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HOLOMORPHIC ISOMETRIC EMBEDDINGS OF COMPLEX GRASSMANNIANS INTO QUADRICS: THE GENERAL CASE

OSCAR MACIA, YASUYUKI NAGATOMO

ABSTRACT. The present article studies holomorphic isometric embeddings of arbitrary complex Grassmannians into quadrics, generalising results in [13]. The moduli spaces of these embeddings up to gauge and image equivalence are discussed using a generalisation of do Carmo–Wallach theory.

1. INTRODUCTION

The topic of holomorphic isometric embeddings between complex manifolds has a long history in complex geometry. For the enriched subclass of Kähler manifolds, the earliest work dates back to Calabi’s seminal 1953 paper [4], and the field has remained active since its inception.

More recently, a general theory describing the moduli space of holomorphic isometric embeddings of Kähler manifolds into Grassmannians has been developed in [14, 15]. The foundations of this theory, a generalisation of ideas of Takahashi [18], and of do Carmo and Wallach [6], are deeply rooted in the principles of gauge theory. Therefore, techniques based on vector bundles and the representation theory of Lie groups play a prominent *rôle* in it.

As a practical application of this generalisation of do Carmo–Wallach theory, the description of the moduli space when the embedded manifold is a complex projective space and the target is a complex hyperquadric (identified as an appropriate real Grassmannian) is by now well-understood: The \mathbf{CP}^1 case has been discussed at length in [11, 12], and the generalisation to \mathbf{CP}^n has been recently considered in [16].

In spite of this, the moduli spaces of holomorphic isometric embeddings of complex Grassmannian manifolds into quadrics are much less familiar. To improve this situation, the authors studied first a simplified version of the problem in [13]: the moduli of holomorphic isomorphic embeddings of the complex Grassmann manifolds $\mathrm{Gr}_m(\mathbf{C}^{m+2})$ of all two-codimensional linear subspaces in \mathbf{C}^{m+2} into quadrics. This particular Hermitian symmetric space is in itself worthy of attention from the intrinsically geometric point of view, as the unique compact, Kähler, quaternionic Kähler manifold with positive scalar curvature [2, 17, 19]. On the other hand that special case is simple enough to provide a model to the intricate representation-theoretic computations that underpin the general theory. In particular, it provides a pathway for the decomposition of the space $\mathrm{GS}(\mathfrak{m}V_0, V_0)$ ([13], §2.5) into irreducible representations.

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While Young *tableaux* and Littlewood–Richardson rules are recognised as the right tools to systematically determine the decomposition of tensor products of $SU(n)$ representations [7], these are nonetheless insufficient to irreducibly decompose $GS(\mathfrak{m}V_0, V_0)$, since Littlewood did not laid a straightforward method to determine which terms of the decomposition belong to the symmetric or skewsymmetric parts [10]. Though more modern algebro–combinatorial techniques (aka *domino tableaux*, [5]) have been developed which would allow to decompose into irreducible representations while keeping track of the general symmetry or skewsymmetry, we have kept with our original approach based on working out the action of the centre of the isotropy subgroup on the highest and lowest–weight vectors, in a way similar to, if more complicated than, that of [13] (compare §3 with *loc.cit.* Lemmas 4.1 to 4.4). The special conditions which, due to the low dimensionality, allow the representations to be decomposed with minimum difficulty disappear in this article, and the parameterisation of the highest–weight modules becomes a delicate issue which is solved in detail in Lemma 3.1.

This article, being a sequel to [11, 12, 16] and a generalisation of the discussion in [13], is based on the same general theory as its predecessors. Therefore, the authors have not included material that could easily be found elsewhere. This theoretical minimum (natural evaluations and identifications, image and gauge equivalences, vector bundles on Hermitian symmetric spaces, standard maps, and do Carmo–Wallach theory, together with some motivation) can be found described in detail, for example, in [13], §§2 and 3, and the reader would do well to consult it to settle the definitions and context of the present work.

The structure of the article is the following: After some preliminary remarks which have been briefly collected in section 2, and which serve as a bridge between the general theory mentioned above and the concrete results of the present work, section 3 is devoted to the detailed computation of the irreducible $SU(p + q)$ representations which make up the moduli, culminating in Theorem 3.6 where its geometry is determined. The relation between the moduli up to gauge and up to image equivalence is discussed in Corollary 3.7. Lastly, Section 4 deals with the special case $p = q$, that is, embeddings of $Gr_q(\mathbf{C}^{2q})$, which generalises the example of the compactified, complexified Minkowski space $Gr_2(\mathbf{C}^4)$ relevant in certain areas of high energy theoretical physics.

The authors intend, in a subsequent report, to further generalise the contents of this article to study the case of holomorphic isometric embeddings of flag varieties into complex hyperquadrics.

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2. PRELIMINARY REMARKS

From this point, familiarity with the generalisation of the do Carmo–Wallach theory, at the level of [13], §§2 and 3, is assumed throughout. The general picture of the

theory is the following: Suppose that a vector bundle $V \rightarrow M$ and a finite-dimensional subspace W of the space of sections $\Gamma(V)$ is given. If the evaluation homomorphism $ev : M \times W \rightarrow V$ is surjective (so that $V \rightarrow M$ is globally generated by W), assigning to each $x \in M$ the p -plane $\text{Ker } ev_x \subset \text{Gr}_p(W)$ defines an induced map $f : M \rightarrow \text{Gr}_p(W)$. Using it to pull the natural exact sequence over $\text{Gr}_p(W)$ back to M it can be proved that $V \rightarrow M$ is naturally identified with $f^*Q \rightarrow M$ where $Q \rightarrow \text{Gr}_p(W)$ stands for the universal quotient bundle. In case that one deals with vector bundles equipped with metrics and connections, additional differential conditions need to be considered (e.g., the gauge condition which states that the connection on $V \rightarrow M$ must be gauge equivalent to the connection on $f^*Q \rightarrow M$). In the present article M stands for a compact, irreducible, Hermitian symmetric space, and $V \rightarrow M$ will be substituted by a complex homogeneous line bundle. The subspace of sections $W \subset \Gamma(V)$ will be the space of holomorphic sections $H^0(V) \subset \Gamma(V)$ since in this context this is the space which determines the *standard maps* (the general definition for standard map being those induced by sets of sections with the same eigenvalue for the Laplace operator acting on sections). Regard W as a real vector space and fix a \mathbf{R}^{n+2} subspace of W , with inclusion map ι . This being a real space of sections \mathbf{R}^{n+2} also generates $V \rightarrow M$ globally. The generalisation of the theory of do Carmo–Wallach shows that this \mathbf{R}^{n+2} can be regarded as the orthogonal complement of the kernel of a certain positive semi-definite symmetric endomorphism T of W , which is indeed positive definite on \mathbf{R}^{n+2} . Specifically,

