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From Slot Mereology To A Mereology Of Slots

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Abstract. In 2013, Bennett proposed a mereological theory in which the parthood relation is defined on the basis of two primitive relations: *a* is a part of *b* iff *a* fills a slot owned by *b*. However, this theory has issues counting how many parts an entity has. We explore the various counting problems and propose a new theory to solve them. Keeping the core idea of Bennett's slots, this theory introduces mereological relations between slots. This theory enables us to solve all known counting problems and to go beyond the limits of Bennett's theory by theorising expected features of mereological theories: supplementation principles and mereological sum and fusion. The theory is illustrated on ontological issues on the nature of structural universals and informational entities.

Keywords: mereology, slot mereology, supplementation, mereological sum, mereological fusion, structural universal, informational entity

1. Introduction

Bennett (2013) proposed a new mereological theory that allows a whole to have the same part multiple times. In this theory, the parthood relation is analysed in terms of two relations: *having a slot*, and *filling a slot*. More precisely, *a* is a part of *b* iff *a* fills a slot of *b*. Thus, *b* can have the same part *a* several times by *a* filling several slots of *b*. Bennett motivated her theory by a problem, exposed by Lewis (1986), that appears with *structural universals*, namely universals composed of other universals. Indeed, a structural universal can have the same universal as a part multiple times. Bennett illustrates her theory by using the example of the methane molecule universal, following Lewis (1986). In her theory, the methane molecule universal CH₄ can have the hydrogen atom universal H as a part four times, reflecting the structure of methane molecules. Structural universals have been debated by, among others, Armstrong (1986), Bigelow (1986), Fisher (2018), Masolo and Vieu (2018) and Garbacz (2020), in articles discussing mereological theories for structural universals, including Bennett's slot mereology. Fisher (2013) and Garbacz (2016) offered the principal analyses of slot mereology following Bennett's seminal work. Note, however, that slot mereologies inspired by Bennett's work are not restricted to structural universals and can be applied to other entities, such as informational entities (see (Barton et al., 2020a), (Barton et al., 2020b) and (Barton et al., 2022)). In some of her examples, Bennett even suggested to apply it to material particulars. This theory is inspired by a role-based analysis in which

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the slots would be identified with roles and the fillers with the role-players, although Bennett does not propose a complete account of roles in terms of slot mereology and does not exploit any well-developed formal ontology of roles (such as (Masolo et al., 2004) or (Loebe, 2007)).¹

In earlier work (Tarbouriech et al., 2021), we revised Bennett’s theory. However, this proposal had problems that we will explore and fix here. Some parts of the present paper are adapted from this previous paper (most of Sections 2 and 3 with a notable exception for Section 3.2.3).

Despite its aim to account for having a part multiple times, as we will show, Bennett’s proposal has counter-intuitive implications involving counting parts. As we will see in Section 3, whatever counting criterion we choose, there are cases for which we get unwanted results, that is, results that do not follow the isomorphism principle between a universal and its instances expressed by Lewis (1986).² We will thus propose a radical revision and an extension of Bennett’s theory that solves such problems. We will illustrate our theory using the METHANE example, occasionally completed by other examples of structural universals and informational entities when needed.

In Section 2, we present Bennett’s theory and use this theory to represent the structure of the METHANE universal. The problems of this theory and of our previous theory are exposed in Section 3. We explore the three identified problems with various examples. Finally, we develop a new solution in Section 4 and show some examples in Section 5. Sections 6 and 7 end this paper with a discussion and a conclusion.

The theorems presented in this paper have been proved using Coq, a proof assistant (for more details, see (Bertot and Casteran, 2004)).³

2. Slot Mereology

Slot mereology separates the parthood relation into two primitive relations: *being a parthood slot of* and *filling a parthood slot*, respectively named P_s and F . As we have just seen, this theory was motivated by the possibility of a whole having the same part multiple times. For example, a structural universal can have the same universal as a part multiple times (e.g. CH_4 having H as a part four times), or an informational entity particular⁴ can have the same informational entity particular as a part multiple times (e.g. the phrase ‘a cat and a dog’ having the word ‘a’ as a part twice). Thus, one would expect this theory to enable a correct count of the number of appearances of each part (even if Bennett does not discuss countability). However, the system fails in that respect, as we will see. Let us first present the system.

¹Appealing to roles in order to represent mereological structure appears in medieval philosophy. Martin (2019) shows how Abaelard, in his various texts, defines the notion of principal part as a part that plays a “functional and structural role in the constitution” of a whole.

²See Section 3 and (Fisher, 2018) for more on the isomorphism principle.

³The Coq file is available at https://gitlab.irit.fr/melodi/tarbouriech/-/blob/main/slot_mereology.v.

⁴The point of view on informational entities adopted here follows the work of Barton et al. (2020a), Barton et al. (2020b) and Barton et al. (2022). Thus, we differentiate the informational entity particular from its concretisations, for instance, when written on paper. The informational entity particulars discussed here (letters, words, sentences) share many properties with universals of non-informational entities - in particular, they can be multi-localised.

2.1. Presentation of the System

Bennet's theory is based on the two primitive relations P_s and F . Five relations are defined on this basis, as presented in Table 1. The definitions, axioms and theorems from Bennett's paper (2013) are respectively identified by "BD", "BA" and "BT", followed by a number.

Table 1
Slot Mereology Definitions

Number	Name	Definition
BD1	Parthood	$P(a, b) \triangleq \exists s(P_s(s, b) \wedge F(a, s))$
BD2	Proper Parthood	$PP(a, b) \triangleq P(a, b) \wedge \neg P(b, a)$
BD3	Overlap	$O(a, b) \triangleq \exists c(P(c, a) \wedge P(c, b))$
BD4	Slot-overlap ⁵	$O_s(a, b) \triangleq \exists s(P_s(s, a) \wedge P_s(s, b))$
BD5	Proper Parthood Slot	$PP_s(s, a) \triangleq P_s(s, a) \wedge \neg F(a, s)$

Eight axioms constrain the system, presented in Table 2. Axiom BA5 makes slots inheritable.

Table 2
Slot Mereology Axioms

Number	Description	Axiom
BA1	Only Slots are Filled	$F(a, s) \rightarrow \exists b(P_s(s, b))$
BA2	Slots Cannot Fill	$F(a, s) \rightarrow \neg \exists b(P_s(a, b))$
BA3	Slots Don't Have Slots	$P_s(s, a) \rightarrow \neg \exists t(P_s(t, s))$
BA4	Improper Parthood Slots	$\exists s(P_s(s, a)) \rightarrow \exists t(P_s(t, a) \wedge F(a, t))$
BA5	Slot Inheritance	$[P_s(s, b) \wedge F(a, s) \wedge P_s(t, a)] \rightarrow P_s(t, b)$
BA6	Mutual Occupancy is Identity	$(P_s(s, b) \wedge F(a, s)) \wedge (P_s(t, a) \wedge F(b, t)) \rightarrow a = b$
BA7	Single Occupancy ⁶	$P_s(s, a) \rightarrow \exists! b(F(b, s))$
BA8	Slot Strong Supplementation	$\exists s(P_s(s, a)) \wedge \exists t(P_s(t, b)) \rightarrow [\neg(\exists u(P_s(u, a) \wedge F(b, u))) \rightarrow \exists v(P_s(v, b) \wedge \neg P_s(v, a))]$

If s is a slot of a , we say that s is a "direct slot" of a iff there is no proper part b of a such that s is also a slot of b . If s is a slot of a that is not a direct slot, we call it a "non-direct slot". Note that in the absence of any discreteness axiom, direct slots are not guaranteed.

Definition 1 (Direct Slot). $DP_s(s, a) \triangleq P_s(s, a) \wedge \neg \exists b(PP(b, a) \wedge P_s(s, b))$

We will call "filler" an entity that fills a slot. Axiom BA4 implies that anything that has at least one slot is also a filler. At this point, we can make some remarks:

- Being a proper slot (Definition BD5) is not an intrinsic property of slots, but a relational property: the same slot can be a proper slot of a filler and an improper slot of another filler.

⁵The relation is called *SO* in Bennett's theory.

⁶A typographical mistake in Bennett's paper has been corrected, following Garbacz (2016).

- Even though every whole that has a proper part also has an improper slot (Axiom BA4), the converse is not true: a filler can have an improper slot without having any proper parts.
- Although Bennett's paper does not mention this possibility, nothing in her system prevents a filler from having several improper slots.
- There are no axioms stating that everything is either a slot or a filler. So there can be other things.

Table 3 presents Bennett's theorems that are used in this paper.

Table 3
Slot Mereology Theorems

Number	Description	Theorem
BT7	Transitivity	$P(a, b) \wedge P(b, c) \rightarrow P(a, c)$
BT8	Anti-Symmetry	$P(a, b) \wedge P(b, a) \rightarrow a = b$
BT9	Conditional Reflexivity	$\exists s(P_s(s, a)) \rightarrow P(a, a)$
BT13	Slot Weak Supplementation	$PP(a, b) \rightarrow \exists s(P_s(s, b) \wedge \neg P_s(s, a))$
BT14	Slot Extensionality ⁶	$\exists c(PP(c, a)) \vee \exists d(PP(d, b)) \rightarrow [a = b \leftrightarrow \forall s(PP_s(s, a) \leftrightarrow PP(s, b))]$

2.2. Representing the Methane Molecule

Any methane molecule particular is composed of five atoms: one carbon and four hydrogens, each bound to the carbon atom. Moreover, any carbon atom particular has six electrons, and each hydrogen atom particular has one electron.⁷ Using slot mereology, the universal METHANE is described as having five proper slots, one for the CARBON universal and four for the HYDROGEN universal. The current structure is described by Facts (1)⁸ and pictured in Figure 1, where $a \leftarrow \overset{s}{\square}$ represents $P_s(s, a)$, $\overset{s}{\square} a$ represents $F(a, s)$ and $\overset{s}{\square} a \leftarrow \overset{s}{\square}$ represents $P_s(s, a) \wedge F(a, s)$ (the same notation will be used in following figures). Note that Axiom BA4 entails that S_0 exists and is an improper slot of METHANE.

$$\begin{array}{ll} P_s(S_i, \text{METHANE}) & 0 \leq i \leq 5 \\ F(\text{METHANE}, S_0) & F(\text{HYDROGEN}, S_i) \quad 2 \leq i \leq 5 \\ & F(\text{CARBON}, S_1) \end{array} \quad (1)$$

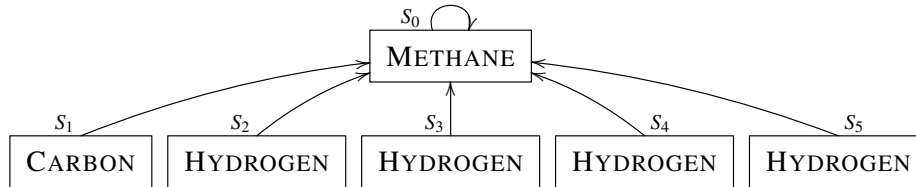


Fig. 1. Representation of METHANE and its parts, CARBON and HYDROGEN

⁷Note that atoms also have other parts, such as nuclei, but for simplicity, we only represent electrons in this example. The reasoning we will develop for electrons also applies to other parts such as nuclei. Furthermore, the specific problems of identity raised by quantum mechanics are ignored (see (French, 2019)).

⁸The formulas that use index i are compact notations for several formulas. This notation is used in the remainder of the paper.

Let us represent the electrons of each atom. Each carbon atom has six electrons, and each hydrogen atom has one. So, in our case, the universal CARBON owns six slots filled with ELECTRON, whereas the universal HYDROGEN owns only one, as described by Facts (2).

$$\begin{aligned} & F(\text{ELECTRON}, S_{12}) \wedge P_s(S_{12}, \text{HYDROGEN}) \\ & F(\text{ELECTRON}, S_i) \wedge P_s(S_i, \text{CARBON}) \quad 6 \leq i \leq 11 \end{aligned} \quad (2)$$

One thing to keep in mind is the fact that even if there are four slots of METHANE filled by HYDROGEN, there is only one universal of HYDROGEN, which has only one slot filled by ELECTRON. Bennett's Axiom BA5 states that wholes inherit slots from their parts. In our case, this means that METHANE inherits from CARBON and HYDROGEN their slots filled by ELECTRON. METHANE inherits six slots from CARBON and only one slot from HYDROGEN. Considering that METHANE has no other slots filled by ELECTRON, METHANE has in total seven slots filled by ELECTRON.

According to Axioms BA4 and BA5, there are two additional slots, called S_{13} and S_{14} , that are improper slots of CARBON and HYDROGEN, respectively (see Facts (3)). For readability, we do not present the full mereological structure of METHANE here. Nonetheless, other examples are presented, with a full representation of the mereological structure, in Section 5.

$$\begin{aligned} & P_s(S_{13}, \text{CARBON}) \quad P_s(S_{14}, \text{HYDROGEN}) \\ & F(\text{CARBON}, S_{13}) \quad F(\text{HYDROGEN}, S_{14}) \end{aligned} \quad (3)$$

Regarding improper slots S_{13} and S_{14} , there are two possibilities: either those two slots are different from the ones previously mentioned, or some of them are identical to some of the previously mentioned slots. In this example, we chose the first possibility: S_{13} and S_{14} are different from all the other slots. All the slots of CARBON and HYDROGEN are inherited by METHANE, due to Axiom BA5: $P_s(S_i, \text{METHANE})$ ($6 \leq i \leq 14$). Note that it is also the case for improper slots.

3. Counting the Parts

3.1. Counting Criteria

“Counting how many times filler A has filler B as a part” means counting the number of appearances of B in A. But what counts as a genuine appearance? As Bennett's theory includes improper slots, we can define two counting criteria. The first counting criterion, **C1**, enables us to count the number of different slots owned by A that are filled by B, whether they are also owned by B (and are thus improper slots of B) or not. The second counting criterion, **C2**, enables us to count the number of different slots owned by A that are filled by B and that are not owned by B (that is, that are not improper slots of B). Note that it means that, according to **C2**, A is part of itself zero times, even though A has improper slots. Therefore **C2** is only relevant when counting proper parts.

The results obtained with the two criteria will be compared to the result obtained when analysing a methane molecule particular, following the isomorphism principle proposed by Lewis (1986). A methane

molecule has as parts one carbon atom particular, four hydrogen atom particulars and ten electron particulars.⁹ Thus, we expect from our mereological theory and counting criterion to lead to the methane molecule universal METHANE having as parts the carbon atom universal CARBON once, the hydrogen atom universal HYDROGEN four times and the electron universal ELECTRON ten times.

3.2. Counting Problems

Bennett's theory raises two issues concerning the countability of parts: the first one stems from the existence of improper slots, and a second one from slots of parts. Ultimately, as we will see, both are caused by the Slot Inheritance Axiom BA5. Finally, we will show that the solution we proposed earlier (Tarbouriech et al., 2021), besides fixing original counting problems, added a new problem.

3.2.1. Improper Slot Problem

Let us say we want to count the number of times HYDROGEN is part of METHANE. METHANE has five different slots filled with HYDROGEN: four direct slots (S_2 to S_5) and one inherited slot (S_{14}), the latter being the improper slot of HYDROGEN itself. HYDROGEN is part of METHANE five times according to **C1** and four times according to **C2**.¹⁰ By comparing those results with the expected result when we count how many hydrogen particulars belong to a particular of methane molecule, we can state that criterion **C2** leads to a correct result, whereas criterion **C1** leads to an incorrect result.

Among the possible models of slot mereology, Figure 2 illustrates three models worthy of interest. In these models, b is proper part of a (thus, they are distinct). Here is an informal description of the three models:

- (2i) a has only one slot s that is filled by b and this slot is not owned by b ;
- (2ii) a has only one slot s that is filled by b and this slot is also owned by b (and thus, is an improper slot of b);
- (2iii) a has exactly two different slots s and t that are filled by b . One of these slots (say t) is also owned by b .

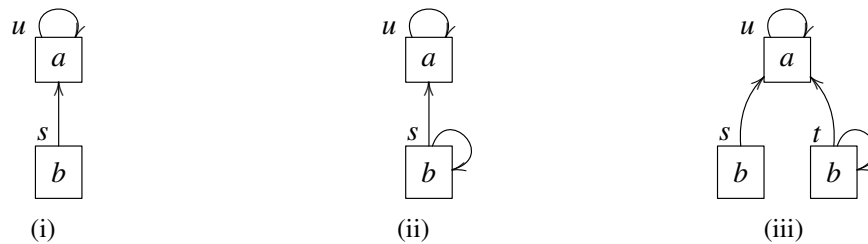


Fig. 2. Three possible models of slot mereology

From the facts represented in Figure 2, we can deduce the following facts:

⁹As noticed by a reviewer, in some common understanding of chemistry, only atoms are considered as bona fide parts of molecules, and electrons are only considered when analysing bonding between atoms. The chemistry examples of this paper mainly serve to illustrate the formal theory, in particular parthood transitivity, and should not be considered as providing a detailed account of mereology in chemistry. If the ELECTRON universal is not to be considered as a part of molecule universals, further work might be needed to address parthood transitivity issues among universals, similarly as when applying classical mereology to represent parthood relations among concrete particulars (see, e.g., (Vieu, 2006)).

¹⁰Different results from the ones presented here are possible if a different representation of the slot structure of the methane molecule is used, as discussed below regarding models 2ii and 2iii in Figure 2.

- in model 2i, according to both **C1** and **C2**, b is part of a once;
- in model 2ii, by **C1**, b is part of a once. However, by **C2**, b is part of a , but zero times;
- in model 2iii, by **C1**, b is part of a twice. However, by **C2**, b is part of a once.¹¹

According to the counting criteria, we get different results for models (2ii) and (2iii). Since it is obviously absurd for b to be part of a , but zero times, **C2** is inappropriate. Thus, we have showed that neither **C1** (by the METHANE example) nor **C2** (by examples of Figure 2) can be compatible with Bennett's theory. Therefore, we will investigate how Bennett's theory can be fixed so that parts can be counted correctly. But we first need to explore a second, related, issue.

3.2.2. Parts of Parts Problem

The second problem stems from the parts of the parts. Let us say we want to count how many times ELECTRON is part of METHANE. If we do so, we will find six slots inherited from CARBON (namely $S_6, S_7, S_8, S_9, S_{10}, S_{11}$) and one slot inherited from HYDROGEN (namely S_{12}), that is, seven slots in total. **C1** would thus lead to METHANE having ELECTRON as a part seven times, whereas **C2** would lead to it having ELECTRON as a part at most seven times (depending on whether some of those slots also are improper slots of ELECTRON, see model (2iii) in Figure 2). This result is different from the expected result of ELECTRON being part of METHANE ten times.

3.2.3. Problems Of Copy-Slot Mechanism

In earlier work (Tarbouriech et al., 2021) we removed Bennett's Slot Inheritance BA5, adopted an anti-inheritance axiom, clarified the axiomatisation of improper slots, and developed the "copy-slot mechanism" as a solution for the two counting problems. The idea we developed in that work is that slots were no longer inherited but rather that the mereological structure of fillers was duplicated every time the filler occupies a slot, i.e., the slots of the filler were duplicated to be slots of the whole. The original slot and the duplicated slot have the same content: only their owners change. We introduced two relations *copied_from* and *copied_through* to implement this copy mechanism. Consider the structure of the dihydrogen, namely a molecule composed of two hydrogen atoms. The representation in Figure 3i, based on Bennett's theory alone, is not satisfactory, as it leads to an incorrect count of the number of occurrences of the ELECTRON and HYDROGENNUCLEUS¹² universal, as we have just seen for METHANE. The structure pictured in Figure 3ii is a model of the theory that encompasses the copy-slot mechanism. The structure of the HYDROGEN universal (namely slots S_3 and S_4) is repeated twice: once for each of the slots in which the HYDROGEN universal occur (slots S_1 and S_2). By further constraining improper slots, we fixed the first problem (*Improper Slot Problem*). With the copy-slot mechanism, we fixed the second counting problem (*Parts of Parts Problem*).

However, the copy-slot mechanism has a flaw. This flaw leads to an excessive duplication of slots, and by extension to a wrong count of parts. Figures 4i and 4ii represent the same structure of the string "cats". The first one does not have copy-slots, the second does.¹³ By looking at the second figure, we see that "cats" owns two slots S_7 and S_{13} filled by "c", whereas it should only have one. This is because of the multiple levels in the mereological structure: S_7 is a slot copied from the structure of "cat", while S_{13} is copied from the structure of "ca". The copy-slot mechanism misses a feature ensuring

¹¹ s and t are filled by the very same universal. In contrast to Bennett's figures in which slots can be drawn one inside another, we chose to separate them, even if it means repeating the filler.

¹²Here, we will not describe the mereological structure of HYDROGENNUCLEUS and consider it as a mereological atom.

¹³Only entities useful for the current reasoning are pictured in the figures. It is possible to consider other entities, such as "at" or "ct", but they are not necessary to expose the problem.

