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Multiscale modeling for a class of high-contrast heterogeneous sign-changing problems

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Abstract

The mathematical formulation of sign-changing problems involves a linear second-order partial differential equation in the divergence form, where the coefficient can assume positive and negative values in different subdomains. These problems find their physical background in negative-index metamaterials, either as inclusions embedded into common materials as the matrix or vice versa. In this paper, we propose a numerical method based on the constraint energy minimizing generalized multiscale finite element method (CEM-GMsFEM) specifically designed for sign-changing problems. The construction of auxiliary spaces in the original CEM-GMsFEM is tailored to accommodate the sign-changing setting. The numerical results demonstrate the effectiveness of the proposed method in handling sophisticated coefficient profiles and the robustness of coefficient contrast ratios. Under several technical assumptions and by applying the T-coercivity theory, we establish the inf-sup stability and provide an a priori error estimate for the proposed method.

1 Introduction

Metamaterials greatly expand the design space of materials by offering unconventional properties that are not found in nature. Typically, metamaterials are created by assembling common materials periodically. The exotic characteristics of metamaterials manifest at the effective medium level, which can strikingly differ from the intrinsic properties of the constituent materials. Some notable examples include auxetics, exhibiting a negative Poisson's ratio [2, 38]; pentamode materials, characterized by a vanishing shear modulus [42, 37]; and negative-index materials, displaying negative electric and magnetic permeability [48, 46]. From a mathematical perspective, the emergence of metamaterials prompts a reconsideration of the well-posedness theory of Partial Differential Equations (PDEs), as the coefficients in PDEs may fall outside the conventional range for coercivity. In this context, we focus on the so-called sign-changing problem, which is rooted in the background that negative-index materials are embedded into a common medium, or vice versa.

The mathematical nature of the sign-changing problem is a linear second-order PDE in the divergence form, where the coefficient allows both positive and negative values in different subdomains, with a discontinuity across the interface between the subdomains. The well-posedness of the model problem is generally questionable due to the absence of uniform strict positivity of the coefficient. To address this, T-coercivity

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was introduced by Bonnet-BenDhia, Ciarlet Jr., and Zwölf in their work [13], providing a systematic approach to study the well-posedness of the model problem. The key idea behind T-coercivity is to devise a bijective map \mathcal{T} that enables the inf-sup (Banach–Necas–Babuška) condition to hold trivially. The construction of \mathcal{T} only relies on the geometric information of interfaces that separate the subdomains with different signs of the coefficient. The main statement of the T-coercivity theory is that if the contrast ratio between the negative and positive coefficients is sufficiently large, then the model problem becomes well-posed. In [10], the sharpest condition of the contrast ratio ensuring well-posedness on certain simple interface problems was derived. Further extensions of the T-coercivity theory encompass Helmholtz-like problems [24], time-harmonic Maxwell equations [12, 11], eigenvalue problems [18], and mixed problems [6].

Although analytical solutions offer us a profound mathematical insight into sign-changing problems, numerical methods are indispensable for practical applications. The T-coercivity theory justifies the well-posedness of the PDE models, which also guides the design of suitable approximations of the original problem. An intriguing aspect lies in the compatibility between \mathcal{T} at the continuous level and the discretization level. As an immediate result, it is emphasized in [20] that, to obtain an optimal convergence rate, meshes near flat interfaces should be symmetric. Later, a new treatment at the corners of interfaces was proposed in [9], leading to meshing rules for an arbitrary polygonal interface. Considering the low regularity of the solution due to the heterogeneous coefficient, a posteriori error analysis was conducted in [44] and in [25], which provides a reliable error estimator for adaptive mesh refinement routines [47]. Beyond classic continuous Galerkin methods, Chung and Ciarlet Jr. introduced a staggered discontinuous Galerkin method in [21], accompanied by stability and convergence analysis.

The nature of heterogeneous coefficients in sign-changing problems motivates the application of multiscale computational methods. Pioneered by Hou and Wu in [32], the methodology of incorporating model information into the construction of finite element spaces, coined as MsFEMs, has garnered significant attention. As a discretization scheme, an advantage of MsFEMs is that the meshes are not required to resolve the heterogeneity of the coefficient, although the implementation of MsFEMs indeed relies on a pair of nested meshes. To relieve the rigidity from boundary conditions in constructing multiscale bases, the oversampling technique is introduced in [32] and subsequently proved to improve convergence rates (ref. [30, 29]). The accuracy of MsFEMs, to some extent, may deteriorate when the coefficient violates the scale-separation assumption, as evidenced by the convergence theories in [33, 51, 43]. To address this issue, the Generalized Multiscale Finite Element Method (GMsFEM) was proposed by Efendiev, Galvis, and Hou in [28]. GMsFEMs leverage spectral decomposition to perform dimension reduction for the online space, exhibiting superior performance when dealing with high-contrast and channel-like coefficient profiles [23]. The first construction of multiscale bases capable of achieving the theoretically best approximation property for general L^∞ coefficients was credited to Målqvist and Peterseim in their celebrating work [40]. This construction, known as Localized Orthogonal Decomposition (LOD), utilizes quasi-interpolation operators to decompose the solution into macroscopic and microscopic components [3, 41]. The combination of GMsFEMs and LOD led to the development of a CEM-GMsFEM by Chung, Efendiev, and Leung in [22], where “CEM” is the acronym for “Constraint Energy Minimizing”. The novelty of CEM-GMsFEMs resides in replacing quasi-interpolation operators in LOD with element-wise eigenspace projections. Moreover, CEM-GMsFEMs introduce a relaxed version of the energy minimization problems to construct multiscale bases, which eliminates the necessity of solving saddle-point linear systems. Our intention here is not to present a comprehensive review of multiscale computational methods from the community, and hence, notable advancements such as heterogeneous multiscale methods [27, 1], generalized finite element methods [4, 5, 39], and variational multiscale methods [34, 35] are not covered.

This article serves as an application of the CEM-GMsFEM to sign-changing problems. While the T-coercivity theory guarantees the well-posedness of the model problem, the heterogeneity of the coefficient leads to a generally low regularity of the solution, resulting in suboptimal convergence rates when using standard finite element methods. The CEM-GMsFEM, being a multiscale computational method, is specifically designed to handle the low regularity of the solution, and the construction of multiscale bases in this paper is tailored to the sign-changing setting. For instance, the auxiliary space forms a core module in the original CEM-GMsFEM and is created by solving generalized eigenvalue problems, where the coefficient enters bilinear forms on both sides. However, this approach is not suitable for sign-changing problems, as

88 the eigenvalues can be negative, rendering the generalized eigenvalue problems ill-defined. To ensure the
 89 positivity of the eigenvalues and the approximation ability of the auxiliary space, we modify the generalized
 90 eigenvalue problems by replacing the coefficient with its absolute value counterpart. It is worth noting
 91 that in building multiscale bases, we adhere to the relaxed version of the energy minimization problems
 92 in this paper, which offers implementation advantages. We conduct numerical experiments to validate the
 93 performance of the proposed method, emphasizing that coarse meshes need not align with the sign-changing
 94 interfaces. Moreover, we demonstrate that contrast robustness, which is an important feature of the original
 95 CEM-GMsFEM, is inherited in the proposed method. Under several assumptions, we prove the existence of
 96 multiscale bases in the T-coercivity framework, the exponential decay property for the oversampling layers,
 97 and the inf-sup stability of the online space. Moreover, we can provide an a priori estimate for the proposed
 98 method, indicating that errors can be bounded in terms of coarse mesh sizes and the number of oversampling
 99 layers, while being independent of the regularity of the solution.

100 Currently, to the best of our knowledge, efforts to apply multiscale computational methods to sign-
 101 changing problems are scarce. An exception can be found in the work by Chaumont-Frelet and Verfürth in
 102 [19], where they employed the LOD framework. Note that contrast ratios play a significant role in the T-
 103 coercivity theory for assessing the well-posedness of the model problem. In comparison, the proposed method
 104 presented in this paper is more general, as it can handle a wider range of coefficient profiles, including those
 105 with high contrast ratios.

106 This paper is organized as follows. In Section 2, we introduce the model problem and present the T-
 107 coercivity theory. The construction of multiscale bases in the proposed method is detailed in Section 3. To
 108 validate the performance of the proposed method, Section 4 presents numerical experiments conducted on
 109 four different models. All theoretical analysis for the proposed method is gathered in Section 5. Finally, in
 110 Section 6, we conclude the paper.

111 2 Preliminaries

112 For simplicity, we consider a 2D Lipschitz domain denoted as Ω , which can be divided into two non-
 113 overlapping subdomains Ω^+ and Ω^- with Γ as the interfaces. The extension of the following discussion to
 114 3D is straightforward. Let σ belong to $L^\infty(\Omega)$ such that the essential infimum of σ over Ω^+ is greater than
 115 zero ($\text{ess inf}_{x \in \Omega^+} \sigma(x) > 0$), and the essential supremum of σ over Ω^- is less than zero ($\text{ess sup}_{x \in \Omega^-} \sigma(x) < 0$).
 116 In particular, σ is discontinuous across the interface separating Ω^+ and Ω^- . We further introduce notations:
 117 $\sigma_{\max}^+ = \text{ess sup}_{x \in \Omega^+} \sigma(x)$, $\sigma_{\min}^+ = \text{ess inf}_{x \in \Omega^+} \sigma(x)$, $\sigma_{\max}^- = \text{ess sup}_{x \in \Omega^-} |\sigma(x)|$ and $\sigma_{\min}^- = \text{ess inf}_{x \in \Omega^-} |\sigma(x)|$.
 118 Hence, we require that $\sigma_{\max}^+ \geq \sigma_{\min}^+ > 0$ and $\sigma_{\max}^- \geq \sigma_{\min}^- > 0$. We redefine V as the conventional Hilbert
 119 space $H_0^1(\Omega)$, and consider a bilinear form $a(\cdot, \cdot)$ on $V \times V$ given by:

$$a(v, w) = \int_{\Omega} \sigma \nabla v \cdot \nabla w \, dx, \quad \forall (v, w) \in V \times V.$$

120 Then for an function f belonging to $L^2(\Omega)$, the following variational form defines the model problem:

$$\text{find } u \in V \text{ s.t. } \forall v \in V, a(u, v) = \int_{\Omega} f v \, dx. \quad (1)$$

121 Certainly, (1) corresponds to a PDE of u , i.e., $-\text{div } \sigma \nabla u = f$ with a boundary condition $u = 0$ on $\partial\Omega$. It is
 122 also convenient to introduce a notation for a norm as $\|v\|_{\bar{a}, \omega} := (\int_{\omega} |\sigma| |\nabla v|^2 \, dx)^{1/2}$, where ω is a subdomain
 123 of Ω . Moreover, we abbreviate $\|\cdot\|_{\bar{a}, \Omega}$ as $\|\cdot\|_{\bar{a}}$, which is exactly the *energy norm* on V .

124 However, the well-posedness of (1) is generally questionable due to the loss of uniform strict positivity
 125 of σ . T-coercivity is based on a fact: if there exists a *bijective* map $\mathcal{T}: V \rightarrow V$ and a positive constant α
 126 such that for all $v \in V$, $|a(v, \mathcal{T}v)| \geq \alpha \|v\|_{\bar{a}}^2$, then the Banach–Něcas–Babuška theory confirms the existence
 127 and uniqueness of the solution. The novelty of T-coercivity lies in providing a systematic construction of
 128 such \mathcal{T} using a “flip” operator, which is essentially dependent on the geometric information of the interfaces
 129 between Ω^+ and Ω^- . Let $V^\pm \subset H^1(\Omega^\pm)$ be the space obtained by restricting functions in V in Ω^\pm and Γ

130 be the interface. Assuming the existence of a bounded map $\mathcal{R}: V^+ \rightarrow V^-$ with $\mathcal{R}v|_\Gamma = v|_\Gamma$ in the sense of
 131 Sobolev traces, the construction of \mathcal{T} is given by

$$\mathcal{T}v = \begin{cases} v_1, & \text{in } \Omega^+, \\ -v_2 + 2\mathcal{R}v_1, & \text{in } \Omega^-, \end{cases} \quad (2)$$

132 where

$$v = \begin{cases} v_1, & \text{in } \Omega^+, \\ v_2, & \text{in } \Omega^-. \end{cases} \quad (3)$$

133 Certainly, the definition of \mathcal{T} in (2) is bijective and bounded. Furthermore, we can show that

$$\begin{aligned} a(v, \mathcal{T}v) &= \int_{\Omega^+} |\sigma| |\nabla v_1 \cdot \nabla v_1| dx - \int_{\Omega^-} |\sigma| |\nabla v_2 \cdot \nabla(-v_2 + 2\mathcal{R}v_1)| dx \\ &\stackrel{(\forall \eta > 0)}{\geq} \int_{\Omega^+} |\sigma| |\nabla v_1|^2 dx - \frac{1}{\eta} \int_{\Omega^-} |\sigma| |\nabla \mathcal{R}v_1|^2 dx + (1 - \eta) \int_{\Omega^-} |\sigma| |\nabla v_2|^2 dx \quad (\text{by Young's inequality}) \\ &\geq \int_{\Omega^+} |\sigma| |\nabla v_1|^2 dx - \frac{\sigma_{\max}^- \|\mathcal{R}\|_1^2}{\eta} \int_{\Omega^+} |\nabla v_1|^2 dx + (1 - \eta) \int_{\Omega^-} |\sigma| |\nabla v_2|^2 dx \\ &\geq \left(1 - \frac{\sigma_{\max}^- \|\mathcal{R}\|_1^2}{\sigma_{\min}^+ \eta}\right) \int_{\Omega^+} |\sigma| |\nabla v_1|^2 dx + (1 - \eta) \int_{\Omega^-} |\sigma| |\nabla v_2|^2 dx \\ &\geq \left(1 - \sqrt{\frac{\sigma_{\max}^-}{\sigma_{\min}^+}} \|\mathcal{R}\|_1\right) \|v\|_a^2, \quad (\text{by choosing } \eta = \sqrt{\sigma_{\max}^- / \sigma_{\min}^+} \|\mathcal{R}\|_1) \end{aligned} \quad (4)$$

134 where $\|\mathcal{R}\|_1$ should be understood as a positive constant such that for all $w \in V^+$,

$$\left(\int_{\Omega^-} |\nabla \mathcal{R}w|^2 dx\right)^{1/2} \leq \|\mathcal{R}\|_1 \left(\int_{\Omega^-} |\nabla w|^2 dx\right)^{1/2}. \quad (5)$$

135 We can observe that (1) will always be well-posed if the ratio $\sigma_{\min}^+ / \sigma_{\max}^-$ is large enough. Note that the
 136 derivation in (4) involves flipping “positive” to “negative” as indicated by the definition of $\mathcal{T}: V^+ \rightarrow V^-$.
 137 It is possible to flip “negative” to “positive”, and the well-posedness of (1) can also be achieved if the
 138 $\sigma_{\min}^- / \sigma_{\max}^+$ is large, as discussed in [10].

139 To gain insight into the operator \mathcal{R} , let us consider a square domain $\Omega = (0, 1) \times (0, 1)$ with $\Omega^+ =$
 140 $(0, 1) \times (\gamma, 1)$ and $\Omega^- = (0, 1) \times (0, \gamma)$, where $0 < \gamma < 1$. For a $w \in V^+$, we can define $\mathcal{R}w$ as follows:

$$(\mathcal{R}w)(x_1, x_2) = w(x_1, 1 - \frac{1-\gamma}{\gamma}x_2), \quad \forall (x_1, x_2) \in \Omega^-.$$

141 It is also straightforward to calculate that $\|\mathcal{R}\|_1 = \max\{\sqrt{(1-\gamma)/\gamma}, 1\}$. Note that the construction of \mathcal{R} is
 142 not unique. As an alternative approach, we can introduce a smooth cut-off function ξ satisfying $\xi = 1$ on Γ
 143 and $\text{supp } \xi w \subset (0, 1) \times (\gamma, a)$ with $\gamma < a < 1$. Then, we can utilize the same technique to design \mathcal{R}' such
 144 that $\text{supp } \mathcal{R}'(\xi w) \subset (0, 1) \times (b, \gamma)$ with $0 \leq b < \gamma$. Naturally, as an operator, $\mathcal{R}'\xi: V^+ \rightarrow V^-$ is also valid.

