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Simple sufficient criteria for second-order extended regular variation in the Gumbel domain of attraction

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Abstract. Statistical theory for the estimators of the location, scale and shape parameters of the approximating Generalized Pareto distribution in the Peaks-Over-Threshold approach to extreme value statistics typically requires a second-order extended regular variation condition on the tail quantile function of the distribution underlying the random variable of interest. Somewhat surprisingly, the existing sufficient criteria ensuring that this condition holds appear not to easily apply to many common light-tailed distributions, from the well-known Gaussian, log-normal and Gamma distributions to more specific examples arising in finance and reliability theory. We provide several sufficient criteria based on the computation of appropriate asymptotic expansions of a distribution function or of its quantile function. A list of examples to which our theory applies is given.

AMS Subject Classifications: 60E05, 62E10

Keywords: Asymptotic expansion, Extreme value condition, Generalized Gamma distribution, Gumbel domain of attraction, Tail quantile function, Weibull-tailed distribution

1 Introduction

In classical extreme value theory, the Peaks-Over-Threshold approach rests upon the assumption, motivated by the Pickands-Balkema-de Haan theorem, that exceedances of the random variable of interest X above a high threshold are approximately Generalized Pareto distributed. Equivalently, if F is the distribution function of X , q is its quantile function, and $U : t \mapsto q(1 - 1/t)$ is its tail quantile function, the standard assumption is that there exists $\gamma \in \mathbb{R}$, called the extreme value index of X , and a positive scale function a such that

$$\forall x > 0, \lim_{t \rightarrow \infty} \frac{U(tx) - U(t)}{a(t)} = \int_1^x s^{\gamma-1} ds.$$

This condition, rewritten as the extrapolation formula $U(tx) \approx U(t) + a(t) \int_1^x s^{\gamma-1} ds$, plays a fundamental role in the construction of semiparametric extreme quantile estimators of X ; this usually rests upon the construction of a triplet of estimators for the location parameter $U(t)$, the scale parameter $a(t)$ and the shape parameter γ , see Chapters 3 and 4 in de Haan and Ferreira (2006) and Sections 5.2, 5.5 and 5.6 in Beirlant et al. (2004) for a detailed exposition.

At the heart of the statistical theory for the best-known estimators of these parameters, such as the moment estimators of Dekkers et al. (1989), the Generalized Pareto maximum likelihood estimators of Smith (1987) and Drees et al. (2004) and the probability-weighted moment estimators of Hosking et al. (1985) and Diebolt et al. (2007), is a refinement of the extreme value condition called the second-order (extended) regular variation condition on the tail quantile function U . It is usually formulated

as follows: there exist a second-order parameter $\rho \leq 0$ and an auxiliary function A having constant sign and converging to 0 at infinity such that

$$\forall x > 0, \lim_{t \rightarrow \infty} \frac{1}{A(t)} \left(\frac{U(tx) - U(t)}{a(t)} - \int_1^x s^{\gamma-1} ds \right) = \int_1^x s^{\gamma-1} \left(\int_1^s u^{\rho-1} du \right) ds.$$

See Definition 2.3.1 p.44 in de Haan and Ferreira (2006) or Equation (5.32) p.163 in Beirlant et al. (2004) and the subsequent discussion therein. This second-order condition quantifies the error made when approximating the extremes of the underlying distribution by those of the Generalized Pareto model, and as such, is key when evaluating the bias of extreme value estimators. Moreover, the function $|A|$ is necessarily regularly varying with index ρ (see for example Theorem 2.3.3 p.44 in de Haan and Ferreira (2006)), so that the value of ρ gives an idea of the difficulty of the extreme value extrapolation problem: the closer ρ is to 0, the slower $A(t)$ converges to 0 and the harder the extreme value problem tends to be.

It is therefore of interest to know whether a given tail quantile function U satisfies the second-order condition, and if that is the case, to quantify the scale function a , the second-order parameter ρ and the auxiliary function A . In the Fréchet and Weibull domains of attraction, that is, when $\gamma \neq 0$, simple sufficient criteria formulated in terms of asymptotic expansions of either U or F have been known for decades, see for instance Exercises 2.12 and 2.14 p.62 in de Haan and Ferreira (2006); within the Fréchet domain of attraction, such criteria typically require the distribution to belong to the so-called Hall class of distributions, see Hall (1982). This idea was later extended to a sufficient criterion on a probability density function in Lemma A.1 in Daouia et al. (2024a), so as to provide a result easier to apply in models whose distribution and quantile functions cannot be expressed in simple closed form, such as the Student and Fisher distributions.

Checking the second-order condition in the Gumbel domain of attraction $\gamma = 0$, however, appears to be more difficult. We are only aware of three criteria: first, Theorem 2.3.8 p.48 in de Haan and Ferreira (2006), which links this condition to an equivalent second-order condition on the distribution function F . Checking the latter requires an ansatz for the function a and, in the examples we are aware of, the computation of an asymptotic expansion for $1 - F$; see for instance the application to the Gaussian distribution in Exercise 2.9 p.61 in de Haan and Ferreira (2006) taken from p.391 in de Haan and Stadtmüller (1996). A second criterion is Theorem 2.3.12 p.49 in de Haan and Ferreira (2006), which states that the second-order condition is satisfied with $\gamma = 0$ if U is twice differentiable and the function $t \mapsto 1 + tU''(t)/U'(t)$ has constant sign, converges to 0 at infinity, and its absolute value is regularly varying with nonnegative index. This criterion is useful when the quantile function is explicit and smooth, which is for example the case for the logistic, Weibull and Gumbel distributions, but it is much harder to use otherwise. A third result is Lemma A.1 in Daouia et al. (2024a), which contains a criterion for the case when U is asymptotically equivalent to a power of the logarithm function, with a remainder term of a fairly specific form, which is unfortunately not the case in many important examples such as, again, the Gaussian and Gamma distributions. Interestingly, the companion, but different, problem of checking the rate of weak convergence of renormalized maxima of light-tailed distributions to the Gumbel distribution seems to have received much more attention: see Chapter 2 in Resnick (1987) and in particular Section 2.4.2 therein.

The contribution of this paper is to give several simple sufficient criteria to check the second-order condition for $\gamma = 0$. Our overarching framework is the class of Weibull-tailed distributions, which define a subclass of light-tailed distributions that has been the subject of a substantial body of extreme value statistical research, from the work of Broniatowski (1993) to later research by, among others, Girard (2004), Gardes et al. (2011), Mercadier and Soulier (2012) and El Methni and Girard (2024). We first consider the case when the distribution of interest in fact has an exponential decay of Generalized Gamma-type in the sense of Stacy (1962), that is, we assume that $1 - F(x)$ satisfies a suitable asymptotic expansion with leading term $cx^\alpha e^{-Cx^\beta}$ as $x \rightarrow \infty$ for some $\alpha \in \mathbb{R}$ and $\beta, c, C > 0$. This makes it possible to derive asymptotic expansions for the tail quantile function U and then to check the second-order condition with explicit values for a and A . Second, we prove a high-level result on the validity of the second-order condition when an asymptotic expansion for U is assumed

to be available in the first place. The intersection of these two settings is not empty: for example, both results apply to the Generalized Gamma and Gaussian distributions, but our theory is crafted to handle a wide range of examples, in which U may be explicitly determined or not. We provide a list of distributions covered by our results, such as the Generalized Gamma distribution, the Gaussian and Variance-Gamma distributions, the Gamma-Gompertz distribution and the logistic-beta distribution.

This paper is organized as follows. Section 2 gives necessary notation, after which Section 3 focuses on the class of Generalized Gamma-type distributions and Section 4 discusses the class of Weibull-tailed distributions in general. The Appendix gathers the proofs of our results.

2 Background and notation

Let F be a distribution function, with associated survival function $\bar{F} = 1 - F$ and quantile function $q : \tau \mapsto \inf\{x \in \mathbb{R} \mid F(x) \geq \tau\}$. The corresponding tail quantile function is $U : t \mapsto q(1 - 1/t)$, for $t > 1$; it is the left-continuous inverse of the function $1/\bar{F}$. We assume throughout that the right endpoint of F is infinite, that is, $q(1) = U(\infty) = \infty$. Our results will focus on distributions belonging to the Gumbel domain of attraction, that is, whose tail quantile function U satisfies the extended regular variation condition with extreme value index $\gamma = 0$:

$$\forall x > 0, \lim_{t \rightarrow \infty} \frac{U(tx) - U(t)}{a(t)} = \log(x) \quad \text{for a certain positive scale function } t \mapsto a(t).$$

Our aim is to provide sufficient conditions guaranteeing that the following, stronger, second-order extended regular variation condition holds:

$\mathcal{C}_2(a, \rho, A)$ There exist a positive scale function a , a second-order parameter $\rho \leq 0$ and an auxiliary function A having constant sign and converging to 0 at infinity such that

$$\forall x > 0, \lim_{t \rightarrow \infty} \frac{1}{A(t)} \left(\frac{U(tx) - U(t)}{a(t)} - \log(x) \right) = \int_1^x s^{-1} \left(\int_1^s u^{\rho-1} du \right) ds.$$

The gist of our approach is that, similarly to what has been found in the Fréchet and Weibull domains of attraction (see Exercises 2.12 and 2.14 p.62 in de Haan and Ferreira (2006)), having a sufficiently precise asymptotic expansion of U at one's disposal is enough in order to check condition $\mathcal{C}_2(a, \rho, A)$ and to calculate the functions a , A and the parameter ρ .

This insight will be the basis for our results. The asymptotic behavior of distributions belonging to the Gumbel domain of attraction can however be quite difficult to characterize, as Theorem 1.2.6 p.22 in de Haan and Ferreira (2006) shows. Throughout this paper, we shall work with the class of Weibull-tailed distributions, which contains the vast majority of the distributions belonging to the Gumbel domain of attraction that are used in practice, and whose definition we recall below.

Definition 1 (Weibull-tailed distribution). We say that a distribution function F is *Weibull-tailed* (or has a Weibull-type tail) with index $\theta > 0$ if $-\log(1 - F)$ is regularly varying with index θ ; in other words,

$$F(x) = 1 - \exp(-x^\theta L(x)) \quad \text{for } x \text{ large enough}$$

where L is a slowly varying function, that is, a positive measurable function such that $L(tx)/L(t) \rightarrow 1$ as $t \rightarrow \infty$ for any $x > 0$.

The most obvious example of a Weibull-tailed distribution is the Weibull distribution itself, that is, $F(x) = 1 - \exp(-Cx^\theta)$ for $x > 0$ given a positive constant C , which arises when the slowly varying function L is constant equal to C . In this case, it is immediate that $U(t) = C^{-1/\theta}(\log(t))^{1/\theta}$. In general, by the asymptotic inversion theorem for regularly varying functions with positive index (see Proposition B.1.9.9 p.367 in de Haan and Ferreira (2006)), the fact that $-\log(1 - F)$ is regularly varying with index $\theta > 0$ is equivalent to $t \mapsto U(e^t)$ being regularly varying with index $1/\theta$. In other

words, if a distribution function F is Weibull-tailed with index $\theta > 0$, there must exist a slowly varying function L_U such that

$$U(t) = (\log(t))^{1/\theta} L_U(\log(t)).$$

The Weibull distribution is recovered when L_U is constant equal to $C^{-1/\theta}$. Our objective is to provide simple sufficient conditions, either on F or directly on L_U , so that $\mathcal{C}_2(a, \rho, A)$ condition holds.

