



**HAL**  
open science

# Harmonizable Multifractional Stable Field: sharp results on sample path behavior

Antoine Ayache, Christophe Louckx

► **To cite this version:**

Antoine Ayache, Christophe Louckx. Harmonizable Multifractional Stable Field: sharp results on sample path behavior. 2024. <hal-04667162>

**HAL Id: hal-04667162**

**<https://hal.science/hal-04667162v1>**

Preprint submitted on 2 Aug 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# Harmonizable Multifractional Stable Field: sharp results on sample path behavior

Antoine Ayache & Christophe Louckx  
Univ. Lille,  
CNRS, UMR 8524 - Laboratoire Paul Painlevé,  
F-59000 Lille, France  
E-mails: antoine.ayache@univ-lille.fr  
christophe.louckx@univ-lille.fr

August 2, 2024

## Abstract

Since about three decades, there is an increasing interest in study of multifractional processes/fields. The paradigmatic example of them is Multifractional Brownian Field (MBF) over  $\mathbb{R}^N$ , which is a Gaussian generalization with varying Hurst parameter (the Hurst function) of the well-known Fractional Brownian Motion (FBM). Harmonizable Multifractional Stable Field (HMSF) is a very natural (and maybe the most natural) extension of MBF to the framework of heavy-tailed Symmetric  $\alpha$ -Stable (S $\alpha$ S) distributions. Many methods related with Gaussian fields fail to work in such a non-Gaussian framework, this is what makes study of HMSF to be difficult. In our article we construct wavelet type random series representations for the S $\alpha$ S stochastic field generating HMSF and for related fields. Then, under weakened versions of the usual Hölder condition on the Hurst function, we obtain sharp results on sample path behavior of HMSF: optimal global and pointwise moduli of continuity, quasi-optimal pointwise modulus of continuity on a universal event of probability 1 not depending on the location, and an estimate of the behavior at infinity which is optimal when the Hurst function has a limit at infinity to which it converges at a logarithmic rate.

**Key Words:** Heavy-tailed stable distributions, varying Hurst parameter, wavelet random series, moduli of continuity, sample path roughness.

**AMS Subject Classification (MSC2020 database):** 60G52, 60G17, 60G22.

# 1 Introduction and background

All the random variables considered in this paper are defined on a same probability space  $\Omega$  which is assumed to be complete. Multifractional stochastic processes/fields (see e.g. [21, 7, 17, 23, 24, 25, 15, 8, 9, 10, 3, 18, 1, 19]) are extensions of the well-known Fractional Brownian Motion (FBM) (see e.g. the two books [14, 22]) whose local sample paths roughness are allowed to change from point to point. More precisely, in contrast with FBM, the pointwise Hölder exponent of multifractional process/field can take different values at different points. Recall that, for a generic real-valued stochastic field  $Y = \{Y(t), t \in \mathbb{R}^N\}$  with continuous and nowhere differentiable sample paths on  $\mathbb{R}^N$  ( $N \in \mathbb{N}$  being arbitrary), their roughness in a neighborhood of any arbitrary fixed point  $\tau \in \mathbb{R}^N$  is usually measured through  $\rho_Y(\tau)$ , the pointwise Hölder exponent of  $Y$  at  $\tau$  defined as

$$\rho_Y(\tau) := \sup \left\{ a \in [0, 1] : \limsup_{t \rightarrow \tau} \frac{|Y(t) - Y(\tau)|}{|t - \tau|^a} < +\infty \right\}; \quad (1.1)$$

for any given  $\omega \in \Omega$ , the closer to zero is  $\rho_Y(\tau, \omega)$ , the rougher is the sample path  $t \mapsto Y(t, \omega)$  in the vicinity of  $\tau$ .

In the present article we focus on the real-valued Harmonizable Multifractional Stable Field (HMSF)  $Z = \{Z(t), t \in \mathbb{R}^N\}$  whose finite-dimensional distributions are Symmetric  $\alpha$ -Stable (S $\alpha$ S), for any arbitrary and fixed  $\alpha \in (0, 2)$ . Apart of the constant deterministic stability parameter  $\alpha$ , the field  $Z$  depends on a deterministic functional parameter denoted by  $H(\cdot)$  and called the Hurst function, since it basically plays a similar role to that of the constant Hurst parameter of FBM; throughout the article (except in Corollary 2.18), the function  $H(\cdot)$  is assumed to be continuous on  $\mathbb{R}^N$  and with real values in an arbitrary fixed compact interval  $[\underline{H}, \overline{H}] \subset (0, 1)$ . The field  $Z$  is defined as

$$Z(t) := \Re \left( \int_{\mathbb{R}^N} \frac{e^{it \cdot \xi} - 1}{|\xi|_2^{H(t) + \frac{N}{\alpha}}} d\widetilde{M}_\alpha(\xi) \right), \quad \text{for all } t \in \mathbb{R}^N, \quad (1.2)$$

where  $t \cdot \xi$  is the usual inner product of  $t$  and  $\xi$ ,  $|\xi|_2$  is the Euclidean norm of  $\xi$ , and  $\widetilde{M}_\alpha$  is a complex-valued rotationally invariant  $\alpha$ -stable random measure with control measure the Lebesgue measure  $\lambda_N$  on  $\mathbb{R}^N$ . A detailed presentation of such a random measure and the corresponding stable stochastic integral and related topics can for instance be found in Chapter 6 of the book [22]. The following remark, which provides two very important properties of this stochastic integral, will play a fundamental role in our article.

**Remark 1.1** (i) *The stable stochastic integral  $\int_{\mathbb{R}^N} (\cdot) d\widetilde{M}_\alpha$  is a linear map on the Lebesgue space  $L^\alpha(\mathbb{R}^N)$  such that, for any deterministic function  $g \in L^\alpha(\mathbb{R}^N)$ , the real part*

$\Re\left(\int_{\mathbb{R}^N} g(\xi) d\widetilde{M}_\alpha(\xi)\right)$  is a real-valued Symmetric  $\alpha$ -Stable (S $\alpha$ S) random variable with a scale parameter satisfying

$$\sigma\left(\Re\left(\int_{\mathbb{R}^N} g(\xi) d\widetilde{M}_\alpha(\xi)\right)\right)^\alpha = \int_{\mathbb{R}^N} |g(\xi)|^\alpha d\xi. \quad (1.3)$$

We recall in passing that an arbitrary real-valued random variable  $Y$  is said to be S $\alpha$ S with scale parameter  $\sigma(Y) \geq 0$ , when the value of its characteristic function, at any  $y \in \mathbb{R}$ , is equal to  $\exp(-\sigma(Y)^\alpha |y|^\alpha)$ ; except in the very special case  $\sigma(Y) = 0$  in which  $Y$  vanishes almost surely, in the generic case  $\sigma(Y) > 0$  the characteristic function of  $Y$  is Lebesgue integrable on  $\mathbb{R}$ , which implies that  $Y$  has a bounded probability density function. Even if the absolute moment of order  $\gamma$  of  $Y$  is infinite when  $\gamma \geq \alpha$ , the scale parameter plays a role which is basically similar to that of a standard deviation.

Thus, the equality (1.3) is reminiscent of the classical isometry property of stochastic Wiener integrals; it implies, among many other things, that  $\Re\left(\int_{\mathbb{R}^N} g_n(\xi) d\widetilde{M}_\alpha(\xi)\right)$  converges to

$\Re\left(\int_{\mathbb{R}^N} g(\xi) d\widetilde{M}_\alpha(\xi)\right)$  in probability, when a sequence  $(g_n)_n$  converges to  $g$  in  $L^\alpha(\mathbb{R})$ .

(ii) Let  $m \in \mathbb{N}$  be arbitrary and let  $f_1, \dots, f_m$  be arbitrary functions of  $L^\alpha(\mathbb{R}^N)$  whose supports are disjoint up to Lebesgue negligible sets, then the real-valued S $\alpha$ S random variables

$\Re\left(\int_{\mathbb{R}^N} f_1(\xi) d\widetilde{M}_\alpha(\xi)\right), \dots, \Re\left(\int_{\mathbb{R}^N} f_m(\xi) d\widetilde{M}_\alpha(\xi)\right)$  are independent.

In the very particular Gaussian case  $\alpha = 2$ , the HMSF in (1.2) reduces to the Multifractional Brownian Field (MBF), that we denote by  $Z_2 = \{Z_2(t), t \in \mathbb{R}^N\}$ , which was initially introduced in [7] and which is the paradigmatic example of a multifractional field. Thus, the HMSF is a very natural (and maybe the most natural) extension of the MBF to the framework of heavy-tailed S $\alpha$ S distributions. When  $H(\cdot)$  is a smooth enough Hölder function, an important result of [7] is that, for all  $\tau \in \mathbb{R}^N$ , the pointwise Hölder exponent of the MBF  $Z_2$  equals  $H(\tau)$ , on an event of probability 1 which a priori depends on  $\tau$ , in other words one has

$$\forall \tau \in \mathbb{R}^N, \mathbb{P}(\rho_{Z_2}(\tau) = H(\tau)) = 1. \quad (1.4)$$

Later, under a similar smoothness condition on  $H(\cdot)$ , the article [4] was able to obtain a significantly stronger result: the equality between  $\rho_{Z_2}(\tau)$  and  $H(\tau)$  holds on a universal event of probability 1 which does not depend on  $\tau$ , that is

$$\mathbb{P}(\forall \tau \in \mathbb{R}^N, \rho_{Z_2}(\tau) = H(\tau)) = 1. \quad (1.5)$$

As far as we know, in the non-Gaussian framework  $\alpha \in (0, 2)$ , the HMSF  $Z$  in (1.2), or slightly different variants of it, has already been considered in two articles in the literature: the paper [13] which restricts its study to the case  $N = 1$  and  $\alpha \in (1, 2)$ , and the paper [11] which introduces a large class of harmonizable multi-operator scaling stable random fields including the HMSF  $Z$ . Theorem 4.6 in [11], which was obtained by using LePage random series expansion for such a random field, and Corollary 4.8 in [11] imply, when  $H(\cdot)$  is a locally Lipschitz function, that, on any non-empty compact box  $T := \prod_{t=1}^N [\mu_t, \nu_t]$  of  $\mathbb{R}^N$ , sample paths of  $Z$  are almost surely Hölder continuous functions of any order  $\gamma < \min_{t \in T} H(t)$ . In the case  $N = 1$  and  $\alpha \in (1, 2)$ , Theorem 2.5 in [13], which was also obtained thanks to LePage random series and by assuming that  $H(\cdot)$  is a Hölder continuous function on  $T$  of order strictly larger than  $\max_{t \in T} H(t)$ , provides a more precise uniform modulus of continuity for  $Z$  which shows that, on any non-empty compact interval  $T := [\mu_1, \nu_1]$  of  $\mathbb{R}$ , sample paths of  $Z$  are, almost surely, Hölder continuous functions of order  $\min_{t \in T} H(t)$ , up to a logarithmic factor raised to the power  $1/\alpha + 1/2 + \eta$ , where  $\eta > 0$  is arbitrarily small. Also, we mention that Corollary 4.10 in [11], which provides results on pointwise Hölder exponents and directional pointwise Hölder exponents of harmonizable multi-operator scaling stable random fields, implies that (1.4) keeps valid when the MBF  $Z_2$  is replaced by an HMSF  $Z$  with an arbitrary stability parameter  $\alpha \in (0, 2)$  and a locally Lipschitz Hurst function parameter  $H(\cdot)$ . Yet, the question to know whether or not the significantly stronger result (1.5) keeps valid in the same situation has remained open. One of the motivation for our article is to give a positive answer to this question under a pointwise Hölder regularity assumption on  $H(\cdot)$ . Another motivation for it is to obtain optimal uniform and pointwise moduli of continuity for  $Z$ , under weaker assumptions on  $H(\cdot)$  than those in [11, 13]; the power of the logarithmic factor in these moduli of continuity is  $1/\alpha + \eta$ , which is better than the power  $1/\alpha + 1/2 + \eta$  earlier obtained in [13]. A third motivation for our article is to derive an almost sure estimate for the asymptotic behavior of  $Z$  at infinity, and to show that this estimate is optimal when the function  $H(\cdot)$  has a limit at infinity to which it converges at a logarithmic rate; there is no need that the rate of convergence be faster than the very slow logarithmic rate.

Our study of the HMSF  $Z = \{Z(t), t \in \mathbb{R}^N\}$ , defined in (1.2), makes an extensive use of wavelet methods, and relies on a useful classical point of view on multifractional processes/fields, which consists in expressing them in terms of more general stochastic fields indexed by the couple  $(u, v) \in \mathbb{R}^N \times (0, 1)$ , the two variables  $u$  (which corresponds to  $t$ ) and  $v$  (which somehow corresponds to  $H(t)$ ) being a priori not related. Thus, we express the HMSF  $Z = \{Z(t), t \in \mathbb{R}^N\}$  as

$$Z(t) = X(t, H(t)), \quad \text{for all } t \in \mathbb{R}^N, \quad (1.6)$$

where the SaS stochastic field  $X = \{X(u, v), (u, v) \in \mathbb{R}^N \times (0, 1)\}$ , which is called *the field generating HMSF*, is defined as

$$X(u, v) := \Re \left( \int_{\mathbb{R}^N} F_\alpha(u, v, \xi) d\widetilde{M}_\alpha(\xi) \right), \quad \text{for all } (u, v) \in \mathbb{R}^N \times (0, 1). \quad (1.7)$$

The kernel function  $F_\alpha$  is given by  $F_\alpha(u, v, 0) := 0$  for all  $(u, v) \in \mathbb{R}^N \times (0, 1)$ , and, more importantly, by

$$F_\alpha(u, v, \xi) := \frac{e^{iu \cdot \xi} - 1}{|\xi|_2^{v + \frac{N}{\alpha}}}, \quad \text{for all } (u, v) \in \mathbb{R}^N \times (0, 1) \text{ and } \xi \in \mathbb{R}^N \setminus \{0\}. \quad (1.8)$$

Let us emphasize that many properties of the HMSF  $Z$  are strongly influenced by those of the field  $X$  which is very closely related to  $Z$  through the equality (1.6).

The remaining of our article is organized as follows. In Section 2, we introduce wavelet type random series representations for the field  $X$  and related fields, which mainly allow us to show that sample paths of  $X$  are continuous on  $\mathbb{R}^N \times (0, 1)$  and infinitely differentiable with respect to the variable  $v$ , and that sample paths of  $\partial_v^m X$  share the same continuity property, for any given  $m \in \mathbb{Z}_+$  (notice that  $\partial_v^0 X = X$ ). In Section 3, mainly, we obtain, for all  $m \in \mathbb{Z}$ , a global modulus of continuity for the field  $\partial_v^m X$  on any arbitrary non-empty compact box of  $\mathbb{R}^N \times (0, 1)$ , as well as an estimate of the asymptotic behavior at infinity of  $\partial_v^m X$  with respect to the variable  $u$ , and uniformly in the variable  $v$  restricted to any compact interval of  $(0, 1)$ ; also, under some conditions on  $H(\cdot)$ , we obtain the counterparts for the HMSF  $Z$  of these two results, as well as a pointwise modulus of continuity for  $Z$ . Section 4 is devoted to study of optimality of results of the previous section. Among other things, under the same global Hölder condition on  $H(\cdot)$  as in the previous section and a weak additional condition, it shows that the global modulus of continuity for  $Z$  is optimal, and that this is also the case for the estimate of the asymptotic behavior of  $Z$  at infinity, provided that its Hurst function has a limit at infinity to which it converges at a logarithmic rate. In addition, under the same pointwise Hölder condition on  $H(\cdot)$ , at an arbitrary given point  $\tau \in \mathbb{R}^N$ , as in the previous section, Section 4 shows that the pointwise modulus of continuity for  $Z$  at  $\tau \in \mathbb{R}^N$  is optimal on an event of probability 1 which depends on  $\tau$ , and also that it is quasi-optimal (optimal up to a logarithmic factor) on a universal event of probability 1 not depending on  $\tau$ , provided that  $H(\cdot)$  satisfies a bit stronger pointwise Hölder condition at  $\tau$ . A consequence of the latter result is that (1.5) keeps valid when the MBF  $Z_2$  is replaced by an HMSF  $Z$  with an arbitrary stability parameter  $\alpha \in (0, 2)$  and an arbitrary continuous Hurst function parameter  $H(\cdot)$  satisfying the latter pointwise Hölder condition at any  $\tau \in \mathbb{R}^N$ . Section 5 is the Appendix which is devoted to the long proof of a crucial lemma which plays a fundamental role in our article.

## 2 Wavelet type random series representations for the field $X$ and related fields

In this section we show that, for any fixed  $(u, v) \in \mathbb{R}^N \times (0, 1)$ , the kernel function  $\xi \mapsto F_\alpha(u, v, \xi)$  defined in (1.8) can be expanded, with convergence in  $L^\alpha(\mathbb{R}^N)$ , on a sequence of functions issued from Meyer wavelets. Then, thanks to the "isometry property" of stable stochastic integral (see Part (i) of Remark 1.1), we obtain wavelet type random series representations for the field  $X$  and related fields. Moreover, we show that these series are almost surely absolutely convergent, and even better: they are almost surely normally convergent in  $(u, v)$  with respect to the uniform semi-norm on any non-empty compact box of  $\mathbb{R}^N \times (0, 1)$ . One of the consequences of the latter result and of (1.6) is that the HMSF  $Z$  has, almost surely continuous sample paths on  $\mathbb{R}^N$  as soon as the Hurst function  $H(\cdot)$  is continuous on  $\mathbb{R}^N$ .

First, we introduce the main ingredients which allow to construct the wavelet type random series representations for the field  $X$  and related fields.

Let  $\Upsilon_* := \{1, \dots, 2^N - 1\}$ , the sequence of functions  $(\psi_{\delta,j,k})_{(\delta,j,k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N}$ , belonging to the Schwartz space  $\mathcal{S}(\mathbb{R}^N)$ , denotes an orthonormal Meyer wavelet basis of the Hilbert space  $L^2(\mathbb{R}^N)$ ; one mentions in passing that two well-known references on wavelet bases and wavelet theory are the two books [20, 12]. Recall that, each real-valued wavelet function  $\psi_{\delta,j,k}$  is a dilated and translated version of the mother wavelet  $\psi_\delta$ , that is,

$$\psi_{\delta,j,k}(x) := 2^{j\frac{N}{2}} \psi_\delta(2^j x - k), \quad \text{for all } x \in \mathbb{R}^N; \quad (2.1)$$

its Fourier transform  $\widehat{\psi}_{\delta,j,k}$ , defined as

$$\widehat{\psi}_{\delta,j,k}(\xi) := (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} \psi_{\delta,j,k}(x) dx, \quad \text{for every } \xi \in \mathbb{R}^N,$$

belongs to  $\mathcal{S}(\mathbb{R}^N)$ , similarly to  $\psi_{\delta,j,k}$  itself, and, in view of (2.1), it is given by

$$\widehat{\psi}_{\delta,j,k}(\xi) = 2^{-j\frac{N}{2}} e^{-i(2^{-j}k) \cdot \xi} \widehat{\psi}_\delta(2^{-j}\xi), \quad \text{for all } \xi \in \mathbb{R}^N. \quad (2.2)$$

Also, recall that, for all  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ , the  $C^\infty$  function  $\widehat{\psi}_{\delta,j,k}$  is compactly supported such that

$$\text{supp}(\widehat{\psi}_{\delta,j,k}) \subseteq \mathcal{K}_j, \quad (2.3)$$

where  $\mathcal{K}_j$  is the compact subset of  $\mathbb{R}^N$  defined as

$$\mathcal{K}_j := \left[ -\frac{2^{j+3}\pi}{3}, \frac{2^{j+3}\pi}{3} \right]^N \setminus \left( -\frac{2^{j+1}\pi}{3}, \frac{2^{j+1}\pi}{3} \right)^N. \quad (2.4)$$

In particular, one has, for all  $\delta \in \Upsilon_*$ ,

$$\text{supp}(\widehat{\psi}_\delta) \subseteq \mathcal{K}_0 := \left[-\frac{8\pi}{3}, \frac{8\pi}{3}\right]^N \setminus \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)^N. \quad (2.5)$$

Our next goal is to show that, for any fixed  $(u, v) \in \mathbb{R}^N \times (0, 1)$ , the kernel function  $F_\alpha(u, v, \cdot)$  can be expanded, with convergence in  $L^\alpha(\mathbb{R}^N)$ , on the sequence of functions  $\overline{(\widehat{\psi}_{\delta,j,k})}_{(\delta,j,k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N}$ . One mentions that the value of anyone of the functions  $\widehat{\psi}_{\delta,j,k}$ , at any  $\xi \in \mathbb{R}^N$ , is  $\widehat{\psi}_{\delta,j,k}(\xi) = \overline{\widehat{\psi}_{\delta,j,k}(-\xi)}$ , the conjugate of the complex number  $\widehat{\psi}_{\delta,j,k}(-\xi)$ . Also, one mentions that, for each  $\alpha \in (0, 2]$ , the space  $L^\alpha(\mathbb{R}^N)$  is equipped with the usual metric  $\Delta_\alpha$  for which it is a complete metric space; setting  $\beta(\alpha) := \min\{1, \alpha\}$  this metric  $\Delta_\alpha$  can then be defined in the following concise way:

$$\Delta_\alpha(g_1; g_2) = \left( \int_{\mathbb{R}^N} |g_1(\xi) - g_2(\xi)|^\alpha d\xi \right)^{\frac{\beta(\alpha)}{\alpha}}, \quad \text{for all } g_1, g_2 \in L^\alpha(\mathbb{R}^N). \quad (2.6)$$

**Proposition 2.1** *Let  $(\mathcal{D}_n)_{n \in \mathbb{N}}$  be an arbitrary ascending sequence (that is  $\mathcal{D}_n \subset \mathcal{D}_{n+1}$ , for every  $n \in \mathbb{N}$ ) of non-empty finite subsets of  $\mathbb{Z} \times \mathbb{Z}^N$  whose union is  $\mathbb{Z} \times \mathbb{Z}^N$  (i.e.  $\mathbb{Z} \times \mathbb{Z}^N = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$ ). For all  $(u, v, \xi) \in \mathbb{R}^N \times (0, 1) \times \mathbb{R}^N$ , one sets*

$$S_n^{(\alpha)}(u, v, \xi) := \sum_{\delta \in \Upsilon_*} \sum_{(j,k) \in \mathcal{D}_n} \left\langle F_\alpha(u, v, \cdot), \overline{\widehat{\psi}_{\delta,j,k}} \right\rangle \overline{\widehat{\psi}_{\delta,j,k}(\xi)}, \quad (2.7)$$

where

$$\left\langle F_\alpha(u, v, \cdot), \overline{\widehat{\psi}_{\delta,j,k}} \right\rangle := \int_{\mathbb{R}^N} F_\alpha(u, v, \xi) \widehat{\psi}_{\delta,j,k}(\xi) d\xi = 2^{-j\frac{N}{2}} \int_{\mathbb{R}^N} \frac{e^{iu \cdot \xi} - 1}{|\xi|_2^{v+\frac{N}{\alpha}}} e^{-ik \cdot (2^{-j}\xi)} \widehat{\psi}_\delta(2^{-j}\xi) d\xi; \quad (2.8)$$

we note in passing that the last equality in (2.8) follows from (1.8) and (2.2). Then, for any arbitrary and fixed  $(u, v) \in \mathbb{R}^N \times (0, 1)$ , one has

$$S_n^{(\alpha)}(u, v, \cdot) \xrightarrow[n \rightarrow +\infty]{L^\alpha(\mathbb{R}^N)} F_\alpha(u, v, \cdot). \quad (2.9)$$

For proving Proposition 2.1 and for later purposes, we need to express  $S_n^{(\alpha)}$  in a convenient way. To this end, we have to introduce some new functions.