145 3 Methods

146 The proposed method relies on a pair of nested meshes \mathcal{K}_h and \mathcal{K}_H . The fine mesh \mathcal{K}_h should be capable
 147 of resolving the heterogeneity of σ . Meanwhile, the construction of multiscale bases, known as the offline
 148 phase, is performed on \mathcal{K}_h . Solving the final linear system is commonly referred to as the online phase, whose
 149 computational complexity is associated with the coarse mesh \mathcal{K}_H . We reserve the index i for enumerating

150 all coarse elements on \mathcal{K}_H , where $1 \leq i \leq N_{\text{elem}}$. The oversampling region K_i^m of a coarse element K_i holds
 151 a vital significance in describing the method. Mathematically, K_i^m is defined by a recursive relation:

$$\forall m \geq 1, \quad K_i^m := \text{int} \left(\cup \{ \bar{K} \mid K \in \mathcal{K}_H \text{ with } \bar{K} \cap \overline{K_i^{m-1}} \neq \emptyset \} \right)$$

152 and $K_i^0 := K_i$. Figure 1 serves as an illustration of the nested meshes, and note that two oversampling
 153 regions $K_{i'}^2$ and $K_{i''}^2$ are colored in gray.

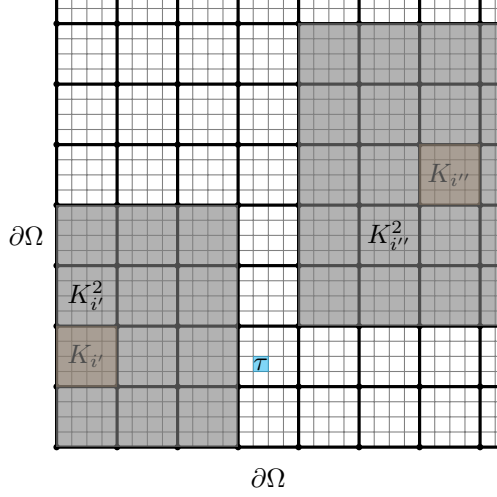


Figure 1: Illustration of the nested meshes \mathcal{K}_h and \mathcal{K}_H . A fine element τ , two coarse elements $K_{i'}$ and $K_{i''}$, accompanied by their corresponding oversampling regions $K_{i'}^2$ and $K_{i''}^2$, are colored differently.

154 In the original CEM-GMsFEM, a generalized eigenvalue problem,

$$\text{find } \lambda \in \mathbb{R} \text{ and } v \in H^1(K_i) \setminus \{0\} \text{ s.t. } \int_{K_i} \sigma \nabla v \cdot \nabla w \, dx = \lambda \int_{K_i} \mu_{\text{msh}} \text{diam}(K_i)^{-2} \sigma v w \, dx, \quad (6)$$

155 is solved on each coarse element K_i , where μ_{msh} is a generic positive constant that depends on the mesh
 156 quality. Then the local auxiliary space V_i^{aux} is formed by collecting leading eigenvectors. However, recalling
 157 that σ is not uniformly positive, the left-hand bilinear form in (6) is not positive semidefinite, and similarly,
 158 the right-hand bilinear form in (6) is not positive definite. Then, determining leading eigenvectors via (6) is
 159 problematic since the eigenvalues could be negative. To address this issue, we instead construct the following
 160 generalized eigenvalue problem on each K_i , which forms the first step of the proposed method:

First step

$$\begin{aligned} & \text{find } \lambda \in \mathbb{R} \text{ and } v \in H^1(K_i) \setminus \{0\} \text{ s.t. } \forall w \in H^1(K_i), \\ & \int_{K_i} |\sigma| \nabla v \cdot \nabla w \, dx = \lambda \int_{K_i} \mu_{\text{msh}} \text{diam}(K_i)^{-2} |\sigma| v w \, dx. \end{aligned} \quad (7)$$

161
 162 One may argue that if σ only takes values in $\{-1, 1\}$, then (7) would yield trivial eigenspaces correspond-
 163 ing to the Laplace operator, thus failing to capture any heterogeneity information of σ . However, according
 164 to T-coercivity, in such cases, the well-posedness of the model problem is violated (ref. [10] section 6). This
 165 inherent issue suggests that the original model problem should require a more sophisticated investigation in
 166 this case, which falls outside the scope of the proposed method. Upon solving the l^* leading eigenvectors,
 167 denoted as $\psi_{i,j}$ with $1 \leq j \leq l^*$, we can construct the local auxiliary space $V_i^{\text{aux}} \subset L^2(K_i)$ as $\text{span}\{\psi_{i,j}\}$

168 for $1 \leq i \leq N_{\text{elem}}$. The global auxiliary space $V^{\text{aux}} \subset L^2(\Omega)$ is defined as $V^{\text{aux}} = \oplus_{i=1}^{N_{\text{elem}}} \tilde{V}_i^{\text{aux}}$, where
 169 $\tilde{V}_i^{\text{aux}} \subset L^2(\Omega)$ is the space by performing zero-extension of functions in V_i^{aux} .

170 We introduce several notations for future reference. Let μ be a function in $L^\infty(\Omega)$ satisfying

$$\mu|_{K_i} = \mu_{\text{msh}} \text{diam}(K_i)^{-2} \sigma|_{K_i}$$

171 for all $K_i \in \mathcal{K}_H$. Next, for a subdomain ω in Ω , we define two bilinear forms:

$$a(v, w)_\omega := \int_\omega \sigma \nabla v \cdot \nabla w \, dx \text{ and } s(v, w)_\omega := \int_\omega \mu v w \, dx.$$

172 Similarly to the definition of $\|\cdot\|_{\tilde{a}, \omega}$, we define $\|v\|_{\tilde{s}, \omega}$ as $(\int_\omega |\mu| |v|^2 \, dx)^{1/2}$. Again, we drop the subscript ω
 173 if $\omega = \Omega$. Additionally, we define the orthogonal projection operator $\mathcal{P}_H: L^2(\Omega) \rightarrow L^2(\Omega)$ under the norm
 174 $\|\cdot\|_{\tilde{s}}$, with $V^{\text{aux}} = \text{im } \mathcal{P}_H$. We also use the shorthand notation V_i^m as $H_0^1(K_i^m)$, and we always implicitly
 175 identify V_i^m as a subspace of V .

176 Functions in V^{aux} may not be continuous in Ω , and thus V^{aux} cannot be used as a conforming finite
 177 element space. In the second step, we will construct a multiscale basis $\phi_{i,j}$ in V_i^m , corresponding to $\psi_{i,j}$
 178 in V_i^{aux} that is obtained in the first step. This construction is based on the following variational problem,
 179 where the righthand bilinear form is defined on the original coarse element K_i :

Second step

$$\begin{aligned} \text{find } \phi_{i,j} \in V_i^m \text{ s.t. } \forall w \in V_i^m, \\ a(\phi_{i,j}, w)_{K_i^m} + s(\mathcal{P}_H \phi_{i,j}, \mathcal{P}_H w)_{K_i^m} = s(\psi_{i,j}, \mathcal{P}_H w)_{K_i}. \end{aligned} \quad (8)$$

180
 181 In the original CEM-GMsFEM [22], the variational form in (8) is derived from a “relaxed” constrained
 182 energy minimization problem:

$$\phi_{i,j} = \text{argmin} \{ a(w, w)_{K_i^m} \mid w \in V_i^m, \mathcal{P}_H w = \psi_{i,j} \}.$$

183 We note that, in the present situation, the bilinear forms $a(\cdot, \cdot)_{K_i^m}$ and $s(\mathcal{P}_H \cdot, \mathcal{P}_H \cdot)_{K_i^m}$ may lack coercivity,
 184 leading to an ill-defined minimization. The multiscale space is formed as

$$V_H^m = \text{span} \{ \phi_{i,j} \mid 1 \leq i \leq N_{\text{elem}}, 0 \leq j < l^* \},$$

185 and the solution of the model problem is approximated by solving the following variational problem:

$$\text{find } u_H \in V_H^m \text{ s.t. } \forall w_H \in V_H^m, a(u_H, w_H) = \int_\Omega f w_H \, dx. \quad (9)$$

186 Note that (7) and (8) are practically solved on the fine mesh \mathcal{K}_h .

187 The success of the proposed method hinges on that the multiscale basis decays rapidly w.r.t. m , the
 188 number of oversampling layers. To demonstrate this, we consider a square domain with a 10×10 periodic
 189 structure, as illustrated in the subplot **(a)** of Fig. 2. Each periodic cell contains a square inclusion that is
 190 centered with a negative coefficient imposed, resulting in the union of all inclusions forming the subdomain
 191 Ω^- . The length ratio of the inclusion to the periodic cell is set to 1/2. The material properties are set as
 192 $\sigma = 1.0$ in Ω^+ and $\sigma = -0.1$ in Ω^- such that T-coercivity is satisfied. The coarse mesh \mathcal{K}_H aligns with
 193 the periodic structure. We select a coarse element marked with red borders in the subplot **(a)** of Fig. 2
 194 and plot the first three eigenfunctions $\psi_{i,1}$, $\psi_{i,2}$ and $\psi_{i,3}$ calculated via (7) in the subplots **(b)** to **(d)**. It is
 195 worth noting that the first eigenvalue is always 0, and the corresponding eigenfunction is always a constant
 196 function, as easily derived in (7) and also validated in the subplot **(b)**. The decay of the multiscale basis $\phi_{i,1}$,
 197 $\phi_{i,2}$ and $\phi_{i,3}$ are solved by (8) with different oversampling layers is demonstrated in Fig. 3, while $a(\cdot, \cdot)_{K_i^m}$
 198 and $s(\mathcal{P}_H \cdot, \mathcal{P}_H \cdot)_{K_i^m}$ are not coercive. In this figure, the first, second, and third rows correspond to the
 199 results of the first, second, and third eigenfunctions, respectively, while the first, second, and third columns

200 display the plots of the multiscale basis with $m = 1, 2,$ and $3,$ respectively. The position of the selected
 201 coarse element determines the maximum value of $m,$ which in this case is $8.$ Consequently, we calculate the
 202 relative differences in both the energy and L^2 norm of the multiscale bases between $m = 8$ and $m = 1, \dots, 7,$
 203 and present the results in the fourth column of Fig. 3. From the plots of multiscale bases, we observe
 204 that multiscale bases vanish away from the selected coarse element. Hence, one may expect to accurately
 205 compute multiscale bases with a small number of oversampling layers. Moreover, the relative differences
 206 in the logarithmic scale are almost linear w.r.t. $m,$ which suggests the exponential decay may still hold in
 207 sign-changing problems.

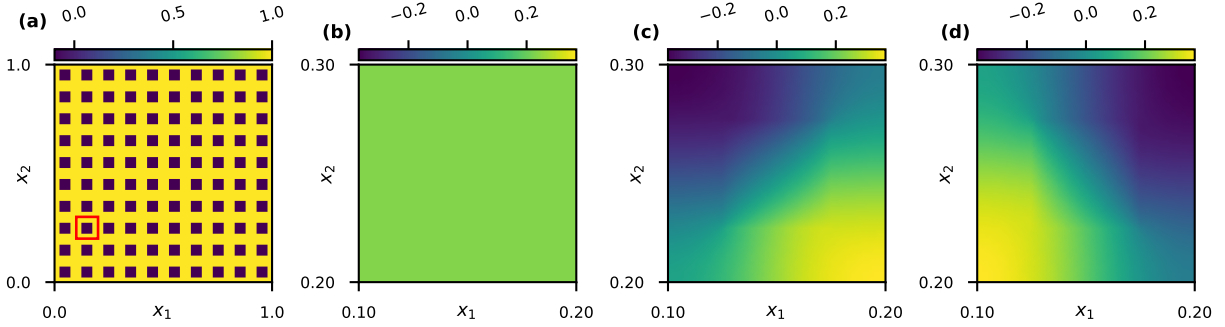


Figure 2: **(a)** The coefficient profile and the marked coarse element. **(b)–(d)** The plot of the first/second/third eigenfunction corresponding to the marked coarse element.

208 4 Numerical experiments

209 We conduct numerical experiments on a square domain $\Omega = (0, 1) \times (0, 1)$. The fine mesh \mathcal{K}_h is generated
 210 by dividing Ω into 400×400 squares. Consequently, the coefficient profile σ is represented by a 400×400
 211 matrix/image, with each element corresponding to a constant value on a fine element. To investigate the
 212 convergence behavior of the proposed method with different coarse mesh sizes $H,$ we consider four different
 213 coarse meshes $\mathcal{K}_H:$ $10 \times 10,$ $20 \times 20,$ $40 \times 40,$ and $80 \times 80.$ The reference solution, auxiliary spaces, and
 214 multiscale bases are all calculated on the fine mesh \mathcal{K}_h using the Q_1 finite element method. We evaluate
 215 the convergence of the proposed method using two relative error indices: the relative energy error and the
 216 relative L^2 error which are defined as follows:

$$\frac{\|e_h\|_{\bar{a}}}{\|u_h\|_{\bar{a}}} \text{ and } \frac{\|e_h\|_{L^2(\Omega)}}{\|u_h\|_{L^2(\Omega)}},$$

217 where u_h is the reference solution calculated by the Q_1 FEM on \mathcal{K}_h or the nodal interpolation of the exact
 218 solution (if available), and e_h is the error between the reference solution and the numerical solution. For
 219 simplicity, we take μ as

$$\mu|_{K_i} = \mu_{\text{msh}} \text{diam}(K_i)^{-2} \sigma|_{K_i} = 24H^{-2} \sigma|_{K_i}$$

220 for all numerical experiments, as suggested in [50]. In the following discussions, we mark statements of
 221 direct observations from numerical experiments with a circled number, e.g., $\textcircled{1}.$ We implement the method
 222 using the Python libraries NumPy and SciPy¹, and all the codes are hosted on Github².

¹Instead of using the default sparse linear system solver in SciPy, we opted to utilize the `pardiso` solver to enhance efficiency.

²<https://github.com/Laphet/sign-changing>

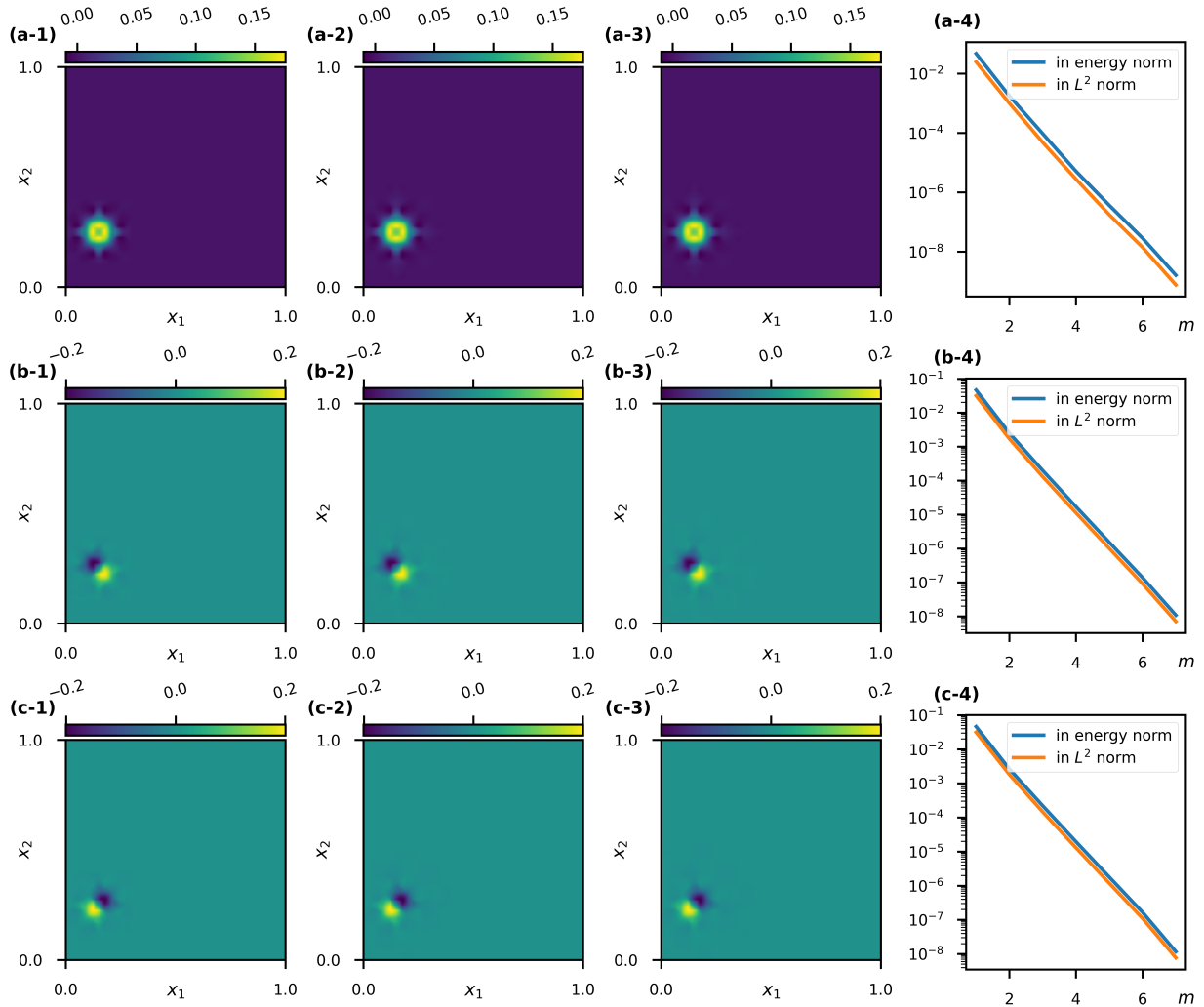


Figure 3: The subplots are marked as $(\mathbf{x}-\mathbf{y})$, where \mathbf{x} can take **a**, **b**, or **c**, corresponding to the results for the first, second, or third eigenfunction, respectively. If \mathbf{y} is **1**, **2**, or **3**, the subplot displays the multiscale basis with m oversampling layers, m equal to \mathbf{y} . Alternatively, if \mathbf{y} is **4**, the subplot shows the relative differences (y-axis) in the energy and L^2 norm of the multiscale bases between $m = 8$ and $m = 1, \dots, 7$ (x-axis).

