilde{E}(v) \right| dv = 8A \int_0^\infty (1 + v) |E(v)| dv. \end{aligned}$$

We next infer from (5.3), (5.5), and Fubini's theorem that

$$\begin{aligned} \int_{\mathbb{R}} (1 + |v|) \left| \tilde{g}(v) \Sigma'_\epsilon(\tilde{E}^\delta(v)) (\rho'_\delta \star \tilde{E})(v) \right| dv &\leq 2 \int_{\mathbb{R}} (1 + |v|) \left| \tilde{g}(v) (\rho'_\delta \star \tilde{E})(v) \right| dv \\ &\leq 2A \int_{\mathbb{R}} \int_{\mathbb{R}} |v| (1 + |v|) |\rho'_\delta(v_1)| |\tilde{E}(v - v_1)| dv_1 dv \\ &\leq \frac{2A}{\delta} \int_{\mathbb{R}} \int_{\mathbb{R}} (1 + |v + \delta v_1|)^2 |\rho'(v_1)| |\tilde{E}(v)| dv_1 dv \\ &\leq \frac{2A}{\delta} \|\rho'\|_{L^1(\mathbb{R})} \int_{\mathbb{R}} (2 + |v|)^2 |\tilde{E}(v)| dv \\ &\leq \frac{16A}{\delta} \|\rho'\|_{L^1(\mathbb{R})} \int_0^\infty (1 + v^2) |E(v)| dv \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}} (1 + |v|) \left| \rho'_\delta \star (\tilde{g} \tilde{E})(v) \right| dv &\leq A \int_{\mathbb{R}} \int_{\mathbb{R}} |v - v_1| (1 + |v|) |\rho'_\delta(v_1)| |\tilde{E}(v - v_1)| dv_1 dv \\ &\leq \frac{A}{\delta} \int_{\mathbb{R}} \int_{\mathbb{R}} (1 + |v + \delta v_1|) |v| |\rho'(v_1)| |\tilde{E}(v)| dv_1 dv \\ &\leq \frac{8A}{\delta} \|\rho'\|_{L^1(\mathbb{R})} \int_0^\infty (1 + v^2) |E(v)| dv. \end{aligned}$$

Collecting the above estimates and recalling (5.1) give (5.11a) and (5.11b).

Next, the estimate (5.11c) readily follows from Lemma 5.3 and Lemma 5.4 while (5.11d) is a straightforward consequence of (5.1), (5.3), (5.6), and Lemma 5.4. Finally, we deduce (5.11e) from (5.7), (5.11a), (5.11b), (5.11c), and (5.11d). \square

Proof of Lemma 5.2. We multiply both sides of the above equation by w_ϵ and integrate with respect to v on \mathbb{R} to obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} w_\epsilon \Sigma_\epsilon(\tilde{E}^\delta) dv + \int_{\mathbb{R}} w_\epsilon \partial_v \left(\tilde{g} \Sigma_\epsilon(\tilde{E}^\delta) \right) dv \\ = \int_{\mathbb{R}} w_\epsilon \left(\Sigma'_\epsilon(\tilde{E}^\delta) D^\delta + \Sigma'_\epsilon(\tilde{E}^\delta) \tilde{S}^\delta + \partial_v \tilde{g} \left[\Sigma_\epsilon(\tilde{E}^\delta) - \tilde{E}^\delta \Sigma'_\epsilon(\tilde{E}^\delta) \right] \right) dv, \end{aligned}$$

observing that all the terms appearing in the above equation are well-defined according to Lemma 5.6 and (5.10). Now, applying the integration by parts formula to the second term of the left-hand side

of the above equation and observing that the boundary terms vanish by (5.3) and Corollary 5.5, we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} w_{\epsilon} \Sigma_{\epsilon}(\tilde{E}^{\delta}) dv - \int_{\mathbb{R}} w'_{\epsilon} \tilde{g} \Sigma_{\epsilon}(\tilde{E}^{\delta}) dv \\ &= \int_{\mathbb{R}} w_{\epsilon} \left(\Sigma'_{\epsilon}(\tilde{E}^{\delta}) D^{\delta} + \Sigma'_{\epsilon}(\tilde{E}^{\delta}) \tilde{S}^{\delta} + \partial_v \tilde{g} \left[\Sigma_{\epsilon}(\tilde{E}^{\delta}) - \tilde{E}^{\delta} \Sigma'_{\epsilon}(\tilde{E}^{\delta}) \right] \right) dv. \end{aligned}$$

