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Observability of the heat equation from very small sets

A. Walton Green*, Kévin Le Balc'h†, Jérémy Martin ‡, Marcu-Antone Orsoni §

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Abstract

We consider the heat equation set on a bounded C^1 domain of \mathbb{R}^n with Dirichlet boundary conditions. The first purpose of this paper is to prove that the heat equation is observable from any measurable set ω with positive $(n - 1 + \delta)$ -Hausdorff content, for $\delta > 0$ arbitrary small. The proof relies on a new spectral estimate for linear combinations of Laplace eigenfunctions, obtained via a Remez type inequality, and the use of the so-called Lebeau-Robbiano's method. Even if this observability result is sharp with respect to the scale of Hausdorff dimension, our second goal is to construct families of sets ω which have codimension greater than or equal to 1 for which the heat equation remains observable.

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1 Introduction

1.1 Observability and null-controllability of the heat equation

Given $T > 0$, Ω a bounded open connected set of \mathbb{R}^n , $\omega \subset \Omega$ a measurable set, this paper aims at establishing observability estimates of the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0 & \text{in } \Omega. \end{cases} \quad (1.1)$$

First, let us recall some basic well-posedness properties of the heat equation (1.1). From [Eva10, Sections 7.1], for every $u_0 \in L^2(\Omega)$, there exists a unique weak solution $u \in L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$ of the parabolic initial/boundary-value problem (1.1). By Sobolev embedding on mixed spaces, we deduce that $u \in C([0, T]; L^2(\Omega))$. Moreover, by the $L^2 - L^\infty$ regularizing effect of the heat equation, see for instance [CH98, Proposition 3.5.7], we have $u(T, \cdot) \in L^\infty(\Omega)$. If we assume furthermore that $u_0 \in L^\infty(\Omega)$, then $u \in L^\infty((0, T) \times \Omega)$ according to the weak maximum principle, see for instance [WYW06, Section 4.2].

With this at hand, let us introduce the following notion of observability of the heat equation (1.1) from ω at time T as follows.

Definition 1.1. *Let $T > 0$ and $\omega \subset \Omega$ be a measurable set. The heat equation (1.1) is said to be observable from ω at time T if there exists a positive constant $C = C(\Omega, \omega, T) > 0$ such that*

$$\|u(T, \cdot)\|_{L^\infty(\Omega)} \leq C \int_0^T \sup_{x \in \omega} |u(t, x)| dt, \quad \forall u_0 \in L^\infty(\Omega). \quad (1.2)$$

All the terms in (1.2) are well-defined (and finite) according to the previous recalled properties of the heat equation. Indeed the left hand side is well-defined according to the property $u(T, \cdot) \in L^\infty(\Omega)$ because of the regularizing effects while the right hand side is also well-defined according to the weak maximum principle that implies $u \in L^\infty((0, T) \times \omega)$.

Note that this notion of observability (1.2) differs from the more standard one, usually called $L^2 - L^2$ observability estimate,

$$\|u(T, \cdot)\|_{L^2(\Omega)} \leq C \left(\int_0^T \int_\omega |u(t, x)|^2 dt dx \right)^{1/2}, \quad \forall u_0 \in L^2(\Omega). \quad (1.3)$$

First because of the Hilbertian structure of $L^2(\Omega)$ and $L^2((0, T) \times \omega)$, (1.3) is well-adapted for employing duality methods. For instance, let us consider the associated controlled heat equation

$$\begin{cases} \partial_t y - \Delta y = h1_\omega & \text{in } (0, T) \times \Omega, \\ y = 0 & \text{in } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y_0 & \text{in } \Omega. \end{cases} \quad (1.4)$$

In the controlled system (1.4), at time $t \in [0, +\infty)$, $y(t, \cdot) : \Omega \rightarrow \mathbb{R}$ is the state and $h(t, \cdot) : \omega \rightarrow \mathbb{R}$ is the control. The following classical result, called Hilbert Uniqueness Method [Cor07, Theorem 2.48], holds.

Theorem 1.2 (H.U.M.). *Let $T > 0$, $\omega \subset \Omega$ be a measurable set and $C_{\text{Cost}} = C(\Omega, \omega, T) > 0$ be a positive constant. The following two assertions are equivalent.*

1. *The controlled heat equation (1.4) is null-controllable at time T from ω with control cost $C = C_{\text{Cost}} > 0$, i.e. for every $y_0 \in L^2(\Omega)$, there exists $h \in L^2((0, T) \times \omega)$ with*

$$\|h\|_{L^2((0, T) \times \omega)} \leq C_{\text{Cost}} \|y_0\|_{L^2(\Omega)}, \quad (1.5)$$

such that the unique weak solution $y \in C([0, T]; L^2(\Omega))$ of (1.4) satisfies $y(T, \cdot) = 0$.

2. The heat equation (1.1) is $L^2 - L^2$ -observable at time T from ω with observability constant $C = C_{\text{Cost}} > 0$, i.e. (1.3) holds true with $C = C_{\text{Cost}} > 0$.

Secondly, (1.3) is also well-adapted when considering open sets ω or even measurable sets ω such that $|\omega| > 0$ where $|\cdot|$ stands for the n -dimensional Lebesgue measure in \mathbb{R}^n . But here because we focus on situations of measurable sets $\omega \subset \Omega$ where possibly $|\omega| = 0$, we will work with the more accurate observability notion (1.2) of Definition 1.1.

From now and in all the following, we assume that Ω satisfies the following hypothesis.

The set Ω is a bounded Lipschitz open connected set of \mathbb{R}^n , locally star-shaped. (H)

See [AEWZ14, Definition 4] for a precise definition of locally star-shaped domains. Note that from [AEWZ14, Remark 6] such domains include C^1 bounded domains, Lipschitz polyhedrons in \mathbb{R}^n with $n \geq 3$ and bounded convex domains.

1.2 Main results

Our goals in this paper are twofold. Our main theorem provides observability results for (1.1) when the Hausdorff dimension of the observable set ω is strictly greater than $n - 1$ (i.e. has codimension strictly less than 1). Let us recall that for $s \geq 0$, the s -Hausdorff content of a set $E \subset \mathbb{R}^n$ is

$$C_{\mathcal{H}}^s(E) = \inf \left\{ \sum_j r_j^s ; E \subset \bigcup_j B(x_j, r_j) \right\}, \quad (1.6)$$

and the Hausdorff dimension of E is defined as

$$\dim_{\mathcal{H}}(E) = \inf \{s \geq 0 ; C_{\mathcal{H}}^s(E) = 0\}, \quad \text{codim}_{\mathcal{H}}(E) = n - \dim_{\mathcal{H}}(E).$$

Our main result is as follows.

Theorem 1.3. *Let $T > 0$ and $\delta > 0$. Let $\omega \subset \Omega$ be a measurable subset satisfying $C_{\mathcal{H}}^{n-1+\delta}(\omega) > 0$, then the heat equation (1.1) is observable from ω at time T .*

More precisely, for measurable subset $\omega \subset \Omega$ satisfying $C_{\mathcal{H}}^{n-1+\delta}(\omega) > 0$, there exists $C = C(\Omega, \delta, \omega) > 0$ such that

$$\|u(T, \cdot)\|_{L^\infty(\Omega)} \leq C e^{C/T} \int_0^T \sup_{x \in \omega} |u(t, x)| dt, \quad \forall u_0 \in L^\infty(\Omega). \quad (1.7)$$

Some comments with respect to the geometrical assumptions on Ω and ω are in order.

- When $\omega \subset \Omega$ is a *nonempty open set*, [FR71] proved (1.3) in the one-dimensional case by moment's method then more than twenty years later, [LR95] and [FI96] generalized independently this result to the multi-dimensional case by using Carleman estimates. When Ω is C^∞ -smooth, [LR95] actually used local elliptic Carleman estimates to prove a spectral estimate for linear combination of eigenfunctions (see [JL99, Theorem 4.3] for a precise statement), then the authors introduced the now so-called Lebeau-Robbiano's method for passing from a spectral estimate to the observability of the heat equation (see for instance [Mil10, Theorem 2.2] or [BPS18, Theorem 2.1]). When Ω is C^2 , [FI96] directly proved observability estimates for the heat equation with the help of global parabolic Carleman estimates. Our proof of Theorem 1.3 follows Lebeau-Robbiano's method, so our crucial points would consist in deriving new spectral estimates.