223 4.1 Flat interface model

224 We first consider a flat interface model described in the ending part of Section 2, i.e., we define $\Omega^+ =$
 225 $(0, 1) \times (\gamma, 1)$ and $\Omega^- = (0, 1) \times (0, \gamma)$ with $0 < \gamma < 1$. In Ω^+ , we assign a fixed value of σ_*^+ to σ^+ , while in
 226 Ω^- , we take $-\sigma_*^-$ for σ^- , where σ_*^+ and σ_*^- are both positive. We can devise an exact solution u as

$$u(x_1, x_2) = \begin{cases} -\sigma_*^- x_1(x_1 - 1)x_2(x_2 - 1)(x_2 - \gamma), & \text{in } \Omega^+, \\ \sigma_*^+ x_1(x_1 - 1)x_2(x_2 - 1)(x_2 - \gamma), & \text{in } \Omega^-, \end{cases}$$

227 which corresponds to a smooth source term f given by

$$f(x_1, x_2) = \sigma_*^- \sigma_*^+ \left(2x_2(x_2 - 1)(x_2 - \gamma) + x_1(x_1 - 1)(6x_2 - 2(\gamma + 1)) \right).$$

228 According to the T-coercivity theory [20], the problem is well-posed if

$$\frac{\sigma_*^-}{\sigma_*^+} \notin \left[\frac{\gamma}{1-\gamma}, 1 \right],$$

229 provided that $\gamma \leq 1/2$.

230 **Case I** We set $\gamma = 0.5$, we can check that now the interface $\Gamma = (0, 1) \times \{\gamma\}$ is *fully resolved by every coarse*
 231 *mesh*. Therefore, we can expect that applying the Q_1 FEM on \mathcal{K}_H directly can yield satisfactory accuracy.
 232 we conducted two groups of experiments with $(\sigma_*^+, \sigma_*^-) = (1.01, 1)$ and $(\sigma_*^+, \sigma_*^-) = (1, 1.01)$, both satisfying
 233 the T-coercivity condition. For the setting of the proposed multiscale method, we fix $l^* = 3$, indicating that
 234 we calculate the first three eigenfunctions in (7) and construct three multiscale bases for each coarse element,
 235 while we vary the oversampling layers m from 1 to 4. We refer to Fig. 4 for the numerical results. ① We
 236 can observe from subplots (a) to (d) that the convergence of the Q_1 FEM manifests a linear pattern w.r.t.
 237 H in the logarithmic scale, consistent with the theoretical expectation. ② We can also see that the number
 238 of oversampling layers m has a significant impact on the accuracy of the proposed method. ③ However,
 239 for the same m , the error decaying w.r.t. H does not always hold, as depicted in subplots (b) and (d). ④
 240 Although, for $m = 4$, the proposed method exhibits higher accuracy than the Q_1 FEM, the computational
 241 cost is significantly higher due to the sophisticated process of constructing multiscale bases. Therefore, the
 242 proposed method seems more suitable for scenarios involving intricate coefficient profiles. Interestingly, we
 243 notice that subplots (a) and (c), as well as (b) and (d), are almost identical, implying that the contrast ratio
 244 σ_*^-/σ_*^+ crossing the critical value 1 does not generate a significant influence on numerical methods.

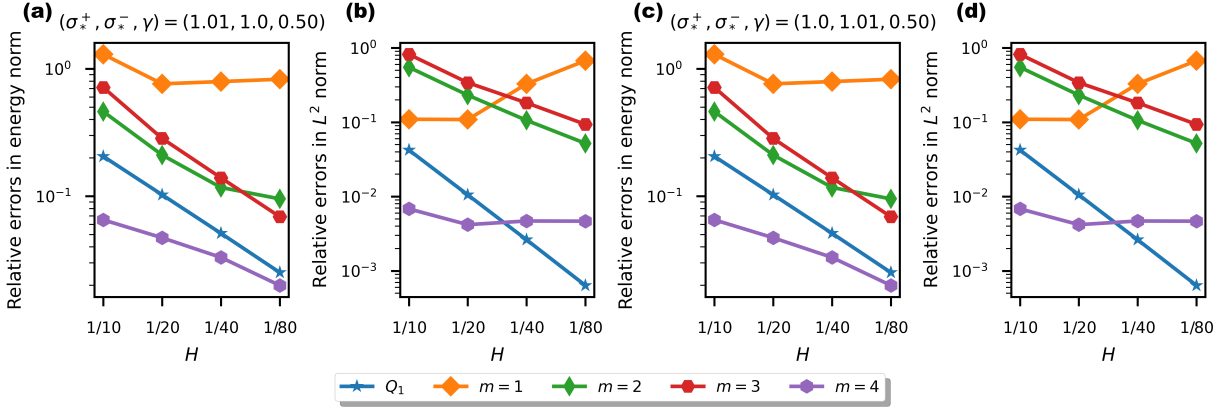


Figure 4: Numerical results for the flat interface model with $\gamma = 1/2$, where the relative errors of the proposed method with different numbers of oversampling layers m and the Q_1 FEM are calculated w.r.t. the coarse mesh size H . Subplots (a) and (b) correspond to $(\sigma_*^+, \sigma_*^-) = (1.01, 1)$, which the relative errors are measured in the energy and L^2 norm, respectively. Similarly, subplots (c) and (d) correspond to the setting $(\sigma_*^+, \sigma_*^-) = (1, 1.01)$, following the same manner.

245 **Case II** Next, we consider the case that $\gamma = 0.49$, resulting in *none of the coarse meshes being capable of*
 246 *resolving the interface*. The results by setting $(\sigma_*^+, \sigma_*^-) = (1, 1.01)$ are reported in Fig. 5, where subplots
 247 (a) and (b) correspond to the relative errors, measured in the energy and L^2 norm respectively. Subplots
 248 (c) and (d) display the actual differences between the reference solution and numerical solutions obtained
 249 by the Q_1 FEM with $H = 1/80$ and the proposed method with $(H, m) = (1/80, 3)$, respectively. ① We
 250 observe that a slight change in γ from 0.5 to 0.49 disrupts the convergence of the Q_1 FEM. ② In contrast,
 251 the proposed method can achieve satisfactory accuracy at a level of 1% for $m = 3$ and of 0.1% for $m = 4$
 252 in the energy norm. ③ Again, for a fixed m , the errors from the proposed method do not always decay w.r.t.

253 H . As shown in subplots (c) and (d), both methods exhibit a concentration of errors near the interface,
 254 while the proposed method outperforms the Q_1 FEM by an order of magnitude of two in terms of pointwise
 255 errors.

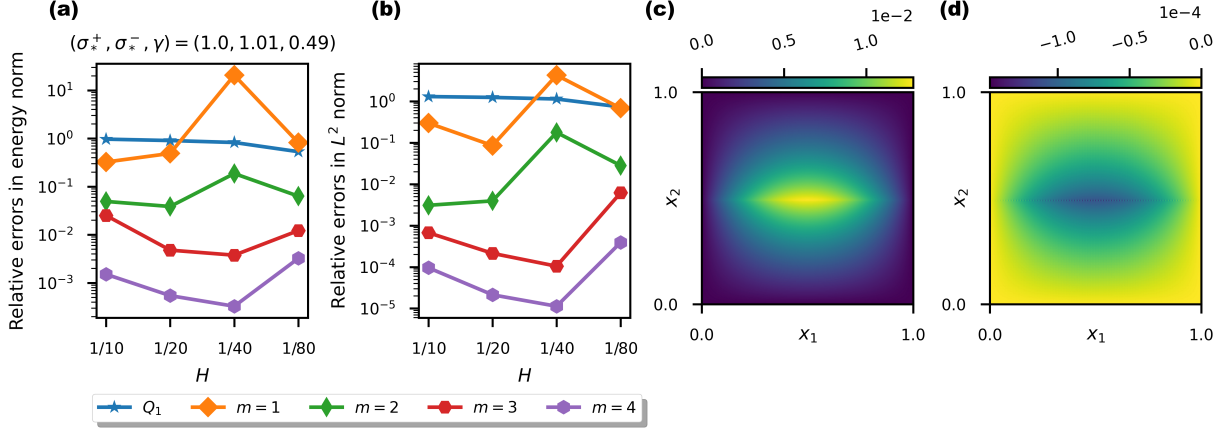


Figure 5: Numerical results for the flat interface model with $(\sigma_*^+, \sigma_*^-, \gamma) = (1, 1.01, 0.49)$. Subplots (a) and (b) show the relative errors of the proposed method with different numbers of oversampling layers m and the Q_1 FEM w.r.t. the coarse mesh size H , but measured in different norms. Subplots (c) and (d) display the actual pointwise differences between the reference solution and numerical solutions obtained by the Q_1 FEM with $H = 1/80$ and the proposed method with $(H, m) = (1/80, 3)$.

256 4.2 Periodic square inclusion model

257 In this subsection, we revisit the periodic square inclusion model described in the ending part of Section 3.
 258 The coefficient profile σ is determined as Fig. 2-(a). Besides the 10×10 periodic configuration shown in
 259 Fig. 2-(a), we also consider the 20×20 periodic configurations. The source term f is constructed as the
 260 superposition of four 2D Gaussian functions centered at $(0.25, 0.25)$, $(0.75, 0.25)$, $(0.25, 0.75)$, and $(0.75, 0.75)$
 261 with a variance of 0.01, as shown in Fig. 6-(a). The reference solutions corresponding to the 10×10 and
 262 20×20 periodic configurations are plotted in Fig. 6-(b) and (c), respectively. We can observe that multiscale
 263 features emerge in the reference solutions, which may inspire future investigations into extending classical
 264 multiscale asymptotic analysis (ref. [7, 26]) to sign-changing problems. Notably, the homogenization theory
 265 for the model was recently completed by Bunoiu and Ramdani in [15, 14, 16, 17]. Meanwhile, numerical
 266 methods such as LOD [19], along with the proposed method, seek a low-dimensional representation of the
 267 solutions and can be regarded as general numerical homogenization techniques beyond the periodic setting.

268 We conduct numerical experiments using the proposed method on the 10×10 and 20×20 periodic
 269 configurations. The relative errors in the energy norm and the L^2 norm are tabulated in Tables 1 and 2,
 270 respectively. We again take $l^* = 3$ for all tests, while varying the number of oversampling layers m from
 271 1 to 4. ① From Tables 1 and 2, when $H = 1/10$, enlarging m from 2 to 4 does not lead to a significant
 272 improvement in accuracy, and relative errors in the energy norm reach a saturation level of 1%. ② In
 273 comparison, for $H = 1/80$, the numerical solutions achieve a relative error of 0.1% in the energy norm for
 274 $m = 3$ and 4. We can conclude that the accuracy of the proposed method is mutually controlled by the
 275 number of oversampling layers m and the coarse mesh size H , and this convergence pattern is consistent
 276 with the flat interface model. ③ Another noteworthy observation is that the scale of models, i.e., $1/10$ and
 277 $1/20$, does not affect the accuracy of the proposed method, as the relative errors in Tables 1 and 2 are almost
 278 at the same level with different H and m . Hence, we can infer that the proposed method is robust to the
 279 scale of models and is not subject to the resonance error phenomenon encountered in the classic MsFEMs
 280 [32, 33, 29].

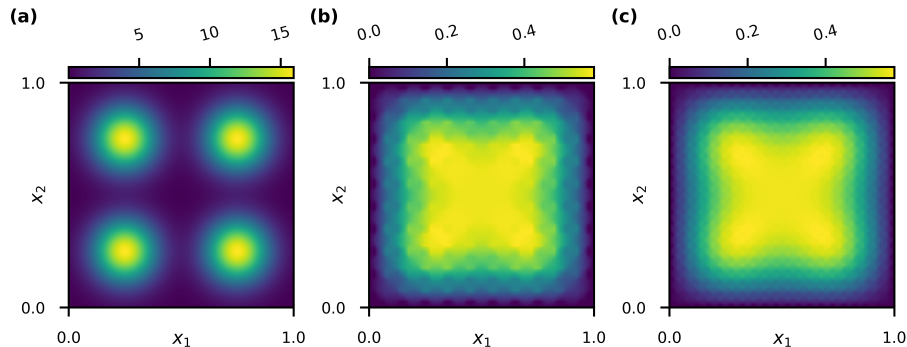


Figure 6: **(a)** The smooth source term f that is constructed as the superposition of four 2D Gaussian functions centered at $(0.25, 0.25)$, $(0.75, 0.25)$, $(0.25, 0.75)$, and $(0.75, 0.75)$ with a variance of 0.01. **(b)-(c)** The reference solutions correspond to the 10×10 and 20×20 periodic configurations, respectively.

Table 1: For the 10×10 periodic square inclusion model, the relative errors in the energy norm (in the columns labeled with $\|\cdot\|_{\bar{a}}$) and in the L^2 norm (in the columns labeled with $\|\cdot\|_{L^2(\Omega)}$).

H	$m = 1$		$m = 2$		$m = 3$		$m = 4$	
	$\ \cdot\ _{\bar{a}}$	$\ \cdot\ _{L^2(\Omega)}$	$\ \cdot\ _{\bar{a}}$	$\ \cdot\ _{L^2(\Omega)}$	$\ \cdot\ _{\bar{a}}$	$\ \cdot\ _{L^2(\Omega)}$	$\ \cdot\ _{\bar{a}}$	$\ \cdot\ _{L^2(\Omega)}$
$\frac{1}{10}$	2.433e-1	8.931e-2	5.162e-2	5.741e-3	5.225e-2	5.785e-3	5.232e-2	5.786e-3
$\frac{1}{20}$	3.785e-1	1.923e-1	4.960e-2	4.981e-3	5.583e-2	5.922e-3	5.662e-2	6.024e-3
$\frac{1}{40}$	6.978e-1	5.833e-1	4.011e-2	2.292e-3	1.753e-3	3.293e-5	1.376e-4	3.075e-6
$\frac{1}{80}$	8.832e-1	8.610e-1	8.895e-2	1.064e-2	3.855e-3	3.991e-5	1.941e-4	1.921e-6

Table 2: For the 20×20 periodic square inclusion model, the relative errors in the energy norm (in the columns labeled with $\|\cdot\|_{\bar{a}}$) and in the L^2 norm (in the columns labeled with $\|\cdot\|_{L^2(\Omega)}$).