Theorem 2.1. *Let (L, h) be a fixed Hermitian line bundle on G/K , and let $f : G/K \rightarrow \text{Gr}_n(\mathbf{R}^{n+2})$ be a full holomorphic map satisfying the gauge condition for (L, h) . Let W denote $H^0(G/K, L)$ regarded as a real vector space equipped with a complex structure. Then, there is a positive semi-definite symmetric endomorphism $T \in S(W)$ such that the pair (W, T) satisfies the following three conditions:*

- (a) *The vector space \mathbf{R}^{n+2} is a subspace of W with the inclusion $\iota : \mathbf{R}^{n+2} \rightarrow W$ preserving the inner products, and $L \rightarrow M$ is globally generated by \mathbf{R}^{n+2} .*
- (b) *As a subspace, $\mathbf{R}^{n+2} = \text{Ker } T^\perp$ and the restriction of T is a positive symmetric endomorphism of \mathbf{R}^{n+2} .*
- (c) *The endomorphism T satisfies*

$$(T^2 - \text{Id}_W, \text{GS}(V_0, V_0))_S = 0, \quad (T^2, \text{GS}(mV_0, V_0))_S = 0.$$

If $\iota^* : W \rightarrow \mathbf{R}^{n+2}$ denotes the adjoint linear map of $\iota : \mathbf{R}^{n+2} \rightarrow W$, then $f : G/K \rightarrow \text{Gr}_n(\mathbf{R}^{n+2})$ is realized as

$$(2.1) \quad f([g]) = (\iota^* T \iota)^{-1} (f_0([g]) \cap \text{Ker } T^\perp),$$

where the orientation of $(\iota^* T \iota)^{-1} (f_0([g]) \cap \text{Ker } T^\perp)_{[g]}$ is given by the orientation of $L_{[g]}$ and \mathbf{R}^{n+2} . Moreover, if the orientation of $\text{Ker } T$ is fixed, then we have a unique holomorphic totally geodesic embedding of $\text{Gr}_n(\mathbf{R}^{n+2})$ into $\text{Gr}_{n'}(W)$ by $\iota(\mathbf{R}^{n+2}) = \text{Ker } T^\perp$, where $n' = n + \dim \text{Ker } T$ and a bundle isomorphism $(ev \circ \iota \circ (\iota^* T \iota))^* : L \rightarrow f^*Q$ as the natural identification by f . Such two maps f_i , ($i = 1, 2$) are gauge equivalent if and only if $\iota^* T_1 \iota = \iota^* T_2 \iota$, where T_i and ι correspond to f_i in (2.1), respectively.

Conversely, suppose that a vector space \mathbf{R}^{n+2} with an inner product and an orientation, and a positive semi-definite symmetric endomorphism $T \in \text{End}(W)$ satisfying conditions (a), (b), (c) are given. Then we have a unique holomorphic totally geodesic embedding of $\text{Gr}_n(\mathbf{R}^{n+2})$ into $\text{Gr}_{n'}(W)$ after fixing the orientation of $\text{Ker}T$ and the map $f : G/K \rightarrow \text{Gr}_p(\mathbf{R}^{n+2})$ defined by (2.1) is a full holomorphic map into $\text{Gr}_n(\mathbf{R}^{n+2})$ satisfying the gauge condition with bundle isomorphism $L \cong f^*Q$ as the natural identification.

Theorem 2.1, originally proved in [14], gives the representation theoretic conditions that such endomorphism T must satisfy (condition (c) in the theorem). Each such T determines a new holomorphic embedding $\text{Gr}_n(\mathbf{R}^{n+2}) \rightarrow \text{Gr}_{n'}(W)$ where $n' = n + \dim \text{ker } T$. The theoretical expression for the holomorphic isometric embedding f constructed from the standard map f_0 , the symmetric endomorphism T , and the inclusion map ι is given by Equation (2.1). Each symmetric endomorphism T determines a unique gauge equivalence class of maps. Therefore, knowledge of the representation spaces where T lives determines all the relevant differential geometric information of the moduli space up to gauge equivalence \mathcal{M}_k , in particular its dimension, and the meaning of the boundary points in the compactification.

In the present context, Theorem 2.1 allows to determine the moduli space \mathcal{M}_k of holomorphic isometric embeddings of degree k modulo the gauge equivalence of maps due to the special characterisation, in terms of vector bundles and pull-back connections, to which these maps are subject in the generalisation of do Carmo–Wallach theory: Let \mathcal{Q}_n denote the complex quadric hypersurface in $\mathbf{C}P^{n+1}$, identified with $\text{Gr}_n(\mathbf{R}^{n+2})$ as in [8], pp.278–282, and let $Q \rightarrow \mathcal{Q}_n$ be the universal quotient bundle over \mathcal{Q}_n . Analogously, let $\tilde{Q} \rightarrow \text{Gr}_p(\mathbf{C}^{p+q})$ denote the universal quotient bundle over $\text{Gr}_p(\mathbf{C}^{p+q})$, which satisfies $\mathcal{O}(1) \cong \wedge^q \tilde{Q}$. Now, it is a fact in the general theory that $f : \text{Gr}_p(\mathbf{C}^{p+q}) \rightarrow \mathcal{Q}_n$ is a holomorphic isometric embedding of degree k if and only if the pull-back by f of the canonical connection on $Q \rightarrow \mathcal{Q}_n$ coincides with the canonical connection on $\mathcal{O}(k) \rightarrow \text{Gr}_p(\mathbf{C}^{p+q})$, that is, if and only if f satisfies the gauge condition for $(\mathcal{O}(k), h_k)$, where h_k denotes the standard Einstein–Hermitian metric on $\mathcal{O}(k) \rightarrow \text{Gr}_p(\mathbf{C}^{p+q})$. Since the canonical connection on $(\mathcal{O}(k), h_k)$ is the Hermitian connection, Theorem 2.1 allows to determine the moduli space \mathcal{M}_k .

