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Existence, uniqueness and positivity of solutions to the Guyon-Lekeufack path-dependent volatility model with general kernels

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Abstract

We show the existence and uniqueness of a continuous solution to a path-dependent volatility model introduced by Guyon and Lekeufack (2023) to model the price of an equity index and its spot volatility. The considered model for the trend and activity features can be written as a Stochastic Volterra Equation (SVE) with non-convolutional and non-bounded kernels as well as non-Lipschitz coefficients. We first prove the existence and uniqueness of a solution to the SVE under integrability and regularity assumptions on the two kernels and under a condition on the second kernel weighting the past squared returns which ensures that the activity feature is bounded from below by a positive constant. Then, assuming in addition that the kernel weighting the past returns is of exponential type and that an inequality relating the logarithmic derivatives of the two kernels with respect to their second variables is satisfied, we show the positivity of the volatility process which is obtained as a non-linear function of the SVE's solution. We show numerically that the choice of an exponential kernel for the kernel weighting the past returns has little impact on the quality of model calibration compared to other choices and the inequality involving the logarithmic derivatives is satisfied by the calibrated kernels. These results extend those of Nutz and Valdevenito (2023).

Keywords: path-dependent volatility, Stochastic Volterra Equations

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $(\mathcal{F}_t)_{t \geq 0}$ and a (\mathcal{F}_t) -adapted Brownian motion $(W_t)_{t \in \mathbb{R}}$, we consider the following model for $t \geq 0$:

$$\left\{ \begin{array}{l} \frac{dS_t}{S_t} = \sigma_t dW_t \\ \sigma_t = \beta_0 + \beta_1 R_{1,t} + \beta_2 \sqrt{R_{2,t}} \\ R_{1,t} = \int_{-\Delta}^t K_1(s, t) \sigma_s dW_s \\ R_{2,t} = \int_{-\Delta}^t K_2(s, t) \sigma_s^2 ds \end{array} \right. \quad \begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

where $\beta_0, \beta_2 \geq 0$, $\beta_1 \leq 0$, $\Delta \in [0, +\infty]$ and $K_1, K_2 : \Gamma \rightarrow \mathbb{R}_+$ are kernels weighting the past returns and the past squared returns respectively with

$$\Gamma := \{(u, t) \in \mathbb{R} \times \mathbb{R}^+ \mid u \leq t\}.$$

We assume that $S_0 = s_0 > 0$ and the past evolutions of R_1 and R_2 between $-\Delta$ and 0 are given by the deterministic initial conditions $(r_{1,u})_{-\Delta \leq u \leq 0}$ and $(r_{2,u})_{-\Delta \leq u \leq 0}$ respectively with $r_{1,u} \in \mathbb{R}$ and $r_{2,u} \geq 0$. This model has its roots in the work of Guyon and Lekeufack (2023). In this paper, they conduct an empirical study on the dependence of the volatility of an equity index with respect to the past path of the associated equity price. To this end, they consider the following model:

$$\text{Volatility}_t = \beta_0 + \beta_1 R_{1,t} + \beta_2 \sqrt{R_{2,t}}. \quad (5)$$

The definition and interpretation of each term is provided below.

- Volatility is a proxy measure of the spot volatility of an equity index. Two measures are considered: the value of an implied volatility index such as the VIX and an estimator of the realized volatility over one day using intraday observations of the equity index.
- R_1 is a trend feature defined by:

$$R_{1,t} = \sum_{t_i \leq t} K_1(t - t_i) r_{t_i}$$

where r_{t_i} is the daily return between day t_{i-1} and day t_i of the equity index and $K_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a decreasing kernel weighting the past returns. Since $\beta_1 \geq 0$, this feature allows to capture the leverage effect, i.e. the fact that volatility tends to rise when prices fall.

- R_2 is an activity or volatility feature defined by:

$$R_{2,t} = \sum_{t_i \leq t} K_2(t - t_i) r_{t_i}^2$$

where K_2 is also a decreasing kernel. Since $\beta_2 \geq 0$, this feature allows to capture the volatility clustering phenomenon, i.e. the fact that periods of large volatility tend to be followed by periods of large volatility, and periods of small volatility tend to be followed by periods of small volatility. Note that Guyon and Lekeufack consider the square root of R_2 in model (5) in order to get a quantity that is homogeneous to a volatility.

The key result of Guyon and Lekeufack is to show that model (5), when calibrated on historical data, allows explaining (in the sense of the R^2 score) more than 80% of the variations of the volatility for several main equity indices. They conclude that volatility is mostly path-dependent and they propose a continuous version (see model (4.1) in Guyon and Lekeufack, 2023) of model (5). The model (1)-(4) is a reproduction of this continuous version but with two differences:

1. we consider general kernels for the weighting of the past returns and the past squared returns instead of convolutional ones and
2. the lower boundary of the integrals defining R_1 and R_2 is parametrized by Δ instead of being fixed to $-\infty$.

Note that Equations (2)-(4) can be rewritten as follows:

$$R_t = g(t) + \int_0^t K_1(s, t) \gamma(R_s) dW_s + \int_0^t K_2(s, t) b(R_s) ds \quad (6)$$

where $R_t := (R_{1,t}, R_{2,t})$ and

$$\begin{aligned} g(t) &= \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} := \begin{pmatrix} \int_{-\Delta}^0 K_1(s, t) (\beta_0 + \beta_1 r_{1,s} + \beta_2 \sqrt{r_{2,s}}) dW_s \\ \int_{-\Delta}^0 K_2(s, t) (\beta_0 + \beta_1 r_{1,s} + \beta_2 \sqrt{r_{2,s}})^2 ds \end{pmatrix} \\ b(r_1, r_2) &= \begin{pmatrix} 0 \\ (\beta_0 + \beta_1 r_1 + \beta_2 \sqrt{r_2})^2 \end{pmatrix} \\ \gamma(r_1, r_2) &= \begin{pmatrix} \beta_0 + \beta_1 r_1 + \beta_2 \sqrt{r_2} \\ 0 \end{pmatrix}. \end{aligned} \quad (7)$$

Thus, in the model defined by Equations (1)-(4), the process (R_1, R_2) is autonomous since σ is a function of R_1 and R_2 only. Moreover, provided that Equation (6) admits a (\mathcal{F}_t) -adapted solution such that $(\sigma_t)_{t \geq 0}$ defined by (2) satisfies:

$$\mathbb{P} \left(\int_0^T \sigma_s^2 ds \right) = 1, \quad \forall T > 0,$$

then Equation (1) has a unique solution given by:

$$S_t = s_0 \exp \left(\int_0^t \sigma_s dW_s - \frac{1}{2} \int_0^t \sigma_s^2 ds \right).$$

The objective of this note is to show global existence and uniqueness of a continuous solution to Equation (6) and to establish sufficient conditions under which the volatility process $(\sigma_t)_{t \geq 0}$ is positive. Note that Equation (6) corresponds to a Stochastic Volterra Equation (SVE). SVEs have been extensively studied in the literature starting with the work of Berger and Mizel (1980a, 1980b) who develop existence and uniqueness results for SVEs driven by a Brownian motion and with Lipschitz coefficients. These results have then been extended in numerous directions. Protter (1985) generalises the existence and uniqueness results to SVEs driven by right continuous semimartingales. Pardoux and Protter (1990) and Alòs and Nualart (1997) consider the case of SVEs with anticipating coefficients. Cochran et al. (1995) and Coutin and Decreusefond (2001) study SVEs with singular kernels. Wang (2008) proves the existence and uniqueness of a solution to an SVE with non-Lipschitz coefficients satisfying a condition similar to the one of Yamada and Watanabe (1971) for stochastic differential equations. Zhang (2010) works on SVEs in Banach spaces with locally Lipschitz coefficients and singular kernels. More recently, there has been a resurgence of interest for SVEs in the mathematical finance literature with the advent of rough volatility models (see e.g. Gatheral et al., 2018 and El Euch et al., 2018) who rely on such equations. Among this new wave of publications, let us mention for instance those of Abi Jaber et al. (2019), Abi Jaber et al. (2021) and Abi Jaber et al. (2024) who address several challenges raised by rough volatility models that could not be dealt with using the earlier results on SVEs. Due to the quadratic growth induced by the square and the fact that the square root function is non-differentiable at 0, the coefficients b and γ in Equation (6) are non-Lipschitz and none of the above results can be used directly (even those considering non-Lipschitz coefficients) to show the existence and uniqueness of a solution to (6). To obtain the existence, we resort to a localization technique which exploits the particular structure of the coefficients b and γ .

Regarding the positivity of the volatility process, the SVE literature only deals with the existence of non-negative solutions (see Alfonsi, 2023 and Alfonsi and Szulda, 2024), while we aim at obtaining the positivity of a non-linear function of the coordinates of the solution of an SVE. In the path-dependent volatility (PDV) literature, Guyon and Lekeufack (2023) first observed that, in the 2-factor Markovian PDV model (corresponding to model (1)-(4) with exponential kernels, i.e. $K_i(s, t) = \lambda_i e^{-\lambda_i(t-s)}$ for $i \in \{1, 2\}$), the volatility process σ is non-negative provided that $\lambda_2 < 2\lambda_1$ because, under this condition, the drift of σ is positive whenever σ reaches 0. This result has then been strengthened by Nutz and Valdevenito (2023) who proved that $\sigma_t \geq \sigma_0 \exp(\beta_1 \lambda_1 W_t - \lambda_1 t - \frac{1}{2} \beta_1^2 \lambda_1^2 t)$ so that the volatility process is bounded from below by a positive continuous stochastic process when its initial value σ_0 is positive. Moreover, they showed existence and uniqueness for the 4-factor Markovian PDV model which essentially corresponds to (1)-(4) where both kernels are convex combinations of two exponential kernels, i.e. $K_i(s, t) = \theta_i \lambda_{1,i} e^{-\lambda_{1,i}(t-s)} + (1 - \theta_i) \lambda_{2,i} e^{-\lambda_{2,i}(t-s)}$ for $i \in \{1, 2\}$. This implies in particular the existence and uniqueness for the 2-factor Markovian PDV model. In fact, in the 4-factor Markovian PDV model, the trend and volatility features are both composed of two factors:

$$\begin{aligned} R_{1,t} &= \theta_1 R_{1,0,t} + (1 - \theta_1) R_{1,1,t} \\ R_{2,t} &= \theta_2 R_{2,0,t} + (1 - \theta_2) R_{2,1,t} \end{aligned}$$

where $\theta_1, \theta_2 \in [0, 1]$ and for $j \in \{0, 1\}$:

$$\begin{aligned} R_{1,j,t} &= \int_{-\Delta}^t \lambda_{1,j} e^{-\lambda_{1,j}(t-s)} \sigma_s dW_s \\ R_{2,j,t} &= \int_{-\Delta}^t \lambda_{2,j} e^{-\lambda_{2,j}(t-s)} \sigma_s^2 ds. \end{aligned}$$

Nutz and Valdevenito (2023) show existence and uniqueness when the initial conditions satisfy: $R_{1,j,0} \in \mathbb{R}$ and $R_{2,j,0} > 0$ for $j \in \{0, 1\}$. Note that the second conditions can be weakened to $(1 - \theta_2)R_{2,1,0} - \theta_2 R_{2,0,0}^- > 0$ when $\lambda_{2,0} \geq \lambda_{2,1}$ (which we can assume without loss of generality because of the symmetrical roles of $R_{2,0}$ and $R_{2,1}$).