3 Distributions having Generalized Gamma-type exponential decay

Let throughout, for any $a > 0$, $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$ denote Euler's Gamma function at the point a . The Generalized Gamma distribution with shape parameters $a > 0$ and $b > 0$ and scale parameter $\lambda > 0$, proposed by Stacy (1962), has probability density function

$$x \mapsto \frac{b\lambda^a}{\Gamma(a/b)} x^{a-1} e^{-(\lambda x)^b}, \quad x > 0.$$

The class of Generalized Gamma distributions contains several classes of standard distributions, such as the Gamma distribution, of course (for $b = 1$), as well as the Weibull distribution (for $a = b$) and in particular the Rayleigh distribution (for $a = b = 2$), and the half-normal distribution (that is, the distribution of the absolute value of a unit Gaussian random variable, obtained for $a = 1$ and $b = 2$). Outside of very specific cases, such as when $a = b$, the Generalized Gamma distribution function is not explicitly determined, but its extreme value behavior can be characterized thanks to the following lemma.

Lemma 1. *Let $\alpha \in \mathbb{R}$, $\beta, C > 0$ be given. Then for any $N \geq 1$,*

$$\int_x^\infty t^\alpha e^{-Ct^\beta} dt = \frac{x^{\alpha-\beta+1} e^{-Cx^\beta}}{\beta C} \left(1 + \sum_{k=1}^{N-1} \left(\prod_{j=1}^k (\alpha - j\beta + 1) \right) \frac{x^{-k\beta}}{(\beta C)^k} + O(x^{-N\beta}) \right) \text{ as } x \rightarrow \infty.$$

In other words, the survival function of the Generalized Gamma distribution with shape parameters $a > 0$ and $b > 0$ satisfies an asymptotic expansion of the form

$$\bar{F}(x) = cx^\alpha e^{-Cx^\beta} \left(1 + \frac{D}{x^\beta} (1 + o(1)) \right) \text{ as } x \rightarrow \infty$$

for some constants $\alpha, D \in \mathbb{R}$ and $\beta, c, C > 0$. The class of distributions satisfying this kind of asymptotic expansion, which will be called throughout this section the *class of distributions having Generalized Gamma-type exponential decay*, is much wider than the class of Generalized Gamma distributions; in particular, it contains any continuous distribution having a probability density function f satisfying

$$f(x) = Kx^{a-1} e^{-Cx^b} \left(1 + \frac{d}{x^b} (1 + o(1)) \right) \text{ as } x \rightarrow \infty$$

where $a, d \in \mathbb{R}$ and $b, C, K > 0$. This encompasses any continuous distribution whose probability density function is asymptotically equivalent to a Generalized Gamma density function (with remainder term converging fast enough), but also, among others, the Gaussian distribution, as well as several other examples we shall study in this section. Writing

$$\bar{F}(x) = cx^\alpha e^{-Cx^\beta} \left(1 + \frac{D}{x^\beta} (1 + o(1)) \right) = \exp(-Cx^\beta(1 + \varepsilon(x)))$$

where $\varepsilon(x) \rightarrow 0$ as $x \rightarrow \infty$, it should also be clear that any distribution having Generalized Gamma-type exponential decay is also a Weibull-tailed distribution.

For a continuous distribution function with infinite right endpoint and having Generalized Gamma-type exponential decay, taking $x = U(t) \rightarrow \infty$ as $t \rightarrow \infty$ and using the identity $-\log \bar{F}(U(t)) = \log(t)$ results in the equation

$$\log(t) = C(U(t))^\beta - \alpha \log(U(t)) - \log c - \frac{D}{(U(t))^\beta} (1 + o(1)) \text{ as } t \rightarrow \infty.$$

Working with successive approximations to $U(t)$, this identity suggests that an asymptotic expansion for the tail quantile function U can be obtained, and therefore that condition $\mathcal{C}_2(a, \rho, A)$ can eventually be checked. This motivates our first result.

Theorem 1 (Distributions with Generalized Gamma-type exponential decay). *Assume that F is a continuous distribution function with infinite right endpoint and having Generalized Gamma-type exponential decay, that is,*

$$\bar{F}(x) = cx^\alpha e^{-Cx^\beta} \left(1 + \frac{D}{x^\beta} (1 + o(1)) \right) \text{ as } x \rightarrow \infty$$

for some $\alpha \in \mathbb{R}$, $\beta, c, C > 0$ and $D \in \mathbb{R}$.

(i) Then

$$\begin{aligned} U(t) = & \left(\frac{1}{C} \log(t) \right)^{1/\beta} \left(1 + \frac{\alpha}{\beta^2} \frac{\log \log(t)}{\log(t)} + \frac{1}{\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right) \frac{1}{\log(t)} \right. \\ & + \frac{\alpha^2(1-\beta) \log^2(\log(t))}{2\beta^4 \log^2(t)} + \frac{\alpha}{\beta^3} \left(\alpha + (1-\beta) \left(\log c - \frac{\alpha}{\beta} \log C \right) \right) \frac{\log \log(t)}{\log^2(t)} \\ & \left. + \frac{1}{\beta} \left(\frac{1-\beta}{2\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right)^2 + \frac{\alpha}{\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right) + CD \right) \frac{1}{\log^2(t)} + o\left(\frac{1}{\log^2(t)} \right) \right) \end{aligned}$$

as $t \rightarrow \infty$.

(ii) If $\beta \neq 1$, then the tail quantile function U satisfies condition $\mathcal{C}_2(a, 0, A)$ with

$$\begin{aligned} a(t) = & \frac{(\log(t))^{1/\beta-1}}{\beta C^{1/\beta}} \left(1 + \frac{\alpha(1-\beta) \log \log(t)}{\beta^2 \log(t)} + \frac{1}{\beta} \left(\alpha + (1-\beta) \left(\log c - \frac{\alpha}{\beta} \log C \right) \right) \frac{1}{\log(t)} \right) \\ \text{and } A(t) = & \frac{1-\beta}{\beta} \frac{1}{\log(t)}. \end{aligned}$$

(iii) Suppose that $\beta = 1$ and $\alpha \neq 0$. If moreover there exists $D' \in \mathbb{R}$ such that

$$\bar{F}(x) = cx^\alpha e^{-Cx} \left(1 + \frac{D}{x} + \frac{D'}{x^2} + o\left(\frac{1}{x^2} \right) \right) \text{ as } x \rightarrow \infty,$$

then the tail quantile function U satisfies condition $\mathcal{C}_2(a, 0, A)$ with

$$\begin{aligned} a(t) = & \frac{1}{C} \left(1 + \frac{\alpha}{\log(t)} - \alpha^2 \frac{\log \log(t)}{\log^2(t)} + (\alpha(\alpha + \alpha \log C - \log c) - CD) \frac{1}{\log^2(t)} \right) \\ \text{and } A(t) = & -\frac{\alpha}{\log^2(t)}. \end{aligned}$$

Results in the spirit of Theorem 1(i), which are of independent interest since they provide rather precise approximations to tail quantiles, have been shown before in specific examples: for example, Appendix A in Ledford and Tawn (1997) features a weaker asymptotic expansion of the tail quantiles of

the standard Gaussian distribution, and Section 4 in Fung and Seneta (2018) contains weaker asymptotic expansions of the tail quantiles of the Gamma, Skew-normal and Variance-Gamma distributions. In fact, one can view Theorem 1(i) as a stronger version of the result in Section 3.2 of Fung and Seneta (2018), whose motivation for finding lower-order properties of tail quantiles was to calculate the rate of convergence of certain bivariate copulas along the diagonal.

We now provide a non-exhaustive list of distributions to which Theorem 1 applies. In each case, we work out suitable values of the functions a and A .

Example 1 (Generalized Gamma, Gamma, and chi-squared distributions). Let $\alpha, \beta > 0$. Recall that the Generalized Gamma distribution with shape parameters α and β and unit scale has probability density function

$$f_{\alpha,\beta}(x) = \frac{\beta}{\Gamma(\alpha/\beta)} x^{\alpha-1} e^{-x^\beta}, \quad x > 0.$$

By Lemma 1, the corresponding survival function $\bar{F}_{\alpha,\beta}$ satisfies the asymptotic expansion

$$\bar{F}_{\alpha,\beta}(x) = \frac{x^{\alpha-\beta} e^{-x^\beta}}{\Gamma(\alpha/\beta)} \left(1 + \frac{\alpha-\beta}{\beta} \frac{1}{x^\beta} + \frac{(\alpha-\beta)(\alpha-2\beta)}{\beta^2} \frac{1}{x^{2\beta}} + o\left(\frac{1}{x^{2\beta}}\right) \right) \text{ as } x \rightarrow \infty.$$

It is then a consequence of Theorem 1(ii) that, for $\beta \neq 1$, the Generalized Gamma distribution with unit scale satisfies condition $\mathcal{C}_2(a, 0, A)$ with

$$a(t) = \frac{(\log(t))^{1/\beta-1}}{\beta} \left(1 + \frac{(\alpha-\beta)(1-\beta)}{\beta^2} \frac{\log \log(t)}{\log(t)} + \frac{1}{\beta} (\alpha-\beta - (1-\beta) \log \Gamma(\alpha/\beta)) \frac{1}{\log(t)} \right)$$

and $A(t) = \frac{1-\beta}{\beta} \frac{1}{\log(t)}.$

For $\beta = 1$, the Generalized Gamma distribution is nothing but the Gamma distribution, and Theorem 1(iii) entails that the Gamma distribution with unit scale and shape parameter $\alpha \neq 1$ satisfies condition $\mathcal{C}_2(a, 0, A)$ with

$$a(t) = 1 + \frac{\alpha-1}{\log(t)} - (\alpha-1)^2 \frac{\log \log(t)}{\log^2(t)} + (\alpha-1) \frac{\log \Gamma(\alpha) + \alpha - 2}{\log^2(t)} \quad \text{and} \quad A(t) = -\frac{\alpha-1}{\log^2(t)}.$$

We therefore find back the result of Lemma A.2 in Daouia et al. (2024b). For $\alpha = \beta = 1$, we find the unit exponential distribution, whose tail quantile function satisfies $U(tx) - U(t) = \log(x)$ for any $t > 1$ and $x > 0$.

It immediately follows that the Gamma distribution with scale parameter λ and shape parameter $\alpha \neq 1$ (*i.e.* with mean α/λ) satisfies the second-order regular variation condition with a replaced by a/λ . In particular, the chi-squared distribution with ν degrees of freedom, having density function

$$f_\nu(x) = \frac{1}{2^{\nu/2} \Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, \quad x > 0,$$

is nothing but the Gamma distribution with scale parameter $1/2$ and shape parameter $\nu/2$, and as such satisfies condition $\mathcal{C}_2(a, 0, A)$ with

$$a(t) = 2 \left(1 + \frac{\nu-2}{2} \frac{1}{\log(t)} - \frac{(\nu-2)^2}{4} \frac{\log \log(t)}{\log^2(t)} + \frac{(\nu-2)(2 \log \Gamma(\nu/2) + \nu - 4)}{4} \frac{1}{\log^2(t)} \right)$$

and $A(t) = -\frac{\nu-2}{2} \frac{1}{\log^2(t)}.$

Example 2 (Benktander type II distribution). Let $\alpha, \beta > 0$. Consider the Benktander type II distribution function, arising in actuarial science (see p.72 in Beirlant et al. (2004)) and defined by

$$F_{\alpha,\beta}(x) = 1 - x^{\beta-1} \exp\left(-\frac{\alpha}{\beta}x^\beta\right), \quad x > 0.$$

If $\beta = 1$, this is nothing but an exponential distribution, discussed in Example 1. When $\beta \neq 1$, it is a consequence of Theorem 1(ii) that the Benktander type II distribution satisfies condition $\mathcal{C}_2(a, 0, A)$ with

$$a(t) = \frac{1}{\beta(\alpha/\beta)^{1/\beta}} (\log(t))^{1/\beta-1} \left(1 - \frac{(1-\beta)^2 \log \log(t)}{\beta^2 \log(t)} - \frac{1-\beta}{\beta} \left(1 - \frac{1-\beta}{\beta} \log(\alpha/\beta)\right) \frac{1}{\log(t)}\right)$$

and $A(t) = \frac{1-\beta}{\beta} \frac{1}{\log(t)}$.