H	$m = 1$		$m = 2$		$m = 3$		$m = 4$	
	$\ \cdot\ _{\bar{a}}$	$\ \cdot\ _{L^2(\Omega)}$	$\ \cdot\ _{\bar{a}}$	$\ \cdot\ _{L^2(\Omega)}$	$\ \cdot\ _{\bar{a}}$	$\ \cdot\ _{L^2(\Omega)}$	$\ \cdot\ _{\bar{a}}$	$\ \cdot\ _{L^2(\Omega)}$
$\frac{1}{10}$	5.571e-1	3.663e-1	6.379e-2	6.159e-3	2.653e-2	1.492e-3	2.637e-2	1.455e-3
$\frac{1}{20}$	4.531e-1	2.884e-1	3.195e-2	1.498e-3	2.598e-2	1.441e-3	2.616e-2	1.442e-3
$\frac{1}{40}$	6.306e-1	4.938e-1	4.544e-2	2.613e-3	2.589e-2	1.361e-3	2.782e-2	1.485e-3
$\frac{1}{80}$	8.803e-1	8.499e-1	8.795e-2	1.057e-2	4.303e-3	4.984e-5	2.862e-4	3.115e-6

4.3 Periodic cross-shaped inclusion model

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In this subsection, we apply the proposed method to a periodic cross-shaped inclusion model. Specifically, in each periodic cell, a cross-shaped inclusion is centered, imposing a negative coefficient, and the width of the cross arms is set to $1/5$ of the periodic cell. We consider two periodic configurations, 10×10 and 20×20 , as shown in Fig. 7-(a) and (b), respectively. Note that cross-shaped inclusions are all connected, leading to long channels crossing the domain. Constructing a “flip” operator \mathcal{R} in this model to validate the T-coercivity condition is nontrivial compared with the flat interface and the periodic square inclusion models. Using the theory developed in [9], one can prove a weak T-coercivity property holds as long as $\sigma_{\min}^+/\sigma_{\max}^-$ or $\sigma_{\min}^-/\sigma_{\max}^+ > 3$. This model is designed to test the capability of the proposed method to handle long and

290 high-contrast channels, and we hence set $(\sigma_*^+, \sigma_*^-) = (1, 10^3)$, where σ_*^+ and σ_*^- are defined as in the flat
 291 interface model. The source term f is again from Fig. 6-(a), and the reference solutions corresponding to
 292 the 10×10 and 20×20 periodic configurations are plotted in Fig. 7-(c) and (d), respectively. From reference
 293 solution plots, we can observe that long and high-contrast channels create more pronounced oscillations
 294 compared to the square inclusion model (cf. Fig. 6-(b) and (c)), challenging numerical methods greatly.

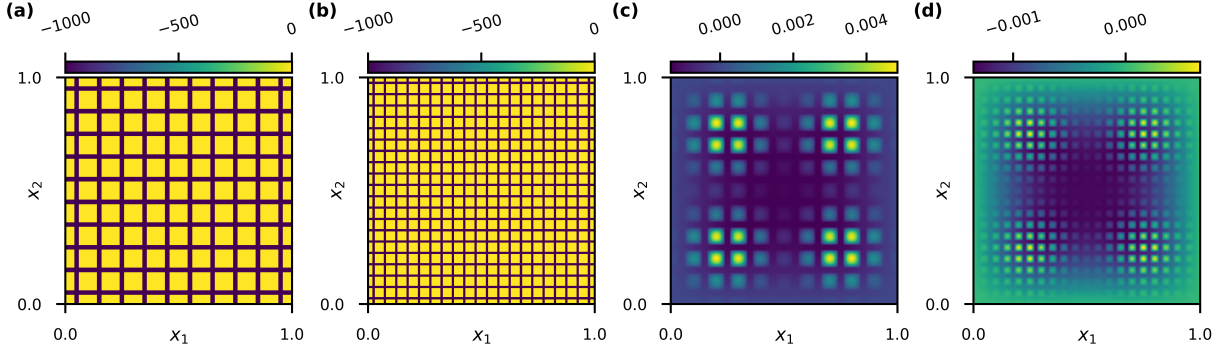


Figure 7: Subplots (a) and (b) display visualizations for the coefficient σ used in the periodic cross-shaped inclusion model with 10×10 and 20×20 periodic configurations, respectively. Subplots (c) and (d) demonstrate the reference solutions for 10×10 and 20×20 periodic configurations, respectively.

295 The numerical results of the Q_1 FEM and the proposed method with $m \in \{1, 2, 3, 4\}$ are presented
 296 in Fig. 8. Subplots (a) and (b) in Fig. 8 share the same setting (corresponding to the 10×10 periodic
 297 configuration) but the relative errors are measured in the different norms, and similarly for subplots (c) and
 298 (d). The Q_1 FEM fails to provide satisfactory accuracy. ① Even with a finer coarse mesh, such as $H = 1/80$,
 299 there is only a slight improvement, yet the relative errors in the energy norm remain close to 50%. The
 300 classical CEM-GMsFEM [22] is proven to be effective in handling long and high-contrast channels, and the
 301 proposed method inherits this advantage. ② By setting $m = 3$, the proposed method can achieve a relative
 302 error of 1% in the energy norm for both the 10×10 and 20×20 periodic configurations. ③ Typically, the
 303 relative errors in the L^2 norm are significantly smaller by an order of magnitude than those in the energy
 304 norm, and the proposed method can achieve a relative error of 0.1% in the L^2 norm for $m = 3$ and 4. ④
 305 Furthermore, comparing subplots (a) and (c), as well as (b) and (d), reveals similar convergence behavior,
 306 indicating that the scale of the models does not affect the accuracy of the proposed method. Note that all
 307 coarse meshes employed cannot resolve the channels, highlighting the capability of the proposed method to
 308 handle complex coefficient profiles.

309 4.4 Random inclusion model

310 In this subsection, we consider a random inclusion model that is utilized in several multiscale methods as
 311 a showcase of the capability of handling nonperiodic coefficient profiles [22, 52, 50, 45]. The subdomains
 312 Ω^+ and Ω^- are demonstrated in Fig. 9-(a), and σ is determined again by (σ_*^+, σ_*^-) . We consider two cases:
 313 $(\sigma_*^+, \sigma_*^-) = (1, 10^{-3})$ and $(\sigma_*^+, \sigma_*^-) = (1, 10^3)$. By setting the source term f as depicted in Fig. 6-(a),
 314 we plot the reference solutions for the two cases, as shown in Fig. 9-(b) and (c). We can observe that
 315 void-type inclusions $((\sigma_*^+, \sigma_*^-) = (1, 10^{-3}))$ and rigid-type inclusions $((\sigma_*^+, \sigma_*^-) = (1, 10^3))$ exhibit distinct
 316 characteristics in the reference solutions. A recent work [31] discussed the asymptotic behavior when the
 317 coefficient in inclusions tends to positive infinity, while the case of negative infinity has not been explored in
 318 the literature. Our aim in studying this model is to investigate the effect of choosing different eigenvectors
 319 l^* .

320 The approximation of eigenspaces relies on the rapid growth of the eigenvalue λ in (7). Ideally for the
 321 Laplace operator, the eigenvalue distribution follows Weyl's law [49, 36]. However, as the contrast ratio

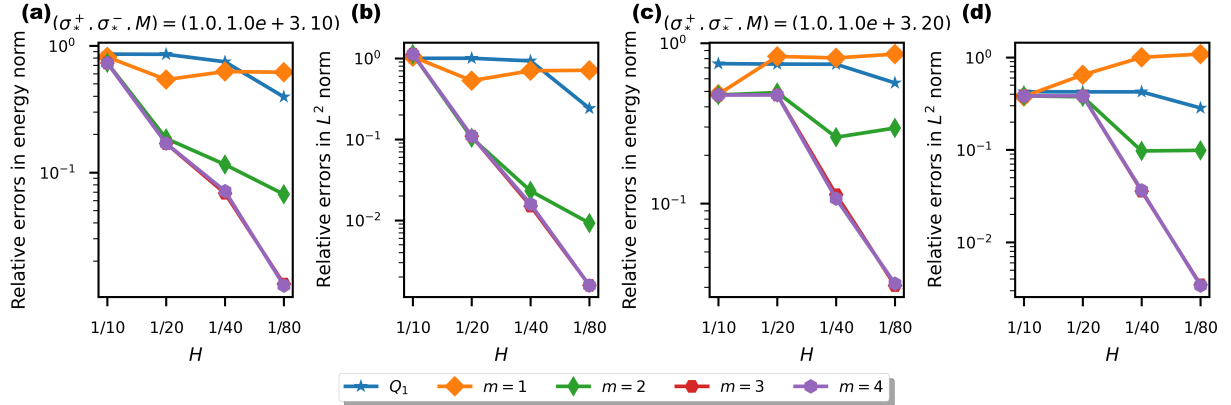


Figure 8: Numerical results for the periodic cross-shaped inclusion model with different periodic configurations. Subplots (a) and (b) show the relative errors of the proposed method for the 10×10 configuration with different numbers of oversampling layers m and the Q_1 FEM w.r.t. the coarse mesh size H , but measured in different norms. Similarly, subplots (c) and (d) correspond to the 20×20 configuration, following the same manner.

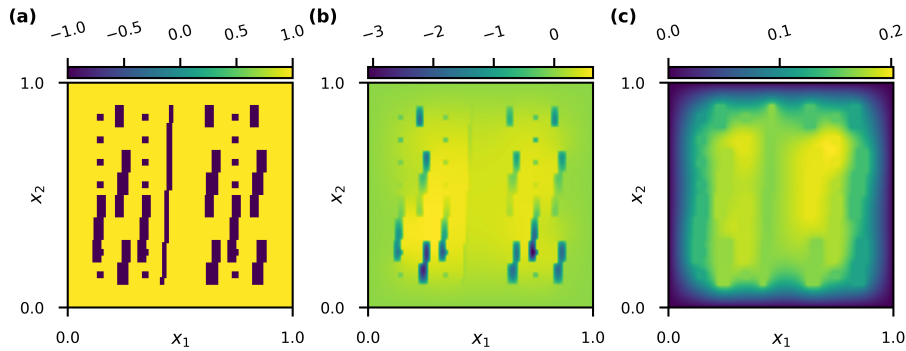


Figure 9: (a) The illustration of inclusions (dark regions) of the random inclusion model. (b) The plot of the reference solution by setting $(\sigma_*^+, \sigma_*^-) = (1, 10^{-3})$. (c) The plot of the reference solution by setting $(\sigma_*^+, \sigma_*^-) = (1, 10^3)$.

322 increases, the growth rate is expected to slow down, leading to a deterioration of the approximation quality.
 323 Surprisingly, this deterioration commonly appears in several leading eigenvalues in (7) where a weighted
 324 L^2 bilinear form is utilized. To investigate this phenomenon, we examine the first four eigenvalues for
 325 $(\sigma_*^+, \sigma_*^-) = (1, 10^{-3})$ and $(\sigma_*^+, \sigma_*^-) = (1, 10^3)$. The results are presented in Tables 3 and 4, where the
 326 minimal and maximal values of eigenvalues are calculated over all coarse elements. Since the first eigenvalue
 327 λ_1 is always 0, we omit the results for λ_1 in Tables 3 and 4. ① We can observe from Table 3 that for λ_2 ,
 328 there exist small eigenvalues that are on the order of 10^{-4} . ② However, for λ_3 and λ_4 , the minimum values
 329 are significantly larger, on the order of 10^{-1} , compared to the minimum values of λ_1 . ③ Interestingly, for
 330 $H = 1/80$, the maximum and minimum values for both cases are nearly the same. We conjecture that this
 331 is because the coarse meshes are fine enough, and the microstructures on each coarse element are simple.
 332 Note that the eigenvalues by setting $(\sigma_*^+, \sigma_*^-) = (1, 10^{-3})$ and $(\sigma_*^+, \sigma_*^-) = (10^3, 1)$ are strictly identical. If
 333 some coarse elements exhibit a “symmetrization” pattern between Ω^+ and Ω^- and also contribute to the
 334 extreme values, we can expect to observe the aforementioned phenomenon.

335 We proceed by presenting the numerical errors of the proposed methods for the two settings in Fig. 10,

Table 3: For the random inclusion model with $(\sigma_*^+, \sigma_*^-) = (1, 10^{-3})$, the minimal and maximal values of the second/third/fourth eigenvalue over total coarse elements.

H	λ_2		λ_3		λ_4	
	min	max	min	max	min	max
$\frac{1}{10}$	6.939e-4	4.114e-1	3.011e-1	4.114e-1	4.121e-1	8.229e-1
$\frac{1}{20}$	7.806e-4	4.121e-1	3.066e-1	8.404e-1	4.129e-1	1.278
$\frac{1}{40}$	1.008e-3	4.146e-1	2.617e-1	7.579e-1	5.091e-1	1.291
$\frac{1}{80}$	1.051e-1	6.755e-1	3.255e-1	8.499e-1	5.228e-1	1.350

Table 4: For the random inclusion model with $(\sigma_*^+, \sigma_*^-) = (1, 10^3)$, the minimal and maximal values of the second/third/fourth eigenvalue over total all coarse elements.

H	λ_2		λ_3		λ_4	
	min	max	min	max	min	max
$\frac{1}{10}$	2.685e-1	4.335e-1	3.707e-1	1.141	6.694e-1	1.553
$\frac{1}{20}$	6.856e-2	4.121e-1	3.254e-1	8.696e-1	5.733e-1	1.264
$\frac{1}{40}$	1.365e-1	6.502e-1	3.292e-1	7.866e-1	5.763e-1	1.300
$\frac{1}{80}$	1.051e-1	6.755e-1	3.255e-1	8.499e-1	5.228e-1	1.350

336 where we fix $m = 3$ and change H and l^* . Subplots **(a)** and **(b)** in Fig. 10 correspond to $(\sigma_*^+, \sigma_*^-) = (1, 10^{-3})$,
337 and subplots **(c)** and **(d)** correspond to $(\sigma_*^+, \sigma_*^-) = (1, 10^3)$. **①** Upon initial observation, we note a decay
338 of numerical errors with increasing l^* for most cases. **②** However, this decay is less pronounced with l^*
339 from 3 to 4. Therefore, it is advisable to consider employing a larger number of eigenvectors as a stabilizing
340 strategy rather than solely relying on it for accuracy improvement. **③** To gain further clarity, the results
341 with $H = 1/10$ in Fig. 10-**(a)** and **(b)** do not show a decreasing pattern in errors with l^* . **④** However, we
342 can observe a more substantial reduction in errors as l^* increases for $H = 1/10$ in Fig. 10-**(c)** and **(d)**. This
343 implies that the proposed method is more sensitive to the number of eigenvectors when the coarse mesh is
344 not fine enough, while $l^* = 3$ is a recommended choice.

345 5 Analysis

346 In this section, we present a rigorous analysis of the proposed method. We begin by examining the global
347 version of the method, where we set K_i^m in (8) as Ω . While the global version can yield optimal error
348 estimates, it is impractical due to its high computational cost. Next, we delve into the local version of the
349 method and replicate several estimates found in the original CEM-GMsFEM. Given the sign-changing setting
350 and the T-coercivity framework, certain assumptions seem to be unavoidable due to technical difficulties,
351 and we will explain the rationale behind these assumptions.

352 5.1 Global version

353 Let Υ be $\sigma_{\min}^+/\sigma_{\max}^-$. As shown in (4), the well-posedness of the model problem can be ensured if Υ is
354 sufficiently large. By replacing σ with $-\sigma$, the case that $\sigma_{\min}^-/\sigma_{\max}^+$ is sufficiently large can also be reduced
355 to the case that we mainly focus on. Essentially, we require that the solution to the model problem be
356 at least unique to discuss the convergence of the numerical method. Therefore, we introduce the following
357 stronger assumption, although it is rarely touched in the subsequent analysis.

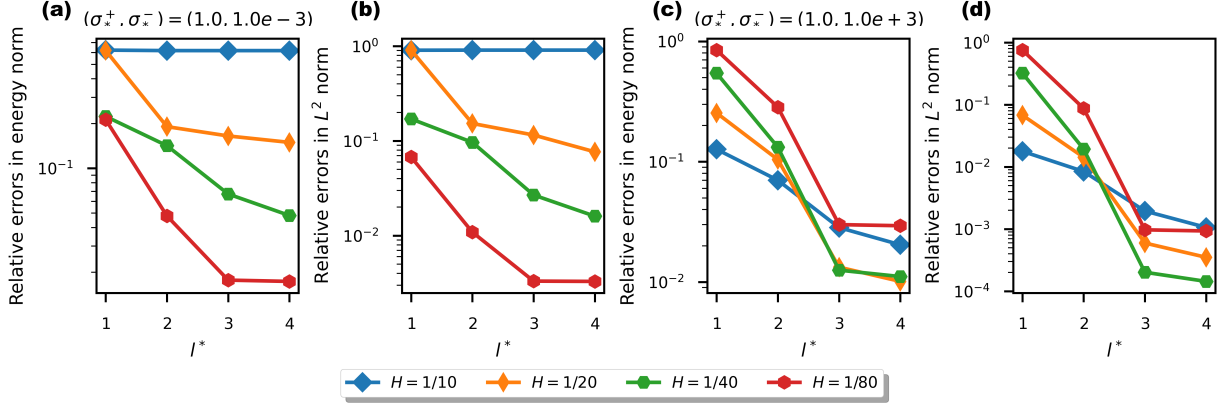


Figure 10: Numerical results for the random inclusion model. Subplots (a) and (b) show the relative errors of the proposed method for the setting $(\sigma_*^+, \sigma_*^-) = (1, 10^{-3})$ with different coarse mesh size H w.r.t. the number of eigenvectors l^* , but measured in different norms. Similarly, subplots (c) and (d) correspond to $(\sigma_*^+, \sigma_*^-) = (1, 10^3)$, following the same manner.