Our contribution is to extend these results to general kernels assuming only some integrability and regularity of the two kernels, in particular the kernels are not required to be of convolutional type or bounded. First, we show existence and uniqueness of a solution to the SVE (6) and then under some additional assumptions on the two kernels, we find a positive stochastic lower bound for the volatility σ by adapting the proof of Nutz and Valdevenito (2023) for the 2-factor PDV model.

Let us now introduce the assumptions of our first theorem.

(I.1) For any $T > 0$,

$$\sup_{t \in [0, T]} \int_{-\Delta}^t (K_1(s, t)^2 + K_2(s, t)) ds < \infty.$$

(I.2) For any $T > 0$,

$$\limsup_{\varepsilon \downarrow 0} \sup_{t \in [0, T]} \int_t^{t+\varepsilon} (K_1(s, t + \varepsilon)^2 + K_2(s, t + \varepsilon)) ds < 1.$$

(I.3)

$$\sup_{s \in (-\Delta, 0]} |\beta_0 + \beta_1 r_{1,s} + \beta_2 \sqrt{r_{2,s}}| < \infty.$$

(I.4) There exists $\alpha_1 > 1$ and $\alpha_2 > 1$ such that for any $T > 0$,

$$\sup_{t \in [0, T]} \int_0^t (K_1(s, t)^{2\alpha_1} + K_2(s, t)^{\alpha_2}) ds < \infty.$$

(I.5) There exists $\gamma > 0$ such that for $0 \leq t < t' \leq T$,

$$\sqrt{\int_{-\Delta}^t (K_1(s, t') - K_1(s, t))^2 ds} + \int_{-\Delta}^t |K_2(s, t') - K_2(s, t)| ds \leq C|t' - t|^\gamma.$$

(I.6) For any $T > 0$,

$$\inf_{t \in [0, T]} g_2(t) > 0.$$

Our first main result is the following:

Theorem 1. *Under the assumptions (I.1)-(I.6), there exists a unique solution to Equation (6). Moreover, the solution is locally γ^* -Hölder continuous for any $\gamma^* \in \left(0, \min\left(\gamma, \frac{1}{2\alpha_1^*}, \frac{1}{\alpha_2^*}\right)\right)$ with $\alpha_i^* = \frac{\alpha_i}{\alpha_i - 1}$.*

Assumptions (I.1)-(I.3) guarantee the existence and uniqueness of a local solution to Equation (6) while assumptions (I.4)-(I.5) provide the regularity properties of the solution. Finally, (I.6) ensures that R_2 is bounded from below by a positive deterministic function which allows dealing with the fact that the square root function is non-differentiable at 0 and in turn to deduce that the local solution is actually global. This assumption is satisfied whenever $s \mapsto \sigma_s^2 \inf_{t \in [0, T]} K_2(s, t)$ is positive on a set with positive Lebesgue measure. When $K_2(s, t) = \theta_2 \lambda_{2,0} e^{-\lambda_{2,0}(t-s)} + (1 - \theta_2) \lambda_{2,1} e^{-\lambda_{2,1}(t-s)}$, this condition is equivalent to the one we gave above ($(1 - \theta_2)R_{2,1,0} - \theta_2 R_{2,0,0}^- > 0$ when $\lambda_{2,0} \geq \lambda_{2,1}$) in the case of the 4-factor Markovian PDV model. Remark that, by Hölder's inequality, assumption (I.4) implies assumption (I.2), but we still write assumption (I.2) to be able to separate the assumptions used to show the existence and uniqueness and those used to show the regularity of the solution. In order to get the positivity of the volatility σ , we need some additional assumptions that are stated below.

(II.1) K_1 and K_2 are continuously differentiable with respect to their second variable and they verify for all $t \geq 0$:

$$\int_0^t K_1(u, u)^2 du + \int_0^t \left(\int_{-\Delta}^v |\partial_v K_1(u, v)|^2 du \right)^{1/2} dv + \int_0^t K_2(u, u) du + \int_0^t \int_{-\Delta}^v |\partial_v K_2(u, v)| dudv < \infty.$$

(II.2) $K_1(s, t) = f(s)e^{h(t)}$ where h is assumed to be differentiable and non-increasing.

(II.3) $\partial_t K_2(s, t) - 2h'(t)K_2(s, t) \geq 0$ for all $t \geq 0$ and $s \in (-\Delta, t]$.

(II.4) $g_2(0) > 0$.

We can now state our second main result.

Theorem 2. *Under assumptions (I.1)-(I.5) and (II.1)-(II.4), there exists a unique solution to Equation (6) with the same Hölder regularity as in Theorem 1. Moreover, the process $(\sigma_t = \beta_0 + \beta_1 R_{1,t} + \beta_2 \sqrt{R_{2,t}})_{t \geq 0}$ is continuous and bounded from below by the process X defined by:*

$$X_t = \sigma_0 \exp \left(\beta_1 \int_0^t K_1(s, s) dW_s + h(t) - h(0) - \frac{1}{2} \beta_1^2 \int_0^t K_1(s, s)^2 ds \right)$$

and thus is positive if $\sigma_0 > 0$.

The assumption (II.1) implies that R_1 and R_2 are Itô processes (see Lemma 4) while the assumptions (II.2) and (II.3) allow to use a comparison result for Itô processes (see Lemma 5), similar to the one of [Karatzas and Shreve (1991), Proposition 2.18] for solutions of SDEs, which gives the positive stochastic lower bound in Theorem 2. Finally, assumption (II.4) combined with assumptions (II.2) and (II.3) implies assumption (I.6) (see Lemma 6) which justifies that (I.6) has not been included in the set of assumptions of Theorem 2. This assumption is satisfied whenever $s \mapsto K_2(s, 0)\sigma_s^2$ is positive on a set with positive Lebesgue measure. We provide below two examples of kernels K_1 satisfying both assumptions (II.1) and (II.2).

Example 1. *The exponential kernel $K_1(s, t) = \lambda e^{-\lambda(t-s)}$ with $\lambda > 0$ is recovered by taking $f(s) = \lambda e^{\lambda s}$ and $h(t) = -\lambda t$.*

Example 2. *By considering $f(s) = (s+\Delta)^a \mathbb{1}_{\{s \geq -\Delta\}}$ and $h(t) = -a \log(t+\Delta)$ for $a > 0$ and $\Delta < \infty$, one obtains the non-convolutional kernel $K_1(s, t) = \left(\frac{s+\Delta}{t+\Delta} \right)^a \mathbb{1}_{\{s \geq -\Delta\}}$ which allows modeling long memory.*

Let us make two remarks regarding assumption (II.3).

Remark 1. *If $K_1(u, t) = \lambda_1 e^{-\lambda_1(t-u)}$ and $K_2(u, t) = \lambda_2 e^{-\lambda_2(t-u)}$, condition (II.3) is equivalent to $2\lambda_1 \geq \lambda_2$ so we recover the condition of Nutz and Valdevenito (2023).*

Remark 2. *Since the condition (II.4) and the definition of $g_2(0)$ imply that $(-\Delta, 0] \ni s \mapsto K_2(s, 0)$ is positive on a set with positive Lebesgue measure and for $s \leq 0 \leq t$, $K_2(s, t) \geq K_2(s, 0) \exp \left(2 \int_0^t h'(u) du \right)$, kernels vanishing for $t - s$ greater than some finite cutoff lag are excluded from our set of assumptions. This can be viewed as a limitation given that Guyon and Lekeufack (2023) truncate the two kernels after some cut-off lag in their numerical experiments. However, we will show in the following that the R^2 scores achieved by the calibration of the PDV model (5) for several choices of kernels are still high (although a bit lower than those reported by Guyon and Lekeufack, 2023) when the kernels are not truncated.*

In their empirical study, Guyon and Lekeufack (2023) choose to work with time-shifted power-law (TSPL) kernels for the two kernels K_1 and K_2 :

$$K_i(u, t) = \frac{Z_{\alpha_i, \delta_i}}{(t - u + \delta_i)^{\alpha_i}}, \quad i = 1, 2,$$

with $\alpha_i, \delta_i > 0$ and Z_{α_i, δ_i} a normalization constant. This choice is motivated by the ability of this parametric form to mix short and long memory, which is a characteristic of historical volatility data.