Example 3 (Gaussian and truncated Gaussian distributions). Consider the standard Gaussian distribution, whose probability density function is defined by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

By Lemma 1, the corresponding distribution function Φ satisfies

$$\bar{\Phi}(x) = 1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x} \left(1 - \frac{1}{x^2} + o\left(\frac{1}{x^2}\right)\right).$$

Conclude, by Theorem 1(iii), that the standard Gaussian distribution satisfies condition $\mathcal{C}_2(a, 0, A)$ with

$$a(t) = \frac{1}{\sqrt{2 \log(t)}} \left(1 + \frac{\log \log(t)}{4 \log(t)} + \left(\frac{1}{2} \log(4\pi) - 1\right) \frac{1}{2 \log(t)}\right) \quad \text{and} \quad A(t) = -\frac{1}{2 \log(t)}.$$

We thus recover the result of Lemma A.3 in Daouia et al. (2024b), which is compatible with Exercise 2.9 p.61 in de Haan and Ferreira (2006). It immediately follows that the Gaussian distribution with mean μ and variance σ^2 satisfies the second-order regular variation condition with a replaced by σa . Meanwhile, the Gaussian distribution truncated to lie above a threshold x_0 , having distribution function

$$F_{\mu,\sigma,x_0}(x) = \frac{\Phi((x-\mu)/\sigma) - \Phi((x_0-\mu)/\sigma)}{1 - \Phi((x_0-\mu)/\sigma)}, \quad x > x_0,$$

satisfies the second-order regular variation condition with a replaced by

$$a_{x_0}(t) = \frac{\sigma}{\sqrt{2 \log(t)}} \left(1 + \frac{\log \log(t)}{4 \log(t)} + \left(\frac{1}{2} \log(4\pi) - 1 + \log(1 - \Phi((x_0-\mu)/\sigma))\right) \frac{1}{2 \log(t)}\right).$$

Example 4 (Log-normal and Johnson S_U distribution). Let $\mu \in \mathbb{R}$ and $\sigma > 0$. Consider first the log-normal distribution with location parameter μ and dispersion parameter σ , whose distribution function is

$$F_{\mu,\sigma}(x) = \Phi\left(\frac{\log(x) - \mu}{\sigma}\right), \quad x > 0,$$

where Φ is the distribution function of the unit Gaussian distribution. The tail quantile function $U = U_{\mu,\sigma}$ of this log-normal distribution is then $U(t) = \exp(\mu + \sigma W(t))$, where W is the tail quantile function of the unit Gaussian distribution. Clearly, for any $t > 1$ and $x > 0$,

$$U(tx) - U(t) = \exp(\mu + \sigma W(t))(\exp(\sigma(W(tx) - W(t))) - 1).$$

Set

$$b(t) = \frac{1}{\sqrt{2\log(t)}} \left(1 + \frac{\log \log(t)}{4\log(t)} + \left(\frac{1}{2} \log(4\pi) - 1 \right) \frac{1}{2\log(t)} \right) \quad \text{and} \quad B(t) = -\frac{1}{2\log(t)}.$$

It was shown in Example 3 that the unit Gaussian distribution satisfies the second-order extended regular variation condition with scale function b and auxiliary function B ; in particular,

$$W(tx) - W(t) = b(t) \log(x) \left(1 + \frac{B(t)}{2} \log(x) + o(|B(t)|) \right).$$

Since $b(t) \sim 1/\sqrt{2\log(t)} \rightarrow 0$ and $|B(t)| = 1/(2\log(t)) = o(b(t))$ as $t \rightarrow \infty$, one has

$$\exp(\sigma(W(tx) - W(t))) - 1 = \sigma b(t) \left(\log(x) + \sigma b(t) \times \frac{1}{2} \log^2(x) + o(b(t)) \right).$$

Conclude that the log-normal distribution with location parameter μ and dispersion parameter σ satisfies condition $\mathcal{C}_2(a, 0, A)$ with

$$a(t) = \sigma b(t) \exp(\mu + \sigma W(t)) \quad \text{and} \quad A(t) = \sigma b(t).$$

We can provide simpler expressions for $a(t)$ and $A(t)$ by noting that the former is only determined asymptotically up to multiplication by terms of the form $1 + o(|A(t)|) = 1 + o(1/\sqrt{\log(t)})$. Recalling the asymptotic expansion of $1 - \Phi(x)$ found in Example 3 and applying Theorem 1(i), we find

$$\begin{aligned} W(t) &= \sqrt{2\log(t)} \left(1 - \frac{\log \log(t)}{4\log(t)} - \frac{\log(4\pi)}{4\log(t)} + o\left(\frac{1}{\log(t)}\right) \right) \\ &= \sqrt{2\log(t)} - \frac{\log \log(t)}{2\sqrt{2\log(t)}} \left(1 + \frac{\log(4\pi)}{\log \log(t)} + o\left(\frac{1}{\log \log(t)}\right) \right) \quad \text{as } t \rightarrow \infty. \end{aligned}$$

As such

$$U(t) = \exp(\mu + \sigma W(t)) = \exp(\mu + \sqrt{2\sigma^2 \log(t)}) \left(1 - \frac{\sigma \log \log(t)}{2\sqrt{2\log(t)}} \left(1 + \frac{\log(4\pi)}{\log \log(t)} \right) + o\left(\frac{1}{\sqrt{\log(t)}}\right) \right).$$

Since in fact

$$b(t) = \frac{1}{\sqrt{2\log(t)}} \left(1 + o\left(\frac{1}{\sqrt{\log(t)}}\right) \right),$$

one can conclude that suitable choices of a and A are

$$a(t) = \frac{\sigma \exp(\mu + \sqrt{2\sigma^2 \log(t)})}{\sqrt{2\log(t)}} \left(1 - \frac{\sigma \log \log(t)}{2\sqrt{2\log(t)}} \left(1 + \frac{\log(4\pi)}{\log \log(t)} \right) \right) \quad \text{and} \quad A(t) = \frac{\sigma}{\sqrt{2\log(t)}}.$$

From this one can deduce the validity of the second-order condition for the Johnson S_U distribution (Johnson, 1949), which has been used in finance to model index returns (Simonato, 2012). Let arsinh denote the inverse of the hyperbolic sine function, that is, $\operatorname{arsinh}(x) = \log(x + \sqrt{x^2 + 1})$ for any x . Fix $m, \mu \in \mathbb{R}$ and $s, \sigma > 0$. The four-parameter Johnson S_U distribution (Johnson, 1949), with parameters m, s, μ, σ , has distribution function

$$G_{m,s,\mu,\sigma}(x) = \Phi \left(\frac{\operatorname{arsinh}(m + sx) - \mu}{\sigma} \right).$$

Equivalently, the tail quantile function $V = V_{m,s,\mu,\sigma}$ of this distribution satisfies

$$V(t) = \frac{\sinh(\mu + \sigma W(t)) - m}{s}, \quad t > 1.$$

Since $\sinh(z) = (e^z - e^{-z})/2$ for any z , the tail quantile function V is in fact linked to the tail quantile function U of the log-normal distribution:

$$V(t) = \frac{1}{2s} (\exp(\mu + \sigma W(t)) - \exp(-\mu - \sigma W(t)) - 2m) = \frac{1}{2s} \left(U(t) - \frac{1}{U(t)} - 2m \right).$$

Then, for any $t > 1$ and $x > 0$,

$$V(tx) - V(t) = \frac{U(tx) - U(t)}{2s} \left(1 + \frac{1}{U(t)U(tx)} \right).$$

Now obviously

$$U(t)U(tx) \sim \exp(2\mu + \sqrt{2\sigma^2 \log(t)}(1 + \sqrt{\log(tx)/\log(t)})) \sim \exp(2\mu + 2\sqrt{2\sigma^2 \log(t)}) \text{ as } t \rightarrow \infty$$

so that

$$V(tx) - V(t) = \frac{U(tx) - U(t)}{2s} \left(1 + O \left(\frac{1}{\exp(2\sqrt{2\sigma^2 \log(t)})} \right) \right) = \frac{U(tx) - U(t)}{2s} (1 + o(A(t))).$$

Conclude that the Johnson S_U distribution satisfies the second-order condition with $\rho = 0$, a replaced by $a/(2s)$ and the same function A as the log-normal distribution.

Example 5 (Fatigue life distribution). Let $\alpha > 0$. The fatigue life distribution with shape parameter α , which is also the Birnbaum-Saunders distribution with unit scale (Birnbaum and Saunders, 1969), has distribution function

$$F_\alpha(x) = \Phi \left(\frac{\sqrt{x} - \sqrt{1/x}}{\alpha} \right), \quad x > 0,$$

where once again Φ is the distribution function of the unit Gaussian distribution. Use Lemma 1 to get

$$\bar{\Phi}(x) = \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x} \left(1 - \frac{1}{x^2} + \frac{3}{x^4} + o \left(\frac{1}{x^4} \right) \right).$$

Straightforward calculations lead to

$$\begin{aligned} \bar{F}_\alpha(x) &= \frac{\alpha \exp(1/\alpha^2)}{\sqrt{2\pi}} \frac{1}{\sqrt{x}} \exp \left(-\frac{x}{2\alpha^2} \right) \\ &\times \left(1 + \left(1 - \alpha^2 - \frac{1}{2\alpha^2} \right) \frac{1}{x} + \left(\frac{3}{2} - 3\alpha^2 + 3\alpha^4 - \frac{1}{2\alpha^2} + \frac{1}{8\alpha^4} \right) \frac{1}{x^2} + o \left(\frac{1}{x^2} \right) \right). \end{aligned}$$

Then, according to Theorem 1(iii), the fatigue life distribution satisfies condition $\mathcal{C}_2(a, 0, A)$ with

$$a(t) = 2\alpha^2 \left(1 - \frac{1}{2 \log(t)} - \frac{\log \log(t)}{4 \log^2(t)} + \frac{1}{4} \left(3 - \log(4\pi) + \frac{1}{\alpha^4} \right) \frac{1}{\log^2(t)} \right) \quad \text{and} \quad A(t) = \frac{1}{2 \log^2(t)}.$$

Example 6 (Skew-normal distribution). Let $\lambda \neq 0$. Consider the skew-normal distribution with asymmetry parameter λ , introduced by O'Hagan and Leonard (1976) in the context of Bayesian estimation of a Gaussian location parameter, whose probability density function is defined by

$$f_\lambda(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2} \Phi(\lambda x) = 2\varphi(x)\Phi(\lambda x)$$

where φ and Φ respectively denote the probability density function and the distribution function of the unit Gaussian distribution. For $\lambda > 0$, it was seen in Example 3 that

$$\Phi(\lambda x) = 1 + O\left(\frac{e^{-\lambda^2 x^2/2}}{x}\right) = 1 + o(x^{-\alpha}) \text{ as } x \rightarrow \infty \text{ for any } \alpha > 0.$$