358 **Assumption I.** The model problem (1) is well-posed in the sense that there exists a positive constant C_{wp} ,
 359 independent of f , such that

$$\|u\|_{\bar{a}} \leq C_{\text{wp}} \|f\|_{L^2(\Omega)}.$$

360 We define the global operator $\mathcal{G}_i^\infty : L^2(\Omega) \rightarrow V = H_0^1(\Omega)$ corresponding to the coarse element K_i via the
 361 following variational problem:

$$\text{find } \mathcal{G}_i^\infty \psi \in V \text{ s.t. } \forall w \in V, a(\mathcal{G}_i^\infty \psi, w) + s(\mathcal{P}_H \mathcal{G}_i^\infty \psi, \mathcal{P}_H w) = s(\mathcal{P}_H \psi, \mathcal{P}_H w)_{K_i}. \quad (10)$$

362 We reiterate several important facts here: the bilinear form is $s(\cdot, \cdot)$ defined on $L^2(\Omega) \times L^2(\Omega)$ as

$$s(v, w) = \int_{\Omega} \mu v w \, dx = \mu_{\text{msh}} \sum_{i=1}^{N_{\text{elem}}} \text{diam}(K_i)^{-2} \int_{K_i} \sigma v w \, dx,$$

363 which is not coercive due to the sign-changing property of μ ; the operator $\mathcal{P}_H : L^2(\Omega) \rightarrow L^2(\Omega)$ is an
 364 orthogonal projection with $\text{im } \mathcal{P}_H = V^{\text{aux}}$, regarding the weighted L^2 inner-product $\int_{\Omega} |\mu| v w \, dx$ but not the
 365 bilinear form $s(\cdot, \cdot)$ or the standard L^2 inner-product. Taking a summation of \mathcal{G}_i^∞ gives $\mathcal{G}^\infty := \sum_{i=1}^{N_{\text{elem}}} \mathcal{G}_i^\infty$,
 366 which is interpreted as the global operator corresponding to the whole domain. Certainly, we are required
 367 to check the well-posedness of (10), which essentially involves proving the inf-sup stability of the bilinear
 368 form:

$$a(v, w) + s(\mathcal{P}_H v, \mathcal{P}_H w), \quad \forall (v, w) \in V \times V.$$

369 Following the T-coercivity approach, we may examine the coercivity of

$$a(v, \mathcal{T}v) + s(\mathcal{P}_H v, \mathcal{P}_H \mathcal{T}v).$$

370 We encounter a challenge here as the properties of $\mathcal{P}_H \mathcal{T}$ are not clear, as \mathcal{P}_H is associated with the coarse
 371 mesh \mathcal{K}_H , while \mathcal{T} depends on the geometric information of Ω^\pm and could be rather complicated. To address
 372 this issue, we introduce the following assumption.

373 **Assumption II.** All coarse elements in \mathcal{K}_H can be categorized into two groups \mathcal{K}_H^+ or \mathcal{K}_H^- , where

$$\mathcal{K}_H^\pm := \{K \mid K \in \mathcal{K}_H \text{ and } K \subset \Omega^\pm\}.$$

374 This assumption is rather stringent, as it implies that the coarse mesh \mathcal{K}_H is capable of resolving Ω^\pm .
 375 On the other hand, such an assumption greatly facilitates the analysis once the orthogonal projection \mathcal{T}_H
 376 comes into play. Another reason for this assumption is shown in proving Lemma 5.5. We emphasize that
 377 several numerical experiments in Section 4 indeed do not align with the assumption, yet the obtained results
 378 remain promising. To facilitate the analysis, we also require the following assumption. We mention that
 379 similar assumptions are also raised in constructing “flip” operators that satisfy weak T-coercivity (ref. [9]
 380 section 4).

381 **Assumption III.** There exists a positive constant $\|\mathcal{R}\|_0$, such that for any $v \in V(\Omega^+)$,

$$\|\mathcal{R}v\|_{0,\Omega^-} \leq \|\mathcal{R}\|_0 \|v\|_{0,\Omega^+},$$

382 where \mathcal{R} is the operator in (2).

383 We introduce this assumption to establish an estimate for $s(\mathcal{P}_H v, \mathcal{P}_H \mathcal{T} v)$. This assumption is valid when
 384 the operator \mathcal{R} acts as a change of variables. Denoting λ_i^* as the $(l^* + 1)$ -th eigenvalue of the generalized
 385 eigenvalue problem in (7), the following lemma will be frequently utilized in the analysis, while its proof is
 386 simply a straightforward application of the properties of eigenspace expansions.

387 **Lemma 5.1.** *On each coarse element K_i and for any $v \in H^1(K_i)$, the following estimates hold:*

$$\|v - \mathcal{P}_H v\|_{\bar{s}, K_i} \leq \frac{1}{\sqrt{\lambda_i^*}} \|v\|_{\bar{a}, K_i}, \quad (11)$$

$$\|v\|_{\bar{s}, K_i}^2 \leq \|\mathcal{P}_H v\|_{\bar{s}, K_i}^2 + \frac{1}{\lambda_i^*} \|v\|_{\bar{a}, K_i}^2. \quad (12)$$

388 We also introduce a notation that $\epsilon := 1/(\max_i \lambda_i^*)$. The following estimates pave the way for proving
 389 the well-posedness of (10).

390 **Lemma 5.2.** *It holds that for any $v \in V$,*

$$a(v, \mathcal{T} v) \geq \left(1 - \|\mathcal{R}\|_1 / \sqrt{\Upsilon}\right) \|v\|_{\bar{a}}^2, \quad (13)$$

$$s(\mathcal{P}_H v, \mathcal{P}_H \mathcal{T} v) \geq \left(1 - \|\mathcal{R}\|_0 / \sqrt{\Upsilon}\right) \|\mathcal{P}_H v\|_{\bar{s}}^2 - \epsilon \|\mathcal{R}\|_0 / \sqrt{\Upsilon} \|v\|_{\bar{a}}^2, \quad (14)$$

$$\|\mathcal{T} v\|_{\bar{a}} \leq \max \left\{ \left(1 + 8\|\mathcal{R}\|_1^2 / \Upsilon\right)^{1/2}, \sqrt{2} \right\} \|v\|_{\bar{a}}, \quad (15)$$

$$\|\mathcal{T} v\|_{\bar{s}} \leq \max \left\{ \left(1 + 8\|\mathcal{R}\|_0^2 / \Upsilon\right)^{1/2}, \sqrt{2} \right\} \|v\|_{\bar{s}}. \quad (16)$$

391 *Proof.* The proof of (13) has already been given in (4). We hence first prove (14). For any $v \in V$ with v_1
 392 and v_2 defined as (3), we have

$$\begin{aligned} s(\mathcal{P}_H v, \mathcal{P}_H \mathcal{T} v) &= \int_{\Omega^+} |\mu| |\mathcal{P}_H v_1|^2 dx + \int_{\Omega^-} |\mu| |\mathcal{P}_H v_2|^2 dx - 2 \int_{\Omega^-} |\mu| \mathcal{P}_H v_2 \mathcal{P}_H \mathcal{R} v_1 dx \\ &\stackrel{(\forall \eta > 0)}{\geq} \int_{\Omega^+} |\mu| |\mathcal{P}_H v_1|^2 dx - \frac{1}{\eta} \int_{\Omega^-} |\mu| |\mathcal{P}_H \mathcal{R} v_1|^2 dx + (1 - \eta) \int_{\Omega^-} |\mu| |\mathcal{P}_H v_2|^2 dx, \end{aligned}$$

393 where we implicitly utilize assumption II. We turn to the estimate of $\int_{\Omega^-} |\mu| |\mathcal{P}_H \mathcal{R} v_1|^2 dx$ as follows:

$$\begin{aligned} \int_{\Omega^-} |\mu| |\mathcal{P}_H \mathcal{R} v_1|^2 dx &\leq \int_{\Omega^-} |\mu| |\mathcal{R} v_1|^2 dx \quad (\mathcal{P}_H \text{ is an orthogonal projection}) \\ &\leq \mu_{\max}^- \int_{\Omega^-} |\mathcal{R} v_1|^2 dx \leq \mu_{\max}^- \|\mathcal{R}\|_0^2 \int_{\Omega^+} |v_1|^2 dx \quad (\text{by assumption III}) \end{aligned}$$

$$\begin{aligned}
&\leq \|\mathcal{R}\|_0^2 \frac{\mu_{\max}^-}{\mu_{\min}^+} \int_{\Omega^+} |\mu| |v_1|^2 dx \\
&\leq \|\mathcal{R}\|_0^2 \frac{\sigma_{\max}^-}{\sigma_{\min}^+} \left(\int_{\Omega^+} |\mu| |\mathcal{P}_H v_1|^2 dx + \epsilon \int_{\Omega^+} |\sigma| |\nabla v_1|^2 dx \right). \quad (\text{by (12)})
\end{aligned}$$

394 By choosing $\eta = \|\mathcal{R}\|_0/\sqrt{\Upsilon}$, we derive that

$$s(\mathcal{P}_H v, \mathcal{P}_H \mathcal{T} v) \geq (1 - \|\mathcal{R}\|_0/\sqrt{\Upsilon}) \|\mathcal{P}_H v\|_{\bar{s}}^2 - \epsilon \|\mathcal{R}\|_0/\sqrt{\Upsilon} \|v\|_{\bar{a}}^2,$$

395 which finishes the proof of (14).

396 The proofs of (15) and (16) follow a similar procedure, and we only provide the proof for the former:

$$\begin{aligned}
\|\mathcal{T} v\|_{\bar{a}}^2 &= \int_{\Omega^+} |\sigma| |\nabla v_1|^2 dx + \int_{\Omega^-} |\sigma| |-\nabla v_2 + 2\nabla \mathcal{R} v_1|^2 dx \\
&\leq \int_{\Omega^+} |\sigma| |\nabla v_1|^2 dx + 2 \int_{\Omega^-} |\sigma| |\nabla v_2|^2 dx + 8 \int_{\Omega^-} |\sigma| |\nabla \mathcal{R} v_1|^2 dx \quad (\text{by the basic inequality}) \\
&\leq \int_{\Omega^+} |\sigma| |\nabla v_1|^2 dx + 2 \int_{\Omega^-} |\sigma| |\nabla v_2|^2 dx + 8\sigma_{\max}^- \|\mathcal{R}\|_1^2 \int_{\Omega^+} |\nabla v_1|^2 dx \\
&\leq \left(1 + 8\|\mathcal{R}\|_1^2 \frac{\sigma_{\max}^-}{\sigma_{\min}^+}\right) \int_{\Omega^+} |\sigma| |\nabla v_1|^2 + 2 \int_{\Omega^-} |\sigma| |\nabla v_2|^2 dx \\
&\leq \max\left\{1 + 8\|\mathcal{R}\|_1^2/\Upsilon, 2\right\} \|v\|_{\bar{a}}^2.
\end{aligned}$$

397 □

398 *Remark.* According to the proof, we can refine $\|v\|_{\bar{a}}^2$ in (14) to $\|v\|_{\bar{a}, \Omega^+}^2$. However, we choose to retain the
399 original form for the sake of simplicity.

400 Now the well-posedness of (10) can be established by combining (13) and (14).

401 **Proposition 5.3.** *There exist Υ' and ϵ' such that for any $\Upsilon \geq \Upsilon'$ and $\epsilon \leq \epsilon'$, the operator \mathcal{G}_i^∞ in (10) is
402 well-posed.*

403 Recalling that $\text{im } \mathcal{P}_H = V^{\text{aux}}$, The following lemma in some sense offers an interpolation operator that
404 maps from $L^2(\Omega)$ to V , such that the projections onto V^{aux} by \mathcal{P}_H are preserved.

405 **Lemma 5.4** (ref. [22]). *There exists a bounded map $\mathcal{Q}_H: L^2(\Omega) \rightarrow V$ and a positive constant C_{inv} such
406 that for all $v \in L^2(\Omega)$, it holds that $\mathcal{P}_H \mathcal{Q}_H v = \mathcal{P}_H v$ and $\|\mathcal{Q}_H v\|_{\bar{a}} \leq C_{\text{inv}} \|\mathcal{P}_H v\|_{\bar{s}}$. Moreover, for each coarse
407 element K_i , $\mathcal{Q}_H v|_{K_i}$ depends only on the data of v in K_i and vanishes on ∂K_i .*

408 We introduce two function spaces as $W := \ker \mathcal{P}_H \cap V$ and $V_H^\infty := \text{im } \mathcal{G}^\infty \subset V$. According to Lemma 5.1,
409 it is clear that

$$\|w\|_{\bar{s}} \leq \sqrt{\epsilon} \|w\|_{\bar{a}}, \quad \forall w \in W. \quad (17)$$

410 The following lemma reveals a relationship of ‘‘orthogonality’’ between W and V_H^∞ concerning the bilinear
411 form $a(\cdot, \cdot)$. However, we must be cautious in using the term ‘‘orthogonal’’ since $a(\cdot, \cdot)$ cannot define an inner
412 product on V .

413 **Lemma 5.5.** *For any $v \in V_H^\infty$ and $w \in W$, it holds that $a(v, w) = 0$. If $w \in V$ and $a(v, w) = 0$ holds for
414 any $v \in V_H^\infty$, then $w \in W$.*

415 *Proof.* The first argument that $a(v, w) = 0$ for any $v \in V_H^\infty$ and $w \in W$ is a direct result of (10). We
416 then prove the second argument step by step. For simplicity, the notation $\sum_{i,j}$ is used to represent the
417 summation over $i = 1, \dots, N_{\text{elem}}$ and $j = 1, \dots, l^*$.

418 **Step1** We first state that the set $\{\mathcal{G}_i^\infty \psi_{i,j} \mid 1 \leq i \leq N_{\text{elem}}, 1 \leq j \leq l^*\}$ is linearly independent, where each
 419 $\psi_{i,j}$ is an eigenfunction by solving (7). Suppose there exists coefficients $\alpha_{i,j}$ such that $\sum_{i,j} \alpha_{i,j} \mathcal{G}_i^\infty \psi_{i,j} = 0$.
 420 Then for any $w \in V$, applying (10), we have

$$0 = a\left(\sum_{i,j} \alpha_{i,j} \mathcal{G}_i^\infty \psi_{i,j}, z\right) + s(\mathcal{P}_H \sum_{i,j} \alpha_{i,j} \mathcal{G}_i^\infty \psi_{i,j}, \mathcal{P}_H z) = s\left(\sum_{i,j} \alpha_{i,j} \psi_{i,j}, \mathcal{P}_H z\right), \quad \forall z \in V.$$

421 Recalling the property of \mathcal{Q}_H from Lemma 5.4, we find $s(\sum_{i,j} \alpha_{i,j} \psi_{i,j}, z') = 0$ for any $z' \in V^{\text{aux}}$. By
 422 assumption II, on each K_i , we have $\mu = |\mu|$ or $-|\mu|$, which yields that

$$\int_{K_i} |\mu| \sum_j \alpha_{i,j} \psi_{i,j} z'' \, dx = \int_{K_i} \mu \sum_j \alpha_{i,j} \psi_{i,j} z'' \, dx = 0, \quad \forall z'' \in V_i^{\text{aux}}.$$

423 Therefore, we derive that $\alpha_{i,j} = 0$ for any i and j , and thus prove the statement.

424 **Step2** We then state that $\{\psi_{i,j} - \mathcal{P}_H \mathcal{G}_i^\infty \psi_{i,j} \mid 1 \leq i \leq N_{\text{elem}}, 1 \leq j \leq l^*\}$ is a linearly independent set.
 425 Once again, from (10), if $\sum_{i,j} \alpha_{i,j} (\psi_{i,j} - \mathcal{P}_H \mathcal{G}_i^\infty \psi_{i,j}) = 0$, we have

$$a\left(\sum_{i,j} \alpha_{i,j} \mathcal{G}_i^\infty \psi_{i,j}, z\right) = s\left(\sum_{i,j} \alpha_{i,j} (\psi_{i,j} - \mathcal{P}_H \mathcal{G}_i^\infty \psi_{i,j}), \mathcal{P}_H z\right) = 0, \quad \forall z \in V.$$

426 According to assumption I, we conclude that $\sum_{i,j} \alpha_{i,j} \mathcal{G}_i^\infty \psi_{i,j} = 0$, which yields $\alpha_{i,j} = 0$ for any i and j from
 427 the previous step.