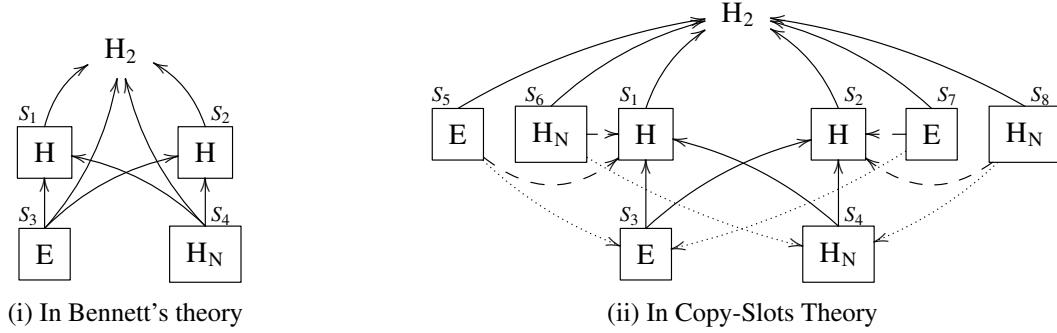


Fig. 3. The mereological structure of DIHYDROGEN represented in Bennett's and Copy-Slots theories. H_2 is DIHYDROGEN, H is HYDROGEN, H_N is the HYDROGENNUCLEUS and E is ELECTRON. (A dotted arrow represents the *copied_from* relation. A dashed arrow represents the *copied_through* relation.)

unicity. As a solution, we will propose a new system. Even though the relations *copied_from* and *copied_through* are discarded, the core idea of the copy-slot mechanism is preserved. This new system also works by considering that the mereological structure of each filler should be duplicated as many times as necessary, as we will see shortly.

$$\begin{array}{llll}
 P_s(S_3, \text{cat}) & F(c, S_3) & \text{copied_from}(S_3, S_1) & \text{copied_through}(S_3, S_5) \\
 P_s(S_4, \text{cat}) & F(a, S_4) & \text{copied_from}(S_4, S_2) & \text{copied_through}(S_4, S_5) \\
 P_s(S_7, \text{cats}) & F(c, S_7) & \text{copied_from}(S_7, S_3) & \text{copied_through}(S_7, S_9) \\
 P_s(S_8, \text{cats}) & F(a, S_8) & \text{copied_from}(S_8, S_4) & \text{copied_through}(S_8, S_9) \\
 P_s(S_{10}, \text{cats}) & F(ca, S_{10}) & \text{copied_from}(S_{10}, S_5) & \text{copied_through}(S_{10}, S_9) \\
 P_s(S_{11}, \text{cats}) & F(t, S_{11}) & \text{copied_from}(S_{11}, S_6) & \text{copied_through}(S_{11}, S_9) \\
 P_s(S_{13}, \text{cats}) & F(c, S_{13}) & \text{copied_from}(S_{13}, S_1) & \text{copied_through}(S_{13}, S_{10}) \\
 P_s(S_{14}, \text{cats}) & F(a, S_{14}) & \text{copied_from}(S_{14}, S_2) & \text{copied_through}(S_{14}, S_{10})
 \end{array} \quad (4)$$

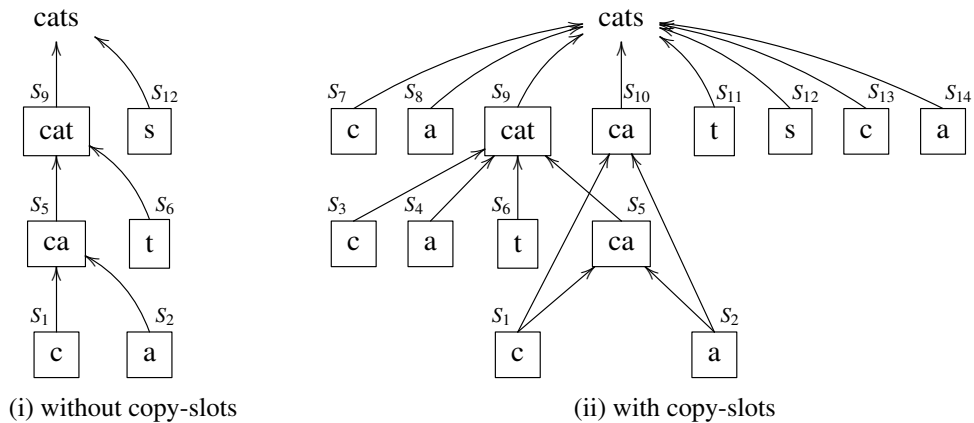


Fig. 4. A partial view on the mereological structure of "cats" without and with copy-slots

3.2.4. Conclusion of the Counting Problems

The two original counting problems are in fact caused by the same axiom of slot inheritance BA5, which i) makes improper slots inheritable and ii) does not make the subparts inheritable multiple times. Since this axiom was presumably introduced by Bennett to allow parthood transitivity, we will need to replace it by alternative axioms that do not lead to the same problems, while still ensuring parthood transitivity. Also, we will make sure that the excessive duplication flaw is fixed.

4. Defining A Mereology Of Slots

Our analysis above suggests that the theory lacks axioms that could ensure correct counting results according to our counting criteria. After giving a more precise and restrictive characterisation of what slots are, we will develop a mereology of slots, that is a theory of mereological relations between slots.

4.1. On the Nature of Slots

4.1.1. Slot Definition

We add the unary predicate S , that states that something is a slot, defined by Definition 2.¹⁴

Definition 2 (Slot). $S(s) \triangleq \exists a(P_s(s, a))$

Bennett defines a slot as “a location in a mereological nexus” and “an aspect of the mereological structure of a whole” (Bennett, 2013, p. 87). We restrict those definitions by stating that a slot is a holistic aspect of the mereological structure of a whole, i.e., a contextual mereological location in a unique whole. Therefore, it depends existentially on this whole, as explained below in Section 4.1.2.

In our theory, we consider slots as inner elements of wholes that characterise different contexts in which parts occur.¹⁵ As inner elements of a whole, that is, as elements of its mereological structure, they are not shareable with other wholes, as imposed by Axiom 1 below. This is in contradiction with Bennett’s axiomatisation since in her theory, slots can have multiple owners because of slot inheritance.

Axiom 1 (Single Owner). $\forall a, b, s(P_s(s, a) \wedge P_s(s, b) \rightarrow a = b)$

We showed in the previous section that slot inheritance is the origin of counting problems. By imposing that slots have a single owner, slot inheritance is no longer possible, as demonstrated by Theorem 1¹⁶ below. Therefore, we reject Bennett’s Slot Inheritance BA5. Consequently, as BT7 is the only theorem of Bennett’s theory that was proved with BA5, we will need to show that it is still a theorem of our theory when the relevant axioms will be added.

Theorem 1 (Anti-Inheritance). $\forall a, b, s, t([a \neq b \wedge P_s(s, b) \wedge F(a, s) \wedge P_s(t, a)] \rightarrow \neg P_s(t, b))$

¹⁴Our definitions, axioms and theorems are denoted using the full word and a number. Therefore, they can be distinguished from Bennett’s.

¹⁵To follow the analogy with roles that inspired Bennett’s theory, those contexts can be seen as relational roles. In this view, we consider that slots correspond to fully saturated roles, such as “President of the United States”, as opposed to the unsaturated role “President”. For more on role saturation, see Masolo et al. (2004).

¹⁶In earlier work (Tarbouriech et al., 2021), a similar formula, introduced as an axiom, did not include $a \neq b$ in its premises. However, this led to only two models: the empty model and the model with one filler and its improper slot.

Proof. Let a and b be two different fillers ($a \neq b$) and s and t two slots such that a is a part of b by filling its slot s ($P_s(s, b) \wedge F(a, s)$) and t is a slot of a ($P_s(t, a)$). We want to prove that t cannot be a slot of b ($\neg P_s(t, b)$). Suppose that t is a slot of b ($P_s(t, b)$). Because t is a slot of a and b , we know by A1 that $a = b$. Contradiction. Therefore, t cannot be a slot of b . \square

In the remainder, we will sometimes call slots “occurring contexts”. Consider the two following definite descriptions: “the universal of hydrogen that fills s_2 ” and “the universal of hydrogen that fills s_3 ”. Those two descriptions are misleading. By reading them, one could assume the existence of two different universals of hydrogen, which does not make sense. Indeed, what is intended here is to refer to two occurrences of the same universal, HYDROGEN, in the slots s_2 and s_3 . Those two different occurrences exist because there are two different slots. Those slots are “occurring contexts” of the universal within a given universal, the owner of the slots. Those occurring contexts are not intrinsic properties of the universals filling the slots. An occurring context characterises the inner structures of an entity larger than the filler of this context.

We add two relations, namely SO and SF , respectively defined by Definition 3 and Definition 4, where $SO(s, t)$ means “ s has the same owner as t ” and $SF(s, t)$ means “ s has the same filler as t ”. These relations state that two slots have the same owner, for the former, and the same filler, for the latter. They are trivially conditionally reflexive, symmetric, and because the owner and the filler of a slot are unique, transitive.

Definition 3 (Same Owner). $SO(s, t) \triangleq \exists a(P_s(s, a) \wedge P_s(t, a))$

Definition 4 (Same Filler). $SF(s, t) \triangleq \exists a(F(a, s) \wedge F(a, t))$

Now that slots only have one owner, the two counting criteria **C1** and **C2** are equivalent when counting proper parts. Note that with anti-inheritance, the distinction between direct and non-direct slots is not relevant anymore: all slots are direct (but see Section 4.2 to see how this distinction is recovered).

Since slot inheritance was instrumental in Bennett’s theory to derive important theorems, we will adopt instead what we call an operation of “contextualisation” between slots. Consider the following example: some entity a has a slot s filled by b , and b owns a slot t filled by c . In Bennett’s theory, with slot inheritance, the slot t will also be a slot of a . Here, instead of inheriting t , we posit that a has a slot u filled by c , which is a copy of t , *contextualised* by s . Contextualisation will fix the flaw of the copy-slot mechanism, as we will see in Section 4.2.

But first, let’s address some questions about slots’ nature that are barely touched in Bennett’s paper: are slots existentially dependent on their owners? On their fillers? And what are improper slots?

4.1.2. Existential Dependencies of Slots

Even though Bennett explores the existential dependencies of entities on their slots and parts, she says nothing about the existential dependencies of slots on their owners and fillers.

In Bennett’s theory, slots can be related to fillers by two relations: P_s and F . Therefore, a specific slot can be owned by some fillers and filled by some other (or identical) fillers. Does a slot existentially depend on its owner? Or on its filler? And if so, what is the nature of this dependency?

Bennett’s theory asserts that all slots have an owner (BA1). As we already said in the previous section, in our theory, a slot is an inner element of a single whole (Axiom 1). Therefore, the facts that slots always have an owner and are elements of their single owner’s mereological structure suggest that slots existentially depend on their unique owner. Thus, although we do not formally account for the modal aspect of dependence in our theory, we can assume that slots are specifically dependent on their owner:

the existence of a specific slot depends on the existence of its owner. If the owner would disappear, the slot would also cease to exist.

We just pointed out that slots are inner elements of their owner. They have another important characteristic in Bennett's theory: they are always filled (BA7). Bennett also points out that the content of a slot can change over time: a slot can have two different fillers at two different times. Thus, a slot does not specifically depend on its filler. This raises the question of whether a slot generically depends on its fillers across time, which we will not address here as we leave the complex issue of mereological change for further work.

4.1.3. Properness Of Slots

Bennett's theory proposes Definition BD5 to define proper slots. The definition states that some slot s is a proper slot of a if it is a slot of a and a does not fill s . Therefore, an improper slot is a slot owned and filled by the same entity, as defined by Definition 5.

Definition 5 (Improper Slot). $IP_s(s, a) \triangleq P_s(s, a) \wedge F(a, s)$

In Bennett's theory, the properness of a slot is a property relative to an owner of this slot. Therefore, a slot can be an improper slot of a filler a and, at the same time, a proper slot of a filler b . However, in our theory, Axiom 1 makes it impossible for a slot to have multiple owners. Therefore, proper and improper slots form a partition of slots, i.e. slots are either proper or improper, but not both, as expressed by Lemma 2, where \oplus is the XOR connective.

Lemma 2 (Either Proper or Improper). $\forall s[S(s) \rightarrow \exists! a(PP_s(s, a) \oplus IP_s(s, a))]$

Proof. Let s be a slot. s has an owner a (Definition 2), and this owner is unique (Axiom 1). Either a fills s , and therefore s is an improper slot, or a does not fill s , and therefore s is a proper slot of a . \square

Bennett does not explicitly state why she admitted improper slots. However, as she clearly aims to mimic classical extensional mereology, she needs some sort of parthood reflexivity and improper parthood relation. Improper slots make the parthood relation conditionally reflexive (BT9), and we note that they are not used for any other purpose in Bennett's paper. Therefore, we can presume that improper slots are introduced only for the conditional reflexivity of parthood

Using Axiom 1 and Definitions of P (BD1) and PP (BD2), we can deduce Lemma 3, which states that a proper part of a is filler of a proper slot of a . This ensures that every proper part of a whole fills a slot of this whole, without owning this slot. This excludes models like the one pictured in Figure 2ii.

Lemma 3 (Proper Parts iff Proper Slots). $\forall a, b(PP(b, a) \leftrightarrow \exists s(PP_s(s, a) \wedge F(b, s)))$

Proof. Let a and b be two fillers.

Let us first suppose that b is a proper part of a . By definition of proper parthood BD2, $P(b, a) \wedge \neg P(a, b)$. From $P(b, a)$ and definition BD1, we deduce the existence of a slot s such that $P_s(s, a) \wedge F(b, s)$. We want to prove that 1) s is a proper slot of a and that 2) b fills s , which is already in the hypotheses. s is a slot of a . If a does not fill s , then s is a proper slot of a . Suppose that a fills s . According to BA7, $a = b$. Contradiction: b is a proper part of a . Therefore a does not fill s .

Let us now suppose that there is some s that is a proper slot of a and that is filled by b . We want to prove that b is a part of a and that a is not a part of b . b fills a slot of a . Thus, by BD1, b is a part of a . Suppose now that a is a part of b . By antisymmetry (BT8), $a = b$. Thus a fills s : contradiction as s is a proper slot of a . Therefore a is not a part of b . \square

Bennett's Axiom BA4 ensures that anything that possesses a slot has an improper slot (and therefore is part of itself). Bennett justifies the conditional reflexivity by stating that "the reflexivity of parthood is restricted to things that have parthood slots. That's because [BA3] and the definition of parthood entail that parthood slots cannot have parts at all" (Bennett, 2013, p. 94). This certainly justifies why slots are not part of themselves, but it does not justify why fillers without slots are not part of themselves. We found no other justifications for this in Bennett's paper. Garbacz (2016) made the same observation. We therefore complement Bennett's Axiom BA4 by adding that every filler has an improper slot, in line with what Garbacz proposed in his Axiom 9: we thus add Axiom 2 to the theory and broaden Bennett's Theorem of conditional reflexivity BT9 as Lemma 4.

Axiom 2 (Additional Improper Slot). $\forall a, s(F(a, s) \rightarrow \exists t(IP_s(t, a)))$

Lemma 4 (General Conditional Reflexivity). $\forall a, s(P_s(s, a) \vee F(a, s) \rightarrow P(a, a))$

Proof. This is a trivial consequence of BD1, BA4 and Axiom 2. \square

In Bennett's theory, nothing excludes that an entity has an improper slot several times. Remember that we determine the number of slots of a universal by considering the number of parts of a particular that would instantiate this universal (isomorphism principle, see Section 3.1). For example, a particular of METHANE has arguably itself as a part only once. From this viewpoint, METHANE should have a unique improper slot. More generally, we add the following Axiom 3 asserting that a filler has at most one improper slot.

Axiom 3 (Unique Improper Slot per Filler). $\forall a, s, t(IP_s(s, a) \wedge IP_s(t, a) \rightarrow s = t)$

With Axiom BA6, Bennett introduces anti-symmetry, i.e., if a is part of b by filling its slot s and b is part of a by filling its slot t , then $a = b$. With Axiom 3, we can assert that, besides $a = b$, we also have $s = t$, as expressed by Theorem 5.

Theorem 5 (Mutual Occupancy is Slot Identity). $\forall a, b, s, t(P_s(s, b) \wedge F(a, s) \wedge P_s(t, a) \wedge F(b, t) \rightarrow s = t)$

Proof. Let s and t be two slots, and a and b two fillers such that $(P_s(s, b) \wedge F(a, s)) \wedge (P_s(t, a) \wedge F(b, t))$. By BA6, we know that $a = b$. Therefore, by Definition 5, s and t are improper slots of a ($IP_s(s, a) \wedge IP_s(t, a)$). By Axiom 3, $s = t$. \square

To sum up, so far we removed slot inheritance by removing Bennett's Axiom BA5 and clarified what slots are in our theory, i.e., non-shareable elements of mereological structures. We also made clear the differences between proper and improper slots. We kept axioms BA1 to BA4 and BA6 to BA8 in our theory, which will be used, with our Axioms 1 to 3 as a basis for the operation of contextualisation between slots and mereology of slots that we will now present.

The theory we propose and that will be developed in the remainder of this paper is summarised in Appendix A. As explained above, the removal of Bennett's axiom BA5 implies that we will need to show that BT7 is a theorem of our theory. Nonetheless, other Bennett's theorems presented in Appendix A are not dependent on BA5 and still hold with the axioms we considered in the remainder of the paper, as do all of Theorems 1 and 5 and Lemmas 2 to 4 demonstrated in this section.

4.2. Slot Contextualisation

Now that we further explained what slots are in our theory, let us look at our solution to make parthood transitive. We keep the core idea developed with the copy-slot mechanism: the mereological structure of a filler should be duplicated as many times as the filler occupies a slot. In this paper, as seen in Section 4.1.1, the idea is to use slots as occurring contexts of parts. Instead of using the relations *copied_from* and *copied_through*, we want a theory in which we can assert about the Figure 3ii that S_5 and S_6 are respectively the results of S_3 and S_4 being contextualised by S_1 . In the same way, we want to say that S_7 and S_8 are the results of S_3 and S_4 being contextualised by S_2 .

In Figure 5, b is a part of a by filling s , and c is part of b by filling t . In this configuration, we want b 's parts to also be parts of a , i.e., we want c to be a part of a . The slot t cannot be inherited by a , as it would be in Bennett's theory: t is a slot of b and only b . However, c should fill a slot of a because it fills a slot of b , which itself fills a slot of a . For this, we introduce the relation of *slot contextualisation*. In our example, slot contextualisation relates a slot u of a to s and t . We say that u is the "contextualisation of t by s ". The relation of slot contextualisation is abbreviated *CoS*: $CoS(u, s, t)$ means that u is the result of the contextualisation of t by s . Furthermore, we want this relation to only hold between slots, we want slots u and s to have the same owner and we want slots u and t to have the same filler. From now on, $\square \leftarrow \bullet \rightarrow \square$ represents, in figures, the contextualisation with the arrow head, the black dot and the arrow tail respectively linked to u , s and t in $CoS(u, s, t)$.

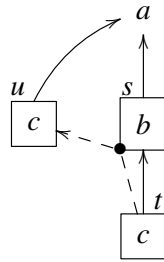


Fig. 5. An example of contextualisation

As already mentioned, the contextualisation is a relation between three slots, as imposed by Axiom 4.

Axiom 4 (Domains of Contextualisation). $\forall s, t, u (CoS(u, s, t) \rightarrow S(u) \wedge S(s) \wedge S(t))$

The existence of the contextualisation is implied by the existence of some filler a such that one of the slots is a slot of a , and the other one is filled by a . This condition of existence is captured by the predicate Cb , defined by Definition 6: $Cb(t, s)$ reads " t is contextualisable by s ". Axiom 5 states that a slot t is contextualisable by a slot s if and only if a contextualisation u of t by s exists.

Definition 6 (Contextualisable by). $Cb(t, s) \triangleq \exists a (F(a, s) \wedge P_s(t, a))$

Axiom 5 (Contextualisable iff Contextualisation Exists). $\forall s, t [Cb(t, s) \leftrightarrow \exists u (CoS(u, s, t))]$

We also impose, with Axiom 6, that the contextualisation is unique. Consequently, we will use the notation $s \circ t$ as a binary function that gives the resulting slot of t contextualised by s , in other words, the contextualisation of t by s .

Axiom 6 (Unicity of Contextualisation). $\forall s, t, u, v (CoS(u, s, t) \wedge CoS(v, s, t) \rightarrow u = v)$

We will now use the notation $u = s \circ t$ instead of the statement $CoS(u, s, t)$, and in the context in which t is contextualisable by s , we will simply use $s \circ t$.¹⁷

We can deduce from this that if $s \circ t$ and $t \circ s$ exist, then $s = t$. In other words, in the general case where s and t are different, if the contextualisation $s \circ t$ exists, then $t \circ s$ does not. This is expressed by Theorem 6.

Theorem 6 (Symmetric Contextualisation is Slot Identity). $\forall s, t, u, v (u = s \circ t \wedge v = t \circ s \rightarrow s = t)$

Proof. Let s, t, u and v be four slots such that $u = s \circ t$ and $v = t \circ s$. By Axiom 5, we know there are a and b such that $F(a, s) \wedge P_s(t, a)$ and $F(b, t) \wedge P_s(s, b)$. By T5, $s = t$. \square

Let us now introduce Axioms 7 and 8 to prevent some unwanted contextualisations. Axiom 7 ensures that if $s \circ t$ and $s \circ u$ are equal, then $t = u$, i.e. it ensures that when an entity a is part of another entity c by filling a slot s , there are as many slots of a as there are slots of c that are the results of a contextualisation by s . Otherwise, it would be possible for the contextualisations of two different slots of the same entity a to be identical, as pictured in Figure 6i. In such a case, a could have two slots filled by b which by contextualisation could lead to only one slot of c filled by b ; this would obviously not lead to the correct result when counting how many times b is part of c .

In addition, we would like to ensure that a contextualisation of an improper slot s by some slot t results in t . Otherwise, in the example pictured by Figure 6ii, a would be part of c three times instead of once. However, it is possible to adopt a weaker axiom, more similar to Axiom 7 and to use it to demonstrate such a proposition. This adopted axiom is Axiom 8 and the resulting theorem is Theorem 8, presented further below.

Axiom 7 (Injectivity to the Left). $\forall s, t, u, v (v = s \circ t \wedge v = s \circ u \rightarrow t = u)$

Axiom 8 (Injectivity to the Right). $\forall s, t, u, v (v = t \circ s \wedge v = u \circ s \wedge \exists a (IP_s(s, a)) \rightarrow t = u)$

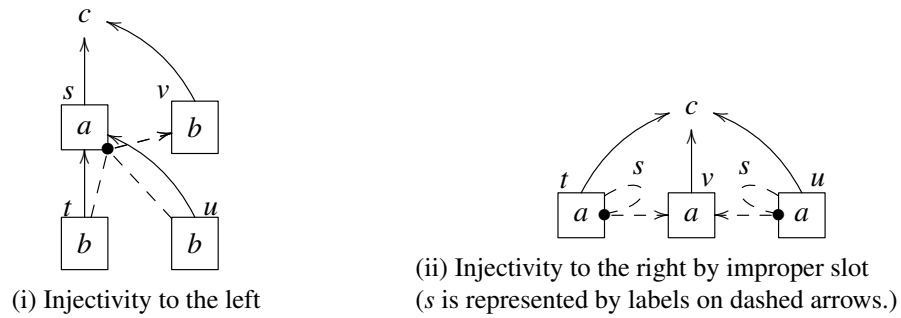


Fig. 6. Injectivities of Contextualisation: unwanted models with $t \neq u$

¹⁷In proofs, we will often omit reference to Axiom 5 for this last use.

Axiom 8 ensures that if s is an improper slot and $v = t \circ s = u \circ s$, then $u = t$. However, this axiom does not ensure that $t = v$. For this, we will introduce Axiom 9, an axiom of associativity. There is moreover another motivation to introduce it. Consider the example given in Figure 4, reinterpreted using contextualisation. We know that $S_7 = S_9 \circ S_3$ and $S_{13} = S_{10} \circ S_1$. Among the slots used in these contextualisations, some are also results of contextualisation: $S_3 = S_5 \circ S_1$ and $S_{10} = S_9 \circ S_5$. By replacing S_3 and S_{10} in the first two equations, we get $S_7 = S_9 \circ (S_5 \circ S_1)$ and $S_{13} = (S_9 \circ S_5) \circ S_1$. If we want to avoid slot duplication, as exposed in the copy-slot mechanism, we want that $S_7 = S_{13}$, i.e., $S_9 \circ (S_5 \circ S_1) = (S_9 \circ S_5) \circ S_1$. This is enabled by Axiom 9, which intuitively states that if $t \circ u$ and $s \circ (t \circ u)$ exist, then $s \circ t$ and $(s \circ t) \circ u$ exist and $s \circ (t \circ u) = (s \circ t) \circ u$, and vice versa.