For $\lambda < 0$, however, one has, from Example 3 again,

$$\Phi(\lambda x) = 1 - \Phi(-\lambda x) = \frac{1}{-\lambda\sqrt{2\pi}} \frac{e^{-\lambda^2 x^2/2}}{x} \left(1 - \frac{1}{\lambda^2 x^2} + o\left(\frac{1}{x^2}\right)\right).$$

Apply then Lemma 1 to obtain that the distribution function F_λ of the skew-normal distribution with asymmetry parameter λ satisfies

$$\bar{F}_\lambda(x) = \begin{cases} \frac{2}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x} \left(1 - \frac{1}{x^2} + o\left(\frac{1}{x^2}\right)\right) & \text{as } x \rightarrow \infty \text{ when } \lambda > 0, \\ \frac{1}{-\lambda\pi(1+\lambda^2)} \frac{e^{-(1+\lambda^2)x^2/2}}{x^2} \left(1 - \frac{1+3\lambda^2}{\lambda^2(1+\lambda^2)} \frac{1}{x^2} + o\left(\frac{1}{x^2}\right)\right) & \text{as } x \rightarrow \infty \text{ when } \lambda < 0. \end{cases}$$

This coincides with Lemma 3.1 in Xiong and Peng (2020) up to terms of order $1/x^2$. It is now a consequence of Theorem 1(ii) that this distribution satisfies condition $\mathcal{C}_2(a, 0, A)$ with

$$a(t) = \begin{cases} \frac{1}{\sqrt{2\log(t)}} \left(1 + \frac{\log \log(t)}{4\log(t)} + \left(\frac{1}{2} \log(\pi) - 1\right) \frac{1}{2\log(t)}\right) & \text{if } \lambda > 0, \\ \frac{1}{\sqrt{2(1+\lambda^2)\log(t)}} \left(1 + \frac{\log \log(t)}{2\log(t)} + (\log(-2\pi\lambda) - 2) \frac{1}{2\log(t)}\right) & \text{if } \lambda < 0, \end{cases}$$

$$\text{and } A(t) = -\frac{1}{2\log(t)}.$$

Likewise, the skew-normal distribution with location parameter μ , scale parameter σ and shape parameter λ , whose probability density function is defined by

$$f_{\mu,\sigma,\lambda}(x) = \frac{2}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \Phi\left(\lambda \frac{x-\mu}{\sigma}\right)$$

also satisfies the second-order regular variation condition with a replaced by σa .

Example 7 (Variance-Gamma distribution). According to the first parametrization in Fischer et al. (2024), the Variance-Gamma distribution with parameters $r > 0$, $\theta \in \mathbb{R}$, $\sigma > 0$ and $\mu \in \mathbb{R}$ has probability density function

$$f(x) = \frac{1}{\sigma\sqrt{\pi}\Gamma(r/2)} \exp\left(\theta \frac{x-\mu}{\sigma^2}\right) \left(\frac{|x-\mu|}{2\sqrt{\theta^2 + \sigma^2}}\right)^{(r-1)/2} K_{(r-1)/2}\left(\frac{\sqrt{\theta^2 + \sigma^2}}{\sigma^2} |x-\mu|\right).$$

Here K_ν denotes a modified Bessel function of the second kind, that is, for any $\nu \in \mathbb{R}$,

$$K_\nu(x) = \int_0^\infty \exp(-x \cosh(t)) \cosh(\nu t) dt, \quad x > 0.$$

This distribution was introduced to the financial literature by Madan and Seneta (1990). To simplify the discussion, we set the location parameter μ to 0 and the scale parameter σ to 1. This yields a two-parameter family of distributions whose probability density function is

$$f_{r,\theta}(x) = \frac{1}{\sqrt{\pi}\Gamma(r/2)} e^{\theta x} \left(\frac{|x|}{2\sqrt{1+\theta^2}}\right)^{(r-1)/2} K_{(r-1)/2}\left(\sqrt{1+\theta^2}|x|\right).$$

According to Formula 9.7.2 p.378 in Abramovitz and Stegun (1964),

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left(1 + \frac{4\nu^2 - 1}{8x} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{128x^2} + o\left(\frac{1}{x^2}\right) \right) \text{ as } x \rightarrow \infty.$$

It follows that

$$f_{r,\theta}(x) = \frac{x^{r/2-1} e^{-x(\sqrt{1+\theta^2}-\theta)}}{(2\sqrt{1+\theta^2})^{r/2} \Gamma(r/2)} \left(1 + \frac{(r-1)^2 - 1}{8\sqrt{1+\theta^2}x} + \frac{((r-1)^2 - 1)((r-1)^2 - 9)}{128(1+\theta^2)x^2} + o\left(\frac{1}{x^2}\right) \right)$$

as $x \rightarrow \infty$. By Lemma 1, the distribution function $F_{r,\theta}$ of this Variance-Gamma distribution satisfies the asymptotic expansion

$$\begin{aligned} \bar{F}_{r,\theta}(x) &= \frac{x^{r/2-1} e^{-x(\sqrt{1+\theta^2}-\theta)}}{(2\sqrt{1+\theta^2})^{r/2} (\sqrt{1+\theta^2} - \theta) \Gamma(r/2)} \left(1 + \left(\frac{r-2}{2(\sqrt{1+\theta^2} - \theta)} + \frac{(r-1)^2 - 1}{8\sqrt{1+\theta^2}} \right) \frac{1}{x} \right. \\ &\quad \left. + \left(\frac{(r-2)(r-4)}{4(\sqrt{1+\theta^2} - \theta)^2} + \frac{((r-1)^2 - 1)(r-4)}{16\sqrt{1+\theta^2}(\sqrt{1+\theta^2} - \theta)} + \frac{((r-1)^2 - 1)((r-1)^2 - 9)}{128(1+\theta^2)} \right) \frac{1}{x^2} + o\left(\frac{1}{x^2}\right) \right). \end{aligned}$$

By Theorem 1(iii) then, when $r \neq 2$, the Variance-Gamma distribution satisfies condition $C_2(a, 0, A)$ with

$$\begin{aligned} a(t) &= \frac{1}{\sqrt{1+\theta^2} - \theta} \left(1 + \frac{r-2}{2\log(t)} - \frac{(r-2)^2 \log \log(t)}{4 \log^2(t)} \right. \\ &\quad \left. + \frac{1}{2} \left((r-2) \left(\frac{r-2}{2} + \frac{r}{2} \log(\sqrt{1+\theta^2} - \theta) + \log((2\sqrt{1+\theta^2})^{r/2} \Gamma(r/2)) \right) \right. \right. \\ &\quad \left. \left. - \left(r-2 + (\sqrt{1+\theta^2} - \theta) \frac{(r-1)^2 - 1}{4\sqrt{1+\theta^2}} \right) \right) \frac{1}{\log^2(t)} \right) \end{aligned}$$

$$\text{and } A(t) = -\frac{r-2}{2\log^2(t)}.$$

For $r = 2$, it is readily seen from the identity $K_{1/2}(x) = e^{-x} \sqrt{\pi/(2x)}$ that the Variance-Gamma distribution is in fact the asymmetric Laplace distribution having probability density function

$$g_\theta(x) = \frac{1}{2\sqrt{1+\theta^2}} \exp\left(\theta x - \sqrt{1+\theta^2}|x|\right).$$

On the positive half-line, this distribution has cumulative distribution function

$$G_\theta(x) = 1 - \frac{1}{2\sqrt{1+\theta^2}(\sqrt{1+\theta^2} - \theta)} e^{-(\sqrt{1+\theta^2}-\theta)x}.$$

From this it immediately follows that, for t large enough, its tail quantile function U_θ is equal to

$$U_\theta(t) = \frac{\log(t) - \log(2\sqrt{1+\theta^2}(\sqrt{1+\theta^2} - \theta))}{\sqrt{1+\theta^2} - \theta},$$

and, finally, that just like for the exponential distribution, one has $U_\theta(tx) - U_\theta(t) = \frac{1}{\sqrt{1+\theta^2}-\theta} \log(x)$ for any $x > 0$ when t is large enough.

4 A general result for Weibull-tailed distributions

The class of distributions having Generalized Gamma-type exponential decay is not closed under certain important operations in extreme value analysis, such as taking minima or maxima of independent

random variables. For example, although the Weibull distribution is certainly a Generalized Gamma distribution, the minimum of two independent Weibull random variables with different shape parameters, whose distribution function can be written as

$$F(x) = 1 - \exp(-C_1 x^{\beta_1} - C_2 x^{\beta_2}), \quad x > 0,$$

for parameters $\beta_1, \beta_2, C_1, C_2 > 0$, does not have a Generalized Gamma-type exponential decay. This distribution, called the Bi-Weibull distribution and discussed in Section 1.1 of Berger and Sun (1993), is a natural distribution in reliability theory and the competing risks literature. There are several such examples we shall highlight later in this section where, despite the fact that the distribution does not have a Generalized Gamma-type exponential decay, it is possible to obtain an asymptotic expansion of its tail quantile function U , either because this function is explicitly determined or because an asymptotic expansion can be found relatively easily. For such examples, it is of interest to have a high-level result linking the validity of the second-order condition $\mathcal{C}_2(\gamma, \rho, A)$ to that asymptotic expansion. Our next theorem goes in this direction.

Theorem 2 (Weibull-tailed distributions). *Assume that F is a continuous distribution function having a Weibull-type tail with index $\theta > 0$. Suppose that there are $C > 0$, $\beta \leq 0$, $\mu \in \mathbb{R}$ and a function ℓ_β , having constant sign and such that $|\ell_\beta|$ is regularly varying with index $-\beta$ and converges to 0 at infinity, such that*

$$U(t) - \mu = \left(\frac{1}{C} \log(t) \right)^{1/\theta} (1 + \ell_\beta(t)) \text{ for large enough } t.$$

(i) *Assume that $\theta \neq 1$. If $\beta < 0$, or if $\beta = 0$ and under the extra assumption that there is a function η , converging to 0 at infinity, such that*

$$\forall x > 0, \ell_0(tx) - \ell_0(t) = \eta(t) \frac{\log(x)}{\log(t)} + o\left(\frac{1}{\log^2(t)}\right) \text{ as } t \rightarrow \infty,$$

then condition $\mathcal{C}_2(a, 0, A)$ holds with

$$a(t) = \begin{cases} \frac{(\log(t))^{1/\theta-1}}{\theta C^{1/\theta}} & \text{if } \beta < 0, \\ \frac{(\log(t))^{1/\theta-1}}{\theta C^{1/\theta}} (1 + \ell_0(t) + \theta \eta(t)) & \text{if } \beta = 0 \end{cases} \quad \text{and} \quad A(t) = \frac{1-\theta}{\theta} \frac{1}{\log(t)}.$$

(ii) *If $\theta = 1$ and $\beta < 0$, then condition $\mathcal{C}_2(a, \rho, A)$ holds with $\rho = -\beta$,*

$$a(t) = \frac{1}{C} (1 - \beta \ell_\beta(t) \log(t)) \quad \text{and} \quad A(t) = \beta^2 \ell_\beta(t) \log(t).$$

Theorem 2 extends the part of Lemma A.1 in Daouia et al. (2024b) devoted to light-tailed distributions by allowing for a more general form of L_U . Note that, for $\beta < 0$, the assumption made in Theorem 2 on the function L_U defined by the equation $U(t) = (\log(t))^{1/\theta} L_U(\log(t))$ is $L_U(t) = C^{-1/\theta} (1 + e^{-\beta t} \mathcal{L}(e^t))$, where \mathcal{L} is slowly varying, which corresponds to the setting where L_U converges to a constant at an exponential rate.