However, by considering a TSPL kernel for K_1 , assumption **(II.2)** is not satisfied so we cannot apply Theorem 2. Therefore, we propose to consider an exponential kernel for K_1 and a TSPL kernel for K_2 , i.e. $K_1(u, t) = \lambda e^{-\lambda(t-u)}$ and $K_2(u, t) = \frac{Z_{\alpha, \delta}}{(t-u+\delta)^\alpha}$. Assuming that $\alpha > 1$ and $2\lambda\delta \geq \alpha$ (corresponding to condition **(II.3)**), one can check that all the assumptions of Theorem 2 are satisfied. Note that if $\Delta < +\infty$, we can have $0 < \alpha \leq 1$. This alternative choice for K_1 and K_2 implying a guarantee of positivity for the volatility process, it is natural to compare its performance to the one of Guyon and Lekeufack (2023), that is two TSPL kernels, as well as two other options:

1. the 2-factor Markovian PDV model (corresponding to exponential kernels for both K_1 and K_2) for which Nutz and Valdevenito (2023) gave a condition guaranteeing the positivity of the volatility;
2. the 4-factor Markovian PDV model (corresponding to a convex combination of two exponential kernels), which was the option recommended by Guyon and Lekeufack (2023) for applications (e.g. solving the joint SPX/VIX smile calibration problem) since it is Markovian and a convex combination of two exponential kernels provides a good approximation of a TSPL kernel.

To perform this comparison, we calibrate model (5) on implied volatility indices (the VIX for the S&P 500, the VSTOXX for the Euro Stoxx 50 and the IVI for the FTSE 100), extracted from Refinitiv¹, and realized volatilities computed using 5-minute returns per trading day coming from Heber et al. (2009). The data of the corresponding indices is also extracted from Refinitiv. Note that the data is split into a train set spanning the period from January 1, 2000 to December 31, 2018 and a test set spanning the period from January 1, 2019 to May 15, 2022 (resp. to December 30, 2021) for the implied volatility indices (resp. for the realized volatilities). The calibration methodology is essentially the one described in Guyon and Lekeufack (2023) but we make some modifications. First, unlike Guyon and Lekeufack who truncate the sums in the formulas of R_1 and R_2 in Equation (5) to 1,000 business days, we use all the past index data (which goes back to January 2, 1980 for the S&P 500, December 31, 1986 for the Euro Stoxx 50 and January 3, 1984 for the FTSE 100). This is motivated by Remark 2 which shows that kernels that vanish for $t - s > C$ for some cut-off lag $C > 0$ do not satisfy the assumptions of Theorem 2. Second, we bound the variables β_0 and β_2 from below by 0 and the variable β_1 from above by 1 in the numerical optimization as this allows to stabilize the results for the VIX. For the same reason, we add an L^2 penalization in the objective function following Andrès et al. (2023) in the case of the two TSPL kernels. The resulting R^2 scores are presented in Table 1. We observe that the compared choices for K_1 and K_2 provide very close results both on the train and the test sets. The best model in terms of R^2 scores is the convex combination of two exponential kernels for K_1 and K_2 but this is also the choice with the largest number of parameters (there is three parameters per kernel versus one for the exponential kernel and two for the TSPL kernel). Besides, there is no condition guaranteeing the positivity of the volatility available so far in this case (Nutz and Valdevenito, 2023, actually show that the volatility can be negative with positive probability if one considers the analogue for the 4-factor PDV model of the positivity condition in the 2-factor PDV model). The same remark applies to the choice of two TSPL kernels which is the second best model on the train set. Finally, the choice of two exponential kernels is the one with the worst scores. These results show that, in view of applications, the choice of an exponential kernel for K_1 and a TSPL kernel for K_2 is an attractive alternative to the choice of two TSPL kernels or two convex combinations of two exponential kernels since it does not deteriorate the fit to historical data, the model is parsimonious in terms of number of parameters and there is a sufficient condition ($2\lambda\delta \geq \alpha$) guaranteeing the positivity of the volatility. Moreover, this sufficient condition is satisfied numerically by the calibrated parameters as shown in Table 2.

The sequel is organized as follows. First, we introduce two theorems from Zhang (2010) regarding the existence and uniqueness of continuous solutions to a Stochastic Volterra Equation. In the next section, we prove Theorem 1. The last section is dedicated to the proof of Theorem 2.

1 Preliminary results

We start by stating an existence and uniqueness result for a stochastic Volterra integral equation with (possibly) singular kernels and Lipschitz coefficients. This result is a simplified version of Theorem 3.1

¹www.lseg.com

Table 1: R^2 scores of model (5) for various choices of kernels K_1 and K_2

		Implied volatility index			Realized volatility		
		VIX	VSTOXX	IVI	SPX	STOXX	FTSE
K_1 and K_2 are TSPL kernels	R^2 train	89.34%	91.36%	92.13%	67.03%	58.86%	61.74%
	R^2 test	83.43%	90.91%	87.97%	68.84%	62.48%	62.72%
K_1 and K_2 are exponential kernels	R^2 train	89.10%	90.28%	90.50%	66.99%	58.53%	61.54%
	R^2 test	74.26%	86.43%	82.37%	63.41%	61.68%	62.48%
K_1 and K_2 are convex combinations of two exponential kernels	R^2 train	90.01%	91.99%	92.37%	67.55%	59.10%	62.14%
	R^2 test	83.01%	91.61%	88.74%	68.86%	62.68%	62.80%
K_1 is an exponential kernel and K_2 is a TSPL kernel	R^2 train	89.86%	91.76%	92.29%	67.12%	58.85%	61.74%
	R^2 test	81.33%	91.03%	87.67%	64.61%	62.78%	62.88%

Table 2: Value of $\frac{2\lambda\delta}{\alpha}$ when we calibrate model (5) with an exponential kernel for K_1 and a TSPL kernel for K_2 .

	VIX	VSTOXX	IVI	SPX	STOXX	FTSE
$\frac{2\lambda\delta}{\alpha}$	4.00	2.84	1.62	2.23	1.97	1.97

of Zhang (2010) who works in a 2-smooth Banach space while we work in \mathbb{R}^d . We are interested in the following stochastic Volterra integral equation in \mathbb{R}^d :

$$X_t = g(t) + \int_0^t K_1(s, t) \gamma(X_s) dW_s + \int_0^t K_2(s, t) b(X_s) ds \quad (8)$$

where $(g(t))_{t \geq 0}$ is a \mathbb{R}^d -valued measurable and (\mathcal{F}_t) -adapted process, K_1 and K_2 are two kernels from Γ to \mathbb{R}_+ , b is a function from \mathbb{R}^d to \mathbb{R}^d , γ is a function from \mathbb{R}^d to $\mathbb{R}^d \times \mathbb{R}^q$ and W is a (\mathcal{F}_t) -adapted q -dimensional Brownian motion.

Theorem 3. Assume that (I.1) and (I.2) as well as the following conditions hold:

(i) There exists $p \geq 2$ such that, for any $T > 0$,

$$\sup_{t \in [0, T]} \mathbb{E}[\|g(t)\|^p] < \infty.$$

(ii) There exists $C > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$\|b(x) - b(y)\| + \|\gamma(x) - \gamma(y)\| \leq C\|x - y\|.$$

Then, there exists a unique (\mathcal{F}_t) -adapted solution X to Equation (8) such that:

$$\sup_{t \in [0, T]} \mathbb{E}[\|X_t\|^p] < \infty. \quad (9)$$

Remark 3. The conclusions of Theorem 3 still hold if the integral is taken from 0 to t in assumption (I.1) instead of from $-\Delta$ to t .

Under additional conditions on the kernels K_1 and K_2 , one can show that the solution to Equation (8) is continuous using the Kolmogorov-Centsov theorem. The following theorem is an adaptation of Theorem 3.3 in Zhang (2010).

Theorem 4. Assume that the conditions of Theorem 3 as well as conditions (I.4) and (I.5) hold. If g is almost surely continuous and satisfies for any $T > 0$ and any $p \geq 2$:

$$\sup_{t \in [0, T]} \mathbb{E} [\|g(t)\|^p] < \infty,$$

then there exists a unique continuous (\mathcal{F}_t) -adapted solution to Equation (8). Moreover, if there exists $\delta > 0$ such that for any $p \geq 2$:

$$\mathbb{E} [\|g(t') - g(t)\|^p] \leq C|t' - t|^{\delta p}, \quad 0 \leq t < t' \leq T,$$

then the solution is locally γ^* -Hölder continuous for any $\gamma^* \in \left(0, \min\left(\frac{1}{2\alpha_1^*}, \frac{1}{\alpha_2^*}, \gamma, \delta\right)\right)$ with $\alpha_i^* = \frac{\alpha_i}{\alpha_i - 1}$.

Remark 4. The conclusion of Theorem 4 still holds if the integral is taken from 0 to t in assumption (I.5) instead of from $-\Delta$ to t .

2 Proof of Theorem 1

The existence and uniqueness theorem (Theorem 3) cannot be used as is for model (6) since the global Lipschitz condition is not satisfied because of the square root and the square. However, by using a localization technique, we can return to the assumptions of Theorem 3 and show the existence of a solution up to some stopping time.

Step 1 of the proof of Theorem 1: existence of a local solution. For $n \in \mathbb{N}^*$, we define,

$$b_n(r_1, r_2) = \begin{pmatrix} 0 \\ \left(-n \vee \left(\beta_0 + \beta_1 r_1 + \beta_2 \sqrt{r_2 \vee \frac{1}{n}}\right) \wedge n\right) \left(\beta_0 + \beta_1 r_1 + \beta_2 \sqrt{r_2 \vee \frac{1}{n}}\right) \end{pmatrix},$$

$$\gamma_n(r_1, r_2) = \begin{pmatrix} \beta_0 + \beta_1 r_1 + \beta_2 \sqrt{r_2 \vee \frac{1}{n}} \\ 0 \end{pmatrix}.$$

One can easily check that b_n and γ_n are globally Lipschitz. Let us verify that $\sup_{t \in [0, T]} \mathbb{E} [\|g(t)\|^p] < \infty$ for all $p \geq 2$. Since the initial conditions $(r_{1,s})_{-\Delta \leq s \leq 0}$ and $(r_{2,s})_{-\Delta \leq s \leq 0}$ are deterministic, we have for all $p \geq 2$:

$$\begin{aligned} \mathbb{E} [|g_1(t)|^p] &= \mathbb{E} [|G|^p] \left(\int_{-\Delta}^0 K_1(s, t)^2 (\beta_0 + \beta_1 r_{1,s} + \beta_2 \sqrt{r_{2,s}})^2 ds \right)^{\frac{p}{2}} \\ &\leq \mathbb{E} [|G|^p] \sup_{s \in (-\Delta, 0]} |\beta_0 + \beta_1 r_{1,s} + \beta_2 \sqrt{r_{2,s}}|^p \left(\int_{-\Delta}^0 K_1(s, t)^2 ds \right)^{\frac{p}{2}} \end{aligned}$$

where G is a standard normal random variable. We deduce that $\mathbb{E} [|g_1(t)|^p]$ is finite by assumptions (I.1) and (I.3). Moreover, still by assumptions (I.1) and (I.3), we have for all $p \geq 2$:

$$\mathbb{E} [|g_2(t)|^p] \leq \sup_{s \in (-\Delta, 0]} |\beta_0 + \beta_1 r_{1,s} + \beta_2 \sqrt{r_{2,s}}|^{2p} \left(\int_{-\Delta}^0 K_2(s, t) ds \right)^p < \infty.$$