Axiom 9 (Contextualisation Associativity).

$$\forall s, t, u, v [\exists w (v = s \circ w \wedge w = t \circ u) \leftrightarrow \exists x (v = x \circ u \wedge x = s \circ t)]$$

This axiom means that there are two “paths” from u to $v = (s \circ t) \circ u = s \circ (t \circ u)$, as pictured respectively by Figures 7i and 7ii.¹⁸ In Figure 7i, t is contextualised by s to get $(s \circ t)$, filled by the same filler as t , which owns u . Thus, $(s \circ t) \circ u$ exists. In Figure 7ii, u is contextualised by t to get $(t \circ u)$. And this slot, owned by the filler of s , is contextualised by s to get $s \circ (t \circ u)$. Associativity implies that $(s \circ t) \circ u$ and $s \circ (t \circ u)$ are identical.

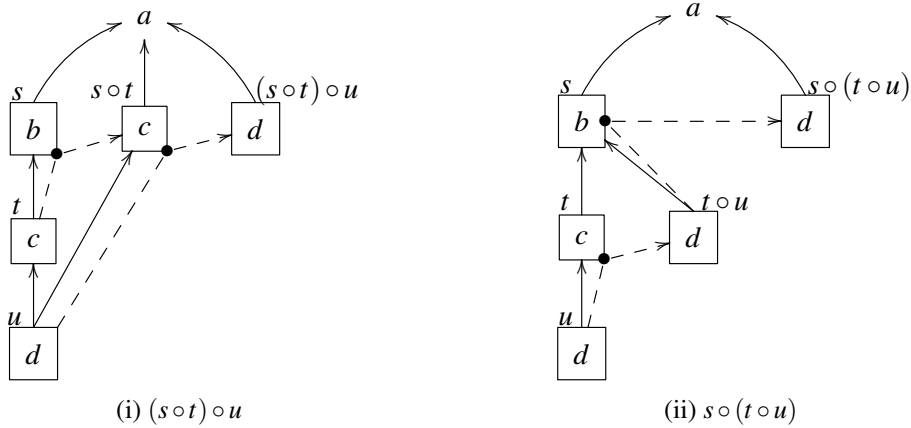


Fig. 7. Associativity: $(s \circ t) \circ u = s \circ (t \circ u)$

Slots of various properness can be contextualised together. We can distinguish three specific non-disjoint cases: 1) both slots are improper slots (Left-and-Right-Improper Contextualisation, see Theorem 7); 2) the first operand is an improper slot (Improper Slot Is Right Neutral Element, see Theorem 8); 3) the second operand is an improper slot (Improper Slot Is Left Neutral Element, see Theorem 9).

The only slots that can be contextualised by themselves are improper slots. This is because of the condition expressed by Axiom 5 (Contextualisable iff Contextualisation Exists). Therefore, a slot s is improper iff $s = s \circ s$ is true, as expressed by Theorem 7.

Theorem 7 (Left-and-Right-Improper Contextualisation). $\forall s (\exists a (IP_s(s, a)) \leftrightarrow s = s \circ s)$

¹⁸Figure 7 is subdivided into two figures for readability: they are two partial views of the same model.

Proof. Left-to-right: let a be a filler and s its improper slot ($IP_s(s, a)$). By definition, this means that $P_s(s, a) \wedge F(a, s)$. According to Axiom 5, the slot $s \circ s$ exists. By Axiom 4, we know that $s \circ s$ is owned by some b ($P_s(s \circ s, b)$). And by Single Occupancy Axiom (BA7), we know that there is a c that fills $s \circ s$ ($F(c, s \circ s)$). We want to show that $s = s \circ s$. We will show that $a = b = c$, and conclude that, as there can only be one improper slot per filler, $s = s \circ s$.

Let us first show that $a = c$. We know by Axiom 2 that c has an improper slot t ($P_s(t, c) \wedge F(c, t)$). With $F(c, s \circ s) \wedge P_s(t, c)$, we know that $(s \circ s) \circ t$ exists. By Contextualisation Associativity (Axiom 9), $s \circ t$ exists. Thus, by Axiom 5, there is a d that fills s and owns t ($F(d, s) \wedge P_s(t, d)$). However, we know that s is already filled by a and that a slot can only be filled by one filler (BA7), therefore $a = d$. We also know that t is owned by c and d , i.e. a . However, there can only be one owner (Axiom 1), therefore, $a = c = d$. One of the consequences is that $s = t$, as s and t are improper slots of a and that there is only one improper slot per filler.

Let us prove that $a = b$. We know that $s \circ (s \circ s)$ exists. Therefore by Axiom 5, there is a e that fills s and owns $s \circ s$. By BA7 (Single Occupancy), $a = e$. By Axiom 1 (Single Owner), $e = b$. So $a = b$.

We know that s and $s \circ s$ are filled and owned by a , i.e. they are a 's improper slots. However, there is only one improper slot per filler. Therefore, $s = s \circ s$.

Right-to-left: let s be a slot such that $s = s \circ s$. By Axiom 5, we know that there is an a such that $P_s(s, a) \wedge F(a, s)$, which is, by definition, $IP_s(s, a)$. \square

Theorems 8 and 9 show that improper slots act as neutral elements when contextualisation is defined. The converses, i.e. neutral elements of contextualisation are improper slots, will be demonstrated below by Theorems 14 and 15.

Theorem 8 (Improper Slot Is Right Neutral Element). $\forall a, s, t (IP_s(s, a) \wedge F(a, t) \rightarrow t = t \circ s)$

Proof. Let s and t be two slots and a be a filler such that s is the improper slot of a ($IP_s(s, a)$) and a fills t ($F(t, a)$). According to Theorem 7, $s = s \circ s$. According to Axiom 5, there is a u such that $u = t \circ s$. So $u = t \circ (s \circ s)$. By associativity (Axiom 9), we know that $u = (t \circ s) \circ s$. So $u = u \circ s$. Finally, as $t \circ s = u \circ s$, by Axiom 8, $t = u$. Thus $t = t \circ s$. \square

Theorem 9 (Improper Slot Is Left Neutral Element). $\forall a, s, t (IP_s(s, a) \wedge P_s(t, a) \rightarrow t = s \circ t)$

Proof. Let s and t be slots, and a a filler, such that s is the improper slot of a ($IP_s(s, a)$) and t is a slot of a ($P_s(t, a)$). With Theorem 7, we know that $s = s \circ s$. By Axiom 5 (Contextualisable iff Contextualisation Exists), there is a u such that $u = s \circ t$. So $u = (s \circ s) \circ t$, which, by Contextualisation Associativity (Axiom 9), gives $u = s \circ (s \circ t)$. So $u = s \circ u$. Finally, as $s \circ t = s \circ u$, by Axiom 7, $t = u$. Thus $t = s \circ t$. \square

Theorem 10 states that if a slot s is the result of a contextualisation of some slot by t and t is the result of a contextualisation of some slot by s , then $s = t$.

Theorem 10 (Mutual Contextualisation is Identity). $\forall s, t, u, v (s = t \circ u \wedge t = s \circ v \rightarrow s = t)$

Proof. Let s, t, u and v be slots such that $s = t \circ u$ and $t = s \circ v$. By replacing s by $t \circ u$ in $t = s \circ v$, we know that $t = (t \circ u) \circ v$. By Contextualisation Associativity (Axiom 9), we know $t = t \circ (u \circ v)$. With the same reasoning, we know $s = s \circ (v \circ u)$. With Theorem 6, $u = v$. So, by replacing v by u , we know that $s = t \circ u$ and $t = t \circ (u \circ u)$. u being an improper slot, we know by Theorem 7 that $u = u \circ u$. So $t = t \circ u$. Therefore by Unicity of Contextualisation (Axiom 6), $s = t$. \square

As said when introducing contextualisation, the resulting slot and the contextualising slot should have the same owner. This is guaranteed by Theorem 11. Furthermore, the resulting slot and the contextualised slot have the same filler, as expressed by Theorem 12.

Theorem 11 (Contextualisation Same Owner). $\forall u, s, t (u = s \circ t \rightarrow SO(u, s))$

Proof. Let u , s and t be such that $u = s \circ t$. By Axiom 4, we know that u , s and t are slots, i.e. they have an owner. We call a the owner of u ($P_s(u, a)$). The owners of s and t are not useful for this proof, so we ignore them. By BA4, we know that a as an improper slot v ($IP_s(v, x)$). By Improper Slot Is Left Neutral Element (Theorem 9), we know that $u = v \circ u$. So $u = v \circ (s \circ t)$. By Contextualisation Associativity (Axiom 9), $u = (v \circ s) \circ t$. As there is a contextualisation $v \circ s$, we know by Axiom 5 that there is a b that fills v ($F(b, v)$) and owns s ($P_s(s, b)$). The slot v is filled by a and b , which gives, by BA4, that $a = b$. This means that s and u are both owned by a , i.e. they have the same owner. \square

Theorem 12 (Contextualisation Same Filler). $\forall u, s, t (u = s \circ t \rightarrow SF(u, t))$

Proof. Let u , s and t be such that $u = s \circ t$. By Axiom 4, we know that u , s and t are slots. With BA7, we know there is a a that fills t . We want to prove that a also fills u .

By Axiom 2, a has an improper slot v . According to Axiom 5, $t \circ v$ exists. According to Axiom 5, there is a b that fills s and owns t . According to Theorem 11, $t \circ v$ and t have the same owner. This owner being unique (Axiom 1), $t \circ v$ is owned by b . Therefore, $s \circ (t \circ v)$ exists. By Contextualisation Associativity (Axiom 9), $s \circ (t \circ v) = (s \circ t) \circ v = u \circ v$. So, by Axiom 5, there is something that fills u and owns v . With Axiom 1, there is only one owner, namely a . Therefore, u is filled by a . As a fills t and u , by Definition 4, $SF(u, t)$. \square

We mentioned earlier that because we removed BA5, we would need to prove BT7 using our new theory. Here is a proof of it, using contextualisation.

Theorem 13 (Parthood Transitivity). $\forall a, b, c (P(a, b) \wedge P(b, c) \rightarrow P(a, c))$

Proof. Let a , b and c be three fillers such that a is a part of b and b is a part of c . We want to prove that a is a part of c , i.e. that there is some slot owned by c and filled by a . By definition of parthood (BD1), there are two slots s and t such that $F(a, s) \wedge P_s(s, b)$ and $F(b, t) \wedge P_s(t, c)$. According to Axiom 5, as b fills t and owns s , there is a slot u such that $u = t \circ s$. By Theorem 12, u and s have the same filler, i.e. a . By Theorem 11, u and t have the same owner, i.e. c . Therefore, u being filled by a and owned c , a is a part of c . \square

As already mentioned before, Theorems 14 and 15 show that neutral elements are improper slots.

Theorem 14 (Right Neutral Element Is Improper Slot). $\forall s, t (t = t \circ s \rightarrow \exists a (IP_s(s, a) \wedge F(a, t)))$

Proof. Let s and t be two slots such that $t = t \circ s$. According to Axiom 5 (Contextualisable iff Contextualisation Exists), there is a filler a that fills t and owns s . According to Theorem 12 (Contextualisation Same Filler) and BA7 (Single Occupancy), s and t have the same filler, i.e. a . The slot s is filled and owned by a , so by Definition 5, s is the improper slot of a . \square

Theorem 15 (Left Neutral Element Is Improper Slot). $\forall s, t (t = s \circ t \rightarrow \exists a (IP_s(s, a) \wedge P_s(t, a)))$

Proof. Let s and t be slots such that $t = s \circ t$. According to Axiom 5 (Contextualisable iff Contextualisation Exists), there is a filler a that fills s and owns t . According to Theorem 11 (Contextualisation Same Owner) and Axiom 1 (Single Owner), s and t have the same owner, i.e. a . The slot s is filled and owned by a , so by Definition 5, s is the improper slot of a . \square

Theorem 16 shows that if s and t are contextualisable by u , then for all v , s is a contextualisation of t by v iff $(u \circ s)$ is a contextualisation of $(u \circ t)$ by v .

Theorem 16 (Contextualisation Stable under Contextualisation).

$$\forall s, t, u (Cb(s, u) \wedge Cb(t, u) \rightarrow \forall v (s = t \circ v \leftrightarrow u \circ s = (u \circ t) \circ v))$$

Proof. Let s , t and u be slots such that s and t are contextualisable by u . Let v be a slot. We want to prove that $s = t \circ v \leftrightarrow u \circ s = (u \circ t) \circ v$.

Left-to-right: suppose that $s = t \circ v$. Let us prove that $u \circ s = (u \circ t) \circ v$. First, let us prove that $(u \circ t) \circ v$ exists. For it to exist, according to Axiom 5, there must be some filler that fills $u \circ t$ and owns v . According to Axiom 5, $s = t \circ v$ implies that there is a filler a that fills t and owns v . Let us show that a fills $u \circ t$. According to Theorem 12 (Contextualisation Same Filler), $u \circ t$ and t have the same filler, i.e. a . Therefore, as there is a filler that fills $u \circ t$ and owns v , $(u \circ t) \circ v$ exists.

Let us now prove that $(u \circ t) \circ v$ equals $u \circ s$. By Axiom 9 (Contextualisation Associativity), $(u \circ t) \circ v = u \circ (t \circ v)$. Using the fact that $s = t \circ v$, we get that $(u \circ t) \circ v = u \circ s$.

Right-to-left: suppose that $u \circ s = (u \circ t) \circ v$. Let us prove that $s = t \circ v$. By Axiom 9 (Contextualisation Associativity), $u \circ s = u \circ (t \circ v)$. By Axiom 7 (Injectivity to the Left), $s = t \circ v$. \square

Definition 1 defines what a direct slot is on the basis of Bennett's theory, in particular by using multiple owners for the same slot. However, since A1 prevents any slot from having multiple owners, this definition is not relevant anymore. Direct slots can be redefined using slot contextualisation: a direct slot is a slot that is not the result of the contextualisation of a proper slot by another proper slot; put differently, a direct slot s can be equal to $t \circ u$ only if t , u , or both are improper slots (which implies that if one of them is not an improper slot, it is identical to s).

Definition 7 (Direct Slot — With Contextualisation).

$$DP'_s(s, a) \triangleq P_s(s, a) \wedge \forall t, u [s = t \circ u \rightarrow (\exists b (IP_s(t, b))) \vee (\exists c (IP_s(u, c)))]$$

We defined the *slot contextualisation* relation and operator in order to recontextualise parts when they should be inherited. After stating the domain and existential conditions of contextualisation, we explained why it should be unique and associative. After exploring special cases of contextualisation, we demonstrated expected properties, such as, in contextualisation $u = s \circ t$, u and s have the same owner, and u and t have the same filler. We also proved that parthood transitivity holds. Exploiting this relation of slot contextualisation, we will define in the next sections basic mereological relations between slots.

4.3. Slot Parthood

We first define slot (general) parthood, then slot proper parthood.

4.3.1. Slot General Parthood

If $u = s \circ t$, as in Figure 5, the two slots s and u of the same owner a cannot be seen as representing two non-overlapping locations in the mereological structure of a . In fact, $u = s \circ t$ counts as a copy of t , which is a slot of the filler of s . Matching the parthood relation between c (the filler of t) and b (the filler of s), we propose to consider u (the contextualisation of t by s) as a part of s and define a parthood relation between slots (*PoS*) on the basis of the contextualisation relation, as expressed by Definition 8. Lemma 17 gives the domain and the range of the relation. As we will see, this will enable the characterisation of a classical mereology among slots of the same owner.

Definition 8 (Part of Slot). $PoS(u, s) \triangleq \exists t (u = s \circ t)$

Lemma 17 (*PoS* Domain and Range). $\forall s, t (PoS(s, t) \rightarrow S(s) \wedge S(t))$

1 **Proof.** By Definition 8 and Axiom 4. \square 1

2
3 The relation PoS is conditionally reflexive, anti-symmetrical and transitive, as respectively expressed 3
4 by Theorems 18 to 20. Except for the restriction to slots in reflexivity, these are the properties expected 4
5 from a parthood relation, such as in classical mereology (Varzi, 2019). 5

6
7 **Theorem 18** (Conditional PoS Reflexivity). $\forall s(S(s) \rightarrow PoS(s,s))$ 7

8
9 **Proof.** Let s be a slot and a its owner ($P_s(s,a)$). By BA7, we know that there is a b that fills s ($F(b,s)$). By Axiom 2, b has an 9
10 improper slot t ($P_s(t,b) \wedge F(b,t)$). By Axiom 5 (Contextualisable iff Contextualisation Exists) and Theorem 8 (Improper Slot 10
11 Is Right Neutral Element), we know that $s \circ t = s$. By Definition 8, $PoS(s,s)$. \square 11

12
13 **Theorem 19** (PoS Anti-Symmetry). $\forall s,t(PoS(s,t) \wedge PoS(t,s) \rightarrow s = t)$ 13

14
15 **Proof.** Derives directly from Theorem 10 (Mutual Contextualisation is Identity). \square 15

16
17 **Theorem 20** (PoS Transitivity). $\forall s,t,u(PoS(s,t) \wedge PoS(t,u) \rightarrow PoS(s,u))$ 17

18
19 **Proof.** Let s, t and u be slots such that $PoS(s,t)$ and $PoS(t,u)$. By Definition 8, there are v and w such that $s = t \circ v$ and $t = u \circ w$. 19
20 By replacing t in the expression of s , we get $s = (u \circ w) \circ v$. According to Axiom 9, the contextualisation is associative, therefore 20
21 $s = u \circ (w \circ v)$. By Definition 8, $PoS(s,u)$. \square 21

22
23 Two slots standing in a PoS relation have the same owner, as expressed by Theorem 21. Furthermore, 23
24 all slots are slots of a iff they are slot-parts of their a 's improper slot, as expressed by Theorem 22. 24

25
26 **Theorem 21** (PoS Same Owner). $\forall s,t(PoS(s,t) \rightarrow SO(s,t))$ 26

27
28 **Proof.** Derives directly from Theorem 11 (Contextualisation Same Owner). \square 28

29
30 **Theorem 22** (Slots iff Slot-Parts of Improper Slot). $\forall a,s(IP_s(s,a) \rightarrow \forall t(P_s(t,a) \leftrightarrow PoS(t,s)))$ 30

31
32 **Proof.** Let a be a filler and s its improper slot ($IP_s(s,a)$). Let t be a slot. 32

33
34 Left-to-right: suppose that t is a slot of a . By Theorem 9 (Improper Slot Is Left Neutral Element), $t = s \circ t$. Therefore, by 34
35 Definition 8, $PoS(t,s)$. 35

36
37 Right-to-left: suppose that t is a slot-part of s . By Theorem 21, s and t have the same owner. Therefore, t is a slot of a . \square 37

38
39 The mereology we are currently defining has an uncommon characteristic. Indeed, the relation of 39
40 slot-parthood PoS is locally restricted. As showed by Theorems 21 and 22, this relation can only hold 40
41 between slots of the same owner and all slots of a filler are parts of its improper slots.¹⁹ Thus, there are 41
42 as many separated mereological structures as slot-owners. Two slots owned by different entities cannot 42

43
44 ¹⁹As we will see in Section 4.7 and especially with Theorem 62, it means that improper slots are local universes. For a given 44
45 owner, there is nothing bigger than its improper slot, i.e. the improper slot is not a proper part of something. Therefore, this 45
46 mereological theory does not have junks, as defined in (Varzi, 2019, § 4.2) or (Cotnoir and Varzi, 2021, pp. 220-229). 46

be mereologically related. Similarly, we will later introduce other mereological relations that can also only hold between slots of the same owner (Theorems 28, 34 and 45).

Moreover, the mereological structure of slots and the mereological structure of fillers constrain each other, as expressed by Theorem 23. For any pair of slots t and s , if t , filled by b , is a slot-part of s , filled by a , then b is a part of a . However, if b is a part of a , it is not true that for every pair of slots respectively filled by b and a , the first slot is a part of the second one. As a matter of fact, these two slots do not necessarily fulfil the contextualisation conditions. As an example, consider two molecule universals: METHANE and CARBONDIOXIDE. Both have the universal CARBON as a part, as it fills a slot of each. The slot of METHANE filled by CARBON is not a slot-part of the improper slot of CARBONDIOXIDE. Reciprocally, the slot of CARBONDIOXIDE filled by CARBON is not a slot-part of the improper slot of METHANE. What we can state is that if b is a part of a , then, there are (at least) two slots respectively filled by b and a with the first one being a part of the second; in particular, this will be satisfied with the improper slot of a .

Theorem 23 (Slot Structure and Filler Structure constrain Each Other).

$$\forall a, b (\exists s, t (PoS(t, s) \wedge F(a, s) \wedge F(b, t)) \leftrightarrow P(b, a))$$

Proof. Left-to-right: let a and b be two fillers, and s and t be two slots, such that $F(b, t)$, $F(a, s)$ and $PoS(t, s)$. By Definition 8, there is a u such that $t = s \circ u$. By Theorem 12 (Contextualisation Same Filler) and BA7 (Single Occupancy), t and u have the same filler, namely b . By Axioms 5 (Contextualisable iff Contextualisation Exists) and BA7, we know that there is an entity filling s and owning u , and this entity is a . Therefore, $P_s(u, a)$ and $F(b, u)$. By BD1, this means that $P(b, a)$.

Right-to-left: let a be a filler and b a part of a ($P(b, a)$). According to BA4, there is a s that is the improper slot of a ($IP_s(s, a)$). By the Parthood Definition (BD1), there is a t such that $F(b, t) \wedge P_s(t, a)$. According to Theorem 22, t is a slot-part of s . \square

Theorem 24 states that PoS is stable under contextualisation, i.e. u is a slot-part of t iff $s \circ u$ is a slot-part of $s \circ t$.

Theorem 24 (PoS Stable under Contextualisation).

$$\forall s, t, u (Cb(t, s) \wedge Cb(u, s) \rightarrow (PoS(u, t) \leftrightarrow PoS(s \circ u, s \circ t)))$$

Proof. Let s, t, u such that t and u are contextualisable by s .