The assumption made on ℓ_0 in the case $\beta = 0$ as part of Theorem 2(i) may seem quite precise; it is, in fact, guided by our experience on examples, where $\ell_0(t)$ can often be expanded as a series of terms involving increasing powers of $\log(t)$, in which case Theorem 2(i) immediately applies. It is interesting to note that, in such situations, the function $t \mapsto U(e^t)$ actually belongs to the Hall class (Hall, 1982), only with a more precise form of the remainder term. We state a corollary of Theorem 2(i) summarizing this discussion.

Corollary 1 (Weibull-tailed distributions II). *Assume that F is a continuous distribution function having a Weibull-type tail with index $\theta > 0$. If $\theta \neq 1$ and there is $\mu \in \mathbb{R}$ with*

$$U(t) - \mu = \left(\frac{1}{C} \log(t) \right)^{1/\theta} \left(1 + \sum_{i=1}^l \frac{D_i}{\log^{\delta_i}(t)} + \sum_{j=1}^m \frac{D'_j}{\log^{1+\delta'_j}(t)} + o\left(\frac{1}{\log^2(t)} \right) \right) \text{ as } t \rightarrow \infty,$$

for some $C > 0$, $D_1, \dots, D_l, D'_1, \dots, D'_m \in \mathbb{R}$, $0 < \delta_1 < \dots < \delta_l \leq 1$ and $0 < \delta'_1 < \dots < \delta'_m \leq 1$, then the tail quantile function U satisfies condition $\mathcal{C}_2(a, 0, A)$ with

$$a(t) = \frac{(\log(t))^{1/\theta-1}}{\theta C^{1/\theta}} \left(1 + \sum_{i=1}^l \frac{D_i(1-\theta\delta_i)}{\log^{\delta_i}(t)} \right) \text{ and } A(t) = \frac{1-\theta}{\theta} \frac{1}{\log(t)}.$$

More general results than Corollary 1 can of course be stated. For example, if the remainder term $\ell_0(t)$ can be written as a linear combination of $\log \log(t)/\log(t)$, $1/\log(t)$, $\log^2(\log(t))/\log^2(t)$, $\log \log(t)/\log^2(t)$ and $1/\log^2(t)$, then one essentially finds back Theorem 1(ii). The difficulty in practice is to come up with such an expansion; this is where results such as Theorem 1(i), which provide an asymptotic expansion of U starting from an asymptotic expansion of the survival function $1 - F$, are interesting.

The reader will have noticed that we did not treat the case $\theta = 1$ and $\beta = 0$ in Theorem 2. This corresponds to the setting where the extremes of the underlying distribution are theoretically close to those of the exponential distribution, but with convergence happening at a very slow rate. In our experience with examples, this situation is quite rare. We nonetheless give a result in the spirit of Corollary 1 that goes towards tackling such cases.

Theorem 3 (Weibull-tailed distributions III). *Assume that F is a continuous distribution function having a Weibull-type tail with index $\theta = 1$. If there are a positive real number $\delta \neq 1$ and $\mu \in \mathbb{R}$ such that*

$$U(t) - \mu = \frac{1}{C} \log(t) \left(1 + \frac{D}{\log^\delta(t)} + \sum_{i=1}^l \frac{D_i}{\log^{\delta+\delta_i}(t)} + \sum_{j=1}^m \frac{D'_j}{\log^{1+\delta+\delta'_j}(t)} + o\left(\frac{1}{\log^{2+\delta}(t)} \right) \right) \text{ as } t \rightarrow \infty,$$

for some $C > 0$, $D \neq 0$, $D_1, \dots, D_l, D'_1, \dots, D'_m \in \mathbb{R}$, $0 < \delta_1 < \dots < \delta_l \leq 1$ and $0 < \delta'_1 < \dots < \delta'_m \leq 1$, then the tail quantile function U satisfies condition $\mathcal{C}_2(a, 0, A)$ with

$$a(t) = \frac{1}{C} \left(1 + \frac{D(1-\delta)}{\log^\delta(t)} + \sum_{i=1}^l \frac{D_i(1-\delta-\delta_i)}{\log^{\delta+\delta_i}(t)} \right) \text{ and } A(t) = D \frac{\delta(\delta-1)}{\log^{1+\delta}(t)}.$$

We now list a few examples which are not distributions with Generalized Gamma-type exponential decay, but where Theorem 2, Corollary 1 and/or Theorem 3 apply.

Example 8 (Exponentiated Weibull distribution). Let $\alpha > 0$ be different from 1 and $\beta > 0$. Consider the exponentiated Weibull distribution with unit scale, introduced to the reliability and lifetime modeling literature by Mudholkar and Srivastava (1993), whose distribution function is

$$F_{\alpha,\beta}(x) = (1 - e^{-x^\beta})^\alpha, \quad x > 0.$$

When $\beta = 1$, this is the exponentiated exponential distribution discussed in Gupta and Kundu (2001); when α is a positive integer, this is the distribution of the lifetime of a parallel system of components having independent and identically distributed Weibull lifetimes with shape parameter β (i.e. the distribution of their maximum). Its tail quantile function is

$$U_{\alpha,\beta}(t) = \left(-\log \left(1 - \left(1 - \frac{1}{t} \right)^{1/\alpha} \right) \right)^{1/\beta}, \quad t > 1.$$

In particular

$$U_{\alpha,\beta}(t) = \left(\log(t) + \log(\alpha) + \frac{1-\alpha}{2\alpha} \frac{1}{t} (1 + o(1)) \right)^{1/\beta} \quad \text{as } t \rightarrow \infty.$$

When $\beta \neq 1$,

$$\begin{aligned} U_{\alpha,\beta}(t) &= (\log(t))^{1/\beta} \left(1 + \frac{\log(\alpha)}{\log(t)} + O\left(\frac{1}{t}\right) \right)^{1/\beta} \\ &= (\log(t))^{1/\beta} \left(1 + \frac{1}{\beta} \frac{\log(\alpha)}{\log(t)} + \frac{1-\beta}{2\beta^2} \frac{\log^2(\alpha)}{\log^2(t)} + o\left(\frac{1}{\log^2(t)}\right) \right). \end{aligned}$$

Theorem 2(i) applies (via Corollary 1) and yields that the exponentiated Weibull distribution with unit scale satisfies condition $\mathcal{C}_2(a, 0, A)$ with

$$a(t) = \frac{(\log(t))^{1/\beta-1}}{\beta} \left(1 + \frac{1-\beta}{\beta} \frac{\log(\alpha)}{\log(t)} \right) \quad \text{and} \quad A(t) = \frac{1-\beta}{\beta} \frac{1}{\log(t)}.$$

When on the contrary $\beta = 1$, that is, when the exponentiated exponential distribution is considered, then in fact

$$U_{\alpha,\beta}(t) - \log(\alpha) = \log(t) \left(1 + \frac{1-\alpha}{2\alpha} \frac{1}{t \log(t)} (1 + o(1)) \right).$$

In this case Theorem 2(ii) applies: the exponentiated exponential distribution with unit scale satisfies condition $\mathcal{C}_2(a, \rho, A)$ with $\rho = -1$,

$$a_1(t) = 1 - \frac{1-\alpha}{2\alpha} t^{-1} \quad \text{and} \quad A_1(t) = \frac{1-\alpha}{2\alpha} t^{-1}.$$

In both situations, the versions with scale parameter $\lambda > 0$, whose distribution function is $F_{\alpha,\beta}(\lambda x)$, satisfy the same respective condition with the scale function divided by λ .

Example 9 (Gamma-Gompertz distribution). Let $s, \beta > 0$ with $\beta \neq 1$. The Gamma-Gompertz distribution with shape parameters β and s and unit scale, studied by Bemmaor and Glady (2012) in the context of marketing applications, has distribution function

$$F_{s,\beta}(x) = 1 - \frac{\beta^s}{(\beta - 1 + e^x)^s}, \quad x > 0.$$

Its tail quantile function is

$$U_{s,\beta}(t) = \log(1 - \beta + \beta t^{1/s}) = \frac{1}{s} \log(t) + \log(\beta) + \log\left(1 + \frac{1-\beta}{\beta} t^{-1/s}\right), \quad t > 1.$$

In particular, an asymptotic expansion of $U_{s,\beta}$ is

$$U_{s,\beta}(t) - \log(\beta) = \frac{1}{s} \log(t) \left(1 + \frac{s(1-\beta)}{\beta} \frac{1}{t^{1/s} \log(t)} (1 + o(1)) \right) \quad \text{as } t \rightarrow \infty.$$

Then Theorem 2(ii) entails that the Gamma-Gompertz distribution with shape parameters β and s and unit scale satisfies condition $\mathcal{C}_2(a, \rho, A)$ with $\rho = -1/s$,

$$a(t) = \frac{1}{s} \left(1 - \frac{1-\beta}{\beta} t^{-1/s} \right) \quad \text{and} \quad A(t) = \frac{1-\beta}{s\beta} t^{-1/s}.$$

The Gamma-Gompertz distribution with scale parameter $\lambda > 0$, having distribution function $x \mapsto 1 - \beta^s / (\beta - 1 + e^{\lambda x})^s$, $x > 0$, then satisfies this same condition with a replaced by a/λ .

Example 10 (Exponential-logarithmic distribution). Let $p \in (0, 1)$. The Exponential-Logarithmic distribution with parameter p and unit scale, introduced in Tahmasbi and Rezaei (2008) as a model for a situation where failure of a device occurs due to the presence of an random number of initial defects of some kind, has distribution function

$$F_p(x) = 1 - \frac{\log(1 - (1 - p)e^{-x})}{\log(p)}, \quad x > 0.$$

Its tail quantile function is

$$U_p(t) = \log\left(\frac{1 - p}{1 - p^{1/t}}\right) = \log(1 - p) - \log(1 - p^{1/t}), \quad t > 1.$$

An asymptotic expansion of U follows as

$$U_p(t) - \log(1 - p) = \log(t) - \log(-\log(p)) - \frac{\log(p)}{2t} + o\left(\frac{1}{t}\right) \text{ as } t \rightarrow \infty.$$

In particular

$$U_p(t) + \log\left(\frac{-\log(p)}{1 - p}\right) = \log(t) \left(1 - \frac{\log(p)}{2t \log(t)}(1 + o(1))\right).$$

By Theorem 2(ii), the Exponential-Logarithmic distribution with parameter p and unit scale satisfies condition $\mathcal{C}_2(a, \rho, A)$ with $\rho = -1$,

$$a(t) = 1 + \frac{\log(p)}{2}t^{-1} \quad \text{and} \quad A(t) = -\frac{\log(p)}{2}t^{-1}.$$

Consequently, the Exponential-Logarithmic distribution with parameter p and scale parameter $\beta > 0$, having distribution function $x \mapsto 1 - \log(1 - (1 - p)e^{-\beta x})/\log(p)$, $x > 0$, satisfies this same condition with a replaced by a/β .