428 **Step3** Now we return to the original argument. If $\mathcal{P}_H w \neq 0$, we assert that there exists a set of $\{\alpha_{i,j}\}$
 429 such that

$$s\left(\sum_{i,j} \alpha_{i,j} (\psi_{i,j} - \mathcal{P}_H \mathcal{G}_i^\infty \psi_{i,j}), \mathcal{P}_H w\right) = 1.$$

430 Otherwise, for all choices of $\{\alpha_{i,j}\}$, we must have $s(\sum_{i,j} \alpha_{i,j} (\psi_{i,j} - \mathcal{P}_H \mathcal{G}_i^\infty \psi_{i,j}), \mathcal{P}_H w) = 0$. Taking $\{\psi_{i,j}\}$
 431 as bases for V^{aux} , we can check that the matrix representation of $s(\cdot, \cdot)$ on V^{aux} is diagonal, which is also
 432 nonsingular thanks to assumption II. From the previous step, we have established that $\{\psi_{i,j} - \mathcal{P}_H \mathcal{G}_i^\infty \psi_{i,j}\}$
 433 spans the finite-dimensional space V^{aux} . However, the relation $s(\sum_{i,j} \alpha_{i,j} (\psi_{i,j} - \mathcal{P}_H \mathcal{G}_i^\infty \psi_{i,j}), \mathcal{P}_H v) = 0$ for
 434 any $\{\alpha_{i,j}\}$ implies that $\mathcal{P}_H v = 0$, which is a contradiction. Therefore, We have

$$1 = a\left(\sum_{i,j} \alpha_{i,j} \mathcal{G}_i^\infty \psi_{i,j}, w\right) = s\left(\sum_{i,j} \alpha_{i,j} (\psi_{i,j} - \mathcal{P}_H \mathcal{G}_i^\infty \psi_{i,j}), \mathcal{P}_H w\right),$$

435 which contradicts to $a(v, w) = 0$ for any $v \in V_H^\infty$. □

436 The global solution is defined by solving the following variational form:

$$\text{find } u_H^\infty \in V_H^\infty, \quad \text{s.t. } \forall w_H \in V_H^\infty, \quad a(u_H, w_H) = \int_{\Omega} f w_H \, dx. \quad (18)$$

437 To guarantee the well-posedness of (18), we need to examine the inf-sup stability of V_H^∞ , i.e., proving the
 438 existence of a uniform lower bound of

$$\inf_{v_H \in V_H^\infty} \sup_{w_H \in V_H^\infty} \frac{a(v_H, w_H)}{\|v_H\|_{\bar{a}} \|w_H\|_{\bar{a}}}.$$

439 The Fortin trick [8] suggests that it suffices to check that

$$\inf_{v \in W} \sup_{w \in W} \frac{a(v, w)}{\|v\|_{\bar{a}} \|w\|_{\bar{a}}} \geq \Lambda_W$$

440 holds. The technique that we utilize here is introducing a modified \mathcal{T} operator, denoted as \mathcal{T}_H , defined on
 441 W as follows:

$$\begin{aligned}\mathcal{T}_H v &= \mathcal{T}v - \mathcal{Q}_H \mathcal{P}_H \mathcal{T}v = \begin{cases} v_1 - \mathcal{Q}_H \mathcal{P}_H v_1, & \text{in } \Omega^+, \\ -v_2 + 2\mathcal{R}v_1 + \mathcal{Q}_H \mathcal{P}_H v_2 - 2\mathcal{Q}_H \mathcal{P}_H \mathcal{R}v_1, & \text{in } \Omega^-. \end{cases} \\ &= \begin{cases} v_1, & \text{in } \Omega^+, \\ -v_2 + 2\mathcal{R}v_1 - 2\mathcal{Q}_H \mathcal{P}_H \mathcal{R}v_1, & \text{in } \Omega^-. \end{cases}\end{aligned}$$

442 Here, the terms $\mathcal{Q}_H \mathcal{P}_H v_1$ and $\mathcal{Q}_H \mathcal{P}_H v_2$ vanish because $\mathcal{P}_H v = 0$ on each coarse element, and meanwhile
 443 $\mathcal{Q}_H \mathcal{P}_H v|_{K_i}$ is only dependent on $\mathcal{P}_H v|_{K_i}$ (see Lemma 5.4). Recalling the boundness of \mathcal{Q}_H from Lemma 5.4,
 444 we can see that \mathcal{T}_H is a bounded operator that maps from W to W . Then, we reduce the proof of the inf-sup
 445 stability to establish the coercivity of $a(v, \mathcal{T}_H v)$ for all $v \in W$. The details of this proof are essentially a
 446 replication of (4). Taking $\mathcal{R}'_H := \mathcal{R} - \mathcal{Q}_H \mathcal{P}_H \mathcal{R}$, we can obtain

$$a(v, \mathcal{T}_H v) \geq \left(1 - \|\mathcal{R}'_H\|_1 / \sqrt{\Upsilon}\right) \|v\|_{\bar{a}}^2,$$

447 where $\|\mathcal{R}'_H\|_1$ is defined similarly as $\|\mathcal{R}\|_1$ in (5). Presenting a complete estimate of $\|\mathcal{R}'_H\|_1$ is complicated.
 448 However, we note that if $\sigma_{\max}^- / \sigma_{\min}^-$ can be bounded, which is a common scenario in practice, then an apriori
 449 estimate of $\|\mathcal{R}'_H\|_1$ could be achieved.

450 After proving the inf-sup stability of V_H^∞ , an error estimate of the global solution can be derived. Taking
 451 $e = u - u_H^\infty$, we have $a(e, w_H) = 0$ for all $w_H \in V_H^\infty$. Therefore, Lemma 5.5 gives that $e \in W$. Then, we can
 452 show that

$$\begin{aligned}\|e\|_{\bar{a}} &\leq \frac{1}{\Lambda_W} \sup_{w \in W} \frac{a(e, w)}{\|w\|_{\bar{a}}} = \frac{1}{\Lambda_W} \sup_{w \in W} \frac{a(u, w)}{\|w\|_{\bar{a}}} = \frac{1}{\Lambda_W} \sup_{w \in W} \frac{\int_{\Omega} f w \, dx}{\|w\|_{\bar{a}}} \\ &\leq \frac{1}{\Lambda_W} \sup_{w \in W} \frac{\|f\|_{\bar{s}^*} \|w\|_{\bar{s}}}{\|w\|_{\bar{a}}} \quad (\text{by the Cauchy-Schwarz inequality}) \\ &\leq \frac{\sqrt{\epsilon}}{\Lambda_W} \|f\|_{\bar{s}^*}, \quad (\text{by the estimate (17)})\end{aligned}$$

453 where

$$\|f\|_{\bar{s}^*} := \left(\int_{\Omega} |\mu|^{-1} |f|^2 \, dx \right)^{1/2} \approx H \max\left\{1/(\sigma_{\min}^+)^{1/2}, 1/(\sigma_{\min}^-)^{1/2}\right\} \|f\|_{L^2(\Omega)}.$$

454 We summarize all results presented above in the following theorem.

455 **Theorem 5.6.** *There exist positive constants Υ' and ϵ' such that for any $\Upsilon \geq \Upsilon'$ and $\epsilon \leq \epsilon'$, the solution*
 456 *u_H^∞ in (18) exists and is unique. Moreover, the following estimate*

$$\|u - u_H^\infty\|_{\bar{a}} \leq \frac{\sqrt{\epsilon}}{\Lambda_W} \|f\|_{\bar{s}^*},$$

457 *holds.*

458 5.2 Local version

459 When turning to the practical method outlined in Section 3, the initial concern pertains to the existence
 460 of multiscale bases as defined in (8). However, a challenge arises due to the incompatibility between the
 461 definition of an oversampling region K_i^m and T-coercivity. Consequently, we cannot obtain a result similar
 462 to Proposition 5.3 for the local version. Nevertheless, it is worth noting that the numerical experiments
 463 presented in Section 4 did not encounter any lack of well-posedness at the discrete level. To proceed with
 464 the analysis, we introduce the following assumption.

465 **Assumption IV.** There exists a list of subdomains $\{\hat{K}_i^1, \hat{K}_i^2, \dots\}$ that fulfills the requirements listed below:

- 466 1. Each \hat{K}_i^m consists of coarse elements from \mathcal{K}_H .
- 467 2. There exists an inclusion relation $K_i \subset \hat{K}_i^1 \subset \hat{K}_i^2 \subset \dots \subset$, such that $\text{dist}(\partial\hat{K}_i^m, \partial\hat{K}_i^{m+1}) \geq C_{\text{msh}}H$.
- 468 3. For any $v \in V$ with $\text{supp } v \subset \overline{\hat{K}_i^m}$ or $\overline{\Omega \setminus \hat{K}_i^m}$, it holds that $\text{supp } \mathcal{T}v \subset \overline{\hat{K}_i^m}$ or $\overline{\Omega \setminus \hat{K}_i^m}$ accordingly.

469 A construction of such subdomains can be found in [19], where a method called symmetrization is
 470 described. However, implementing such a construction (i.e., replacing K_i^m in (8)) is highly impractical
 471 due to the lack of clarity associated with the definition of the operator \mathcal{T} . The fulfillment of the first and
 472 second requirements allows us to employ conventional cut-off functions. Furthermore, the third requirement
 473 suggests that, to some extent, T-coercivity can be preserved within these subdomains. In a sense, this
 474 complies with the definition of geometrically-based T-coercivity operators [44, 10, 20]. To distinguish the
 475 local problems utilized for analysis from the practical method in Section 3, we add a hat notation “ $\hat{\cdot}$ ” (e.g.,
 476 $\hat{V}_i^m = H_0^1(\hat{K}_i^m)$) to indicate that this term is associated with \hat{K}_i^m rather than K_i^m . Applying the same
 477 technique as in establishing Proposition 5.3, we can obtain the well-posedness of the modified local operator
 478 $\hat{\mathcal{G}}_i^m$ associated with the coarse element K_i :

$$\begin{aligned} & \text{find } \hat{\mathcal{G}}_i^m \psi \in \hat{V}_i^m \text{ s.t. } \forall w \in \hat{V}_i^m, \\ & a(\hat{\mathcal{G}}_i^m \psi, w)_{\hat{K}_i^m} + s(\mathcal{P}_H \hat{\mathcal{G}}_i^m \psi, \mathcal{P}_H w)_{\hat{K}_i^m} = s(\mathcal{P}_H \psi, \mathcal{P}_H w)_{K_i}. \end{aligned} \quad (19)$$

479 To facilitate the analysis, we introduce the following notations for constants that are from (13) to (16):

$$\begin{aligned} C_0 &:= \max \left\{ \left(1 + 8\|\mathcal{R}\|_0^2/\Upsilon\right)^{1/2}, \sqrt{2} \right\}, \\ C_1 &:= \max \left\{ \left(1 + 8\|\mathcal{R}\|_1^2/\Upsilon\right)^{1/2}, \sqrt{2} \right\}, \\ C_2 &:= \min \left\{ 1 - \|\mathcal{R}\|_1/\sqrt{\Upsilon} - \epsilon\|\mathcal{R}\|_0/\sqrt{\Upsilon}, 1 - \|\mathcal{R}\|_0/\sqrt{\Upsilon} \right\}. \end{aligned}$$

480 We can see that C_0 , C_1 , and C_2 can all be bounded from above and below providing that Υ is sufficiently
 481 large and ϵ is sufficiently small, which has also been stated in Proposition 5.3 and Theorem 5.6. A key
 482 ingredient employed in the analysis is multiplying by a cut-off function $\hat{\chi}_i^{m-1,m} \in C^{0,1}(\Omega)$ defined as

$$\hat{\chi}_i^{m-1,m} = \begin{cases} 1, & \text{in } \hat{K}_i^{m-1}, \\ 0, & \text{in } \Omega \setminus \hat{K}_i^m, \end{cases}$$

483 with $0 \leq \hat{\chi}_i^{m-1,m} \leq 1$ in $\hat{K}_i^m \setminus \hat{K}_i^{m-1}$. By carefully controlling the parameter μ_{msh} , we can derive the
 484 inequality

$$\left| \nabla \hat{\chi}_i^{m-1,m} \right|^2 |\sigma| \leq |\mu|$$

485 on Ω for any i and m . Taking $\hat{\mathcal{G}}^m := \sum_{i=1}^{N_{\text{elem}}} \hat{\mathcal{G}}_i^m$, We shall prove an estimate of $\mathcal{G}^\infty - \hat{\mathcal{G}}^m$ in Theorem 5.9 below
 486 as the first main result in this subsection, while Lemmas 5.7 and 5.8 as presented in [50] are reformulated
 487 next to reflect the current sign-changing context.

488 **Lemma 5.7.** *There exists a positive constant θ with $\theta < 1$ that depends on C_0 , C_1 , and C_2 such that for*
 489 *any $m \geq 1$, $1 \leq i \leq N_{\text{elem}}$ and $\psi \in L^2(\Omega)$,*

$$\|\mathcal{G}_i^\infty \psi\|_{\hat{a}, \Omega \setminus \hat{K}_i^m}^2 + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\hat{s}, \Omega \setminus \hat{K}_i^m}^2 \leq \theta^m \left(\|\mathcal{G}_i^\infty \psi\|_{\hat{a}}^2 + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\hat{s}}^2 \right).$$

490 **Lemma 5.8.** *It holds that for any $m \geq 1$, $1 \leq i \leq N_{\text{elem}}$ and $\psi \in L^2(\Omega)$,*

$$\|(\mathcal{G}_i^\infty - \hat{\mathcal{G}}_i^m)\psi\|_{\bar{a}}^2 + \|\mathcal{P}_H(\mathcal{G}_i^\infty - \hat{\mathcal{G}}_i^m)\psi\|_{\bar{s}}^2 \leq C\theta^{m-1} \left(\|\mathcal{G}_i^\infty\psi\|_{\bar{a}}^2 + \|\mathcal{P}_H\mathcal{G}_i^\infty\psi\|_{\bar{s}}^2 \right),$$

491 *where θ here is identical to the one in Lemma 5.7, and C is a positive constant that depends on C_0 , C_1 , and*
 492 *C_2 .*

493 **Theorem 5.9.** *There exist positive constants Υ' and ϵ' such that for any $\Upsilon \geq \Upsilon'$ and $\epsilon \leq \epsilon'$, it holds that*
 494 *for any $\psi \in L^2(\Omega)$,*

$$\|(\mathcal{G}^\infty - \hat{\mathcal{G}}^m)\psi\|_{\bar{a}}^2 + \|\mathcal{P}_H(\mathcal{G}^\infty - \hat{\mathcal{G}}^m)\psi\|_{\bar{s}}^2 \leq C(m+1)^d\theta^{m-1}\|\mathcal{P}_H\psi\|_{\bar{s}}^2,$$

495 *where θ and C depend on C_0 , C_1 , and C_2 with $\theta < 1$.*

496 In the proof of Theorem 5.9, we require an assumption that specifies the growth of the size of \hat{K}_i^m with
 497 m .

498 **Assumption V.** The number of coarse elements within \hat{K}_i^m satisfies a relation:

$$\#\{K \in \mathcal{K}_H \mid K \subset \hat{K}_i^m\} \leq C_{\text{ol}}m^d,$$

499 where C_{ol} depends solely on the mesh quality.