- When ω is only assumed to be measurable with **positive Lebesgue measure** i.e. $|\omega| > 0$ and Ω is Lipschitz and locally star-shaped, [AEWZ14] proved (1.3). Their result is actually based on the obtaining of spectral estimate of Jerison-Lebeau's type together with the analytic properties of linear combination of eigenfunctions inside the domain Ω . It is worth mentioning that as far as we know the obtaining of (1.3) for general bounded Lipschitz domains Ω with observation localized in measurable sets ω such that $|\omega| > 0$ or even for nonempty open sets ω is open (see [BP20] for some progress in that direction).
- When Ω is $W^{2,\infty}$, [BM23] proved (1.2) for measurable subsets ω with possibly zero-Lebesgue measure but with **positive Hausdorff content** satisfying $\mathcal{C}_{\mathcal{H}}^{n-1+\delta}(\omega) > 0$ for a $\delta \in (0, 1)$ that is a priori close to 1 by using recent quantitative propagation of smallness results for gradient of solutions to A -harmonic functions from [LM18]. In particular, they went a little bit beyond [AEWZ14] proving the observability from measurable sets that are of zero n -dimensional Lebesgue measure. Their result is also valid replacing the flat Laplacian $-\Delta$ by a divergence elliptic operator with Lipschitz coefficients, i.e. $-\operatorname{div}(A(x)\nabla\cdot)$ with a matrix-valued Lipschitz and uniformly elliptic diffusion A . Note also that it was the first observability result for the heat equation from a measurable set with zero-Lebesgue measure in the multi-dimensional case.
- In the one-dimensional case, i.e. $n = 1$, [Zhu23] exactly obtained Theorem 1.3 replacing the flat Laplacian $-\partial_{xx}^2$ by a rough divergence elliptic operator, $-\partial_x(a(x)\partial_x\cdot)$ where $a \in L^\infty(\Omega)$ and uniformly elliptic. Their main result is a [LM18]-like quantitative propagation of smallness result for gradient of a -harmonic functions in the two-dimensional case for sets with Hausdorff dimensions $\dim_{\mathcal{H}}(E) = \delta > 0$ for arbitrary $\delta > 0$. It is obtained exploiting the very specific relation between a -harmonic functions and holomorphic functions in the complex plane.
- The assumption made in Theorem 1.3 on ω , satisfying $\mathcal{C}_{\mathcal{H}}^{n-1+\delta}(\omega) > 0$ for arbitrary $\delta > 0$, is sharp in the following sense: there exists a subset ω with $\dim_{\mathcal{H}}(\omega) = n - 1$ such that (1.2) does not hold from ω . Indeed, assume that Ω is C^1 and let $\omega := \{x \in \Omega ; \varphi_\lambda(x) = 0\}$ be a nodal set of a Laplace eigenfunction φ_λ associated to the eigenvalue $\lambda > 0$ with Dirichlet boundary conditions on $\partial\Omega$ i.e.

$$\begin{cases} -\Delta\varphi_\lambda = \lambda\varphi_\lambda & \text{in } \Omega, \\ \varphi_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.8)$$

such that $\|\varphi_\lambda\|_{L^\infty(\Omega)} = 1$. Then it is a well-known fact that $\mathcal{C}_{\mathcal{H}}^{n-1}(\omega) > 0$, actually one can prove that $\mathcal{C}_{\mathcal{H}}^{n-1}(\omega) \geq c\sqrt{\lambda}$ for λ sufficiently large, see [LMNN21]. Taking φ_λ as a initial data, we obtain that the solution u of (1.1) is given by $u(t, x) = e^{-\lambda t}\varphi_\lambda(x)$ so the left hand side of the observability estimate (1.2) is equal to $e^{-\lambda T}$, while the right hand side vanishes. Hence, the inequality (1.2) cannot hold for this specific ω .

- Note that the observability estimate (1.7) is quantitative in function of T only. If one wants to obtain a quantitative estimate in function of the Hausdorff content of ω , we can establish the following statement in the spirit of [AEWZ14]. For $x_0 \in \Omega$, $R \in (0, 1]$ be such that $B(x_0, 4R) \subset \Omega$, for every $m > 0$, there exists $C = C(\Omega, \delta, R, m) > 0$ such that for every measurable subset $\omega \subset B(x_0, R)$ with $\mathcal{C}_{\mathcal{H}}^{n-1+\delta}(\omega) \geq m$, one has

$$\|u(T, \cdot)\|_{L^\infty(\Omega)} \leq Ce^{C/T} \int_0^T \sup_{x \in \omega} |u(t, x)| dt, \quad \forall u_0 \in L^\infty(\Omega). \quad (1.9)$$

From Theorem 1.3 and a duality argument in the spirit of Theorem 1.2 we can deduce a null-controllability result for (1.4). In the following, we denote by $\mathcal{M}(\Omega)$ the space of

Borel measures on Ω . Recall that the norm of $\mathcal{M}(\Omega)$ is defined as

$$\|\mu\|_{\mathcal{M}(\Omega)} = \sup_{f \in C_c^0(\Omega)} \frac{|\int_{\Omega} f d\mu|}{\|f\|_{\infty}}. \quad (1.10)$$

Theorem 1.4. *Let $T > 0$ and $\delta > 0$. Let $\omega \subset \Omega$ be a measurable subset satisfying $\mathcal{C}_{\mathcal{H}}^{n-1+\delta}(\omega) > 0$, then (1.4) is null-controllable at time T , that is for every $y_0 \in L^2(\Omega)$, there exists $h \in L^\infty(0, T; \mathcal{M}(\Omega))$ such that the (very)-weak solution y of (1.4) satisfies $y(T, \cdot) = 0$.*

More precisely, for measurable subset $\omega \subset \Omega$ satisfying $\mathcal{C}_{\mathcal{H}}^{n-1+\delta}(\omega) > 0$, there exists $C = C(\Omega, \delta, \omega) > 0$, for every $y_0 \in L^2(\Omega)$, there exists $h \in L^\infty(0, T; \mathcal{M}(\Omega))$ satisfying

$$\sup_{t \in (0, T)} \|h(t)\|_{\mathcal{M}(\Omega)} \leq C e^{\frac{C}{T}} \|y_0\|_{L^2(\Omega)}, \quad (1.11)$$

such that the (very)-weak solution y of (1.4) satisfies $y(T, \cdot) = 0$.

The proof of Theorem 1.4 will be omitted because it is a standard adaptation of [BM23, Section 5] starting from Theorem 1.3. We insist on the fact that a first difficulty in the analysis is to give a meaning of the concept of (very)-weak solution to (1.4) with a control $h = h(t)$, supported in ω and taking its values in $\mathcal{M}(\Omega)$.

Notwithstanding the sharpness in the scale of Hausdorff dimension of Theorem 1.3 just demonstrated, our second goal is to exemplify specially constructed families of sets which have codimension greater than or equal to 1 for which the heat equation remains observable. The main examples of interest are first, the cartesian product of any bounded set $\omega' \subset \mathbb{R}^{n-1}$ with $\dim_{\mathcal{H}}(\omega') > n - 2$ and a single, specially chosen point x_0 ; see Corollary 3.5 below. Second, we can in fact construct closed observation sets of zero dimension, see Corollary 3.8 below. Therein, we show that the set of points

$$\{(x_0, k^{-\alpha}) : k = 1, 2, \dots\}$$

forms an observation set for the heat equation on $(0, 1)^2$ for any $\alpha > 0$ and a suitably chosen point x_0 .

A few natural natural open questions immediately arise. First, is $\{(x_0, e^{-k}) : k = 1, 2, \dots\}$ an observation set in $(0, 1)^2$? Second, a potential replacement in higher dimensions for a single point with prescribed algebraic properties would be certain algebraic curves. At this point it is even open whether the diagonal line $\{(x, y) \in (0, 1)^2 : x = y\}$ is an observation set for the heat equation on $(0, 1)^2$. See the discussion around Question 3.10 below for one possible path to solve this problem.

2 Proof of the main observability estimate, i.e. Theorem 1.3

In order to prove Theorem 1.3, we will follow the classical Lebeau-Robbiano's method, [LR95], consisting in establishing first spectral estimate. The unbounded operator $-\Delta$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ is self-adjoint with compact resolvent so it admits an orthonormal basis of eigenfunctions $(\varphi_k)_{k \geq 1}$, associated to the sequence of real eigenvalues $(\lambda_k)_{k \geq 1}$. Given $\Lambda > 0$, we introduce the spectral projector Π_Λ as follows

$$\Pi_\Lambda u = \sum_{\lambda_k \leq \Lambda} \langle u, \varphi_k \rangle_{L^2} \varphi_k, \quad \forall u \in L^2(\Omega). \quad (2.1)$$

2.1 The spectral estimate

The main result of this section states the following spectral estimates.

Theorem 2.1. *For all $\delta > 0$, for every $\omega \subset \Omega$ such that $\mathcal{C}_{\mathcal{H}}^{n-1+\delta}(\omega) > 0$, there exists $C = C(\Omega, \delta, \omega) > 0$ such that for every $\Lambda > 0$, we have*

$$\|\Pi_{\Lambda} u\|_{L^{\infty}(\Omega)} \leq C e^{C\sqrt{\Lambda}} \sup_{x \in \omega} |(\Pi_{\Lambda} u)(x)| \quad \forall u \in L^2(\Omega). \quad (2.2)$$

On the one hand, Theorem 2.1 improves the spectral estimates from [AEWZ14], [AE13], [BM23] with respect to the Hausdorff dimension of the observation set and extends [Zhu23] to the multi-dimensional setting. On the other hand, Theorem 2.1 is only valid for an elliptic operator with real analytic metric while the spectral estimate of [BM23] is valid for $W^{1,\infty}$ metric and the one of [Zhu23] is valid for L^{∞} metric. We focus our attention on the case of the pure Laplacian (constant coefficients), but we will point out the changes needed to handle the case of real-analytic coefficients in the Appendix when we discuss an alternative proof of Theorem 2.1. Therein, we will show that Theorem 2.1 actually follows in the now-standard way from a propagation of smallness result for gradients of harmonic functions due to Malinnikova [Mal04], and the well-known lifting strategy of Jerison and Lebeau [JL99]. However, here we give a direct and elementary proof of (2.2) which allows us to deal with the linear combinations of eigenfunctions directly, only relying on the Bernstein inequality (2.4), see below, in the vein of e.g. [Kov01, JM22, NSV04, LM19].