Example 11 (Logistic-beta distribution). Let $\alpha, \beta > 0$. The logistic-beta distribution, also referred to as the type IV generalized logistic distribution in Chapter 23.10 in Johnson et al. (1995), has probability density function

$$f_{\alpha, \beta}(x) = \frac{1}{B(\alpha, \beta)} \left(\frac{1}{1 + e^{-x}}\right)^\alpha \left(1 - \frac{1}{1 + e^{-x}}\right)^\beta = \frac{1}{B(\alpha, \beta)} \frac{e^{-\beta x}}{(1 + e^{-x})^{\alpha + \beta}},$$

where $B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$ denotes Euler's Beta function. Clearly

$$f_{\alpha, \beta}(x) = \frac{e^{-\beta x}}{B(\alpha, \beta)} (1 - (\alpha + \beta)e^{-x} + o(e^{-x})) \text{ as } x \rightarrow \infty$$

so that the distribution function $F_{\alpha, \beta}$ satisfies

$$\bar{F}_{\alpha, \beta}(x) = \frac{e^{-\beta x}}{\beta B(\alpha, \beta)} \left(1 - \frac{\beta(\alpha + \beta)}{\beta + 1}e^{-x} + o(e^{-x})\right).$$

To apply Theorem 2, it remains to find an asymptotic expansion of $U(t)$. We adapt the idea of the proof of Theorem 1(i): taking the logarithm, plugging in $x = U(t)$ and letting $t \rightarrow \infty$ results in the equation

$$\log(t) = \beta U(t) + \log(\beta B(\alpha, \beta)) + \frac{\beta(\alpha + \beta)}{\beta + 1}e^{-U(t)}(1 + o(1)) \text{ as } t \rightarrow \infty.$$

This successively yields

$$U(t) = \frac{1}{\beta} \log(t) - \frac{\log(\beta B(\alpha, \beta))}{\beta} + o(1)$$

and

$$\begin{aligned} U(t) + \frac{\log(\beta B(\alpha, \beta))}{\beta} &= \frac{1}{\beta} \log(t) - \frac{\alpha + \beta}{\beta + 1} (\beta B(\alpha, \beta))^{1/\beta} \frac{1}{t^{1/\beta}} (1 + o(1)) \\ &= \frac{1}{\beta} \log(t) \left(1 - \frac{\beta(\alpha + \beta)}{\beta + 1} (\beta B(\alpha, \beta))^{1/\beta} \frac{1}{t^{1/\beta} \log(t)} (1 + o(1)) \right). \end{aligned}$$

By Theorem 2(ii), the logistic-beta distribution with parameters α and β satisfies condition $\mathcal{C}_2(a, \rho, A)$ with $\rho = -1/\beta$,

$$a(t) = \beta \left(1 + \frac{\alpha + \beta}{\beta + 1} (\beta B(\alpha, \beta))^{1/\beta} t^{-1/\beta} \right) \quad \text{and} \quad A(t) = -\frac{\alpha + \beta}{\beta(\beta + 1)} (\beta B(\alpha, \beta))^{1/\beta} t^{-1/\beta}.$$

Example 12 (Bi-Weibull distribution). Let $\lambda_1, \lambda_2 > 0$ and $\beta_1, \beta_2 > 0$ with $\beta_2 < \beta_1$. Consider the Bi-Weibull distribution $\text{BiWeibull}(\lambda_1, \lambda_2, \beta_1, \beta_2)$ having distribution function

$$F_{\lambda_1, \lambda_2, \beta_1, \beta_2}(x) = 1 - \exp \left(- \left(\frac{x}{\lambda_1} \right)^{\beta_1} - \left(\frac{x}{\lambda_2} \right)^{\beta_2} \right), \quad x > 0.$$

Taking logarithms and plugging in $x = U(t)$ results in the equation

$$\log(t) = \left(\frac{U(t)}{\lambda_1} \right)^{\beta_1} + \left(\frac{U(t)}{\lambda_2} \right)^{\beta_2}, \quad t > 1,$$

or equivalently

$$U(t) = \lambda_1 (\log(t))^{1/\beta_1} \left(1 + \frac{\lambda_1^{\beta_1}}{\lambda_2^{\beta_2}} (U(t))^{\beta_2 - \beta_1} \right)^{-1/\beta_1}, \quad t > 1.$$

Letting $t \rightarrow \infty$ entails $U(t) \sim \lambda_1 (\log(t))^{1/\beta_1}$, and then

$$U(t) = \lambda_1 (\log(t))^{1/\beta_1} \left(1 - \frac{(\lambda_1/\lambda_2)^{\beta_2}}{\beta_1} \frac{1}{(\log(t))^{1 - \beta_2/\beta_1}} + o \left(\frac{1}{(\log(t))^{1 - \beta_2/\beta_1}} \right) \right) \quad \text{as } t \rightarrow \infty,$$

and, by successive approximations, the above identity theoretically makes it possible to obtain an asymptotic expansion of $U(t)$ formulated in terms of inverse powers of $\log(t)$, in the spirit of Corollary 1 and Theorem 3. It is, however, very difficult to give such an asymptotic expansion up to terms of order $1/\log^2(t)$ or higher in the greatest generality, because the successive terms appearing in the approximation will converge very slowly if β_2 is very close to β_1 . We illustrate these calculations for $\beta_1 = 1$ and $\beta_2 = 1/4$; we take the scale parameter λ_1 to be 1 for the sake of simplicity, and we write $\lambda = \lambda_2$. The equation satisfied by $U(t)$ becomes

$$U(t) = \log(t) \left(1 + \frac{\lambda^{-1/4}}{(U(t))^{3/4}} \right)^{-1}, \quad t > 1.$$

We then successively find

$$\begin{aligned} U(t) &= \log(t) \left(1 - \frac{\lambda^{-1/4}}{(\log(t))^{3/4}} + O \left(\frac{1}{(\log(t))^{3/2}} \right) \right), \\ U(t) &= \log(t) \left(1 - \frac{\lambda^{-1/4}}{(\log(t))^{3/4}} + \frac{1}{4} \frac{\lambda^{-1/2}}{(\log(t))^{3/2}} + O \left(\frac{1}{(\log(t))^{9/4}} \right) \right) \\ \text{and } U(t) &= \log(t) \left(1 - \frac{\lambda^{-1/4}}{(\log(t))^{3/4}} + \frac{1}{4} \frac{\lambda^{-1/2}}{(\log(t))^{3/2}} + \frac{1}{32} \frac{\lambda^{-3/4}}{(\log(t))^{9/4}} + O \left(\frac{1}{(\log(t))^3} \right) \right). \end{aligned}$$

Applying Theorem 3, we find that the $\text{BiWeibull}(1, \lambda, 1, 1/4)$ distribution satisfies condition $\mathcal{C}_2(a, 0, A)$ with

$$a(t) = 1 - \frac{\lambda^{-1/4}}{4(\log(t))^{3/4}} - \frac{\lambda^{-1/2}}{8(\log(t))^{3/2}} \quad \text{and} \quad A(t) = \frac{3\lambda^{-1/4}}{16} \frac{1}{(\log(t))^{7/4}}.$$

Example 13 (Polynomial-exponential distribution). Let $\alpha > 0$ be different from 1. Define a continuous function H_α on the positive half-line by $H_\alpha(x) = x(x^\alpha - 1)/(x - 1)$ if $x \neq 1$ and α if $x = 1$. The polynomial-exponential distribution with shape parameter α and unit scale, introduced by Chesneau et al. (2022) to model the occurrence of zero values in survival and environmental data, has distribution function $F_\alpha(x) = 1 - e^{-H_\alpha(x)}$ for $x > 0$. Once again, taking logarithms, plugging in $x = U(t)$ and rearranging results in the equation

$$U(t) = (\log(t))^{1/\alpha} \left(1 + \frac{1}{\log(t)} - \frac{1}{U(t)} \right)^{1/\alpha} \quad \text{for } t \text{ large enough.}$$

A Taylor expansion and a straightforward proof by induction shows that $U(t)$ can indeed be expanded using sums of inverse powers of $\log(t)$ as in Corollary 1. For instance, when $\alpha < 4$, straightforward calculations in the spirit of those carried out in Example 12 entail that there exist $m \geq 0$, $D'_1, \dots, D'_m \in \mathbb{R}$ and $0 < \delta'_1 < \dots < \delta'_m \leq 1$ such that

$$U(t) = (\log(t))^{1/\alpha} \left(1 - \frac{1}{\alpha(\log(t))^{1/\alpha}} - \frac{1 + \alpha}{2\alpha^2(\log(t))^{2/\alpha}} - \frac{(\alpha + 1)(\alpha + 2)}{3\alpha^3(\log(t))^{3/\alpha}} + \frac{1}{\alpha \log(t)} + \sum_{j=1}^m \frac{D'_j}{\log^{1+\delta'_j}(t)} + o\left(\frac{1}{\log^2(t)}\right) \right) \text{ as } t \rightarrow \infty.$$

Following Corollary 1, the polynomial-exponential distribution distribution satisfies condition $\mathcal{C}_2(a, 0, A)$ with

$$a(t) = \begin{cases} \frac{(\log(t))^{1/\alpha-1}}{\alpha} \left(1 + \frac{1 - \alpha}{\alpha \log(t)} \right) & \text{if } 0 < \alpha < 2, \\ \frac{(\log(t))^{1/\alpha-1}}{\alpha} \left(1 + \frac{1 + \alpha}{2\alpha^2(\log(t))^{2/\alpha}} + \frac{1 - \alpha}{\alpha \log(t)} \right) & \text{if } 2 \leq \alpha < 3, \\ \frac{(\log(t))^{1/\alpha-1}}{\alpha} \left(1 + \frac{1 + \alpha}{2\alpha^2(\log(t))^{2/\alpha}} + \frac{2(\alpha + 1)(\alpha + 2)}{3\alpha^3(\log(t))^{3/\alpha}} + \frac{1 - \alpha}{\alpha \log(t)} \right) & \text{if } 3 \leq \alpha < 4, \end{cases}$$

and $A(t) = \frac{1 - \alpha}{\alpha \log(t)}$.

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Appendix: Proofs of our results

Proof of Lemma 1. Let, for any $x > 0$,

$$I_{\alpha,\beta,C}(x) = \int_x^\infty t^\alpha e^{-Ct^\beta} dt.$$

An integration by parts yields

$$I_{\alpha,\beta,C}(x) = \frac{x^{\alpha-\beta+1}e^{-Cx^\beta}}{\beta C} + \frac{\alpha - \beta + 1}{\beta C} I_{\alpha-\beta,\beta,C}(x).$$

Since

$$I_{\alpha-\beta,\beta,C}(x) = \int_x^\infty t^{\alpha-\beta} e^{-Ct^\beta} dt \leq x^{-\beta} \int_x^\infty t^\alpha e^{-Ct^\beta} dt = x^{-\beta} I_{\alpha,\beta,C}(x),$$

one clearly has

$$I_{\alpha,\beta,C}(x) = \frac{x^{\alpha-\beta+1}e^{-Cx^\beta}}{\beta C} + \frac{\alpha - \beta + 1}{\beta C} I_{\alpha-\beta,\beta,C}(x) = \frac{x^{\alpha-\beta+1}e^{-Cx^\beta}}{\beta C} (1 + O(x^{-\beta})) \text{ as } x \rightarrow \infty.$$

A straightforward proof by induction then entails

$$\begin{aligned} I_{\alpha,\beta,C}(x) &= \frac{x^{\alpha-\beta+1}e^{-Cx^\beta}}{\beta C} \left(1 + \sum_{k=1}^{N-1} \left(\prod_{j=1}^k (\alpha - j\beta + 1) \right) \frac{x^{-k\beta}}{(\beta C)^k} \right) + O(I_{\alpha-N\beta,\beta,C}(x)) \\ &= \frac{x^{\alpha-\beta+1}e^{-Cx^\beta}}{\beta C} \left(1 + \sum_{k=1}^{N-1} \left(\prod_{j=1}^k (\alpha - j\beta + 1) \right) \frac{x^{-k\beta}}{(\beta C)^k} + O(x^{-N\beta}) \right) \text{ as } x \rightarrow \infty \end{aligned}$$

for any $N \geq 1$, as required. \square

Proof of Theorem 1. (i) The distribution function F has an infinite right endpoint, $U(t) \rightarrow \infty$ as $t \rightarrow \infty$, and is continuous, so $F(U(t)) = 1/t$ for any $t > 1$. Plugging $x = U(t)$ in the asymptotic expansion of F leads to

$$\log(t) = C(U(t))^\beta - \alpha \log(U(t)) - \log c - \frac{D}{(U(t))^\beta} (1 + o(1)) \text{ as } t \rightarrow \infty. \quad (1)$$

We repeatedly use (1) in order to get the desired asymptotic expansion of $U(t)$.