Left-to-right: suppose that u is a slot-part of t ($PoS(u, t)$). Therefore, there is a v such that $u = t \circ v$. Thus, by Theorem 16, $s \circ u = s \circ (t \circ v)$. By Contextualisation Associativity (Axiom 9), $s \circ u = (s \circ t) \circ v$. By Definition 8, this means that $PoS(s \circ u, s \circ t)$.

Right-to-left: suppose that $s \circ u$ is a slot-part of $s \circ t$ ($PoS(s \circ u, s \circ t)$). Therefore, there is a v such that $s \circ u = (s \circ t) \circ v$. By associativity, $s \circ u = s \circ (t \circ v)$, i.e. by Theorem 16, $u = t \circ v$. By Definition 8, this means that $PoS(u, t)$. \square

4.3.2. Slot Proper Parthood

With the relation PoS , we can define the relation of proper parthood between slots $PPoS$, as expressed in Definition 9. This definition follows that of proper parthood in classical mereology.

Definition 9 (Proper Part of Slot). $PPoS(s, t) \triangleq PoS(s, t) \wedge s \neq t$

As is the case in classical mereology, this relation is irreflexive, asymmetrical and transitive, as expressed by Theorems 25 to 27. Furthermore, two slots in a $PPoS$ -relation share the same owner, as stated by Theorem 28.

Theorem 25 (*PPoS Irreflexivity*). $\forall s(\neg PPoS(s,s))$

Proof. Derives directly from Definition 9. \square

Theorem 26 (*PPoS Asymmetry*). $\forall s,t(PPoS(s,t) \rightarrow \neg PPoS(t,s))$

Proof. Derives directly from Definition 9 and Theorem 19. \square

Theorem 27 (*PPoS Transitivity*). $\forall s,t,u(PPoS(s,t) \wedge PPoS(t,u) \rightarrow PPoS(s,u))$

Proof. Derives directly from Definition 9 and Theorems 19 and 20. \square

Theorem 28 (*PPoS Same Owner*). $\forall s,t(PPoS(s,t) \rightarrow SO(s,t))$

Proof. Derives directly from Definition 9 and Theorem 21. \square

A slot can be in relation *PPoS* with either a proper or an improper slot. These two possibilities are pictured in Figure 8. Figure 8i is identical to Figure 5, adding the *PPoS*(*u,s*). Figure 8ii highlights an important feature of contextualisation by improper slots: every proper slot of a filler *a* is a slot-proper-part of *a*'s improper slot, as expressed by Theorem 29.



(i) $PPoS(u,s)$: proper part of proper slot ($u = s \circ t$)

(ii) $PPoS(t,s)$: proper part of improper slot ($t = s \circ t$)

Fig. 8. *PPoS* relation according to whole properness (dotted arrows represent the *PPoS* relation)

Theorem 29 (Proper Slots iff Proper Parts Of Improper Slot).

$$\forall a,s(IP_s(s,a) \rightarrow \forall t(PP_s(t,a) \leftrightarrow PPoS(t,s)))$$

Proof. Let *a* be a filler and *s* its improper slot ($IP_s(s,a)$). Let *t* be a slot. We want to prove that $PP_s(t,a) \leftrightarrow PPoS(t,s)$.

Left-to-right: suppose *t* is a proper slot of *a* ($PP_s(t,a)$). We want to prove $PPoS(t,s) \wedge s \neq t$. With Theorem 22 (Slots iff Slot-Parts of Improper Slot), $PPoS(t,s) \wedge s \neq t$. *s* and *t* cannot be identical, as the same slot would be proper and improper, which is not possible according to Lemma 2 (Either Proper or Improper). Thus $s \neq t$.

Right-to-left: suppose *t* is proper slot-part of *s* ($PPoS(t,s)$). We want to prove that *t* is a proper slot of *a*, i.e. it is a slot of *a*, but it is not filled by *a*. By Theorem 28, *s* and *t* have the same owner, thus $PP_s(t,a)$. Suppose that *t* is filled by *a*, making *t* an

improper slot of a . By Axiom 3, $s = t$. However, as $PPoS(t, s)$, $s \neq t$: contradiction. Therefore, t is not filled by a . \square

Theorem 30 shows that proper parthood relations between slots and proper parthood relations between fillers constrain each other.

Theorem 30 (Slot Structure and Filler Structure constrain Each Other — Proper Part).

$$\forall a, b (\exists s, t (PPoS(t, s) \wedge F(a, s) \wedge F(b, t)) \leftrightarrow PP(b, a))$$

Proof. Let a and b be two fillers.

Left-to-right: let s and t be two slots such that t is a slot-proper-part of s , a fills s and b fills t . According to Theorem 23, b is a part of a . According to BD2, we have to prove that a is not a part of b to finish the proof. Suppose that a is a part of b . With BT8, P is anti-symmetric. Therefore, $a = b$. According to Definition 9, as $PPoS(t, s)$, $PoS(t, s)$ and $s \neq t$. Furthermore, with Definition 8, there is some u such that $t = s \circ u$. With Axiom 5 and BA7, there is a single entity that fills s and owns u , i.e. a . With Theorem 12 and BA7, t and u are filled by the same entity, i.e. a . The slot u , being owned and filled by a , is its improper slot. By Theorem 8, $s = s \circ u$. Finally, with Axiom 7, $s = t$: contradiction. Therefore a is not a part of b , and thus, $PP(b, a)$.

Right-to-left: suppose that b is a proper part of a . According to BD2, b is a part of a and a is not a part of b . According to Theorem 23 and $P(b, a)$, there are two slots s and t such that s is a slot-part of t , a fills s and b fills t . According to Definition 9, we have to prove $s \neq t$ to fulfil the proof. Suppose that $s = t$. Therefore, because $F(a, s)$, $F(b, t)$ and Single Occupancy (BA7), $a = b$. Contradiction: b is a proper part of a and they cannot be identical. Therefore, $s \neq t$. \square

Theorem 31 states that $PPoS$ is stable under contextualisation, i.e. if u is a slot-proper-part of t , then $s \circ u$ is a slot-proper-part of $s \circ t$.

Theorem 31 ($PPoS$ Stable under Contextualisation).

$$\forall s, t, u (Cb(t, s) \wedge Cb(u, s) \rightarrow (PPoS(u, t) \leftrightarrow PPoS(s \circ u, s \circ t)))$$

Proof. Let s , t and u be slots such that t and u are contextualisable by s .

Left-to-right: suppose that u is a slot-proper-part of t . By Definition 9, u is a slot-part of t and is different from t . We want to prove that $s \circ u$ is a slot-proper-part of $s \circ t$, i.e. that $s \circ u$ is a slot-part of $s \circ t$ and that $s \circ t \neq s \circ u$. By Theorem 24 (PoS Stable under Contextualisation), we know that PoS is stable under contextualisation. Therefore, $s \circ u$ is a slot-proper-part of $s \circ t$. Suppose now that $s \circ t = s \circ u$. By Axiom 7 (Injectivity to the Left), we infer that $t = u$. Contradiction. Therefore, $s \circ t \neq s \circ u$, and $PPoS(s \circ u, s \circ t)$.

Right-to-left: suppose that $s \circ u$ is a slot-proper-part of $s \circ t$. We want to prove that u is a slot-proper-part of t , i.e. that u is a slot-part of t and that u and t are different. By Definition 9, $s \circ u$ is a slot-part of $s \circ t$. Therefore, by Theorem 24, u is a slot-part of t . We want to prove $u \neq t$. By Definition 9, $s \circ u \neq s \circ t$. Suppose that $u = t$. Thus $s \circ t \neq s \circ t$: contradiction. Therefore, $u \neq t$. \square

With slot contextualisation, we defined the relations of slot-parthood and slot-proper-parthood. Those relations have the same properties as the relations of ground mereology, as Varzi (1996) calls it: slot-parthood is (conditionally) reflexive, antisymmetric and transitive, while slot-proper-parthood is irreflexive, asymmetric and transitive. We also showed that both relations also have the same-owner property and are stable under contextualisation. Finally, we showed that slot-mereological structure and filler-mereological structure are linked and partially constrain each other. With these relations, we can enrich further the mereology of slots, by defining the slot-overlap relation.

4.4. Overlap

4.4.1. Bennett's Theory

In her theory, Bennett defines two overlap relations: the overlap on fillers BD3 and the overlap on slots BD4. The first relation is the classic overlap relation of mereology. However, in the examples we are interested in, this relation is not that relevant. Indeed, if we consider the structural universals of molecules, all those universals are overlapping, because they all have as a part the universal of ELECTRON.²⁰ The second relation introduced by Bennett, at first sight, seems more interesting: two fillers are slot-overlapping if they share a slot. However, the relevance of this relation is also questionable: remember that in Bennett's theory, the improper slots are also inherited. So, in our examples of universals of molecules, each of them inherits the improper slot of the ELECTRON universal. Therefore, they are all slot-overlapping on this improper slot. One could posit that the universal of ELECTRON does not have an improper slot. But the problem remains: the universal of ELECTRON fills some slots of, for example, CARBON. Therefore, all universals of molecules that have the universal of CARBON as a part slot-overlap with each other. Once again, the problem here ultimately lies in the Slot Inheritance Axiom BA5. In our theory, as a slot has a unique owner, Bennett's relation of slot-overlap between fillers collapses to the identity on fillers. Fortunately, the slot-mereological structure we have just introduced enables us to grasp a more adequate notion of overlap.

A second problem with Bennett's overlap is about model ambiguity. Consider the two strings "xyz" and "xyyz". The mereological structure pictured in Figure 9 is compatible with Bennett's theory, where X can be either "xyz" or "xyyz". "xy" and "yz" are overlapping in the general sense of having a common part (namely "y"). But from the mereological structure pictured on this figure, there is no way to know whether "xy" and "yz" are overlapping in the sense of sharing the same occurrence of "y" (as in the whole "xyz") or not (as in the whole "xyyz"). While a model in which there are two different slots filled by "y", one owned by "xy" and the other by "yz", can exclusively represent "xyyz", we cannot find a model exclusive to "xyz" that contains slots filled by "xy", "y", and "yz". The model pictured in Figure 9 seems to be such a model for "xyz", but it is ambiguous, as previously showed.

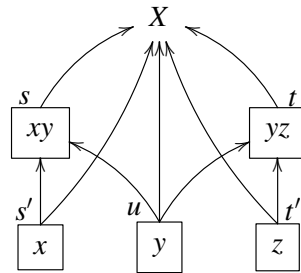


Fig. 9. Bennett's slot-overlap (and overlap) between *xy* and *yz*

4.4.2. Slot-Overlap

With the slot parthood relation *PoS*, we can define the relation of overlap between slots *OoS*, as expressed in Definition 10, following the classical definition of overlap (see for example (Varzi, 2019)).

²⁰See Footnote 9 for questioning this assertion. Anyway, the very same reasoning applies to other groups of universals, e.g. all organic molecules overlap over CARBON.

Definition 10 (Overlap of Slot). $OoS(s,t) \triangleq \exists u(PoS(u,s) \wedge PoS(u,t))$

Figure 10 pictures two configurations in which s and t slot-overlap on u : in Figure 10i, s and t are filled by the same entity b . On the contrary, in Figure 10ii, s and t have different fillers, namely b and d . Unlike Bennett's slot-overlap, this overlap of slots is less pervasive as it is restricted to slots of the same whole and corresponds more closely to our preformal intuition.

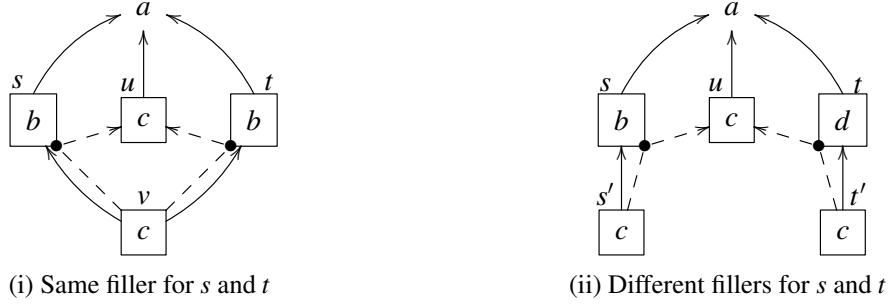


Fig. 10. Slot-overlap between s and t

With our relation of overlap of slots and with the existence of contextualisations, the model ambiguity depicted in Figure 9 with Bennett's theory disappears: we have either the model in Figure 11i for the string "xyz" or the model in Figure 11ii for the string "xyyz" (slots for parts "x" and "z" in "xy" and "yz" and their contextualisations are omitted for readability). The model in Figure 11i cannot represent "xyyz" as there is only one slot filled by "y" in the whole.

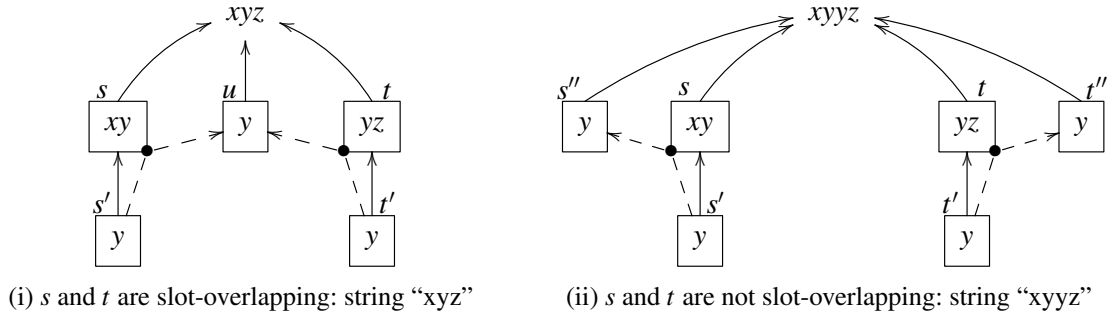


Fig. 11. Non-ambiguous models for "xyz" and "xyyz"

This relation is conditionally reflexive and symmetrical, as expressed by Theorems 32 and 33. Except for the premise in Theorem 32, these theorems are common theorems of classical mereology.

Theorem 32 (Conditional OoS Reflexivity). $\forall s(S(s) \rightarrow OoS(s,s))$

Proof. Relation PoS is reflexive, therefore, by definition OoS is also reflexive. \square

Theorem 33 (OoS Symmetry). $\forall s,t(OoS(s,t) \rightarrow OoS(t,s))$

1 **Proof.** According to the Definition 10 and the commutativity of the AND logical operator. \square 1

2
3 Two overlapping slots have the same owner, as showed by Theorem 34. 3

4
5 **Theorem 34** (*OoS Same Owner*). $\forall s, t (OoS(s, t) \rightarrow SO(s, t))$ 5

6
7 **Proof.** Let s and t be two slots such that $OoS(s, t)$. According to Definition 10, we know $\exists u (PoS(u, s) \wedge PoS(u, t))$. With 7
8 Theorem 11 (Contextualisation Same Owner) and Axiom 1 (Single Owner), $SO(s, t)$. \square 8

9
10 As expressed by Lemma 35, if two slots overlap, they also overlap with the slots the other one is part 10
11 of. 11

12
13 **Lemma 35** (Overlap with Part Implies Overlap with Whole). 13

$$14 \quad \forall s, t, u (OoS(u, t) \wedge PoS(t, s) \rightarrow OoS(u, s)) \quad 14$$

15
16 **Proof.** Let s , t and u be three slots such that $OoS(u, t) \wedge PoS(t, s)$. By Definition 10, we know that there is a v such that 15
16 $PoS(v, u) \wedge PoS(v, t)$. By Theorem 20 (*PoS Transitivity*), we know that $PoS(v, s)$. Therefore, there is a common slot-part between 16
17 u and s , namely v . By definition, $OoS(u, s)$. \square 17

18
19 Lemma 36 expresses the fact that all the slots of any filler a overlap with a 's improper slot. 19

20
21 **Lemma 36** (Slot-Overlap With Improper Slot). $\forall a, s, t (IP_s(s, a) \wedge P_s(t, a) \rightarrow OoS(s, t))$ 21

22
23 **Proof.** Let a be a filler, s its improper slot ($IP_s(s, a)$) and t one of its slots ($P_s(t, a)$). According to Theorem 18 (Conditional 22
23 *PoS Reflexivity*), $PoS(t, t)$. According to Theorem 22 (Slots iff Slot-Parts of Improper Slot), $PoS(t, s)$. By Definition 10, 23
24 $OoS(s, t)$. \square 24

25
26 From the definition of *OoS*, we can deduce that every slot s that is a slot-part of a slot t overlaps with 25
26 that slot, as stated by Lemma 37. 26

27
28 **Lemma 37** (*PoS Implies OoS*). $\forall s, t (PoS(s, t) \rightarrow OoS(s, t))$ 27

29
30 **Proof.** Let s and t be slots such that $PoS(s, t)$. We want to prove that there is a slot that is a slot-part of s and t . The slot s 29
30 fulfils these requirements. It is a slot-part of t by hypothesis. And the slot-part relation being reflexive, it is also a slot-part of 30
31 itself. \square 31

32
33 Theorem 38 states that *OoS* is stable under contextualisation, i.e. if t and u overlap, then $s \circ t$ and $s \circ u$ 32
33 overlap. 33

34
35 **Theorem 38** (*OoS Stable under Contextualisation*). 34

$$35 \quad \forall s, t, u (Cb(t, s) \wedge Cb(u, s) \rightarrow (OoS(t, u) \leftrightarrow OoS(s \circ t, s \circ u))) \quad 35$$

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46

Proof. Let s , t and u be slots such that t and u are contextualisable by s .

Left-to-right: suppose that u slot-overlaps with t ($OoS(t, u)$). By Definition of OoS (Definition 10), we know there is a v such that $PoS(v, t) \wedge Pos(v, u)$. By Theorem 21 (PoS Same Owner), v has the same owner as t , thus v is contextualisable by s . By Theorem 24 (PoS Stable under Contextualisation) and $PoS(v, t) \wedge Pos(v, u)$, we deduce $PoS(s \circ v, s \circ t) \wedge Pos(s \circ v, s \circ u)$. Therefore, by Definition 10, $OoS(s \circ t, s \circ u)$.

Right-to-left: suppose that $s \circ t$ overlaps with $s \circ u$. By Definition D10, there is a v' such that $PoS(v', s \circ t)$ and $PoS(v', s \circ u)$. By Definition 8, there is v_t and v_u such that $v' = (s \circ t) \circ v_t = (s \circ u) \circ v_u$. By Contextualisation Associativity (Axiom 9), $s \circ (t \circ v_t) = s \circ (u \circ v_u)$. Thus, by Axiom 7, $t \circ v_t = u \circ v_u$. Let us call v the slot equal to $t \circ v_t$ and $u \circ v_u$. By Definition 8, v is a slot-part of t and u . Thus, t and u overlap. \square

With Theorem 23, we showed cross-constraints between PoS relations between slots and P relations between fillers. Theorem 39 expresses a constraint from the OoS relations between slots towards the O relations between fillers. However, unlike Theorem 23, we did not find interesting constraints from O relations between fillers towards OoS relations between slots. Indeed, if two entities a and b overlap, it means that they have a common part c , but in two different slots s and t , as slots cannot have multiple owners. Nothing ensures that there are two slots u and v such that $u \circ s = v \circ t$.

Theorem 39 (Slot-Overlap Constrains Overlap between Fillers).

$$\forall a, b, s, t (OoS(s, t) \wedge F(a, s) \wedge F(b, t) \rightarrow O(a, b))$$

Proof. Let s and t be two slots and a and b their respective fillers. Suppose that s and t slot-overlap ($OoS(s, t)$). By Definition 10, it means that there is a slot u that is a slot-part of both s and t . By Lemma 17 and BA7, there is some filler c that fills u . By Theorem 23, c is a part of a and b . Therefore, by BD3, a and b are overlapping. \square

We defined a slot-overlap relation, which has the same properties as in classical mereology: (conditionally) reflexive and symmetric. Furthermore, this relation also has the same-owner property and is stable under contextualisation. Those basic mereological relations PoS and OoS will now be used to introduce supplementation principles in the next section.

4.5. Supplementation Principles

Bennett's BA8 is an adaptation of classical strong supplementation into slot mereology. It is used in the proofs of Slot Weak Supplementation (BT13) and Slot Extensionality (BT14). However, Garbacz (2016) showed that this move has multiple problems, as BT13 does not capture the idea of Weak Supplementation and BT14 actually is not a theorem of Bennett's theory. Therefore, Garbacz proposed a revision of the theory. We can even go further than Garbacz's reasoning by noticing that BA8 is actually a theorem of the theory.²¹

In the following sections, we first analyse Garbacz's proposal, then we propose an axiomatisation of supplementation in our own theory.

²¹Here follows a proof that BA8 is a theorem, using only Bennett's axioms.

Proof. Let a and b be two fillers such that both have a slot. Suppose that there are no slot owned by a and filled by b , i.e. suppose that b is not a part of a . Let us call this H1. We want to prove that there is some slot that is a slot of b but not a slot of a . From the assumptions, b has a slot. Thus, by BA4, there is a slot u that is an improper slot of b . Suppose that u is owned by a : contradiction with H1. Therefore, u is owned by b , but not by a . \square

First of all, let us note that the main motivation behind Bennett’s slot mereology and ours is to represent entities that can have a same part multiple times. Therefore, we do not want supplementation over fillers. Indeed, the DIHYDROGEN molecule universal has as proper part the HYDROGEN atom universal (twice), but there is no proper part of DIHYDROGEN that does not overlap with HYDROGEN.

4.5.1. Bennett’s Theory and Garbacz’s Revisions

Bennett’s Slot Strong Supplementation (BA8)²² intuitively states that “if a and b have parthood slots, and b is not a part of a , then b has a parthood slot that isn’t a parthood slot of a ” (Bennett, 2013, p. 96). Cotnoir (2015) states that this axiom is too weak, and sees it as a slot version of Varzi’s Strong Company (2019). This axiom is used in the proof of BT13.

Garbacz (2016) showed why Bennett’s proof of BT13 is invalid: the premise of BT13 ($PP(a, b)$) does not imply the premises of BA8 ($\exists s(P_s(s, a)) \wedge \exists s(P_s(s, b))$). Indeed, a being a proper part of b implies that b has a slot, but tells us nothing about a having a slot. Garbacz thus offers two solutions: weaken BA8 or introduce a new axiom that states that every filler has a slot, which can be improper. We already discussed the second solution and consequently added Axiom A2.