**Definition 2.2** *For each  $\delta \in \Upsilon_*$ , one denotes by  $\Psi_\delta^{(\alpha)}$  the real-valued function on  $\mathbb{R}^N \times \mathbb{R}$  defined as*

$$\Psi_\delta^{(\alpha)}(x, v) := \int_{\mathbb{R}^N} \frac{e^{ix \cdot \eta}}{|\eta|_2^{v+\frac{N}{\alpha}}} \widehat{\psi}_\delta(\eta) d\eta, \quad \text{for all } (x, v) \in \mathbb{R}^N \times \mathbb{R}; \quad (2.10)$$

notice that  $\Psi_\delta^{(\alpha)}$  is well-defined thanks to (2.5) and to the fact that  $\widehat{\psi}_\delta \in \mathcal{S}(\mathbb{R}^N)$ . For every  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ , one denotes by  $\psi_{\delta, j, k}^{(\alpha)}$  the real-valued function on  $\mathbb{R}^N$  defined as

$$\psi_{\delta, j, k}^{(\alpha)}(x) = 2^{jN(1-\frac{1}{\alpha})} \psi_\delta(2^j x - k), \quad \text{for all } x \in \mathbb{R}^N. \quad (2.11)$$

Observe that the Fourier transform of  $\psi_{\delta, j, k}^{(\alpha)}$  is given by

$$\widehat{\psi}_{\delta, j, k}^{(\alpha)}(\xi) = 2^{-j\frac{N}{\alpha}} e^{-i(2^{-j}k) \cdot \xi} \widehat{\psi}_\delta(2^{-j}\xi), \quad \text{for all } \xi \in \mathbb{R}^N. \quad (2.12)$$

**Remark 2.3** In view of (2.10), setting in (2.8)  $\eta = 2^{-j}\xi$ , we obtain, for all  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$  and  $(u, v) \in \mathbb{R}^N \times \mathbb{R}$ , that

$$\left\langle F_\alpha(u, v, \cdot), \overline{\widehat{\psi}_{\delta, j, k}} \right\rangle = 2^{j(\frac{N}{2} - \frac{N}{\alpha} - v)} \left[ \Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v) \right]. \quad (2.13)$$

Then, combining (2.2) and (2.12) with (2.13), we get, for all  $\xi \in \mathbb{R}^N$ , that

$$\left\langle F_\alpha(u, v, \cdot), \overline{\widehat{\psi}_{\delta, j, k}} \right\rangle \overline{\widehat{\psi}_{\delta, j, k}(\xi)} = 2^{-jv} \left[ \Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v) \right] \overline{\widehat{\psi}_{\delta, j, k}^{(\alpha)}(\xi)}. \quad (2.14)$$

The equality (2.14) allows to express  $S_n^{(\alpha)}$ , defined in (2.7), in terms of the functions  $\Psi_\delta^{(\alpha)}$  and  $\overline{\widehat{\psi}_{\delta, j, k}^{(\alpha)}}$ . We point out that these functions have nice properties: the integral  $\int_{\mathbb{R}^N} |\overline{\widehat{\psi}_{\delta, j, k}^{(\alpha)}}(\xi)|^\alpha d\xi$  does not depend on  $(j, k)$ , since (2.12) clearly entails that

$$\int_{\mathbb{R}^N} |\overline{\widehat{\psi}_{\delta, j, k}^{(\alpha)}}(\xi)|^\alpha d\xi = \int_{\mathbb{R}^N} |\widehat{\psi}_\delta(\xi)|^\alpha d\xi, \quad \text{for all } (\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N. \quad (2.15)$$

Moreover, one knows from Propositions 5.10 and 5.11 in the book [1] that:

**Lemma 2.4** ([1]) *For every  $\delta \in \Upsilon_*$ , the function  $\Psi_\delta^{(\alpha)}$ , defined in (2.10), is infinitely differentiable on  $\mathbb{R}^N \times \mathbb{R}$  and, for all  $\gamma := (\gamma_1, \dots, \gamma_N) \in \mathbb{Z}_+^N$  and  $m \in \mathbb{Z}_+$ , its partial derivative  $\partial_x^\gamma \partial_v^m \Psi_\delta^{(\alpha)} := \frac{\partial^{\gamma_1 + \dots + \gamma_N + m} \Psi_\delta^{(\alpha)}}{(\partial x_1)^{\gamma_1} \dots (\partial x_N)^{\gamma_N} (\partial v)^m}$ ,  $x_1, \dots, x_N$  being the coordinates of the variable  $x$ , is given by*

$$(\partial_x^\gamma \partial_v^m \Psi_\delta^{(\alpha)})(x, v) = \int_{\mathbb{R}^N} e^{ix \cdot \eta} (i\eta)^\gamma \frac{(-\log |\eta|_2)^m}{|\eta|_2^{v + \frac{N}{\alpha}}} \widehat{\psi}_\delta(\eta) d\eta, \quad \text{for all } (x, v) \in \mathbb{R}^N \times \mathbb{R}, \quad (2.16)$$

with the convention that  $(i\eta)^\gamma := \prod_{r=1}^N (i\eta_r)^{\gamma_r}$ , for each  $\eta = (\eta_1, \dots, \eta_N) \in \mathbb{R}^N$ . Moreover, the function  $\partial_x^\gamma \partial_v^m \Psi_\delta^{(\alpha)}$  is well-localized in the variable  $x \in \mathbb{R}^N$ , uniformly in the variable  $v$  restricted to any arbitrary non-empty compact interval of  $\mathbb{R}$ ; namely, for every fixed real numbers  $L \geq 0$ ,  $M > 0$  and  $T \geq 0$ , one has

$$\sup \left\{ \prod_{r=1}^N (1 + T + |x_r|)^L |(\partial_x^\gamma \partial_v^m \Psi_\delta^{(\alpha)})(x, v)|, (u, v) \in \mathbb{R}^N \times [-M, M] \right\} < +\infty. \quad (2.17)$$

Before proving Proposition 2.1, let us emphasize that a straightforward consequence of it, of the equality (2.14) and of the "isometry property" of stable stochastic integral (see Part (i) of Remark 1.1) is the following important result, which provides a wavelet type random series representation for the field  $X$ .

**Theorem 2.5** *We use the same notations as in Definition 2.2. Let  $(\varepsilon_{\delta,j,k}^{(\alpha)})_{(\delta,j,k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N}$  be the sequence of the  $S\alpha S$  random variables defined as*

$$\varepsilon_{\delta,j,k}^{(\alpha)} := \Re \left( \int_{\mathbb{R}^N} \overline{\widehat{\psi}_{\delta,j,k}^{(\alpha)}(\xi)} d\widetilde{M}_\alpha(\xi) \right), \quad \text{for all } (\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N. \quad (2.18)$$

Then, the  $S\alpha S$  stochastic field  $X$ , which was introduced in (1.7), can be expressed, for all fixed  $(u, v) \in \mathbb{R}^N \times (0, 1)$ , as

$$X(u, v) = \sum_{(\delta,j,k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N} 2^{-jv} [\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)] \varepsilon_{\delta,j,k}^{(\alpha)}, \quad (2.19)$$

where the random series is convergent in probability, for any choice of the sequence of its partial sums. More precisely, for any arbitrary ascending sequence  $(\mathcal{D}_n)_{n \in \mathbb{N}}$  of non-empty finite subsets of  $\mathbb{Z} \times \mathbb{Z}^N$  whose union is  $\mathbb{Z} \times \mathbb{Z}^N$ , the random variable

$$X_n(u, v) := \sum_{\delta \in \Upsilon_*} \sum_{(j,k) \in \mathcal{D}_n} 2^{-jv} [\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)] \varepsilon_{\delta,j,k}^{(\alpha)} \quad (2.20)$$

converges in probability to  $X(u, v)$ .

Let us now prove Proposition 2.1.

**Proof of Proposition 2.1** Throughout the proof  $(u, v) \in \mathbb{R}^N \times (0, 1)$  is arbitrary and fixed.

First step: We show that the sequence  $(S_n^\alpha(u, v, \cdot))_{n \in \mathbb{N}}$ , defined in (2.7) (see also (2.14)), converges in the complete metric space  $L^\alpha(\mathbb{R}^N)$ , equipped with the metric  $\Delta_\alpha$  (see (2.6)), to some function denoted by  $\widetilde{F}_\alpha(u, v, \cdot)$ . To this end, it is enough to show that  $(S_n^\alpha(u, v, \cdot))_{n \in \mathbb{N}}$  is a Cauchy sequence in this complete metric space. The latter result can be obtained by proving that

$$\sum_{\delta \in \Upsilon_*} \sum_{(j,k) \in \mathbb{Z} \times \mathbb{Z}^N} \Delta_\alpha \left( 2^{-jv} [\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)] \overline{\widehat{\psi}_{\delta,j,k}^{(\alpha)}(\cdot; 0)} \right) < +\infty. \quad (2.21)$$

Indeed, for all  $(n, p) \in \mathbb{N}^2$ , (2.7), (2.14), (2.6), the inclusion  $\mathcal{D}_n \subset \mathcal{D}_{n+p}$  and the triangle inequality imply that

$$\begin{aligned} & \Delta_\alpha \left( S_{n+p}^\alpha(u, v, \cdot); S_n^\alpha(u, v, \cdot) \right) \\ & \leq \sum_{\delta \in \Upsilon_*} \sum_{(j,k) \in (\mathbb{Z} \times \mathbb{Z}^N) \setminus \mathcal{D}_n} \Delta_\alpha \left( 2^{-jv} [\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)] \overline{\widehat{\psi}_{\delta,j,k}^{(\alpha)}(\cdot; 0)} \right). \end{aligned} \quad (2.22)$$

Moreover, (2.21) and the fact that the ascending sequence  $(\mathcal{D}_n)_{n \in \mathbb{N}}$  satisfies  $\bigcup_{n \in \mathbb{N}} \mathcal{D}_n = \mathbb{Z} \times \mathbb{Z}^N$ , entail that the sum in the right-hand side of (2.22) converges to 0 when  $n$  goes to  $+\infty$ . Let us now prove that (2.21) holds. It follows from (2.6) and (2.15) that, for all  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z}^N \times \mathbb{Z}$ , one has

$$\begin{aligned} \Delta_\alpha \left( 2^{-jv} [\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)] \overline{\widehat{\psi}_{\delta, j, k}^{(\alpha)}(\cdot; 0)} \right) \\ = 2^{-jv\beta(\alpha)} |\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)|^{\beta(\alpha)} \Delta_\alpha(\widehat{\psi}_\delta; 0). \end{aligned} \quad (2.23)$$

Let us provide an appropriate upper bound for  $|\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)|^{\beta(\alpha)}$ . We set  $|u|_\infty := \max_{1 \leq r \leq N} |u_r|$ . When  $j < 0$ , using the mean value theorem, the triangle inequality and (2.17) with  $M = 1$  and  $T = |u|_\infty$ , and denoting by  $\partial_{x_r}$  the partial derivative operator of order 1 with respect to the  $r$ th coordinate of the variable  $x$  of the function  $\Psi_\delta^{(\alpha)}$  (see (2.10)), we obtain that

$$\begin{aligned} & |\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)|^{\beta(\alpha)} \\ & \leq 2^{j\beta(\alpha)} |u|_\infty^{\beta(\alpha)} \sum_{r=1}^N \max_{s \in [-|u|_\infty, |u|_\infty]^N} |(\partial_{x_r} \Psi_\delta^{(\alpha)})(2^j s - k, v)|^{\beta(\alpha)} \\ & \leq c_1 2^{j\beta(\alpha)} \max_{s \in [-|u|_\infty, |u|_\infty]^N} \prod_{r=1}^N (1 + |u|_\infty + |2^j s_r - k_r|)^{-L\beta(\alpha)} \\ & \leq c_1 2^{j\beta(\alpha)} \max_{s \in [-|u|_\infty, |u|_\infty]^N} \prod_{r=1}^N (1 + |u|_\infty - |2^{-j} s_r| + |k_r|)^{-L\beta(\alpha)} \leq c_1 2^{j\beta(\alpha)} \prod_{r=1}^N (1 + |k_r|)^{-L\beta(\alpha)}, \end{aligned} \quad (2.24)$$

where the finite constant  $c_1$  does not depend on  $(\delta, j, k)$ . When,  $j \geq 0$ , using the triangle inequality and (2.17) with  $M = 1$  and  $T = 0$ , we get that

$$\begin{aligned} & |\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)|^{\beta(\alpha)} \leq |\Psi_\delta^{(\alpha)}(2^j u - k, v)|^{\beta(\alpha)} + |\Psi_\delta^{(\alpha)}(-k, v)|^{\beta(\alpha)} \\ & \leq c_2 \left( \prod_{r=1}^N (1 + |2^j u_r - k_r|)^{-L\beta(\alpha)} + \prod_{r=1}^N (1 + |k_r|)^{-L\beta(\alpha)} \right), \end{aligned} \quad (2.25)$$

where the finite constant  $c_2$  does not depend on  $(\delta, j, k)$ . Next, notice that since the non-negative fixed real number  $L$  is arbitrary, one can assume that it is large enough so that  $L\beta(\alpha) > 1$ , which implies that

$$c_3 := \sup_{x \in \mathbb{R}^N} \left\{ \sum_{k \in \mathbb{Z}^N} \prod_{r=1}^N (1 + |x_r - k_r|)^{-L\beta(\alpha)} \right\} < +\infty. \quad (2.26)$$

Finally, putting together (2.23), (2.24), (2.25) and (2.26), it follows that (2.21) holds.

*Second step:* We show that  $\widetilde{F}_\alpha(u, v, \xi) = F_\alpha(u, v, \xi)$ , for Lebesgue almost all  $\xi \in \mathbb{R}^N$ . For all  $m \in \mathbb{Z}_+$ , let  $\mathcal{A}_m$  be the closed subset of  $\mathbb{R}^N$  defined as

$$\mathcal{A}_m = \mathbb{R}^N \setminus \left( \frac{-2^{-m+1}\pi}{3}, \frac{2^{-m+1}\pi}{3} \right)^N. \quad (2.27)$$

Using (1.8), (2.27) and the inequality  $2(v + N/\alpha) > N$ , one has

$$\int_{\mathbb{R}^N} |F_\alpha(u, v, \xi)|^2 \mathbf{1}_{\mathcal{A}_m}(\xi) d\xi \leq 4 \int_{\mathcal{A}_m} |\xi|_2^{-2(v + \frac{N}{\alpha})} d\xi < +\infty,$$

which shows that  $F_\alpha(u, v, \cdot) \mathbf{1}_{\mathcal{A}_m}(\cdot) \in L^2(\mathbb{R}^N)$ . Thus, using the fact that  $(\widehat{\psi}_{\delta,j,k})_{(\delta,j,k)}$  is an orthonormal basis of  $L^2(\mathbb{R}^N)$ , one gets, for all fixed  $m \in \mathbb{Z}_+$ , that

$$\sum_{\delta \in \Upsilon_*} \sum_{(\delta,j,k) \in \mathcal{D}_n} \left( \int_{\mathcal{A}_m \cap \mathcal{K}_j} F_\alpha(u, v, \xi) \widehat{\psi}_{\delta,j,k}(\xi) d\xi \right) \overline{\widehat{\psi}_{\delta,j,k}(\cdot)} \xrightarrow[n \rightarrow +\infty]{L^2(\mathbb{R}^N)} F_\alpha(u, v, \cdot) \mathbf{1}_{\mathcal{A}_m}(\cdot), \quad (2.28)$$

where  $\mathcal{K}_j$  is as in (2.4). Next, for each  $m \in \mathbb{Z}_+$ , let  $\widetilde{\mathcal{K}}_m$  be the compact subset of  $\mathbb{R}^N$  defined as

$$\widetilde{\mathcal{K}}_m = \left[ -\frac{2^{m+3}\pi}{3}, \frac{2^{m+3}\pi}{3} \right]^N \setminus \left( -\frac{2^{-m+3}\pi}{3}, \frac{2^{-m+3}\pi}{3} \right)^N. \quad (2.29)$$

Observe that, for all  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ ,  $m \in \mathbb{Z}_+$  and  $\xi \in \mathbb{R}^N$ , one has

$$\left( \int_{\mathcal{A}_m \cap \mathcal{K}_j} F_\alpha(u, v, \xi) \widehat{\psi}_{\delta,j,k}(\xi) d\xi \right) \overline{\widehat{\psi}_{\delta,j,k}(\xi)} \mathbf{1}_{\widetilde{\mathcal{K}}_m}(\xi) = \left( \int_{\mathbb{R}^N} F_\alpha(u, v, \xi) \widehat{\psi}_{\delta,j,k}(\xi) d\xi \right) \overline{\widehat{\psi}_{\delta,j,k}(\xi)} \mathbf{1}_{\widetilde{\mathcal{K}}_m}(\xi). \quad (2.30)$$

Indeed, when  $\widetilde{\mathcal{K}}_m \cap \mathcal{K}_j = \emptyset$  the equality (2.30) is satisfied, since its left and right hand sides vanish due to (2.3); the equality (2.30) also holds when  $\widetilde{\mathcal{K}}_m \cap \mathcal{K}_j \neq \emptyset$ , since one then necessarily has that  $-m \leq j \leq m+2$  which implies that  $\mathcal{A}_m \cap \mathcal{K}_j = \mathcal{K}_j$  and thus, one knows from (2.3), that the integrals at both sides of (2.30) are the same.

Putting together (2.7), (2.8), (2.28), the inclusion  $\widetilde{\mathcal{K}}_m \subset \mathcal{A}_m$  and (2.30), one obtain, for all fixed  $m \in \mathbb{Z}_+$ , that

$$S_n^{(\alpha)}(u, v, \cdot) \mathbf{1}_{\widetilde{\mathcal{K}}_m}(\cdot) \xrightarrow[n \rightarrow +\infty]{L^2(\mathbb{R}^N)} F_\alpha(u, v, \cdot) \mathbf{1}_{\widetilde{\mathcal{K}}_m}(\cdot). \quad (2.31)$$

Moreover, since  $\alpha \in (0, 2)$  and  $\widetilde{\mathcal{K}}_m$  is a compact subset of  $\mathbb{R}^N$ , using Hölder inequality, one can derive from (2.31) that, for all fixed  $m \in \mathbb{Z}_+$ ,

$$S_n^{(\alpha)}(u, v, \cdot) \mathbf{1}_{\widetilde{\mathcal{K}}_m}(\cdot) \xrightarrow[n \rightarrow +\infty]{L^\alpha(\mathbb{R}^N)} F_\alpha(u, v, \cdot) \mathbf{1}_{\widetilde{\mathcal{K}}_m}(\cdot). \quad (2.32)$$

On another hand, one knows from the result obtained in the first step that

$$S_n^{(\alpha)}(u, v, \cdot) \xrightarrow[n \rightarrow +\infty]{L^\alpha(\mathbb{R}^N)} \widetilde{F}_\alpha(u, v, \cdot),$$

and consequently that, for all fixed  $m \in \mathbb{Z}_+$ ,

$$S_n^{(\alpha)}(u, v, \cdot) \mathbb{1}_{\tilde{\mathcal{K}}_m}(\cdot) \xrightarrow[n \rightarrow +\infty]{L^\alpha(\mathbb{R}^N)} \tilde{F}_\alpha(u, v, \cdot) \mathbb{1}_{\tilde{\mathcal{K}}_m}(\cdot). \quad (2.33)$$

Finally, combining (2.32) and (2.33), it follows that, for all  $m \in \mathbb{Z}_+$  and for Lebesgue almost all  $\xi \in \mathbb{R}^N$ , one has  $\tilde{F}_\alpha(u, v, \xi) \mathbb{1}_{\tilde{\mathcal{K}}_m}(\xi) = F_\alpha(u, v, \xi) \mathbb{1}_{\tilde{\mathcal{K}}_m}(\xi)$ . Then, using the fact that  $\bigcup_{m \in \mathbb{Z}_+} \tilde{\mathcal{K}}_m = \mathbb{R}^N \setminus \{0\}$ , one gets, for Lebesgue almost all  $\xi \in \mathbb{R}^N$ , that  $\tilde{F}_\alpha(u, v, \xi) = F_\alpha(u, v, \xi)$ . ■

Now, we are going to show that the random series in (2.19) is convergent in a much stronger way than the one described in Theorem 2.5, to this end we will need the following very crucial lemma, whose long proof is postponed to Section 5 (the Appendix).

**Lemma 2.6** *For each  $\alpha \in (0, 2)$ , there is a universal event  $\Omega_\alpha^*$  of probability 1 for which one has what follows. For all  $\eta > 0$ , there exists a positive finite random variable  $C$ , only depending on  $\alpha$  and  $\eta$ , such that for all  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ , the following inequality, in which  $\lfloor \alpha \rfloor$  denotes the integer part of  $\alpha$ , holds on  $\Omega_\alpha^*$ :*

$$|\varepsilon_{\delta, j, k}^{(\alpha)}| \leq C(1 + |j|)^{\frac{1}{\alpha} + \eta} \log^{\frac{\lfloor \alpha \rfloor}{2}}(3 + |j| + |k|_1), \quad (2.34)$$

where  $|k|_1 := \sum_{r=1}^N |k_r|$ . Notice that, when  $\alpha \in (0, 1)$  is arbitrary, the logarithmic factor in the right-hand side of (2.34) disappears since  $\lfloor \alpha \rfloor = 0$ . On another hand, when  $\alpha \in [1, 2)$  is arbitrary,  $\vartheta > 0$  is an arbitrary finite fixed constant and one restricts to arbitrary  $j \in \mathbb{Z}_+$  and  $k \in \mathbb{Z}^N$  such that

$$|k|_\infty := \sup\{|k_1|, \dots, |k_N|\} \leq \vartheta 2^j, \quad (2.35)$$

then, for some positive finite random variable  $C'$  not depending  $(\delta, j, k)$ , the following significantly improved version of the inequality (2.34) holds on  $\Omega_\alpha^*$ :

$$|\varepsilon_{\delta, j, k}^{(\alpha)}| \leq C'(1 + j)^{\frac{1}{\alpha} + \eta}. \quad (2.36)$$

We mention in passing that the inequality (2.34) is reminiscent of the two inequalities (2.35) and (2.36) in [2]. Also, we mention that the inequality (2.36) is a generalization to  $k \in \mathbb{Z}^N$  of the inequality (1.11) in [6] for which  $k \in \mathbb{Z}$ .

The following proposition, whose proof relies on Lemma 2.6, shows that the random series in (2.19) is convergent in a much stronger way than the one described in Theorem 2.5.

**Proposition 2.7** *For all  $(u, v, \omega) \in \mathbb{R}^N \times (0, 1) \times \Omega_\alpha^*$ , the series of real numbers*

$$\sum_{(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N} 2^{-jv} [\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)] \varepsilon_{\delta, j, k}^{(\alpha)}(\omega) \quad (2.37)$$

*is absolutely convergent.*

Before proving Proposition 2.7, we mention that we will need in its proof, as well as in many other places in the remaining of our article, the inequality

$$\sqrt{\log(3 + |x| + |y|)} \leq \sqrt{\log(3 + |x|)}\sqrt{\log(3 + |y|)}, \quad \text{for all } (x, y) \in \mathbb{R}^2. \quad (2.38)$$

**Proof of Proposition 2.7** Let  $(u, v, \omega) \in \mathbb{R}^N \times (0, 1) \times \Omega_\alpha^*$  be arbitrary and fixed, one sets

$$\begin{aligned} Y(u, v, \omega) &:= \sum_{(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N} \left| 2^{-jv} [\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)] \right| |\varepsilon_{\delta, j, k}^{(\alpha)}(\omega)| \\ &= Y^-(u, v, \omega) + Y^+(u, v, \omega), \end{aligned} \quad (2.39)$$

where

$$Y^-(u, v, \omega) := \sum_{\delta \in \Upsilon_*} \sum_{(j, k) \in \mathbb{N} \times \mathbb{Z}^N} 2^{jv} |\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)| |\varepsilon_{\delta, -j, k}^{(\alpha)}(\omega)| \quad (2.40)$$

and

$$Y^+(u, v, \omega) := \sum_{\delta \in \Upsilon_*} \sum_{(j, k) \in \mathbb{Z}_+ \times \mathbb{Z}^N} 2^{-jv} |\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)| |\varepsilon_{\delta, j, k}^{(\alpha)}(\omega)|. \quad (2.41)$$

First, one focuses on  $Y^-(u, v, \omega)$ . One can derive from (2.24) with  $L > 1$ , (2.34) and (2.38) that, for some positive finite random variable  $C_1$ , one has

$$Y^-(u, v, \omega) \leq C_1(\omega) \left( \sum_{j=1}^{+\infty} 2^{j(v-1)} (1+j)^{\frac{1}{\alpha} + \eta} \log^{\frac{1}{2}}(3+j) \right) \left( \sum_{k \in \mathbb{Z}} \frac{\log^{\frac{1}{2}}(3 + |k|)}{(1 + |k|)^L} \right)^N < +\infty. \quad (2.42)$$

Let us now focus on  $Y^+(u, v, \omega)$ . One can derive from the triangle inequality that

$$Y^+(u, v, \omega) \leq A(u, v, \omega) + A(0, v, \omega), \quad (2.43)$$

where, for all  $(y, v, \omega) \in \mathbb{R}^N \times (0, 1) \times \Omega_\alpha^*$ ,

$$A(y, v, \omega) := \sum_{\delta \in \Upsilon_*} \sum_{j \in \mathbb{Z}_+} 2^{-jv} \sum_{k \in \mathbb{Z}^N} |\Psi_\delta^{(\alpha)}(2^j y - k, v)| |\varepsilon_{\delta, j, k}^{(\alpha)}(\omega)|. \quad (2.44)$$

Moreover, (2.17), (2.34) and (2.38) imply, for all  $(\delta, j) \in \Upsilon_* \times \mathbb{Z}_+$  and  $(y, v, \omega) \in \mathbb{R}^N \times (0, 1) \times$