Asymptotic equivalent of $U(t)$: The leading term in the right-hand side of (1) is obviously $C(U(t))^\beta$, so that $U(t) \sim (\frac{1}{C} \log(t))^{1/\beta}$.

First term in the asymptotic expansion of $U(t)$: Write

$$U(t) = \left(\frac{1}{C} \log(t) \right)^{1/\beta} (1 + \varepsilon_1(t))$$

with $\varepsilon_1(t) \rightarrow 0$ as $t \rightarrow \infty$. It follows from (1) that

$$0 = \beta \log(t) \varepsilon_1(t) (1 + o(1)) - \frac{\alpha}{\beta} \log \log(t) + O(1)$$

and therefore

$$\lim_{t \rightarrow \infty} \frac{\log(t)}{\log \log(t)} \varepsilon_1(t) = \frac{\alpha}{\beta^2}.$$

Second term in the asymptotic expansion of $U(t)$: Write

$$U(t) = \left(\frac{1}{C} \log(t) \right)^{1/\beta} \left(1 + \frac{\alpha}{\beta^2} \frac{\log \log(t)}{\log(t)} + \frac{\log \log(t)}{\log(t)} \varepsilon_2(t) \right),$$

with $\varepsilon_2(t) \rightarrow 0$ as $t \rightarrow \infty$. Use (1) to get

$$0 = \beta \log \log(t) \varepsilon_2(t) + \frac{\alpha}{\beta} \log C - \log c + o(1)$$

and therefore

$$\lim_{t \rightarrow \infty} \log \log(t) \varepsilon_2(t) = \frac{1}{\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right).$$

Third term in the asymptotic expansion of $U(t)$: Write

$$U(t) = \left(\frac{1}{C} \log(t) \right)^{1/\beta} \left(1 + \frac{\alpha}{\beta^2} \frac{\log \log(t)}{\log(t)} + \frac{1}{\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right) \frac{1}{\log(t)} + \frac{1}{\log(t)} \varepsilon_3(t) \right),$$

with $\varepsilon_3(t) \rightarrow 0$ as $t \rightarrow \infty$. Then

$$C(U(t))^\beta - \log(t) = \frac{\alpha}{\beta} \log \log(t) + \log c - \frac{\alpha}{\beta} \log C + \beta \varepsilon_3(t) + \frac{\log^2(\log(t))}{\log(t)} \left(\frac{\alpha^2(\beta-1)}{2\beta^3} + o(1) \right)$$

and

$$\alpha \log(U(t)) = \frac{\alpha}{\beta} \log \log(t) - \frac{\alpha}{\beta} \log C + o\left(\frac{\log^2(\log(t))}{\log(t)}\right).$$

Plugging these two asymptotic expansions along with the fact that $1/(U(t))^\beta = O(1/\log(t))$ into (1) results in

$$0 = \beta \varepsilon_3(t) + \frac{\log^2(\log(t))}{\log(t)} \left(\frac{\alpha^2(\beta-1)}{2\beta^3} + o(1) \right)$$

and therefore

$$\lim_{t \rightarrow \infty} \frac{\log(t)}{\log^2(\log(t))} \varepsilon_3(t) = \frac{\alpha^2(1-\beta)}{2\beta^4}.$$

Fourth term in the asymptotic expansion of $U(t)$: Write

$$U(t) = \left(\frac{1}{C} \log(t) \right)^{1/\beta} \left(1 + \frac{\alpha \log \log(t)}{\beta^2 \log(t)} + \frac{1}{\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right) \frac{1}{\log(t)} + \frac{\alpha^2(1-\beta) \log^2(\log(t))}{2\beta^4 \log^2(t)} + \frac{\log^2(\log(t))}{\log^2(t)} \varepsilon_4(t) \right),$$

with $\varepsilon_4(t) \rightarrow 0$ as $t \rightarrow \infty$. One has

$$C(U(t))^\beta - \log(t) = \frac{\alpha}{\beta} \log \log(t) + \log c - \frac{\alpha}{\beta} \log C + \beta \frac{\log^2(\log(t))}{\log(t)} \varepsilon_4(t) + \frac{\log \log(t)}{\log(t)} \left(\frac{\alpha(\beta-1)}{\beta^2} \left(\log c - \frac{\alpha}{\beta} \log C \right) + o(1) \right)$$

and

$$\alpha \log(U(t)) = \frac{\alpha}{\beta} \log \log(t) - \frac{\alpha}{\beta} \log C + \frac{\alpha^2 \log \log(t)}{\beta^2 \log(t)} + o\left(\frac{\log \log(t)}{\log(t)}\right).$$

Plugging these two asymptotic expansions along with the asymptotic equivalent $1/(U(t))^\beta = O(1/\log(t))$ into (1) results in

$$0 = \beta \frac{\log^2(\log(t))}{\log(t)} \varepsilon_4(t) + \frac{\log \log(t)}{\log(t)} \left(\frac{\alpha(\beta-1)}{\beta^2} \left(\log c - \frac{\alpha}{\beta} \log C \right) - \frac{\alpha^2}{\beta^2} + o(1) \right)$$

and therefore

$$\lim_{t \rightarrow \infty} \log \log(t) \varepsilon_4(t) = \frac{\alpha}{\beta^3} \left(\alpha + (1-\beta) \left(\log c - \frac{\alpha}{\beta} \log C \right) \right).$$

Fifth term in the asymptotic expansion of $U(t)$: Write finally

$$U(t) = \left(\frac{1}{C} \log(t) \right)^{1/\beta} \left(1 + \frac{\alpha \log \log(t)}{\beta^2 \log(t)} + \frac{1}{\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right) \frac{1}{\log(t)} + \frac{\alpha^2(1-\beta) \log^2(\log(t))}{2\beta^4 \log^2(t)} + \frac{\alpha}{\beta^3} \left(\alpha + (1-\beta) \left(\log c - \frac{\alpha}{\beta} \log C \right) \right) \frac{\log \log(t)}{\log^2(t)} + \frac{\log \log(t)}{\log^2(t)} \varepsilon_5(t) \right),$$

with $\varepsilon_5(t) \rightarrow 0$ as $t \rightarrow \infty$. One has

$$C(U(t))^\beta - \log(t) = \frac{\alpha}{\beta} \log \log(t) + \log c - \frac{\alpha}{\beta} \log C + \frac{\alpha^2 \log \log(t)}{\beta^2 \log(t)} + \beta \frac{\log \log(t)}{\log(t)} \varepsilon_5(t) + \frac{1}{\log(t)} \left(\frac{\beta-1}{2\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right)^2 + o(1) \right)$$

and

$$\alpha \log(U(t)) = \frac{\alpha}{\beta} \log \log(t) - \frac{\alpha}{\beta} \log C + \frac{\alpha^2 \log \log(t)}{\beta^2 \log(t)} + \frac{\alpha}{\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right) \frac{1}{\log(t)} + o\left(\frac{1}{\log(t)}\right).$$

Plugging these two asymptotic expansions along with the asymptotic equivalent $1/(U(t))^\beta \sim C/\log(t)$ into (1) results in

$$0 = \beta \frac{\log \log(t)}{\log(t)} \varepsilon_5(t) + \left(\frac{\beta-1}{2\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right)^2 - \frac{\alpha}{\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right) - CD + o(1) \right) \frac{1}{\log(t)}$$

and therefore

$$\lim_{t \rightarrow \infty} \log \log(t) \varepsilon_5(t) = \frac{1}{\beta} \left(\frac{1-\beta}{2\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right)^2 + \frac{\alpha}{\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right) + CD \right).$$

This yields the desired asymptotic expansion of U .

(ii) We first remark that for any $x > 0$,

$$\frac{\log \log(tx)}{\log(tx)} - \frac{\log \log(t)}{\log(t)} = \left(-\frac{\log \log(t)}{\log^2(t)} + \frac{1}{\log^2(t)} \right) \log(x) + o\left(\frac{1}{\log^2(t)}\right)$$

and

$$\frac{1}{\log(tx)} - \frac{1}{\log(t)} = -\frac{\log(x)}{\log^2(t)} + o\left(\frac{1}{\log^2(t)}\right)$$

as $t \rightarrow \infty$. It then follows from the asymptotic expansion of U shown in (i) that

$$\begin{aligned} U(tx) &= \left(\frac{1}{C} \log(t) \right)^{1/\beta} \left(1 + \frac{1}{\beta} \frac{\log(x)}{\log(t)} + \frac{1-\beta}{2\beta^2} \frac{\log^2(x)}{\log^2(t)} + o\left(\frac{1}{\log^2(t)}\right) \right) \\ &\times \left(1 + \frac{\alpha}{\beta^2} \frac{\log \log(t)}{\log(t)} + \frac{1}{\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right) \frac{1}{\log(t)} \right. \\ &\quad \left. + \frac{\alpha^2(1-\beta)}{2\beta^4} \frac{\log^2(\log(t))}{\log^2(t)} + \frac{\alpha}{\beta^3} \left(\alpha + (1-\beta) \left(\log c - \frac{\alpha}{\beta} \log C \right) \right) \frac{\log \log(t)}{\log^2(t)} \right. \\ &\quad \left. + \frac{1}{\beta} \left(\frac{1-\beta}{2\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right)^2 + \frac{\alpha}{\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right) + CD \right) \frac{1}{\log^2(t)} + o\left(\frac{1}{\log^2(t)}\right) \right) \\ &+ \left(\frac{1}{C} \log(t) \right)^{1/\beta} \left(\frac{\alpha}{\beta^2} \left(-\frac{\log \log(t)}{\log^2(t)} + \frac{1}{\log^2(t)} \right) - \frac{1}{\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right) \frac{1}{\log^2(t)} \right) \log(x) \end{aligned}$$

as $t \rightarrow \infty$. Consequently

$$\begin{aligned} U(tx) - U(t) &= \frac{(\log(t))^{1/\beta-1}}{\beta C^{1/\beta}} \left(\log(x) + \frac{1-\beta}{2\beta} \frac{\log^2(x)}{\log(t)} + o\left(\frac{1}{\log(t)}\right) \right) \\ &\times \left(1 + \frac{\alpha}{\beta^2} \frac{\log \log(t)}{\log(t)} + \frac{1}{\beta} \left(\log c - \frac{\alpha}{\beta} \log C \right) \frac{1}{\log(t)} + o\left(\frac{1}{\log(t)}\right) \right) \\ &+ \frac{(\log(t))^{1/\beta-1}}{\beta C^{1/\beta}} \left(\frac{\alpha}{\beta} \left(-\frac{\log \log(t)}{\log(t)} + \frac{1}{\log(t)} \right) - \left(\log c - \frac{\alpha}{\beta} \log C \right) \frac{1}{\log(t)} \right) \log(x) \\ &= \frac{(\log(t))^{1/\beta-1}}{\beta C^{1/\beta}} \left(a_1(t) \log(x) + \frac{1-\beta}{\beta} \frac{1}{\log(t)} \times \frac{1}{2} \log^2(x) + o\left(\frac{1}{\log(t)}\right) \right) \end{aligned}$$

where

$$a_1(t) = 1 + \frac{\alpha(1-\beta)}{\beta^2} \frac{\log \log(t)}{\log(t)} + \frac{1}{\beta} \left(\alpha + (1-\beta) \left(\log c - \frac{\alpha}{\beta} \log C \right) \right) \frac{1}{\log(t)}$$

converges to 1 as $t \rightarrow \infty$. Note that

$$a(t) = \frac{(\log(t))^{1/\beta-1}}{\beta C^{1/\beta}} a_1(t) \sim \frac{(\log(t))^{1/\beta-1}}{\beta C^{1/\beta}} \text{ as } t \rightarrow \infty$$

to complete the proof.