Integrating with respect to time over $(0, t)$ for $t \in [0, T]$ gives

$$\begin{aligned} \int_{\mathbb{R}} w_{\epsilon} \Sigma_{\epsilon}(\tilde{E}^{\delta}(t)) dv &= \int_{\mathbb{R}} w_{\epsilon} \Sigma_{\epsilon}(\tilde{E}^{\delta}(0)) dv + \int_0^t \int_{\mathbb{R}} w'_{\epsilon} \tilde{g} \Sigma_{\epsilon}(\tilde{E}^{\delta}) dv ds \\ &+ \int_0^t \int_{\mathbb{R}} w_{\epsilon} \left(\Sigma'_{\epsilon}(\tilde{E}^{\delta}) D^{\delta} + \Sigma'_{\epsilon}(\tilde{E}^{\delta}) \tilde{S}^{\delta} + \partial_v \tilde{g} \left[\Sigma_{\epsilon}(\tilde{E}^{\delta}) - \tilde{E}^{\delta} \Sigma'_{\epsilon}(\tilde{E}^{\delta}) \right] \right) dv ds. \end{aligned} \quad (5.13)$$

At this point, we note that, by (5.3) and (5.5),

$$\begin{aligned} I_1 &:= \left| \int_{\mathbb{R}} w'_{\epsilon}(v) \tilde{g}(v) \Sigma_{\epsilon}(\tilde{E}^{\delta}(v)) dv \right| \leq 2A \int_{\mathbb{R}} |v w'_{\epsilon}(v)| \left| \tilde{E}^{\delta}(v) \right| dv \\ &\leq 2A(1 + \beta) \int_{\mathbb{R}} w_{\epsilon}(v) \left| \tilde{E}^{\delta}(v) \right| dv. \end{aligned}$$

Since

$$\begin{aligned} \int_{\mathbb{R}} w_{\epsilon}(v) \left| \tilde{E}^{\delta}(v) \right| dv &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} w_{\epsilon}(v) \rho_{\delta}(v_1) \left| \tilde{E}(v - v_1) \right| dv_1 dv \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} w_{\epsilon}(v + \delta v_1) \rho(v_1) \left| \tilde{E}(v) \right| dv_1 dv \\ &\leq \int_{\mathbb{R}} [w_{\epsilon}(v) + (1 + \beta \epsilon^{-(\beta+1)/2}) \delta] \left| \tilde{E}(v) \right| dv \end{aligned} \quad (5.14)$$

by (5.9), we conclude that

$$I_1 \leq 2A(1 + \beta) \int_{\mathbb{R}} [w_{\epsilon}(v) + (1 + \beta \epsilon^{-(\beta+1)/2}) \delta] \left| \tilde{E}(v) \right| dv. \quad (5.15)$$

Next, recalling (5.12), we infer from (5.3) and (5.5) that

$$\begin{aligned} I_2 &:= \left| \int_{\mathbb{R}} w_{\epsilon}(v) \Sigma'_{\epsilon}(\tilde{E}^{\delta}(v)) D^{\delta}(v) dv \right| \\ &\leq 2 \int_{\mathbb{R}} w_{\epsilon}(v) \left| \tilde{g}(v) (\rho'_{\delta} \star \tilde{E})(v) - \rho'_{\delta} \star (\tilde{g} \tilde{E})(v) \right| dv + 2A \int_{\mathbb{R}} w_{\epsilon}(v) \left| \tilde{E}^{\delta}(v) \right| dv. \end{aligned}$$