$\Omega_\alpha^*$ , that

$$\begin{aligned}
& \sum_{k \in \mathbb{Z}^N} |\Psi_\delta^{(\alpha)}(2^j y - k, v)| |\varepsilon_{\delta, j, k}^{(\alpha)}(\omega)| \\
& \leq C_2(\omega) \sum_{k \in \mathbb{Z}^N} \left( \prod_{r=1}^N (2 + |2^j y_r - k_r|)^{-L} \right) (1+j)^{\frac{1}{\alpha} + \eta} \log^{\frac{1}{2}} \left( 3 + j + \sum_{r=1}^N |k_r| \right) \\
& = C_2(\omega) \sum_{k \in \mathbb{Z}^N} \left( \prod_{r=1}^N (2 + |2^j y_r - \lfloor 2^j y_r \rfloor - k_r|)^{-L} \right) (1+j)^{\frac{1}{\alpha} + \eta} \log^{\frac{1}{2}} \left( 3 + j + \sum_{r=1}^N |\lfloor 2^j y_r \rfloor + k_r| \right) \\
& \leq C_2(\omega) (1+j)^{\frac{1}{\alpha} + \eta} \sum_{k \in \mathbb{Z}^N} \left( \prod_{r=1}^N (1 + |k_r|)^{-L} \right) \log^{\frac{1}{2}} \left( 3 + j + N + \sum_{r=1}^N (|2^j y_r| + |k_r|) \right) \\
& \leq C_2(\omega) (1+j)^{\frac{1}{\alpha} + \eta} \log^{\frac{1}{2}} (N + 3 + j + 2^j |y|_1) \sum_{k \in \mathbb{Z}^N} \prod_{r=1}^N (1 + |k_r|)^{-L} \log^{\frac{1}{2}} (3 + |k_r|) \\
& = C_3(\omega) (1+j)^{\frac{1}{\alpha} + \eta} \log^{\frac{1}{2}} (N + 3 + j + 2^j |y|_1), \tag{2.45}
\end{aligned}$$

where  $L > 1$  is fixed, and the positive finite random variables  $C_2$  and  $C_3$  do not depend on  $j$ ,  $y$  and  $v$ . Combining (2.44) and (2.45), one obtains, for all  $(y, v, \omega) \in \mathbb{R}^N \times (0, 1) \times \Omega_\alpha^*$ , that  $A(y, v, \omega) < +\infty$ . Then, (2.43) entails that  $Y^+(u, v, \omega) < +\infty$ . Finally, combining the latter result with (2.42) and (2.39), one gets that  $Y(u, v, \omega) < +\infty$ , which shows that the proposition is satisfied.  $\blacksquare$

**Remark 2.8** *It results from Theorem 2.5 and Proposition 2.7 that the almost surely absolutely convergent random series in (2.37) provides a modification of the field*

$X = \{X(u, v), (u, v) \in \mathbb{R}^N \times (0, 1)\}$  *to which it will be systematically identified from now on. Thus,  $X$  can be expressed, for all  $(u, v, \omega) \in \mathbb{R}^N \times (0, 1) \times \Omega_\alpha^*$ , through the absolutely convergent series*

$$X(u, v, \omega) = \sum_{j=-\infty}^{+\infty} X_j(u, v, \omega), \tag{2.46}$$

where the  $S\alpha S$  stochastic fields  $X_j = \{X_j(u, v), (u, v) \in \mathbb{R}^N \times (0, 1)\}$ ,  $j \in \mathbb{Z}$ , are defined, for every  $(u, v, \omega) \in \mathbb{R}^N \times (0, 1) \times \Omega_\alpha^*$ , through the absolutely convergent series

$$X_j(u, v, \omega) := \sum_{(\delta, k) \in \mathbb{Y}_* \times \mathbb{Z}^N} 2^{-jv} [\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)] \varepsilon_{\delta, j, k}^{(\alpha)}(\omega). \tag{2.47}$$

Also,  $X$  can be expressed, for all  $(u, v, \omega) \in \mathbb{R}^N \times (0, 1) \times \Omega_\alpha^*$ , as

$$X(u, v, \omega) = X^-(u, v, \omega) + X^+(u, v, \omega). \tag{2.48}$$

The two S $\alpha$ S stochastic fields  $X^- = \{X^-(u, v), (u, v) \in \mathbb{R}^N \times (0, 1)\}$  and  $X^+ = \{X^+(u, v), (u, v) \in \mathbb{R}^N \times (0, 1)\}$  are respectively called the low frequency part and the high frequency part of  $X$ . They are defined, for every  $(u, v, \omega) \in \mathbb{R}^N \times (0, 1) \times \Omega_\alpha^*$ , through the absolutely convergent series

$$X^-(u, v, \omega) := \sum_{j=-\infty}^{-1} X_j(u, v, \omega) \quad (2.49)$$

and

$$X^+(u, v, \omega) := \sum_{j=0}^{+\infty} X_j(u, v, \omega). \quad (2.50)$$

Notice that, later we will show that the series in (2.46), (2.47), (2.49) and (2.50) are also normally convergent in  $(u, v)$  with respect to the uniform semi-norm on any compact box of  $\mathbb{R}^N \times (0, 1)$ .

We are now going to derive some results on sample paths regularity of the S $\alpha$ S stochastic fields introduced in Remark 2.8. To this end, we need the following definition.

**Definition 2.9** For any fixed real numbers  $\varrho > 0$  and  $0 < a < b < 1$ , we denote by  $Q_{\varrho, a, b}$  the compact box of  $\mathbb{R}^N \times (0, 1)$  defined as

$$Q_{\varrho, a, b} := [-\varrho, \varrho]^N \times [a, b]. \quad (2.51)$$

Moreover, for any real-valued function  $g$  defined on  $\mathbb{R}^N \times (0, 1)$ , we denote by  $\|g\|_{\varrho, a, b} = \|g(\bullet, \cdot)\|_{\varrho, a, b}$  the non-negative quantity (which may be infinite) defined as

$$\|g\|_{\varrho, a, b} = \|g(\bullet, \cdot)\|_{\varrho, a, b} := \sup_{(x, v) \in Q_{\varrho, a, b}} |g(x, v)|. \quad (2.52)$$

Also, we need the following lemma.

**Lemma 2.10** Let  $\varrho > 0$  and  $0 < a < b < 1$  be arbitrary and fixed. For all  $j \in \mathbb{Z}$ ,  $m \in \mathbb{Z}_+$ ,  $\gamma \in \mathbb{Z}_+^N$  and  $\omega \in \Omega_\alpha^*$  one has

$$\sum_{(\delta, k) \in \Upsilon_* \times \mathbb{Z}^N} \|(\partial_x^\gamma \partial_v^m \Psi_\delta^{(\alpha)})(\bullet - k, \cdot)\|_{\varrho, a, b} |\varepsilon_{\delta, j, k}^{(\alpha)}(\omega)| < +\infty. \quad (2.53)$$

Therefore, the S $\alpha$ S stochastic fields  $\mathcal{B}_j = \{\mathcal{B}_j(x, v), (x, v) \in \mathbb{R}^N \times (0, 1)\}$ ,  $j \in \mathbb{Z}$ , defined, for every  $(x, v, \omega) \in \mathbb{R}^N \times (0, 1) \times \Omega_\alpha^*$ , as

$$\mathcal{B}_j(x, v, \omega) := \sum_{(\delta, k) \in \Upsilon_* \times \mathbb{Z}^N} \Psi_\delta^{(\alpha)}(x - k, v) \varepsilon_{\delta, j, k}^{(\alpha)}(\omega), \quad (2.54)$$

have  $C^\infty$  sample paths on  $\mathbb{R}^N \times (0, 1)$ , which satisfy, for all  $j \in \mathbb{Z}$ ,  $m \in \mathbb{Z}_+$ ,  $\gamma \in \mathbb{Z}_+^N$  and  $(x, v, \omega) \in \mathbb{R}^N \times (0, 1) \times \Omega_\alpha^*$ ,

$$(\partial_x^\gamma \partial_v^m \mathcal{B}_j)(x, v, \omega) = \sum_{(\delta, k) \in \Upsilon_* \times \mathbb{Z}^N} (\partial_x^\gamma \partial_v^m \Psi_\delta^{(\alpha)})(x - k, v) \varepsilon_{\delta, j, k}^{(\alpha)}(\omega), \quad (2.55)$$

where the series is normally convergent in  $(x, v)$  with respect to the uniform semi-norm on each compact box  $\mathbb{R}^N \times (0, 1)$ . Moreover, for any fixed  $\eta > 0$ , there is a positive finite random variable  $C$ , which does not depend on  $j$ , such that, for every  $(j, \omega) \in \mathbb{Z} \times \Omega_\alpha^*$ , one has

$$\|(\partial_x^\gamma \partial_v^m \mathcal{B}_j)(\bullet, \cdot, \omega)\|_{2^j \varrho, a, b} \leq C(\omega)(1 + |j|)^{\frac{1}{\alpha} + \eta}. \quad (2.56)$$

**Proof** The inequality (2.53) easily follows from (2.52), (2.17) with  $L > 1$ ,  $M = 1$  and  $T = \varrho$ , (2.34) and (2.38). Observe that, since  $\varrho > 0$ ,  $0 < a < b < 1$ ,  $\gamma \in \mathbb{Z}_+^N$  and  $m \in \mathbb{Z}_+$  are arbitrary, (2.53) implies that the series in (2.54) and all its term by term partial derivatives of any order are uniformly convergent in  $(x, v)$  with respect to the uniform semi-norm on any compact box of  $\mathbb{R}^N \times (0, 1)$ . A consequence of the latter fact is that, for each  $j \in \mathbb{Z}$  and  $\omega \in \Omega_\alpha^*$ ,  $\mathcal{B}_j(\bullet, \cdot, \omega)$  is a  $C^\infty$  function on  $\mathbb{R}^N \times (0, 1)$  satisfying (2.55).

Let us now show that (2.56) holds. First, we assume that  $j < 0$ . Using (2.55), (2.17) with  $L > 1$ ,  $M = 1$  and  $T = \varrho$ , (2.34) with  $\eta$  replaced by  $\eta/2$ , and (2.38), it follows, for all  $(u, v) \in Q_{\varrho, a, b}$  and  $\omega \in \Omega_\alpha^*$ , that

$$\begin{aligned} & |(\partial_x^\gamma \partial_v^m \mathcal{B}_j)(2^j u, v, \omega)| \\ & \leq C_1(\omega)(1 + |j|)^{\frac{1}{\alpha} + \frac{\eta}{2}} \sum_{k \in \mathbb{Z}^N} \left( \prod_{r=1}^N (1 + \varrho + |2^j u_r - k_r|)^{-L} \right) \log^{\frac{1}{2}} \left( 3 + |j| + \sum_{r=1}^N |k_r| \right) \\ & \leq C_1(\omega)(1 + |j|)^{\frac{1}{\alpha} + \frac{\eta}{2}} \log^{\frac{1}{2}}(3 + |j|) \sum_{k \in \mathbb{Z}^N} \left( \prod_{r=1}^N (1 + \varrho + |k_r| - 2^j |u_r|)^{-L} \log^{\frac{1}{2}}(3 + |k_r|) \right) \\ & \leq C_2(\omega)(1 + |j|)^{\frac{1}{\alpha} + \eta} \sum_{k \in \mathbb{Z}^N} \left( \prod_{r=1}^N (1 + |k_r|)^{-L} \log^{\frac{1}{2}}(3 + |k_r|) \right) \\ & = C_3(\omega)(1 + |j|)^{\frac{1}{\alpha} + \eta}, \end{aligned} \quad (2.57)$$

where  $C_1$ ,  $C_2$  and  $C_3$  are positive finite random variables not depending on  $j$  and  $(u, v)$ . From now on, we assume that  $j \geq 0$ . Let  $\mathcal{K}_j^{(\varrho)}$  and  $\overline{\mathcal{K}}_j^{(\varrho)}$  be the two disjoint sets defined as

$$\mathcal{K}_j^{(\varrho)} := \{k \in \mathbb{Z}^N, |k|_\infty \leq 2^{j+1} \varrho\} \quad \text{and} \quad \overline{\mathcal{K}}_j^{(\varrho)} := \{k \in \mathbb{Z}^N, |k|_\infty > 2^{j+1} \varrho\}. \quad (2.58)$$

Then, one can derive from (2.55), the equality  $\mathbb{Z}^N = \mathcal{K}_j^{(\varrho)} \cup \overline{\mathcal{K}}_j^{(\varrho)}$  and the triangle inequality that, for all  $j \geq 0$ ,  $(u, v) \in Q_{\varrho, a, b}$  and  $\omega \in \Omega_\alpha^*$ ,

$$\begin{aligned} |(\partial_x^\gamma \partial_v^m \mathcal{B}_j)(2^j u, v, \omega)| &\leq \sum_{(\delta, k) \in \Upsilon_* \times \mathcal{K}_j^{(\varrho)}} |(\partial_x^\gamma \partial_v^m \Psi_\delta^{(\alpha)})(2^j u - k, v)| |\varepsilon_{\delta, j, k}^{(\alpha)}(\omega)| \\ &\quad + \sum_{(\delta, k) \in \Upsilon_* \times \overline{\mathcal{K}}_j^{(\varrho)}} |(\partial_x^\gamma \partial_v^m \Psi_\delta^{(\alpha)})(2^j u - k, v)| |\varepsilon_{\delta, j, k}^{(\alpha)}(\omega)|. \end{aligned} \quad (2.59)$$

Let us conveniently bound each one of the two sums in the right-hand side of (2.59). Observe that the inequality (2.36) is valid when  $k \in \mathcal{K}_j^{(\varrho)}$ , since the condition (2.35) holds with  $\vartheta = 2\rho$ . Thus, using (2.36), (2.17) with  $L > 1$ ,  $M = 1$  and  $T = 0$  and (2.26) with  $L\beta(\alpha)$  replaced by  $L > 1$ , one obtains that

$$\begin{aligned} &\sum_{(\delta, k) \in \Upsilon_* \times \mathcal{K}_j^{(\varrho)}} |(\partial_x^\gamma \partial_v^m \Psi_\delta^{(\alpha)})(2^j u - k, v)| |\varepsilon_{\delta, j, k}^{(\alpha)}(\omega)| \\ &\leq C_4(\omega)(1+j)^{\frac{1}{\alpha} + \eta} \sum_{k \in \mathcal{K}_j^{(\varrho)}} \left( \prod_{r=1}^N (1 + |2^j u_r - k_r|)^{-L} \right) \leq C_5(\omega)(1+j)^{\frac{1}{\alpha} + \eta}, \end{aligned} \quad (2.60)$$

where  $C_4$  and  $C_5$  are two positive finite random variables not depending on  $j$  and  $(u, v)$ . On another hand, using (2.34), (2.17) with  $L > 4$ ,  $M = 1$  and  $T = 0$ , and (2.38), one gets that

$$\begin{aligned} &\sum_{(\delta, k) \in \Upsilon_* \times \overline{\mathcal{K}}_j^{(\varrho)}} |(\partial_x^\gamma \partial_v^m \Psi_\delta^{(\alpha)})(2^j u - k, v)| |\varepsilon_{\delta, j, k}^{(\alpha)}(\omega)| \\ &\leq C_6(\omega)(1+j)^{\frac{1}{\alpha} + \eta} \log^{\frac{1}{2}}(3+j) \sum_{k \in \overline{\mathcal{K}}_j^{(\varrho)}} \left( \prod_{r=1}^N (1 + |2^j u_r - k_r|)^{-L} \log^{\frac{1}{2}}(3 + |k_r|) \right) \\ &\leq C_7(\omega)(1+j)^{\frac{1}{\alpha} + 2\eta} \sum_{n=1}^N \sum_{k \in \overline{\mathcal{K}}_{j, n}^{(\varrho)}} \left( \prod_{r=1}^N (1 + |2^j u_r - k_r|)^{-L} \log^{\frac{1}{2}}(3 + |k_r|) \right), \end{aligned} \quad (2.61)$$

where  $C_6$  and  $C_7$  are two positive finite random variables not depending on  $j$  and  $(u, v)$ , and

$$\overline{\mathcal{K}}_{j, n}^{(\varrho)} := \{k = (k_1, \dots, k_N) \in \mathbb{Z}^N, |k_n| > 2^{j+1} \varrho\}. \quad (2.62)$$

Moreover, using arguments rather similar to those which have allowed to derive (2.45), the fact that  $\sup_{1 \leq r \leq N} |u_r| \leq \varrho$ , the inequality  $L > 4$  and (2.62), one has, for every  $n \in \{1, \dots, N\}$ ,

that

$$\begin{aligned}
& \sum_{k \in \bar{\mathcal{K}}_{j,n}^{(\varrho)}} \left( \prod_{r=1}^N (1 + |2^j u_r - k_r|)^{-L} \log^{\frac{1}{2}}(3 + |k_r|) \right) \\
& \leq c_8 \log^{\frac{N-1}{2}}(4 + 2^j \varrho) \sum_{|k_n| > 2^{j+1} \varrho} (1 + |2^j u_n - k_n|)^{-L} \log^{\frac{1}{2}}(3 + |k_n|) \\
& \leq 2^L c_8 \log^{\frac{N-1}{2}}(4 + 2^j \varrho) \sum_{|k_n| > 2^{j+1} \varrho} (1 + |k_n|)^{-L} \log^{\frac{1}{2}}(3 + |k_n|) \\
& \leq c_9 \log^{\frac{N-1}{2}}(4 + 2^j \varrho) \sum_{k_n > 2^{j+1} \varrho} (1 + k_n)^{-L+1} \\
& \leq c_9 \log^{\frac{N-1}{2}}(4 + 2^j \varrho) \int_{2^{j+1} \varrho}^{+\infty} y^{-L+1} dy \leq c_{10} (1 + j)^{\frac{N-1}{2}} 2^{-j(L-2)} \leq c_{11} 2^{-2j}, \tag{2.63}
\end{aligned}$$

where  $c_8, \dots, c_{11}$  are finite deterministic constants not depending on  $j$  and  $(u, v)$ . Next, putting together (2.59) to (2.63), it follows that one has, for all  $j \geq 0$ ,  $(u, v) \in Q_{\varrho, a, b}$  and  $\omega \in \Omega_{\alpha}^*$ ,

$$|(\partial_x^\gamma \partial_v^m \mathcal{B}_j)(2^j u, v, \omega)| \leq C_{12}(\omega) (1 + j)^{\frac{1}{\alpha} + \eta}, \tag{2.64}$$

where  $C_{12}$  is a positive finite random variable not depending on  $j$  and  $(u, v)$ . Finally, (2.57) and (2.64) show that (2.56) holds.  $\blacksquare$

**Lemma 2.11** *In view of (2.54), the stochastic fields  $X_j$ ,  $j \in \mathbb{Z}$ , introduced in (2.47), can be expressed, for all  $j \in \mathbb{Z}$  and  $(u, v, \omega) \in \mathbb{R}^N \times (0, 1) \times \Omega_{\alpha}^*$ , as*

$$X_j(u, v, \omega) = 2^{-jv} (\mathcal{B}_j(2^j u, v, \omega) - \mathcal{B}_j(0, v, \omega)). \tag{2.65}$$

Their, sample paths are  $C^\infty$  functions on  $\mathbb{R}^N \times (0, 1)$  satisfying, for all  $j \in \mathbb{Z}$ ,  $(u, v, \omega) \in \mathbb{R}^N \times (0, 1) \times \Omega_{\alpha}^*$  and  $m \in \mathbb{Z}_+$ ,

$$(\partial_v^m X_j)(u, v, \omega) = \sum_{p=0}^m \binom{m}{p} (-j \log(2))^{m-p} 2^{-jv} ((\partial_v^p \mathcal{B}_j)(2^j u, v) - (\partial_v^p \mathcal{B}_j)(0, v)) \tag{2.66}$$

and, for every  $\gamma = (\gamma_1, \dots, \gamma_N) \in \mathbb{Z}_+^N \setminus \{0\}$ ,

$$(\partial_u^\gamma \partial_v^m X_j)(u, v, \omega) = \sum_{p=0}^m \binom{m}{p} (-j \log(2))^{m-p} 2^{j(\gamma_1 + \dots + \gamma_N - v)} (\partial_x^\gamma \partial_v^p \mathcal{B}_j)(2^j u, v), \tag{2.67}$$

with the convention that  $0^0 = 1$ .

**Proof** The equality (2.65) is a straightforward consequence of (2.54) and (2.47). The fact that, for each  $j \in \mathbb{Z}$ , sample paths of the field  $X_j$  are  $\mathcal{C}^\infty$  functions on  $\mathbb{R}^N \times (0, 1)$  satisfying (2.66) and (2.67) easily results from (2.65), Lemma 2.10 and the general Leibniz rule for calculating the derivative of any order  $m$  of the product of two  $m$ -times differentiable functions.  $\blacksquare$

**Remark 2.12** Let  $\varrho > 0$ ,  $0 < a < b < 1$ ,  $p \in \mathbb{Z}_+$  and  $\omega \in \Omega_\alpha^*$  be arbitrary and fixed. One can derive from the mean value theorem that, for all  $j \in \mathbb{Z}$ ,  $(u^{(1)}, u^{(2)}) \in [-\varrho, \varrho]^N \times [-\varrho, \varrho]^N$  and  $v \in [a, b]$ ,

$$\begin{aligned} & |(\partial_v^p \mathcal{B}_j)(2^j u^{(1)}, v, \omega) - (\partial_v^p \mathcal{B}_j)(2^j u^{(2)}, v, \omega)| \\ & \leq 2^j |u^{(1)} - u^{(2)}|_1 \sup_{1 \leq n \leq N} \|(\partial_{x_n} \partial_v^p \mathcal{B}_j)(\bullet, \cdot, \omega)\|_{2^j \varrho, a, b}, \end{aligned} \quad (2.68)$$

where, for any  $n \in \{1, \dots, N\}$ ,  $\partial_{x_n}$  is the partial derivative operator of order 1 with respect to the  $n$ th coordinate of the variable  $x \in \mathbb{R}^N$ . Then (2.68) and (2.56) imply that, for any fixed  $\eta > 0$ ,

$$|(\partial_v^p \mathcal{B}_j)(2^j u^{(1)}, v, \omega) - (\partial_v^p \mathcal{B}_j)(2^j u^{(2)}, v, \omega)| \leq C(\omega) 2^j (1 + |j|)^{\frac{1}{\alpha} + \eta} |u^{(1)} - u^{(2)}|_1, \quad (2.69)$$

where the positive finite random variable  $C$  does not depend on  $j$  and  $(u^{(1)}, u^{(2)})$ . Moreover, setting  $u^{(2)} = 0$  and  $C'(\omega) := N \varrho C$ , it results from (2.69) that

$$\|(\partial_v^p \mathcal{B}_j)(\bullet, \cdot, \omega) - (\partial_v^p \mathcal{B}_j)(0, \cdot, \omega)\|_{2^j \varrho, a, b} \leq C'(\omega) 2^j (1 + |j|)^{\frac{1}{\alpha} + \eta}. \quad (2.70)$$

**Proposition 2.13** For all  $m \in \mathbb{Z}_+$ ,  $\gamma \in \mathbb{Z}_+^N$ ,  $\varrho > 0$ ,  $0 < a < b < 1$ , and  $\omega \in \Omega_\alpha^*$ , one has

$$\sum_{j=-\infty}^{-1} \|(\partial_u^\gamma \partial_v^m X_j)(\bullet, \cdot, \omega)\|_{\varrho, a, b} < +\infty. \quad (2.71)$$

Therefore the S $\alpha$ S stochastic field  $X^-$ , called the low frequency part of the field  $X$  and defined through (2.49), has  $\mathcal{C}^\infty$  sample paths on  $\mathbb{R}^N \times (0, 1)$ , which satisfy, for all  $\gamma \in \mathbb{Z}_+^N$ ,  $m \in \mathbb{Z}_+$  and  $(u, v, \omega) \in \mathbb{R}^N \times (0, 1) \times \Omega_\alpha^*$ ,

$$(\partial_u^\gamma \partial_v^m X^-)(u, v, \omega) = \sum_{j=-\infty}^{-1} (\partial_u^\gamma \partial_v^m X_j)(u, v, \omega), \quad (2.72)$$

where the series is normally convergent in  $(u, v)$  with respect to the uniform semi-norm on each compact box of  $\mathbb{R}^N \times (0, 1)$ .

**Proof** The inequality (2.71) easily results from (2.52), (2.66), (2.67), (2.70) and (2.56). It clearly follows from (2.71) that, for all  $\gamma \in \mathbb{Z}_+^N$  and  $m \in \mathbb{Z}_+$ , the series in the right-hand side of (2.72) is normally convergent in  $(u, v)$  with respect to the uniform semi-norm on each compact box of  $\mathbb{R}^N \times (0, 1)$ . Then, one can derive from (2.49) that, for every  $\omega \in \Omega_\alpha^*$ ,  $X^-(\bullet, \cdot, \omega)$  is a  $C^\infty$  function on  $\mathbb{R}^N \times (0, 1)$  whose partial derivatives of any order are given by (2.72).  $\blacksquare$

**Proposition 2.14** For all  $m \in \mathbb{Z}_+$ ,  $\varrho > 0$ ,  $0 < a < b < 1$ , and  $\omega \in \Omega_\alpha^*$ , one has

$$\sum_{j=0}^{+\infty} \|(\partial_v^m X_j)(\bullet, \cdot, \omega)\|_{\varrho, a, b} < +\infty. \quad (2.73)$$

Therefore the SoS stochastic field  $X^+$ , called the high frequency part of the field  $X$  and defined through (2.50), has continuous sample paths on  $\mathbb{R}^N \times (0, 1)$  which also are, for every fixed  $u \in \mathbb{R}^N$ , infinitely differentiable in the variable  $v \in (0, 1)$ . Moreover, for each  $m \in \mathbb{Z}_+$  and  $\omega \in \Omega_\alpha^*$ , the function  $(\partial_v^m X^+)(\bullet, \cdot, \omega)$  is continuous on  $\mathbb{R}^N \times (0, 1)$  and satisfies, for all  $(u, v) \in \mathbb{R}^N \times (0, 1)$ ,

$$(\partial_v^m X^+)(u, v, \omega) = \sum_{j=0}^{+\infty} (\partial_v^m X_j)(u, v, \omega), \quad (2.74)$$

where the series is normally convergent in  $(u, v)$  with respect to the uniform semi-norm on each compact box of  $\mathbb{R}^N \times (0, 1)$ .