Garbacz noted that even though Bennett’s proof is not correct, BT13 still is a theorem of the theory. Indeed, the Slot Weak Supplementation Theorem states that “if a is a proper part of b , then b has a parthood slot s that isn’t a parthood slot of a ” (Bennett, 2013, p. 97). As a is a proper part of b , it occupies one of the slots of b . According to Axiom BA4, b has an improper slot (let us call it t). Slot t cannot be owned by a , otherwise, by Axiom BA6 (Mutual Occupancy Is Identity), a and b would be equal, which would contradict the fact that a is a proper part of b . So, in every case, Slot Weak Supplementation is satisfied thanks to the existence of the improper slot of b . Note that if the slot s of b filled by a is not an improper slot of a (that is, it is not owned by a), it is true that $P_s(s, b) \wedge \neg P_s(s, a)$, and thus the Slot Weak Supplementation is satisfied. As Garbacz noted, “instead of being a form of supplementation, [BT13] is a consequence of the specific form of the reflexivity of slot parthood: each object that has slots fills a slot in itself”.

Moreover, Garbacz (2016) showed that BT14 is not a theorem of Bennett’s theory. Indeed, Garbacz gives a model of the theory, pictured in Figure 12, in which all of Bennett’s axioms are satisfied, but not BT14. Consequently, Garbacz proposed three revisions of BA8 to be able to prove BT14.

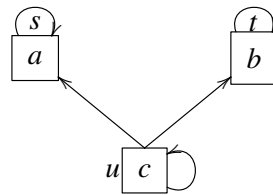


Fig. 12. Garbacz’s counter-example

Garbacz’s final revision of BA8 is exposed in (5). Intuitively, it means that if b has a proper slot and b is not a part of a , then there is a proper slot of b that is not a proper slot of a .

$$\forall a, b [\exists s(PP_s(s, b)) \rightarrow (\nexists t(P_s(t, a) \wedge F(b, t)) \rightarrow \exists u(PP_s(u, b) \wedge \neg PP_s(u, a))] \quad (5)$$

²²As proved above, BA8 is an unnecessary axiom of Bennett’s theory. However, in order to correctly render Cotnoir’s and Garbacz’s views, BA8 is treated as an axiom in this section. Furthermore, as BA8 is a theorem, Garbacz did not actually propose a revision, but an extension of Bennett’s theory.

4.5.2. Slot Supplementation

Our Single Owner Axiom (Axiom 1) makes Garbacz's revisions of BA8 vacuously true theorems. Therefore, we have to find a new axiomatic formulation for slot strong supplementation. Like in classical mereology, we have introduced here notions of parthood, proper parthood and overlap relations between slots. This means that we can adapt the Strong Supplementation axiom of classical mereology, as expressed by Axiom 10 below. We previously explained that in our mereology, the mereological relations can only hold between slots of the same owner. However, in the Slot Strong Supplementation Axiom, we need not impose that s and t have the same owner. Indeed, if they do not have the same owner, none of the possible slot-parts of t can overlap with s , and in particular, t does not overlap with s . Thus, the axiom is trivially true in that case.

Axiom 10 (Slot Strong Supplementation).

$$\forall s, t [S(s) \wedge S(t) \rightarrow (\neg PoS(t, s) \rightarrow \exists u (PoS(u, t) \wedge \neg OoS(u, s)))]$$

From this axiom, we can deduce a theorem of Slot Weak Supplementation, expressed by T40. This theorem states that if a whole has a proper slot, then it must have another slot that does not overlap with the first one.

Theorem 40 (Slot Weak Supplementation). $\forall s, t (PPoS(s, t) \rightarrow \exists u (PoS(u, t) \wedge \neg OoS(u, s))$

Proof. Let s and t be two slots such that $PPoS(s, t)$, i.e. $PoS(s, t) \wedge s \neq t$. From $PoS(s, t)$, and PoS Anti-Symmetry (Theorem 19), we deduce $\neg PoS(t, s)$. From $\neg PoS(t, s)$ and Slot Strong Supplementation (Axiom 10), we deduce $\exists u (PoS(u, t) \wedge \neg OoS(u, s))$. \square

We can also prove theorems of OoS -Extensionality (Theorem 41) and $PPoS$ -Extensionality (Theorem 42). For Theorem 41, the premise $S(s) \wedge S(t)$ is necessary: without it, $\forall u, OoS(s, u) \leftrightarrow OoS(t, u)$ would be vacuously true for every pair of fillers, leading to all fillers being equal.

Theorem 41 (OoS -Extensionality). $\forall s, t [S(s) \wedge S(t) \rightarrow (\forall u, OoS(s, u) \leftrightarrow OoS(t, u)) \rightarrow s = t]$

Proof. Let s and t be two slots such that $\forall u, OoS(s, u) \leftrightarrow OoS(t, u)$ (let us call this H1). We want to prove that $s = t$. Suppose that s is different from t ($s \neq t$). Either $PoS(t, s)$ or $\neg PoS(t, s)$.

Suppose first that $PoS(t, s)$. By Definition 9, we know $PPoS(t, s)$. According to the Slot Weak Supplementation Theorem 40, there is a u such that $PoS(u, s) \wedge \neg OoS(u, t)$. With $PoS(u, s)$ and Lemma 37 (PoS Implies OoS), we know that $OoS(u, s)$. By H1, we deduce that $OoS(u, t)$. Contradiction: u does and does not overlap t . Suppose now that $\neg PoS(t, s)$. By Slot Strong Supplementation Axiom 10, there is an u such that $PoS(u, t) \wedge \neg OoS(u, s)$. By the same reasoning, we also get a contradiction: u does and does not overlap s .

Thus, $s = t$. \square

Theorem 42 ($PPoS$ -Extensionality).

$$\forall s, t [\exists u (PPoS(u, s) \vee PPoS(u, t)) \rightarrow (\forall u, PPoS(u, s) \leftrightarrow PPoS(u, t)) \rightarrow s = t]$$

Proof. Let s and t be two slots such that one of them as a slot-proper-part u , and that $\forall u, PPOs(u, s) \leftrightarrow PPOs(u, t)$ (let us call this H1). We want to prove that $s = t$. Suppose that s is different from t . Either $POs(t, s)$ or $\neg POs(t, s)$.

Suppose first that $POs(t, s)$. Therefore, by Definition of $PPoS$ (Definition 9), $PPoS(t, s)$, and by H1, $PPoS(t, t)$. Contradiction, as $PPoS$ is irreflexive (Theorem 25).

Suppose thus that $\neg POs(t, s)$. According to Slot Strong Supplementation (Axiom 10), there is a v such that $POs(v, t) \wedge \neg OOs(v, s)$. Suppose that t and v are different. By definition, $PPoS(v, t)$, and by H1, $PPoS(v, s)$. However, v does not overlap s ($\neg OOs(v, s)$). Contradiction. Thus $t = v$, and $\neg OOs(t, s)$. However, u is a slot-part of s and t : by hypothesis, it is a slot-part of one of them, and by H1, it is also a slot-part of the other one. Therefore, $OOS(t, s)$. Contradiction.

Thus $s = t$. \square

We added the Slot Strong Supplementation axiom and proved Slot Weak Supplementation and OOS -Extensionality. Thanks to our mereological relations among slots, this axiom and these theorems are syntactically similar or identical to those of classical mereology. Compared to Bennett's theory, our supplementation correctly captures the spirit of supplementation and extensionality. In the next section, we will develop further our mereology of slots and introduce the sum of slots.

4.6. Sum of Slots

4.6.1. Bennett's Theory

In her theory, Bennett did not propose any sum operator. However, she made three remarks about sums.

First, she noted that a fully unrestricted axiom of unrestricted sum is not reasonable. Indeed, she argues that in her theory, slots cannot be parts of anything, so there cannot be sums involving slots. In our theory, slots can be parts of other slots. Yet, they cannot be parts of fillers. Therefore, we follow Bennett here: unrestricted sum is off the table.

Second, slots being put aside, she argues that sums of fillers cannot be unique. Bennett gives an example of two things a and b that might have multiple sums: $a + b$, $a + a + b$. Moreover, keeping our molecule universal examples, we could think of molecules that are not bonded in the same way. As pointed out by Lewis (1986) and McFarland (2018), butane and isobutane have the same chemical formula, i.e. the same parts (C_4H_{10}), but are isomers, i.e. their parts are arranged in different ways. Also, as illustrated by Bennett's example above, there are entities that have the same parts but not the same number of them. For example, METHANE and ETHANE (CH_4 and C_2H_6) have the same atom universals as parts, but in different quantities. If a sum of fillers were to exist, it cannot be idempotent, as it would make METHANE and ETHANE the same entity. Nonetheless, even a non-idempotent sum would not be enough as it would not enable us to differentiate BUTANE and ISOBUTANE universals.

Third, she argues that a non-idempotent sum of fillers would entail that the world is junky (i.e. everything is a proper part of something else), and thus, there is no universe. She gives the following example (Bennett, 2013, p. 99):

“To see this, imagine a world with two simple fillers, a and b . Almost unrestricted composition entails that they have a fusion; call it c . But if *every* two or more things have a fusion, then every composite must fuse with each and all of its own proper parts. So c and a must compose something, as must c and b , and c and a and b (also known as d).”

As she argues, in classical extensional mereology, Strong Supplementation entails that all “composites” of a and b are identical. Importantly, our axiom of Slot Strong Supplementation does not entail the same conclusion. Indeed, in our theory, different entities which own different slots may have the

same entities as part, whether with the same multiplicities (like BUTANE or ISOBUTANE) or not (like METHANE and ETHANE).

For these reasons, Bennett's formal system is silent on sums. We agree with Bennett on her entire reasoning about sums of fillers. However, there is one point where our visions diverge: Bennett's theory does not have a parthood relation over slots, but our theory does. In the following section, we will show how sums over slots can be introduced in our theory.

4.6.2. Slot-Sum

We introduce the relation SoS_1 to capture the intuition of sum of slots. $SoS_1(u, s, t)$ means that u is a sum of s and t . We first define SoS_1 and SoS_2 following Propositions 39₂ and 39₃ in Varzi (2019). The definitions are given by Definitions 11 and 12. We will show that these two definitions are equivalent, as proved by Theorem 44. The domains of SoS_1 is given by Lemma 43.

Definition 11 (Sum with Parthood).

$$SoS_1(u, s, t) \triangleq PoS(s, u) \wedge PoS(t, u) \wedge \forall v (PoS(v, u) \rightarrow OoS(s, v) \vee OoS(t, v))$$

Definition 12 (Sum with Overlap). $SoS_2(u, s, t) \triangleq \forall v (OoS(u, v) \leftrightarrow OoS(s, v) \vee OoS(t, v))$

Lemma 43 (Domains of Sum). $\forall s, t, u (SoS_1(u, s, t) \rightarrow S(s) \wedge S(t) \wedge S(u))$

Proof. Trivially by Definition 11 and Lemma 17. \square

Theorem 44 (SoS_1 and SoS_2 are Equivalent). $\forall s, t [SO(s, t) \rightarrow \forall u (SoS_1(u, s, t) \leftrightarrow SoS_2(u, s, t))]$

Proof. Let s and t be two slots with the same owner. Let u be a slot.

Left-to-right: suppose $SoS_1(u, s, t)$. By Definition 11, s and t are slot-parts of u and $\forall v (PoS(v, u) \rightarrow OoS(s, v) \vee OoS(t, v))$ (let us call this H). We want to prove $\forall v (OoS(u, v) \leftrightarrow OoS(s, v) \vee OoS(t, v))$. Let w be a slot. We want to prove that 1) $OoS(u, w) \rightarrow OoS(s, w) \vee OoS(t, w)$ and 2) $OoS(s, w) \vee OoS(t, w) \rightarrow OoS(u, w)$.

1) Suppose that u and w overlap. By Definition 10, there is a x that is slot-part of u and w . According to H, x overlaps s or t . Suppose that x overlaps s . Therefore, by Lemma 35 (Overlap with Part Implies Overlap with Whole), $OoS(s, x)$ and $PoS(x, w)$, we deduce that $OoS(s, w)$. Similarly, we can show that if x overlaps t , then w overlaps t .

2) Suppose that s overlaps w . By Lemma 35 and $PoS(s, u)$, we deduce that w and u overlap. The same reasoning holds if we suppose that t and w overlap: w overlap u .

Right-to-left: suppose $SoS_2(u, s, t)$, i.e. $\forall v (OoS(v, u) \leftrightarrow OoS(v, s) \vee OoS(v, t))$ (let us call this H). We want to prove that 1) s and t are slot-parts of u and that 2) $\forall v (PoS(v, u) \rightarrow OoS(s, v) \vee OoS(t, v))$.

1) Let us prove that s is a slot-part of u . Suppose that s is not a slot-part of u . Therefore, by the Slot Strong Supplementation (Axiom 10), there is a v that is a slot-part of s and that does not overlap u . v being a slot-part of s , both are overlapping, by Lemma 37. According to H, as v overlaps s , it overlaps u . Contradiction: by Slot Strong Supplementation, it cannot overlap u . Therefore, s is a slot-part of u . A similar reasoning proves that t is slot-part of u .

2) Let v be a slot-part of u . By Lemma 37, v and u are overlapping. With H, $OoS(v, s) \vee OoS(v, t)$. \square

SoS_1 and SoS_2 being equivalent, we will use SoS_1 in the remainder. However, every true proposition involving SoS_1 is also true when substituting SoS_2 for SoS_1 . Axiom 11 is the only axiom we will accept involving the sum relation. It ensures that a sum of two slots exists if they have the same owner. The addition of this axiom will be enough to derive classical properties of binary sum.

Axiom 11 (Sum Existence). $\forall s, t (SO(s, t) \rightarrow \exists u (SoS_1(u, s, t)))$

1 Slots can only be summed within the same mereological structure, i.e. if they have the same owner. 1
2 Therefore the resulting slot also has the same owner. This is stated by Theorem T45. 2

3
4 **Theorem 45** (Sum Same Owner). $\forall s, t, u [SoS_1(u, s, t) \rightarrow \exists a (P_s(u, a) \wedge P_s(s, a) \wedge P_s(t, a))]$ 3
4

5
6 **Proof.** Let u, s and t be slots such that $SoS_1(u, s, t)$. By Definition 11, we know $PoS(s, u) \wedge PoS(t, u)$. By Theorem 21 (PoS 6
7 Same Owner) and Axiom 1 (Single Owner), we know that s, t and u have the same owner. \square 7
8

9 Slot Strong Supplementation ensures the unicity of the sum, as showed by Theorem 46. The unicity 9
10 being proved, we will use the operator $+$ to represent sums of slots: $s + t$ is a binary function that returns 10
11 the sum of s and t . Furthermore, we will use the notation $u = s + t$ instead of $SoS_1(u, s, t)$. 11
12

13 **Theorem 46** (Sum Unicity). $\forall s, t, u, v (SoS_1(u, s, t) \wedge SoS_1(v, s, t) \rightarrow u = v)$ 13
14

15 **Proof.** Let s, t, u and v be four slots such that u and v are both sums of s and t . We want to prove $u = v$. According to OoS - 15
16 Extensionality (Theorem 41), if we prove $\forall w (OoS(u, w) \leftrightarrow OoS(v, w))$, then we have $u = v$. Both sums being equivalent, let 16
17 us use the Definition 12: we know that $\forall w (OoS(u, w) \leftrightarrow OoS(s, w) \vee OoS(t, w))$ and $\forall w (OoS(v, w) \leftrightarrow OoS(s, w) \vee OoS(t, w))$. 17
18 Therefore, we know that $\forall w (OoS(u, w) \leftrightarrow OoS(v, w))$, and that $u = v$. \square 18
19

20 The sum of slots is idempotent and commutative, as expressed by Theorems 47 and 48. Those prop- 20
21 erties are standard results of mereology (Propositions 41, 42 and 43 of Varzi (2019)). 21
22

23 **Theorem 47** (Sum Idempotence). $\forall s (S(s) \rightarrow s + s = s)$ 23
24

25 **Proof.** Let s be a slot and a its owner ($P_s(s, a)$). We have to prove that $s + s = s$, which means, by Definition 12, $\forall v (OoS(v, s) \leftrightarrow$ 25
26 $OoS(v, s) \vee OoS(v, s))$, which is tautologically true. Thus $s + s = s$. \square 26
27

28 **Theorem 48** (Sum Commutativity). $\forall s, t (SO(s, t) \rightarrow s + t = t + s)$ 28
29

30
31 **Proof.** By Definition 12 and the commutativity of the OR operand. \square 31
32

33 The following explores the various existing theorems resulting from the use of the slot-sum and con- 33
34 textualisation operators with the relations PoS and OoS . We consider the \circ operator to have a higher 34
35 priority than the $+$ operator. Therefore $s \circ t + u$ has to be interpreted as $(s \circ t) + u$. The first four lemmas 35
36 (Lemmas 49 to 52) are similar to Theorems 44, 45, 46 and 47 of Varzi (2019). They illustrate how slot- 36
37 sum and PoS work together. Lemma 49 states that if $s + t$ exists, then s is a slot-part of $s + t$. Lemma 50 37
38 states that if $t + u$ exists and s is a slot-part of t , then s is a slot-part of $t + u$. Lemma 51 states that if $s + t$ 38
39 is a slot-part of u , then s is a slot-part of u . And Lemma 52 states that s is a slot-part of t if and only if 39
40 $s + t = t$. 40
41

42 **Lemma 49.** $\forall s, t (SO(s, t) \rightarrow PoS(s, s + t))$ 42
43
44
45
46

1 **Proof.** By Definition 11. \square

2
3 **Lemma 50.** $\forall t, u [SO(t, u) \rightarrow \forall s (PoS(s, t) \rightarrow PoS(s, t + u))]$

4
5
6 **Proof.** Let s, t and u be three slots such that t and u have the same owner and that s is part of t . By Lemma 49, t is slot-part of $t + u$. Thus, by PoS Transitivity (Theorem 20), $PoS(s, t + u)$. \square

7
8
9 **Lemma 51.** $\forall s, t [SO(s, t) \rightarrow \forall u (PoS(s + t, u) \rightarrow PoS(s, u))]$

10
11 **Proof.** Let s, t and u be three slots such that s and t have the same owner and $s + t$ is slot-part of u . By Definition 11, $PoS(s, s + t)$. By PoS Transitivity (Theorem 20), $PoS(s, u)$. \square

12
13
14 **Lemma 52.** $\forall s, t (PoS(s, t) \leftrightarrow s + t = t)$

15
16
17 **Proof.** Left-to-right: let s be a slot-part of t ($PoS(s, t)$). We want to prove $s + t = t$, that is, by Definition 12, $\forall v (OoS(v, t) \leftrightarrow OoS(v, s) \vee OoS(v, t))$, which is tautologically true. Thus $s + t = t$.

18
19 Right-to-left: let s and t be slots such that $s + t = t$. By Definition 11, $PoS(s, t)$. \square

20
21 The following Theorem 53 describes the behaviour of overlap combined with sums. It states that
22 something overlaps the contextualisation of the sum of two operands by a slot s iff it overlaps the con-
23 textualisation of one of the operands by s .

24
25 **Theorem 53** (Overlaps the Contextualised Sum iff Overlaps one of the Contextualised Operands).

$$26 \quad \forall s, t, u [Cb(t, s) \wedge Cb(u, s) \rightarrow \forall v (OoS(v, s \circ (t + u)) \leftrightarrow OoS(v, s \circ t) \vee OoS(v, s \circ u))]$$

27
28
29 **Proof.** Let s, t and u be slots such that t and u are contextualisable by s . By Axiom 5 (Contextualisable iff Contextualisation Exists), $s \circ t$ and $s \circ u$ exist. By Axiom 11 (Sum Existence), $t + u$ exists, and by Theorem 45, $t + u$ is a slot of a . Finally, by Axiom 5, $s \circ (t + u)$ exists. Let v be a slot.

30
31 Left-to-right: suppose that v and $s \circ (t + u)$ overlap. We want to prove that v overlaps $s \circ t$ or $s \circ u$. By Definition 10 (OoS), there is a w that is a slot-part of v and $s \circ (t + u)$. w being a slot-part of $s \circ (t + u)$, we know by Definition 8 (PoS), that there is a w_1 such that $w = (s \circ (t + u)) \circ w_1$, i.e., by Contextualisation Associativity (Axiom 9), $w = s \circ ((t + u) \circ w_1)$. Let w_2 be $(t + u) \circ w_1$. By definition, w_2 is a slot-part of $t + u$. The definition of sum (Definition 11) states that every slot-part of $t + u$ overlaps t or u . So w_2 overlaps (1) t or (2) u .

32
33 (1) If w_2 overlaps t : with Theorem 38 (OoS Stable under Contextualisation), as we know $OoS(w_2, t)$, we know that $OoS(s \circ w_2, s \circ t)$, i.e. $OoS(w, s \circ t)$. By definition of OoS , there is a w_3 , slot-part of w and $s \circ t$. Let us prove that w_3 is a slot-part of v . We know that w_3 is slot-part of w , and that w is slot-part of v , therefore, by PoS Transitivity (Theorem 20), w_3 is slot-part of v . Thus $OoS(v, s \circ t)$.

34
35 (2) By the same reasoning applied to u , $OoS(v, s \circ u)$.

36
37 That is, in every cases, $OoS(v, s \circ t) \vee OoS(v, s \circ u)$.

38
39 Right-to-left: suppose that v overlaps $s \circ t$, i.e. there is a w slot-part of v and $s \circ t$. We want to prove that v and $s \circ (t + u)$ overlap, i.e. that there is a shared slot-part. Let us prove that this shared slot-part is w . As w being a slot-part of v is one of the hypotheses, we only have to prove that w is a slot-part of $s \circ (t + u)$. As w is a slot-part of $s \circ t$, we know that there is a w' such that $w = (s \circ t) \circ w'$. With Contextualisation Associativity (Axiom 9), we get $w = s \circ (t \circ w')$. t being a slot-part of $t + u$, there is a t' such that $t = (t + u) \circ t'$. So, we can write, by replacing t by $(t + u) \circ t'$ in $w = s \circ (t \circ w')$, that $w = s \circ (((t + u) \circ t') \circ w')$. Finally, by Associativity, we get that $w = (s \circ (t + u)) \circ (t' \circ w')$, which is, by Definition 8, that w is a slot-part of $s \circ (t + u)$. A similar reasoning leads to the fact that if $OoS(v, s \circ u)$, then $OoS(v, s \circ (t + u))$. \square

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Let us focus on the distributivity of contextualisation over the sum. In the first step, we prove that left distributivity holds, presented by Theorem 54. In the second step, we discuss why right distributivity is unwanted in our theory.

Theorem 54 (Left Distributivity). $\forall s, t, u [Cb(t, s) \wedge Cb(u, s) \rightarrow (s \circ (t + u) = s \circ t + s \circ u)]$

Proof. Let s, t and u be slots such that t and u are contextualisable by s . We want to prove that $s \circ (t + u)$ and $s \circ t + s \circ u$ are identical. To do so, we will use the *OoS-Extensionality* (Theorem 41). Therefore, we have to prove that for every v that overlaps one, it also overlaps the other one ($\forall v (OoS(v, s \circ (t + u)) \leftrightarrow OoS(v, s \circ t + s \circ u))$). Let v be a slot.