2.2 Proof of the spectral estimate

The main step for obtaining Theorem 2.1, which relies on the simple Lagrange interpolation formula and a slicing/rotation argument for capacity, is the following proposition which is proved at the end of this part.

Proposition 2.2. *Let $r, \delta, m > 0$. There exists $C = C(r, \delta, m) > 0$ such that for any ball $B \subset \mathbb{R}^n$ of radius r , any $E \subset B$ with $\mathcal{C}_{\mathcal{H}}^{n-1+\delta}(E) \geq m$, any $(M_{\ell})_{\ell \geq 0} \subset (0, \infty)^{\mathbb{N}}$ with*

$$\frac{M_{\ell}}{\ell!} \rightarrow 0 \text{ as } \ell \rightarrow +\infty, \quad (2.3)$$

any $f \in C^{\infty}(B)$ satisfying

$$\sup_B |f^{(\ell)}| \leq \frac{M_{\ell}}{r^{\ell}} \sup_B |f|, \quad \ell \geq 0, \quad (2.4)$$

then for ℓ_0 defined as

$$\ell_0 := \inf\{\ell \in \mathbb{N} ; 2M_{\ell} \leq \ell!\} \in \mathbb{N}, \quad (2.5)$$

there holds

$$\sup_B |f| \leq C^{\ell_0} \sup_E |f|.$$

For now, let us prove Theorem 2.1 using Proposition 2.2.

Proof of Theorem 2.1 using Proposition 2.2. First, from the assumption on ω , we can find a ball $B \subset \Omega$ of radius $r > 0$ and $c = c(\Omega, \delta, \omega) > 0$ such that

$$\mathcal{C}_{\mathcal{H}}^{n-1+\delta}(B \cap \omega) \geq c.$$

Then, from [AEWZ14, Theorem 3], there exists $C = C(\Omega, \delta, \omega) > 0$ such that

$$\|\Pi_{\Lambda} u\|_{L^2(\Omega)} \leq C e^{C\sqrt{\Lambda}} \sup_B |\Pi_{\Lambda} u|. \quad (2.6)$$

Henceforth C may change in each instance, yet always remains only dependent on Ω, δ, ω . For simplicity, let us now fix $\Lambda > 0$ and set $v = \Pi_\Lambda u$. Then, for any $p \geq 0$,

$$(-\Delta)^p v = \sum_{\lambda_k \leq \Lambda} \lambda_k^p \langle v, \varphi_k \rangle \varphi_k \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega, \quad (2.7)$$

so Sobolev embeddings, orthogonality of $(\varphi_k)_{k \geq 0}$, and (2.6) imply that

$$\|\nabla^\ell v\|_{L^\infty(\Omega)} \leq C \|v\|_{H^{\ell+n}(\Omega)} \leq C \Lambda^{\frac{\ell+n}{2}} \|v\|_{L^2(\Omega)} \leq \sqrt{\Lambda}^\ell C e^{C\sqrt{\Lambda}} \sup_B |v| \quad \forall \ell \geq 0. \quad (2.8)$$

Now, we are in the position to apply Proposition 2.2 with $E = B \cap \omega$, noticing that v satisfies (2.4) with

$$M_\ell = C^{\sqrt{\Lambda}} \left(C\sqrt{\Lambda} \right)^\ell.$$

By Stirling's formula, (2.3) holds and $M_\ell/\ell! < \frac{1}{2}$ once ℓ is larger than $A\sqrt{\Lambda}$ for some $A = A(\Omega, \delta, \omega)$ a large absolute constant multiple of C . Therefore, Proposition 2.2 implies

$$\sup_B |v| \leq C^{A\sqrt{\Lambda}} \sup_\omega |v|,$$

which when combined with (2.8) with $\ell = 0$ establishes (2.2). \square

2.3 Proof of the Remez inequality

In this part, we prove Proposition 2.2.

2.3.1 One-dimensional case

We begin by the one-dimensional case. Our starting point is the following very simple Bang-type estimate for analytic functions. It is a simplified version of [NSV04, Estimate (4.1)] or [JM22, Lemma 4.7] with one small modification, replacing the Lebesgue measure with the following quantity related to transfinite diameter, see [Pom92].

For any set $E \subset [0, 1]$, define

$$\gamma_k(E) = \sup_{\{x_i\}_{i=1}^k \subset E} \min_i \prod_{j \neq i} |x_i - x_j|. \quad (2.9)$$

The following one-dimensional Remez estimate holds.

Proposition 2.3. *Let $(M_\ell)_{\ell \geq 0} \subset (0, \infty)^\mathbb{N}$ satisfying (2.3), $f \in C^\infty([0, 1])$ satisfying*

$$\sup_{[0,1]} |f^{(\ell)}| \leq M_\ell \sup_{[0,1]} |f|, \quad \ell \geq 0. \quad (2.10)$$

Then for any $E \subset [0, 1]$, for ℓ_0 as in (2.5),

$$\sup_{[0,1]} |f| \leq \frac{4\ell_0}{\gamma_{\ell_0}(E)} \sup_E |f|. \quad (2.11)$$

Proof. The proof is quite elementary, relying chiefly on the following Lagrange interpolation formula (see for instance [Dem16, Chapitre 2, Section 1.2]). For any $x, x_1 < \dots < x_k \in [0, 1]$, there exists $\xi \in [0, 1]$ such that

$$f(x) = \sum_{i=1}^k f(x_i) \prod_{j \neq i} \frac{(x - x_j)}{(x_i - x_j)} + \frac{f^{(k)}(\xi)}{k!} \prod_j (x - x_j). \quad (2.12)$$

Choosing $\{x_i\}_{i=1}^k \subset E$ such that $2 \min_i \prod_{j \neq i} |x_i - x_j| \geq \gamma_k(E)$, it is immediate that

$$\left| \sum_{i=1}^n f(x_i) \prod_{j \neq i} \frac{(x - x_j)}{(x_i - x_j)} \right| \leq \frac{2k \cdot \sup_E |f|}{\gamma_k(E)}$$

To estimate the second term in (2.12) we simply apply (2.10) to obtain

$$\left| \frac{f^{(k)}(\xi)}{k!} \prod_j (x - x_j) \right| \leq \frac{M_k}{k!} \sup_{[0,1]} |f|.$$

which is less than $\frac{1}{2} \sup_{[0,1]} |f|$ for $k = \ell_0$ hence (2.11) holds. \square

The transfinite diameter of E is the limit of the k -th root of $\gamma_k(E)$ as k approaches infinity. An equivalent characterization is also through

$$\Delta_k(E) = \sup_{\{x_i\}_{i=1}^k \subset E} \prod_{i=1}^k \prod_{j \neq i} |x_i - x_j|,$$

and the identity

$$\lim_{k \rightarrow \infty} \Delta_k(E)^{\frac{1}{k(k-1)}} = \lim_{k \rightarrow \infty} \gamma_k(E)^{\frac{1}{k}},$$

holds for any compact set E , see [Pom92, Proposition 9.6].

To use Proposition 2.3, we recall some basic facts from the theory of capacity (see e.g. [Mat95, Chapter 8] and [Pom92, Chapters 9 and 10]). For any $0 < s \leq n$ and any probability Radon measure μ on \mathbb{R}^n , define then s -energy of μ by

$$I_s(\mu) = \iint |x - y|^{-s} d\mu(x) d\mu(y),$$

with the modification for $s = 0$ to define $I_s(\mu)$ with $|x - y|^{-s}$ replaced by $-\log|x - y|$. Letting $\mathcal{P}(E)$ denote the set of all probability Radon measures supported on E , and define the s -capacity of E by

$$\text{cap}_s(E) = \begin{cases} \sup \{I_s(\mu)^{-1} : \mu \in \mathcal{P}(E)\}, & s > 0; \\ \sup \{\exp(-I_0(\mu)) : \mu \in \mathcal{P}(E)\}, & s = 0. \end{cases}$$

The concavity of the logarithm and Jensen's inequality show that

$$\text{cap}_0(E)^s \geq \text{cap}_s(E). \quad (2.13)$$

Furthermore, $\text{cap}_0(E)$ coincides with the transfinite diameter of E for E compact (see [Ahl10, Theorem 2.3, page 28]), which means

$$\text{cap}_0(E) = \lim_{k \rightarrow \infty} \Delta_k(E)^{\frac{2}{k(k-1)}} = \lim_{k \rightarrow \infty} \gamma_k(E)^{\frac{1}{k}},$$

and in fact,

$$\gamma_k(E) \geq (\text{cap}_0 E)^{k-1}. \quad (2.14)$$

See [Pom92, Proposition 9.6 and Equation (16)].

Capacities will be more natural to work with than the closely related, and more familiar Hausdorff content, defined above in (1.6). This relationship between capacity and Hausdorff content is facilitated by Frostman's lemma [Mat95, Theorem 8.8] which implies the following result.