In general terms, if (L, h) is a given Hermitian holomorphic line bundle over a Kähler manifold, $H^0(M, L)$ inherits a complex structure commuting with the evaluation $ev : H^0(M, L) \rightarrow L$. Any complex subspace of $H^0(M, L)$ can also be regarded as a real vector space U equipped with this complex structure and an inner product inherited from the L^2 -Hermitian product on $H^0(M, L)$, and the couple $(L \rightarrow M, U)$ induces a map of M into a quadric. If this map is a holomorphic isometric immersion, commutativity between the complex structure and the evaluation map allows to regard it as a holomorphic isometric immersion into a totally geodesic complex projective space contained in the target quadric. If the Kodaira embedding into a complex projective space, which is the induced map by $(\mathcal{O}(k) \rightarrow \text{Gr}_p(\mathbf{C}^{p+q}), H^0(\text{Gr}_p(\mathbf{C}^{p+q}), \mathcal{O}(k)))$ is a holomorphic isometric embedding, then it is rigid [4]. When both the bundle and the space of sections inducing the map are regarded as real and oriented, the Kodaira embedding can also be viewed as a holomorphic isometric embedding into a quadric. Therefore, the characterisation of the moduli space of full holomorphic isometric embeddings $\text{Gr}_p(\mathbf{C}^{p+q}) \rightarrow \mathcal{Q}_n$ now follows from Theorem 6.10 of [15],

Theorem 2.2. *Let (L, h) be a Hermitian holomorphic line bundle over a Kähler manifold M . Let \mathcal{M} be the moduli space of full holomorphic isometric immersions of M into a quadric $\text{Gr}_n(\mathbf{R}^{n+2})$ with the gauge condition for (L, h) modulo gauge equivalence relation of maps.*

Suppose that \mathbf{R}^{n+2} is a complex subspace of $H^0(M, L)$ and the inner product on \mathbf{R}^{n+2} is compatible with the complex structure. If there exists $f \in \mathcal{M}$ such that the evaluation map $ev : \mathbf{R}^{n+2} \rightarrow L$ by f satisfies $J_L \circ ev = ev \circ J$, then \mathcal{M} has the induced complex structure and is an open submanifold of a complex subspace of $H(\mathbf{R}^{n+2})^\perp$.

As a consequence of the previous discussion, the moduli up to gauge equivalence of maps, of full holomorphic isometric embeddings $\text{Gr}_p(\mathbf{C}^{p+q}) \rightarrow \mathcal{Q}_n$ satisfying the gauge condition for $(\mathcal{O}(k) \rightarrow \mathbf{C}P^m, h_k)$ is an open submanifold which according to the general theory lies in the orthogonal complement $H(\mathbf{R}^{n+2})^\perp$ of the complex subspace of Hermitian endomorphisms of the space of symmetric endomorphisms $\text{Sym}(\mathbf{R}^{n+2})$, where $U \equiv \mathbf{R}^{n+2} = H^0(\text{Gr}_p(\mathbf{C}^{p+q}), \mathcal{O}(k))$.

3. MODULI SPACES

In the following paragraphs we compute the specific $\text{SU}(p+q)$ representations which, according to Theorem 2.1 and §2 make up the moduli space up to gauge equivalence. In order to apply the generalisation of the do Carmo–Wallach theory a detailed understanding of the symmetric endomorphisms of the space $W = H^0(\text{G/K}, L)$ of holomorphic sections of certain line bundles over G/K is essential.

The Grassmannian of p -planes in \mathbf{C}^{p+q} is described as a Hermitian symmetric space by the symmetric pair $(\text{G}, \text{K}) = (\text{SU}(p+q), \text{S}(\text{U}(p) \times \text{U}(q)))$ such that

$$\text{Gr}_p(\mathbf{C}^{p+q}) = \frac{\text{SU}(p+q)}{\text{S}(\text{U}(p) \times \text{U}(q))}$$

The isotropy group $\text{K} = \text{S}(\text{U}(p) \times \text{U}(q))$ is given by matrices of the form

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix}, \quad A \in \text{U}(p), \quad B \in \text{U}(q), \quad |A||B| = 1.$$

Let us fix a maximal torus T^n inside $\text{U}(n)$ defined as the subgroup of diagonal matrices,

$$\text{diag}(x_1, \dots, x_n) = \begin{pmatrix} x_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$$

and denote by $\mathbf{V}_n(\lambda)$ an irreducible complex representation of $\text{U}(n)$ with the highest weight $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are integers. The action of the maximal torus on the highest-weight vector $\hat{w} \in \mathbf{V}_n(\lambda)$ is by definition

$$\text{diag}(x_1, \dots, x_n) \cdot \hat{w} = x_1^{\lambda_1} \dots x_n^{\lambda_n} \hat{w}.$$

If $\mathbf{V}_n(\lambda)$ is regarded as an $\text{SU}(n)$ representation, it will satisfy $\mathbf{V}_n(\lambda) = \mathbf{V}_n(\lambda + t(1, \dots, 1))$, $t \in \mathbf{Z}$, so by convention we will choose $t = -\lambda_n$ so the last weight becomes zero. The *dominant integral weights* of $\text{SU}(n)$, denoted by ϖ_i ($1 \leq i \leq n-1$), are defined by $\lambda_1 = \dots = \lambda_i = 1$, $\lambda_{i+1} = \dots = \lambda_n = 0$. The $\text{SU}(n)$ irreducible representation with highest weight $\varpi = \sum_i k_i \varpi_i$ will be denoted by $\mathbf{F}_n(\varpi)$. The relation between