Left-to-right: suppose v overlaps $s \circ (t + u)$. We want to prove that v overlaps with $s \circ t + s \circ u$. By Theorem 53, we know that v overlaps $s \circ t$ or $s \circ u$. In both cases, by Definition 12, we know that v overlaps $s \circ t + s \circ u$.

Right-to-left: suppose that v overlaps $s \circ t + s \circ u$. We want to prove that v overlaps with $s \circ (t + u)$. By Definition 12, we know that v overlaps $s \circ t$ or $s \circ u$. In both cases, by Theorem 53, we know that v overlaps $s \circ (t + u)$.

We proved that $\forall v (OoS(v, s \circ (t + u)) \leftrightarrow OoS(v, s \circ t + s \circ u))$, therefore by *OoS-Extensionality*, $s \circ (t + u) = s \circ t + s \circ u$. \square

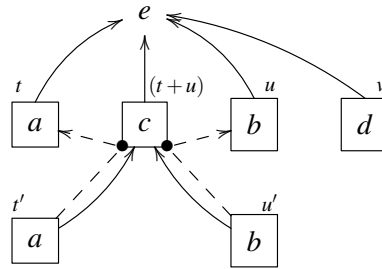


Fig. 13. Counter-example to right distributivity

In this theory, we do not want right distributivity of contextualisation over the sum. Take the example pictured by Figure 13. Consider the sum of t and u , i.e. the slot $(t + u)$. This slot is filled by an entity c , that has two slots t' and u' , such that $t = (t + u) \circ t'$ and $u = (t + u) \circ u'$. From the first equality, if we admit right distributivity, $t \circ t'$ and $u \circ t'$ would exist and t would be identical to $t \circ t' + u \circ t'$. However, neither $t \circ t'$ nor $u \circ t'$ exist, as the conditions of existence are not fulfilled: there is no entity that fills t or u and owns t' .

In fact, the only case where the right distributivity holds, i.e. $(s + t) \circ u = s \circ u + t \circ u$ (call this H) is true, is when $s = t$. If so, $s + t = s + s = s$, and by replacing t , we get $s \circ u = s \circ u + s \circ u$. The sum being idempotent, this is a tautology. However, it is possible to go further: if $(s + t) \circ u$, $s \circ u$ and $t \circ u$ exist (so without positing H) then $s = t$. This is the result of Theorem 56. Before proving it, consider Lemma 55 that shows that if s and a sum of s and another slot are filled by the same filler, they are equal.

Lemma 55 (Same Filler of Operand and Sum Implies Identity). $\forall s, t (SF(s, s + t) \rightarrow s = s + t)$

Proof. Let s and t be two slots such that they have the same filler. Let us call this filler a . By Definition 11, s is a slot-part of $s + t$. So by Definition 8, there is a slot s' such that $s = (s + t) \circ s'$. According to Theorem 12 (Contextualisation Same Filler) and BA7 (Single Occupancy), s and s' have the same filler, i.e. a . According to Axioms 5 (Contextualisable iff Contextualisation Exists) and BA7, there is something that fills $s + t$ and owns s' , i.e. a . The slot s' is owned and filled by a . So by Definition 5, s' is the improper slot of a . Finally, by Theorem 8 (Improper Slot Is Right Neutral Element), as s' is an improper slot, we deduce, from $s = (s + t) \circ s'$, that $s = s + t$. \square

Theorem 56 (Right Distributivity Is Trivial). $\forall s, t, u (Cb(s+t, u) \wedge Cb(s, u) \wedge Cb(t, u) \rightarrow s = t)$

Proof. Let s, t and u be three slots such that $(s+t) \circ u, s \circ u$ and $t \circ u$ exist. We want to prove that $s = t$. According to Axiom 5 (Contextualisable iff Contextualisation Exists), there are three fillers a, b and c such that a fills $s+t$ and owns u, b fills s and owns u and c fills t and owns u . Furthermore, by Axiom 1 (Single Owner), $a = b = c$. According to Lemma 55, as s and $s+t$ are filled by the same filler, $s = s+t$. In the same way, we deduce that $t = s+t$. Therefore, $s = t$. \square

We show with Lemma 57 that if s and t are slot-parts of u , then $s+t$ is a slot-part of u .

Lemma 57 (Sum is Slot-Part if Operands are Slot-Parts). $\forall s, t, u (PoS(s, u) \wedge PoS(t, u) \rightarrow PoS(s+t, u))$

Proof. Let s, t and u be three slots such that s and t are slot-parts of u ($PoS(s, u) \wedge PoS(t, u)$), i.e. there are s' and t' such that $s = u \circ s'$ and $t = u \circ t'$. According to Theorem 21 (PoS Same Owner) and Axiom 1 (Single Owner), s, t and u share the same owner. Therefore, according to Axiom 11 (Sum Existence), $s+t$ exists. We know that $s+t = (u \circ s') + (u \circ t')$. With Left Distributivity (Theorem 54), we get that $s+t = u \circ (s' + t')$, i.e. that $PoS(s+t, u)$. \square

We then prove that slot-sum is associative, as demonstrated by Theorem 58.

Theorem 58 (Sum Associativity). $\forall s, t, u [SO(s, t) \wedge SO(t, u) \rightarrow (s+t) + u = s + (t+u)]$

Proof. Let s, t and u be three slots that have the same owner. SoS_1 and SoS_2 being equivalent, let us use SoS_2 . By Definition 12, $(s+t) + u$ is:

- (a) $\forall v (OoS(v, s+t) \leftrightarrow OoS(v, s) \vee OoS(v, t))$
- (b) $\forall v (OoS(v, (s+t) + u) \leftrightarrow OoS(v, s+t) \vee OoS(v, u))$

By using (a) to partially rewrite (b), we get $\forall v (OoS(v, (s+t) + u) \leftrightarrow OoS(v, s) \vee OoS(v, t) \vee OoS(v, u))$ (1).

Similarly, from $s + (t+u)$, we get $\forall v (OoS(v, s + (t+u)) \leftrightarrow OoS(v, s) \vee OoS(v, t) \vee OoS(v, u))$ (2).

From (1) and (2), we obtain $\forall v (OoS(v, (s+t) + u) \leftrightarrow OoS(v, s + (t+u)))$, which, according to Theorem 41, entails $(s+t) + u = s + (t+u)$. \square

We can prove that sums are stable under contextualisation, i.e. that for some slots u, s and t that can be contextualised by v , it is true that $u = s+t$ iff $v \circ u = v \circ s + v \circ t$, as showed by Theorem 59.

Theorem 59 (Sum Stable under Contextualisation).

$$\forall s, t, u, v [Cb(u, v) \wedge Cb(s, v) \wedge Cb(t, v) \rightarrow (u = s+t \leftrightarrow (v \circ u) = (v \circ s) + (v \circ t))]$$

Proof. Let s, t, u and v be slots such that u, s and t are contextualisable by v . By Definition 6 and Axiom 1, there is a unique a that is the filler of v and the owner of u, s and t . According to Theorem 24 (PoS Stable under Contextualisation), we have $PoS(s, u) \leftrightarrow PoS(v \circ s, v \circ u)$ and $PoS(t, u) \leftrightarrow PoS(v \circ t, v \circ u)$ (call them H1 and H2). We want to prove that $u = s+t \leftrightarrow (v \circ u) = (v \circ s) + (v \circ t)$.

Using Definition 11, it is $[PoS(s, u) \wedge PoS(t, u) \wedge \forall v' (PoS(v', u) \rightarrow OoS(s, v') \vee OoS(t, v'))] \leftrightarrow [PoS(v \circ s, v \circ u) \wedge PoS(v \circ t, v \circ u) \wedge \forall v' (PoS(v', v \circ u) \rightarrow OoS(v \circ s, v') \vee OoS(v \circ t, v'))]$. Using H1 and H2, it can be rewritten as $[PoS(v \circ s, v \circ u) \wedge PoS(v \circ t, v \circ u) \wedge \forall v' (PoS(v', v \circ u) \rightarrow OoS(s, v') \vee OoS(t, v'))] \leftrightarrow [PoS(v \circ s, v \circ u) \wedge PoS(v \circ t, v \circ u) \wedge \forall v' (PoS(v', v \circ u) \rightarrow OoS(v \circ s, v') \vee OoS(v \circ t, v'))]$, which can be simplified into $\forall v' (PoS(v', v \circ u) \rightarrow OoS(s, v') \vee OoS(t, v')) \leftrightarrow \forall v' (PoS(v', v \circ u) \rightarrow OoS(v \circ s, v') \vee OoS(v \circ t, v'))$

Left-to-right: suppose that $\forall v' (PoS(v', v \circ u) \rightarrow OoS(s, v') \vee OoS(t, v'))$ (call this H). Let w' be a slot such that $PoS(w', v \circ u)$, i.e. there is a x such that $w' = (v \circ u) \circ x$. We want to prove $OoS(v \circ s, w') \vee OoS(v \circ t, w')$. We have $w' = (v \circ u) \circ x$. By Axiom 9, we get $w' = v \circ (u \circ x)$. Let w be $u \circ x$. Given $w' = v \circ w$, by Theorem 38, the goal $OoS(v \circ s, v \circ w) \vee OoS(v \circ t, v \circ w)$ is equivalent to $OoS(s, w) \vee OoS(t, w)$. By Definition 8, $w = u \circ x$ gives $PoS(w, u)$. By H and $PoS(w, u)$, we prove the goal

1 $OoS(s, w) \vee OoS(t, w)$.

2 Right-to-left: suppose that $\forall v'(PoS(v', v \circ u) \rightarrow OoS(v \circ s, v') \vee OoS(v \circ t, v'))$ (call this H). We want to prove that
 3 $\forall v'(PoS(v', u) \rightarrow OoS(s, v') \vee OoS(t, v'))$. Let w be a slot such that $PoS(w, u)$. Let us prove that $OoS(s, w) \vee OoS(t, w)$. Ac-
 4 cording to Theorem 21, u and w have the same owner, i.e. a . Therefore, w is contextualisable by v . By Theorem 38, the
 5 goal $OoS(s, w) \vee OoS(t, w)$ is equivalent to $OoS(v \circ s, v \circ w) \vee OoS(v \circ t, v \circ w)$. By Theorem 24, $PoS(w, u)$ is equivalent to
 6 $PoS(v \circ w, v \circ u)$, which, by H, gives $OoS(v \circ s, v \circ w) \vee OoS(v \circ t, v \circ w)$, i.e. the goal. \square

7 We added a new relation of slot-sum, syntactically based on Varzi's 39₂. After axiomatising the ex-
 8 istence condition of the sum of slots, we highlighted the properties of the sum of slots: unicity, same-
 9 owner, idempotence, commutativity, left distributivity, associativity and stability under contextualisation.
 10 In the next section, we will take a step further by defining the fusion of slots.

11 4.7. Fusion of Slots

12 After having axiomatised the sum of slots, we look at mereological fusion. This has two goals. First,
 13 fusion is a classical construct of mereological theories: adding it to our theory extends the coverage of
 14 mereological concepts. Second, we expect the fusion to make provable one of our pre-formal intuitions,
 15 left unproved until now: the fact that an improper slot is the “union”²³ of all the slots of its filler.

16 While binary sum can be understood as an operation over a finite collection of entities where the
 17 operands must be given in an extensional way (e. g. $a + b + c$), fusion operates over a collection of
 18 entities given in an intensional way (e. g. the fusion of all the cats, or the fusion of all the slots of a given
 19 filler), such a collection possibly being infinite.

20 As is the case with sum, there are multiple definitions of fusion in the literature (see (Varzi, 2019)
 21 and (Cotnoir and Varzi, 2021)). While other definitions exist,²⁴ we choose to use the definition of the
 22 so-called Leśniewski fusions, as given by Definition Schema 13. In this schema, ϕ is a formula in which
 23 the variable w should occur free. ϕ itself is not a variable in our (first-order) language; indeed, Definition
 24 Schema 13 is not a simple definition but a definition schema. For readability, we use a predicate-like
 25 notation, such as $\phi(w)$, even though ϕ is not to be understood as a predicate variable. The formula
 26 $FoS_\phi(z)$ should read as “ z is the fusion of all the ϕ -ers”, where the term “ ϕ -er” refers to an entity for
 27 which the formula ϕ holds.

28 What does it mean for z to be the fusion of all ϕ -ers? According to the definition schema, which
 29 contains two conjuncts, z is the fusion of all ϕ -ers iff 1) every ϕ -er is a slot-part of z and 2) every
 30 slot-part of z overlaps with some ϕ -er.

31 Definition Schema 13 (Fusion of Slots).

$$32 FoS_\phi(z) \triangleq \forall w(\phi(w) \rightarrow PoS(w, z)) \wedge \forall v(PoS(v, z) \rightarrow \exists w(\phi(w) \wedge OoS(v, w)))$$

33 In the literature, multiple formulas enforcing the existence of fusion are often taken as axioms. The
 34 Unrestricted Existence, presented in (6), ensures the existence of the fusion of the ϕ -ers, as long as there
 35 is at least one ϕ -er. This condition avoids positing the existence of the fusion of a collection of zero
 36 elements.

37 ²³The word “union” is used here to avoid using both the word “sum”, since our binary sum is not applicable to any (possibly
 38 infinite) number of slots, as well as the word “fusion” which is yet to be formally defined.

39 ²⁴Other existing definitions are equivalent to the definition we choose under some conditions: for more details, see (Cotnoir
 40 and Varzi, 2021, pp. 160-174).

$$\exists w(\phi(w)) \rightarrow \exists z(FoS_\phi(z)) \quad (6)$$

This formula is problematic for multiple reasons: it enables the fusion of fillers, the fusion of slots and fillers, and the fusion of slots that do not have the same owner. Indeed, as explored in Section 4.3 and as stated by Theorems 21 and 34, our theory's mereological relations only holds between slots that have the same owner. Therefore, we conclude that Unrestricted Existence is not desirable to our theory.

The Restricted Existence, presented in (7), generally contains another premise, which states that all ϕ -ers must satisfy some condition ψ .

$$(\exists w(\phi(w)) \wedge \forall w(\phi(w) \rightarrow \psi(w)) \rightarrow \exists z(FoS_\phi(z)) \quad (7)$$

In our case, we identified two conditions: each ϕ -er must be a slot, and all ϕ -ers must have the same owner. While a formula of the form $\forall w(\phi(w) \rightarrow \psi(w))$ can easily take into account that every w is a slot, we do not see what formula can say that all slots have the same owner. Therefore, we propose a slightly modified version as follows:

$$\exists w(\phi(w)) \wedge \forall w(\phi(w) \rightarrow S(w)) \wedge \exists w(\phi(w) \wedge \forall v(\phi(v) \rightarrow SO(w, v))) \rightarrow \exists z(FoS_\phi(z))$$

This formula states that 1) there is a ϕ -er, 2) every ϕ -er is a slot, and 3) there is a ϕ -er that has the same owner as all other ϕ -ers. This formula can be simplified as in Axiom Schema 12, i. e. if there is a ϕ -er and every ϕ -er have the same owner, then the fusion of these ϕ -ers exists.

Axiom Schema 12 (Fusion Existence). $\exists w(\phi(w) \wedge \forall v(\phi(v) \rightarrow SO(v, w))) \rightarrow \exists s(FoS_\phi(s))$

The fusion of the ϕ -ers is unique, as proved by Theorem Schema 60.

Theorem Schema 60 (Fusion Unicity). $\exists w(\phi(w)) \rightarrow \forall s, t(FoS_\phi(s) \wedge FoS_\phi(t) \rightarrow s = t)$

Proof. Consider w such that $\phi(w)$ holds. Let s and t be fusions of the ϕ -ers. We want to show that s and t are equal. To do so, let us use *OoS*-Extensionality: we now have to prove that s and t are slots and that $\forall u, OoS(s, u) \leftrightarrow OoS(t, u)$. By the first conjunct of Definition Schema 13 and $\phi(w)$, w is a slot-part of s and t . According to Lemma 17 (*PoS* Domain and Range), s and t are slots. We still have to prove $\forall u, OoS(s, u) \leftrightarrow OoS(t, u)$.

Let u be a slot. The proof is similar from left-to-right and right-to-left: suppose that s (resp. t) and u are overlapping. Therefore, by Definition 10, there is some slot a such that a is a slot-part of s (resp. t) and u . According to the second conjunct of Definition Schema 13, as a is a slot-part of s (resp. t), there exists some ϕ -er b that overlaps with a , i.e. there is some slot c that is a slot-part of a and b . According to the first conjunct of Definition Schema 13, as b is a ϕ -er, it is a slot-part of t (resp. s). At this point, we know that c is a slot-part of a and b , which are respectively parts of u and t (resp. s). Therefore, by *PoS* Transitivity, c is a common slot-part of u and t (resp. s), i.e. u and t (resp. s) are overlapping. \square

Theorem 61 shows that slot-sum, defined in Section 4.6.2, is equivalent to the fusion of a collection of two slots. This means that binary slot-sum is a particular of fusion, and therefore that Axiom 11 is an instantiation of Axiom Schema 12 and so redundant.

Theorem 61 (Sum is a special Case of Fusion). $\forall s, t, u(FoS_{w=s \vee w=t}(u) \leftrightarrow SoS_1(u, s, t))$

Proof. Let s , t and u be slots. We want to prove that $FoS_{w=s \vee w=t}(u) \leftrightarrow SoS_1(u, s, t)$.

Left-to-right: suppose that u is the fusion of ϕ -ers such that ϕ is $w = s \vee w = t$. By Definition Schema 13, $\forall w(w = s \vee w = t) \rightarrow PoS(w, u)$ (call this H1) and $\forall v(PoS(v, u) \rightarrow \exists w((w = s \vee w = t) \wedge OoS(v, w)))$ (call this H2).

We want to prove that $SoS_1(u, s, t)$, i.e. $PoS(s, u)$, $PoS(t, u)$ and $\forall v(PoS(v, u) \rightarrow OoS(s, v) \vee OoS(t, v))$. $PoS(s, u)$ and $PoS(t, u)$ are trivial using H1: if w is s or t , then $PoS(w, u)$. Let us prove that $\forall v(PoS(v, u) \rightarrow OoS(s, v) \vee OoS(t, v))$. Let v be a slot such that $PoS(v, u)$. By H1, there is w such that $(w = s \vee w = t) \wedge OoS(v, w)$. By rewriting it, we got $(w = s \wedge OoS(v, w)) \vee (w = t \wedge OoS(v, w))$, i.e. $OoS(s, v) \vee OoS(t, v)$.

Right-to-left: suppose that $u = s + t$, i.e. $PoS(s, u)$, $PoS(t, u)$ and $\forall v(PoS(v, u) \rightarrow OoS(s, v) \vee OoS(t, v))$ (call this H). We want to prove that u is the fusion of the ϕ -ers, i.e. $\forall w(w = s \vee w = t) \rightarrow PoS(w, u)$ (call this G1) and $\forall v(PoS(v, u) \rightarrow \exists w((w = s \vee w = t) \wedge OoS(v, w)))$ (call this G2).

G1: let w be a slot such that $w = s \vee w = t$. We want to prove that $PoS(w, u)$. This is trivial, given that both s and t are slot-parts of u .

G2: let v be a slot such that $PoS(v, u)$. We want to prove that there is some w such that $(w = s \vee w = t) \wedge OoS(v, w)$, or, rewritten the same way it was previously, $OoS(v, s) \vee OoS(v, t)$. This is trivial, given that $PoS(v, u)$ and H. \square

Theorem 62 shows that if an entity a has some slot, its improper slot is the sum of its slots. The proposition $\forall t(IP_s(t, a) \leftrightarrow FoS_{P_s(w, a)}(t))$ is equivalent to the current consequent of the theorem, as both improper slot and fusion are unique, by Axiom 3 and Theorem Schema 60.

Theorem 62 (Improper Slot is Fusion of Filler's Slots).

$$\forall a(\exists s(P_s(s, a)) \rightarrow \exists t(IP_s(t, a) \wedge FoS_{P_s(w, a)}(t)))$$

Proof. Let a be a filler and s a slot of a . We want to prove that the improper slot of a is the fusion of a 's slots. Because a owns s , we deduce by BA4 that there is a slot t that is the improper slot of a . Let us prove that t is the fusion of a 's slots. According to Fusion's Definition Schema 13, t is the fusion of a 's slots iff 1) $\forall w(P_s(w, a) \rightarrow PoS(w, t))$ and 2) $\forall v(PoS(v, t) \rightarrow \exists w(P_s(w, a) \wedge OoS(v, w)))$.

1) Let w be a slot of a . According to Theorem 22 (Slots iff Slot-Parts of Improper Slot), w is a slot-part of a 's improper slot, i.e. w is a slot-part of t .

2) Let v be a slot such that v is a slot-part of t . Let us prove that there is a slot that is a slot of a and that slot-overlaps with v . As a 's improper slot, t is a slot of a . Furthermore, as v is a slot-part of t , by Lemma 37, it overlaps with t .

In conclusion, if a has a slot, its improper slot is the fusion of its slots. \square

Finally, Theorem Schema 63 shows that fusion is stable under contextualisation. It states that if s' is the contextualisation of s by t and that every ϕ -er can be contextualised by t , then s is the fusion of the ϕ -ers iff s' is the fusion of the contextualisations by t of the ϕ -ers.

Theorem Schema 63 (Fusion Stable under Contextualisation).

$$\forall s, t[Cb(s, t) \wedge \forall w(\phi(w) \rightarrow Cb(w, t)) \rightarrow (FoS_\phi(s) \leftrightarrow FoS_{\exists w'(w=t \circ w' \wedge \phi(w'))}(t \circ s))]$$

Proof. Let s and t be slots such that s is contextualisable by t and $\forall w(\phi(w) \rightarrow Cb(w, t))$ (call this H). We want to prove $FoS_\phi(s) \leftrightarrow FoS_{\exists w'(w=t \circ w' \wedge \phi(w'))}(t \circ s)$.

Left-to-right: suppose that s is the fusion of the ϕ -ers. By Definition Schema 13, $\forall w(\phi(w) \rightarrow PoS(w, s))$ and $\forall v(PoS(v, s) \rightarrow \exists w(\phi(w) \wedge OoS(v, w)))$ (call them H1 and H2). We want to prove that $FoS_{\exists w'(w=t \circ w' \wedge \phi(w'))}(t \circ s)$, i.e. that $\forall w(\exists w'(w=t \circ w' \wedge \phi(w')) \rightarrow PoS(w, t \circ s))$ and $\forall v(PoS(v, t \circ s) \rightarrow \exists w(\exists w'(w=t \circ w' \wedge \phi(w')) \wedge OoS(v, w)))$ (call them G1 and G2).