Now, thanks to (5.3) and (5.9),

$$\begin{aligned}
& \int_{\mathbb{R}} w_{\epsilon}(v) \left| \tilde{g}(v) (\rho'_{\delta} \star \tilde{E})(v) - \rho'_{\delta} \star (\tilde{g}\tilde{E})(v) \right| dv \\
& \leq \int_{\mathbb{R}} w_{\epsilon}(v) \left| \int_{\mathbb{R}} \left[\tilde{g}(v) \rho'_{\delta}(v_1) \tilde{E}(v - v_1) - \rho'_{\delta}(v_1) \tilde{g}(v - v_1) \tilde{E}(v - v_1) \right] dv_1 \right| dv \\
& \leq \int_{\mathbb{R}} \int_{\mathbb{R}} w_{\epsilon}(v) |\tilde{g}(v) - \tilde{g}(v - v_1)| |\rho'_{\delta}(v_1)| \left| \tilde{E}(v - v_1) \right| dv_1 dv \\
& \leq A \int_{\mathbb{R}} \int_{\mathbb{R}} w_{\epsilon}(v) |v_1 \rho'_{\delta}(v_1)| \left| \tilde{E}(v - v_1) \right| dv_1 dv = A \int_{\mathbb{R}} \int_{\mathbb{R}} w_{\epsilon}(v + \delta v_1) |v_1 \rho'(v_1)| \left| \tilde{E}(v) \right| dv_1 dv \\
& \leq AI(\rho) \int_{\mathbb{R}} [w_{\epsilon}(v) + (1 + \beta \epsilon^{-(\beta+1)/2}) \delta] \left| \tilde{E}(v) \right| dv,
\end{aligned}$$

with

$$I(\rho) := \int_{\mathbb{R}} |v \rho'(v)| dv.$$

Combining the above two estimates with (5.14) leads us to

$$I_2 \leq (2 + I(\rho))A \int_{\mathbb{R}} [w_{\epsilon}(v) + (1 + \beta \epsilon^{-(\beta+1)/2}) \delta] \left| \tilde{E}(v) \right| dv. \quad (5.16)$$

We next use once more (5.3) and (5.14), along with (5.6), to obtain

$$\begin{aligned}
I_3 & := \left| \int_{\mathbb{R}} w_{\epsilon}(v) \partial_v \tilde{g}(v) \left[\Sigma_{\epsilon}(\tilde{E}^{\delta}(v)) - \tilde{E}^{\delta}(v) \Sigma'_{\epsilon}(\tilde{E}^{\delta}(v)) \right] dv \right| \\
& \leq A \int_{\mathbb{R}} w_{\epsilon}(v) \left| \tilde{E}^{\delta}(v) \right| dv \\
& \leq A \int_{\mathbb{R}} [w_{\epsilon}(v) + (1 + \beta \epsilon^{-(\beta+1)/2}) \delta] \left| \tilde{E}(v) \right| dv.
\end{aligned} \quad (5.17)$$

Collecting (5.15), (5.16), and (5.17) and using again (5.14), we infer from (5.13) that there is $L_0 > 0$ depending only on A and $I(\rho)$ such that

$$\begin{aligned}
\int_{\mathbb{R}} w_{\epsilon} \Sigma_{\epsilon}(\tilde{E}^{\delta}(t)) dv & \leq \int_{\mathbb{R}} [w_{\epsilon} + (1 + \beta \epsilon^{-(\beta+1)/2}) \delta] \left| \tilde{E}_0 \right| dv + \int_0^t \int_{\mathbb{R}} w_{\epsilon} \tilde{S}^{\delta}(s) \Sigma_{\epsilon}(\tilde{E}^{\delta}(s)) dv ds \\
& \quad + L_0 \int_0^t \int_{\mathbb{R}} [w_{\epsilon} + (1 + \beta \epsilon^{-(\beta+1)/2}) \delta] \left| \tilde{E}(s) \right| dv ds.
\end{aligned} \quad (5.18)$$