Lemma 2.4. For each $s > t \geq 0$ and $m > 0$ there exists $p > 0$ such that

$$\mathcal{C}_{\mathcal{H}}^s(E) \geq m \implies \text{cap}_t(E) \geq p. \quad (2.15)$$

Consequently,

$$\dim_{\mathcal{H}} E = \sup \{s > 0 : \text{cap}_s E > 0\}. \quad (2.16)$$

With these preliminaries behind us, we are ready to state and prove the promised corollary for sets of positive Hausdorff dimension.

Corollary 2.5. Let $(M_\ell)_{\ell \geq 0} \subset (0, \infty)^{\mathbb{N}}$ satisfying (2.3), $f \in C^\infty([0, 1])$ satisfying (2.10). Then for any compact set $E \subset [0, 1]$, for ℓ_0 as in (2.5),

$$\sup_{[0,1]} |f| \lesssim \left(\frac{1}{\text{cap}_0 E} \right)^{\ell_0} \sup_E |f|. \quad (2.17)$$

Furthermore, for each $s, m > 0$, there exists a constant $C(s, m) > 0$ such that if $\mathcal{C}_{\mathcal{H}}^s(E) \geq m$,

$$\sup_{[0,1]} |f| \lesssim \left(\frac{1}{C(s, m)} \right)^{\ell_0} \sup_E |f|. \quad (2.18)$$

Proof. (2.17) follows immediately from Proposition 2.3 and (2.14). The second inequality (2.18) follows from the first statement (2.15) with in Lemma 2.4 with $t = 0$ and the application of (2.17). \square

2.3.2 Higher Dimensions

First, we prove the following Lebesgue estimate in higher dimensions, which well-known on its own for analytic functions [Bru99], but we outline its standard extension from Proposition 2.3.

Proposition 2.6. For any ball $B \subset \mathbb{R}^n$, any $E \subset \frac{1}{2}B$ with $|E| > 0$, any $(M_\ell)_{\ell \geq 0} \subset (0, \infty)^{\mathbb{N}}$ satisfying (2.3), any $f \in C^\infty(B)$ satisfying (2.4) then for ℓ_0 as in (2.5), the following estimate holds

$$\sup_{\frac{1}{2}B} |f| \leq \left(\frac{c_n |B|}{|E|} \right)^{\ell_0} \sup_E |f|, \quad (2.19)$$

for some $c_n > 0$ a positive constant depending on the dimensional constant n .

Proof. Rescaling, we can assume the radius of B is one. Let $x_0 \in \frac{1}{2}B$ such that $\sup_{\frac{1}{2}B} |f| \leq 2|f(x_0)|$. By using polar coordinates, there exists a line L through x_0 such that

$$\mathcal{L}^1(E \cap L) \geq c_n |E|,$$

for some dimensional constant $c_n > 0$, where \mathcal{L}^1 is the one-dimensional Lebesgue measure on the line L . Restricting f to L , the restriction satisfies the assumptions of Proposition 2.3. Moreover it is not difficult to see that there is an absolute constant $c > 0$ such that

$$\gamma_{\ell_0}(E \cap L_\theta) \geq (c \mathcal{L}_\theta^1(E \cap L_\theta))^{\ell_0}. \quad (2.20)$$

Indeed, we can find $x_1, \dots, x_{\ell_0} \in \overline{E} \cap L_\theta$ such that $|x_j - \theta|$ (let θ be either point on $L \cap (\mathbb{S}^{n-1} + x_0)$) is increasing and $|x_j - x_{j+1}| \geq \mathcal{L}_\theta^1(E \cap L_\theta) \ell_0^{-1}$. Such points can be

constructed as in [AE13, Lemma 1]. Then (2.20) follows from Stirling's formula and the binomial theorem as in [NSV04, p. 69]:

$$\begin{aligned} \prod_{j \neq i} |x_i - x_j| &\geq \frac{(i-1)!(\ell_0 - i)!}{\ell_0^{\ell_0 - 1}} \mathcal{L}_\theta^1(E \cap L_\theta)^{\ell_0 - 1} \geq \frac{i!(\ell_0 - i)!}{\ell_0^{\ell_0}} (\mathcal{L}_\theta^1(E \cap L_\theta))^{\ell_0} \\ &\geq \frac{i!(\ell_0 - i)!}{\ell_0!} \left(\frac{\mathcal{L}_\theta^1(E \cap L_\theta)}{e} \right)^{\ell_0} \geq \left(\frac{\mathcal{L}_\theta^1(E \cap L_\theta)}{2e} \right)^{\ell_0}. \end{aligned}$$

Therefore (2.19) follows from the estimate (2.11) in Proposition 2.3. \square

To obtain the result for sets $E \subset B$ with $\dim_{\mathcal{H}} E > n - 1$, we want to repeat this technique of slicing our set E , but for lower dimensional sets, going to polar coordinates becomes more difficult. Nonetheless, such coarea-type formulas do persist for capacity (see e.g. [Mat95, §10.9]). The particular consequence we will need is from [Mal04, Lemma 5.3]. We then obtain the following result leading to Proposition 2.2.

Proposition 2.7. *Let $R, s > 0$. There exists $C = C(R, s, n) > 0$ such that for any ball $B \subset \mathbb{R}^n$ of radius $r \leq R$, any $E \subset \frac{1}{2}B$ with $\text{cap}_{n-1+s}(E) > 0$, any $(M_\ell)_{\ell \geq 0} \subset (0, \infty)^\mathbb{N}$ satisfying (2.3), and any $f \in C^\infty(B)$ satisfying (2.4), there holds, for ℓ_0 as in (2.5), the following estimate*

$$\sup_{\frac{1}{2}B} |f| \leq \left(\frac{Cr^{n-1+s}}{\text{cap}_{n-1+s}(E)} \right)^{\frac{\ell_0(s+1)}{s}} \sup_E |f|. \quad (2.21)$$

Proof. Let us assume B is centered at the origin. By [Mal04, Lemma 5.3], there exist dimensional constants $c_1 > 0, c_2 > 0$ and a line L through the origin such that

$$G := \left\{ a \in L^\perp : \text{cap}_s(E \cap (L + a)) \geq c_1 r^{1-n} \text{cap}_{n-1+s}(E) \right\}$$

has $\mathcal{L}^{n-1}(G) \geq c_2 r^{-s} \text{cap}_{n-1+s}(E)$, where L^\perp is the $(n-1)$ -dimensional hyperplane perpendicular to L through the origin and \mathcal{L}^{n-1} is the $(n-1)$ -dimensional Lebesgue measure on L^\perp . Define now for each $a \in L^\perp$ the line segment $L_a^* := \{x + a : x \in L, |x| \leq \frac{1}{4}r\}$. Then we construct

$$F = \bigcup_{a \in G} L_a^*,$$

so that

$$|F| \gtrsim r \mathcal{L}^{n-1}(G) \gtrsim r^{1-s} \text{cap}_{n-1+s}(E).$$

On the other hand, for each $x \in F$, there exists a line segment L' , of length $\frac{1}{2}r$ passing through x , such that $\text{cap}_0(E \cap L')^s \geq \text{cap}_s(E \cap L') \geq c_1 r^{1-n} \text{cap}_{n-1+s}(E)$ so by (2.17),

$$|f(x)| \leq \sup_{L'} |f| \lesssim \left(\frac{r}{\text{cap}_0(E \cap L')} \right)^{\ell_0} \sup_E |f| \lesssim \left(\frac{Cr^{n-1+s}}{\text{cap}_{n-1+s}(E)} \right)^{\ell_0/s} \sup_E |f|.$$

Then, the result follows from applying Proposition 2.6 to the set F leading to

$$\sup_{\frac{1}{2}B} |f| \leq \left(\frac{c_n |B|}{|F|} \right)^{\ell_0} \sup_F |f| \lesssim \left(\frac{Cr^n}{r^{1-s} \text{cap}_{n-1+s}(E)} \right)^{\ell_0} \left(\frac{Cr^{n-1+s}}{\text{cap}_{n-1+s}(E)} \right)^{\ell_0/s} \sup_E |f|,$$

then in particular (2.21). \square

2.4 Adaptation of the Lebeau-Robbiano's method

With the help of the spectral estimate of Theorem 2.1, one can now prove Theorem 1.3.

Proof of Theorem 1.3. First, an immediate corollary of Theorem 2.1 is the following one. For all $\delta > 0$, for every $\omega \subset \Omega$ such that $C_{\mathcal{H}}^{n-1+\delta}(\omega) > 0$, there exists $C = C(\Omega, \delta, \omega) > 0$ such that for every $\Lambda > 0$, we have

$$\|\Pi_{\Lambda} u\|_{L^2(\Omega)} \leq C e^{C\sqrt{\Lambda}} \sup_{x \in \omega} |(\Pi_{\Lambda} u)(x)| \quad \forall u \in L^2(\Omega). \quad (2.22)$$

By using the spectral estimates (2.22), the proof of the observability estimate (1.7) is by now quite classical and originally comes from the Lebeau-Robbiano's method for obtaining the null-controllability of the heat equation starting from a spectral estimate, see [LR95] and [LRL12, Section 6]. This strategy was latter extended by Miller in [Mil10]. Note that it has to be adapted to the particular functional setting of the $L^2 - L^\infty$ spectral estimate (2.22). Here we follow the arguments of [BM23, Proof of Theorem 3 in Section 4] and [AEWZ14, Remark 2 in Section 2].