Thus, Theorem 3 guarantees the existence of a unique solution to the following equation on $[0, T]$:

$$R_t^{(n)} = g(t) + \int_0^t K_2(s, t) b_n \left(R_s^{(n)} \right) ds + \int_0^t K_1(s, t) \gamma_n \left(R_s^{(n)} \right) dW_s \quad (10)$$

verifying for all $p \geq 2$:

$$\sup_{t \in [0, T]} \mathbb{E} \left[\left\| R_t^{(n)} \right\|^p \right] < \infty. \quad (11)$$

Moreover, using assumptions **(I.3)** and **(I.5)**, we have for any $p > 0$ and $0 \leq t < t' \leq T$:

$$\begin{aligned} \mathbb{E} \left[\|g(t') - g(t)\|^p \right] &\leq C(|g_1(t') - g_1(t)|^p + |g_2(t') - g_2(t)|^p) \\ &\leq C \left(\mathbb{E} \left[\left\| \int_{-\Delta}^0 (K_1(s, t') - K_1(s, t))^2 (\beta_0 + \beta_1 r_{1,s} + \beta_2 \sqrt{r_{2,s}})^2 ds \right\|^{\frac{p}{2}} \right] \right. \\ &\quad \left. + \mathbb{E} \left[\left\| \int_{-\Delta}^0 (K_2(s, t') - K_2(s, t)) (\beta_0 + \beta_1 r_{1,s} + \beta_2 \sqrt{r_{2,s}})^2 ds \right\|^p \right] \right) \\ &\leq C|t' - t|^{\gamma p} \end{aligned}$$

where C is a constant that can change from line to line. We deduce that the solution to Equation (10) is locally γ^* -Hölder continuous for any $\gamma^* \in \left(0, \min\left(\frac{1}{2\alpha_1^*}, \frac{1}{\alpha_2^*}, \gamma\right)\right)$ by Theorem 4.

We now show that if τ is a stopping time, then the uniqueness of the solution to (10) holds on $[0, \tau)$. Let $R^{(n)}$ and $\tilde{R}^{(n)}$ be two solutions of (10) on $[0, \tau)$ and set $Z_t^{(n)} = R_t^{(n)} - \tilde{R}_t^{(n)}$. We have:

$$\begin{aligned} \mathbb{E} \left[\|Z_t^{(n)} \mathbb{1}_{\{t < \tau\}}\|^2 \right] &\leq 2 \left(\mathbb{E} \left[\left\| \int_0^{t \wedge \tau} K_2(s, t) \left(b_n \left(R_s^{(n)} \right) - b_n \left(\tilde{R}_s^{(n)} \right) \right) ds \right\|^2 \right] \right. \\ &\quad \left. + \mathbb{E} \left[\left\| \int_0^{t \wedge \tau} K_1(s, t) \left(\gamma_n \left(R_s^{(n)} \right) - \gamma_n \left(\tilde{R}_s^{(n)} \right) \right) dW_s \right\|^2 \right] \right). \end{aligned}$$

By the Cauchy-Schwarz inequality, the Lipschitz property of b_n , assumption **(I.1)** and Fubini's theorem for the first term, we obtain:

$$\begin{aligned} &\mathbb{E} \left[\left\| \int_0^{t \wedge \tau} K_2(s, t) \left(b_n \left(R_s^{(n)} \right) - b_n \left(\tilde{R}_s^{(n)} \right) \right) ds \right\|^2 \right] \\ &\leq C \int_0^t K_2(s, t) ds \mathbb{E} \left[\int_0^{t \wedge \tau} K_2(s, t) \left\| b_n \left(R_s^{(n)} \right) - b_n \left(\tilde{R}_s^{(n)} \right) \right\|^2 ds \right] \\ &\leq C \int_0^t K_2(s, t) \mathbb{E} \left[\left\| Z_s^{(n)} \mathbb{1}_{\{s < \tau\}} \right\|^2 \right] ds. \end{aligned}$$

Similarly, by Doob's inequality for the martingale $\left(\int_0^r K_1(s, t) \left(\gamma_n \left(R_s^{(n)} \right) - \gamma_n \left(\tilde{R}_s^{(n)} \right) \right) dW_s \right)_{r \in [0, t]}$, the Lipschitz property of γ_n and Fubini's theorem, we get:

$$\begin{aligned} &\mathbb{E} \left[\left\| \int_0^{t \wedge \tau} K_1(s, t) \left(\gamma_n \left(R_s^{(n)} \right) - \gamma_n \left(\tilde{R}_s^{(n)} \right) \right) dW_s \right\|^2 \right] \\ &\leq C \int_0^t K_1(s, t)^2 \mathbb{E} \left[\left\| Z_s^{(n)} \mathbb{1}_{\{s < \tau\}} \right\|^2 \right] ds. \end{aligned}$$

Thus,

$$\mathbb{E} \left[\left\| Z_t^{(n)} \mathbb{1}_{\{t < \tau\}} \right\|^2 \right] \leq C \int_0^t (K_2(s, t) + K_1(s, t)^2) \mathbb{E} \left[\left\| Z_s^{(n)} \mathbb{1}_{\{s < \tau\}} \right\|^2 \right] ds.$$

By assumption **(I.2)**, the generalised Grönwall lemma of Zhang (2010) (Lemma 2.2) can be applied which gives for all $t \in [0, T]$:

$$\mathbb{E} \left[\left\| Z_t^{(n)} \mathbb{1}_{\{t < \tau\}} \right\|^2 \right] = 0$$

which, by the arbitrariness of t and the continuity of $R^{(n)}$ and $\tilde{R}^{(n)}$, implies that $R^{(n)}|_{[0, \tau)} = \tilde{R}^{(n)}|_{[0, \tau)}$.

At this stage, we can construct a solution R to Equation (6) up to a positive stopping time τ . We define two stopping times τ_n and $\tilde{\tau}_n$ as:

$$\tau_n := \tau_n^1 \wedge \tau_n^2 \text{ and } \tilde{\tau}_n := \tilde{\tau}_n^1 \wedge \tilde{\tau}_n^2$$

where:

$$\begin{aligned} \tau_n^1 &= \inf \left\{ t \geq 0 : \left| \beta_0 + \beta_1 R_{1,t}^{(n)} + \beta_2 \sqrt{R_{2,t}^{(n)}} \right| \geq n \right\}, \\ \tau_n^2 &= \inf \left\{ t \geq 0 : R_{2,t}^{(n)} \leq 1/n \right\}, \\ \tilde{\tau}_n^1 &= \inf \left\{ t \geq 0 : \left| \beta_0 + \beta_1 R_{1,t}^{(n+1)} + \beta_2 \sqrt{R_{2,t}^{(n+1)}} \right| \geq n \right\}, \\ \tilde{\tau}_n^2 &= \inf \left\{ t \geq 0 : R_{2,t}^{(n+1)} \leq 1/n \right\}. \end{aligned}$$

Since $R^{(n)}$ and $R^{(n+1)}$ solve the same equation on $[0, \tau_n \wedge \tilde{\tau}_n)$ by definition of b_n and γ_n , we deduce that $R^{(n)}|_{[0, \tau_n \wedge \tilde{\tau}_n)} = R^{(n+1)}|_{[0, \tau_n \wedge \tilde{\tau}_n)}$ from the above claim. Hence, $\tau_n = \tilde{\tau}_n \leq \tau_{n+1}$ a.s. Therefore, we can define $\tau = \lim_{n \rightarrow +\infty} \tau_n$ and for all $t < \tau$, $R_t := R_t^{(n)}$ if $t < \tau_n$. The constructed process R is a solution of Equation 6 up to τ . Since $R^{(n)}$ is continuous for all $n \in \mathbb{N}^*$, $R^{(n)}|_{[0, \tau_n \wedge \tilde{\tau}_n)} = R^{(n+1)}|_{[0, \tau_n \wedge \tilde{\tau}_n)}$ and $\tau_n = \tilde{\tau}_n$, we deduce that $R_{\tau_n}^{(n)} = R_{\tau_n}^{(n+1)}$ when τ_n is finite. Thus, R is locally γ^* -Hölder continuous on $[0, \tau)$ for any $\gamma^* \in \left(0, \min\left(\frac{1}{2\alpha_1^*}, \frac{1}{\alpha_2^*}, \gamma\right)\right)$.

At this stage, we would like to show that $\mathbb{P}(\tau < \infty) = 0$, i.e. that the constructed solution is global. This relies on the following lemma, the proof of which is postponed after Step 3:

Lemma 1. *Under the assumptions (I.1)-(I.5), we have for all $T > 0$, $p \geq 0$ and $n \in \mathbb{N}^*$:*

$$\sup_{n \in \mathbb{N}^*} \mathbb{E} \left[\sup_{t \in [0, \hat{\tau}_n \wedge T]} (R_{1,t}^{2p} + R_{2,t}^p) \right] \leq C_{T,p}$$

where $C_{T,p}$ is a constant depending only on T and p while $\hat{\tau}_n := \hat{\tau}_n^1 \wedge \hat{\tau}_n^2$ with:

$$\begin{aligned} \hat{\tau}_n^1 &= \inf \left\{ t \geq 0 : \left| \beta_0 + \beta_1 R_{1,t} + \beta_2 \sqrt{R_{2,t}} \right| \geq n \right\}, \\ \hat{\tau}_n^2 &= \inf \left\{ t \geq 0 : R_{2,t} \leq \frac{1}{n} \right\}, \end{aligned}$$

R being the solution constructed in Step 1.