In order to pass the limit $\delta \rightarrow 0$ on both sides of (5.18), we consider each term separately. Using (5.5), let us first consider the following term as

$$\left| \int_{\mathbb{R}} w_{\epsilon}(v) \left[\Sigma_{\epsilon}(\tilde{E}^{\delta}(t, v)) - \Sigma_{\epsilon}(\tilde{E}(t, v)) \right] dv \right| \leq 2 \int_{\mathbb{R}} w_{\epsilon}(v) \left| \tilde{E}^{\delta}(t, v) - \tilde{E}(t, v) \right| dv.$$

It readily follows from (5.10), Lemma 5.4, and Lebesgue's dominated convergence theorem that we can let $\delta \rightarrow 0$ in the above inequality and find

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}} w_{\epsilon}(v) \Sigma_{\epsilon}(\tilde{E}^{\delta}(t, v)) dv = \int_{\mathbb{R}} w_{\epsilon}(v) \Sigma_{\epsilon}(\tilde{E}(t, v)) dv = \int_0^{\infty} w_{\epsilon}(v) \Sigma_{\epsilon}(E(t, v)) dv. \quad (5.19)$$

Similarly,

$$\begin{aligned}
& \left| \int_0^t \int_{\mathbb{R}} w_\epsilon(v) \left[\Sigma'_\epsilon(\tilde{E}^\delta(s, v)) \tilde{S}^\delta(s, v) - \Sigma'_\epsilon(\tilde{E}(s, v)) \tilde{S}(s, v) \right] dv ds \right| \\
& \leq \int_0^t \int_{\mathbb{R}} w_\epsilon(v) \left| \Sigma'_\epsilon(\tilde{E}^\delta(s, v)) \right| \left| \tilde{S}^\delta(s, v) - \tilde{S}(s, v) \right| dv ds \\
& \quad + \int_0^t \int_{\mathbb{R}} w_\epsilon(v) \left| \tilde{S}(s, v) \right| \left| \Sigma'_\epsilon(\tilde{E}^\delta(s, v)) - \Sigma'_\epsilon(\tilde{E}(s, v)) \right| dv ds.
\end{aligned} \tag{5.20}$$

On the one hand, by (5.5),

$$\int_0^t \int_{\mathbb{R}} w_\epsilon(v) \left| \Sigma'_\epsilon(\tilde{E}^\delta(s, v)) \right| \left| \tilde{S}^\delta(s, v) - \tilde{S}(s, v) \right| dv ds \leq 2 \int_0^t \int_{\mathbb{R}} w_\epsilon(v) \left| \tilde{S}^\delta(s, v) - \tilde{S}(s, v) \right| dv ds,$$

and we use again (5.10), Lemma 5.4, and Lebesgue's dominated convergence theorem, together with Lemma 5.3, to conclude that

$$\lim_{\delta \rightarrow 0} \int_0^t \int_{\mathbb{R}} w_\epsilon(v) \left| \Sigma'_\epsilon(\tilde{E}^\delta(s, v)) \right| \left| \tilde{S}^\delta(s, v) - \tilde{S}(s, v) \right| dv ds = 0. \tag{5.21}$$

On the other hand, $w_\epsilon \tilde{S}$ belongs to $L^1((0, T) \times \mathbb{R})$ thanks to Lemma 5.3 and (5.8), while Lemma 5.4, the Lipschitz continuity of Σ'_ϵ , and (5.5) ensure that, at least for a subsequence (not relabeled),

$$\lim_{\delta \rightarrow 0} \left| \Sigma'_\epsilon(\tilde{E}^\delta(s, v)) - \Sigma'_\epsilon(\tilde{E}(s, v)) \right| = 0 \quad \text{for a.e. } (s, v) \in (0, t) \times \mathbb{R}$$

with

$$\left| \Sigma'_\epsilon(\tilde{E}^\delta(s, v)) - \Sigma'_\epsilon(\tilde{E}(s, v)) \right| \leq 4 \quad \text{for a.e. } (s, v) \in (0, t) \times \mathbb{R}.$$