$\mathbf{F}_n(\varpi)$ and $\mathbf{V}_n(\lambda)$ is

$$(3.1) \quad \mathbf{F}_n \left(\sum_{i=1}^{n-1} k_i \varpi_i \right) = \mathbf{V}_n \left(\sum_{i=1}^{n-1} k_i, \sum_{i=2}^{n-1} k_i, \dots, k_{n-2} + k_{n-1}, k_{n-1}, 0 \right).$$

As it is customary, the subindex n in F_n , V_n will be usually neglected when it is clear enough. If $\varpi = \sum_i k_i \varpi_i$ is a dominant integral weight, associated to the $SU(n)$ representation $\mathbf{F}(\varpi)$, the action of an element of the maximal torus of $U(n)$ on the highest weight vector $\hat{w} \in \mathbf{F}(\varpi)$ is given by

$$\text{diag}(x_1, \dots, x_n) \cdot \hat{w} = x_1^{\lambda_1} \dots x_{n-1}^{\lambda_{n-1}} \hat{w}$$

The action of the torus of $U(n)$ on the representation dual to $\mathbf{V}(\lambda)$ is determined as follows: Since the dual representation of $\mathbf{V}(\lambda)$ has lowest weight equal to $-\lambda$, the torus of $U(n)$ acts on the lowest weight vector $\check{w} \in \mathbf{F}(\varpi)$ as it does on the highest weight vector of $\mathbf{F}(-\varpi')$ with

$$\varpi' = - \sum_{i=1}^{n-1} k_i \varpi_{n-i} = - \sum_{j=1}^{n-1} k_{n-j} \varpi_j.$$

Therefore

$$\begin{aligned} \text{diag}(x_1, \dots, x_n) \cdot \check{w} &= x_1^{-\sum_{j=1}^{n-1} k_{n-j}} x_2^{-\sum_{j=2}^{n-1} k_{n-j}} \dots x_{n-2}^{-k_2 - k_1} x_{n-1}^{-k_1} \check{w} \\ &= x_1^{-\sum_{j=1}^{n-1} k_j} x_2^{-\sum_{j=1}^{n-2} k_j} \dots x_{n-2}^{-k_2 - k_1} x_{n-1}^{-k_1} \check{w}. \end{aligned}$$

Recalling that for our purposes $n = p + q$, denote by Y an element of the center $U(1)$ of $K = S(U(p) \times U(q)) = SU(p) \times SU(q) \times U(1)$, the second equality holding only at the Lie algebra level:

$$Y = \text{diag} \left(\underbrace{y^{\frac{1}{p}}, \dots, y^{\frac{1}{p}}}_p, \underbrace{y^{-\frac{1}{q}}, \dots, y^{-\frac{1}{q}}}_q \right).$$

Hence, the action of Y on \check{w} is given by

$$\begin{aligned} (3.2) \quad Y \cdot \check{w} &= \underbrace{y^{-\frac{1}{p} \sum_{j=1}^{p+q-1} k_j} y^{-\frac{1}{p} \sum_{j=1}^{p+q-2} k_j} \dots y^{-\frac{1}{p} (k_{q+1} + k_q + \dots + k_1)}}_p y^{-\frac{1}{p} (k_q + \dots + k_1)} \\ &\times \underbrace{y^{\frac{1}{q} \sum_{j=1}^{q-1} k_j} y^{\frac{1}{q} \sum_{j=1}^{q-2} k_j} \dots y^{\frac{1}{q} (k_2 + k_1)}}_{q-1} y^{\frac{1}{q} k_1} \check{w} \\ &= y^{-\frac{p}{p} (k_q + \dots + k_1) - \frac{(p-1)k_{q+1}}{p} + \frac{(p-2)k_{q+2}}{p} - \dots - \frac{2k_{p+q-2}}{p} - \frac{k_{p+q-1}}{p}} \\ &\times y^{\frac{q-1}{q} k_1 + \frac{(q-2)k_2}{q} + \frac{(q-3)k_3}{q} + \dots + \frac{2k_{q-2}}{q} + \frac{k_{q-1}}{q}} \check{w} \\ &= y^{-\frac{1}{q} k_1 - \frac{2}{q} k_2 - \dots - \frac{q-1}{q} k_{q-1} - k_q - \frac{1}{p} \sum_{i=1}^{p-1} (p-i) k_{q+i}} \check{w} \\ &= y^{-\frac{1}{q} \sum_{i=1}^q i k_i - \frac{1}{p} \sum_{j=1}^{p-1} k_{q+j}} \check{w}. \end{aligned}$$

In the present situation (compare with the setting in [13]) $H^0(M; \mathcal{O}(k)) = \mathbf{F}(k\varpi_q)$ in virtue of the Bott–Borel–Weil Theorem [3]. The moduli space of interest is now a subset of $S^2(\mathbf{F}(k\varpi_q))$, which was denoted by $H(W)^\perp$ in §2. Since $S^2(\mathbf{F}(k\varpi_q)) \subset \otimes^2 \mathbf{F}(k\varpi_q)$ we first apply Littlewood–Richardson rules on Young tableaux with q rows and k columns [7],

to decompose $\otimes^2 \mathbf{F}(k\varpi_q)$ into irreducible representations under the action of $\mathrm{SU}(p+q)$. We have that

Lemma 3.1.

$$\begin{aligned}
(3.3) \quad & \mathbf{F}(k\varpi_q) \otimes_{\mathbf{C}} \mathbf{F}(k\varpi_q) \\
&= \bigoplus_{\substack{k \geq \sum_{\alpha=1}^q i_{\alpha} \\ i_1, i_2, \dots, i_q=0}} \mathbf{V}(2k - i_q, 2k - (i_q + i_{q-1}), \dots, 2k - \sum_{\alpha=1}^q i_{\alpha}, \\
&\quad \sum_{\alpha=1}^q i_{\alpha}, \sum_{\alpha=2}^q i_{\alpha}, \dots, i_{q-1} + i_q, i_q, 0, \dots, 0) \\
&= \bigoplus_{i_{\alpha}=0}^{k \geq \sum_{\alpha=1}^q i_{\alpha}} \mathbf{F}(\underbrace{i_{q-1}, i_{q-2}, \dots, i_2, i_1}_{q-1}, 2k - 2 \sum_{\alpha=1}^q i_{\alpha}, \underbrace{i_1, i_2, \dots, i_{q-1}, i_q}_q, 0, \dots, 0)
\end{aligned}$$