Step 2 of the proof of Theorem 1: proof of $\mathbb{P}(\tau < \infty) = 0$. Let $T > 0$, we have by Markov's inequality and Lemma 1:

$$\begin{aligned} \mathbb{P}(\hat{\tau}_n \leq T) &\leq \mathbb{P} \left(\sup_{t \in [0, T \wedge \hat{\tau}_n]} \left| \beta_0 + \beta_1 R_{1,t} + \beta_2 \sqrt{R_{2,t}} \right| \geq n \right) + \mathbb{P} \left(\inf_{t \in [0, T \wedge \hat{\tau}_n]} R_{2,t} \leq \frac{1}{n} \right) \\ &\leq \frac{1}{n} \mathbb{E} \left[\sup_{t \in [0, T \wedge \hat{\tau}_n]} \left| \beta_0 + \beta_1 R_{1,t} + \beta_2 \sqrt{R_{2,t}} \right| \right] + \mathbb{P} \left(\inf_{t \in [0, T \wedge \hat{\tau}_n]} R_{2,t} \leq \frac{1}{n} \right) \\ &\leq \frac{C_{T,p}}{n} + \mathbb{P} \left(\inf_{t \in [0, T \wedge \hat{\tau}_n]} R_{2,t} \leq \frac{1}{n} \right). \end{aligned}$$

Besides, by assumption (I.6), we have the following deterministic lower bound which does not depend on t :

$$\begin{aligned} R_{2,t} &= \int_{-\Delta}^0 K_2(s, t) \sigma_s^2 ds + \int_0^t K_2(s, t) \sigma_s^2 ds \\ &\geq \inf_{t' \in [0, T]} \underbrace{\int_{-\Delta}^0 K_2(s, t') \sigma_s^2 ds}_{g_2(t')} > 0. \end{aligned}$$

We deduce that $\mathbb{P}(\inf_{t \in [0, T \wedge \hat{\tau}_n]} R_{2,t} \leq \frac{1}{n}) = 0$ for n large enough and consequently that $\mathbb{P}(\hat{\tau}_n \leq T) \xrightarrow[n \rightarrow +\infty]{} 0$ for all $T > 0$. Hence, $\mathbb{P}(\tau < \infty) = 0$ since $\hat{\tau}_n \nearrow \tau$.

Step 3: solution's uniqueness. Let R be the global solution to Equation (6) that we have constructed and let \tilde{R} be another global solution. Let us set $\nu_n = \nu_n^1 \wedge \nu_n^2$, $\tilde{\nu}_n = \tilde{\nu}_n^1 \wedge \tilde{\nu}_n^2$ and $S_n = \nu_n \wedge \tilde{\nu}_n$ where:

$$\begin{aligned} \nu_n^1 &= \inf \left\{ t \geq 0 : \left| \beta_0 + \beta_1 R_{1,t} + \beta_2 \sqrt{R_{2,t}} \right| \geq n \right\}, \\ \nu_n^2 &= \inf \left\{ t \geq 0 : R_{2,t} \leq \frac{1}{n} \right\} \end{aligned}$$

and where $\tilde{\nu}_n^1, \tilde{\nu}_n^2$ are similarly defined for \tilde{R} instead of R . Let $0 \leq t \leq T$, we have:

$$\begin{aligned} \mathbb{E} \left[\|R_t - \tilde{R}_t\|^2 \mathbb{1}_{\{t \leq S_n\}} \right] &\leq C \underbrace{\left(\mathbb{E} \left[\left| \int_0^t K_1(s,t) \left(\beta_1 (R_{1,s} - \tilde{R}_{1,s}) + \beta_2 (\sqrt{R_{2,s}} - \sqrt{\tilde{R}_{2,s}}) \right) \mathbb{1}_{\{s \leq S_n\}} dW_s \right|^2 \right] \right)}_{I_1} \\ &+ \underbrace{\mathbb{E} \left[\left| \int_0^t K_2(s,t) \left(\left(\beta_0 + \beta_1 R_{1,s} + \beta_2 \sqrt{R_{2,s}} \right)^2 - \left(\beta_0 + \beta_1 \tilde{R}_{1,s} + \beta_2 \sqrt{\tilde{R}_{2,s}} \right)^2 \right) \mathbb{1}_{\{s \leq S_n\}} ds \right|^2 \right]}_{I_2} \end{aligned}$$

where C is a constant. In the sequel, this constant can change from line to line and may depend on n but not on t . By Burkholder-Davis-Gundy's inequality (abbreviated BDG in the sequel) to the local martingale $\left(\int_0^r K_1(s,t) \left(\beta_1 (R_{1,s} - \tilde{R}_{1,s}) + \beta_2 (\sqrt{R_{2,s}} - \sqrt{\tilde{R}_{2,s}}) \right) \mathbb{1}_{\{s \leq S_n\}} dW_s \right)_{r \in [0,t]}$ and the fact that $\sqrt{R_{2,s}}$ and $\sqrt{\tilde{R}_{2,s}}$ are bounded from below by $1/\sqrt{n}$ for $s \leq S_n$, we have:

$$\begin{aligned} I_1 &\leq C \mathbb{E} \left[\int_0^t K_1(s,t)^2 \left(\beta_1 (R_{1,s} - \tilde{R}_{1,s}) + \beta_2 (\sqrt{R_{2,s}} - \sqrt{\tilde{R}_{2,s}}) \right)^2 \mathbb{1}_{\{s \leq S_n\}} ds \right] \\ &\leq C \mathbb{E} \left[\int_0^t 2K_1(s,t)^2 \left(\beta_1^2 (R_{1,s} - \tilde{R}_{1,s})^2 + \frac{\beta_2^2 n}{4} (R_{2,s} - \tilde{R}_{2,s})^2 \right) \mathbb{1}_{\{s \leq S_n\}} ds \right] \\ &\leq C \int_0^t K_1(s,t)^2 \mathbb{E} \left[\|R_s - \tilde{R}_s\|^2 \mathbb{1}_{\{s \leq S_n\}} \right] ds. \end{aligned}$$

Moreover, using the fact that $2\beta_0 + \beta_1 (R_{1,s} + \tilde{R}_{1,s}) + \beta_2 (\sqrt{R_{2,s}} + \sqrt{\tilde{R}_{2,s}}) \leq 2n$ and $\sqrt{R_{2,s}} + \sqrt{\tilde{R}_{2,s}} \geq 2/\sqrt{n}$ for $s \leq S_n$, by the Cauchy-Schwarz inequality and by assumption **(I.1)**, we get:

$$\begin{aligned} I_2 &\leq 4n^2 \mathbb{E} \left[\left| \int_0^t K_2(s,t) \left(\beta_1 (R_{1,s} - \tilde{R}_{1,s}) + \beta_2 (\sqrt{R_{2,s}} - \sqrt{\tilde{R}_{2,s}}) \right) \mathbb{1}_{\{s \leq S_n\}} ds \right|^2 \right] \\ &\leq 4n^2 \int_0^t K_2(s,t) ds \times \mathbb{E} \left[\int_0^t K_2(s,t) \left(\beta_1 (R_{1,s} - \tilde{R}_{1,s}) + \beta_2 (\sqrt{R_{2,s}} - \sqrt{\tilde{R}_{2,s}}) \right)^2 \mathbb{1}_{\{s \leq S_n\}} ds \right] \\ &\leq C \int_0^t K_2(s,t) \mathbb{E} \left[\|R_s - \tilde{R}_s\|^2 \mathbb{1}_{\{s \leq S_n\}} \right] ds. \end{aligned}$$

We deduce that:

$$\mathbb{E} \left[\|R_t - \tilde{R}_t\|^2 \mathbb{1}_{\{t \leq S_n\}} \right] \leq C \int_0^t (K_1(s,t)^2 + K_2(s,t)) \mathbb{E} \left[\|R_s - \tilde{R}_s\|^2 \mathbb{1}_{\{s \leq S_n\}} \right] ds$$

Using the generalized Grönwall's lemma in Zhang (2010) (Lemma 2.2), we obtain that $\mathbb{E} \left[\|R_t - \tilde{R}_t\|^2 \mathbb{1}_{\{t \leq S_n\}} \right] = 0$. Thus, almost surely,

$$(R_t - \tilde{R}_t) \mathbb{1}_{\{t \leq S_n\}} = 0.$$

When S_n is finite, since R and \tilde{R} coincide on $[0, S_n]$ and either

$$|\beta_0 + \beta_1 R_{1,S_n} + \beta_2 R_{2,S_n}| = |\beta_0 + \beta_1 \tilde{R}_{1,S_n} + \beta_2 \tilde{R}_{2,S_n}| \geq n$$

or

$$R_{2,S_n} = \tilde{R}_{2,S_n} \leq \frac{1}{n},$$

then we have necessarily $S_n = \nu_n = \tilde{\nu}_n$ by definition of ν_n and $\tilde{\nu}_n$. According to the second step of the proof of Theorem 1, $\nu_n \xrightarrow[n \rightarrow +\infty]{a.s.} +\infty$. Hence, $S_n \xrightarrow[n \rightarrow +\infty]{a.s.} +\infty$ and $R_t = \tilde{R}_t$ for all $t \geq 0$.

We conclude this section with the proof of Lemma 1 which relies on the two next lemmas.

Lemma 2. *Under the assumptions (I.1)-(I.5), we have for all $T > 0$, $p \geq 0$ and $n \in \mathbb{N}^*$:*

$$\sup_{t \in [0, T]} \mathbb{E} [(|R_{1,t}|^{2p} + R_{2,t}^p) \mathbb{1}_{\{t \leq \hat{\tau}_n\}}] \leq C_{T,p}$$

where R and $\hat{\tau}_n$ are defined respectively in Step 1 and 2, and $C_{T,p}$ is a constant depending only on T and p .