We are then in a position to apply Lebesgue's dominated convergence theorem and obtain

$$\lim_{\delta \rightarrow 0} \int_0^t \int_{\mathbb{R}} w_\epsilon(v) \left| \tilde{S}(s, v) \right| \left| \Sigma'_\epsilon(\tilde{E}^\delta(s, v)) - \Sigma'_\epsilon(\tilde{E}(s, v)) \right| dv ds = 0. \tag{5.22}$$

Gathering (5.20), (5.21), and (5.22), we end up with

$$\begin{aligned}
\lim_{\delta \rightarrow 0} \int_0^t \int_{\mathbb{R}} w_\epsilon(v) \Sigma'_\epsilon(\tilde{E}^\delta(s, v)) \tilde{S}^\delta(s, v) dv ds &= \int_0^t \int_{\mathbb{R}} w_\epsilon(v) \Sigma'_\epsilon(\tilde{E}(s, v)) \tilde{S}(s, v) dv ds \\
&= \int_0^t \int_0^\infty w_\epsilon(v) \Sigma'_\epsilon(E(s, v)) S(s, v) dv ds.
\end{aligned} \tag{5.23}$$

Owing to (5.19) and (5.23), we can pass to the limit $\delta \rightarrow 0$ in (5.18) and end up with

$$\begin{aligned}
\int_0^\infty w_\epsilon \Sigma_\epsilon(E(t)) dv &\leq \int_0^\infty w_\epsilon |E_0| dv + \int_0^t \int_0^\infty w_\epsilon S(s) \Sigma_\epsilon(E(s)) dv ds \\
&\quad + L_0 \int_0^t \int_0^\infty w_\epsilon |E(s)| dv ds.
\end{aligned} \tag{5.24}$$

We now perform the limit $\epsilon \rightarrow 0$ in (5.24). First, since

$$\lim_{\epsilon \rightarrow 0} w_\epsilon(v) = w(v) = v + v^{-\beta}, \quad v \in (0, \infty),$$

and $E \in L^\infty(0, T; L^1_{-2\beta, 2}(0, \infty))$ by Lemma 5.3, we readily infer from (5.5) and Lebesgue's dominated convergence theorem that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_0^\infty w_\epsilon(v) \Sigma_\epsilon(E(t, v)) dv &= \int_0^\infty w(v) |E(t, v)| dv, \\ \lim_{\epsilon \rightarrow 0} \int_0^\infty w_\epsilon(v) \Sigma_\epsilon(E_0(v)) dv &= \int_0^\infty w(v) |E_0(v)| dv, \\ \lim_{\epsilon \rightarrow 0} \int_0^t \int_0^\infty w_\epsilon(v) \Sigma_\epsilon(E(s, v)) dv ds &= \int_0^t \int_0^\infty w(v) |E(s, v)| dv ds. \end{aligned}$$

Next, we recall that

$$\lim_{\epsilon \rightarrow 0} w_\epsilon(v) S(s, v) \Sigma'_\epsilon(E(s, v)) = w(v) S(s, v) \operatorname{sign}(E(s, v)) \quad \text{for a.e. } (s, v) \in (0, t) \times (0, \infty),$$

with

$$|w_\epsilon(v) S(s, v) \Sigma'_\epsilon(E(s, v))| \leq (v + \sqrt{\epsilon} + v^{-\beta}) |S(s, v)| \leq 2w(v) |S(s, v)|,$$

due to (5.5) and (5.8). A further application of Lebesgue's dominated convergence theorem gives

$$\lim_{\delta \rightarrow 0} \int_0^t \int_0^\infty w_\epsilon S(s) \Sigma_\epsilon(E(s)) dv ds = \int_0^t \int_0^\infty w S(s) \operatorname{sign}(E(s)) dv ds.$$

We may then let $\epsilon \rightarrow 0$ in (5.24) to obtain Lemma 5.2. \square

Owing to Lemma 5.2, we are left with estimating the contribution of the coagulation term to complete the proof of the continuous dependence, a computation which is by now classical, see [4, Section 8.2.5] for instance, but which we sketch below for the sake of completeness.