From (2.22) and [BM23, Corollary 4.1], we have the following result. There exists $A > 0$, $C > 0$ such that that for all $0 < t_1 < t_2 \leq T$, $u_0 \in L^\infty(\Omega)$, the solution u of (1.1) satisfies

$$e^{-\frac{A}{(t_2-t_1)}} \|u(t_2, \cdot)\|_{L^2(\Omega)} - e^{-\frac{2A}{(t_2-t_1)}} \|u(t_1, \cdot)\|_{L^2(\Omega)} \leq C \int_{t_1}^{t_2} \|u(s, \cdot)\|_{L^\infty(\omega)} ds. \quad (2.23)$$

We now define $l = T/2$, $l_1 = 3T/4$ and

$$l_{m+1} - l = 2^{-m}(l_1 - l) \quad m \geq 1.$$

We apply (2.23) with $t_1 = l_{m+1}$ and $t_2 = l_m$,

$$e^{-\frac{A}{l_m - l_{m+1}}} \|u(l_m, \cdot)\|_{L^2(\Omega)} - e^{-\frac{2A}{(l_m - l_{m+1})}} \|u(l_{m+1}, \cdot)\|_{L^2(\Omega)} \leq C \int_{l_{m+1}}^{l_m} \|u(s, \cdot)\|_{L^\infty(\omega)} ds.$$

Noticing that $\frac{2}{(l_m - l_{m+1})} = \frac{1}{(l_{m+1} - l_{m+2})}$, we get

$$e^{-\frac{A}{l_m - l_{m+1}}} \|u(l_m, \cdot)\|_{L^2(\Omega)} - e^{-\frac{A}{l_{m+1} - l_{m+2}}} \|u(l_{m+1}, \cdot)\|_{L^2(\Omega)} \leq C \int_{l_{m+1}}^{l_m} \|u(s, \cdot)\|_{L^\infty(\omega)} ds, \quad (2.24)$$

then summing the telescopic series (2.24), and using that

$$\lim_{m \rightarrow +\infty} e^{-\frac{A}{l_{m+1} - l_{m+1}}} = 0,$$

we get

$$e^{-\frac{A}{l_1 - l_2}} \|u(l_1, \cdot)\|_{L^2(\Omega)} \leq C \int_l^{l_1} \|u(s, \cdot)\|_{L^\infty(\omega)} ds.$$

From the definition of l, l_1, l_2 we immediately deduce

$$\|u(3T/4, \cdot)\|_{L^2(\Omega)} \leq C e^{C/T} \int_0^T \|u(s, \cdot)\|_{L^\infty(\omega)} ds,$$

then from the $L^2 - L^\infty$ regularizing effect of the heat equation, see for instance [CH98, Proposition 3.5.7], we obtain

$$\|u(T, \cdot)\|_{L^\infty(\Omega)} \leq C T^{-n/4} \|u(3T/4, \cdot)\|_{L^2(\Omega)} \leq C e^{C/T} \int_0^T \|u(s, \cdot)\|_{L^\infty(\omega)} ds,$$

so the quantitative observability estimate (1.7). \square

3 Observability from measurable sets with lower Hausdorff dimension

This section is devoted to study the observability of (1.1) in some particular cases when the observable sets ω has a Hausdorff dimension lower than $n - 1$. In fact, we exemplify cases of zero dimension in arbitrarily large ambient dimension.

3.1 Observability from a finite subset

To set the stage, we recall that in the case of the one-dimensional heat equation posed on $\Omega = (0, 1)$, Dolecki fully described in [Dol73] the situation when $\omega = \{x_0\}$ in terms of certain algebraic properties of the number x_0 . Dolecki's result in fact precisely describes the gap between nodal sets of the eigenfunctions and observation sets. Namely, observation points are characterized by their not being too close to a nodal set, which in Dolecki's formulation means x_0 is not well-approximated by rational numbers. In [Sam15], Samb precised this statement by showing that, in the case when x_0 is an algebraic number of order $d > 1$, the observability inequality holds at any time $T > 0$. He also provided an estimation of the observability cost, which we record for later use.

Theorem 3.1 ([Sam15]). *Let $x_0 \in (0, 1)$ be an algebraic number of order $d > 1$. There exists a positive constant $C > 0$ such that for any $T > 0$,*

$$\|u(T, \cdot)\|_{L^2(0,1)}^2 \leq C e^{\frac{C}{T}} \int_0^T |u(t, x_0)|^2 dt \quad \forall u_0 \in L^2(0, 1),$$

where u is the weak solution of (1.1).

From Samb's proof, it is clear that the result holds for every diophantine number (i.e. non Liouville number) since it only involves that an irrational algebraic number is a diophantine number. We will not use this fact in the rest of the article, so we will stick to Samb's formulation of Theorem 3.1.

A straightforward dimension-counting argument shows that such a result cannot hold on domains for which the dimension of the eigenspaces $E_r(-\Delta)$ is unbounded as $r \rightarrow \infty$, in particular when $\Omega = (0, 1)^n$ for $n \geq 2$.

Proposition 3.2. *Let $\Omega = (0, 1)^n$ with $n \geq 2$ and $T > 0$. For any finite subset $\omega \subset \Omega$, the heat equation (1.1) is not observable from ω at time T .*

This proposition shows that the results of [Dol73] and [Sam15] are very specific to the one-dimensional case. This difference with the multi-dimensional case comes from the fact that, when $n \geq 2$, the degree of degeneracy of the energy levels of the operator $-\Delta$ tends to $+\infty$. Let us recall that in this case, the spectrum of $-\Delta$ is given by

$$\text{Sp}(-\Delta) = \{r_1^2 + \dots + r_n^2; (r_1, \dots, r_n) \in (\mathbb{N}^*)^n\} \quad (3.1)$$

and for all $r \in \text{Sp}(-\Delta)$, the eigenspace $E_r(-\Delta)$ associated to r is

$$E_r(-\Delta) = \text{Span}_{\mathbb{C}} \{\varphi_{r_1} \otimes \varphi_{r_2} \otimes \dots \otimes \varphi_{r_n}; r_1^2 + \dots + r_n^2 = r\},$$

with dimension equal to the sum of squares counting function

$$s_n(r) = \#\{(r_1, \dots, r_n) \in (\mathbb{N}^*)^n; r_1^2 + \dots + r_n^2 = r\}.$$

If $n \geq 2$, then

$$\limsup_{r \rightarrow \infty} s_n(r) = \infty. \quad (3.2)$$

Indeed, for $n = 2$ one can explicitly compute $s_2(5^{2r-1}) = r$ by [Bei64, pp.140-142]. For $n \geq 2$, of course $s_n(r) \geq s_2(r)$ hence (3.2) holds.

Proof of Proposition 3.2. Let $\omega = \{x_1, \dots, x_N\} \subset (0, 1)^n$. By (3.2), we can find r such that $s_n(r) \geq N + 1$. Since $\text{Dim}(E_r(-\Delta)) = s_d(r)$, it follows that the linear application $T : f \in E_r(-\Delta) \mapsto (f(x_1), \dots, f(x_N)) \in \mathbb{R}^N$ is not one-to-one. This provides an eigenfunction ψ , associated to the eigenvalue r , such that

$$\psi(x_1) = \dots = \psi(x_N) = 0.$$

Notice that for all $t \in [0, T]$, $e^{t\Delta}\psi = e^{-rt}\psi$. This leads to

$$\int_0^T \sup_{x \in \omega} |e^{t\Delta}\psi(x)| dt = \int_0^T e^{-rt} \sup\{|\psi(x_1)|, \dots, |\psi(x_N)|\} dt = 0.$$

Since $\|\psi\|_{L^\infty(\Omega)} \neq 0$, the observability of (1.1) does not hold from ω at time $T > 0$. This ends the proof of Proposition 3.2. \square

We will construct some families on low-dimensional observation sets by taking the Cartesian product of other sets of low dimension. The first, simple principle along these lines is we can take the cartesian product of spectral inequalities; see Proposition 3.3 below. The second, more involved argument is that we can take cartesian product of an observability estimate (with good control cost) and a spectral inequality. Naturally, one may wonder if one could take the cartesian product of two observability inequalities. Proposition 3.2 above answers this question in the negative in general since $\{x_0\} \times \{x_0\} \subset (0, 1)^2$ is not an observation set, while $\{x_0\} \subset (0, 1)$ is.

3.2 Spectral inequalities for Cartesian products

To succinctly describe the spectral inequalities of the form (2.2), we say $\text{spec}(\Omega, \omega, C, \beta)$ holds if

$$\forall \Lambda > 0, \forall u \in L^2(\Omega), \quad \|\Pi_\Lambda u\|_{L^\infty(\Omega)} \leq C \exp(C\Lambda^\beta) \|\Pi_\Lambda u\|_{L^\infty(\omega)},$$

where Π_Λ is the spectral projection defined in (2.1). Here we only deal with $-\Delta$ and L^∞ estimate but in spirit these argument works for general L^p and general elliptic operators A .