Proof. Let $T > 0$, $t \in [0, T]$, $n \in \mathbb{N}^*$ and $p \geq \max(\alpha_1^*, \alpha_2^*)$ where $\alpha_i^* := \frac{\alpha_i}{\alpha_i - 1}$ and α_1, α_2 are defined in assumption (I.4). On the one hand, we have for R_1 :

$$\mathbb{E} [|R_{1,t}|^{2p} \mathbb{1}_{\{t \leq \hat{\tau}_n\}}] \leq 2^{p-1} \left(\mathbb{E} [|g_1(t)|^{2p}] + \mathbb{E} \left[\left| \int_0^{t \wedge \hat{\tau}_n} K_1(s, t) (\beta_0 + \beta_1 R_{1,s} + \beta_2 \sqrt{R_{2,s}}) dW_s \right|^{2p} \right] \right).$$

Applying BDG's inequality and twice Hölder's inequality, we get for the second term:

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^{t \wedge \hat{\tau}_n} K_1(s, t) (\beta_0 + \beta_1 R_{1,s} + \beta_2 \sqrt{R_{2,s}}) dW_s \right|^{2p} \right] \\ & \leq \mathbb{E} \left[\left(\int_0^{t \wedge \hat{\tau}_n} K_1(s, t)^2 (\beta_0 + \beta_1 R_{1,s} + \beta_2 \sqrt{R_{2,s}})^2 ds \right)^p \right] \\ & \leq \left(\int_0^t K_1(s, t)^{2\alpha_1} \right)^{\frac{p}{\alpha_1}} \mathbb{E} \left[\left(\int_0^{t \wedge \hat{\tau}_n} |\beta_0 + \beta_1 R_{1,s} + \beta_2 \sqrt{R_{2,s}}|^{2\alpha_1^*} ds \right)^{\frac{p}{\alpha_1^*}} \right] \\ & \leq \left(\int_0^t K_1(s, t)^{2\alpha_1} \right)^{\frac{p}{\alpha_1}} T^{\frac{p}{\alpha_1^*} - 1} \int_0^t \mathbb{E} \left[|\beta_0 + \beta_1 R_{1,s} \mathbb{1}_{\{s \leq \hat{\tau}_n\}} + \beta_2 \sqrt{R_{2,s} \mathbb{1}_{\{s \leq \hat{\tau}_n\}}} |^{2p} \right] ds \\ & \leq C_{p,T} \left(1 + \int_0^t \mathbb{E} [(|R_{1,s}|^{2p} + R_{2,s}^p) \mathbb{1}_{\{s \leq \hat{\tau}_n\}}] ds \right) \end{aligned}$$

where we used assumption (I.4) for the last line. On the other hand, we have for R_2 :

$$\mathbb{E} [R_{2,t}^p \mathbb{1}_{\{t \leq \hat{\tau}_n\}}] \leq 2^{p-1} \left(\mathbb{E} [g_2(t)^p] + \mathbb{E} \left[\left(\int_0^{t \wedge \hat{\tau}_n} K_2(s, t) (\beta_0 + \beta_1 R_{1,s} + \beta_2 \sqrt{R_{2,s}})^2 ds \right)^p \right] \right).$$

Applying again twice Hölder's inequality and using assumption (I.4) for the second term, we obtain similarly:

$$\mathbb{E} \left[\left(\int_0^{t \wedge \hat{\tau}_n} K_2(s, t) (\beta_0 + \beta_1 R_{1,s} + \beta_2 \sqrt{R_{2,s}})^2 ds \right)^p \right] \leq C_{T,p} \left(1 + \int_0^t \mathbb{E} [(|R_{1,s}|^{2p} + R_{2,s}^p) \mathbb{1}_{\{s \leq \hat{\tau}_n\}}] ds \right).$$

Bringing together the four previous equations and using the fact that $\sup_{t \in [0, T]} \mathbb{E} [\|g(t)\|^p] < \infty$ by conditions (I.1) and (I.3), we deduce that:

$$\mathbb{E} [(|R_{1,t}|^{2p} + R_{2,t}^p) \mathbb{1}_{\{t \leq \hat{\tau}_n\}}] \leq C_{T,p} \left(1 + \int_0^t \mathbb{E} [(|R_{1,s}|^{2p} + R_{2,s}^p) \mathbb{1}_{\{s \leq \hat{\tau}_n\}}] ds \right).$$

Since

$$\mathbb{E}[(|R_{1,s}|^{2p} + R_{2,s}^p)\mathbb{1}_{\{s \leq \hat{\tau}_n\}}] = \mathbb{E}\left[\left(\left|R_{1,s}^{(n)}\right|^{2p} + \left|R_{2,s}^{(n)}\right|^p\right)\mathbb{1}_{\{s \leq \hat{\tau}_n\}}\right] \leq \sup_{s \in [0, T]} \mathbb{E}\left[\left\|R_t^{(n)}\right\|^{2p} + \left\|R_t^{(n)}\right\|^p\right]$$

which is finite by (11), we can apply Grönwall's lemma to obtain:

$$\mathbb{E}\left[(|R_{1,t}|^{2p} + R_{2,t}^p)\mathbb{1}_{\{t \leq \hat{\tau}_n\}}\right] \leq C_{T,p}.$$

□

The following lemma is a consequence of the Garsia-Rodemich-Rumsey inequality (Garsia et al., 1970/71) and is inspired from Appendix A.3 in Nualart (2006).

Lemma 3. *Let $(X_t)_{t \geq 0}$ be a real-valued process and τ be a stopping time bounded by a constant $T > 0$. If there exists $p > 0$, $C > 0$ and $\varepsilon > 0$ such that for all $s, t \geq 0$:*

$$\mathbb{E}\left[|X_t - X_s|^p \mathbb{1}_{\{s, t \in [0, \tau]\}}\right] \leq C|t - s|^{1+\varepsilon}, \quad (12)$$

and $\mathbb{E}[|X_{t^*}|^p] < \infty$ for some t^* such that $\mathbb{P}(t^* \leq \tau) = 1$, then:

$$\mathbb{E}\left[\sup_{t \in [0, \tau]} |X_t|^p\right] < \infty.$$

Proof. Define $\psi(x) = |x|^p$ and $q(x) = |x|^{\frac{\gamma}{p}}$ for $0 < \gamma < (2 + \varepsilon) \wedge 2(p + 1)$. We have:

$$\begin{aligned} B &:= \int_{[0, \tau]^2} \psi\left(\frac{X_t - X_s}{q(t-s)}\right) ds dt \\ &= \int_{[0, T]^2} \frac{|X_t - X_s|^p}{|t-s|^\gamma} \mathbb{1}_{\{s, t \in [0, \tau]\}} ds dt. \end{aligned}$$

Taking the expectation on both sides, we get by Fubini's theorem and inequality (12):

$$\mathbb{E}[B] \leq \int_{[0, T]^2} |t-s|^{1+\varepsilon-\gamma} ds dt.$$

The right hand side is finite since $0 < \gamma < 2 + \varepsilon$, therefore $\mathbb{E}[B]$ is finite as well and we deduce that B is almost surely finite. By the Garsia-Rodemich-Rumsey inequality (Garsia et al., 1970/71), we obtain that almost surely for $s, t \in [0, \tau]$:

$$|X_t - X_s| \leq 8 \int_0^{|t-s|} \psi^{-1}\left(\frac{4B}{u^2}\right) q(du)$$

where $\psi^{-1}(u) := \sup\{v : \psi(v) \leq u\} = |u|^{1/p}$. Noting that $q(du) = \frac{\gamma}{p} u^{\frac{\gamma}{p}-1} du$ for $u > 0$, we get:

$$\begin{aligned} |X_t - X_s| &\leq C_{p,\gamma} B^{\frac{1}{p}} \int_0^{|t-s|} u^{\frac{\gamma-2}{p}-1} du \\ &\leq C_{p,\gamma,T} B^{\frac{1}{p}} \end{aligned}$$

where $C_{p,\gamma,T}$ is a deterministic constant depending only on p, γ and T . We deduce that:

$$\sup_{t \in [0, \tau]} |X_t|^p \leq |X_{t^*}|^p + C_{p,\gamma,T} B.$$

Hence $\mathbb{E}\left[\sup_{t \in [0, \tau]} |X_t|^p\right] < \infty$ since $\mathbb{E}[|X_{t^*}|^p] < \infty$ and $\mathbb{E}[B] < \infty$. □

We can now prove Lemma 1.

Proof of Lemma 1. The idea of the proof is to show that:

$$\mathbb{E}[|R_{1,t'} - R_{1,t}|^{2p} \mathbb{1}_{\{t,t' \in [0, \hat{\tau}_n \wedge T]\}}] + \mathbb{E}[|R_{2,t'} - R_{2,t}|^p \mathbb{1}_{\{t,t' \in [0, \hat{\tau}_n \wedge T]\}}] \leq C|t' - t|^{1+\varepsilon}$$

for $0 \leq t < t' \leq T$ with $C, \varepsilon > 0$ so that we can conclude by applying Lemma 3 since $\mathbb{E}[|R_{1,0}|^{2p} + R_{2,0}^p] = \mathbb{E}[|g_1(0)|^{2p} + g_2(0)^p] < \infty$ (see Step 1). We start with the first term. Let $0 \leq t < t' \leq T$ and $p > \min\left(\frac{1}{\gamma}, \alpha_1^*, \alpha_2^*\right)$ where $\alpha_i^* := \frac{\alpha_i}{\alpha_i - 1}$ and $\alpha_1, \alpha_2, \gamma$ are defined in assumptions **(I.4)** and **(I.5)**. We have:

$$\begin{aligned} \mathbb{E}[|R_{1,t'} - R_{1,t}|^{2p} \mathbb{1}_{\{t,t' \in [0, \hat{\tau}_n \wedge T]\}}] &\leq C \left(\underbrace{\mathbb{E}[|g_1(t') - g_1(t)|^{2p}]}_{I_1(t,t')} + \underbrace{\mathbb{E}\left[\left|\int_0^t (K_1(s,t') - K_1(s,t))\gamma(R_s)\mathbb{1}_{\{s \leq \hat{\tau}_n\}}dW_s\right|^{2p}\right]}_{I_2(t,t')} \right. \\ &\quad \left. + \underbrace{\mathbb{E}\left[\left|\int_t^{t'} K_1(s,t')\gamma(R_s)\mathbb{1}_{\{s \leq \hat{\tau}_n\}}dW_s\right|^{2p}\right]}_{I_3(t,t')} \right) \end{aligned}$$

where C is a constant that can change from line to line in the sequel but which does not depend on n . We already showed in the Step 1 that:

$$\begin{aligned} I_1(t,t') &\leq C \mathbb{E}\left[\left|\int_{-\Delta}^0 (K_1(s,t') - K_1(s,t))^2 (\beta_0 + \beta_1 r_{1,s} + \beta_2 \sqrt{r_{2,s}})^2 ds\right|^p\right] \\ &\leq C|t' - t|^{2\gamma p}. \end{aligned}$$