Proof of Proposition 5.1. Let us consider the last term on the right-hand side of (5.2):

$$\begin{aligned} & \int_0^t \int_0^\infty w(v) \operatorname{sign}(E(s, v)) S(s, v) dv ds \\ &= \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty K(v_1, v_2) W(s, v_1, v_2) (c_1 + c_2)(s, v_1) E(s, v_2) dv_2 dv_1 ds, \end{aligned}$$

where

$$W(s, v_1, v_2) = w(v_1 + v_2) \operatorname{sign}(E(s, v_1 + v_2)) - w(v_1) \operatorname{sign}(E(s, v_1)) - w(v_2) \operatorname{sign}(E(s, v_2)).$$

Using the properties of the sign function, we get

$$\begin{aligned} W(s, v_1, v_2) E(s, v_2) &\leq (v_1 + v_2 + (v_1 + v_2)^{-\beta}) |E(s, v_2)| + (v_1 + v_1^{-\beta}) |E(s, v_2)| \\ &\quad - (v_2 + v_2^{-\beta}) |E(s, v_2)| \\ &\leq (2v_1 + v_1^{-\beta}) |E(s, v_2)|. \end{aligned}$$

Consequently,

$$\begin{aligned}
& \int_0^t \int_0^\infty \int_0^\infty K(v_1, v_2) W(s, v_1, v_2) (c_1 + c_2)(s, v_1) E(s, v_2) dv_2 dv_1 ds \\
& \leq \int_0^t \int_0^\infty \int_0^\infty K(v_1, v_2) \left(2v_1 + v_1^{-\beta}\right) (c_1 + c_2)(s, v_1) |E(s, v_2)| dv_2 dv_1 ds \\
& \leq k \int_0^t \int_0^1 \int_0^1 (v_1 v_2)^{-\beta} \left(2v_1 + v_1^{-\beta}\right) (c_1 + c_2)(s, v_1) |E(s, v_2)| dv_2 dv_1 ds \\
& \quad + 2k \int_0^t \int_0^1 \int_1^\infty v_1^{-\beta} v_2 \left(2v_1 + v_1^{-\beta}\right) (c_1 + c_2)(s, v_1) |E(s, v_2)| dv_2 dv_1 ds \\
& \quad + k \int_0^t \int_1^\infty \int_1^\infty (v_1 + v_2) \left(2v_1 + v_1^{-\beta}\right) (c_1 + c_2)(s, v_1) |E(s, v_2)| dv_2 dv_1 ds \\
& \leq 6k \int_0^t [M_{-2\beta}((c_1 + c_2)(s)) + M_2((c_1 + c_2)(s))] \int_0^\infty w(v) |E(t, v)| dv ds.
\end{aligned}$$

Combining the above inequality with (5.2) leads us to the following differential inequality for ν :

$$\nu(t) \leq \nu(0) + (6k + L_0) \int_0^t [1 + M_{-2\beta}((c_1 + c_2)(s)) + M_2((c_1 + c_2)(s))] \nu(s) ds, \quad t \in [0, T].$$

Recalling (5.1) and applying Gronwall's lemma gives the stated result. \square

Proof of Theorem 2.4. Let c_1 and c_2 be two weak solutions to (1.3)–(1.4) in the sense of Theorem 2.4 corresponding to the initial data $c_{1,0}$ and $c_{2,0}$, respectively. Assume that $c_{1,0} = c_{2,0}$ a.e. in $(0, \infty)$. It readily follows from Proposition 5.1 that

$$\int_0^\infty (v^{-\beta} + v) |c_1(t, v) - c_2(t, v)| dv = 0,$$

which implies that $c_1(t) = c_2(t)$ for all t in $[0, T]$. This completes the proof of Theorem 2.4. \square

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