**Proof** The inequality (2.73) easily results from (2.52), (2.66), the triangle inequality and (2.56). It clearly follows from (2.73) that, for all  $m \in \mathbb{Z}_+$ , the series in the right-hand side of (2.74) is normally convergent in  $(u, v)$  with respect to the uniform semi-norm on each compact box of  $\mathbb{R}^N \times (0, 1)$ . Then, one can derive from (2.50) that, for every  $\omega \in \Omega_\alpha^*$ ,  $(u, v) \mapsto X^+(u, v, \omega)$  is a continuous function on  $\mathbb{R}^N \times (0, 1)$ , which is also, for each fixed  $u \in \mathbb{R}^N$ , infinitely differentiable in the variable  $v \in (0, 1)$ , and that, for every  $m \in \mathbb{Z}_+$ ,  $(u, v) \mapsto (\partial_v^m X^+)(u, v, \omega)$  is a continuous function on  $\mathbb{R}^N \times (0, 1)$  satisfying (2.74).  $\blacksquare$

**Remark 2.15** A straightforward consequence of Proposition 2.14 and the mean value theorem is that, for any fixed  $m \in \mathbb{Z}_+$ ,  $\varrho > 0$ ,  $0 < a < b < 1$  and  $\omega \in \Omega_\alpha^*$ , one has

$$\sup_{u \in [-\varrho, \varrho]^N} |(\partial_v^m X^+)(u, v_1, \omega) - (\partial_v^m X^+)(u, v_2, \omega)| \leq C(\omega) |v_1 - v_2|, \quad \text{for all } (v_1, v_2) \in [a, b]^2, \quad (2.75)$$

where  $C$  is a positive finite random variable only depending on  $m$ ,  $\varrho$ ,  $a$  and  $b$ .

The following theorem is a straightforward consequence of Remark 2.8 and Propositions 2.13 and 2.14.

**Theorem 2.16** *The S $\alpha$ S stochastic field  $X$ , called the field generating HMSF and defined through (2.46) or through (1.7) (recall that the two modifications of  $X$  provided by (2.46) and (1.7) are always identified), has continuous sample paths on  $\mathbb{R}^N \times (0, 1)$  which also are, for every fixed  $u \in \mathbb{R}^N$ , infinitely differentiable in the variable  $v \in (0, 1)$ . Moreover, for each  $m \in \mathbb{Z}_+$  and  $\omega \in \Omega_\alpha^*$ , the function  $(\partial_v^m X)(\bullet, \cdot, \omega)$  is continuous on  $\mathbb{R}^N \times (0, 1)$  and satisfies, for all  $(u, v) \in \mathbb{R}^N \times (0, 1)$ ,*

$$(\partial_v^m X)(u, v, \omega) = \sum_{j=-\infty}^{+\infty} (\partial_v^m X_j)(u, v, \omega) = (\partial_v^m X^-)(u, v, \omega) + (\partial_v^m X^+)(u, v, \omega), \quad (2.76)$$

where the series is normally convergent in  $(u, v)$  with respect to the uniform semi-norm on each compact box of  $\mathbb{R}^N \times (0, 1)$ .

**Remark 2.17** *A straightforward consequence of Theorem 2.16 and the mean value theorem is that, for any fixed  $m \in \mathbb{Z}_+$ ,  $\varrho > 0$ ,  $0 < a < b < 1$  and  $\omega \in \Omega_\alpha^*$ , one has*

$$\sup_{u \in [-\varrho, \varrho]^N} |(\partial_v^m X)(u, v_1, \omega) - (\partial_v^m X)(u, v_2, \omega)| \leq C(\omega)|v_1 - v_2|, \quad \text{for all } (v_1, v_2) \in [a, b]^2, \quad (2.77)$$

where  $C$  is a positive finite random variable only depending on  $m$ ,  $\varrho$ ,  $a$  and  $b$ .

**Corollary 2.18** *A sufficient condition for the HMSF  $Z = \{Z(t), t \in \mathbb{R}^N\}$ , defined through (1.6), to have almost surely continuous sample paths on  $\mathbb{R}^N$  is that the Hurst function  $H(\cdot)$  be continuous on  $\mathbb{R}^N$ . Moreover, when  $H(\cdot)$  is discontinuous at some point  $\tau \in \mathbb{R}^N \setminus \{0\}$ , then, with probability 1, sample paths of  $Z$  are discontinuous functions at  $\tau$ .*

**Proof** In view of (1.6) and Theorem 2.16, it is clear that the continuity of  $H(\cdot)$  on  $\mathbb{R}^N$  is a sufficient condition for having, almost surely, the continuity of sample paths of  $Z$  on  $\mathbb{R}^N$ . On another hand, when  $H(\cdot)$  is discontinuous at some point  $\tau \in \mathbb{R}^N \setminus \{0\}$ , there necessarily exist two sequences  $(t'_n)_{n \in \mathbb{N}}$  and  $(t''_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}^N$  such that

$$\lim_{n \rightarrow +\infty} t'_n = \lim_{n \rightarrow +\infty} t''_n = \tau \quad \text{and} \quad H' := \lim_{n \rightarrow +\infty} H(t'_n) \neq H'' := \lim_{n \rightarrow +\infty} H(t''_n), \quad (2.78)$$

where  $H'$  and  $H''$  are in the compact interval  $[\underline{H}, \overline{H}] \subset (0, 1)$  to which all the values of the function  $H(\cdot)$  belong. Then, one can derive from (1.6), Theorem 2.16 and (2.78) that one has, almost surely,

$$\lim_{n \rightarrow +\infty} Z(t'_n) = X(\tau, H') \quad \text{and} \quad \lim_{n \rightarrow +\infty} Z(t''_n) = X(\tau, H''). \quad (2.79)$$

Moreover, since  $\tau \neq 0$  and  $H' \neq H''$ , using (1.7), (1.8) and (1.3), it can be shown that the S $\alpha$ S random variable  $X(\tau, H') - X(\tau, H'')$  has a non-vanishing scale parameter, which implies that  $X(\tau, H') \neq X(\tau, H'')$  almost surely. Combining the latter fact with (2.78) and (2.79), it follows that sample paths of  $Z$  are with probability 1 discontinuous functions  $\tau$ .  $\blacksquare$

### 3 Results on path behavior

First, we state the main results of the section and then we give their proofs. In their statements we use the conventions that  $0 \times (\pm\infty) = 0$  and  $0/0$  is some bounded quantity which in fact does not need to be specified any more. All of them are stated in terms of the  $\ell^1$  norm  $|y|_1 := \sum_{r=1}^N |y_r|$  on  $\mathbb{R}^N$  and remain valid for any other norm on  $\mathbb{R}^N$ . All of them hold on the event  $\Omega_\alpha^*$  of probability 1 which was introduced in Lemma 2.6.

#### Theorem 3.1 (Global modulus of continuity for the field $\partial_v^m X$ )

Let  $m \in \mathbb{Z}_+$ ,  $\varrho > 0$  and  $0 < a < b < 1$  be arbitrary and fixed, and let  $Q_{\varrho, a, b}$  be as in (2.51). Then, one has on the event  $\Omega_\alpha^*$  of probability 1, for all  $\eta > 0$ ,

$$\sup_{(u^{(1)}, v_1), (u^{(2)}, v_2) \in Q_{\varrho, a, b}} \frac{|(\partial_v^m X)(u^{(1)}, v_1) - (\partial_v^m X)(u^{(2)}, v_2)|}{|u^{(1)} - u^{(2)}|_1^{v_1 \vee v_2} \log^{\frac{1}{\alpha} + \eta + m} (1 + |u^{(1)} - u^{(2)}|_1^{-1}) + |v_1 - v_2|} < +\infty, \quad (3.1)$$

where  $v_1 \vee v_2 := \sup\{v_1, v_2\}$ .

#### Corollary 3.2 (Global modulus of continuity for the HMSF $Z$ )

Let  $I$  be an arbitrary non-empty fixed compact box of  $\mathbb{R}^N$ , and let

$$\underline{H}(I) := \min_{t \in I} H(t). \quad (3.2)$$

Assume that the continuous Hurst function  $H(\cdot)$  satisfies, for some finite constant  $c$ ,

$$|H(t^{(1)}) - H(t^{(2)})| \leq c |t^{(1)} - t^{(2)}|_1^{\underline{H}(I)} \log^{\frac{1}{\alpha}} (1 + |t^{(1)} - t^{(2)}|_1^{-1}), \quad \text{for all } (t^{(1)}, t^{(2)}) \in I^2. \quad (3.3)$$

Then, on the event  $\Omega_\alpha^*$  of probability 1, one has, for all  $\eta > 0$ ,

$$\sup_{(t^{(1)}, t^{(2)}) \in I^2} \frac{|Z(t^{(1)}) - Z(t^{(2)})|}{|t^{(1)} - t^{(2)}|_1^{\underline{H}(I)} \log^{\frac{1}{\alpha} + \eta} (1 + |t^{(1)} - t^{(2)}|_1^{-1})} < +\infty. \quad (3.4)$$

#### Corollary 3.3 (Pointwise modulus of continuity for the HFSM $Z$ )

Let  $\tau \in \mathbb{R}^N$  be an arbitrary fixed point. Assume that the continuous Hurst function  $H(\cdot)$

satisfies, for some finite constant  $c$  (which may depend on  $\tau$ ) and for all point  $t$  of  $\mathbb{R}^N$  in a neighborhood of  $\tau$  (or equivalently for all  $t \in \mathbb{R}^N$ ),

$$|H(t) - H(\tau)| \leq c|t - \tau|_1^{H(\tau)} \log^{\frac{1}{\alpha}}(1 + |t - \tau|_1^{-1}). \quad (3.5)$$

Then, on the event  $\Omega_\alpha^*$  of probability 1, one has, for all  $\varrho > 0$  and  $\eta > 0$ ,

$$\sup_{|t - \tau|_1 \leq \varrho} \frac{|Z(t) - Z(\tau)|}{|t - \tau|_1^{H(\tau)} \log^{\frac{1}{\alpha} + \eta}(1 + |t - \tau|_1^{-1})} < +\infty. \quad (3.6)$$

**Theorem 3.4 (Estimate of the asymptotic behavior at infinity of the field  $\partial_v^m X$ )**

Let  $m \in \mathbb{Z}_+$ ,  $\varrho > 0$  and  $0 < a < b < 1$  be arbitrary and fixed, and let  $\mathcal{C}_\varrho := \{x \in \mathbb{R}^N, |x|_1 \geq \varrho\}$ . Then, on the event  $\Omega_\alpha^*$  of probability 1, one has, for all  $\eta > 0$ ,

$$\sup_{(u,v) \in \mathcal{C}_\varrho \times [a,b]} \frac{|(\partial_v^m X)(u, v)|}{|u|_1^v \log^{\frac{1}{\alpha} + \eta + m}(1 + |u|_1)} < +\infty. \quad (3.7)$$

**Corollary 3.5 (Estimate of the asymptotic behavior at infinity of the HMSF  $Z$ )**

For any fixed  $\varrho > 0$ , let  $\mathcal{C}_\varrho$  be as in Theorem 3.4. Then on the event  $\Omega_\alpha^*$  of probability 1, one has, for all  $\eta > 0$ ,

$$\sup_{t \in \mathcal{C}_\varrho} \frac{|Z(t)|}{|t|_1^{H(t)} \log^{\frac{1}{\alpha} + \eta}(1 + |t|_1)} < +\infty. \quad (3.8)$$

Moreover, when the Hurst function  $H(\cdot)$  has a (finite) limit at infinity to which it converges at a logarithmic rate, that is there are two finite constants  $H_\infty \in [\underline{H}, \overline{H}] \subset (0, 1)$  and  $c > 0$  such that

$$|H(t) - H_\infty| \leq c \left( \log(3 + |t|_1) \right)^{-1}, \quad \text{for all } t \in \mathbb{R}^N. \quad (3.9)$$

Then, (3.8) can equivalently be reformulated as: on the event  $\Omega_\alpha^*$  of probability 1, one has, for all  $\eta > 0$ ,

$$\sup_{t \in \mathcal{C}_\varrho} \frac{|Z(t)|}{|t|_1^{H_\infty} \log^{\frac{1}{\alpha} + \eta}(1 + |t|_1)} < +\infty. \quad (3.10)$$

**Proof of Theorem 3.1** Let  $m \in \mathbb{Z}_+$  be arbitrary and fixed. Recall that, one knows from Proposition 2.13 that, on the event  $\Omega_\alpha^*$ , the random function  $(\partial_v^m X^-)(\bullet, \cdot) : (u, v) \mapsto (\partial_v^m X^-)(u, v)$  is  $\mathcal{C}^\infty$  on  $\mathbb{R}^N \times (0, 1)$ , which clearly implies that (3.1) holds when  $\partial_v^m X$  in it is replaced by  $\partial_v^m X^-$ . Thus, in view of (2.48), it is enough to show that (3.1) holds when  $\partial_v^m X$  in it is replaced by  $\partial_v^m X^+$ .

Let  $(u^{(1)}, v_1), (u^{(2)}, v_2) \in Q_{\rho, a, b}$  be arbitrary, there is no restriction to assume that  $0 < |u^{(1)} - u^{(2)}| \leq 1$  and  $v_1 \vee v_2 = v_1$ . Using the triangle inequality and (2.75) one gets, on the event  $\Omega_\alpha^*$ , that

$$\begin{aligned} & |(\partial_v^m X^+)(u^{(1)}, v_1) - (\partial_v^m X^+)(u^{(2)}, v_2)| \\ & \leq |(\partial_v^m X^+)(u^{(1)}, v_1) - (\partial_v^m X^+)(u^{(2)}, v_1)| + |(\partial_v^m X^+)(u^{(2)}, v_1) - (\partial_v^m X^+)(u^{(2)}, v_2)| \\ & \leq |(\partial_v^m X^+)(u^{(1)}, v_1) - (\partial_v^m X^+)(u^{(2)}, v_1)| + C_1 |v_1 - v_2|, \end{aligned} \quad (3.11)$$

where  $C_1$  is a positive finite random variable not depending on  $(u^{(1)}, v_1)$  and  $(u^{(2)}, v_2)$ . In view of (3.11) and of the fact that  $v_1 \vee v_2 = v_1$ , it turns out that for proving that (3.1) holds when  $\partial_v^m X$  in it is replaced by  $\partial_v^m X^+$ , it is enough to show that, for some positive finite random variable  $C_2$ , not depending on  $(u^{(1)}, v_1)$  and  $(u^{(2)}, v_2)$ , one has on  $\Omega_\alpha^*$ ,

$$|(\partial_v^m X^+)(u^{(1)}, v_1) - (\partial_v^m X^+)(u^{(2)}, v_1)| \leq C_2 |u^{(1)} - u^{(2)}|^{v_1} \log_\alpha^{\frac{1}{\alpha} + \eta + m} (1 + |u^{(1)} - u^{(2)}|_1^{-1}). \quad (3.12)$$

Since  $0 < |u^{(1)} - u^{(2)}| \leq 1$  there is a unique  $j_0 \in \mathbb{Z}_+$  satisfying

$$2^{-(j_0+1)} < |u^{(1)} - u^{(2)}|_1 \leq 2^{-j_0}. \quad (3.13)$$

In other words,  $j_0$  is the unique non-negative integer such that

$$j_0 \leq \frac{\log(|u^{(1)} - u^{(2)}|_1^{-1})}{\log(2)} < j_0 + 1. \quad (3.14)$$

Next, notice that, using (2.74) and the triangle inequality one has that

$$|(\partial_v^m X^+)(u^{(1)}, v_1) - (\partial_v^m X^+)(u^{(2)}, v_1)| \leq R_{j_0}(u^{(1)}, u^{(2)}, v_1) + S_{j_0}(u^{(1)}, u^{(2)}, v_1), \quad (3.15)$$

where

$$R_{j_0}(u^{(1)}, u^{(2)}, v_1) := \sum_{j=0}^{j_0} |(\partial_v^m X_j)(u^{(1)}, v_1) - (\partial_v^m X_j)(u^{(2)}, v_1)| \quad (3.16)$$

and

$$S_{j_0}(u^{(1)}, u^{(2)}, v_1) := \sum_{j=j_0+1}^{+\infty} |(\partial_v^m X_j)(u^{(1)}, v_1) - (\partial_v^m X_j)(u^{(2)}, v_1)|. \quad (3.17)$$

One can derive from (3.16), (2.66), the triangle inequality, (2.69), the inequalities  $v \leq b < 1$ , (3.13) and (3.14) that

$$\begin{aligned} R_{j_0}(u^{(1)}, u^{(2)}, v_1) & \leq C_3 |u^{(1)} - u^{(2)}|_1 \sum_{j=0}^{j_0} 2^{j(1-v)} (1+j)^{m+\frac{1}{\alpha}+\eta} \\ & \leq (2^{1-b} - 1)^{-1} C_3 |u^{(1)} - u^{(2)}|_1 2^{(j_0+1)(1-v)} (1+j_0)^{m+\frac{1}{\alpha}+\eta} \\ & \leq C_4 |u^{(1)} - u^{(2)}|^{v_1} \log_\alpha^{\frac{1}{\alpha} + \eta + m} (1 + |u^{(1)} - u^{(2)}|_1^{-1}), \end{aligned} \quad (3.18)$$

where the positive finite random variables  $C_3$  and  $C_4$  do not depend on  $(u^{(1)}, v_1)$ ,  $(u^{(2)}, v_2)$  and  $j_0$ . Moreover, it follows from (3.17), (2.66), the triangle inequality, (2.56), the inequalities  $1 < a \leq v$ , (3.13) and (3.14) that

$$\begin{aligned} S_{j_0}(u^{(1)}, u^{(2)}, v_1) &\leq C_5 \sum_{j=j_0+1}^{+\infty} 2^{-jv} (1+j)^{m+\frac{1}{\alpha}+\eta} \\ &\leq C_5 2^{-(j_0+1)v} (1+j_0)^{m+\frac{1}{\alpha}+\eta} \sum_{p=0}^{+\infty} 2^{-pa} \left(1 + \frac{1+p}{1+j_0}\right)^{m+\frac{1}{\alpha}+\eta} \\ &\leq C_6 |u^{(1)} - u^{(2)}|^{v_1} \log^{\frac{1}{\alpha}+\eta+m} (1 + |u^{(1)} - u^{(2)}|_1^{-1}), \end{aligned} \quad (3.19)$$

where the positive finite random variables  $C_3$  and  $C_4$  do not depend on  $(u^{(1)}, v_1)$ ,  $(u^{(2)}, v_2)$  and  $j_0$ . Finally, combining (3.18) and (3.19) with (3.15), one obtains (3.12). ■

**Proof of Corollary 3.2** Using (1.6) and Theorem 3.1 with  $m = 0$ ,  $a = \underline{H}$ ,  $b = \overline{H}$ , a fixed  $\varrho \geq 1$  such that  $I \subset [-\varrho, \varrho]^N$ , and any fixed  $\eta > 0$ , it follows that, for some positive finite random variable  $C_1$  one has, on the event  $\Omega_\alpha^*$ , for all  $(t^{(1)}, t^{(2)}) \in I^2$ ,

$$\begin{aligned} &|Z((t^{(1)})) - Z(t^{(2)})| \quad (3.20) \\ &\leq C_1 \left( |t^{(1)} - t^{(2)}|_1^{H(t^{(1)}) \vee H(t^{(2)})} \log^{\frac{1}{\alpha}+\eta} (1 + |t^{(1)} - t^{(2)}|_1^{-1}) + |H(t^{(1)}) - H(t^{(2)})| \right). \end{aligned}$$

Moreover, since  $(2N\varrho)^{-1} |t^{(1)} - t^{(2)}|_1 \leq 1$ , one can derive from (3.2) that

$$\begin{aligned} &|t^{(1)} - t^{(2)}|_1^{H(t^{(1)}) \vee H(t^{(2)})} = (2N\varrho)^{H(t^{(1)}) \vee H(t^{(2)})} \left( (2N\varrho)^{-1} |t^{(1)} - t^{(2)}|_1 \right)^{H(t^{(1)}) \vee H(t^{(2)})} \\ &\leq (2N\varrho)^{\overline{H}-\underline{H}} |t^{(1)} - t^{(2)}|^{\underline{H}(I)}. \end{aligned} \quad (3.21)$$

Finally combining (3.20) and (3.21) with (3.3), one obtains (3.4). ■

**Proof of Corollary 3.3** The proof can be done similarly to that of Corollary 3.2. ■

Our next goal is to show that Theorem 3.4 holds. In fact this theorem is a straightforward consequence of the following more technical proposition.

**Proposition 3.6** *Let  $\Psi_\delta^{(\alpha)}$ ,  $\delta \in \Upsilon_*$ , be the same functions as in Definition 2.2. For each  $(m, q) \in \mathbb{Z}_+^2$ ,  $j \in \mathbb{Z}$ ,  $\eta > 0$  and  $(u, v) \in \mathbb{R}^N \times (0, 1)$ , one sets*

$$\begin{aligned} \mathcal{A}_{\eta, j}^{m, q}(u, v) &:= 2^{-jv} (1 + |j|)^{m+\frac{1}{\alpha}+\eta} \quad (3.22) \\ &\times \sum_{(\delta, k) \in \Upsilon_* \times \mathbb{Z}^N} |(\partial_v^q \Psi_\delta^{(\alpha)})(2^j u - k, v) - (\partial_v^q \Psi_\delta^{(\alpha)})(-k, v)| \log^{\frac{1}{2}} (3 + |k|_1), \end{aligned}$$

$$\mathcal{A}_\eta^{m,q}(u,v) := \sum_{j \in \mathbb{Z}} \mathcal{A}_{\eta,j}^{m,q}(u,v) \quad (3.23)$$

and

$$\mathcal{A}_\eta^m(u,v) := \sum_{q=0}^m \mathcal{A}_\eta^{m,q}(u,v). \quad (3.24)$$

Then, for each  $m \in \mathbb{Z}_+$  and  $\eta > 0$ , there is a positive finite random variable  $C$ , such that, on the event  $\Omega_\alpha^*$  of probability 1, one has

$$|(\partial_v^m X)(u,v)| \leq C \mathcal{A}_\eta^m(u,v), \quad \text{for all } (u,v) \in \mathbb{R}^N \times (0,1). \quad (3.25)$$

Moreover, for all  $m \in \mathbb{Z}_+$ ,  $\eta > 0$ ,  $\varrho > 0$  and  $0 < a < b < 1$ , the following inequality holds:

$$\sup_{(u,v) \in \mathcal{C}_\varrho \times [a,b]} \frac{\mathcal{A}_\eta^m(u,v)}{|u|_1^v \log^{\frac{1}{\alpha} + \eta + m}(1 + |u|_1)} < +\infty. \quad (3.26)$$

In order to show that Proposition 3.6 is satisfied, we need the following lemma.

**Lemma 3.7** For each  $(m,q) \in \mathbb{Z}_+^2$ , there is a finite constant  $c$  such that, for all  $\eta > 0$ ,  $(u,v) \in \mathbb{R}^N \times (0,1)$  and  $j \in \mathbb{Z}$ , one has

$$\mathcal{A}_{\eta,j}^{m,q}(u,v) \leq c 2^{j(1-v)} (1 + |j|)^{m + \frac{1}{\alpha} + \eta} |u|_1, \quad \text{when } 2^j |u|_1 \leq 1, \quad (3.27)$$

and

$$\mathcal{A}_{\eta,j}^{m,q}(u,v) \leq c 2^{-jv} (1 + |j|)^{m + \frac{1}{\alpha} + \eta} \log^{\frac{1}{2}}(3 + 2^j |u|_1), \quad \text{when } 2^j |u|_1 > 1. \quad (3.28)$$

**Proof** Throughout the proof,  $q \in \mathbb{Z}_+$  and  $(u,v) \in \mathbb{R}^N \times (0,1)$  are arbitrary and fixed. First we show that (3.27) is satisfied. So, let  $j \in \mathbb{Z}$  be arbitrary and such that

$$2^j |u|_1 \leq 1. \quad (3.29)$$

Using the mean value theorem, one has, for all  $\delta \in \Upsilon_*$ ,  $k \in \mathbb{Z}^N$  and  $q \in \mathbb{Z}_+$ , and for some  $\theta \in (0,1)$ ,

$$|(\partial_v^q \Psi_\delta^{(\alpha)})(2^j u - k, v) - (\partial_v^q \Psi_\delta^{(\alpha)})(-k, v)| \leq 2^j |u|_1 \sup_{1 \leq n \leq N} |(\partial_{x_n} \partial_v^q \Psi_\delta^{(\alpha)})(2^j \theta u - k, v)|. \quad (3.30)$$

Then, one can derive from (2.17) with  $L = 2$ ,  $M = 1$  and  $T = 1$ , and from (3.29) and (3.30) that

$$|(\partial_v^q \Psi_\delta^{(\alpha)})(2^j u - k, v) - (\partial_v^q \Psi_\delta^{(\alpha)})(-k, v)| \leq c_1 2^j |u|_1 \prod_{r=1}^N (1 + |k_r|)^{-2}, \quad (3.31)$$

where the finite constant  $c_1$  does not depend on  $\delta$ ,  $j$ ,  $k$  and  $(u, v)$ . Then, it follows from (3.22) and (3.31) that (3.27) holds.