Remark 1. Before entering into the details of the proof, let us introduce some notation. Recall that by Eqn.(3.1) the weight $k\varpi_q$ is $\lambda = (k, \dots, k, 0, \dots, 0)$ containing q k 's, followed by $p-1$ zeroes. To the representation $\mathbf{V}(\lambda)$ with this highest weight it is associated a Young *tableau* [7], a diagram with q rows, each with k columns, or ‘boxes’. The tensor product $\mathbf{V}(\lambda) \otimes \mathbf{V}(\lambda)$ becomes $\oplus_{\mu} \mathbf{V}(\mu)$ where each of the admissible μ is determined diagrammatically using the Littlewood–Richardson rules [10], which give a list of admissible rearrangements of the $2qk$ boxes appearing in the Young *tableau* of the $\mathbf{V}(\lambda)$ factors of the tensor product. In order to keep track of which boxes of the second diagram are added to each row of the first diagram we introduce variables x_j^i , where $1 \leq i \leq q$, and $1 \leq j \leq 2q$ which stands for the number x of boxes from the i -th row of the second diagram which are to be attached to the right to the j -th row of the first diagram: eg the quantity $k + x_5^1 + x_5^3$ represents the total number of boxes in row 5 after x_5^1 boxes from the first row of the second diagram and x_5^3 boxes from the third row of the second diagram have been attached to the right of the k boxes of the fifth row in the first diagram.

With this notation, the relevant combinatorial Littlewood rules determining admissible arrangements can be stated as follows

$$(3.4) \quad x_q^q \leq x_{q-1}^{q-1} \leq \dots \leq x_1^1$$

$$(3.5) \quad x_i^i + x_{q+1}^i \leq x_{i-1}^{i-1} \quad (1 \leq i \leq q)$$

$$(3.6) \quad x_i^i + \sum_{r=1}^s x_{q+r}^i \leq x_{i-1}^{i-1} + \sum_{r=1}^{s-1} x_{q+r}^i \quad (2 \leq s \leq i-1)$$

$$(3.7) \quad x_i^i + \sum_{r=1}^i x_{q+r}^i = x_{i-1}^{i-1} + \sum_{r=1}^{i-1} x_{q+r}^{i-1} = k$$

$$(3.8) \quad \sum_{i=1}^q x_j^i \leq \sum_{i=1}^q x_{j-1}^i \quad (1 \leq j \leq 2q)$$

this last rule sometimes referred to as the ‘row-length rule’.

Proof. After the boxes of the right diagram have been added to the boxes of the left diagram, consider the number of boxes in the q -th row. The equation (3.7) follows from the total number of boxes in each row before the rearrangement. Applying (3.7) for $i = q$ (that is equal to consider the q -th and $q - 1$ -th rows) we obtain

$$(3.9) \quad x_q^q = k - \sum_{r=1}^q x_{q+r}^q, \quad x_{q-1}^{q-1} = k - \sum_{r=1}^{q-1} x_{q+r}^{q-1}$$

Substituting the values obtained for x_q^q , x_{q-1}^{q-1} into (3.6) for $s = q - 1$, leads to

$$\left(k - \sum_{r=1}^q x_{q+r}^q \right) + \sum_{r=1}^{q-1} x_{q+r}^q \leq \left(k - \sum_{r=1}^{q-1} x_{q+r}^{q-1} \right) + \sum_{r=1}^{q-2} x_{q+r}^{q-1}$$

after cancellation, this leads to

$$k - x_{2q}^q \leq k - x_{2q-1}^{q-1} \implies x_{2q}^q \geq x_{2q-1}^{q-1}$$

By the row-length rule (3.8) it is implied that $x_{2q}^q \leq x_{2q-1}^{q-1}$. Altogether we are lead to

$$(3.10) \quad x_{2q-1}^{q-1} = x_{2q}^q$$

Repeating this same argument now for the $(q - 1)$ -th row, that is fixing $i = q - 1$ in (3.7), we have

$$x_{q-1}^{q-1} = k - \sum_{r=1}^{q-1} x_{q+r}^{q-1}, \quad x_{q-2}^{q-2} = k - \sum_{r=1}^{q-2} x_{q+r}^{q-2}$$

and substituting back into (3.6) for $s = q - 2$ leads to

$$\left(k - \sum_{r=1}^{q-1} x_{q+r}^{q-1} \right) + \sum_{r=1}^{q-2} x_{q+r}^{q-1} \leq \left(k - \sum_{r=1}^{q-2} x_{q+r}^{q-2} \right) + \sum_{r=1}^{q-3} x_{q+r}^{q-2}$$

which after cancellations simplifies to

$$k - x_{2q-1}^{q-1} \leq k - x_{2q-2}^{q-2} \implies x_{2q-1}^{q-1} \geq x_{2q-2}^{q-2}$$

Since the row-length rule (3.8) implies that $x_{2q-2}^{q-2} \geq x_{2q-1}^{q-1}$ therefore

$$(3.11) \quad x_{2q-2}^{q-2} = x_{2q-1}^{q-1}.$$

Steady iteration of the same argument leads to a series of relations similar to the previous ones

$$(3.12) \quad x_{q+1}^1 = x_{q+2}^2 = \dots = x_{2q-2}^{q-2} = x_{2q-1}^{q-1} = x_{2q}^q$$

which allows to set x_{2q}^q as a parameter for the arrangement of the other boxes.