500 We first prove Lemma 5.7.

501 *Proof.* Taking $z_i := (1 - \hat{\chi}_i^{m-1,m})\mathcal{G}_i^\infty\psi$, we can see that z_i is supported in $\Omega \setminus \hat{K}_i^{m-1}$ and hence $(\text{supp } \mathcal{T}z_i) \cap$
 502 $K_i = \emptyset$ according to assumption IV.3. Substituting $\mathcal{T}z_i$ for w in the variational form (10), we have

$$a(\mathcal{G}_i^\infty\psi, \mathcal{T}z_i) + s(\mathcal{P}_H\mathcal{G}_i^\infty\psi, \mathcal{P}_H\mathcal{T}z_i) = 0.$$

503 As a result, we can formulate a decomposition as follows:

$$\begin{aligned} a(z_i, \mathcal{T}z_i) + s(\mathcal{P}_Hz_i, \mathcal{P}_H\mathcal{T}z_i) &= -a(\hat{\chi}_i^{m-1,m}\mathcal{G}_i^\infty\psi, \mathcal{T}z_i) - s(\mathcal{P}_H\hat{\chi}_i^{m-1,m}\mathcal{G}_i^\infty\psi, \mathcal{P}_H\mathcal{T}z_i) \\ &= - \underbrace{\int_{\Omega} \sigma \mathcal{G}_i^\infty\psi \nabla \hat{\chi}_i^{m-1,m} \cdot \nabla \mathcal{T}z_i \, dx}_{J_1} - \underbrace{\int_{\Omega} \sigma \hat{\chi}_i^{m-1,m} \nabla \mathcal{G}_i^\infty\psi \cdot \nabla \mathcal{T}z_i \, dx}_{J_2} - \underbrace{\int_{\Omega} \mu \mathcal{P}_H(\hat{\chi}_i^{m-1,m}\mathcal{G}_i^\infty\psi) \mathcal{P}_H\mathcal{T}z_i \, dx}_{J_3}. \end{aligned}$$

504 For J_1 , by the Cauchy-Schwarz inequality, it is evident that

$$\begin{aligned} |J_1| &\leq \left(\int_{\Omega} |\sigma| \left| \nabla \hat{\chi}_i^{m-1,m} \right|^2 |\mathcal{G}_i^\infty\psi|^2 \, dx \right)^{1/2} \left(\int_{\Omega} |\sigma| |\nabla \mathcal{T}z_i|^2 \, dx \right)^{1/2} \\ &\leq \|\mathcal{G}_i^\infty\psi\|_{\bar{s}, \hat{K}_i^m \setminus \hat{K}_i^{m-1}} \|\mathcal{T}z_i\|_{\bar{a}, \Omega \setminus \hat{K}_i^{m-1}} \\ &\leq C_0 \left(\|\mathcal{P}_H\mathcal{G}_i^\infty\psi\|_{\bar{s}, \hat{K}_i^m \setminus \hat{K}_i^{m-1}}^2 + \epsilon \|\mathcal{G}_i^\infty\psi\|_{\bar{a}, \hat{K}_i^m \setminus \hat{K}_i^{m-1}}^2 \right)^{1/2} \|z_i\|_{\bar{a}, \Omega \setminus \hat{K}_i^{m-1}}, \end{aligned}$$

505 where the last line follows from (15) and Lemma 5.1. For J_2 , we have

$$\begin{aligned} |J_2| &\leq \left(\int_{\hat{K}_i^m \setminus \hat{K}_i^{m-1}} |\sigma| \left| \hat{\chi}_i^{m-1,m} \right|^2 |\nabla \mathcal{G}_i^\infty\psi|^2 \, dx \right)^{1/2} \left(\int_{\Omega} |\sigma| |\nabla \mathcal{T}z_i|^2 \, dx \right)^{1/2} \\ &\leq \|\mathcal{G}_i^\infty\psi\|_{\bar{a}, \hat{K}_i^m \setminus \hat{K}_i^{m-1}} \|\mathcal{T}z_i\|_{\bar{a}, \Omega \setminus \hat{K}_i^{m-1}} \\ &\leq C_0 \|\mathcal{G}_i^\infty\psi\|_{\bar{a}, \hat{K}_i^m \setminus \hat{K}_i^{m-1}} \|z_i\|_{\bar{a}, \Omega \setminus \hat{K}_i^{m-1}}. \end{aligned}$$

506 For J_3 , we can similarly show that

$$\begin{aligned}
|J_3| &\leq \left(\int_{\hat{K}_i^m \setminus \hat{K}_i^{m-1}} |\mu| \left| \mathcal{P}_H \left(\hat{\chi}_i^{m-1, m} \mathcal{G}_i^\infty \psi \right) \right|^2 dx \right)^{1/2} \left(\int_{\Omega} |\mu| |\mathcal{P}_H \mathcal{T} z_i|^2 dx \right)^{1/2} \\
&= \|\mathcal{P}_H \left(\hat{\chi}_i^{m-1, m} \mathcal{G}_i^\infty \psi \right)\|_{\bar{s}, \hat{K}_i^m \setminus \hat{K}_i^{m-1}} \|\mathcal{P}_H \mathcal{T} z_i\|_{\bar{s}, \Omega \setminus \hat{K}_i^{m-1}} \quad (\text{By assumption IV.3}) \\
&\leq \|\hat{\chi}_i^{m-1, m} \mathcal{G}_i^\infty \psi\|_{\bar{s}, \hat{K}_i^m \setminus \hat{K}_i^{m-1}} \|\mathcal{T} z_i\|_{\bar{s}, \Omega \setminus \hat{K}_i^{m-1}} \\
&\leq C_1 \|\mathcal{G}_i^\infty \psi\|_{\bar{s}, \hat{K}_i^m \setminus \hat{K}_i^{m-1}} \|z_i\|_{\bar{s}, \Omega \setminus \hat{K}_i^{m-1}} \\
&\leq C_1 \left(\|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}, \hat{K}_i^m \setminus \hat{K}_i^{m-1}}^2 + \epsilon \|\mathcal{G}_i^\infty \psi\|_{\bar{a}, \hat{K}_i^m \setminus \hat{K}_i^{m-1}}^2 \right)^{1/2} \left(\|\mathcal{P}_H z_i\|_{\bar{s}, \Omega \setminus \hat{K}_i^{m-1}}^2 + \epsilon \|z_i\|_{\bar{a}, \Omega \setminus \hat{K}_i^{m-1}}^2 \right)^{1/2}.
\end{aligned}$$

507 Recalling (13) and (14), we can obtain that

$$a(z_i, \mathcal{T} z_i) + s(\mathcal{P}_H z_i, \mathcal{P}_H \mathcal{T} z_i) \geq C_2 \left(\|z_i\|_{\bar{a}, \Omega \setminus \hat{K}_i^{m-1}}^2 + \|\mathcal{P}_H z_i\|_{\bar{s}, \Omega \setminus \hat{K}_i^{m-1}}^2 \right)$$

508 Providing that ϵ is small enough ($\epsilon \leq 1$), we have

$$\begin{aligned}
&\left(\|\mathcal{G}_i^\infty \psi\|_{\bar{a}, \Omega \setminus \hat{K}_i^m}^2 + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}, \Omega \setminus \hat{K}_i^m}^2 \right)^{1/2} \leq \left(\|z_i\|_{\bar{a}, \Omega \setminus \hat{K}_i^{m-1}}^2 + \|\mathcal{P}_H z_i\|_{\bar{s}, \Omega \setminus \hat{K}_i^{m-1}}^2 \right)^{1/2} \\
&\leq \frac{2C_0 + C_1}{C_2} \left(\|\mathcal{G}_i^\infty \psi\|_{\bar{a}, \hat{K}_i^m \setminus \hat{K}_i^{m-1}}^2 + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}, \hat{K}_i^m \setminus \hat{K}_i^{m-1}}^2 \right)^{1/2}.
\end{aligned}$$

509 Note that

$$\begin{aligned}
&\|\mathcal{G}_i^\infty \psi\|_{\bar{a}, \Omega \setminus \hat{K}_i^{m-1}}^2 + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}, \Omega \setminus \hat{K}_i^{m-1}}^2 \\
&= \|\mathcal{G}_i^\infty \psi\|_{\bar{a}, \Omega \setminus \hat{K}_i^m}^2 + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}, \Omega \setminus \hat{K}_i^m}^2 + \|\mathcal{G}_i^\infty \psi\|_{\bar{a}, \hat{K}_i^m \setminus \hat{K}_i^{m-1}}^2 + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}, \hat{K}_i^m \setminus \hat{K}_i^{m-1}}^2.
\end{aligned}$$

510 We hence derive an iteration relation

$$\begin{aligned}
&\|\mathcal{G}_i^\infty \psi\|_{\bar{a}, \Omega \setminus \hat{K}_i^m}^2 + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}, \Omega \setminus \hat{K}_i^m}^2 \\
&\leq \left(1 + \left(\frac{C_2}{2C_0 + C_1} \right)^2 \right)^{-1} \left(\|\mathcal{G}_i^\infty \psi\|_{\bar{a}, \Omega \setminus \hat{K}_i^{m-1}}^2 + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}, \Omega \setminus \hat{K}_i^{m-1}}^2 \right),
\end{aligned}$$

511 which completes the proof. \square

512 We then prove Lemma 5.8.

513 *Proof.* We now take $z_i = (\mathcal{G}_i^\infty - \hat{\mathcal{G}}_i^m) \psi$ and introduce a decomposition

$$z_i = \underbrace{(1 - \hat{\chi}_i^{m-1, m}) \mathcal{G}_i^\infty \psi}_{z_i'} + \underbrace{(\hat{\chi}_i^{m-1, m} - 1) \hat{\mathcal{G}}_i^m \psi + \hat{\chi}_i^{m-1, m} z_i}_{z_i''}.$$

514 We can observe that $z_i'' \in \hat{V}_i^m$, which implies $\mathcal{T} z_i'' \in \hat{V}_i^m$. Therefore, it is easy to see that $a(z_i, \mathcal{T} z_i'') +$
515 $s(\mathcal{P}_H z_i, \mathcal{P}_H \mathcal{T} z_i'') = 0$ based on the definitions of \mathcal{G}_i^∞ and $\hat{\mathcal{G}}_i^m$ in (10) and (19). Now, our task is to estimate
516 $a(z_i, \mathcal{T} z_i') + s(\mathcal{P}_H z_i, \mathcal{P}_H \mathcal{T} z_i')$, and a starting step is

$$a(z_i, \mathcal{T} z_i') + s(\mathcal{P}_H z_i, \mathcal{P}_H \mathcal{T} z_i') \leq \|z_i\|_{\bar{a}} \|\mathcal{T} z_i'\|_{\bar{a}} + \|\mathcal{P}_H z_i\|_{\bar{s}} \|\mathcal{P}_H \mathcal{T} z_i'\|_{\bar{s}}.$$

517 Recalling that $\|\mathcal{T} z_i'\|_{\bar{a}} \leq C_0 \|z_i'\|_{\bar{a}}$ and using a similar technique as in the proof of Lemma 5.7, we can obtain
518 that

$$\|z_i'\|_{\bar{a}}^2 \leq 2 \int_{\Omega \setminus \hat{K}_i^{m-1}} |\sigma| \left| 1 - \hat{\chi}_i^{m-1, m} \right|^2 |\nabla \mathcal{G}_i^\infty \psi|^2 dx + 2 \int_{\hat{K}_i^m \setminus \hat{K}_i^{m-1}} |\sigma| \left| \nabla \hat{\chi}_i^{m-1, m} \right|^2 |\mathcal{G}_i^\infty \psi|^2 dx$$

$$\begin{aligned}
&\leq 2\|\mathcal{G}_i^\infty \psi\|_{\bar{a}, \Omega \setminus \hat{K}_i^{m-1}}^2 + 2 \int_{\Omega \setminus \hat{K}_i^{m-1}} |\mu| |\mathcal{G}_i^\infty \psi|^2 dx \quad (\text{by assumption IV.2}) \\
&\leq 2(1 + \epsilon) \|\mathcal{G}_i^\infty \psi\|_{\bar{a}, \Omega \setminus \hat{K}_i^{m-1}}^2 + 4 \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}, \Omega \setminus \hat{K}_i^{m-1}}^2.
\end{aligned}$$

519 Moreover, we can see that $\|\mathcal{P}_H \mathcal{T} z'_i\|_{\bar{s}} \leq \|\mathcal{T} z'_i\|_{\bar{s}} \leq C_1 \|z'_i\|_{\bar{s}}$ and

$$\begin{aligned}
\|z'_i\|_{\bar{s}}^2 &= \int_{\Omega} |\mu| \left| 1 - \hat{\chi}_i^{m-1, m} \right|^2 |\mathcal{G}_i^\infty \psi|^2 dx \leq \|\mathcal{G}_i^\infty \psi\|_{\bar{s}, \Omega \setminus \hat{K}_i^{m-1}}^2 \\
&\leq \epsilon \|\mathcal{G}_i^\infty \psi\|_{\bar{a}, \Omega \setminus \hat{K}_i^{m-1}}^2 + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}, \Omega \setminus \hat{K}_i^{m-1}}^2.
\end{aligned}$$

520 Then, we are close to the target estimate as

$$\begin{aligned}
C_2 \left(\|z_i\|_{\bar{a}}^2 + \|\mathcal{P}_H z_i\|_{\bar{s}}^2 \right) &\leq a(z_i, \mathcal{T} z_i) + s(\mathcal{P}_H z_i, \mathcal{P}_H \mathcal{T} z_i) = a(z_i, \mathcal{T} z'_i) + s(\mathcal{P}_H z_i, \mathcal{P}_H \mathcal{T} z'_i) \\
&\leq \left(\|z_i\|_{\bar{a}}^2 + \|\mathcal{P}_H z_i\|_{\bar{s}}^2 \right)^{1/2} \left(\|\mathcal{T} z'_i\|_{\bar{a}}^2 + \|\mathcal{P}_H \mathcal{T} z'_i\|_{\bar{s}}^2 \right)^{1/2}.
\end{aligned}$$

521 Suppose that ϵ is small enough, we can obtain that

$$\|z_i\|_{\bar{a}}^2 + \|\mathcal{P}_H z_i\|_{\bar{s}}^2 \leq \left(\frac{4C_0 + C_1}{C_2} \right)^2 \left(\|\mathcal{G}_i^\infty \psi\|_{\bar{a}, \Omega \setminus \hat{K}_i^{m-1}}^2 + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}, \Omega \setminus \hat{K}_i^{m-1}}^2 \right),$$

522 and we hence derive the desired result by utilizing Lemma 5.7. \square

523 We are now ready to prove Theorem 5.9.

524 *Proof.* We denote $z_j := (\mathcal{G}_j^\infty - \hat{\mathcal{G}}_j^m) \psi$ and $z := \sum_{j=1}^{N_{\text{elem}}} z_j$. We take a decomposition of z as

$$z = \underbrace{(1 - \hat{\chi}_i^{m, m+1}) z_i}_{z'} + \underbrace{\hat{\chi}_i^{m, m+1} z}_{z''},$$

525 where i is arbitrary chosen from $1, \dots, N_{\text{elem}}$. Again, we can observe that $\text{supp}(\mathcal{T} z') \cap K_i = \emptyset$ and hence
526 have

$$a(\mathcal{G}_i^\infty \psi, \mathcal{T} z') + s(\mathcal{P}_H \mathcal{G}_i^\infty \psi, \mathcal{P}_H \mathcal{T} z') = a(\hat{\mathcal{G}}_i^m \psi, \mathcal{T} z') + s(\mathcal{P}_H \hat{\mathcal{G}}_i^m \psi, \mathcal{P}_H \mathcal{T} z') = 0,$$

527 which leads to

$$a(z_i, \mathcal{T} z) + s(\mathcal{P}_H z_i, \mathcal{P}_H \mathcal{T} z) = a(z_i, \mathcal{T} z'') + s(\mathcal{P}_H z_i, \mathcal{P}_H \mathcal{T} z'') \leq \|\mathcal{T} z''\|_{\bar{a}} \|z_i\|_{\bar{a}} + \|\mathcal{P}_H \mathcal{T} z''\|_{\bar{s}} \|\mathcal{P}_H z_i\|_{\bar{s}}.$$

528 Similarly, we can estimate $\|\mathcal{T} z''\|_{\bar{a}}$ and $\|\mathcal{P}_H \mathcal{T} z''\|_{\bar{s}}$ as

$$\|\mathcal{T} z''\|_{\bar{a}} \leq C_0 \|\hat{\chi}_i^{m, m+1} z\|_{\bar{a}} \leq C_0 \left(\|z\|_{\bar{a}, \hat{K}_i^{m+1}} + \|z\|_{\bar{s}, \hat{K}_i^{m+1}} \right) \leq C_0 \left((1 + \sqrt{\epsilon}) \|z\|_{\bar{a}, \hat{K}_i^{m+1}} + \|\mathcal{P}_H z\|_{\bar{s}, \hat{K}_i^{m+1}} \right)$$