Let us now show that (3.28) is satisfied. Similarly to (2.45), it can be shown that

$$\sum_{(\delta, k) \in \Upsilon_* \times \mathbb{Z}^N} |(\partial_v^q \Psi_\delta^{(\alpha)})(2^j u - k, v)| \log^{\frac{1}{2}}(3 + |k|_1) \leq c_2 \log^{\frac{1}{2}}(N + 3 + 2^j |u|_1), \quad (3.32)$$

where the finite constant  $c_2$  does not depend on  $\delta$ ,  $j$  and  $(u, v)$ . Then, it results from (3.22), the triangle inequality and (3.32) (which clearly remains valid when  $u = 0$ ) that (3.28) holds. ■

**Proof of Proposition 3.6** There is no serious difficulty in the proof of the inequality (3.25). It can be obtained through standard calculations using the first equality in (2.76), (2.66), (2.55), the triangle inequality, (2.34) with  $\eta$  replaced by  $\eta/2$ , (2.38), (3.22), (3.23) and (3.24).

From now on, one focuses on the proof of (3.26). Let  $(u, v) \in \mathcal{C}_\rho \times [a, b]$  be arbitrary and fixed, there is no restriction to assume that  $|u|_1 \geq 3$ . Thus, there exists a unique positive integer  $j_1$  such that

$$2^{j_1} < |u|_1 \leq 2^{j_1+1}, \quad (3.33)$$

in other words,  $-j_1 - 1$  is the largest integer  $j$  such that  $2^j |u|_1 \leq 1$ . Thus, one can derive from (3.27) and (3.33) that

$$\begin{aligned} \sum_{j=-\infty}^{-j_1-1} \sum_{q=0}^m \mathcal{A}_{\eta, j}^{m, q}(u, v) &\leq c_1 |u|_1 \sum_{j=-\infty}^{-j_1-1} 2^{j(1-v)} (1 + |j|)^{m + \frac{1}{\alpha} + \eta} \\ &= c_1 |u|_1 \sum_{j=j_1+1}^{+\infty} 2^{-j(1-v)} (1 + j)^{m + \frac{1}{\alpha} + \eta} \\ &\leq c_1 |u|_1 2^{-(j_1+1)(1-v)} (1 + j_1)^{m + \frac{1}{\alpha} + \eta} \sum_{p=0}^{+\infty} 2^{-p(1-b)} \left(1 + \frac{1+p}{1+j_1}\right)^{m + \frac{1}{\alpha} + \eta} \\ &\leq c_2 |u|_1 2^{-(j_1+1)(1-v)} (1 + j_1)^{m + \frac{1}{\alpha} + \eta} \leq c_3 |u|_1^v \log^{\frac{1}{\alpha} + \eta + m}(1 + |u|_1), \end{aligned} \quad (3.34)$$

where the finite constants  $c_1, \dots, c_3$  do not depend on  $(u, v)$  and  $j_1$ . On another hand, it

results from (3.28) and (3.33) that

$$\begin{aligned}
\sum_{j=-j_1}^{-1} \sum_{q=0}^m \mathcal{A}_{\eta,j}^{m,q}(u,v) &\leq c_4 \sum_{j=-j_1}^{-1} 2^{-jv} (1+|j|)^{m+\frac{1}{\alpha}+\eta} \log^{\frac{1}{2}}(3+2^j|u|_1) \\
&= c_4 \sum_{j=1}^{j_1} 2^{jv} (1+j)^{m+\frac{1}{\alpha}+\eta} \log^{\frac{1}{2}}(3+2^{-j}|u|_1) \\
&\leq 2^b c_4 |u|^v \sum_{j=1}^{j_1} 2^{-(j_1+1-j)v} (1+j)^{m+\frac{1}{\alpha}+\eta} \log^{\frac{1}{2}}(3+2^{j_1+1-j}) \\
&= 2^b c_4 |u|^v \sum_{j=1}^{j_1} 2^{-jv} (2+j_1-j)^{m+\frac{1}{\alpha}+\eta} \log^{\frac{1}{2}}(3+2^j) \\
&\leq 2^b c_4 |u|^v (1+j_1)^{m+\frac{1}{\alpha}+\eta} \sum_{j=1}^{+\infty} 2^{-ja} \log^{\frac{1}{2}}(3+2^j) \\
&\leq c_5 |u|_1^v \log^{\frac{1}{\alpha}+\eta+m}(1+|u|_1), \tag{3.35}
\end{aligned}$$

where the finite constants  $c_4$  and  $c_5$  do not depend on  $(u, v)$  and  $j_1$ . Also, it follows from (3.28), (3.33) and the inequality  $\log^{\frac{1}{2}}(3+2^j|u|_1) \leq \log^{\frac{1}{2}}(3+2^j) + \log^{\frac{1}{2}}(3+|u|_1)$ , that

$$\sum_{j=0}^{+\infty} \sum_{q=0}^m \mathcal{A}_{\eta,j}^{m,q}(u,v) \leq c_6 \sum_{j=0}^{+\infty} 2^{-ja} (1+j)^{m+\frac{1}{\alpha}+\eta} \log^{\frac{1}{2}}(3+2^j|u|_1) \leq c_7 \log^{\frac{1}{2}}(3+|u|_1), \tag{3.36}$$

where the finite constants  $c_6$  and  $c_7$  do not depend on  $(u, v)$  and  $j_1$ . Finally, putting together (3.24), (3.23), (3.34), (3.35) and (3.36), one obtains (3.26).  $\blacksquare$

**Proof of Corollary 3.5** The inequality (3.8) is a straightforward consequence of (1.6) and of Theorem 3.4 with  $m = 0$ ,  $a = \underline{H}$  and  $b = \overline{H}$ . Let us show that the two inequalities (3.8) and (3.10) are equivalent when the condition (3.9) holds. There is no restriction to assume that  $\varrho \geq 1$ . Then, for any  $t \in \mathcal{C}_\varrho$ , using the inequalities  $|t|_1 \geq \varrho \geq 1$  and the equality  $|t|_1^{H(t)} = |t|_1^{(H(t)-H_\infty)+H_\infty}$ , one obtains that

$$(|t|_1^{-|H(t)-H_\infty|})|t|_1^{H_\infty} \leq |t|_1^{H(t)} \leq (|t|_1^{|H(t)-H_\infty|})|t|_1^{H_\infty}. \tag{3.37}$$

Moreover, one can derive from the condition (3.9) and the inequality  $\log(|t|_1) \geq 0$ , that

$$|t|_1^{-|H(t)-H_\infty|} := \exp\left(-\log(|t|_1)|H(t)-H_\infty|\right) \geq \exp\left(-c_1 \frac{\log(|t|_1)}{\log(3+|t|_1)}\right) \geq c_2 > 0 \tag{3.38}$$

and

$$|t|_1^{|H(t)-H_\infty|} := \exp\left(\log(|t|_1)|H(t)-H_\infty|\right) \leq \exp\left(c_1 \frac{\log(|t|_1)}{\log(3+|t|_1)}\right) \leq c_3 < +\infty, \tag{3.39}$$

where the positive finite constant  $c_1$  denotes the constant  $c$  in (3.9), and the two positive finite constants  $c_2$  and  $c_3$  do not depend on  $t$ . Combining (3.37) with (3.38) and (3.39), it follows that, for any fixed  $\eta > 0$ , and for all  $t \in \mathcal{C}_\varrho$ ,

$$c_2 |t|_1^{H_\infty} \log^{\frac{1}{\alpha} + \eta}(1 + |t|_1) \leq |t|_1^{H(t)} \log^{\frac{1}{\alpha} + \eta}(1 + |t|_1) \leq c_3 |t|_1^{H_\infty} \log^{\frac{1}{\alpha} + \eta}(1 + |t|_1),$$

which shows that the two inequalities (3.8) and (3.10) are equivalent when the condition (3.9) holds.  $\blacksquare$

## 4 Study of optimality of results on path behavior

It seems natural to wonder whether the results on sample path behavior of the HMSF  $Z$ , provided by Corollaries 3.2, 3.3 and 3.5 of the previous section, are optimal. Studying this issue is the main goal of the present section. First we state the main results of the section, and then we give their proofs. All of them are stated in terms of the  $\ell^1$  norm  $|y|_1 := \sum_{r=1}^N |y_r|$  on  $\mathbb{R}^N$  and remain valid for any other norm on  $\mathbb{R}^N$ .

**Theorem 4.1** *For any fixed  $(\tilde{u}, \tilde{v}) \in \mathbb{R}^N \times (0, 1)$ , there exists an event  $\tilde{\Omega}_{\alpha, \tilde{u}} \subset \Omega_\alpha^*$  of probability 1, which depends on  $\alpha$  and  $\tilde{u}$  but not on  $\tilde{v}$ , such that, on  $\tilde{\Omega}_{\alpha, \tilde{u}}$ , one has*

$$\limsup_{u \rightarrow \tilde{u}} \frac{|X(u, \tilde{v}) - X(\tilde{u}, \tilde{v})|}{|u - \tilde{u}|_1 \tilde{v} \log^{\frac{1}{\alpha}}(1 + |u - \tilde{u}|_1^{-1})} = +\infty. \quad (4.1)$$

Recall that, for any arbitrary real-valued function  $f$  defined on  $\mathbb{R}^N \setminus \{\tilde{u}\}$ ,

$$\limsup_{u \rightarrow \tilde{u}} f(u) := \lim_{\varrho \rightarrow 0^+} \left( \sup \left\{ f(u), u \in \mathbb{R}^N \setminus \{\tilde{u}\} \text{ and } |\tilde{u} - u|_1 \leq \varrho \right\} \right). \quad (4.2)$$

The following corollary shows that the pointwise modulus of continuity for HMSF at an arbitrary point  $\tau \in \mathbb{R}^N$  satisfying the condition (3.5), provided by Corollary 3.3, is optimal on an event of probability 1 depending on  $\tau$ .

**Corollary 4.2** *For any fixed  $\tau \in \mathbb{R}^N$  satisfying the condition (3.5), there exists an event  $\tilde{\Omega}_{\alpha, \tau} \subset \Omega_\alpha^*$  of probability 1, which depends on  $\alpha$  and  $\tau$  such that, on  $\tilde{\Omega}_{\alpha, \tau}$ , one has*

$$\limsup_{t \rightarrow \tau} \frac{|Z(t) - Z(\tau)|}{|t - \tau|_1^{H(\tau)} \log^{\frac{1}{\alpha}}(1 + |t - \tau|_1^{-1})} = +\infty. \quad (4.3)$$

The following corollary shows, under the additional weak condition (4.4), that the global modulus of continuity for HMSF on an arbitrary compact box of  $\mathbb{R}^N$ , provided by Corollary 3.2, is optimal.

**Corollary 4.3** *Let  $I$  be an arbitrary non-empty compact box of  $\mathbb{R}^N$ , and let  $\underline{H}(I)$  be as in (3.2). The topological interior of  $I$  is denoted by  $\mathring{I}$ , and one assumes that there exists some point  $\tau^{(0)} \in \mathring{I}$  such that*

$$H(\tau^{(0)}) = \underline{H}(I) := \min_{t \in I} H(t). \quad (4.4)$$

*Also, one assumes that the condition (3.3) holds. Then, one has almost surely,*

$$\sup_{(t^{(1)}, t^{(2)}) \in I^2} \frac{|Z(t^{(1)}) - Z(t^{(2)})|}{|t^{(1)} - t^{(2)}|_1^{\frac{H(I)}{1}} \log^{\frac{1}{\alpha}}(1 + |t^{(1)} - t^{(2)}|_1^{-1})} = +\infty. \quad (4.5)$$

**Theorem 4.4** *There exists a universal event  $\tilde{\Omega}_\alpha \subset \Omega_\alpha^*$  of probability 1, which does not depend on  $\tilde{v}$ , such that, on  $\tilde{\Omega}_\alpha$ , one has, for all  $\tilde{v} \in (0, 1)$ ,*

$$\limsup_{|u|_1 \rightarrow +\infty} \frac{|X(u, \tilde{v})|}{|u|_1^{\tilde{v}} \log^{\frac{1}{\alpha}}(1 + |u|_1)} = +\infty. \quad (4.6)$$

*Recall that, for any arbitrary real-valued function  $f$  defined on  $\mathbb{R}^N \setminus \{0\}$ ,*

$$\limsup_{|u|_1 \rightarrow +\infty} f(u) := \lim_{\varrho \rightarrow +\infty} \left( \sup \left\{ f(u), u \in \mathbb{R}^N \text{ and } |u|_1 \geq \varrho \right\} \right). \quad (4.7)$$

The following corollary shows that the bound for the behavior at infinity of HMSF, provided by (3.10) in Corollary 3.5, is optimal when the Hurst function  $H(\cdot)$  satisfies a bit stronger condition than the one in (3.9).

**Corollary 4.5** *Assume that there are three finite constants  $H_\infty \in [\underline{H}, \overline{H}]$ ,  $\eta_\infty > 0$  and  $c > 0$  such that*

$$|H(t) - H_\infty| \leq c \left( \log(3 + |t|_1) \right)^{-1-\eta_\infty}, \quad \text{for all } t \in \mathbb{R}^N. \quad (4.8)$$

*Then, on the event  $\tilde{\Omega}_\alpha$  of probability 1, introduced in Theorem 4.4, one has*

$$\limsup_{|t|_1 \rightarrow +\infty} \frac{|Z(t)|}{|t|_1^{H_\infty} \log^{\frac{1}{\alpha}}(1 + |t|_1)} = +\infty. \quad (4.9)$$

**Remark 4.6** *Assume that the condition (3.5) is satisfied for all  $\tau \in \mathbb{R}^N$ , which is, for instance, clearly the case when  $H(\cdot)$  is a locally Lipschitz function on  $\mathbb{R}^N$ . Then, in view of (1.1), Corollaries 3.3 and 4.2 imply that (1.4) remains valid when the MBF  $Z_2$  in it is replaced by an HMSF  $Z$  with an arbitrary parameter  $\alpha \in (0, 2)$ . Notice that Corollary 3.3 further implies that  $\mathbb{P}(\forall \tau \in \mathbb{R}^N, \rho_Z(\tau) \geq H(\tau)) = 1$ , since the event  $\Omega_\alpha^*$  of probability 1, on which (3.6) is valid, does not depend on  $\tau$ . Thus, for proving that (1.5) keeps valid when  $Z_2$  in it is replaced by  $Z$ , which is one of the main motivations of our article, it remains to show that*

$$\mathbb{P}(\forall \tau \in \mathbb{R}^N, \rho_Z(\tau) \leq H(\tau)) = 1. \quad (4.10)$$

*The latter equality will be a consequence of the corollary of the following theorem.*

**Theorem 4.7** *There exists a universal event  $\widehat{\Omega}_\alpha \subset \Omega_\alpha^*$  of probability 1, which does not depend on  $(\tilde{u}, \tilde{v})$ , such that, on  $\widehat{\Omega}_\alpha$ , one has, for all  $(\tilde{u}, \tilde{v}) \in \mathbb{R}^N \times (0, 1)$ ,*

$$\limsup_{u \rightarrow \tilde{u}} \frac{|X(u, \tilde{v}) - X(\tilde{u}, \tilde{v})|}{|u - \tilde{u}|_1^{\tilde{v}}} \geq \widehat{c}(\tilde{v}) > 0, \quad (4.11)$$

where  $\widehat{c}(\tilde{v})$  is a positive finite deterministic constant which depends on  $\tilde{v}$  but not on  $\tilde{u}$ .

The following corollary shows that the pointwise modulus of continuity for HMSF at an arbitrary point  $\tau \in \mathbb{R}^N$  satisfying a bit stronger condition than (3.5), provided by Corollary 3.3, is quasi-optimal (that is optimal up to a logarithmic factor) on the universal event  $\widehat{\Omega}_\alpha$  of probability 1 which does not depend on  $\tau$ .

**Corollary 4.8** *Let  $\widehat{\Omega}_\alpha$  be the same universal event of probability 1 as in Theorem 4.7. Then, for all point  $\tau \in \mathbb{R}^N$  satisfying the condition*

$$\lim_{t \rightarrow \tau} \frac{|H(t) - H(\tau)|}{|t - \tau|_1^{H(\tau)}} = 0, \quad (4.12)$$

which is a bit stronger than the condition (3.5), one has on  $\widehat{\Omega}_\alpha$ ,

$$\lim_{t \rightarrow \tau} \frac{|Z(t) - Z(\tau)|}{|t - \tau|_1^{H(\tau)}} \geq \widehat{c}(H(\tau)) > 0, \quad (4.13)$$

where  $\widehat{c}(H(\tau))$  is the positive finite deterministic constant  $\widehat{c}(\tilde{v})$ , introduced in Theorem 4.7, with  $\tilde{v} = H(\tau)$ .

**Remark 4.9** *Notice that Corollary 4.8 is a strictly stronger result than the one mentioned in (4.10), under the assumption that the Hurst function  $H(\cdot)$  is a locally Lipschitz function on  $\mathbb{R}^N$ . Indeed, under the latter assumption, or more generally when  $H(\cdot)$  is a locally Hölder function on  $\mathbb{R}^N$  of any arbitrary order  $\gamma \in (\bar{H}, 1]$ , recall that  $\bar{H} := \sup_{t \in \mathbb{R}^N} H(t) < 1$ , then the condition (4.12) is satisfied by all point  $\tau \in \mathbb{R}^N$ , thus (4.10) results from Corollary 4.8 and (1.1).*

From now on, our goal is to prove the main results of the section that we have stated. To this end, we need to introduce the functions  $\widetilde{\Psi}_\delta^{(\alpha)}$ ,  $\delta \in \Upsilon_*$ , and to derive some lemmas related to them.

**Definition 4.10** *Let  $\Psi_\delta^{(\alpha)}$ ,  $\delta \in \Upsilon_*$ , be the functions which were introduced in Definition 2.2. The real-valued  $C^\infty$  functions  $\widetilde{\Psi}_\delta^{(\alpha)}$ ,  $\delta \in \Upsilon_*$ , on  $\mathbb{R}^N \times \mathbb{R}$  are defined, for all  $(x, v) \in \mathbb{R}^N \times \mathbb{R}$ , as*

$$\widetilde{\Psi}_\delta^{(\alpha)}(x, v) := (2\pi)^{-N} \Psi_\delta^{(\alpha)}(x, -v - 2N/\alpha) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{ix \cdot \eta} |\eta|_2^{v + \frac{N}{\alpha}} \widehat{\psi}_\delta(\eta) d\eta. \quad (4.14)$$

**Remark 4.11** One knows from (2.17) that there is a finite constant  $c$  such that

$$|\tilde{\Psi}_\delta^{(\alpha)}(x, v)| \leq c \prod_{r=1}^N (1 + |x_r|)^{-4}, \quad \text{for all } (\delta, x, v) \in \Upsilon_* \times \mathbb{R}^N \times [0, 1]. \quad (4.15)$$

Also, one knows from (4.14), (2.5) and a basic property of Fourier transform and inverse Fourier transform on the Schwartz space  $\mathcal{S}(\mathbb{R}^N)$ , that  $\int_{\mathbb{R}^N} e^{-i\xi \cdot x} \tilde{\Psi}_\delta^{(\alpha)}(x, v) dx = |\xi|_2^{v + \frac{N}{\alpha}} \widehat{\psi}_\delta(\xi)$ , for all  $\xi \in \mathbb{R}^N$  and  $v \in \mathbb{R}$ , and consequently that

$$\int_{\mathbb{R}^N} \tilde{\Psi}_\delta^{(\alpha)}(x, v) dx = 0, \quad \text{for all } v \in \mathbb{R}. \quad (4.16)$$

Moreover, similarly to Part (i) of Proposition 5.13 [1], using (2.10), (4.14), Plancherel Theorem and the fact that the Meyer wavelets  $\psi_{\delta, j, k}$ ,  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ , defined in (2.1), are orthonormal, it can be shown that, for all  $v \in \mathbb{R}$ ,  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$  and  $(\delta', j', k') \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ ,

$$2^{jN} \int_{\mathbb{R}^N} \tilde{\Psi}_\delta^{(\alpha)}(2^j u - k, v) \Psi_{\delta'}^{(\alpha)}(2^{j'} u - k', v) du = \begin{cases} 1, & \text{if } (\delta, j, k) = (\delta', j', k'), \\ 0, & \text{else.} \end{cases} \quad (4.17)$$

The following proposition provides a nice Lebesgue integral expression for the SaS random variables  $\varepsilon_{\delta, j, k}^{(\alpha)}$ ,  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ , introduced in (2.18).

**Proposition 4.12** On the event  $\Omega_\alpha^*$  of probability 1, one has, for all  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$  and  $v \in (0, 1)$ ,

$$\varepsilon_{\delta, j, k}^{(\alpha)} = 2^{j(N+v)} \int_{\mathbb{R}^N} \tilde{\Psi}_\delta^{(\alpha)}(2^j u - k, v) X(u, v) du. \quad (4.18)$$

**Proof** Let  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ ,  $v \in (0, 1)$  and  $\omega \in \Omega_\alpha^*$  be arbitrary and fixed. Observe that, in view of (4.15), the continuity on  $\mathbb{R}^N$  of the function  $u \mapsto \tilde{\Psi}_\delta^{(\alpha)}(2^j u - k, v) X(u, v, \omega)$  and Theorem 3.4, the Lebesgue integral in (4.18) is well-defined and finite. Next, for each  $u \in \mathbb{R}^N$  and  $n \in \mathbb{N}$ , let  $X_n(u, v)$  be the same random variable as in (2.20). Observe that, in view of (4.17) and (4.16), for proving the lemma it is enough to show that

$$\int_{\mathbb{R}^N} \tilde{\Psi}_\delta^{(\alpha)}(2^j u - k, v) X(u, v, \omega) du = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \tilde{\Psi}_\delta^{(\alpha)}(2^j u - k, v) X_n(u, v, \omega) du. \quad (4.19)$$

The equality (4.19) can be obtained by using the Lebesgue dominated convergence Theorem, since one knows from Proposition 2.7 that

$$X(u, v, \omega) = \lim_{n \rightarrow +\infty} X_n(u, v, \omega),$$

and one knows from (4.15) and Proposition 3.6 with  $m = 0$ , that the function  $u \mapsto |\tilde{\Psi}_\delta^{(\alpha)}(2^j u - k, v) X_n(u, v, \omega)|$  can be bounded from above, uniformly in  $n \in \mathbb{N}$ , by a positive function not depending on  $n$ , which is Lebesgue integrable over  $\mathbb{R}^N$  in the variable  $u$ .  $\blacksquare$

The following proposition, which will be one of the two main ingredients of the proof of Theorem 4.1, provides a lower bound for the lim sup in (4.1) in terms of some of the random variables  $\varepsilon_{\delta, j, k}^{(\alpha)}$ .

**Proposition 4.13** *For any fixed  $(\tilde{u}, \tilde{v}) \in \mathbb{R}^N \times (0, 1)$ , there is a positive finite deterministic constant  $c(\tilde{v})$ , only depending on  $\tilde{v}$ , such that on the event  $\Omega_\alpha^*$  of probability 1, one has, for all  $\delta \in \Upsilon_*$ ,*

$$\limsup_{j \rightarrow +\infty} \frac{|\varepsilon_{\delta, j, [2^j \tilde{u}]}^{(\alpha)}|}{(1+j)^{\frac{1}{\alpha}}} \leq c(\tilde{v}) \limsup_{u \rightarrow \tilde{u}} \frac{|X(u, \tilde{v}) - X(\tilde{u}, \tilde{v})|}{|u - \tilde{u}|_1^{\frac{1}{\alpha}} \log^{\frac{1}{\alpha}}(1 + |u - \tilde{u}|_1^{-1})}, \quad (4.20)$$

where

$$[2^j \tilde{u}] := ([2^j \tilde{u}_1], \dots, [2^j \tilde{u}_N]), \quad (4.21)$$

Recall that  $[\cdot]$  is the integer part function and that, for any sequence  $(w_j)_{j \in \mathbb{Z}_+}$  of real numbers,

$$\limsup_{j \rightarrow +\infty} w_j := \lim_{J \rightarrow +\infty} \left( \sup_{j \geq J} w_j \right). \quad (4.22)$$

For proving Proposition 4.13, we need the following lemma.