By BDG's inequality, Hölder's inequality, Lemma 2 and assumption **(I.5)**, we get:

$$\begin{aligned} I_2(t,t') &\leq C \mathbb{E}\left[\left|\int_0^t (K_1(s,t') - K_1(s,t))^2 (\beta_0 + \beta_1 R_{1,s} + \beta_2 \sqrt{R_{2,s}})^2 \mathbb{1}_{\{s \leq \hat{\tau}_n\}} ds\right|^p\right] \\ &\leq C \left(\int_0^t (K_1(s,t') - K_1(s,t))^2 ds\right)^{p-1} \\ &\quad \times \int_0^t (K_1(s,t') - K_1(s,t))^2 \mathbb{E}\left[|\beta_0 + \beta_1 R_{1,s} + \beta_2 \sqrt{R_{2,s}}|^{2p} \mathbb{1}_{\{s \leq \hat{\tau}_n\}}\right] ds \\ &\leq C \left(\int_0^t (K_1(s,t') - K_1(s,t))^2 ds\right)^p \\ &\leq C|t' - t|^{2\gamma p}. \end{aligned}$$

Finally, by BDG's inequality, twice Hölder's inequality, assumption **(I.4)** and Lemma 2, we have:

$$\begin{aligned} I_3(t,t') &\leq C \mathbb{E}\left[\left|\int_t^{t'} K_1(s,t')^2 (\beta_0 + \beta_1 R_{1,s} + \beta_2 \sqrt{R_{2,s}})^2 \mathbb{1}_{\{s \leq \hat{\tau}_n\}} ds\right|^p\right] \\ &\leq C \left(\int_t^{t'} K_1(s,t')^{2\alpha_1} ds\right)^{\frac{p}{\alpha_1}} \mathbb{E}\left[\left|\int_t^{t'} (\beta_0 + \beta_1 R_{1,s} + \beta_2 \sqrt{R_{2,s}})^{2\alpha_1^*} \mathbb{1}_{\{s \leq \hat{\tau}_n\}} ds\right|^{\frac{p}{\alpha_1^*}}\right] \\ &\leq C|t' - t|^{\frac{p}{\alpha_1^*} - 1} \int_t^{t'} \mathbb{E}\left[(\beta_0 + \beta_1 R_{1,s} + \beta_2 \sqrt{R_{2,s}})^{2p} \mathbb{1}_{\{s \leq \hat{\tau}_n\}}\right] ds \\ &\leq C|t' - t|^{\frac{p}{\alpha_1^*}}. \end{aligned}$$

Combining the four previous equations, we obtain:

$$\mathbb{E}[|R_{1,t'} - R_{1,t}|^{2p} \mathbb{1}_{\{t,t' \in [0, \hat{\tau}_n \wedge T]\}}] \leq C|t' - t|^{\xi_1}$$

where $\xi_1 = 2\gamma p \wedge \frac{p}{\alpha_1^*} > 1$. Using similar arguments, we can show that:

$$\mathbb{E}[|R_{2,t'} - R_{2,t}|^p \mathbb{1}_{\{t,t' \in [0, \hat{\tau}_n \wedge T]\}}] \leq C|t' - t|^{\xi_2}$$

where $\xi_2 = \gamma p \wedge \frac{p}{\alpha_2^*} > 1$. □

3 Proof of Theorem 2

In this section, $R = (R_1, R_2)$ is the local continuous solution that has been constructed in Step 1 of the proof of Theorem 1 up to a stopping time τ (note that this construction requires only assumption **(I.1)**-**(I.5)**). We will first prove that the volatility σ is bounded from below by a positive stochastic process, and then we will show that the condition **(I.6)** ensuring that the solution is global is implied by the additional assumptions we have made to get the positivity of the volatility. We first introduce two lemmas.

Lemma 4. *Under assumptions **(I.1)**-**(I.5)** and **(II.1)**, R_1 and R_2 are Itô processes on $[0, \tau)$ with decompositions:*

$$\begin{aligned} dR_{1,t} &= \left(\int_{-\Delta}^t \partial_t K_1(u, t) \sigma_u dW_u \right) dt + K_1(t, t) \sigma_t dW_t, \\ dR_{2,t} &= \left(\int_{-\Delta}^t \partial_t K_2(u, t) \sigma_u^2 du \right) dt + K_2(t, t) \sigma_t^2 dt \end{aligned}$$

where we recall that $\sigma_t := \beta_0 + \beta_1 R_{1,t} + \beta_2 \sqrt{R_{2,t}}$.

Proof. We have by definition:

$$R_{1,t} = \int_{-\Delta}^t K_1(u, t) \sigma_u dW_u = \int_{-\Delta}^0 K_1(u, t) \sigma_u dW_u + \int_0^t K_1(u, t) \sigma_u dW_u,$$

By the fundamental theorem of calculus, we can write $K_1(u, t) = K_1(u, s) + \int_s^t \partial_v K_1(u, v) dv$ for all $0 \leq u \leq s \leq t$. Thus,

$$R_{1,t} = \int_{-\Delta}^0 \left(K_1(u, 0) + \int_0^t \partial_v K_1(u, v) dv \right) \sigma_u dW_u + \int_0^t \left(K_1(u, u) + \int_u^t \partial_v K_1(u, v) dv \right) \sigma_u dW_u. \quad (13)$$

Using assumption **(I.3)** and **(II.1)**, we have almost surely:

$$\int_0^t \left(\int_{-\Delta}^0 |\partial_v K_1(u, v)|^2 \sigma_u^2 du \right)^{1/2} dv \leq \sup_{s \in (-\Delta, 0]} \sigma_s \times \int_0^t \left(\int_{-\Delta}^0 |\partial_v K_1(u, v)|^2 dv \right)^{1/2} du < \infty.$$

Similarly, using the continuity of σ deduced from the continuity of R , we can show that:

$$\int_0^t \left(\int_0^v |\partial_v K_1(u, v)|^2 \sigma_u^2 du \right)^{1/2} dv < \infty \text{ a.s.}$$

Hence, we can apply the stochastic Fubini's theorem (see Veraar, 2012) in Equation (13) to obtain:

$$\begin{aligned} R_{1,t} &= r_{1,0} + \int_0^t \left(\int_{-\Delta}^0 \partial_v K_1(u, v) \sigma_u dW_u \right) dv + \int_0^t K_1(u, u) \sigma_u dW_u + \int_0^t \left(\int_0^v \partial_v K_1(u, v) \sigma_u dW_u \right) dv \\ &= r_{1,0} + \int_0^t K_1(u, u) \sigma_u dW_u + \int_0^t \left(\int_{-\Delta}^v \partial_v K_1(u, v) \sigma_u dW_u \right) dv. \end{aligned}$$

Using similar arguments, we also obtain:

$$R_{2,t} = r_{2,0} + \int_0^t K_2(u, u) \sigma_u^2 du + \int_0^t \int_{-\Delta}^v \partial_v K_2(u, v) \sigma_u^2 dudv.$$

□

The following lemma is an extension of the comparison result for solutions of SDEs of [Karatzas and Shreve (1991), Proposition 2.18].

Lemma 5. Let X and Y be real-valued continuous Itô processes with decompositions:

$$\begin{aligned} X_t &= X_0 + \int_0^t b_X(s, X_s) ds + \int_0^t \gamma(s, X_s) dW_s \\ Y_t &= Y_0 + \int_0^t G_s ds + \int_0^t \gamma(s, Y_s) dW_s \end{aligned}$$

Under the following assumptions:

(i) $X_0 \leq Y_0$ a.s.;

(ii) b_X is a real-valued function on $\mathbb{R}_+ \times \mathbb{R}$ such that:

$$|b_X(t, x) - b_X(t, y)| \leq C|x - y| \quad \forall t \in \mathbb{R}_+, \forall x, y \in \mathbb{R}$$

for a positive constant C ;

(iii) G satisfies:

$$b_X(t, Y_t) \leq G_t \quad \forall t \geq 0 \text{ a.s.};$$

(iv) γ is a real-valued function on $\mathbb{R}_+ \times \mathbb{R}$ such that:

$$|\gamma(t, x) - \gamma(t, y)| \leq \ell(t)h(|x - y|) \quad \forall t \geq 0, \forall (x, y) \in \mathbb{R}^2$$

for $\ell : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a locally square-integrable function and $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a strictly increasing function with $h(0) = 0$ and

$$\int_0^\varepsilon h^{-2}(u) du = +\infty, \quad \forall \varepsilon > 0;$$

we have $\mathbb{P}(X_t \leq Y_t, \forall t \geq 0) = 1$.