529 and

$$\|\mathcal{P}_H \mathcal{T} z''\|_{\bar{s}} \leq \|\mathcal{T} z''\|_{\bar{s}} \leq C_1 \|\hat{\chi}_i^{m, m+1} z\|_{\bar{s}} \leq C_1 \|z\|_{\bar{s}, \hat{K}_i^{m+1}} \leq C_1 \left(\sqrt{\epsilon} \|z\|_{\bar{a}, \hat{K}_i^{m+1}} + \|\mathcal{P}_H z\|_{\bar{s}, \hat{K}_i^{m+1}} \right).$$

530 To simplify the expression, we assume that $\epsilon \leq 1$ and derive that

$$a(z_i, \mathcal{T} z) + s(\mathcal{P}_H z_i, \mathcal{P}_H \mathcal{T} z) \leq c_0 \left(\|z\|_{\bar{a}, \hat{K}_i^{m+1}}^2 + \|\mathcal{P}_H z\|_{\bar{s}, \hat{K}_i^{m+1}}^2 \right)^{1/2} \left(\|z_i\|_{\bar{a}}^2 + \|\mathcal{P}_H z_i\|_{\bar{s}}^2 \right)^{1/2},$$

531 where c_0 depends on C_0 , C_1 , and C_2 . Based on assumption V, it holds that

$$\sum_{i=1}^{N_{\text{elem}}} \left(\|z\|_{\bar{a}, \hat{K}_i^{m+1}}^2 + \|\mathcal{P}_H z\|_{\bar{s}, \hat{K}_i^{m+1}}^2 \right) \leq C_{\text{ol}} (m+1)^d \left(\|z\|_{\bar{a}}^2 + \|\mathcal{P}_H z\|_{\bar{s}}^2 \right).$$

532 Therefore, applying the Cauchy–Schwarz inequality,

$$\begin{aligned}
& C_2 \left(\|z\|_{\bar{a}}^2 + \|\mathcal{P}_H z\|_{\bar{s}}^2 \right) \leq \sum_{i=1}^{N_{\text{elem}}} (a(z_i, \mathcal{T}z) + s(\mathcal{P}_H z_i, \mathcal{P}_H \mathcal{T}z)) \\
& \leq c_0 \left(\sum_{i=1}^{N_{\text{elem}}} \left(\|z\|_{\bar{a}, \hat{K}_i^{m+1}}^2 + \|\mathcal{P}_H z\|_{\bar{s}, \hat{K}_i^{m+1}}^2 \right) \right)^{1/2} \left(\sum_{i=1}^{N_{\text{elem}}} \left(\|z_i\|_{\bar{a}}^2 + \|\mathcal{P}_H z_i\|_{\bar{s}}^2 \right) \right)^{1/2} \\
& \leq c_0 \sqrt{C_{\text{ol}}(m+1)^{d/2}} \left(\|z\|_{\bar{a}}^2 + \|\mathcal{P}_H z\|_{\bar{s}}^2 \right)^{1/2} \left(\sum_{i=1}^{N_{\text{elem}}} \left(\|z_i\|_{\bar{a}}^2 + \|\mathcal{P}_H z_i\|_{\bar{s}}^2 \right) \right)^{1/2}.
\end{aligned}$$

533 It has been shown in Lemma 5.8 that, for any i ,

$$\|z_i\|_{\bar{a}}^2 + \|\mathcal{P}_H z_i\|_{\bar{s}}^2 \leq C_* \theta^{m-1} \left(\|\mathcal{G}_i^\infty \psi\|_{\bar{a}}^2 + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}}^2 \right),$$

534 and we turn to provide a bound on $\sum_{i=1}^{N_{\text{elem}}} \|\mathcal{G}_i^\infty \psi\|_{\bar{a}}^2 + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}}^2$. Note that by the definition of (19),

$$\begin{aligned}
& C_2 \left\{ \|\mathcal{G}_i^\infty \psi\|_{\bar{a}}^2 + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}}^2 \right\} \leq a(\mathcal{G}_i^\infty \psi, \mathcal{T} \mathcal{G}_i^\infty \psi) + s(\mathcal{P}_H \mathcal{G}_i^\infty \psi, \mathcal{P}_H \mathcal{T} \mathcal{G}_i^\infty \psi) \\
& = s(\mathcal{P}_H \psi, \mathcal{P}_H \mathcal{T} \mathcal{G}_i^\infty \psi)_{K_i} \leq \|\mathcal{P}_H \psi\|_{\bar{s}, K_i} \|\mathcal{P}_H \mathcal{T} \mathcal{G}_i^\infty \psi\|_{\bar{s}, K_i} \leq \|\mathcal{P}_H \psi\|_{\bar{s}, K_i} \|\mathcal{T} \mathcal{G}_i^\infty \psi\|_{\bar{s}, K_i} \\
& \leq \|\mathcal{P}_H \psi\|_{\bar{s}, K_i} \|\mathcal{T} \mathcal{G}_i^\infty \psi\|_{\bar{s}} \leq C_1 \|\mathcal{P}_H \psi\|_{\bar{s}, K_i} \|\mathcal{G}_i^\infty \psi\|_{\bar{s}} \\
& \leq C_1 \|\mathcal{P}_H \psi\|_{\bar{s}, K_i} \left(\sqrt{\epsilon} \|\mathcal{G}_i^\infty \psi\|_{\bar{a}} + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}} \right),
\end{aligned}$$

535 which gives that

$$\|\mathcal{G}_i^\infty \psi\|_{\bar{a}}^2 + \|\mathcal{P}_H \mathcal{G}_i^\infty \psi\|_{\bar{s}}^2 \leq c_1 \|\mathcal{P}_H \psi\|_{\bar{s}, K_i}^2.$$

536 where c_1 depends on C_1 and C_2 . We finally completed the proof by collecting all the estimates obtained
537 above. \square

538 We similarly denote $\hat{\mathcal{G}}^m := \sum_{i=1}^{N_{\text{elem}}} \hat{\mathcal{G}}_i^m$, where we implicitly lift the image of $\hat{\mathcal{G}}_i^m$ to V . We now define
539 the multiscale solution in the local version as the solution to the following problem: for $m \geq 1$,

$$\text{find } \hat{u}_H^m \in \hat{V}_H^m \text{ s.t. } \forall w_H \in \hat{V}_H^m, a(\hat{u}_H^m, w_H) = \int_{\Omega} f w_H \, dx, \quad (20)$$

540 where $\hat{V}_H^m := \text{im } \hat{\mathcal{G}}^m \subset V$. The remainder of this section is dedicated to proving an error estimate for the
541 multiscale solution, namely Theorem 5.10 below. Once again, we need to verify the well-posedness of (20),
542 or equivalently, demonstrate that

$$\inf_{v_H \in \hat{V}_H^m} \sup_{w_H \in \hat{V}_H^m} \frac{a(v_H, w_H)}{\|v_H\|_{\bar{a}} \|w_H\|_{\bar{a}}}$$

543 can be bounded from below by a positive constant that is independent of m and H . Thanks to Theorem 5.6,
544 the inf-sup condition holds for the global problem, while a global basis and a local basis are connected
545 through ψ . Those are the main ingredients to prove the inf-sup condition for the local problem. First, for
546 any $v_H \in \hat{V}_H^m$, we can find ψ such that $v_H = \hat{\mathcal{G}}^m \psi$. Next, We choose $v'_H = \mathcal{G}^\infty \psi \in V_H^\infty$. Recalling the
547 inf-sup stability on V_H^∞ , we can find $w'_H = \mathcal{G}^\infty \phi \in V_H^\infty$ such that $a(v'_H, w'_H) \geq \Lambda^\infty \|v'_H\|_{\bar{a}} \|w'_H\|_{\bar{a}}$. Similarly,
548 if we denote $w_H = \hat{\mathcal{G}}^m \phi$, we have

$$\begin{aligned}
a(v_H, w_H) &= a(v'_H, w'_H) + a(v'_H, w_H - w'_H) + a(v_H - v'_H, w_H) \\
&\geq \Lambda^\infty \|v'_H\|_{\bar{a}} \|w'_H\|_{\bar{a}} - \|v'_H\|_{\bar{a}} \|w_H - w'_H\|_{\bar{a}} - \|v_H - v'_H\|_{\bar{a}} \|w_H\|_{\bar{a}}.
\end{aligned}$$

549 Therefore, if we can show that

$$\|(\mathcal{G}^\infty - \hat{\mathcal{G}}^m) \psi\|_{\bar{a}} \leq \eta \|\mathcal{G}^\infty \psi\|_{\bar{a}}, \quad (21)$$

550 where η is a small constant, we can finish the proof. The reason is that we now have $1/2\|v_H\|_{\bar{a}} \leq \|v'_H\|_{\bar{a}} \leq$
 551 $3/2\|v_H\|_{\bar{a}}$ and $1/2\|w_H\|_{\bar{a}} \leq \|w'_H\|_{\bar{a}} \leq 3/2\|w_H\|_{\bar{a}}$ by the smallness of η . Consequently, we can deduce that

$$a(v_H, w_H) \geq (\Lambda^\infty/4 - 9\eta/4 - 3\eta/2)\|v_H\|_{\bar{a}}\|w_H\|_{\bar{a}}.$$

552 By comparing (21) with Theorem 5.9, our remaining task is to demonstrate $\|\mathcal{P}_H\psi\|_{\bar{s}}$ can be bounded by
 553 $\|\mathcal{G}^\infty\psi\|_{\bar{a}}$. According to Lemma 5.4, it not harmful to assume that $\psi \in V$ with $\|\psi\|_{\bar{a}} \leq C_{\text{inv}}\|\mathcal{P}_H\psi\|_{\bar{s}}$. Taking
 554 $\mathcal{T}\psi$ as a test function, we can obtain that

$$a(\mathcal{G}^\infty\psi, \mathcal{T}\psi) + s(\mathcal{P}_H\mathcal{G}^\infty\psi, \mathcal{P}_H\mathcal{T}\psi) = s(\mathcal{P}_H\psi, \mathcal{P}_H\mathcal{T}\psi).$$

555 The estimate (14) says that

$$C'\|\mathcal{P}_H\psi\|_{\bar{s}}^2 - C''\|\psi\|_{\bar{a}}^2 \leq s(\mathcal{P}_H\psi, \mathcal{P}_H\mathcal{T}\psi)$$

556 where $C' \rightarrow 1$ and $C'' \rightarrow 0$ as $\Upsilon \rightarrow \infty$. Therefore, if Υ is large enough, we can show that

$$\begin{aligned} c_0\|\mathcal{P}_H\psi\|_{\bar{s}}^2 &\leq (C' - C''C_{\text{inv}}^2)\|\mathcal{P}_H\psi\|_{\bar{s}}^2 \leq C'\|\mathcal{P}_H\psi\|_{\bar{s}}^2 - C''\|\psi\|_{\bar{a}}^2 \\ &\leq s(\mathcal{P}_H\psi, \mathcal{P}_H\mathcal{T}\psi) \leq \|\mathcal{G}^\infty\psi\|_{\bar{a}}\|\mathcal{T}\psi\|_{\bar{a}} + \|\mathcal{P}_H\mathcal{G}^\infty\psi\|_{\bar{s}}\|\mathcal{P}_H\mathcal{T}\psi\|_{\bar{s}} \\ &\leq c_1\sqrt{1 + C_{\text{inv}}^2}\left(\|\mathcal{G}^\infty\psi\|_{\bar{a}}^2 + \|\mathcal{P}_H\mathcal{G}^\infty\psi\|_{\bar{s}}^2\right)^{1/2}\left(\|\psi\|_{\bar{a}}^2 + \|\mathcal{P}_H\psi\|_{\bar{s}}^2\right)^{1/2}, \end{aligned}$$

557 where c_0 and c_1 are positive constants and the last line follows the techniques in the previous proofs. We
 558 hence obtain that

$$\|\psi\|_{\bar{a}}/C_{\text{inv}} \leq \|\mathcal{P}_H\psi\|_{\bar{s}} \leq c_2\left(\|\mathcal{G}^\infty\psi\|_{\bar{a}}^2 + \|\mathcal{P}_H\mathcal{G}^\infty\psi\|_{\bar{s}}^2\right)^{1/2}.$$

559 Meanwhile, utilizing Poincaré inequality, we can see that

$$\|\mathcal{P}_H\mathcal{G}^\infty\psi\|_{\bar{s}} \leq \|\mathcal{G}^\infty\psi\|_{\bar{s}} \leq C_{\text{po}}H^{-1}\|\mathcal{G}^\infty\psi\|_{\bar{a}}.$$

560 Hence, in conjunction with Theorem 5.9, we establish the existence of such η in (21). Once the inf-sup
 561 condition is validated, the existence and uniqueness of the multiscale solution are guaranteed. Moreover, we
 562 can employ Céa's lemma to derive the error estimate. Specifically, we obtain

$$\|u - \hat{u}_H^m\|_{\bar{a}} \leq c_3\|u - v_H\|_{\bar{a}} \leq c_3\left(\|u - u_H^\infty\|_{\bar{a}} + \|u_H^\infty - v_H\|_{\bar{a}}\right),$$

563 where $v_H = \hat{\mathcal{G}}^m\psi \in \hat{V}_H^m$ if $u_H^\infty = \mathcal{G}^\infty\psi$. We have already shown that

$$\|u_H^\infty - v_H\|_{\bar{a}} \leq c_4H^{-1}(m+1)^{d/2}\theta^{(m-1)/2}\|u_H^\infty\|_{\bar{a}}.$$

564 Finally, we present the main theorem of this section.

565 **Theorem 5.10.** *There exist positive constants Υ' and ϵ' such that for any $\Upsilon \geq \Upsilon'$ and $\epsilon \leq \epsilon'$, the solution*
 566 *to (20) exists and is unique. Moreover, the following error estimate holds:*

$$\|u - \hat{u}_H^m\|_{\bar{a}} \leq C_*\left(1 + H^{-2}(m+1)^{d/2}\theta^{(m-1)/2}\right)\|f\|_{\bar{s}^*},$$

567 where the positive constant C_* and θ is independent of H and m with $\theta < 1$.

568 6 Conclusions

569 We have proposed a multiscale computational method for solving sign-changing problems, utilizing the
 570 framework of CEM-GMsFEMs in the construction of multiscale basis functions. However, a direct applica-
 571 tion of the original CEM-GMsFEM encounters an immediate challenge during the construction of auxiliary
 572 spaces, as the generalized spectral problems can become ill-defined due to a non-positive definite right-hand

573 bilinear form. We have addressed this issue by replacing the coefficient with its absolute value in this step
574 and explained that this modification complies with the T-coercivity theory. Moreover, we have focused on
575 the relaxed version of the CEM-GMsFEM, which is more implementation-friendly as it eliminates the need
576 to solve saddle-point problems. The numerical experiments conducted have highlighted several advantages
577 of the proposed method: (1) the flexibility that coarse meshes do not require to resolve with interfaces, (2)
578 the accuracy that remains stable under low regular exact solutions, and (3) the robustness in high contrast
579 coefficient profiles. Due to technical difficulties, such as the nonlocality of the reflection operator \mathcal{T} , the final
580 error estimates have been proved under several stringent assumptions, primarily related to coarse element
581 partitions and coefficient contrast. However, we emphasize that the potential of the proposed method has
582 been demonstrated through numerical experiments, and we firmly believe that the theoretical results can
583 be further improved by relaxing the assumptions and developing more sophisticated analysis techniques.

584 The current implementation of the proposed method did not exploit the parallelism in constructing multi-
585 scale basis functions, and hence the computation time in the offline stage is awkwardly long, even longer
586 than generating reference solutions on the fine mesh. Therefore, we plan to investigate the parallelization
587 in the shared memory architecture of the offline stage in the future. We emphasize that the application of
588 computational multiscale methods to sign-changing problems is much more appealing in large-scale simula-
589 tions. As a matter of fact, the algebraic linear systems in this situation are no longer positive definite, which
590 renders iterative solvers less efficient, even with preconditioning techniques. Consequently, in this context of
591 direct solvers, the reduction of degrees of freedom by multiscale methods is expected to be more beneficial.

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