Proposition 3.3. *Let $\Omega_j \subset \mathbb{R}^{n_j}$, $j = 1, 2$ and $\omega \subset \Omega_1 \times \Omega_2$. Let $\omega_2 = \pi_2(\omega)$ be the projection of ω on Ω_2 . For each $y \in \omega_2$, we denote by*

$$\omega_y := \pi_1(\omega \cap (\Omega_1 \times \{y\}))$$

the horizontal slices of ω . If there exists $C, \beta > 0$ such that $\text{spec}(\Omega_2, \omega_2, C, \beta)$ holds and for every $y \in \omega_2$, $\text{spec}(\Omega_1, \omega_y, C, \beta)$ holds, then there exists C' such that

$$\text{spec}(\Omega_1 \times \Omega_2, \omega, C', \beta), \quad \text{holds.}$$

When $\omega = \omega_1 \times \omega_2$, notice $\omega_y = \omega_1$ for all $y \in \omega_2$, thus in this case we have the corollary

$$\left(\text{spec}(\Omega_j, \omega_j, C, \beta) \text{ holds, } j = 1, 2 \right) \implies \text{spec}(\Omega_1 \times \Omega_2, \omega_1 \times \omega_2, C', \beta) \text{ holds.}$$

Proof. Let $\Lambda > 0$ and $u \in L^2(\Omega_1 \times \Omega_2)$. Let $(\varphi_k^1)_{k \in \mathbb{N}}$ (resp. $(\varphi_k^2)_{k \in \mathbb{N}}$) be an orthonormal basis of $L^2(\Omega_1)$ (resp. $L^2(\Omega_2)$) formed by eigenfunctions of $-\Delta_{\Omega_1}$ (resp. $-\Delta_{\Omega_2}$). Then

$$\Pi_\Lambda u(x, y) = \sum_{\lambda_p^1 + \lambda_q^2 \leq \Lambda} C_{p,q} \varphi_p^1(x) \varphi_q^2(y).$$

The key observation is that for any $x \in \Omega_1$ and $y \in \Omega_2$,

$$v_x = \Pi_\Lambda u(x, \cdot), \quad w_y = \Pi_\Lambda u(\cdot, y)$$

are each linear combinations of eigenfunctions of energy level less than Λ to which the spectral inequalities apply. Thus, for any $x \in \Omega_1$, using that $\text{spec}(\Omega_2, \omega_2, C, \beta)$ holds,

$$\sup_{y \in \Omega_2} |\Pi_\Lambda u(x, y)| = \sup_{\Omega_2} |v_x| \leq C \exp(C\Lambda^\beta) \sup_{\omega_2} |v_x| = C \exp(C\Lambda^\beta) \sup_{y \in \omega_2} |\Pi_\Lambda u(x, y)|.$$

And for any $y \in \omega_2$, $x \in \Omega_1$, using $\text{spec}(\Omega_1, \omega_y, C, \beta)$ holds,

$$|\Pi_\Lambda u(x, y)| = |w_y(x)| \leq C \exp(C\Lambda^\beta) \sup_{z \in \omega_y} |w_y(z)| \leq C \exp(C\Lambda^\beta) \sup_{\omega} |\Pi_\Lambda u|.$$

Combining these two displays and using the fact that $x \in \Omega_1$ was arbitrary concludes the proof. \square

3.3 Observability inequalities for Cartesian products

Proposition 3.4. *Let $T > 0$, $\omega_j \subset \Omega_j \subset \mathbb{R}^{n_j}$, $j = 1, 2$. Let $\omega_1 \subset \Omega_1$ and $\omega_2 \subset \Omega_2$. Suppose*

- (a) *there exists $C_2 > 0$ and $\beta \in (0, 1)$ such that $\text{spec}(\Omega_2, \omega_2, C_2, \beta)$ holds,*
- (b) *there exists $C_1 > 0$ such that for $\alpha = \frac{\beta}{1-\beta}$,*

$$\forall 0 < \tau \leq T, \|v(\tau, \cdot)\|_{L^2(\Omega_1)}^2 \leq C_1 \exp\left(\frac{C_1}{\tau^\alpha}\right) \int_0^\tau \sup_{\omega_1} |v(t, x)|^2 dt$$

where v is the solution of (1.1) on Ω_1 .

Then there exists $C > 0$ such that

$$\|u(T, \cdot)\|_{L^2(\Omega_1 \times \Omega_2)}^2 \leq C \exp\left(\frac{C}{T^\alpha}\right) \int_0^T \sup_{\omega_1 \times \omega_2} |u(t, x)|^2 dt, \quad (3.3)$$

where u is the solution of (1.1) on $\Omega_1 \times \Omega_2$.

Note that (3.3) differs from the functional spaces of the observability notion (1.2) in Definition 1.1 but this is because of the assumption (b). By using the $L^2 - L^\infty$ regularizing effect of the heat equation, at least the left hand side of (3.3) can be transformed into $\|u(T, \cdot)\|_{L^\infty(\Omega_1 \times \Omega_2)}^2$.

The following proof is adapted from [Sam15, Theorem 1.1] who was considering the case of $\{x_0\} \times \omega \subset (0, 1)^2$ where x_0 is an algebraic number of degree $d > 1$ and ω an open interval of $(0, 1)$. A generalization of this type of result, encompassing the case where the minimal time of null-controllability $T(x_0)$ is strictly positive, has been recently considered in the recent preprint [AKBGB⁺24].

Proof. The proof consists of applying the observability estimate on ω_1 before using spectral estimates and then performing a Lebeau-Robbiano argument type. Let $0 < \tau \leq 1$, $u_0 \in L^2(\Omega_1 \times \Omega_2)$ and $\Lambda > 0$ to be chosen later. Let us write the space variable $z = (x, y) \in \Omega_1 \times \Omega_2$ so that $\Delta = \Delta_{\Omega_1} + \Delta_{\Omega_2}$. As before, let $(\varphi_k^j)_{k \in \mathbb{N}}$ be an orthonormal basis of $L^2(\Omega_j)$, for $j = 1, 2$, consisting of eigenfunctions of $-\Delta$ with Dirichlet boundary conditions. If u is the mild solution of (1.1), we can therefore write

$$\Pi_\Lambda^2 u(t, x, y) = \sum_{\lambda_k^2 \leq \Lambda} u_k(t, x) e^{-t\lambda_k^2} \varphi_k^2(y),$$

where the u_k are solutions of the heat equation on Ω_1 and Π_Λ^j is the spectral projection (2.1) on Ω_j .

So for $y \in \Omega_2$ fixed, $v_y(t, x) := \sum_{\lambda_k^2 \leq \lambda} u_k(t, x) e^{-\tau \lambda_k^2} \varphi_k^2(y)$ is also a solution of the heat equation on Ω_1 , whence we have the observability inequality

$$\|v_y(\tau)\|_{L^2(\Omega_1)}^2 \leq C_1 \exp\left(\frac{C_1}{\tau^\alpha}\right) \int_{\frac{\tau}{2}}^{\tau} \sup_{x \in \omega_1} |v_y(t, x)|^2 dt \quad (3.4)$$

for any $y \in \Omega_2$. Furthermore, by the Cauchy-Schwarz inequality, for $x \in \omega_1$ and $t \in (\frac{\tau}{2}, \tau]$,

$$|v_y(t, x)|^2 \leq \left(\sum_{\lambda_k^2 \leq \lambda} |u_k(t, x)|^2 e^{-2t\lambda_k^2} \right) \left(\sum_{\lambda_k^2 \leq \lambda} |\varphi_k^2(y)|^2 \right).$$

By the Weyl's law, integrating the second factor over Ω_2 contributes polynomial blow-up in Λ . This is of no consequence to us since the first factor, by Parseval's identity and the spectral inequality, satisfies

$$\begin{aligned} \sup_{x \in \omega_1} \sum_{\lambda_k^2 \leq \lambda} |u_k(t, x)|^2 e^{-2t\lambda_k^2} &= \sup_{x \in \omega_1} \int_{\Omega_2} |\Pi_\Lambda^2 u(t, x, y)|^2 dy \\ &\leq C_2 \exp\left(C_2 \Lambda^\beta\right) \sup_{(x, y) \in \omega_1 \times \omega_2} |\Pi_\Lambda^2 u(t, x, y)|^2. \end{aligned}$$

Integrate (3.4) over $y \in \Omega_2$ and combine it with the above display and the identity

$$v_y(\tau, x) = \Pi_\Lambda^2 u(\tau, x, y)$$

to obtain the low-frequency observability inequality

$$\|\Pi_\Lambda^2 u(\tau)\|_{L^2(\Omega_1 \times \Omega_2)}^2 \leq C_3 \exp\left(\frac{C_1}{\tau^\alpha} + C_3 \Lambda^\beta\right) \int_{\frac{\tau}{2}}^{\tau} \sup_{(x, y) \in \omega_1 \times \omega_2} |\Pi_\Lambda^2 u(t, x, y)|^2 dt. \quad (3.5)$$

(C_3 is introduced in order to absorb the polynomial growth in Λ).