**Lemma 4.14** *The real numbers  $0 < a < b < 1$  and the non-empty compact  $I' \subset \mathbb{R}^N$  are arbitrary and fixed. For all  $\omega \in \Omega_\alpha^*$  and  $\delta \in \Upsilon_*$ , one has*

$$\sup \left\{ \theta^2 \int_{\{|y|_1 > \theta\}} |X(u + \theta^{-2} y, v, \omega)| |\tilde{\Psi}_\delta^{(\alpha)}(y, v)| dy, (\theta, u, v) \in [1, +\infty) \times I' \times [a, b] \right\} < +\infty. \quad (4.23)$$

**Proof** Let  $\omega \in \Omega_\alpha^*$ ,  $\delta \in \Upsilon_*$ , a non-empty compact  $I' \subset \mathbb{R}^N$  and two real numbers  $0 < a < b < 1$  be arbitrary and fixed. Observe that, one knows from the continuity on  $\mathbb{R}^N \times [a, b]$  of the function  $X(\bullet, \cdot, \omega)$ , Theorem 3.4 with  $m = 0$ , and (2.38), that there exists a positive finite random variable  $C_1$ , such that, for all  $(z, v) \in \mathbb{R}^N \times [a, b]$ ,

$$|X(z, v, \omega)| \leq C_1(\omega) \prod_{r=1}^N (1 + |z_r|).$$

Thus, setting  $C_2(\omega) := C_1(\omega)(1 + \sup_{u \in I'} |u|_\infty)^N < +\infty$ , one obtains that

$$|X(u + \theta^{-2}y, v, \omega)| \leq C_2(\omega) \prod_{r=1}^N (1 + |y_r|), \quad \text{for all } (y, \theta, u, v) \in \mathbb{R}^N \times [1, +\infty) \times I' \times [a, b].$$

Then, it results from (4.15) and the inequality  $N|y|_\infty \geq |y|_1$  that

$$\begin{aligned} \int_{\{|y|_1 > \theta\}} |X(u + \theta^{-2}y, v, \omega)| |\tilde{\Psi}_\delta^{(\alpha)}(y, v)| dy &\leq C_3(\omega) \int_{\{N|y|_\infty > \theta\}} \prod_{r=1}^N (1 + |y_r|)^{-3} dy \\ &\leq 2NC_3(\omega) \left( \int_{\mathbb{R}} (1 + |r|)^{-3} dr \right)^{N-1} \int_{\theta/N}^{+\infty} (1 + s)^{-3} ds \leq C_4(\omega) \theta^{-2}, \end{aligned} \quad (4.24)$$

where  $C_3$  and  $C_4$  are two positive finite random variables not depending on  $(\theta, u, v)$ . It clearly follows from (4.24) that (4.23) holds.  $\blacksquare$

**Proof of Proposition 4.13** First observe that, using (4.18), the change of  $y = 2^j u - \lfloor 2^j \tilde{u} \rfloor$  and (4.16), one has on the event  $\Omega_\alpha^*$ , for all  $j \in \mathbb{Z}_+$ ,

$$\varepsilon_{\delta, j, \lfloor 2^j \tilde{u} \rfloor}^{(\alpha)} = 2^{j\tilde{v}} \int_{\mathbb{R}^N} \tilde{\Psi}_\delta^{(\alpha)}(y, \tilde{v}) \left( X(2^{-j} \lfloor 2^j \tilde{u} \rfloor + 2^{-j}y, \tilde{v}) - X(2^{-j} \lfloor 2^j \tilde{u} \rfloor, \tilde{v}) \right) dy. \quad (4.25)$$

Also observe that, one can easily derives from (4.21) that

$$|2^{-j} \lfloor 2^j \tilde{u} \rfloor - \tilde{u}|_1 \leq N2^{-j}, \quad \text{for all } j \in \mathbb{Z}_+, \quad (4.26)$$

which implies, for all  $j \in \mathbb{Z}_+$ , that  $2^{-j} \lfloor 2^j \tilde{u} \rfloor \in I' := \{z \in \mathbb{R}^N, |z - \tilde{u}|_1 \leq N\}$ . Since  $I'$  is a non-empty compact subset of  $\mathbb{R}^N$  not depending on  $j$ , using the triangle inequality, Lemma 4.14, the fact that  $\sup_{z \in I'} |X(z, \tilde{v})| < +\infty$  (which results from the continuity property of the field  $X$ ) and (4.15), one gets on  $\Omega_\alpha^*$ , for some positive finite random variable  $C_1$  and for every  $j \in \mathbb{Z}_+$ , that

$$\int_{\{|y|_1 > 2^{j/2}\}} |\tilde{\Psi}_\delta^{(\alpha)}(y, \tilde{v})| \left| X(2^{-j} \lfloor 2^j \tilde{u} \rfloor + 2^{-j}y, \tilde{v}) - X(2^{-j} \lfloor 2^j \tilde{u} \rfloor, \tilde{v}) \right| dy \leq C_1 2^{-j}.$$

Then, it results from (4.25) that on  $\Omega_\alpha^*$ , one has, for all  $j \in \mathbb{Z}_+$ ,

$$\begin{aligned} |\varepsilon_{\delta, j, \lfloor 2^j \tilde{u} \rfloor}^{(\alpha)}| &\leq 2^{j\tilde{v}} \int_{\{|y|_1 \leq 2^{j/2}\}} |\tilde{\Psi}_\delta^{(\alpha)}(y, \tilde{v})| \left| X(2^{-j} \lfloor 2^j \tilde{u} \rfloor + 2^{-j}y, \tilde{v}) - X(2^{-j} \lfloor 2^j \tilde{u} \rfloor, \tilde{v}) \right| dy \\ &\quad + C_1 2^{-j(1-\tilde{v})}. \end{aligned} \quad (4.27)$$

In order to conveniently bound the integral in the right-hand side of (4.27), one introduces, on  $\Omega_\alpha^*$ , the positive non-decreasing random function  $\mathcal{M}_{\tilde{u}, \tilde{v}}^{(X)}$ , defined for each  $\varrho \in (0, +\infty)$ , as

$$\mathcal{M}_{\tilde{u}, \tilde{v}}^{(X)}(\varrho) := \sup \left\{ \frac{|X(z, \tilde{v}) - X(\tilde{u}, \tilde{v})|}{|z - \tilde{u}|_1^{\tilde{v}} \log^{\frac{1}{\alpha}}(1 + |z - \tilde{u}|_1^{-1})}, \quad 0 < |z - \tilde{u}|_1 \leq \varrho \right\}. \quad (4.28)$$

Then, setting  $c_2 := N + 1$ , one can derive from the triangle inequality, (4.28) and (4.26), that on  $\Omega_\alpha^*$ , one has, for all  $j \in \mathbb{Z}_+$  and  $y \in \mathbb{R}^N$  satisfying  $|y|_1 \leq 2^{j/2}$ ,

$$\begin{aligned}
& 2^{j\tilde{v}} \left| X(2^{-j} \lfloor 2^j \tilde{u} \rfloor + 2^{-j} y, \tilde{v}) - X(2^{-j} \lfloor 2^j \tilde{u} \rfloor, \tilde{v}) \right| \\
& \leq 2^{j\tilde{v}} \left( \left| X(2^{-j} \lfloor 2^j \tilde{u} \rfloor + 2^{-j} y, \tilde{v}) - X(\tilde{u}, \tilde{v}) \right| + \left| X(2^{-j} \lfloor 2^j \tilde{u} \rfloor, \tilde{v}) - X(\tilde{u}, \tilde{v}) \right| \right) \\
& \leq 2^{j\tilde{v}} \mathcal{M}_{\tilde{u}, \tilde{v}}^{(X)}(c_2 2^{-j/2}) \left( \left| 2^{-j} y + 2^{-j} \lfloor 2^j \tilde{u} \rfloor - \tilde{u} \right|_1^{\tilde{v}} \log^{\frac{1}{\alpha}} \left( 1 + \left| 2^{-j} y + 2^{-j} \lfloor 2^j \tilde{u} \rfloor - \tilde{u} \right|_1^{-1} \right) \right. \\
& \quad \left. + \left| 2^{-j} \lfloor 2^j \tilde{u} \rfloor - \tilde{u} \right|_1^{\tilde{v}} \log^{\frac{1}{\alpha}} \left( 1 + \left| 2^{-j} \lfloor 2^j \tilde{u} \rfloor - \tilde{u} \right|_1^{-1} \right) \right) \quad (4.29) \\
& \leq (1 + j)^{\frac{1}{\alpha}} \mathcal{M}_{\tilde{u}, \tilde{v}}^{(X)}(c_2 2^{-j/2}) \left( \left| y + \lfloor 2^j \tilde{u} \rfloor - 2^j \tilde{u} \right|_1^{\tilde{v}} \log^{\frac{1}{\alpha}} \left( 3 + \left| y + \lfloor 2^j \tilde{u} \rfloor - 2^j \tilde{u} \right|_1^{-1} \right) + c_3 \right),
\end{aligned}$$

where the positive finite constant  $c_3 := \sup_{|w|_1 \leq 3N} |w|_1^{\tilde{v}} \log^{\frac{1}{\alpha}}(3 + |w|_1^{-1})$ . Moreover, setting

$$c_4 := c_3 \int_{\mathbb{R}^N} |\tilde{\Psi}_\delta^{(\alpha)}(y, \tilde{v})| dy < +\infty, \quad (4.30)$$

and using (4.26), one has, for every  $j \in \mathbb{Z}_+$ , that

$$\begin{aligned}
& \int_{\{|y|_1 \leq 2^{j/2}\}} |\tilde{\Psi}_\delta^{(\alpha)}(y, \tilde{v})| \left| y + \lfloor 2^j \tilde{u} \rfloor - 2^j \tilde{u} \right|_1^{\tilde{v}} \log^{\frac{1}{\alpha}} \left( 3 + \left| y + \lfloor 2^j \tilde{u} \rfloor - 2^j \tilde{u} \right|_1^{-1} \right) dy \\
& \leq \int_{\{|y|_1 \leq 2N\}} |\tilde{\Psi}_\delta^{(\alpha)}(y, \tilde{v})| \left| y + \lfloor 2^j \tilde{u} \rfloor - 2^j \tilde{u} \right|_1^{\tilde{v}} \log^{\frac{1}{\alpha}} \left( 3 + \left| y + \lfloor 2^j \tilde{u} \rfloor - 2^j \tilde{u} \right|_1^{-1} \right) dy \\
& \quad + \int_{\{2N < |y|_1 \leq 2^{j/2}\}} |\tilde{\Psi}_\delta^{(\alpha)}(y, \tilde{v})| \left| y + \lfloor 2^j \tilde{u} \rfloor - 2^j \tilde{u} \right|_1^{\tilde{v}} \log^{\frac{1}{\alpha}} \left( 3 + \left| y + \lfloor 2^j \tilde{u} \rfloor - 2^j \tilde{u} \right|_1^{-1} \right) dy \\
& \leq c_4 + \int_{\{2N < |y|_1 \leq 2^{j/2}\}} |\tilde{\Psi}_\delta^{(\alpha)}(y, \tilde{v})| (|y|_1 + N)^{\tilde{v}} \log^{\frac{1}{\alpha}} \left( 3 + (|y|_1 - \left| \lfloor 2^j \tilde{u} \rfloor - 2^j \tilde{u} \right|_1)^{-1} \right) dy \\
& \leq c_5 := c_4 + \int_{\{2N < |y|_1\}} |\tilde{\Psi}_\delta^{(\alpha)}(y, \tilde{v})| (|y|_1 + N)^{\tilde{v}} \log^{\frac{1}{\alpha}} \left( 3 + 2|y|_1^{-1} \right) dy < +\infty. \quad (4.31)
\end{aligned}$$

Next, let  $c_6$  be the positive finite constant, not depending on  $j$ , defined as  $c_6 := c_5 + c_4$ . Then, it follows from (4.27), (4.29), (4.30) and (4.31), that, on  $\Omega_\alpha^*$ , one has, for all  $j \in \mathbb{Z}_+$ ,

$$\frac{|\varepsilon_{\delta, j, \lfloor 2^j \tilde{u} \rfloor}^{(\alpha)}|}{(1 + j)^{\frac{1}{\alpha}}} \leq c_6 \mathcal{M}_{\tilde{u}, \tilde{v}}^{(X)}(c_2 2^{-j/2}) + C_1 2^{-j(1-\tilde{v})} (1 + j)^{-\frac{1}{\alpha}},$$

which implies that

$$\limsup_{j \rightarrow +\infty} \frac{|\varepsilon_{\delta, j, \lfloor 2^j \tilde{u} \rfloor}^{(\alpha)}|}{(1 + j)^{\frac{1}{\alpha}}} \leq c_6 \lim_{j \rightarrow +\infty} \mathcal{M}_{\tilde{u}, \tilde{v}}^{(X)}(c_2 2^{-j/2}). \quad (4.32)$$

Finally, in view of (4.2) and (4.28), one clearly has that

$$\lim_{j \rightarrow +\infty} \mathcal{M}_{\tilde{u}, \tilde{v}}^{(X)}(c_2 2^{-j/2}) = \limsup_{u \rightarrow \tilde{u}} \frac{|X(u, \tilde{v}) - X(\tilde{u}, \tilde{v})|}{|u - \tilde{u}| |\tilde{v}|_1^{\frac{1}{\alpha}} (1 + |u - \tilde{u}|_1^{-1})}.$$

Thus, (4.32) shows that (4.20) holds on  $\Omega_\alpha^*$ .  $\blacksquare$

The second main ingredient of the proof of Theorem 4.1 is the following proposition. On other hand, this proposition is also a major ingredient of the proof of Theorem 4.4.

**Proposition 4.15** *Let  $(k_j)_{j \in \mathbb{N}}$  be an arbitrary sequence of elements of  $\mathbb{Z}^N$ . One has almost surely, for all  $\delta \in \Upsilon_*$ ,*

$$\limsup_{j \rightarrow +\infty} \frac{|\varepsilon_{\delta, j, k_j}^{(\alpha)}|}{(1+j)^{\frac{1}{\alpha}}} = +\infty \quad \text{and} \quad \limsup_{j \rightarrow +\infty} \frac{|\varepsilon_{\delta, -j, k_j}^{(\alpha)}|}{(1+j)^{\frac{1}{\alpha}}} = +\infty. \quad (4.33)$$

For proving this proposition, we need the following remark.

**Remark 4.16** *Let  $\delta \in \Upsilon_*$  be arbitrary and fixed. One knows from Part (i) of Remark 1.1 and (2.12) that the S $\alpha$ S real-valued random variables  $\varepsilon_{\delta, j, k}^{(\alpha)}$ ,  $(j, k) \in \mathbb{Z} \times \mathbb{Z}^N$ , defined in (2.18), are identically distributed and that the common value of their scale parameter is  $\|\widehat{\psi}_\delta\|_\alpha := \left( \int_{\mathbb{R}^N} |\widehat{\psi}_\delta(\eta)|^\alpha d\eta \right)^{1/\alpha} > 0$ . Thus, one can derive from (1.2.10) on page 17 in [22], that there are two universal constants  $0 < c'_\alpha < c''_\alpha < +\infty$ , only depending on  $\alpha$ , such that, for every  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ , one has*

$$c'_\alpha \|\widehat{\psi}_\delta\|_\alpha^\alpha x^{-\alpha} \leq \mathbb{P}(|\varepsilon_{\delta, j, k}^{(\alpha)}| \geq x) \leq c''_\alpha \|\widehat{\psi}_\delta\|_\alpha^\alpha x^{-\alpha}, \quad \text{for all } x \in [1, +\infty). \quad (4.34)$$

*On another hand, denoting by  $\mathbb{Z}_e$  (resp.  $\mathbb{Z}_o$ ) the set of the even (resp. odd) integers, one knows from Part (ii) of Remark 1.1, (2.12), (2.4) and (2.3), that for any given integer  $m \geq 2$ , and any distinct integers  $j_1, j_2, \dots, j_m$  belonging to  $\mathbb{Z}_e$  (resp.  $\mathbb{Z}_o$ ), the  $m$  sequences of random variables  $\{\varepsilon_{\delta, j_1, k}\}_{(\delta, k) \in \Upsilon_* \times \mathbb{Z}^N}$ ,  $\{\varepsilon_{\delta, j_2, k}\}_{(\delta, k) \in \Upsilon_* \times \mathbb{Z}^N}$ ,  $\dots$ ,  $\{\varepsilon_{\delta, j_m, k}\}_{(\delta, k) \in \Upsilon_* \times \mathbb{Z}^N}$  are independent.*

**Proof of Proposition 4.15** We only show that the first equality in (4.33) holds, the second equality in it can be obtained in the same way. Let  $\mathbb{N}_e$  be the set of the positive even integers. One knows from Remark 4.16, that for any given  $\delta \in \Upsilon_*$ ,  $\{\varepsilon_{\delta, j, k_j}\}_{j \in \mathbb{N}_e}$  is a sequence of *independent* and identically distributed real-valued S $\alpha$ S random variables. Thus, using the second part of the Borel-Cantelli Lemma, it is turns out that for proving the first equality in (4.33), it is enough to show that

$$\sum_{j \in \mathbb{N}_e} \mathbb{P}\left(|\varepsilon_{\delta, j, k_j}^{(\alpha)}| \geq (1+j)^{\frac{1}{\alpha}} \log^{\frac{1}{\alpha}}(1+j)\right) = +\infty. \quad (4.35)$$

The equality (4.35) easily follows from the first inequality in (4.34).  $\blacksquare$

We are now in position to prove Theorem 4.1 and its two corollaries.

**Proof of Theorem 4.1** Using (4.20) and the first equality in (4.33) with  $k_j = \lfloor 2^j \tilde{u} \rfloor$ , for all  $j \in \mathbb{N}$ , one obtains (4.1).  $\blacksquare$

**Proof of Corollary 4.2** It follows from (1.6) and the triangle inequality that, on the event  $\Omega_\alpha^*$  of probability 1,

$$\begin{aligned} & \limsup_{t \rightarrow \tau} \frac{|Z(t) - Z(\tau)|}{|t - \tau|_1^{H(\tau)} \log^{\frac{1}{\alpha}}(1 + |t - \tau|_1^{-1})} \\ & \geq \limsup_{t \rightarrow \tau} \frac{|X(t, H(\tau)) - X(\tau, H(\tau))|}{|t - \tau|_1^{H(\tau)} \log^{\frac{1}{\alpha}}(1 + |t - \tau|_1^{-1})} - \limsup_{t \rightarrow \tau} \frac{|X(t, H(t)) - X(t, H(\tau))|}{|t - \tau|_1^{H(\tau)} \log^{\frac{1}{\alpha}}(1 + |t - \tau|_1^{-1})}. \end{aligned} \quad (4.36)$$

Moreover, one can derive from (3.5) and Remark 2.77 with  $m = 0$ ,  $\varrho = |\tau|_\infty + 1$ ,  $a = \underline{H}$  and  $b = \overline{H}$  that, on  $\Omega_\alpha^*$ ,

$$\limsup_{t \rightarrow \tau} \frac{|X(t, H(t)) - X(t, H(\tau))|}{|t - \tau|_1^{H(\tau)} \log^{\frac{1}{\alpha}}(1 + |t - \tau|_1^{-1})} < +\infty. \quad (4.37)$$

Finally, putting together (4.36), (4.1) with  $\tilde{u} = \tau$  and  $\tilde{v} = H(\tau)$ , and (4.37), one obtains the corollary.  $\blacksquare$

**Proof of Corollary 4.3** Using (4.4) and the fact that  $\tau^{(0)}$  belongs to the topological interior of the box  $I$ , one gets that

$$\sup_{(t^{(1)}, t^{(2)}) \in I^2} \frac{|Z(t^{(1)}) - Z(t^{(2)})|}{|t^{(1)} - t^{(2)}|_1^{H(I)} \log^{\frac{1}{\alpha}}(1 + |t^{(1)} - t^{(2)}|_1^{-1})} \geq \limsup_{t \rightarrow \tau^{(0)}} \frac{|Z(t) - Z(\tau^{(0)})|}{|t - \tau^{(0)}|_1^{H(\tau)} \log^{\frac{1}{\alpha}}(1 + |t - \tau^{(0)}|_1^{-1})}.$$

Thus, Corollary 4.3 results from Corollary 4.2.  $\blacksquare$

Our next goal is to show that Theorem 4.4 and its corollary hold. We already mentioned that Proposition 4.15 is a major ingredient of the proof of Theorem 4.4. The other major ingredient of it is the following result.

**Proposition 4.17** *One denotes by  $\langle 0 \rangle$  the vector of  $\mathbb{Z}^N$  whose coordinates are all equal to 0, that is  $\langle 0 \rangle := (0, 0, \dots, 0) \in \mathbb{Z}^N$ . For any fixed  $\tilde{v} \in (0, 1)$ , there exists a positive finite deterministic constant  $c'(\tilde{v})$ , only depending on  $\tilde{v}$ , such that on the event  $\Omega_\alpha^*$  of probability 1, one has, for all  $\delta \in \Upsilon_*$ ,*

$$\limsup_{j \rightarrow +\infty} \frac{|\varepsilon_{\delta, -j, \langle 0 \rangle}^{(\alpha)}|}{(1 + j)^{\frac{1}{\alpha}}} \leq c'(\tilde{v}) \limsup_{|u|_1 \rightarrow +\infty} \frac{|X(u, \tilde{v})|}{|u|_1^{\tilde{v}} \log^{\frac{1}{\alpha}}(1 + |u|_1)}. \quad (4.38)$$

**Proof** First observe that, it easily follows from (4.18) that one has on the event  $\Omega_\alpha^*$ , for all  $j \in \mathbb{N}$ ,

$$|\varepsilon_{\delta, -j, (0)}^{(\alpha)}| \leq \mathcal{W}_{\delta, j}(\tilde{v}) + \tilde{\mathcal{W}}_{\delta, j}(\tilde{v}), \quad (4.39)$$

where

$$\mathcal{W}_{\delta, j}(\tilde{v}) := 2^{-j\tilde{v}} \int_{\{|y|_1 < 2^{-j/2}\}} |\tilde{\Psi}_\delta^{(\alpha)}(y, \tilde{v})| |X(2^j y, \tilde{v})| dy \quad (4.40)$$

and

$$\tilde{\mathcal{W}}_{\delta, j}(\tilde{v}) := 2^{-j\tilde{v}} \int_{\{|y|_1 \geq 2^{-j/2}\}} |\tilde{\Psi}_\delta^{(\alpha)}(y, \tilde{v})| |X(2^j y, \tilde{v})| dy. \quad (4.41)$$

Let us now show that, for any fixed  $\eta > 0$ , there is a positive finite random variable  $C_1$ , such that on  $\Omega_\alpha^*$ , the following inequality holds:

$$\mathcal{W}_{\delta, j}(\tilde{v}) \leq C_1 2^{-j\tilde{v}/2} j^{\frac{1}{\alpha} + \eta}, \quad \text{for all } j \in \mathbb{N}. \quad (4.42)$$

Observe that, one knows from the continuity on  $\mathbb{R}^N$  of the function  $X(\bullet, \tilde{v}, \omega)$  (where  $\omega \in \Omega_\alpha^*$  is arbitrary and fixed), and Theorem 3.4 with  $m = 0$ , that, for some positive finite random variable  $C_2$ , one has on  $\Omega_\alpha^*$ ,

$$|X(u, \tilde{v})| \leq C_2 (1 + |u|_1^{\tilde{v}}) \log^{\frac{1}{\alpha} + \eta} (3 + |u|_1), \quad \text{for all } u \in \mathbb{R}^N.$$

Thus, for all  $j \in \mathbb{N}$  and  $y \in \mathbb{R}^N$  satisfying  $|y|_1 < 2^{-j/2}$ , one gets that

$$|X(2^j y, \tilde{v})| \leq 2 C_2 2^{j\tilde{v}/2} \log^{\frac{1}{\alpha} + \eta} (3 + 2^{j/2}).$$

Then, (4.40) implies, for all  $j \in \mathbb{N}$ , that

$$\mathcal{W}_{\delta, j}(\tilde{v}) \leq \left( 2 C_2 \int_{\mathbb{R}^N} |\tilde{\Psi}_\delta^{(\alpha)}(y, \tilde{v})| dy \right) 2^{-j\tilde{v}/2} \log^{\frac{1}{\alpha} + \eta} (3 + 2^{j/2}). \quad (4.43)$$

Observe that, one knows from (4.15) that the integral in the right-hand side of the inequality (4.43) is finite. Therefore, it results from the latter inequality that (4.42) is satisfied.