G1: let w and w' be such that $w = t \circ w'$ and $\phi(w')$. We want to prove $PoS(w, t \circ s)$. According to H1, as w' is a ϕ -er, then $PoS(w', s)$. Given that $w = t \circ w'$ and $PoS(w', s)$, by Theorem 24 (PoS Stable under Contextualisation), we get $PoS(w, t \circ s)$.

G2: let v be such that $PoS(v, s')$, i.e. by Definition 8, there is v' such that $v = s' \circ v'$. We want to prove $\exists w(\exists w'(w = t \circ w' \wedge \phi(w')) \wedge OoS(v, w))$. By Axiom 9, $v = (t \circ s) \circ v' = t \circ (s \circ v')$. Let v'' be $s \circ v'$. By Definition 8, $PoS(v'', s)$. By H2, we

get that there is w'' such that $\phi(w'')$ and $OoS(v'', w'')$. Given that w'' is a ϕ -er, by H1, $PoS(w'', s)$. By Theorem 21, there is some a such that $P_s(w'', a)$ and $P_s(s, a)$. With Axiom 5 and $s' = t \circ s$, there is some b such that $F(b, t)$ and $P_s(s, b)$. Because of Axiom 1 (Single Owner), $a = b$. As a owns w'' and fills t , by Axiom 5, there is a w''' such that $w''' = t \circ w''$. Our goal is $\exists w(\exists w'(w = t \circ w' \wedge \phi(w')) \wedge OoS(v, w))$. There are w''' and w'' such that $w''' = t \circ w''$ and $\phi(w'')$. We only need to prove that $OoS(v, w''')$. Given that $v = t \circ v''$ and $w''' = t \circ w''$, by Theorem 38, we need to prove that $OoS(v'', w'')$. This as already proven.

Right-to-left: suppose that s' is the fusion of the contextualisations by t of the ϕ -ers. By Definition Schema 13, $\forall w(\exists w'(w = t \circ w' \wedge \phi(w')) \rightarrow PoS(w, s'))$ and $\forall v(PoS(v, s') \rightarrow \exists w(\exists w'(w = t \circ w' \wedge \phi(w')) \wedge OoS(v, w)))$ (call them H1 and H2). We want to prove that $FoS_\phi(s)$, i.e. $\forall w(\phi(w) \rightarrow PoS(w, s))$ and $\forall v(PoS(v, s) \rightarrow \exists w(\phi(w) \wedge OoS(v, w)))$ (call them G1 and G2).

G1: let w be a ϕ -er. We want to prove that $PoS(w, s)$. With H, there is a w' such that $w' = t \circ w$. Given that $s' = t \circ s$ and $w' = t \circ w$, by Theorem 24, we get that $PoS(w, s) \leftrightarrow PoS(w', s')$. Therefore, to prove $PoS(w, s)$, it is enough to prove $PoS(w', s')$. This can be proven using H1: w is a ϕ -er such that $w' = t \circ w$, therefore $PoS(w', s')$.

G2: let v' be a slot such that it is a slot-part of s , i.e. there is a v such that $v' = s \circ v$. We want to prove that there is a ϕ -er that overlaps with v' . s being contextualisable by t , let v'' be $v'' = t \circ v' = s' \circ v = t \circ s \circ v$. By Definition 8, v'' is a slot-part of s' . Therefore, with H2, there are w and w' such that $w = t \circ w'$, w' is a ϕ -er and v'' overlaps with w . As w' is a ϕ -er, let us prove that it overlaps with v' to complete the proof. As $OoS(v'', w)$, i.e. $OoS(t \circ v', t \circ w')$, by Theorem 38, $OoS(v', w')$. \square

5. Back to the Original Problems

Until now, the theory developed in this paper has been illustrated using simple, often partial, models. In this section, we propose to apply it to more realistic examples, in particular those used in Section 3, where we exposed the counting problems. Section 5.1 explores the DIHYDROGEN universal as a first and basic example. Section 5.2 illustrates how overlap works with the universal of two rooms with an adjoining wall. Finally, Section 5.3 shows how to deal with overduplications, illustrated with the informational entity “cats”. The first example will also be used to give examples of interesting properties of slot-sums. The examples will be presented step by step: we will first show the mereological structures with only direct slots; then we will apply contextualisation to represent the full structure.

5.1. The DIHYDROGEN Universal

This section illustrates the theory using the DIHYDROGEN, previously presented in Figure 3. The DIHYDROGEN universal has two proper slots S_1 and S_2 filled by the HYDROGEN universal and one improper slot S_{H_2} . The HYDROGEN universal has two proper slots S_3 and S_4 , respectively filled by the ELECTRON and the HYDROGENNUCLEUS universals, and one improper slot S_H . Finally, both ELECTRON and HYDROGENNUCLEUS only have an improper slot (as before, we consider HYDROGENNUCLEUS as a mereological atom). Their four mereological structures are pictured in Figure 14, where only improper and direct slots are represented.

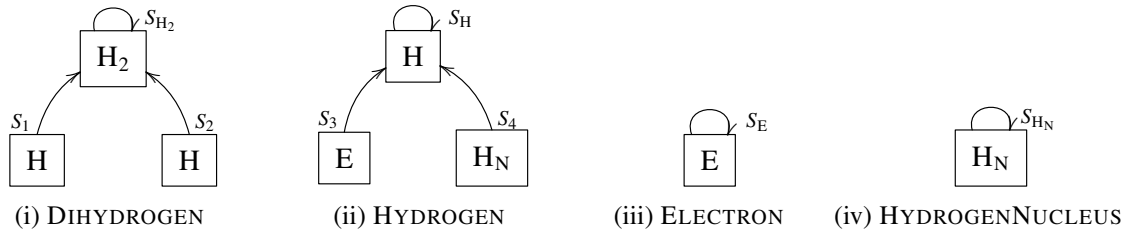


Fig. 14. Direct slots of DIHYDROGEN (H₂), HYDROGEN (H), ELECTRON (E) and HYDROGENNUCLEUS (H_N)

A partial view on the mereological structure of DIHYDROGEN is pictured in Figure 15. To simplify the figure, neither the improper slots nor the contextualisations in which those improper slots are involved are displayed. Some sums, like $S_5 + S_7$ are not represented.

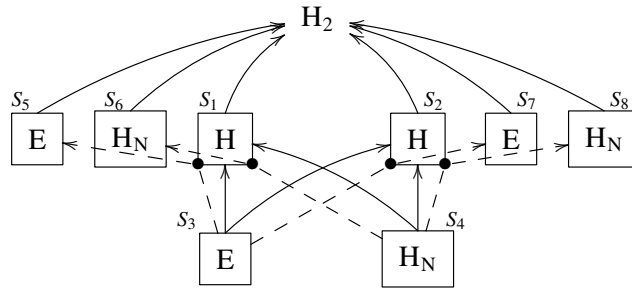


Fig. 15. A partial view on the mereological structure of DIHYDROGEN

In this example, the DIHYDROGEN universal has HYDROGEN as a part twice, and ELECTRON and HYDROGENNUCLEUS two times each, as expected. Furthermore, there are no overduplications. However, this partial view does not represent all the slots of the DIHYDROGEN universal. In Figure 16, all

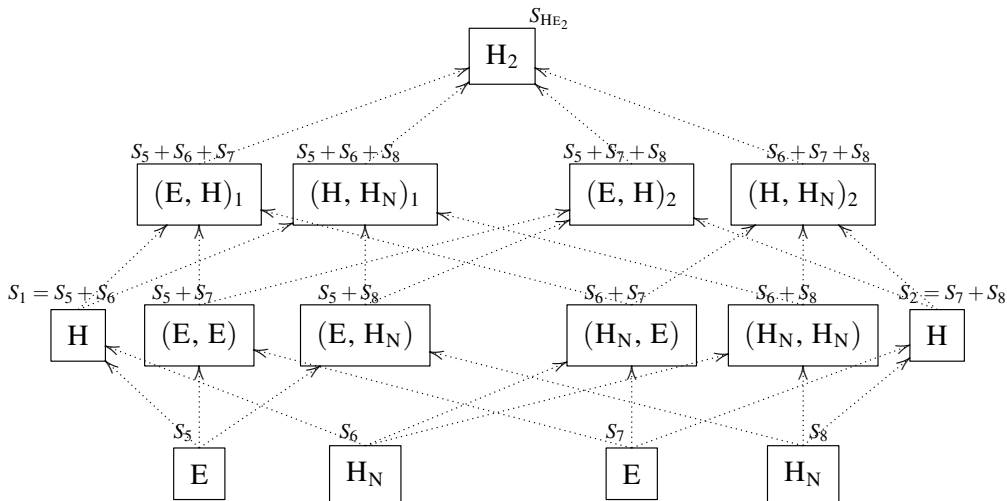


Fig. 16. A semi-lattice of sums of slots (dotted arrows represent the $PPoS$ relation)

the slots of DIHYDROGEN are represented. The relation pictured by dotted arrows is not P_s , but $PPoS$. Slots linked to the same slot all sum together to that targeted slot. For example, summing the slots S_1 , $S_5 + S_7$ and $S_6 + S_7$ results in slot $S_5 + S_6 + S_7$. Some slots are filled by entities whose nature is unknown. For example, slot $S_5 + S_8$ is filled by something that has ELECTRON and HYDROGENNUCLEUS as parts once each.²⁵ However, this filler of $S_5 + S_8$ is not HYDROGEN. Indeed, we can adapt Bennett (2017)'s argumentation for her *Spatial Separation* case. She argued convincingly that the electron of one hydrogen atom and the nucleus of another hydrogen atom do not sum up to an hydrogen atom. Similarly, in

²⁵Our theory does not include any mereological sum on fillers. Such a sum would be by no means trivial, as developed in Section 4.6.1. Therefore, the content of these slots in this example is simply written as a list of their parts. This notation does not belong to the language of our theory.

the current theory, a slot filled by ELECTRON from one contextualisation of HYDROGEN (S_1) and a slot filled by HYDROGENNUCLEUS from a different contextualisation of HYDROGEN (S_2) do not sum up to a slot filled by HYDROGEN, as they do not come from the same contextualisation.²⁶

Also, as it is unknown whether the filler of $S_5 + S_8$ and the one of $S_6 + S_7$ are identical, indexes have been added to differentiate them. Those numbers only have a differentiating use, and not an ordering one. The sum of all the slots of DIHYDROGEN is its improper slot S_{H_2} .

5.2. The TWOADJOININGROOM Universal

Consider the universal whose instances are two two-wall rooms²⁷ separated by an adjoining wall. We consider the universals WALL, ROOM and TWOADJOININGROOM. The first one is atomic and only has an improper slot S_{WALL} . The second one has an improper slot S_{ROOM} and two slots S_1 and S_2 filled by WALL. Finally, the last one has an improper slot S_{TAR} and two slots S_3 and S_4 filled by ROOM. The mereological structures are pictured in Figure 17 where only improper slots and direct slots are represented.

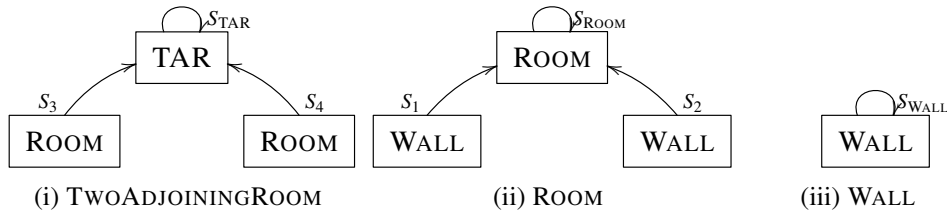


Fig. 17. Direct slots of TWOADJOININGROOM (TAR), ROOM and WALL

There are four slots resulting of the contextualisation of a proper slot of ROOM by a proper slot of TWOADJOININGROOM, namely S_5 to S_8 . Those contextualisations are pictured in Figure 18. Improper slots and contextualisations involving them are not displayed.

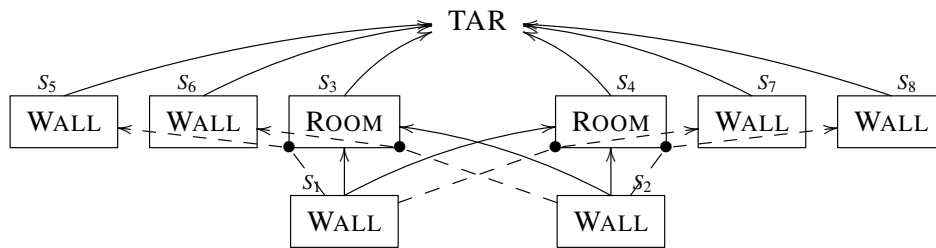


Fig. 18. Full structure of TWOADJOININGROOM

If we want a universal isomorphic to its instances, the two occurrences of the universal ROOM within the universal TWOADJOININGROOM are supposed to overlap. To do so, a slot resulting of a contextualisation of some slot by S_3 has to be the same as a slot resulting of a contextualisation of some slot by

²⁶This conclusion is supported by the fact that for $S_5 + S_8$ to be filled by HYDROGEN, $S_5 + S_8$ must be the improper slot of HYDROGEN (S_H) or one of its contextualisation. But it does not seem that we can rewrite $S_5 + S_8 = (S_1 \circ S_3) + (S_2 \circ S_4)$ using S_H . Contrast this with $S_5 + S_6 = S_1 \circ S_3 + S_1 \circ S_4$. By Left Distributivity, it is $S_1 \circ (S_3 + S_4)$, i.e. $S_1 \circ S_H$. The latter being a contextualisation of S_H , it is filled by HYDROGEN.

²⁷Although our world does not abound with two-wall rooms, this makes for a simple example. One may imagine a semi-circular wall closed by a straight wall.

1 S_4 . Let us say that $S_6 = S_7$. Figure 19 represents the correct structure of TWOADJOININGROOM, with
 2 S_3 and S_4 overlapping on S_6 .

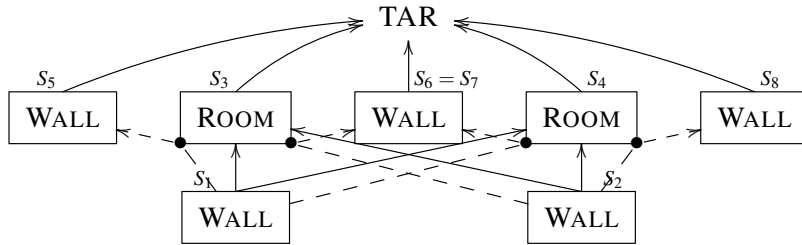


Fig. 19. Full structure of TWOADJOININGROOM with overlapping slots

5.3. The Informational Entity “cats”

For this example, let us recursively decompose the string “cats” into two parts: all the letters except the last one and the last letter, i.e. “cat” and “s”. The string “cat” is itself decomposed into “ca” and “t”. Finally, “ca” is decomposed into “c” and “a”. The four letters only own an improper slot each, respectively S_c , S_a , S_t and S_s . The string “ca” has an improper slot S_{ca} and two proper slots S_1 and S_2 respectively filled by “c” and “a”. The string “cat” has an improper slot S_{cat} and two direct proper slots S_3 and S_4 filled by “ca” and “t”. Finally, the string “cats” has an improper slot S_{cats} and two direct proper slots S_5 and S_6 filled by “cat” and “s”. The mereological structures are pictured in Figure 20 where only improper slots and direct slots are represented. In this figure and the following ones, some slots are not pictured. For example, we can consider that “cats” also decomposes into “ca” and “ts”. However, for simplicity, we do not explore every possibility here.

To explain the role of Axiom 9 (Contextualisation Associativity), in Figure 21, we first do not accept it and show a model with overduplicated slots. Then, we show in Figure 22 how overduplications are solved thanks to the adoption of Axiom 9. In these two last figures, contextualisations are not represented as usual. For readability, the black dots used in all previous figures to represent the contextualising slots are replaced by labels between square brackets on the dashed arrows ($\square \leftarrow [s] \square$).

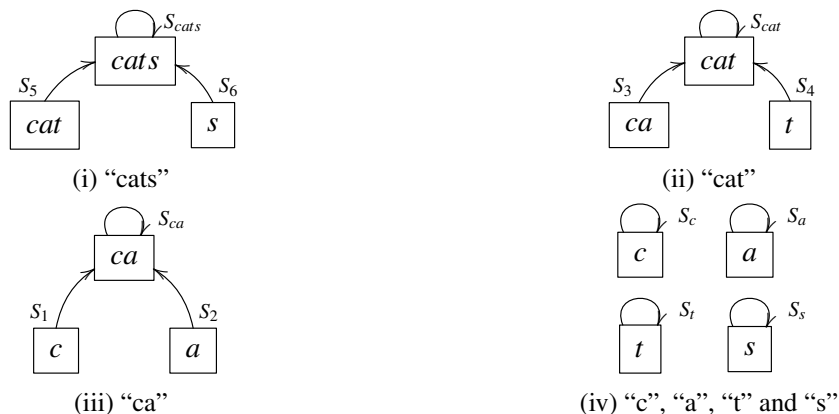


Fig. 20. Some relevant direct and improper slots of the informational entities

As in the previous examples, improper slots and contextualisations involving them are not displayed. The indirect slots of “cat” are S_7 and S_8 respectively equal to $S_3 \circ S_1$ and $S_3 \circ S_2$. Those of “cats” are $S_9 = S_5 \circ S_3$, $S_{10} = S_9 \circ S_1$, $S_{11} = S_9 \circ S_2$, $S_{12} = S_5 \circ S_7$, $S_{13} = S_5 \circ S_8$, and $S_{14} = S_5 \circ S_4$. These are all the possible contextualisations by the proper slots initially considered.

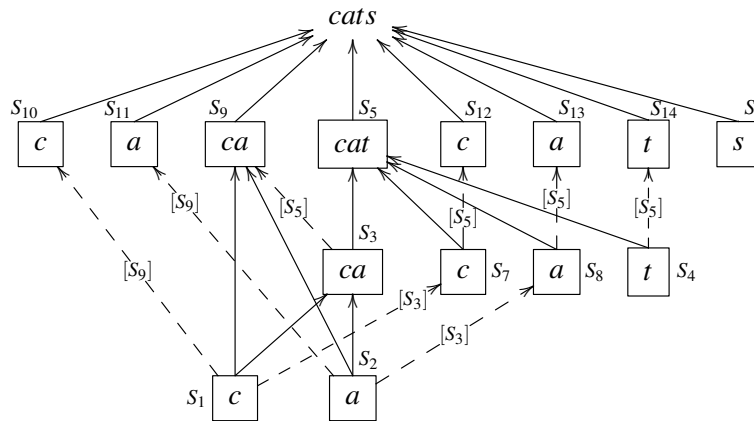


Fig. 21. A partial view of the structure of “cats” without the Associativity Axiom

However, in Figure 21, “cats” has two slots filled by “c” and two slots filled by “a”, where it should only have one of each. This is because Contextualisation Associativity (Axiom 9) is not accepted here. If we accept it, $S_{10} = (S_5 \circ S_3) \circ S_1 = S_5 \circ (S_3 \circ S_1) = S_{12}$ and $S_{11} = (S_5 \circ S_3) \circ S_2 = S_5 \circ (S_3 \circ S_2) = S_{13}$, as pictured in Figure 22.

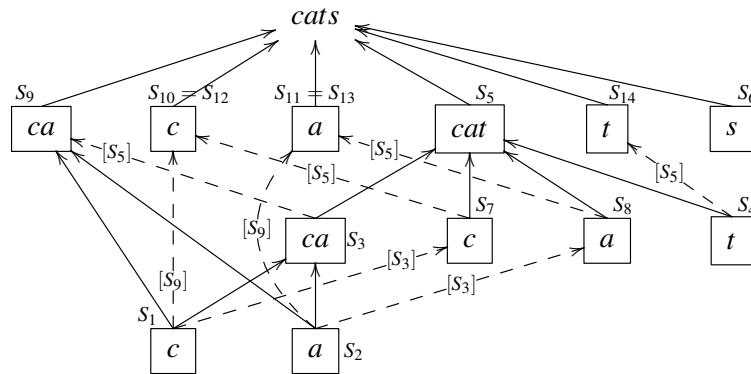


Fig. 22. A partial view of the structure of “cats” with the Associativity Axiom

6. Discussion

6.1. Identity Based on Non-Mereological Relations

When it comes to identity criteria in classical mereological theories, the basic principle used is extensionality: if two things have exactly the same proper parts or overlap exactly the same entities, they

are identical. However, for the entities considered in this paper (e.g. structural universals and informational entities), this is not adequate. Indeed, as pointed out above, METHANE, BUTANE and ISOBUTANE have exactly the same parts: CARBON and HYDROGEN. Similarly, the words “cat”, “act” and “tact” are composed of the same letters: “a”, “c” and “t”.

In the context of a mereological analysis of words, Carrara and Smid (2022) identify three identity principles: *Kind*, *Number* and *Order*. Those principles are defined as follows (Carrara and Smid, 2022, p. 178):

- *Kind*: “If two words do not have all the same letters, they are distinct.”
- *Number*: “If the number of letter instances in one word is different from the number of letter instances in another, they are distinct.”
- *Order*: “If the order of letters in one word is different from the order of letters in another word, they are distinct.”

In classic mereological theories, the negation of the premise of the generalisation of the first principle to any entity is usually accepted as a necessary and sufficient identity condition. However, according to Carrara and Smid (2022), this is not enough for word types: the negations of the three premises are “individually necessary and jointly sufficient for the identity of word types”. With this joint criterion, we can differentiate our previous examples: “cat”, “act” and “tact”. The two principles *Kind* and *Number* are enough to differentiate “cat” and “tact”, but to differentiate “cat” and “act”, the three are needed.

These identity principles have been introduced for informational entities, but they can be adapted to differentiate structural universals too. In fact, even though they were not clearly expressed, Lewis discussed similar principles. Lewis (1986, p. 37) uses a variant of the *Number* principle to differentiate METHANE and BUTANE:

“We might restore the talk of parts many times over; agree that two different things cannot be made of the very same parts taken once each, but insist that two things can be made of the same parts if there is a difference in how many times over some part is taken. Such is the difference between the structural universals *methane* and *butane*.”

Furthermore, on page 38, Lewis describes the problem of identity between BUTANE and ISOBUTANE:

“Consider isobutane. Where butane has a straight chain, isobutane branches. Its molecules consist of a central carbon atom bonded to three outlying carbon atoms; the central carbon atom is bonded also to one hydrogen atom, and each outlying carbon atom is bonded to three hydrogen atoms. So the structural universal *isobutane* consists of the universal *carbon* four times over, the universal *hydrogen* ten times over, and the dyadic universal *bonded* thirteen times over — just like the universal *butane*. But these two structural universals are different, as witness the different molecules that instantiate them. Even if our adverbial differences made sense, they would not solve our problem.”