It is classical that since $u^+ := (I - \Pi_\Lambda^2)u$ satisfies the heat equation on $\Omega = \Omega_1 \times \Omega_2$, we have from the $L^2 - L^\infty$ regularizing effect of the heat equation, see for instance [CH98, Proposition 3.5.7]

$$\forall t > \frac{\tau}{2}, \quad \sup_{\Omega} |u^+(t, \cdot)| \leq \frac{C}{\tau^{\frac{n}{4}}} \left\| u^+\left(\frac{\tau}{4}, \cdot\right) \right\|_{L^2(\Omega)} \leq \frac{C}{\tau^{\frac{n}{4}}} e^{-\Lambda \frac{\tau}{4}} \|u_0\|_{L^2(\Omega)}, \quad (3.6)$$

for some positive constant $C > 0$. So by applying the triangle inequality, (3.5), triangle inequality again, followed by (3.6) to absorb the error, we obtain that for all $0 < \tau \leq 1$ and $\lambda > 0$

$$\|u(\tau)\|_{L^2(\Omega)}^2 \leq C_3 \exp\left(\frac{C_1}{\tau^\alpha} + C_3 \Lambda^\beta\right) \left[\int_{\frac{\tau}{2}}^{\tau} \sup_{\omega_1 \times \omega_2} |u(t, \cdot)|^2 dt + \tau^{-\frac{n}{4}} C \exp\left(-\frac{\tau}{4} \Lambda\right) \|u_0\|_{L^2(\Omega)}^2 \right].$$

One can find a positive parameter $\gamma > 0$ such that $\gamma \geq 10(C_1 + C_3 \gamma^\beta)$. Let us choose then $\Lambda = \frac{\gamma}{\tau^{\alpha+1}}$, noting the key relationship $\beta(\alpha + 1) = \alpha$, so that

$$\exp\left(\frac{C_1}{\tau^\alpha} + C_3 \Lambda^\beta\right) \leq \exp\left(\frac{\gamma}{10\tau^\alpha}\right), \quad \tau^{-\frac{n}{4}} \exp\left(-\frac{\tau}{4} \Lambda\right) \leq C_5 \exp\left(-\frac{\gamma}{5\tau^\alpha}\right).$$

Therefore,

$$f(\tau) \|u(\tau)\|_{L^2(\Omega)}^2 \leq \int_{\frac{\tau}{2}}^{\tau} \sup_{\omega_1 \times \omega_2} |u(t, \cdot)|^2 dt + f(\eta\tau) \|u_0\|_{L^2(\Omega)}^2, \quad (3.7)$$

with $f(\tau) = C_6 \exp\left(-\frac{\gamma}{10\tau^\alpha}\right)$, and $\eta = 2^{-\frac{1}{\alpha}} < 1$. Let us assume that $0 < T \leq 1$ and define the following sequence:

$$T_0 = T \quad \text{and} \quad \forall k \in \mathbb{N}, T_{k+1} = T_k - (1 - \eta)\eta^k T.$$

By applying (3.7) with initial data $u_0 = u(T_{k+1})$ and running until time T_k (τ then represents the elapsed time $T_k - T_{k+1} = (1 - \eta)\eta^{k+1}T$), we obtain

$$f\left((1 - \eta)\eta^k T\right) \|u(T_k)\|_{L^2(\Omega)}^2 \leq \int_{T_{k+1}}^{T_k} \sup_{\omega_1 \times \omega_2} |u(t, \cdot)|^2 dt + f\left((1 - \eta)\eta^{k+1} T\right) \|u(T_{k+1})\|_{L^2(\Omega)}^2.$$

Since $f\left((1 - \eta)\eta^k T\right) \rightarrow 0$ as $k \rightarrow \infty$, we can rearrange the above display so the LHS is a telescoping series, yielding

$$f\left((1 - \eta)T\right) \|u(T)\|_{L^2(\Omega)}^2 \leq \int_0^T \sup_{\omega_1 \times \omega_2} |u(t, \cdot)|^2 dt,$$

leading to (3.3) recalling the form of f . □

3.4 Examples

As demonstrated from the results of [Dol73] and [Sam15], there are zero-dimensional observation sets in dimension 1. We will now use Propositions 3.3 and 3.4 to supply examples in higher dimensions. Combining Theorem 2.1, Theorem 3.1, and Proposition 3.4 exhibits observation sets of dimension $n - 2 + \delta$.

Corollary 3.5. *Let $T > 0$, $\delta > 0$ and $\Omega = (0, 1) \times \Omega'$ where $\Omega' \subset \mathbb{R}^{n-1}$. Let $x_0 \in (0, 1)$ be an algebraic number of order $d > 1$ and $\omega \subset \Omega'$ be a measurable subset satisfying $C_{\mathcal{H}}^{n-2+\delta}(\omega) > 0$. The heat equation (1.1) is observable from $\{x_0\} \times \omega$ at time T .*

As we generate sets of even smaller dimension we make the following comment. Since the eigenfunctions of $-\Delta$ are continuous, $\Pi_\Lambda u$ is always continuous which entails

$$\sup_{x \in E} |\Pi_\Lambda u(x)| = \sup_{x \in \bar{E}} |\Pi_\Lambda u(x)|. \quad (3.8)$$

Therefore, to generate nontrivial examples, we should restrict our attention to sets whose closure is low-dimensional (or to closed sets). An example to motivate this restriction is the zero-dimensional set $\omega = \mathbb{Q}^d \cap \Omega$ for which the spectral inequality trivially holds by (3.8). We now introduce a family of zero-dimensional sets, indexed by $\alpha > 0$,

$$\omega_\alpha = \{i^{-\alpha} : i = 1, 2, \dots\}.$$

We have the following crude estimate for $\gamma_k(\omega_\alpha)$ (recall γ_k from (2.9)) which suffices for our purposes.

Lemma 3.6. *For each $\alpha > 0$ there exists $c_1, c_2 > 0$ such that*

$$\gamma_k(\omega_\alpha) \geq c_1 e^{c_2 k \log k}. \quad (3.9)$$

Proof. By simply selecting the first k terms of the sequence $(i^{-\alpha})_{i \in \mathbb{N}^*}$, we have

$$\gamma_k(\omega_\alpha) \geq \inf_{i=1, \dots, k} \prod_{\substack{j=1 \\ j \neq i}}^k |j^{-\alpha} - i^{-\alpha}|.$$

Rewrite

$$\prod_{\substack{j=1 \\ j \neq i}}^k |j^{-\alpha} - i^{-\alpha}| = \prod_{\substack{j=1 \\ j \neq i}}^k j^{-\alpha} i^{-\alpha} |j^{\alpha} - i^{\alpha}|.$$

Obviously $|j^{\alpha} - i^{\alpha}|$ is minimized when j is close to i so by the mean value theorem $|j^{\alpha} - i^{\alpha}| \gtrsim i^{\alpha-1}$. Therefore,

$$\gamma_k(\omega_{\alpha}) \gtrsim \inf_{i=1, \dots, k} \prod_{j=1}^k j^{-\alpha} i^{-1} = \inf_{i=1, \dots, k} (k!)^{-\alpha} i^{-k}.$$

Clearly this infimum is attained when $i = k$ and then the form (3.9) follows by Stirling's formula. \square

By combining Proposition 2.3 and Lemma 3.6 together with the methodology of the proof of Theorem 2.1 using Proposition 2.2, we obtain the following corollary.

Corollary 3.7. *For any $\beta \in (\frac{1}{2}, 1)$ there exists $C > 0$ such that $\text{spec}((0, 1), \omega_{\alpha}, C, \beta)$ holds.*

Finally, from Corollary 3.7 and Proposition 3.4 we conclude that the heat equation is indeed observable from $\{x_0\} \times \omega_{\alpha}$, which is a zero-dimensional set. In fact that the cartesian product $\omega_{\alpha} \times \dots \times \omega_{\alpha}$ is countable and therefore must have Hausdorff dimension zero.

Corollary 3.8. *Let $\alpha, T > 0$, $n \geq 2$, and x_0 an algebraic number of degree greater than one. Then the heat equation (1.1) on $\Omega = (0, 1)^n$ is observable from $\{x_0\} \times \omega_{\alpha} \times \dots \times \omega_{\alpha}$ at time T .*

Another indication of the smallness of this observation set is that when $n = 2$ it only has one accumulation point (which is the minimal number in a bounded domain), namely

$$(x_0, 0),$$

and the set becomes even more concentrated there as $\alpha \rightarrow \infty$. To this end we pose the following open question.

Question 3.9. *Is the heat equation (1.1) on $\Omega = (0, 1)^2$ observable from*

$$\{x_0\} \times \left\{ e^{-k} : k = 1, 2, \dots \right\}$$

for some point $x_0 \in (0, 1)$?

Is is relatively straightforward to check that for

$$\omega_{\text{exp}} = \left\{ e^{-k} : k = 1, 2, \dots \right\}, \quad \gamma_k(\omega_{\text{exp}}) \sim e^{-k^2},$$

which does not yield $\text{spec}((0, 1), \omega_{\text{exp}}, C, \beta)$ except when $\beta = 1$. To conclude observability of the heat equation of course we need $\beta < 1$.

Second, we were unable to prove Proposition 3.4 in the full generality of Proposition 3.3 using sections and projections of general non-product-type observation sets. If one were able to do so then a large family of curves in $(0, 1)^2$ would be observation sets. Of course, Corollary 3.5 shows easily that a line parallel to an axis with distance x_0 from that axis is indeed an observation set. But for any line not parallel to one of the axes, we do not know if the heat equation is observable. Our desired extension of Proposition 3.4 discussed above would prove this, but we do not know if such an extension is possible.

Question 3.10. *Is the heat equation on $(0, 1)^2$ observable from any line not parallel to one of the axes?*

A Another proof of the spectral estimate

The goal of this part is to present another proof of Theorem 2.1 that is more in the spirit of [BM23] starting from Malinnikova's result on the propagation of smallness for solutions to generalized Cauchy-Riemann systems that are factor of the Laplacian.