In order to conveniently bound  $\tilde{\mathcal{W}}_{\delta, j}(\tilde{v})$ , one introduces, on  $\Omega_\alpha^*$ , the positive non-increasing random function  $\tilde{\mathcal{M}}_{\tilde{v}}^{(X)}$ , defined for each  $\varrho \in (0, +\infty)$ , as

$$\tilde{\mathcal{M}}_{\tilde{v}}^{(X)}(\varrho) := \sup \left\{ \frac{|X(z, \tilde{v})|}{|z|_1^{\tilde{v}} \log^{\frac{1}{\alpha}} (1 + |z|_1)}, |z|_1 \geq \varrho \right\}. \quad (4.44)$$

Then, it results from (4.41) and (4.44) that, on  $\Omega_\alpha^*$ , one has, for all  $j \in \mathbb{N}$ ,

$$\begin{aligned} \tilde{\mathcal{W}}_{\delta, j}(\tilde{v}) &\leq \tilde{\mathcal{M}}_{\tilde{v}}^{(X)}(2^{j/2}) 2^{-j\tilde{v}} \int_{\{|y|_1 \geq 2^{-j/2}\}} |\tilde{\Psi}_\delta^{(\alpha)}(y, \tilde{v})| |2^j y|_1^{\tilde{v}} \log^{\frac{1}{\alpha}} (1 + |2^j y|_1) dy \\ &\leq c'(\tilde{v}) \tilde{\mathcal{M}}_{\tilde{v}}^{(X)}(2^{j/2}) (1 + j)^{\frac{1}{\alpha}}, \end{aligned} \quad (4.45)$$

where the deterministic finite constant  $c'(\tilde{v}) := \int_{\mathbb{R}} |\tilde{\Psi}_\delta^{(\alpha)}(y, \tilde{v})| |y|_1^{\tilde{v}} \log^{\frac{1}{\alpha}} (3 + |y|_1) dy$ .

Finally, in view of (4.7), putting together (4.39), (4.42) and (4.45), one obtains (4.38). ■

We are now ready to prove Theorem 4.4 and its corollary.

**Proof of Theorem 4.4** Using (4.38) and the second equality in (4.33) with  $k_j = \langle 0 \rangle$ , for all  $j \in \mathbb{N}$ , one obtains (4.6).  $\blacksquare$

**Proof of Corollary 4.5** It follows from (1.6) and the triangle inequality that, on the event  $\Omega_\alpha^*$  of probability 1,

$$\begin{aligned} \limsup_{|t|_1 \rightarrow +\infty} \frac{|Z(t)|}{|t|_1^{H_\infty} \log^{\frac{1}{\alpha}}(1 + |t|_1)} &\geq \limsup_{|t|_1 \rightarrow +\infty} \frac{|X(t, H_\infty)|}{|t|_1^{H_\infty} \log^{\frac{1}{\alpha}}(1 + |t|_1)} \\ &\quad - \limsup_{|t|_1 \rightarrow +\infty} \frac{|X(t, H(t)) - X(t, H_\infty)|}{|t|_1^{H_\infty} \log^{\frac{1}{\alpha}}(1 + |t|_1)}. \end{aligned}$$

Moreover, one knows from Theorem 4.4, that on the event  $\tilde{\Omega}_\alpha \subset \Omega_\alpha^*$  of probability 1, one has

$$\limsup_{|t|_1 \rightarrow +\infty} \frac{|X(t, H_\infty)|}{|t|_1^{H_\infty} \log^{\frac{1}{\alpha}}(1 + |t|_1)} = +\infty.$$

Thus, it turns out that, for proving (4.9), it is enough to show that the following equality holds on the event  $\Omega_\alpha^*$  of probability 1:

$$\limsup_{|t|_1 \rightarrow +\infty} \frac{|X(t, H(t)) - X(t, H_\infty)|}{|t|_1^{H_\infty} \log^{\frac{1}{\alpha}}(1 + |t|_1)} = 0. \quad (4.46)$$

Let us fix an arbitrary  $t \in \mathbb{R}^N$  such that  $|t|_1 \geq 1$ . Since the random function  $v \mapsto X(t, v)$  is  $\mathcal{C}^\infty$  on the interval  $(0, 1)$  (see Theorem 2.16), one can use the mean value theorem to get that

$$X(t, H(t)) - X(t, H_\infty) = (\partial_\nu X)(t, \nu)(H(t) - H_\infty), \quad (4.47)$$

where  $\nu$  is such that  $0 < \underline{H} \leq \min\{H(t), H_\infty\} \leq \nu \leq \max\{H(t), H_\infty\} \leq \overline{H} < 1$ , which entails that

$$|\nu - H_\infty| \leq |H(t) - H_\infty| \leq c_1 \left( \log(3 + |t|_1) \right)^{-1 - \eta_\infty}, \quad (4.48)$$

where the last inequality follows from (4.8); notice that the finite constant  $c_1$  does not depend on  $t$ . Next one recalls that, one knows from Theorem 3.4 with  $m = 1$ ,  $\eta = \eta_\infty/2$ ,  $\varrho = 1$ ,  $a = \underline{H}$  and  $b = \overline{H}$ , that, there is a positive finite random variable  $C_2$  such that, on  $\Omega_\alpha^*$  one has

$$|(\partial_\nu X)(u, v)| \leq C_2 |u|_1^\nu \log^{\frac{1}{\alpha} + \frac{\eta_\infty}{2} + 1}(1 + |u|_1), \quad \text{for all } (u, v) \in \mathcal{C}_1 \times [\underline{H}, \overline{H}],$$

which, in particular, implies that

$$|(\partial_\nu X)(t, \nu)| \leq C_2 |t|_1^\nu \log^{\frac{1}{\alpha} + \frac{\eta_\infty}{2} + 1}(1 + |t|_1). \quad (4.49)$$

Observe that, using the inequalities  $|t|_1 \geq 1$  and (4.48), one gets that

$$\begin{aligned} |t|_1^\nu &= |t|_1^{H_\infty - H_\infty + \nu} \leq |t|_1^{H_\infty + |\nu - H_\infty|} = |t|_1^{H_\infty} \exp(|\nu - H_\infty| \log(|t|_1)) \\ &\leq |t|_1^{H_\infty} \exp(c_1 \log^{-\eta_\infty}(3 + |t|_1)) \leq \exp(c_1) |t|_1^{H_\infty}. \end{aligned} \quad (4.50)$$

Thus, setting  $C_3 := C_2 \exp(c_1)$ , one can derive from (4.49) and (4.50) that

$$|(\partial_\nu X)(t, \nu)| \leq C_3 |t|_1^{H_\infty} \log^{\frac{1}{\alpha} + \frac{\eta_\infty}{2} + 1}(1 + |t|_1).$$

Then (4.47) and the second inequality in (4.48) imply that, on  $\Omega_\alpha^*$ , for all  $t \in \mathbb{R}^N$  with  $|t|_1 \geq 1$ .

$$|X(t, H(t)) - X(t, H_\infty)| \leq c_1 C_2 |t|_1^{H_\infty} \log^{\frac{1}{\alpha} - \frac{\eta_\infty}{2}}(1 + |t|_1). \quad (4.51)$$

One can derive from (4.51), that for every real number  $\varrho \geq 1$ ,

$$\sup \left\{ \frac{|X(t, H(t)) - X(t, H_\infty)|}{|t|_1^{H_\infty} \log^{\frac{1}{\alpha}}(1 + |t|_1)}, t \in \mathbb{R}^N \text{ and } |t|_1 \geq \varrho \right\} \leq c_1 C_2 \log^{-\frac{\eta_\infty}{2}}(1 + \varrho). \quad (4.52)$$

Thus, in view of (4.7), letting  $\varrho$  in (4.52) go to  $+\infty$ , one obtains (4.46).  $\blacksquare$

Our next and final goal is to show that Theorem 4.7 and its corollary hold. The following two propositions are the two main ingredients of the proof of Theorem 4.7. We skip the proof of the first one of them since it is very similar to that of Proposition 4.13.

**Proposition 4.18** *For any fixed  $(\tilde{u}, \tilde{v}) \in \mathbb{R}^N \times (0, 1)$ , there is a positive finite deterministic constant  $c''(\tilde{v})$ , only depending on  $\tilde{v}$ , such that on the event  $\Omega_\alpha^*$  of probability 1, one has, for all  $\delta \in \Upsilon_*$ ,*

$$\limsup_{j \rightarrow +\infty} |\varepsilon_{\delta, j, [2^j \tilde{u}]}^{(\alpha)}| \leq c''(\tilde{v}) \limsup_{u \rightarrow \tilde{u}} \frac{|X(u, \tilde{v}) - X(\tilde{u}, \tilde{v})|}{|u - \tilde{u}|_1^{\tilde{v}}}. \quad (4.53)$$

**Proposition 4.19** *There exist a universal event  $\widehat{\Omega}_\alpha \subset \Omega_\alpha^*$  of probability 1 (not depending on  $\tilde{u}$ ) and a universal deterministic strictly positive finite constant  $\widehat{c}_0$  (not depending on  $\tilde{u}$ ), such that on  $\widehat{\Omega}_\alpha$ , one has, for all  $\delta \in \Upsilon_*$  and  $\tilde{u} \in \mathbb{R}^N$ ,*

$$\limsup_{j \rightarrow +\infty} |\varepsilon_{\delta, j, [2^j \tilde{u}]}^{(\alpha)}| \geq \widehat{c}_0 > 0. \quad (4.54)$$

**Proof** First, recall that one knows from Remark 4.16 that the real-valued SaS random variables  $\|\widehat{\psi}_\delta\|_\alpha^{-1} \varepsilon_{\delta,j,k}^{(\alpha)}$ ,  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ , are identically distributed and that the common value of their scale parameter is equal to 1. Thus, since  $\min_{\delta \in \Upsilon_*} \|\widehat{\psi}_\delta\|_\alpha > 0$  and the common probability distribution of these random variables is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}$ , for any given arbitrarily small real number  $\theta \in (0, 1)$ , there exists a finite constant  $\widehat{c}_0(\theta) > 0$  such that

$$\mathbb{P}(|\varepsilon_{\delta,j,k}^{(\alpha)}| < \widehat{c}_0(\theta)) \leq \theta, \quad \text{for all } (\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N. \quad (4.55)$$

Having made this first remark, in the sequel, for any integer  $m \in \mathbb{Z}$ , one denotes by  $\langle m \rangle$  the vector of  $\mathbb{Z}^N$  whose coordinates are all equal to  $m$ . Observe the collection of the cubes  $[\ell, \ell + \langle 1 \rangle) := \prod_{r=1}^N [\ell_r, \ell_r + 1)$ ,  $\ell \in \mathbb{Z}^N$ , forms a partition of  $\mathbb{R}^N$ . Since this collection of cubes is countable, it is enough to show that the proposition holds for all  $\tilde{u} \in [\ell, \ell + \langle 1 \rangle)$ , where  $\ell \in \mathbb{Z}^N$  is arbitrary and fixed. In the sequel, for avoiding heavy notations, one assumes that this fixed  $\ell$  is equal to  $\langle 0 \rangle$ ; the proof can be done in the same way for any other fixed  $\ell$ . For all level  $j \in \mathbb{Z}_+$ , and  $k \in \mathcal{K}_j := \{0, \dots, 2^j - 1\}^N$ , one denotes by  $I_{j,k}$  the dyadic cube of level  $j$  defined as  $I_{j,k} := [2^{-j}k, 2^{-j}(k + \langle 1 \rangle)) := \prod_{r=1}^N [2^{-j}k_r, 2^{-j}(k_r + 1))$ . Observe that, for every  $j \in \mathbb{Z}_+$ , one has  $I_{j,k'} \cap I_{j,k''} = \emptyset$  when  $k' \neq k''$ , and that

$$[\langle 0 \rangle, \langle 1 \rangle) = \bigcup_{k \in \mathcal{K}_j} I_{j,k}.$$

Also observe that being given such a cube  $I_{j,k}$ , for all level  $j' \in \{0, \dots, j\}$ , there exists a unique  $k' = k'(j, k, j') \in \mathcal{K}_{j'}$ , which depends on  $j, k$  and  $j'$ , such that  $I_{j,k} \subseteq I_{j',k'}$ ; moreover when  $k = \lfloor 2^j \tilde{u} \rfloor$ , for any fixed  $\tilde{u} \in [\langle 0 \rangle, \langle 1 \rangle)$ , then  $k' = \lfloor 2^{j'} \tilde{u} \rfloor$ . Having made these observations, let us denote by  $\mathbb{Z}_{+,e}$  the set of the non-negative even integers and, for any given  $j \in \mathbb{Z}_{+,e}$  and  $k \in \mathcal{K}_{3j}$ , let us denote by  $\Lambda_{3j,k}$  the finite set of cardinality  $j + 1$  defined as

$$\Lambda_{3j,k} := \{(j', k') \in \mathbb{Z}_{+,e} \times \mathbb{Z}^N, j \leq j' \leq 3j, k' \in \mathcal{K}_{j'} \text{ and } I_{3j,k} \subseteq I_{j',k'}\}; \quad (4.56)$$

notice that, thanks to our previous observations, it results from (4.56) that, for every  $j \in \mathbb{Z}_{+,e}$  and  $\tilde{u} \in [\langle 0 \rangle, \langle 1 \rangle)$ , the set  $\Lambda_{3j, \lfloor 2^{3j} \tilde{u} \rfloor}$  can be expressed as

$$\Lambda_{3j, \lfloor 2^{3j} \tilde{u} \rfloor} = \{(j', \lfloor 2^{j'} \tilde{u} \rfloor), j' \in \mathbb{Z}_{+,e} \text{ and } j \leq j' \leq 3j\}. \quad (4.57)$$

For each  $\delta \in \Upsilon_*$ ,  $j \in \mathbb{Z}_{+,e}$  and  $k \in \mathcal{K}_{3j}$ , one defines the event  $\Theta_{3j,k}^\delta$  as

$$\Theta_{3j,k}^\delta := \bigcap_{(j', k') \in \Lambda_{3j,k}} \{|\varepsilon_{\delta,j',k'}^{(\alpha)}| < \widehat{c}_0(\theta)\}, \quad (4.58)$$

where  $\widehat{c}_0(\theta) > 0$  is the same constant as in (4.55). Notice that, in view of (4.56) and our previous observations, there are no two distinct elements  $(j'_1, k'_1)$  and  $(j'_2, k'_2)$  of  $\Lambda_{3j,k}$  which are such that  $j'_1 = j'_2$  and  $k'_1 \neq k'_2$ . Thus, one knows from Remark 4.16 that the random variables  $\varepsilon_{\delta, j', k'}^{(\alpha)}$ ,  $(j', k') \in \Lambda_{3j,k}$ , are independent. Then one derive from (4.58), (4.55) and the fact that  $j + 1$  is the cardinality of  $\Lambda_{3j,k}$ , that, for all  $\delta \in \Upsilon_*$ ,  $j \in \mathbb{Z}_{+,e}$  and  $k \in \mathcal{K}_{3j}$ ,

$$\mathbb{P}(\Theta_{3j,k}^\delta) \leq \theta^{j+1}. \quad (4.59)$$

Next, for every  $j \in \mathbb{Z}_{+,e}$ , let  $\Theta_{3j}$  be the event defined as

$$\Theta_{3j} := \bigcup_{\delta \in \Upsilon_*} \bigcup_{k \in \mathcal{K}_{3j}} \Theta_{3j,k}^\delta. \quad (4.60)$$

Since  $\Upsilon_* = \{1, \dots, 2^N - 1\}$  and  $\mathcal{K}_{3j} = \{0, \dots, 8^j - 1\}^N$ , it results from (4.60) and (4.59) that, for all  $j \in \mathbb{Z}_{+,e}$ ,

$$\mathbb{P}(\Theta_{3j}) \leq \sum_{\delta \in \Upsilon_*} \sum_{k \in \mathcal{K}_{3j}} \mathbb{P}(\Theta_{3j,k}^\delta) \leq \theta(2^N - 1)(8^N \theta)^j. \quad (4.61)$$

Since,  $\theta \in (0, 1)$  can be arbitrarily small, one can assume that it is chosen so that  $8^N \theta < 1$ . Then, it results from (4.61) that

$$\sum_{j \in \mathbb{Z}_{+,e}} \mathbb{P}(\Theta_{3j}) < +\infty.$$

Therefore, Borel-Cantelli Lemma entails that the probability of the event

$$\Theta := \bigcap_{J \in \mathbb{Z}_{+,e}} \bigcup_{j \in \mathbb{Z}_{+,e}, j \geq J} \Theta_{3j}$$

is equal to 0, and consequently that the probability of the opposite event

$$\bar{\Theta} := \bigcup_{J \in \mathbb{Z}_{+,e}} \bigcap_{j \in \mathbb{Z}_{+,e}, j \geq J} \bar{\Theta}_{3j} \quad (4.62)$$

is equal to 1. Notice that  $\bar{\Theta}_{3j}$  is the opposite event to  $\Theta_{3j}$ . Thus, using (4.60) and (4.58), one gets that

$$\bar{\Theta}_{3j} = \left\{ \inf_{(\delta, k) \in \Upsilon_* \times \mathcal{K}_{3j}} \sup_{(j', k') \in \Lambda_{3j,k}} |\varepsilon_{\delta, j', k'}^{(\alpha)}| \geq \widehat{c}_0(\theta) \right\}. \quad (4.63)$$

Moreover, it follows from (4.57) that, for all  $j \in \mathbb{Z}_{+,e}$  and  $\tilde{u} \in [\langle 0 \rangle, \langle 1 \rangle]$ ,

$$\sup_{(j', k') \in \Lambda_{3j, [2^{3j} \tilde{u}]}} |\varepsilon_{\delta, j', k'}^{(\alpha)}| = \sup \{ |\varepsilon_{\delta, j', [2^{j'} \tilde{u}]}^{(\alpha)}|, j' \in \mathbb{Z}_{+,e} \text{ and } j \leq j' \leq 3j \}. \quad (4.64)$$

Then, using the fact that  $[2^{3j}\tilde{u}] \in \mathcal{K}_{3j}$ , for all  $j \in \mathbb{Z}_{+,e}$  and  $\tilde{u} \in [\langle 0 \rangle, \langle 1 \rangle]$ , one can derive from (4.62), (4.63) and (4.64), that on the event  $\bar{\Theta}$  of probability 1 (which does not depend on  $\delta$  and  $\tilde{u}$ ), (4.54) holds, for every  $\delta \in \Upsilon_*$  and  $\tilde{u} \in [\langle 0 \rangle, \langle 1 \rangle]$ .

As we already mentioned, at the beginning of the proof, similarly to what we have done in the case where  $\ell = \langle 0 \rangle$ , in the general case where  $\ell \in \mathbb{Z}^N$  is arbitrary and fixed, by replacing set  $\mathcal{K}_j$  by the set  $2^j\ell + \mathcal{K}_j := \{2^j\ell + k, k \in \mathcal{K}_j\}$ , it can be shown that there exists an event  $\bar{\Theta}^{(\ell)}$  of probability 1 (which does not depend on  $\delta$  and  $\tilde{u}$ ), such that (4.54) holds on  $\bar{\Theta}^{(\ell)}$ , for every  $\delta \in \Upsilon_*$  and  $\tilde{u} \in [\ell, \ell + \langle 1 \rangle]$ , and with the same strictly positive deterministic constant  $\hat{c}_0$  as in the case  $\ell = \langle 0 \rangle$ .

Finally, letting  $\hat{\Omega}_\alpha$  be the event of probability 1 defined as

$$\hat{\Omega}_\alpha := \Omega_\alpha^* \cap \left( \bigcap_{\ell \in \mathbb{Z}^N} \bar{\Theta}^{(\ell)} \right),$$

with  $\bar{\Theta}^{(0)} = \bar{\Theta}$ , one obtains the proposition. ■

We are now in position to prove Theorem 4.7 and its corollary.

**Proof of Theorem 4.7** This theorem is a straightforward consequence of Propositions 4.18 and 4.19 ■

**Proof of Corollary 4.8** It follows from (1.6) and the triangle inequality that, on the event  $\Omega_\alpha^*$  of probability 1,

$$\begin{aligned} & \limsup_{t \rightarrow \tau} \frac{|Z(t) - Z(\tau)|}{|t - \tau|_1^{H(\tau)}} \\ & \geq \limsup_{t \rightarrow \tau} \frac{|X(t, H(\tau)) - X(\tau, H(\tau))|}{|t - \tau|_1^{H(\tau)}} - \limsup_{t \rightarrow \tau} \frac{|X(t, H(t)) - X(t, H(\tau))|}{|t - \tau|_1^{H(\tau)}}. \end{aligned} \tag{4.65}$$

Moreover, one can derive from (4.12) and Remark 2.77 with  $m = 0$ ,  $\varrho = |\tau|_\infty + 1$ ,  $a = \underline{H}$  and  $b = \overline{H}$  that, on  $\Omega_\alpha^*$ ,

$$\limsup_{t \rightarrow \tau} \frac{|X(t, H(t)) - X(t, H(\tau))|}{|t - \tau|_1^{H(\tau)}} = 0. \tag{4.66}$$

Finally, putting together (4.65), (4.11) with  $\tilde{u} = \tau$  and  $\tilde{v} = H(\tau)$ , and (4.66), one obtains the corollary. ■

## 5 Appendix

This section is devoted to the proof of Lemma 2.6. To this end, we need some preliminary results. Let us point out that throughout the section the fixed real numbers  $\alpha \in (0, 2)$  and  $\eta > 0$  are the same as in the statement of Lemma 2.6.

The following lemma can be obtained by using hyperspherical coordinates on  $\mathbb{R}^N$  and classical calculations without serious difficulties, this is why we skip its proof.

**Lemma 5.1** *For each arbitrary and fixed integer  $N \geq 1$  and real number  $\eta > 0$ , one sets*

$$b_{N,\eta} := \begin{cases} \frac{\alpha\eta}{4}, & \text{if } N = 1, \\ \frac{\alpha\eta}{4\pi} \left( \prod_{k=1}^{N-2} \int_0^\pi \sin^k \theta d\theta \right)^{-1}, & \text{if } N \geq 2, \end{cases} \quad (5.1)$$

with the convention that a product over the empty set is equal to 1. Let  $\phi_\eta$  be the non-negative function on  $\mathbb{R}^N$  defined as  $\phi_\eta(0) := 0$  and, more importantly, as

$$\phi_\eta(\xi) := b_{N,\eta} |\xi|_2^{-N} \left( 1 + |\log |\xi|_2| \right)^{-1-\alpha\eta}, \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\}, \quad (5.2)$$

Then, one has that  $\int_{\mathbb{R}^N} \phi_\eta(\xi) d\xi = 1$ , which means that  $\phi_\eta$  is a probability density function on  $\mathbb{R}^N$ .

The following lemma provides a random LePage series representation for the complex-valued  $\alpha$ -stable stochastic process

$$\left\{ \widetilde{\varepsilon}_{\delta,j,k}^{(\alpha)} \right\}_{(\delta,j,k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N} := \left\{ \int_{\mathbb{R}^N} \overline{\widehat{\psi}_{\delta,j,k}^{(\alpha)}}(\xi) d\widetilde{M}_\alpha(\xi) \right\}_{(\delta,j,k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N}, \quad (5.3)$$

which is very closely related to the real-valued SaS stochastic process  $\left\{ \varepsilon_{\delta,j,k}^{(\alpha)} \right\}_{(\delta,j,k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N}$  defined through (2.18). The proof of the lemma has been omitted since it is rather similar to that of Theorem 4.2 in [16].

**Lemma 5.2** *Let  $(\kappa^{(m)})_{m \in \mathbb{N}}$ ,  $(\Gamma_m)_{m \in \mathbb{N}}$  and  $(g_m)_{m \in \mathbb{N}}$  be three mutually independent sequences of random variables, defined on the same probability space, and satisfying the following three properties.*

1. *The  $\kappa^{(m)}$ 's,  $m \in \mathbb{N}$ , are  $\mathbb{R}^N$ -valued, independent and identically distributed with the probability density function  $\phi_\eta$  defined in (5.2).*

2. The  $\Gamma_m$ 's,  $m \in \mathbb{N}$ , are Poisson arrival times with unit rate; in other words there is a sequence  $(\mathcal{E}_n)_{n \in \mathbb{N}}$  of independent and identically distributed exponential random variables with parameter equals to 1 such that, for all  $m \in \mathbb{N}$ ,

$$\Gamma_m = \sum_{n=1}^m \mathcal{E}_n. \quad (5.4)$$

3. The  $g_m$ 's,  $m \in \mathbb{N}$ , are complex-valued, independent and identically distributed with a centered and rotationally invariant Gaussian distribution satisfying  $\mathbb{E}(|\Re(g_1)|^\alpha) = 1$ .

Let the positive finite constant  $a_\alpha := \left( \int_0^{+\infty} x^{-\alpha} \sin x \, dx \right)^{-\frac{1}{\alpha}}$ . Then, one has that

$$\left\{ \widehat{\varepsilon}_{\delta,j,k}^{(\alpha)} \right\}_{(\delta,j,k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N} \stackrel{(d)}{=} \left\{ a_\alpha \sum_{m=1}^{+\infty} \Gamma_m^{-\frac{1}{\alpha}} \times g_m \left[ \phi_\eta(\kappa^{(m)}) \right]^{-\frac{1}{\alpha}} \overline{\widehat{\psi}_{\delta,j,k}^{(\alpha)}(\kappa^{(m)})} \right\}_{(\delta,j,k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N}, \quad (5.5)$$

where " $\stackrel{(d)}{=}$ " means equality of all finite-dimensional distributions. Also, notice that the random series in the right-hand side of (5.5) are almost surely convergent.