(iii) When $\beta = 1$, one has the following refinement of (1):

$$\log(t) = CU(t) - \alpha \log(U(t)) - \log c - \log \left(1 + \frac{D}{U(t)} + \frac{D'}{(U(t))^2} \right) + o \left(\frac{1}{(U(t))^2} \right) \text{ as } t \rightarrow \infty. \quad (2)$$

It also follows from (i) that

$$\begin{aligned} U(t) &= \frac{1}{C} \log(t) \left(1 + \alpha \frac{\log \log(t)}{\log(t)} + (\log c - \alpha \log C) \frac{1}{\log(t)} + \alpha^2 \frac{\log \log(t)}{\log^2(t)} \right. \\ &\quad \left. + (\alpha(\log c - \alpha \log C) + CD) \frac{1}{\log^2(t)} + o \left(\frac{1}{\log^2(t)} \right) \right) \end{aligned}$$

as $t \rightarrow \infty$. As a consequence, for any $x > 0$,

$$\begin{aligned} &\frac{CU(tx)}{\log(tx)} - \frac{CU(t)}{\log(t)} \\ &= \alpha \left(\frac{\log \log(tx)}{\log(tx)} - \frac{\log \log(t)}{\log(t)} \right) + (\log c - \alpha \log C) \left(\frac{1}{\log(tx)} - \frac{1}{\log(t)} \right) + o \left(\frac{1}{\log^2(t)} \right) \\ &= \left(-\alpha \frac{\log \log(t)}{\log^2(t)} + \frac{\alpha + \alpha \log C - \log c}{\log^2(t)} \right) \log(x) + o \left(\frac{1}{\log^2(t)} \right). \end{aligned} \quad (3)$$

Moreover, using the fact that $CU(t) \sim \log(t)$ and since $C(U(tx) - U(t)) \rightarrow \log(x)$ as $t \rightarrow \infty$, one has

$$\frac{1}{U(tx)} - \frac{1}{U(t)} = -C \frac{\log(x)}{\log^2(t)} + o \left(\frac{1}{\log^2(t)} \right) \text{ and } \frac{1}{(U(tx))^2} - \frac{1}{(U(t))^2} = o \left(\frac{1}{\log^2(t)} \right).$$

We then find that

$$\begin{aligned} &\log \left(1 + \frac{D}{U(tx)} + \frac{D'}{(U(tx))^2} \right) - \log \left(1 + \frac{D}{U(t)} + \frac{D'}{(U(t))^2} \right) \\ &= \log \left(1 - CD \frac{\log(x)}{\log^2(t)} + o \left(\frac{1}{\log^2(t)} \right) \right) = -CD \frac{\log(x)}{\log^2(t)} + o \left(\frac{1}{\log^2(t)} \right). \end{aligned} \quad (4)$$

Combining (2), (3) and (4) yields

$$\begin{aligned} U(tx) - U(t) &= \frac{1}{C} \log(x) + \frac{\alpha}{C} \log \left(1 + \frac{\log(x)}{\log(t)} \right) \\ &\quad + \frac{\alpha}{C} \log \left(1 + \frac{\log(t)}{CU(t)} \left(\frac{CU(tx)}{\log(tx)} - \frac{CU(t)}{\log(t)} \right) \right) \\ &\quad + \frac{1}{C} \left(\log \left(1 + \frac{D}{U(tx)} + \frac{D'}{(U(tx))^2} \right) - \log \left(1 + \frac{D}{U(t)} + \frac{D'}{(U(t))^2} \right) \right) + o \left(\frac{1}{(U(t))^2} \right) \\ &= \frac{1}{C} \log(x) \left(1 + \frac{\alpha}{\log(t)} - \alpha^2 \frac{\log \log(t)}{\log^2(t)} + (\alpha(\alpha + \alpha \log C - \log c) - CD) \frac{1}{\log^2(t)} \right) \\ &\quad - \frac{1}{C} \times \frac{\alpha}{\log^2(t)} \times \frac{1}{2} \log^2(x) + o \left(\frac{1}{\log^2(t)} \right). \end{aligned}$$

The conclusion follows immediately. \square

Proof of Theorem 2. (i) In the case $\theta \neq 1$ and $\beta < 0$, one has $\ell_\beta(tx) = O(|\ell_\beta(t)|) = o(t^{-\beta/2}) = o(1/\log^2(t))$ as $t \rightarrow \infty$ for any $x > 0$, because $|\ell_\beta|$ is regularly varying with index β (use Potter bounds, see Proposition B.1.9.5 p.367 in de Haan and Ferreira (2006)). Then

$$U(tx) - \mu = \left(\frac{1}{C} \log(t)\right)^{1/\theta} \left(1 + \frac{1}{\theta} \frac{\log(x)}{\log(t)} + \frac{1-\theta}{2\theta^2} \frac{\log^2(x)}{\log^2(t)} + o\left(\frac{1}{\log^2(t)}\right)\right)$$

and therefore

$$U(tx) - U(t) = \frac{(\log(t))^{1/\theta-1}}{\theta C^{1/\theta}} \left(\log(x) + \frac{1-\theta}{\theta} \frac{1}{\log(t)} \times \frac{1}{2} \log^2(x) + o\left(\frac{1}{\log(t)}\right)\right)$$

as required. When $\beta = 0$, write instead

$$\begin{aligned} U(tx) - U(t) &= \left(\frac{1}{C} \log(t)\right)^{1/\theta} \left(\left(1 + \frac{\log(x)}{\log(t)}\right)^{1/\theta} (\ell_0(tx) - \ell_0(t)) + \left(\left(1 + \frac{\log(x)}{\log(t)}\right)^{1/\theta} - 1 \right) (1 + \ell_0(t)) \right) \\ &= \frac{(\log(t))^{1/\theta-1}}{\theta C^{1/\theta}} \left((1 + \ell_0(t) + \theta \eta(t)) \log(x) + \frac{1-\theta}{\theta} \frac{1}{\log(t)} \times \frac{1}{2} \log^2(x) + o\left(\frac{1}{\log^2(t)}\right) \right) \end{aligned}$$

from which the result follows immediately.

(ii) When $\theta = 1$ and $\beta < 0$, write, for any $x > 0$ and t large enough,

$$\begin{aligned} C(U(tx) - U(t)) &= (1 - \beta \ell_\beta(t) \log(t)) \log(x) + \beta^2 \ell_\beta(t) \log(t) \left(\frac{1}{-\beta} \left(\frac{x^{-\beta} - 1}{-\beta} - \log(x) \right) \right) \\ &\quad + \ell_\beta(tx) \log(x) + \left(\frac{\ell_\beta(tx)}{\ell_\beta(t)} - x^{-\beta} \right) \ell_\beta(t) \log(t). \end{aligned}$$

Now

$$\ell_\beta(tx) = O(|\ell_\beta(t)|) \text{ and } \lim_{t \rightarrow \infty} \frac{\ell_\beta(tx)}{\ell_\beta(t)} - x^{-\beta} = 0$$

because $|\ell_\beta|$ is regularly varying with index β and has constant sign. Therefore

$$\begin{aligned} U(tx) - U(t) &= \frac{1}{C} \left((1 - \beta \ell_\beta(t) \log(t)) \log(x) + \beta^2 \ell_\beta(t) \log(t) \int_1^x s^{-1} \left(\int_1^s u^{-\beta-1} du \right) ds \right) \\ &\quad + o(|\ell_\beta(t)| \log(t)). \end{aligned}$$

The desired conclusion follows. \square

Proof of Corollary 1. The case when $D_1 = \dots = D_l = D'_1 = \dots = D'_m = 0$ is handled directly. Otherwise, one has

$$U(t) - \mu = \left(\frac{1}{C} \log(t)\right)^{1/\theta} (1 + \ell_0(t))$$

where

$$\ell_0(t) = \sum_{i=1}^l \frac{D_i}{\log^{\delta_i}(t)} + \sum_{j=1}^m \frac{D'_j}{\log^{1+\delta'_j}(t)} + o\left(\frac{1}{\log^2(t)}\right)$$

defines a function having (up to alteration on a compact interval) constant sign, converging to 0, and whose absolute value is slowly varying. Besides, since for any $r \in \mathbb{R}$ and $x > 0$,

$$\log^r(tx) - \log^r(t) = r \log^{r-1}(t) \log(x) \left(1 + \frac{r-1}{2} \frac{\log(x)}{\log(t)} (1 + o(1))\right) \text{ as } t \rightarrow \infty, \quad (5)$$

one has

$$\ell_0(tx) - \ell_0(t) = \left(- \sum_{i=1}^l \frac{D_i \delta_i}{\log^{\delta_i}(t)} \right) \frac{\log(x)}{\log(t)} + o\left(\frac{1}{\log^2(t)} \right) \text{ as } t \rightarrow \infty.$$

Conclude by applying Theorem 2(i) and recalling that $a(t)$ is uniquely determined only up to terms of order smaller than or equal to $A(t)$. \square

Proof of Theorem 3. Using (5) again, one has for any $x > 0$ and large enough t ,

$$\begin{aligned} U(tx) - \mu &= \frac{1}{C} \log(t) \left(1 + \frac{\log(x)}{\log(t)} \right) \\ &\times \left(1 + \frac{D}{\log^\delta(t)} - \frac{D\delta}{\log^{1+\delta}(t)} \log(x) + \frac{D\delta(\delta+1)}{\log^{2+\delta}(t)} \times \frac{1}{2} \log^2(x) \right. \\ &\left. + \sum_{i=1}^l \frac{D_i}{\log^{\delta+\delta_i}(t)} - \sum_{i=1}^l \frac{D_i(\delta+\delta_i)}{\log^{1+\delta+\delta_i}(t)} \log(x) + \sum_{j=1}^m \frac{D'_j}{\log^{1+\delta+\delta'_j}(t)} + o\left(\frac{1}{\log^{2+\delta}(t)} \right) \right). \end{aligned}$$

Conclude that

$$\begin{aligned} U(tx) - U(t) &= \frac{1}{C} \left(\left(1 + \frac{D(1-\delta)}{\log^\delta(t)} + \sum_{i=1}^l \frac{D_i(1-\delta-\delta_i)}{\log^{\delta+\delta_i}(t)} \right) \log(x) + \frac{D\delta(\delta-1)}{\log^{1+\delta}(t)} \times \frac{1}{2} \log^2(x) + o\left(\frac{1}{\log^{1+\delta}(t)} \right) \right) \end{aligned}$$

as $t \rightarrow \infty$ in order to conclude the proof. \square