Proof. Using the assumptions on the function h , we can construct a strictly decreasing sequence $(a_n)_{n \in \mathbb{N}}$ with $a_0 = 1$, $\lim_{n \rightarrow +\infty} a_n = 0$ and $\int_{a_n}^{a_{n-1}} h^{-2}(u) du = n$ for all $n \geq 1$. Moreover, we can construct a sequence $(\rho_n)_{n \geq 1}$ of continuous functions on \mathbb{R} with support in (a_n, a_{n-1}) such that:

- $0 \leq \rho_n(x) \leq \frac{2}{nh^2(x)}$ for all $x > 0$ and
- $\int_{a_n}^{a_{n-1}} \rho_n(x) dx = 1$

because the upper bound in the first item integrates to 2 over (a_n, a_{n-1}) . Let us now define the sequence $(\psi_n)_{n \geq 1}$ as:

$$\psi_n(x) = \int_0^x \int_0^y \rho_n(u) du dy \quad \text{for } x \in \mathbb{R}.$$

For all $n \geq 1$, ψ_n is twice continuously differentiable ($\psi_n(x) = 0$ for $x \leq a_n$) with $\psi'_n(x) = \int_0^x \rho_n(u) du \in [0, 1]$. Besides, $\psi_n(x) \xrightarrow[n \rightarrow +\infty]{} x^+$ since $\psi_n(x) = 0$ if $x \leq 0$, $\psi_n(x) \leq \int_0^x dy = x$ and for $x \geq a_{n-1}$,

$$\begin{aligned} \psi_n(x) &= \int_{a_n}^x \int_{a_n}^y \rho_n(u) du dy \\ &= \underbrace{\int_{a_n}^{a_{n-1}} \int_{a_n}^y \rho_n(u) du dy}_{\geq 0} + \underbrace{\int_{a_{n-1}}^x \int_{a_n}^y \rho_n(u) du dy}_{=1} \\ &\geq \int_{a_{n-1}}^x du = x - a_{n-1}. \end{aligned}$$

Using a similar decomposition, we can also show that $\psi_{n+1}(x) - \psi_n(x) \geq 0$ for all $x \in \mathbb{R}$, i.e. that the sequence $(\psi_n)_{n \geq 1}$ is non-decreasing. By a localization argument, we can assume that without loss of generality:

$$\mathbb{E} \left[\int_0^t |\gamma(s, X_s)|^2 + |\gamma(s, Y_s)|^2 ds \right] < \infty \quad (14)$$

since $\int_0^t \gamma(s, X_s)^2 + \gamma(s, Y_s)^2 ds$ is finite for any $t \geq 0$ almost surely. We have:

$$\begin{aligned}\Delta_t &:= X_t - Y_t \\ &= X_0 - Y_0 + \int_0^t (b_X(s, X_s) - G_s) ds + \int_0^t (\gamma(s, X_s) - \gamma(s, Y_s)) dW_s.\end{aligned}$$

Thus, by Itô's formula,

$$\begin{aligned}\psi_n(\Delta_t) &= \psi_n(X_0 - Y_0) + \int_0^t \psi'_n(\Delta_s) (b_X(s, X_s) - G_s) ds \\ &\quad + \int_0^t \psi'_n(\Delta_s) (\gamma(s, X_s) - \gamma(s, Y_s)) dW_s + \frac{1}{2} \int_0^t \psi''_n(\Delta_s) (\gamma(s, X_s) - \gamma(s, Y_s))^2 ds.\end{aligned}$$

By assumption (14) and $0 \leq \psi'_n \leq 1$, the stochastic integral is a martingale so its expectation is 0. Besides, $\psi_n(X_0 - Y_0) \leq 0$ since ψ_n is non-decreasing. Thus, taking the expectation on both sides in the above equation, we get:

$$\begin{aligned}\mathbb{E}[\psi_n(\Delta_t)] &\leq \mathbb{E} \left[\int_0^t \psi'_n(\Delta_s) (b_X(s, X_s) - G_s) ds \right] \\ &\quad + \frac{1}{2} \mathbb{E} \left[\int_0^t \psi''_n(\Delta_s) (\gamma(s, X_s) - \gamma(s, Y_s))^2 ds \right].\end{aligned}$$

Using that $\psi'_n(x) = \rho_n(|x|)$, $\rho_n(x) \leq \frac{2}{nh^2(x)}$ and assumption (iv), we obtain:

$$\frac{1}{2} \mathbb{E} \left[\int_0^t \psi''_n(\Delta_s) (\gamma(s, X_s) - \gamma(s, Y_s))^2 ds \right] \leq \frac{1}{n} \int_0^t \ell(s)^2 ds.$$

Finally, by assumption (ii), remark that:

$$\begin{aligned}b_X(s, X_s) - G_s &= b_X(s, X_s) - b_X(s, Y_s) + \underbrace{b_X(s, Y_s) - G_s}_{\leq 0} \\ &\leq C|\Delta_s|.\end{aligned}$$

Since $\psi'_n(x) = 0$ for $x \leq 0$ and $\psi'_n \leq 1$, we have therefore:

$$\psi'_n(\Delta_s) (b_X(s, X_s) - G_s) \leq C\Delta_s^+.$$

Hence,

$$\mathbb{E}[\psi_n(\Delta_t)] \leq C \int_0^t \mathbb{E}[\Delta_s^+] ds + \frac{1}{n} \int_0^t \ell(s)^2 ds.$$

Taking the limit $n \rightarrow +\infty$, we deduce from the monotone convergence theorem that $\mathbb{E}[\Delta_t^+] \leq C \int_0^t \mathbb{E}[\Delta_s^+] ds$. Again, without loss of generality, we can assume that $\mathbb{E}[\int_0^t \Delta_s^+ ds] < \infty$ because if this assumption does not hold, we can use a localization argument to return to a situation where it holds. Grönwall's lemma yields $\mathbb{E}[\Delta_t^+] = 0$. Thus, $X_t - Y_t \leq 0$ for all $t \geq 0$ a.s. \square

Endowed with these two lemmas, we are able to establish a result generalizing the one of Nutz and Valdevenito (2023).

Proposition 1. *Under assumptions (I.1)-(I.5) and (II.1)-(II.3), we have:*

$$\sigma_t \geq \sigma_0 \exp \left(\beta_1 \int_0^t K_1(s, s) dW_s + h(t) - h(0) - \frac{1}{2} \beta_1^2 \int_0^t K_1(s, s)^2 ds \right)$$

for all $0 \leq t < \tau$ almost surely.

Proof. Let $0 \leq t < \tau$. There exists $n \geq 1$ such that $t < \hat{\tau}_n$ where $\hat{\tau}_n$ is defined in the second step of the proof of Theorem 1. Because on $[0, \hat{\tau}_n)$, $R_{2,t} > 1/n$ and because the square root function is twice continuously differentiable on \mathbb{R}_+^* , we can apply Itô's lemma, which combined with Lemma 4 yields:

$$d\sqrt{R_{2,t}} = \frac{1}{2\sqrt{R_{2,t}}} \left(\int_{-\Delta}^t \partial_t K_2(u, t) \sigma_u^2 du \right) dt + \frac{1}{2\sqrt{R_{2,t}}} K_2(t, t) \sigma_t^2 dt.$$

Noting that $\partial_t K_1(u, t) = h'(t)K_1(u, t)$, we also have by Lemma 4:

$$dR_{1,t} = h'(t)R_{1,t}dt + K_1(t, t)\sigma_t dW_t.$$

Thus, by definition of σ_t :

$$\sigma_t = \sigma_0 + \int_0^t G_s ds + \int_0^t \gamma(s, \sigma_s) dW_s$$

where $\gamma(s, x) = \beta_1 K_1(s, s)x$ for $(s, x) \in \mathbb{R}_+ \times \mathbb{R}$ and

$$\begin{aligned} G_t &= \beta_1 h'(t)R_{1,t} + \frac{\beta_2}{2\sqrt{R_{2,t}}} \left(\int_{-\Delta}^t \partial_t K_2(u, t) \sigma_u^2 du \right) + \frac{\beta_2}{2\sqrt{R_{2,t}}} K_2(t, t) \sigma_t^2 \\ &= h'(t)\sigma_t - h'(t)\beta_0 - h'(t)\beta_2\sqrt{R_{2,t}} + \frac{\beta_2}{2\sqrt{R_{2,t}}} \left(\int_{-\Delta}^t \partial_t K_2(u, t) \sigma_u^2 du \right) + \frac{\beta_2}{2\sqrt{R_{2,t}}} K_2(t, t) \sigma_t^2 \\ &= h'(t)\sigma_t - h'(t)\beta_0 + \frac{\beta_2}{\sqrt{R_{2,t}}} \int_{-\Delta}^t \left(-h'(t)K_2(u, t) + \frac{1}{2}\partial_t K_2(u, t) \right) \sigma_u^2 du + \frac{\beta_2}{2\sqrt{R_{2,t}}} K_2(t, t) \sigma_t^2. \end{aligned}$$

Using that $h' \leq 0$, $\beta_0, \beta_2 \geq 0$ and assumption **(II.3)**, we have $G_t \geq h'(t)\sigma_t$. Moreover, γ satisfies assumption (iv) of Lemma 5 since $t \mapsto K_1(t, t)$ is locally square-integrable by assumption **(II.1)**. According to Lemma 5, we therefore have $\sigma_t \geq X_t$ for all $t \in [0, \tau)$ almost surely where X is solution of the following SDE:

$$dX_t = h'(t)X_t dt + \beta_1 K_1(t, t)X_t dW_t$$

with $X_0 = \sigma_0$. The solution of this SDE is $X_t = \sigma_0 \exp\left(\beta_1 \int_0^t K_1(s, s) dW_s + h(t) - h(0) - \frac{1}{2}\beta_1^2 \int_0^t K_1(s, s)^2 ds\right)$ which concludes the proof. \square

We conclude the proof of Theorem 2 by showing that condition **(I.6)** is implied by the conditions of Theorem 2 so that we can apply Theorem 1 to get the global existence and uniqueness of a solution (R_1, R_2) to Equation (6).

Lemma 6. *Assumption **(I.6)** holds if assumptions **(II.2)**, **(II.3)** and **(II.4)** are satisfied.*

Proof. Let $T > 0$ and $t \in [0, T]$. As already observed in Remark 2, assumption **(II.3)** implies that:

$$K_2(s, t) \geq K_2(s, 0) e^{2 \int_0^t h'(u) du} \quad \forall s \in (-\Delta, 0].$$

Thus,

$$\begin{aligned} g_2(t) &:= \int_{-\Delta}^0 K_2(s, t) (\beta_0 + \beta_1 r_{1,s} + \beta_2 \sqrt{r_{2,s}})^2 ds \\ &\geq e^{2 \int_0^t h'(u) du} \underbrace{\int_{-\Delta}^0 K_2(s, 0) (\beta_0 + \beta_1 r_{1,s} + \beta_2 \sqrt{r_{2,s}})^2 ds}_{g_2(0)}. \end{aligned}$$

Hence,

$$\inf_{t \in [0, T]} g_2(t) \geq e^{2 \int_0^T h'(u) du} g_2(0) > 0$$

by assumptions **(II.2)** and **(II.3)**. \square

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