Lemma A.1. *Let $\delta > 0$, $\mathcal{K} \subset\subset B_1$, $\mathcal{E} \subset\subset B_1$ be a subset of an hyperplane such that $\mathcal{C}_{\mathcal{H}}^{n-2+\delta}(\mathcal{E}) > 0$. Then, there exist $C = C(\mathcal{K}, \mathcal{E}, \delta) > 0$ and $\alpha = \alpha(\mathcal{K}, \mathcal{E}, \delta) \in (0, 1)$ such that for every $u : B_1 \rightarrow \mathbb{R}$ satisfying $\Delta u = 0$ in B_1 then*

$$\sup_{x \in \mathcal{K}} |\nabla u(x)| \leq C \sup_{x \in \mathcal{E}} |\nabla u(x)|^\alpha \|\nabla u\|_{L^\infty(B_1)}^{1-\alpha}. \quad (\text{A.1})$$

In other words, Lemma A.1 tells us that we are able to propagate smallness for gradient of harmonic functions from sets of Hausdorff dimension $d - 2 + \delta$ for every $\delta > 0$. Note that [LM18] obtains the same result for A -harmonic function with A satisfying elliptic and Lipschitz assumptions and for measurable sets \mathcal{E} satisfying $\mathcal{C}_{\mathcal{H}}^{n-1-\delta}(\mathcal{E}) = m$ for some $\delta > 0$ a priori small. As we will see during the proof of , Lemma A.1 is a straightforward corollary of [Mal04, Main Result]. Finally, as stated in [Mal09, Section 3.2], one can certainly replace the standard Laplace operator Δ in Lemma A.1 by a second order uniformly elliptic operator L with analytic coefficients and propagation of smallness of gradient of solutions to $Lu = 0$ still hold. This will explain why Theorem 2.1 still hold for $L = \text{div}(A\nabla \cdot)$ with A , a matrix-valued uniformly elliptic with real-analytic entries.

Proof. Let us remark that

$$F = (F_1, \dots, F_n) = \nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right),$$

is a solution to the following system

$$\sum_{j=1}^n \frac{\partial F_j}{\partial x_j} = 0, \quad \frac{\partial F_j}{\partial x_k} = \frac{\partial F_k}{\partial x_j}, \quad 1 \leq j < k \leq n. \quad (\text{A.2})$$

Indeed, the first equation of (A.2) corresponds to $\text{div}(\nabla u) = 0$ while the second corresponds to Schwarz's theorem on mixed derivatives by using the fact that because u is harmonic on B_1 then it is C^∞ -smooth on B_1 . By defining the operator A as

$$A : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad AF = \left(\sum_{j=1}^n \frac{\partial F_j}{\partial x_j}, \left(\frac{\partial F_j}{\partial x_k} - \frac{\partial F_k}{\partial x_j} \right)_{1 \leq j < k \leq n} \right), \quad (\text{A.3})$$

with $m = 1 + (n(n-1))/2$. The last equations of $AF = 0$ imply by Poincaré lemma that F is locally a gradient. Then the first equation of $AF = 0$ implies that F is locally a gradient of a harmonic function so each component of F is harmonic then the system of equations (A.2) is a generalized Cauchy-Riemann system in the sense of [Mal04, Section 4.1]. Moreover, the operator A defined in (A.3) is a factor of the Laplacian in the sense of [Mal04, Definition 4.1] because

$$\begin{aligned} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^n \frac{\partial F_j}{\partial x_j} \right) - \sum_{k < i \leq n} \frac{\partial}{\partial x_k} \left(\frac{\partial F_k}{\partial x_i} - \frac{\partial F_i}{\partial x_k} \right) + \sum_{i < k \leq n} \frac{\partial}{\partial x_k} \left(\frac{\partial F_i}{\partial x_k} - \frac{\partial F_k}{\partial x_i} \right) \\ = \Delta F_i, \quad 1 \leq i \leq n. \end{aligned}$$

Moreover, Lemma 2.4 furnishes that $\mathcal{C}_{\mathcal{H}}^{n-2+\delta}(\mathcal{E}) > 0 \Rightarrow \text{cap}_{n-2+\delta/2}(\mathcal{E}) > 0$ so one can apply directly the propagation of smallness result from [Mal04], i.e. [Mal04, Main Result] to $F = \nabla u$ to get (A.1). \square

With Lemma A.1 at hand, we can prove Theorem 2.1.

Proof of Theorem 2.1. We split the proof into two steps.

Step 1: Observation on a suitable ball. Let $\Lambda \geq 1$ and let us define

$$u = \Pi_\Lambda u. \quad (\text{A.4})$$

First, let us take $B(x_0, r)$ be such that $B(4r, x_0) \subset \Omega$ and

$$m := \mathcal{C}^{d-2+\delta}(\omega \cap B(x_0, r)) > 0.$$

Such a ball exists because $\mathcal{C}^{d-2+\delta}(\omega) > 0$. Then we have from [AEWZ14, Theorem 3],

$$\|u\|_{L^2(\Omega)} \leq C e^{C\sqrt{\Lambda}} \sup_{x \in B(x_0, r)} |u(x)|. \quad (\text{A.5})$$

Note that we have used [AEWZ14] instead of the more classical reference [JL99] because we are only assuming that Ω is a bounded Lipschitz domain, locally star-shaped. Moreover, we have

$$-\Delta u = \sum_{\lambda_k \leq \Lambda} \lambda_k \langle u, \phi_k \rangle \phi_k \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (\text{A.6})$$

so by elliptic regularity and Sobolev embeddings we have

$$\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{H^d(\Omega)} \leq C e^{C\sqrt{\Lambda}} \|u\|_{L^2(\Omega)}, \quad (\text{A.7})$$

hence by (2.6),

$$\|u\|_{L^\infty(\Omega)} \leq C e^{C\sqrt{\Lambda}} \sup_{x \in B(x_0, r)} |u(x)|. \quad (\text{A.8})$$

Step 2: Observation on ω . Let us take

$$\hat{u}(x, t) = \sum_{\lambda_k \leq \Lambda} u_k \frac{\sinh(\sqrt{\lambda_k} t)}{\sqrt{\lambda_k}} \phi_k(x) \quad (t, x) \in (-1, +1) \times \Omega. \quad (\text{A.9})$$

One can check that

$$\Delta_{t,x} \hat{u} = 0 \quad (t, x) \in (-1, +1) \times \Omega. \quad (\text{A.10})$$

Up to a translation, let us assume that $x_0 = 0$ and reducing $r > 0$ if necessary, let us consider B_1 the unit ball in \mathbb{R}^{n+1} and $\mathcal{K} \subset \mathbb{R}^{n+1}$ such that

$$B(x_0, r) \times (-1/2, +1/2) \subset \subset \mathcal{K} \subset \subset B_1 \subset \subset \Omega \times (-1, +1),$$

and

$$\mathcal{E} = (\omega \cap B(x_0, r)) \times \{0\} \subset \mathbb{R}^n \times \{0\}.$$

Then we can apply the propagation of smallness for gradients of harmonic functions i.e. (A.1) from Lemma A.1,

$$\sup_{(x,t) \in \mathcal{K}} |\nabla \hat{u}| \leq C \sup_{(x,t) \in \mathcal{E}} |\nabla \hat{u}|^\alpha \|\nabla \hat{u}\|_{L^\infty(B)}^{1-\alpha}. \quad (\text{A.11})$$

In particular we get

$$\sup_{x \in B(x_0, r)} |\partial_t \hat{u}(x, 0)| \leq C \left(\sup_{x \in \omega} |\partial_t \hat{u}(x, 0)| \right)^\alpha \|\hat{u}\|_{W_{t,x}^{1,\infty}(B)}^{1-\alpha}. \quad (\text{A.12})$$

The left hand side of (A.12) exactly gives

$$\sup_{x \in B(x_0, r)} |\partial_t \hat{u}(x, 0)| = \sup_{x \in B(x_0, r)} |u(x)|. \quad (\text{A.13})$$

The first right hand side term of (A.12) exactly gives

$$\sup_{x \in \omega \cap B(x_0, r)} |\partial_t \hat{u}(x, 0)| = \sup_{x \in \omega \cap B(x_0, r)} |u(x)| \leq \sup_{x \in \omega} |u(x)|. \quad (\text{A.14})$$

Then by elliptic regularity estimates on (A.10), the second right hand side term of (A.12) can be estimated as

$$\|\hat{u}\|_{W_{t,x}^{1,\infty}(B)} \leq C \|\hat{u}\|_{L^2((-1,+1) \times \Omega)} \leq C e^{C\sqrt{\lambda}} \|u\|_{L^\infty(\Omega)}. \quad (\text{A.15})$$

Therefore, we have from (A.12), (A.13), (A.14) and (A.15),

$$\sup_{x \in B(x_0, r)} |u(x)| \leq C e^{C\sqrt{\lambda}} (\sup_{x \in \omega} |u(x)|)^\alpha \|u\|_{L^\infty(\Omega)}^{1-\alpha}. \quad (\text{A.16})$$

We conclude the proof of (2.2) by gathering (A.8) and (A.16). This ends the proof of Theorem 2.1. \square

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