Next, substituting the expressions (3.9) into (3.6), this time fixing $s = q - 2$ we obtain

$$\left(k - \sum_{r=1}^q x_{q+r}^q \right) + \sum_{r=1}^{q-2} x_{q+r}^q \leq \left(k - \sum_{r=1}^{q-1} x_{q+r}^{q-1} \right) + \sum_{r=1}^{q-3} x_{q+r}^{q-1}$$

that is

$$k - x_{2q-1}^q - x_{2q}^q \leq k - x_{2q}^{q-1} \leq k - x_{2q-1}^{q-1} - x_{2q-2}^{q-1}$$

Using the previously deduced equality of $x_{2q}^q = x_{2q-1}^{q-1}$ and simplifying one gets $x_{2q-1}^q \geq x_{2q-2}^{q-1}$. Then, using the row-length rule (3.8), together with the fact that $x_{2q-2}^{q-2} = x_{2q-1}^{q-1}$ we have that $x_{2q-2}^{q-1} \geq x_{2q-1}^q$, hence

$$(3.13) \quad x_{2q-2}^{q-1} = x_{2q-1}^q.$$

Iteration of the argument as before leads to the identification of x_{2q-1}^q as a new parameter through relations

$$(3.14) \quad x_{q+1}^2 = x_{q+2}^3 = \cdots = x_{2q-2}^{q-1} = x_{2q-1}^q$$

Orderly application of the same method each time for lower s in (3.6) leads to a sequence of relations in which $\{x_{q+r}^q : r = 1, \dots, q\}$ become the parameters and determine all other entries

$$(3.15) \quad x_{q+\mu}^\lambda = x_{q+r}^q \quad \text{if} \quad (q+\mu) - \lambda = r, \quad 1 \leq \lambda \leq \mu \leq q.$$

The remaining variables x_i^i , $i = 1, 2, \dots, q$ are determined by the corresponding Littlewood rule (3.7), hence

$$x_1^1 = k - x_{2q}^q, \quad x_2^2 = k - x_{2q-1}^q - x_{2q}^q, \quad \dots$$

etc., that is

$$(3.16) \quad x_i^i = k - \sum_{r=q+1-i}^q x_{q+r}^q$$

The corresponding irreducible decomposition for $\mathbf{F}(k\varpi_q) \otimes_{\mathbf{C}} \mathbf{F}(k\varpi_q)$, that is, $\mathbf{V}(\lambda) \otimes_{\mathbf{C}} \mathbf{V}(\lambda)$ where $\lambda = (k, \dots, k, 0, \dots, 0)$ will be $\oplus_{\lambda'} \mathbf{V}(\lambda')$ where λ' is any of the admissible highest-weights

$$\lambda' = \left(\underbrace{k + x_1^1, k + x_2^2, \dots, k + x_q^q, \sum_{r=1}^q x_{q+1}^r, \sum_{r=2}^q x_{q+2}^r, \dots, x_{2q}^q, 0, \dots, 0}_{2q} \right).$$

To summarise, from our previous computation

$$\begin{aligned} \lambda'_i &= 2k - \sum_{r=q+1-i}^q x_{q+r}^q & (i = 1, \dots, q) \\ \lambda'_{q+j} &= \sum_{r=j}^q x_{q+r}^q & (j = 1, \dots, q) \end{aligned}$$

Finally, renaming the variables to define $i_\alpha = x_{q+\alpha}^q$ with $\alpha = 1, 2, \dots, q$ the statement of the Lemma follows. \square

It will be convenient, to achieve greater simplicity in later expressions, to rearrange the variables so to have increasing indices in Eqn. (3.3). Hence, define $j_\alpha = i_{q-\alpha}$, $\alpha =$

$0, 1, \dots, q-1$ such that

$$\begin{aligned} & \mathbf{F}(i_{q-1}, i_{q-2}, \dots, i_1, 2k-2 \sum_{\alpha=1}^q i_\alpha, i_1, i_2, \dots, i_q, 0, \dots, 0) \\ &= \mathbf{F}(j_1, j_2, \dots, j_{q-1}, 2k-2 \sum_{\alpha=0}^{q-1} j_\alpha, j_{q-1}, j_{q-2}, j_0, 0, \dots, 0) \\ &= \mathbf{F}\left(\sum_{i=1}^{q-1} j_i \varpi_i + \left(2k-2 \sum_{\alpha=0}^{q-1} j_\alpha\right) \varpi_q + \sum_{i=1}^q j_{q-i} \varpi_{q+i}\right). \end{aligned}$$

Lemma 3.2. *The lowest weight of*

$$\mathbf{F}\left(\sum_{i=1}^{q-1} j_i \varpi_i + \left(2k-2 \sum_{\alpha=0}^{q-1} j_\alpha\right) \varpi_q + \sum_{i=1}^q j_{q-i} \varpi_{q+i}\right)$$

is

$$-2k + \frac{p+q}{pq} \sum_{i=1}^{q-1} (q-i)j_i + \left(1 + \frac{q}{p}\right) j_0$$

Proof. If \check{w} is the lowest weight vector in the aforesaid $\mathrm{SU}(p+q)$ representation, then following (3.2)

$$\begin{aligned} Y \cdot \check{w} &= y^{-\frac{1}{q} \sum_{i=1}^q i k_i - \frac{1}{p} \sum_{j=1}^{p-1} (p-j) k_{q+j}} \check{w} \\ &= y^{-\frac{1}{q} \sum_{\alpha=1}^{q-1} \alpha j_\alpha - (2k-2 \sum_{\alpha=0}^{q-1} j_\alpha) - \frac{1}{p} \sum_{\alpha=1}^{q-1} (p-\alpha) j_{q-\alpha} - \frac{p-q}{p} j_0} \check{w} \\ &= y^{-2k - \frac{1}{q} \sum_{\alpha=1}^{q-1} \alpha j_\alpha + 2 \sum_{\alpha=0}^{q-1} j_\alpha - \frac{1}{p} \sum_{i=1}^{q-1} (p-(q-l)) j_l - \frac{p-q}{p} j_0} \check{w} \\ &= y^{-2k - \sum_{i=1}^{q-1} \left(\frac{i}{q} - 2 + \frac{p-q+i}{p}\right) j_i + \left(1 + \frac{q}{p}\right) j_0} \check{w}. \end{aligned}$$