Lewis clearly states that they have the same parts, and the same number of each parts, thus failing to be different with the principles *Kind* and *Number*. Lewis states what the difference between BUTANE and ISOBUTANE should be: “Where butane has a straight chain, isobutane branches”. Their difference is thus based on how the parts are spatially arranged. Lewis examines whether this difference could be accounted for mereologically, considering the BONDED universal as a part of the molecule universals. But as he observes, considering as parts these relations between those entities does not solve the problem: assuming that those universals have slots filled with BONDED, both BUTANE and ISOBUTANE would have BONDED as a part the same number of times, i.e. 13 times. In fact, such a mereological approach

1 does not fully capture the idea that the parts arrangement is relevant for the identity of the whole (Lewis
2 doesn't draw this conclusion and simply dismisses the very idea of structural universals).

3 The identity of a whole is based on which entities are related with non-mereological relations such as
4 BONDED for molecule universals, and letter ordering for words. But for the same reason that motivates
5 Bennett to introduce slots to represent mereological relations, i.e. the lack of intrinsic contexts of atom
6 universals or letters (see Section 4.1.1), we observe that non-mereological relations like BONDED cannot
7 hold between fillers. Nevertheless, as previously explained, slots, being the occurring contexts for their
8 fillers within a whole, could instead be the relata of those non-mereological relations. This means that
9 our theory has all the potential required to formalise the general idea behind the third principle of Carrara
10 and Smid (2022).

11 In our theory, the generalised principle *Kind* could read “If two fillers do not have all the same parts,
12 they are distinct”, and the generalised principle *Number* could read “If two fillers have a different number
13 of slots filled by the same entity, they are distinct”. To be able to generalise the third principle *Order*, the
14 signature of our theory would need to be extended with non-mereological relations between slots.

15 Each domain needs its relevant non-mereological relations. An extension of our theory for the
16 molecule and atom universals with the BONDED relation requires an appropriate analysis of this re-
17 lation, which is not within the scope of this paper. For the order within informational entities, adapting
18 existing theories of mereology and order over intervals, such as Allen's theory (1983), can be considered
19 for future work.

21 6.2. Slot Mereology, Particulars and Universals

22
23 We have used slot mereology to represent the structure of special entities, such as structural universals
24 and informational entities. The common property between these entities is the fact that they can have
25 the same entity as a part multiple times. However, we haven't discussed the relevance of using slot
26 mereologies for entities that do not have this property, like material individuals. It might indeed be
27 considered as desirable to have a single theory for both universals and particulars. But one can wonder
28 whether it is feasible. We do not see any feature of the theory that would make impossible its use to
29 represent the structure of material individuals. However, an important advantage of slot mereologies is
30 to represent a context of appearance of an entity that is not intrinsically associated with this context,
31 as discussed in Section 4.1.1. This is not the case of material individuals, as such individuals can only
32 appear in one mereological context at a given time. Unless slots are reused for something else than
33 mereological purposes, for instance to represent roles, as suggested by Bennett, using slot mereologies
34 appears unnecessary for material individuals.

35 Nonetheless, slot mereologies enable the representation of the mereological structure of structural
36 universals. Now that mereological structures of universals and their instances are representable, it opens
37 up a field of investigation on the relations between the structures of universals and the structures of their
38 instances. This raises questions such as: is the structure of a universal really isomorphic to the structure
39 of its instances, as assumed by Lewis (1986) (see Section 3.1)? How could we ensure such isomorphism?
40 Further, how could we ensure some flexibility in this isomorphism, in particular to account for essential
41 and optional parts and for mereological change (e.g. if a cat loses its tail and has additional thumbs, does
42 it still instantiate the universal of CAT)? We could also consider questions related to counting parts: how
43 far does the number of parts of a universal constrain the number of parts of its instances?

44 This theory could also be linked to “silent change”, as presented by Grewe et al. (2016), in which an
45 assertion remains true at the universal level while the involved particulars may change: in this theory,
46

1 a particular could keep the same slot structure while the fillers filling those slots might change. As 1
2 said above, we leave the complex issues of mereological change and of questioning the isomorphism 2
3 principle for further work. 3
4

5 6 **7. Conclusion**

7
8 In this paper we discussed Bennett's slot mereology (Bennett, 2013), mainly illustrated on standard 8
9 examples of structural universals such as METHANE. We showed how Bennett's theory tackles the prob- 9
10 lem of structural universals by splitting the parthood relation into two relations: P_s and F . However, we 10
11 also showed that this theory has counting problems: although it was intended to manage entities with the 11
12 same part multiple times, it generally leads to the wrong counting results. Bennett was aware that the 12
13 theory she proposed was a first sketch. And indeed, it raised some issues, such as the counting problems 13
14 and the difficulty to express some mereological principles, like supplementation, using slots. 14

15 In a previous paper, we proposed a theory that fixed Bennett's counting problems, but which came 15
16 with its own problem. That theory added the claim that slots are inner elements of entities, i.e. they 16
17 cannot be shared or inherited. To compensate the removal of slot inheritance, we developed the copy- 17
18 slot mechanism, i.e. slots are copied as many times as needed to count correctly. However, our theory 18
19 presented an excessive amount of copies, which ultimately led to the wrong counting results. 19

20 In the present paper, we kept both ideas: Bennett's idea of splitting parthood and our copy-of-inner- 20
21 elements idea. We first discussed the nature of slots: they have a single owner on which they are existen- 21
22 tially dependent and are either proper or improper. We then introduced the *slot contextualisation* relation 22
23 that relates together three slots. The use of this relation fixes the counting problems. However, we pushed 23
24 further by developing a mereological theory in which relations hold between slots. Therefore, we devel- 24
25 oped slot-(proper-)parthood, slot-overlap, slot-sum and slot-fusion. Finally, we were able to propose a 25
26 principle of slot strong supplementation that captures the idea of classical strong supplementation that 26
27 Bennett did not succeed to capture with hybrid relations between slots and fillers. Our theory can be ap- 27
28 plied to various domains: it offers a sound and richer representation framework for structural universals, 28
29 and it can be used to represent informational entities. 29

30 This work can be extended in several directions. In particular, relations between slots could help 30
31 capturing the structure of universals beyond mereology. The articulation between this theory and the one 31
32 developed by Barton et al. (2022) to account for the mereology of informational entities should also be 32
33 investigated. 33
34

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37
38 We are grateful for the insightful comments received on a previous version of this paper from Anthony 38
39 Fisher, Claudio Masolo and an anonymous reviewer. 39
40

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Appendix A. Overview of the Theory

This theory has three primitives: P_s , F and the contextualisation CoS .

Number	Name	Definition
BD1	Parthood	$P(a, b) \triangleq \exists s(P_s(s, b) \wedge F(a, s))$
BD2	Proper Parthood	$PP(a, b) \triangleq P(a, b) \wedge \neg P(b, a)$
BD3	Overlap	$O(a, b) \triangleq \exists c(P(c, a) \wedge P(c, b))$
BD4	Slot-overlap	$O_s(a, b) \triangleq \exists s(P_s(s, a) \wedge P_s(s, b))$
BD5	Proper Parthood Slot	$PP_s(s, a) \triangleq P_s(s, a) \wedge \neg F(a, s)$
D1	Direct Slot	$DP_s(s, a) \triangleq P_s(s, a) \wedge \neg \exists b(PP(b, a) \wedge P_s(s, b))$
D2	Slot	$S(s) \triangleq \exists a(P_s(s, a))$
D3	Same Owner	$SO(s, t) \triangleq \exists a(P_s(s, a) \wedge P_s(t, a))$
D4	Same Filler	$SF(s, t) \triangleq \exists a(F(a, s) \wedge F(a, t))$
D5	Improper Slot	$IP_s(s, a) \triangleq P_s(s, a) \wedge F(a, s)$
D6	Contextualisable by	$Cb(t, s) \triangleq \exists a(F(a, s) \wedge P_s(t, a))$
D7	Direct Slot — With Contextualisation	$DP'_s(s, a) \triangleq P_s(s, a) \wedge \forall t, u[s = t \circ u \rightarrow (\exists b(IP_s(t, b))) \vee (\exists c(IP_s(u, c)))]$
D8	Part of Slot	$PoS(u, s) \triangleq \exists t(u = s \circ t)$
D9	Proper Part of Slot	$PPoS(s, t) \triangleq PoS(s, t) \wedge s \neq t$
D10	Overlap of Slot	$OoS(s, t) \triangleq \exists u(PoS(u, s) \wedge PoS(u, t))$
D11	Sum with Parthood	$SoS_1(u, s, t) \triangleq PoS(s, u) \wedge PoS(t, u) \wedge \forall v(PoS(v, u) \rightarrow OoS(s, v) \vee OoS(t, v))$
D12	Sum with Overlap	$SoS_2(u, s, t) \triangleq \forall v(OoS(u, v) \leftrightarrow OoS(s, v) \vee OoS(t, v))$
DS13	Fusion of Slots	$FoS_\phi(z) \triangleq \forall w(\phi(w) \rightarrow PoS(w, z)) \wedge \forall v(PoS(v, z) \rightarrow \exists w(\phi(w) \wedge OoS(v, w)))$

Number	Description	Axiom
BA1	Only Slots are Filled	$F(a, s) \rightarrow \exists b(P_s(s, b))$
BA2	Slots Cannot Fill	$F(a, s) \rightarrow \neg \exists b(P_s(a, b))$
BA3	Slots Don't Have Slots	$P_s(s, a) \rightarrow \neg \exists t(P_s(t, s))$
BA4	Improper Parthood Slots	$\exists s(P_s(s, a)) \rightarrow \exists t(P_s(t, a) \wedge F(a, t))$
BA6	Mutual Occupancy is Identity	$(P_s(s, b) \wedge F(a, s)) \wedge (P_s(t, a) \wedge F(b, t)) \rightarrow a = b$
BA7	Single Occupancy	$P_s(s, a) \rightarrow \exists! b(F(b, s))$
A1	Single Owner	$P_s(s, a) \wedge P_s(s, b) \rightarrow a = b$
A2	Additional Improper Slot	$F(a, s) \rightarrow \exists t(IP_s(t, a))$
A3	Unique Improper Slot per Filler	$IP_s(s, a) \wedge IP_s(t, a) \rightarrow s = t$
A4	Domains of Contextualisation	$CoS(u, s, t) \rightarrow S(u) \wedge S(s) \wedge S(t)$
A5	Contextualisable iff Contextualisation Exists	$Cb(t, s) \rightarrow \exists u(CoS(u, s, t))$
A6	Unicity of Contextualisation	$CoS(u, s, t) \wedge CoS(v, s, t) \rightarrow u = v$
A7	Injectivity to the Left	$v = s \circ t \wedge v = s \circ u \rightarrow t = u$
A8	Injectivity to the Right	$v = t \circ s \wedge v = u \circ s \wedge \exists a(IP_s(s, a)) \rightarrow t = u$
A9	Contextualisation Associativity	$\exists w(v = s \circ w \wedge w = t \circ u) \leftrightarrow \exists x(v = x \circ u \wedge x = s \circ t)$
A10	Slot Strong Supplementation	$S(s) \wedge S(t) \rightarrow (\neg PoS(t, s) \rightarrow \exists u(PoS(u, t) \wedge \neg OoS(u, s)))$
A11	Sum Existence	$SO(s, t) \rightarrow \exists u(SoS_1(u, s, t))$
AS12	Fusion Existence	$\exists w(\phi(w) \wedge \forall v(\phi(v) \rightarrow SO(v, w))) \rightarrow \exists s(FoS_\phi(s))$

Number	Description	Theorem/Lemma
BT8	Anti-Symmetry	$P(a,b) \wedge P(b,a) \rightarrow a = b$
BT9	Conditional Reflexivity	$\exists s(P_s(s,a)) \rightarrow P(a,a)$
BT13	Slot Weak Supplementation	$PP(a,b) \rightarrow \exists s(P_s(s,b) \wedge \neg P_s(s,a))$
T1	Anti-Inheritance	$\forall a,b,s,t([a \neq b \wedge P_s(s,b) \wedge F(a,s) \wedge P_s(t,a)] \rightarrow \neg P_s(t,b))$
L2	Either Proper or Improper	$\forall s[S(s) \rightarrow \exists! a(PP_s(s,a) \oplus IP_s(s,a))]$
L3	Proper Parts iff Proper Slots	$\forall a,b(PP(b,a) \leftrightarrow \exists s(PP_s(s,a) \wedge F(b,s)))$
L4	General Conditional Reflexivity	$\forall a,s(P_s(s,a) \vee F(a,s) \rightarrow P(a,a))$
T5	Mutual Occupancy is Slot Identity	$\forall a,b,s,t(P_s(s,b) \wedge F(a,s) \wedge P_s(t,a) \wedge F(b,t) \rightarrow s = t)$
T6	Symmetric Contextualisation is Slot Identity	$\forall s,t,u,v(u = s \circ t \wedge v = t \circ s \rightarrow s = t)$
T7	Left-and-Right-Improper Contextualisation	$\forall s(\exists a(IP_s(s,a)) \leftrightarrow s = s \circ s)$
T8	Improper Slot Is Right Neutral Element	$\forall a,s,t(IP_s(s,a) \wedge F(a,t) \rightarrow t = t \circ s)$
T9	Improper Slot Is Left Neutral Element	$\forall a,s,t(IP_s(s,a) \wedge P_s(t,a) \rightarrow t = s \circ t)$
T10	Mutual Contextualisation is Identity	$\forall s,t,u,v(s = t \circ u \wedge t = s \circ v \rightarrow s = t)$
T11	Contextualisation Same Owner	$\forall u,s,t(u = s \circ t \rightarrow SO(u,s))$
T12	Contextualisation Same Filler	$\forall u,s,t(u = s \circ t \rightarrow SF(u,t))$
T13	Parthood Transitivity	$\forall a,b,c(P(a,b) \wedge P(b,c) \rightarrow P(a,c))$
T14	Right Neutral Element Is Improper Slot	$\forall s,t(t = t \circ s \rightarrow \exists a(IP_s(s,a) \wedge F(a,t)))$
T15	Left Neutral Element Is Improper Slot	$\forall s,t(t = s \circ t \rightarrow \exists a(IP_s(s,a) \wedge P_s(t,a)))$
T16	Contextualisation Stable under Contextualisation	$\forall s,t,u,s',t'(Cb(s,u) \wedge Cb(t,u) \rightarrow \forall v(s = t \circ v \leftrightarrow u \circ s = (u \circ t) \circ v))$
L17	PoS Domain and Range	$\forall s,t(PoS(s,t) \rightarrow S(s) \wedge S(t))$
T18	Conditional PoS Reflexivity	$\forall s(S(s) \rightarrow PoS(s,s))$
T19	PoS Anti-Symmetry	$\forall s,t(PoS(s,t) \wedge PoS(t,s) \rightarrow s = t)$
T20	PoS Transitivity	$\forall s,t,u(PoS(s,t) \wedge PoS(t,u) \rightarrow PoS(s,u))$
T21	PoS Same Owner	$\forall s,t(PoS(s,t) \rightarrow SO(s,t))$
T22	Slots iff Slot-Parts of Improper Slot	$\forall a,s(IP_s(s,a) \rightarrow \forall t(P_s(t,a) \leftrightarrow PoS(t,s)))$
T23	Slot Structure and Filler Structure constrain Each Other	$\forall a,b(\exists s,t(PoS(t,s) \wedge F(a,s) \wedge F(b,t)) \leftrightarrow P(b,a))$
T24	PoS Stable under Contextualisation	$\forall s,t,u(Cb(t,s) \wedge Cb(u,s) \rightarrow (PoS(u,t) \leftrightarrow PoS(s \circ u, s \circ t)))$
T25	PPoS Irreflexivity	$\forall s(\neg PPOs(s,s))$
T26	PPoS Asymmetry	$\forall s,t(PPOs(s,t) \rightarrow \neg PPOs(t,s))$
T27	PPoS Transitivity	$\forall s,t,u(PPOs(s,t) \wedge PPOs(t,u) \rightarrow PPOs(s,u))$
T28	PPoS Same Owner	$\forall s,t(PPOs(s,t) \rightarrow SO(s,t))$
T29	Proper Slots iff Proper Parts Of Improper Slot	$\forall a,s(IP_s(s,a) \rightarrow \forall t(PP_s(t,a) \leftrightarrow PPOs(t,s)))$
T30	Slot Structure and Filler Structure constrain Each Other — Proper Part	$\forall a,b(\exists s,t(PPOs(t,s) \wedge F(a,s) \wedge F(b,t)) \leftrightarrow PP(b,a))$
T31	PPoS Stable under Contextualisation	$\forall s,t,u(Cb(t,s) \wedge Cb(u,s) \rightarrow (PPOs(u,t) \leftrightarrow PPOs(s \circ u, s \circ t)))$

Number	Description	Theorem/Lemma
T32	Conditional <i>OoS</i> Reflexivity	$\forall s(S(s) \rightarrow OoS(s,s))$
T33	<i>OoS</i> Symmetry	$\forall s,t(OoS(s,t) \rightarrow OoS(t,s))$
T34	<i>OoS</i> Same Owner	$\forall s,t(OoS(s,t) \rightarrow SO(s,t))$
L35	Overlap with Part Implies Overlap with Whole	$\forall s,t,u(OoS(u,t) \wedge PoS(t,s) \rightarrow OoS(u,s))$
L36	Slot-Overlap With Improper Slot	$\forall a,s,t(IP_s(s,a) \wedge P_s(t,a) \rightarrow OoS(s,t))$
L37	<i>PoS</i> Implies <i>OoS</i>	$\forall s,t(PoS(s,t) \rightarrow OoS(s,t))$
T38	<i>OoS</i> Stable under Contextualisation	$\forall s,t,u(Cb(t,s) \wedge Cb(u,s) \rightarrow (OoS(t,u) \leftrightarrow OoS(s \circ t, s \circ u)))$
T39	Slot-Overlap Constrains Overlap between Fillers	$\forall a,b,s,t(OoS(s,t) \wedge F(a,s) \wedge F(b,t) \rightarrow O(a,b))$
T40	Slot Weak Supplementation	$\forall s,t(PPoS(s,t) \rightarrow \exists u(PoS(u,t) \wedge \neg OoS(u,s))$
T41	<i>OoS</i> -Extensionality	$\forall s,t[S(s) \wedge S(t) \rightarrow (\forall u, OoS(s,u) \leftrightarrow OoS(t,u)) \rightarrow s = t]$
T42	<i>PPoS</i> -Extensionality	$\forall s,t[\exists u(PPoS(u,s) \vee PPoS(u,t)) \rightarrow (\forall u, PPoS(u,s) \leftrightarrow PPoS(u,t)) \rightarrow s = t]$
L43	Domains of Sum	$\forall s,t,u(SoS_1(u,s,t) \rightarrow S(s) \wedge S(t) \wedge S(u))$
T44	<i>SoS</i> ₁ and <i>SoS</i> ₂ are Equivalent	$\forall s,t[SO(s,t) \rightarrow \forall u(SoS_1(u,s,t) \leftrightarrow SoS_2(u,s,t))]$
T45	Sum Same Owner	$\forall s,t,u[SoS_1(u,s,t) \rightarrow \exists a(P_s(u,a) \wedge P_s(s,a) \wedge P_s(t,a))]$
T46	Sum Unicity	$\forall s,t,u,v(SoS_1(u,s,t) \wedge SoS_1(v,s,t) \rightarrow u = v)$
T47	Sum Idempotence	$\forall s(S(s) \rightarrow s + s = s)$
T48	Sum Commutativity	$\forall s,t(SO(s,t) \rightarrow s + t = t + s)$
L49	-	$\forall s,t(SO(s,t) \rightarrow PoS(s, s + t))$
L50	-	$\forall t,u[SO(t,u) \rightarrow \forall s(PoS(s,t) \rightarrow PoS(s, t + u))]$
L51	-	$\forall s,t[SO(s,t) \rightarrow \forall u(PoS(s + t, u) \rightarrow PoS(s, u))]$
L52	-	$\forall s,t(PoS(s,t) \leftrightarrow s + t = t)$
T53	Overlaps the Contextualised Sum iff Overlaps one of the Contextualised Operands	$\forall s,t,u[\exists a(F(a,s) \wedge P_s(t,a) \wedge P_s(u,a)) \rightarrow \forall v(OoS(v, s \circ (t + u)) \leftrightarrow OoS(v, s \circ t) \vee OoS(v, s \circ u))]$
T54	Left Distributivity	$\forall s,t,u[\exists a(F(a,s) \wedge P_s(t,a) \wedge P_s(u,a)) \rightarrow (s \circ (t + u) = s \circ t + s \circ u)]$
L55	Same Filler of Operand and Sum Implies Identity	$\forall s,t(SF(s, s + t) \rightarrow s = s + t)$
T56	Right Distributivity Is Trivial	$\forall s,t,u((s + t) \circ u = s + t \circ u \wedge s \circ u = s \circ u \wedge t \circ u = t \circ u \rightarrow s = t)$
L57	Sum is Slot-Part if Operands are Slot-Parts	$\forall s,t,u(PoS(s,u) \wedge PoS(t,u) \rightarrow PoS(s + t, u))$
T58	Sum Associativity	$\forall s,t,u[SO(s,t) \wedge SO(t,u) \rightarrow (s + t) + u = s + (t + u)]$
T59	Sum Stable under Contextualisation	$\forall s,t,u,v(Cb(u,v) \wedge Cb(s,v) \wedge Cb(t,v) \rightarrow (u = s + t \leftrightarrow (v \circ u) = (v \circ s) + (v \circ t)))$
TS60	Fusion Unicity	$\exists w(\phi(w)) \rightarrow \forall s,t(FoS_\phi(s) \wedge FoS_\phi(t) \rightarrow s = t)$
T61	Sum is a special Case of Fusion	$\forall s,t,u(FoS_{w=s \vee w=t}(u) \leftrightarrow SoS_1(u, s, t))$
T62	Improper Slot is Fusion of Filler's Slots	$\forall a(\exists s(P_s(s,a)) \rightarrow \forall t(IP_s(t,a) \leftrightarrow FoS_{P_s(w,a)}(t)))$
TS63	Fusion Stable under Contextualisation	$\forall s,s',t(Cb(s,t) \wedge \forall w(\phi(w) \rightarrow Cb(w,t)) \rightarrow (FoS_\phi(s) \leftrightarrow FoS_{\exists w'(w=t \circ w' \wedge \phi(w'))}(t \circ s))$