**Remark 5.3** From now on, the stochastic process  $\left\{ \widehat{\varepsilon}_{\delta,j,k}^{(\alpha)} \right\}_{(\delta,j,k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N}$ , defined in (5.3), will completely be identified to the one in the right-hand side of the equality in distribution (5.5).

**Remark 5.4** There exist four strictly positive and finite random variables  $C_1, \dots, C_4$  such that one has almost surely

$$|g_m| \leq C_1 \log^{\frac{1}{2}}(3+m), \quad \text{for all } m \in \mathbb{N}, \quad (5.6)$$

$$\mathcal{E}_n \leq C_2 \log(2+n), \quad \text{for all } n \in \mathbb{N}, \quad (5.7)$$

and

$$C_3 m \leq \Gamma_m \leq C_4 m, \quad \text{for all } m \in \mathbb{N}. \quad (5.8)$$

Notice that the inequality (5.6) follows from e.g. Lemma 1 in [5], the inequality (5.7) is borrowed from Remark 2.10 in [6], and the inequality (5.8) can easily be derived from (5.4) and the strong law of large numbers.

**Lemma 5.5** Let  $\mathcal{K}_0$  be the same compact subset of  $\mathbb{R}^N$  as in (2.5) and, for every  $j \in \mathbb{Z}$ , let  $(\beta_n^{(j)})_{n \in \mathbb{N}}$  be the sequence of the independent and identically distributed Bernoulli random variables defined, for all  $n \in \mathbb{N}$ , as

$$\beta_n^{(j)} := \mathbf{1}_{\mathcal{K}_0} \left( 2^{-j} \kappa^{(n)} \right). \quad (5.9)$$

Then, one has, for every  $(n, \delta, j, k) \in \mathbb{N} \times \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ ,

$$\left| [\phi_\eta(\kappa^{(n)})]^{-\frac{1}{\alpha}} \overline{\widehat{\psi}_{\delta,j,k}^{(\alpha)}(\kappa^{(n)})} \right| \leq \mu_{\alpha,\eta} (1 + |j|)^{\frac{1}{\alpha} + \eta} \beta_n^{(j)}, \quad (5.10)$$

where the deterministic positive finite constant  $\mu_{\alpha,\eta}$  is defined as

$$\mu_{\alpha,\eta} := \max_{\delta \in \Upsilon_*} \sup_{\xi \in \mathcal{K}_0} [\phi_\eta(\xi)]^{-\frac{1}{\alpha}} |\widehat{\psi}_\delta(\xi)|. \quad (5.11)$$

**Proof** one can derive from (5.2), (2.12), the inclusion in (2.5), (5.9), the triangle inequality and (5.11), that, for every  $(n, \delta, j, k) \in \mathbb{N} \times \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ ,

$$\begin{aligned} \left| [\phi_\eta(\kappa^{(n)})]^{-\frac{1}{\alpha}} \overline{\widehat{\psi}_{\delta,j,k}^{(\alpha)}(\kappa^{(n)})} \right| &\leq b_{N,\eta}^{-\frac{1}{\alpha}} |2^{-j} \kappa^{(n)}|_2^{\frac{N}{\alpha}} \left( 1 + |j| + |\log |2^{-j} \kappa^{(n)}|_2| \right)^{\frac{1}{\alpha} + \eta} \left| \widehat{\psi}_\delta(2^{-j} \kappa^{(n)}) \right| \beta_n^{(j)} \\ &\leq (1 + |j|)^{\frac{1}{\alpha} + \eta} \left[ \phi_\eta(2^{-j} \kappa^{(n)}) \right]^{-\frac{1}{\alpha}} \left| \widehat{\psi}_\delta(2^{-j} \kappa^{(n)}) \right| \beta_n^{(j)} \leq \mu_{\alpha,\eta} (1 + |j|)^{\frac{1}{\alpha} + \eta} \beta_n^{(j)}, \end{aligned}$$

which shows that (5.10) is satisfied. ■

We are now in position to prove Lemma 2.6 when  $\alpha \in (0, 1)$ .

**Proof of Lemma 2.6 when  $\alpha \in (0, 1)$**  In view of (2.18), (5.3) and Remark 5.3, one has, almost surely for all  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ ,

$$|\varepsilon_{\delta,j,k}^{(\alpha)}| \leq |\widehat{\varepsilon}_{\delta,j,k}^{(\alpha)}| \leq a_\alpha \sum_{m=1}^{+\infty} \Gamma_m^{-\frac{1}{\alpha}} \times |g_m| \left| [\phi_\eta(\kappa^{(m)})]^{-\frac{1}{\alpha}} \overline{\widehat{\psi}_{\delta,j,k}^{(\alpha)}(\kappa^{(m)})} \right|. \quad (5.12)$$

Then, one can derive from (5.12), (5.6), the first inequality in (5.8) and (5.10), that, for some positive finite random variable  $C_1$  not depending on  $(\delta, j, k)$ , almost surely,

$$|\varepsilon_{\delta,j,k}^{(\alpha)}| \leq C_1 (1 + |j|)^{\frac{1}{\alpha} + \eta} \sum_{m=1}^{+\infty} m^{-\frac{1}{\alpha}} \log^{\frac{1}{2}}(3 + m). \quad (5.13)$$

Then, noticing that

$$\sum_{m=1}^{+\infty} m^{-\frac{1}{\alpha}} \log^{\frac{1}{2}}(3 + m) < +\infty, \quad \text{when } \alpha \in (0, 1), \quad (5.14)$$

it results from (5.13) that the inequality (2.34) holds when  $\alpha \in (0, 1)$ , since one then has  $[\alpha] = 0$ . ■

Having proved Lemma 2.6 when  $\alpha \in (0, 1)$ , from now on, we focus on its proof in the case where  $\alpha \in [1, 2)$ , we mention in passing that a major difficulty in the latter case is that the series in (5.14) fails to be convergent.

First, we need to introduce some additional notations. For all  $m \in \mathbb{N}$ , the real-valued, centered and identically distributed Gaussian random variables  $g_{0,m}$  and  $g_{1,m}$ ,  $m \in \mathbb{N}$ , are defined as

$$g_{0,m} = \Re\mathfrak{e}(g_m), \quad g_{1,m} = \Im\mathfrak{m}(g_m). \quad (5.15)$$

Moreover, for every  $(\delta, j, k, m) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N \times \mathbb{N}$ , the real-valued random variables  $\lambda_{0,m}^{(\delta,j,k)}$  and  $\lambda_{1,m}^{(\delta,j,k)}$  are defined as

$$\lambda_{0,m}^{(\delta,j,k)} := \Re\mathfrak{e} \left\{ [\phi_\eta(\kappa^{(m)})]^{-\frac{1}{\alpha}} \overline{\widehat{\psi}_{\delta,j,k}^{(\alpha)}(\kappa^{(m)})} \right\}, \quad \lambda_{1,m}^{(\delta,j,k)} := \Im\mathfrak{m} \left\{ [\phi_\eta(\kappa^{(m)})]^{-\frac{1}{\alpha}} \overline{\widehat{\psi}_{\delta,j,k}^{(\alpha)}(\kappa^{(m)})} \right\}. \quad (5.16)$$

Then, one clearly has that

$$\Re\mathfrak{e} \left\{ g_m [\phi_\eta(\kappa^{(m)})]^{-\frac{1}{\alpha}} \overline{\widehat{\psi}_{\delta,j,k}^{(\alpha)}(\kappa^{(m)})} \right\} = \lambda_{0,m}^{(\delta,j,k)} g_{0,m} - \lambda_{1,m}^{(\delta,j,k)} g_{1,m}, \quad (5.17)$$

and one can derive from (2.18), (5.3) and Remark 5.3, that, for every  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ ,

$$\varepsilon_{\delta,j,k}^{(\alpha)} = \Re\mathfrak{e}(\varepsilon_{\delta,j,k}^{(\alpha)}) = a_\alpha \sum_{m=1}^{+\infty} \Gamma_m^{-\frac{1}{\alpha}} (\lambda_{0,m}^{(\delta,j,k)} g_{0,m} - \lambda_{1,m}^{(\delta,j,k)} g_{1,m}), \quad (5.18)$$

where the random series is almost surely convergent, since the random series in the right-hand side of (5.5) have this convergence property. Next, for all  $(m, l) \in \mathbb{N} \times \{0, 1\}$  and  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ , one sets

$$S_{l,m}^{(\delta,j,k)} := \sum_{n=1}^m \lambda_{l,n}^{(\delta,j,k)} g_{l,n}. \quad (5.19)$$

The following lemma, which provides the first upper bound for  $S_{l,m}^{(\delta,j,k)}$ , is a straightforward consequence of (5.16), Lemma 5.5, (5.19), (5.6) and (5.15).

**Lemma 5.6** *Using the same notations as in (5.16), Lemma 5.5 and (5.19), one has, for all  $(n, l, \delta, j, k) \in \mathbb{N} \times \{0, 1\} \times \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ ,*

$$|\lambda_{l,n}^{(\delta,j,k)}| \leq \mu_{\alpha,\eta} (1 + |j|)^{\frac{1}{\alpha} + \eta} \beta_n^{(j)}. \quad (5.20)$$

*Moreover, there exists a positive finite random variable  $C'$ , such that, one has almost surely, for every  $(m, l, \delta, j, k) \in \mathbb{N} \times \{0, 1\} \times \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ ,*

$$|S_{l,m}^{(\delta,j,k)}| \leq C' (1 + |j|)^{\frac{1}{\alpha} + \eta} B_m^{(j)} \log^{\frac{1}{2}}(3 + m), \quad (5.21)$$

where, for each  $(j, m) \in \mathbb{Z} \times \mathbb{N}$ , the binomial random variable  $B_m^{(j)}$  is defined as

$$B_m^{(j)} := \sum_{n=1}^m \beta_n^{(j)}. \quad (5.22)$$

The following lemma, which provides the second upper bound for  $S_{l,m}^{(\delta,j,k)}$ , can be proved by following the main lines of the proof of Lemma 2.8 in [6]; we skip its proof.

**Lemma 5.7** *There exists a positive finite random variable  $C''$  such that, one has almost surely, for all  $(m, l, \delta, j, k) \in \mathbb{N} \times \{0, 1\} \times \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ ,*

$$|S_{l,m}^{(\delta,j,k)}| \leq C'' (1 + |j|)^{\frac{1}{\alpha} + \eta} \sqrt{B_m^{(j)} \log(3 + |j| + |k|_1 + m)}. \quad (5.23)$$

Also, we skip the proofs of the two following lemmas since they can be done in almost the same ways as those of Lemmas 2.11 and 2.12 in [6].

**Lemma 5.8** *For all  $(l, \delta, j, k) \in \{0, 1\} \times \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ , the random series*

$$\chi_l^{(\delta,j,k)} := \sum_{m=1}^{+\infty} \left( \Gamma_m^{-\frac{1}{\alpha}} - \Gamma_{m+1}^{-\frac{1}{\alpha}} \right) S_{l,m}^{(\delta,j,k)} \quad (5.24)$$

*is almost surely absolutely convergent. Moreover, letting  $a_\alpha$  be the same deterministic finite constant as in Lemma 5.2, one has almost surely, for every  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ ,*

$$\varepsilon_{\delta,j,k}^{(\alpha)} = a_\alpha \left( \chi_0^{(\delta,j,k)} - \chi_1^{(\delta,j,k)} \right). \quad (5.25)$$

**Lemma 5.9** *For each  $j \in \mathbb{Z}$ , the probability  $p_j \in (0, 1)$ , which in fact corresponds to the value of the parameter of the Bernoulli random variable in (5.9), is given by*

$$p_j := \mathbb{P} \left( 2^{-j} \kappa^{(1)} \in \mathcal{K}_0 \right). \quad (5.26)$$

*Since  $\sum_{j \in \mathbb{Z}} p_j < +\infty$ , the binomial random variables  $B_m^{(j)}$ ,  $(j, m) \in \mathbb{Z} \times \mathbb{N}$ , defined in (5.22), satisfy the following property: for any fixed real number  $\theta \in (1/2, 1)$ , there is a positive finite random variable  $C_\theta$  such that one has, almost surely,*

$$B_m^{(j)} \leq C_\theta (p_j m + m^\theta), \quad \text{for all } (j, m) \in \mathbb{Z} \times \mathbb{N}. \quad (5.27)$$

We are now in position to prove Lemma 2.6 when  $\alpha \in [1, 2)$ .

**Proof of Lemma 2.6 when  $\alpha \in [1, 2)$**  First one shows that the inequality (2.34) is satisfied.

One knows from (5.4) and the inequality (2.25) in [6] (whose proof relies on (5.4), (5.7) and (5.8)) that there is a positive finite random variable  $C_1$ , such that, one has almost surely, for all  $m \in \mathbb{N}$ ,

$$0 < \Gamma_m^{-\frac{1}{\alpha}} - \Gamma_{m+1}^{-\frac{1}{\alpha}} \leq C_1 m^{-(\frac{1}{\alpha}+1)} \log(3+m). \quad (5.28)$$

Since  $1 \geq 1/\alpha > 1/2$ , one can choose  $\theta_0$  such that

$$\theta_0 \in (1/2, 1) \text{ and } 1/\alpha + 1 - \theta_0 > 1. \quad (5.29)$$

For each  $j \in \mathbb{Z}$ , let  $\mathcal{M}_j$  and  $\overline{\mathcal{M}}_j$  be the two non-empty disjoint sets which form a partition of  $\mathbb{N}$ , and which are defined as

$$\mathcal{M}_j = \{m \in \mathbb{N}, p_j m \geq m^{\theta_0}\} \quad \text{and} \quad \overline{\mathcal{M}}_j = \{m \in \mathbb{N}, p_j m < m^{\theta_0}\} \quad (5.30)$$

Observe that

$$\begin{cases} m \in \mathcal{M}_j \implies p_j m + m^{\theta_0} \leq 2p_j m \\ m \in \overline{\mathcal{M}}_j \implies p_j m + m^{\theta_0} < 2m^{\theta_0}. \end{cases} \quad (5.31)$$

Since  $\mathbb{N} = \mathcal{M}_j \cup \overline{\mathcal{M}}_j$ , one knows from Lemma 5.8 and the triangle inequality that almost surely, for all  $(\delta, j, k) \in \{0, 1\} \times \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ ,

$$|\varepsilon_{\delta, j, k}^{(\alpha)}| \leq a_\alpha \sum_{l=0}^1 \left( \sum_{m \in \mathcal{M}_j} (\Gamma_m^{-\frac{1}{\alpha}} - \Gamma_{m+1}^{-\frac{1}{\alpha}}) |S_{l, m}^{(\delta, j, k)}| + \sum_{m \in \overline{\mathcal{M}}_j} (\Gamma_m^{-\frac{1}{\alpha}} - \Gamma_{m+1}^{-\frac{1}{\alpha}}) |S_{l, m}^{(\delta, j, k)}| \right). \quad (5.32)$$

Observe that, (5.28), (5.23), (2.38), (5.27) with  $\theta = \theta_0$ , and the first inequality in (5.31) imply almost surely, for all  $(l, \delta, j, k) \in \{0, 1\} \times \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ , that

$$\begin{aligned} & \sum_{m \in \mathcal{M}_j} (\Gamma_m^{-\frac{1}{\alpha}} - \Gamma_{m+1}^{-\frac{1}{\alpha}}) |S_{l, m}^{(\delta, j, k)}| \\ & \leq C_2 (1 + |j|)^{\frac{1}{\alpha} + \eta} \sqrt{\log(3 + |j| + |k|_1)} \sum_{m \in \mathcal{M}_j} m^{-(\frac{1}{\alpha}+1)} \log^{\frac{3}{2}}(3+m) \sqrt{B_m^{(j)}} \\ & \leq C_3 (1 + |j|)^{\frac{1}{\alpha} + \eta} \sqrt{p_j \log(3 + |j| + |k|_1)} \sum_{m \in \mathcal{M}_j} m^{-(\frac{1}{\alpha} + \frac{1}{2})} \log^{\frac{3}{2}}(3+m) \\ & \leq C_4 (1 + |j|)^{\frac{1}{\alpha} + \eta} \sqrt{p_j \log(3 + |j| + |k|_1)}, \end{aligned} \quad (5.33)$$

where  $C_2$  and  $C_3$  are two positive finite random variables not depending on  $(l, \delta, j, k)$ , and

$$C_4 := C_3 \sum_{m=1}^{+\infty} m^{-(\frac{1}{\alpha} + \frac{1}{2})} \log^{\frac{3}{2}}(3+m) < +\infty.$$

Also observe that, one can derive from (5.28), (5.21), (5.27) with  $\theta = \theta_0$ , the second inequality in (5.31), (5.29) and the fact that

$$\sum_{m=1}^{+\infty} m^{-1-\frac{1}{\alpha}+\theta_0} \log^{\frac{3}{2}}(3+m) < +\infty,$$

that one has almost surely, for all  $(l, \delta, j, k) \in \{0, 1\} \times \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ ,

$$\sum_{m \in \overline{\mathcal{M}}_j} (\Gamma_m^{-\frac{1}{\alpha}} - \Gamma_{m+1}^{-\frac{1}{\alpha}}) |S_{l,m}^{(\delta,j,k)}| \leq C_5 (1 + |j|)^{\frac{1}{\alpha} + \eta}, \quad (5.34)$$

where the positive finite random variable  $C_5$  does not depend on  $(l, \delta, j, k)$ . Next, combining (5.33) and (5.34) with (5.32), one obtains, almost surely, for all  $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$ , that

$$|\varepsilon_{\delta,j,k}^{(\alpha)}| \leq C_6 (1 + |j|)^{\frac{1}{\alpha} + \eta} \left( 1 + \sqrt{p_j \log(3 + |j| + |k|_1)} \right), \quad (5.35)$$

where the positive finite random variable  $C_6$  does not depend on  $(\delta, j, k)$ . Moreover, one knows from (5.26) that  $p_j \in (0, 1)$ , for every  $j \in \mathbb{Z}$ . Thus, (5.35) shows that the inequality (2.34) holds when  $\alpha \in [1, 2)$  i.e. when  $\lfloor \alpha \rfloor = 1$ .

Let us now prove that, under the condition (2.35), the inequality (2.36) is satisfied. So, from now on, one restricts to arbitrary  $j \in \mathbb{Z}_+$  and  $k \in \mathbb{Z}^N$  satisfying this condition, which entails that

$$\log(3 + j + |k|_1) \leq \log((4 + N\vartheta)2^j) \leq (j + 1) \log(4 + N\vartheta). \quad (5.36)$$

Moreover, using (5.26), the equality in (2.5) and the fact that the probability density function of the random variable  $\kappa^{(1)}$  is the function  $\phi_\eta$  defined in (5.2), one get that

$$\begin{aligned} p_j &= b_{N,\eta} \int_{2^j \mathcal{K}_0} |\xi|_2^{-N} \left( 1 + |\log |\xi|_2| \right)^{-1-\alpha\eta} d\xi \\ &\leq b_{N,\eta} \left( \frac{2^{j+1}\pi}{3} \right)^{-N} \left( 1 + \log \left( \frac{2^{j+1}\pi}{3} \right) \right)^{-1-\alpha\eta} \lambda_N(2^j \mathcal{K}_0) \\ &\leq b_{N,\eta} \lambda_N(\mathcal{K}_0) (\log(2))^{-1-\alpha\eta} (2 + j)^{-1-\alpha\eta}, \end{aligned} \quad (5.37)$$

where  $2^j \mathcal{K}_0$  is the compact subset of  $\mathbb{R}^N$  such that  $2^j \mathcal{K}_0 := \{2^j z, z \in \mathcal{K}_0\}$ , and  $\lambda_N$  denotes the Lebesgue measure on  $\mathbb{R}^N$ . Next, let the positive finite deterministic constant

$$c_7 := b_{N,\eta} \lambda_N(\mathcal{K}_0) (\log(2))^{-1-\alpha\eta} \log(4 + N\vartheta).$$

Then, it clearly follows from (5.36) and (5.37) that, for all  $j \in \mathbb{Z}_+$  and  $k \in \mathbb{Z}^N$  satisfying the condition (2.35), one has

$$p_j \log(3 + j + |k|_1) \leq c_7.$$

Therefore, one can derive from (5.35) that the inequality (2.36) holds, for all  $\delta \in \Upsilon_*$  and for any such  $j$  and  $k$ . ■

## Acknowledgements

This work has been partially supported by the Labex CEMPI (ANR-11-LABX-0007-01) and the Australian Research Council's Discovery Projects funding scheme (project number DP220101680).

## References

- [1] A. Ayache. *Multifractional stochastic fields: wavelet strategies in multifractional frameworks*. World Scientific, 2019.
- [2] A. Ayache and G. Boutard. Stationary increment harmonizable stable fields: upper estimates on path behavior. *Journal of Theoretical Probability*, 30:1369–1423, 2017.
- [3] A. Ayache and J. Hamonier. Linear multifractional stable motion: fine path properties. *Revista Matemática Iberoamericana*, 30(4):1301–1354, 2014.
- [4] A. Ayache, S. Jaffard, and M.S. Taqqu. Wavelet construction of Generalized Multifractional Processes. *Revista Matemática Iberoamericana*, 23(1):327–370, 2007.
- [5] A. Ayache and M.S. Taqqu. Rate optimality of wavelet series approximations of Fractional Brownian Motion. *Journal of Fourier Analysis and Applications*, 9(5):451–471, 2003.
- [6] A. Ayache and Y. Xiao. An optimal uniform modulus of continuity for harmonizable fractional stable motion. *Transactions of the American Mathematical Society*, to appear.
- [7] A. Benassi, S. Jaffard, and D. Roux. Elliptic Gaussian random processes. *Revista Matemática Iberoamericana*, 13(1):19–90, 1997.
- [8] S. Bianchi, A. Pantanella, and A. Pianese. Modeling and simulation of currency exchange rates using MPRE. *International Journal of Modeling and Optimization*, 2(3):309–314, 2012.
- [9] S. Bianchi, A. Pantanella, and A. Pianese. Modeling stock prices by multifractional Brownian motion: an improved estimation of the pointwise regularity. *Quantitative Finance*, 13(8):1317–1330, 2013.
- [10] S. Bianchi and A. Pianese. Multifractional processes in finance. *Risk and Decision Analysis*, 5(1):1–22, 2014.

- [11] H. Biermé, C. Lacaux, and H.P. Scheffler. Multi-operator scaling random fields. *Stochastic Processes and their Applications*, 121(11):2642–2677, 2011.
- [12] I. Daubechies. *Ten lectures on wavelets*, volume 61. Society for Industrial Mathematics, 1992.
- [13] M. Dozzi and G. Shevchenko. Real harmonizable multifractional stable process and its local properties. *Stochastic Processes and their Applications*, 121(7):1509–1523, 2011.
- [14] P. Embrechts and M. Maejima. *Selfsimilar processes*. Princeton University Press, 2002.
- [15] K.J. Falconer and J. Lévy Véhel. Multifractional, multistable, and other processes with prescribed local form. *Journal of Theoretical Probability*, 22(2):375–401, 2009.
- [16] N. Kôno and M. Maejima. Self-similar stable processes with stationary increments. In S. Cambanis, G. Samorodnitsky, and M.S Taqqu, editors, *Stable Processes and Related Topics*, volume 25 of *Progress in Probability*, pages 275–295. Birkhäuser, Boston, 1991.
- [17] C. Lacaux. Real harmonizable multifractional Lévy motions. *Annales de l’Institut Henri Poincaré (B) Probabilités et Statistiques*, 40(3):259–277, 2004.
- [18] J. Lebovits and M. Podolskij. Estimation of the global regularity of a multifractional Brownian motion. *Electronic Journal of Statistics*, 11(1):78–98, 2017.
- [19] D. Loboda, F. Mies, and A. Steland. Regularity of multifractional moving average processes with random Hurst exponent. *Stochastic Processes and their Applications*, 140:21–48, 2021.
- [20] Y. Meyer. *Wavelets and operators*, volume 37. Cambridge University Press, 1992.
- [21] R. Peltier and J. Lévy Véhel. Multifractional Brownian motion: definition and preliminary results. *Rapport de recherche INRIA* 2645, 1995.
- [22] G. Samorodnitsky and M.S. Taqqu. *Stable non-Gaussian random processes: stochastic models with infinite variance*. Chapman and Hall, 1994.
- [23] S. Stoev and M.S. Taqqu. Path properties of the linear multifractional stable motion. *Fractals*, 13(2):157–178, 2005.
- [24] S. Stoev and M.S. Taqqu. How rich is the class of multifractional Brownian motions? *Stochastic Processes and their Applications*, 116(2):200–221, 2006.

- [25] D. Surgailis. Nonhomogeneous fractional integration and multifractional processes. *Stochastic Processes and their Applications*, 118(2):171–198, 2008.