We can see that

$$\begin{aligned} & \sum_{i=1}^{q-1} \left(\frac{i}{q} - 2 + \frac{p-q+i}{p}\right) j_i = \sum_{i=1}^{q-1} \left(\frac{i}{q} - 1 + \frac{-q+i}{p}\right) j_i \\ &= \sum_{i=1}^{q-1} \left(\frac{pi - pq + q(-q+i)}{pq}\right) j_i = \sum_{i=1}^{q-1} \left(\frac{(p+q)i - q(p+q)}{pq}\right) j_i \\ &= \frac{p+q}{pq} \sum_{i=1}^{q-1} (i-q) j_i. \end{aligned}$$

Hece,

$$Y \cdot \check{w} = y^{-2k + \frac{p+q}{pq} \sum_{i=1}^{q-1} (q-i) j_i + \left(1 + \frac{q}{p}\right) j_0} \check{w}.$$

□

In order to describe explicitly the moduli up to gauge equivalence of maps of holomorphic isometric embeddings $\mathrm{Gr}_p(\mathbf{C}^{p+q}) \rightarrow \mathcal{Q}_N = \mathrm{Gr}_N(W)$ of degree k , with $p \geq q$, the generalisation of do Carmo–Wallach theory, Theorems 2.1 and 2.2, requires to determine $\mathrm{GS}(\mathfrak{m}V_0, V_0) \cap \mathbf{H}(W)^\perp$.

The universal quotient bundle $Q \rightarrow \text{Gr}_p(\mathbf{C}^{p+q})$ of the p -plane Grassmannian is of rank q , and therefore is related to the holomorphic line bundle $\mathcal{O}(1) \rightarrow \text{Gr}_p(\mathbf{C}^{p+q})$ by $\Lambda^q Q \cong \mathcal{O}(1)$. At the reference point $o \in \text{Gr}_p(\mathbf{C}^{p+q})$, the center $U(1)$ of the isotropy subgroup $S(U(p) \times U(q))$, generated by Y , acts on the fibre $\mathcal{O}(1)_o$ with weight -1 . We will denote this standard fibre by \mathbf{C}_{-1} . Analogously, \mathbf{C}_k will stand for the irreducible representation of $U(1)$ with highest weight k . With this notation the homogeneous bundle $SU(p+q) \times_{S(U(p) \times U(q))} \mathbf{C}_{-k}$ is the complex line bundle $\mathcal{O}(k) \rightarrow \text{Gr}_p(\mathbf{C}^{p+q})$, $p \geq q$, of degree k . Notice that, by Theorem 2.1, W is identified with $H^0(\text{Gr}_p(\mathbf{C}^{p+q}), \mathcal{O}(k))$.

Lemma 3.3.

$$\mathfrak{m}V_0 = \mathbf{C}^p \otimes \mathbf{C}^{q*} \otimes \mathbf{C}_{-k+\frac{1}{p}+\frac{1}{q}}.$$

Proof. If $\mathbf{C}^p = \mathbf{V}(1, 0, \dots, 0)$ denotes the standard representation of $U(p+q)$, the tautological vector bundle $S \rightarrow \text{Gr}_p(\mathbf{C}^{p+q})$ can be identified with the homogeneous bundle $SU(p+q) \times_{S(U(p) \times U(q))} \mathbf{C}^p$. The universal quotient bundle $Q \rightarrow \text{Gr}_p(\mathbf{C}^{p+q})$ is defined in the standard way, by the exact sequence $0 \rightarrow S \rightarrow \mathbf{C}^{p+q} \rightarrow Q \rightarrow 0$. Altogether, they define the holomorphic tangent bundle, $T \rightarrow \text{Gr}_p(\mathbf{C}^{p+q})$, $T = S^* \otimes Q$, which is a homogeneous bundle with standard fibre

$$\left(\mathbf{C}^p \otimes \mathbf{C}_{\frac{1}{p}}\right)^* \otimes \left(\mathbf{C}^q \otimes \mathbf{C}_{-\frac{1}{q}}\right) = \mathbf{C}^{p*} \otimes \mathbf{C}^q \otimes \mathbf{C}_{-\frac{1}{p}-\frac{1}{q}}.$$

Analogously, the standard fibre of the cotangent bundle $T^* \rightarrow \text{Gr}_p(\mathbf{C}^{p+q})$ is given by

$$T^* = \mathbf{C}^p \otimes \mathbf{C}^{q*} \otimes \mathbf{C}_{\frac{1}{p}+\frac{1}{q}}.$$

Now, $T^c \cong \mathfrak{m}^c$, and V_0 is spanned by the lowest-weight vector of $H^0(\text{Gr}_p(\mathbf{C}^{p+q}), \mathcal{O}(k))$, then

$$T^c \otimes V_0 = \mathbf{C}^p \otimes \mathbf{C}^{q*} \otimes \mathbf{C}_{\frac{1}{p}+\frac{1}{q}} \otimes \mathbf{C}_{-k} = \mathbf{C}^p \otimes \mathbf{C}^{q*} \otimes \mathbf{C}_{-k+\frac{1}{p}+\frac{1}{q}}.$$

□

Lemma 3.4.

$$\text{GS}(\mathfrak{m}V_0, V_0) \cap H(W)^\perp = \mathbf{F}(2k\varpi_q).$$

Proof. As a consequence of the previous Lemma 3.3, the lowest-weight of $\text{GS}(\mathfrak{m}V_0, V_0)$ is

$$-k - k + \frac{1}{p} + \frac{1}{q} = -2k + \frac{1}{p} + \frac{1}{q}.$$

On the other hand, by Lemma 3.2, the lowest-weight of

$$F \left(\sum_{i=1}^{q-1} j_i \varpi_i + \left(2k - 2 \sum_{\alpha=0}^{q-1} j_\alpha \right) \varpi_q + \sum_{i=1}^q j_{q-i} \varpi_{q+i} \right)$$

is lower than the lowest-weight of $\text{GS}(\mathfrak{m}V_0, V_0)$ in $S^2(\mathbf{F}(k\varpi_q))$ if and only if

$$-2k + \frac{p+q}{pq} \sum_{i=1}^{q-1} (q-i)j_i + \left(1 + \frac{q}{p}\right) j_0 \leq -2k + \frac{1}{p} + \frac{1}{q},$$

or, equivalently, if and only if

$$\frac{p+q}{pq} \left(-1 + \sum_{i=1}^{q-1} (q-i)j_i \right) + \left(1 + \frac{q}{p}\right) j_0 \leq 0,$$

if and only if

$$-1 + \sum_{i=1}^{q-1} (q-i)j_i \leq 0, \quad j_0 = 0,$$

that is

$$(q-1)j_1 + (q-2)j_2 + \cdots + 2j_{q-2} + j_{q-1} \leq 1, \quad j_0 = 0.$$

It follows that $j_{q-1} = \{0, 1\}$ and all the other terms vanish. Therefore,

$$\text{GS}(\mathfrak{m}V_0, V_0) \subseteq \underbrace{\mathbf{F}(2k\varpi_q)}_{j_{q-1}=0} \oplus \underbrace{\mathbf{F}((2k-2)\varpi_q + \varpi_{q+1})}_{j_{q-1}=1}.$$

However, since

$$(3.17) \quad \mathbf{F}(2k\varpi_q) \subset \mathbf{S}^2(\mathbf{F}(k\varpi_q))$$

while

$$(3.18) \quad \mathbf{F}((2k-2)\varpi_q + \varpi_{q+1}) \subset \Lambda^2(\mathbf{F}(k\varpi_q))$$

we have that

$$\text{GS}(\mathfrak{m}V_0, V_0) \cap \mathbf{H}(W)^\perp = \mathbf{F}(2k\varpi_q).$$

The inclusions 3.17 and 3.18 are determined as follows: Let \hat{w} denote as usual the highest-weight vector of $\mathbf{F}(k\varpi_q)$, and denote by \hat{w}_- the vector in $\mathbf{F}(k\varpi_q)$ with weight just below that of \hat{w} . It is clear that $\hat{w} \otimes \hat{w}$ is contained in $\mathbf{S}^2\mathbf{F}(k\varpi_q)$ and that it is the highest-weight vector in the irreducible component $\mathbf{F}(2k\varpi_q)$. Therefore, the inclusion 3.17 follows. Analogously, $\hat{w} \wedge \hat{w}_-$ belongs to $\Lambda^2(\mathbf{F}(k\varpi_q))$, and it is the highest-weight vector in the irreducible component $\mathbf{F}((2k-2)\varpi_q + \varpi_{q+1})$, implying the inclusion 3.18, and completing the proof. \square

Corollary 3.5. *The orthogonal complement to $\text{GS}(\mathfrak{m}V_0, V_0) \oplus \mathbf{R}Id$ in $\text{Sym}(W)$ is*

$$V_k = \mathbf{S}^2(\mathbf{F}(k\varpi_q)) - \mathbf{F}(2k\varpi_q)$$

Remark 2. Although hook-length formulae for the dimension $\mathbf{d}_{p+q}(\lambda)$ of each $\text{SU}(p+q)$ -representation $\mathbf{V}(\lambda)$ contributing for the moduli exist, the explicit expression is too cumbersome to be useful in itself. The dimension of the corresponding moduli is more easily computed simply as $\dim V_k = \dim \mathbf{S}^2\mathbf{F}(k\varpi_q) - \dim \mathbf{F}(2k\varpi_q)$; explicitly

$$(3.19) \quad \dim V_k = \frac{1}{2} \mathbf{d}_{p+q}(k\varpi_q)(\mathbf{d}_{p+q}(k\varpi_q) + 1) - \mathbf{d}_{p+q}(2k\varpi_q)$$

where the needed hook-rule formula is given by

$$\mathbf{d}_a(b\varpi_c) = \prod_{i=1}^c \prod_{j=1}^b \frac{a-i+j}{1+(c-i)+(b-j)}.$$

As a consequence of the previous Lemmas and Corollary, Theorem 2.1 allows to give a clear picture of the desired moduli space, information which is summarised in the next Theorem, and which generalises to arbitrary Grassmannian manifolds the main result in [13]:

Theorem 3.6. *If $f : \text{Gr}_p(\mathbf{C}^{p+q}) \rightarrow \text{Gr}_n(\mathbf{R}^{n+2})$ is a full holomorphic isometric embedding of degree k , then $n + 2 \leq \dim V_k$.*

Let \mathcal{M}_k be the moduli space of full holomorphic isometric embeddings of degree k of $\text{Gr}_p(\mathbf{C}^{p+q})$ into $\text{Gr}_N(\mathbf{R}^{N+2})$ by the gauge equivalence of maps, where $N + 2 = \dim V_k$. Then, \mathcal{M}_k can be regarded as an open bounded convex body in V_k .

Let $\overline{\mathcal{M}_k}$ be the closure of the moduli \mathcal{M}_k by the topology induced from the inner product. Every boundary point of $\overline{\mathcal{M}_k}$ distinguishes a subspace \mathbf{R}^{h+2} of \mathbf{R}^{N+2} and describes a full holomorphic isometric embedding into $\text{Gr}_h(\mathbf{R}^{h+2})$ which can be regarded as totally geodesic submanifold of $\text{Gr}_N(\mathbf{R}^{N+2})$. The inner product on \mathbf{R}^{N+2} determines the orthogonal decomposition of $\mathbf{R}^{N+2} : \mathbf{R}^{N+2} = \mathbf{R}^{h+2} \oplus (\mathbf{R}^{h+2})^\perp$. Then the totally geodesic submanifold $\text{Gr}_h(\mathbf{R}^{h+2})$ can be obtained as the common zero set of sections of $Q \rightarrow \text{Gr}_N(\mathbf{R}^{N+2})$, which belongs to $(\mathbf{R}^{h+2})^\perp$.

The formal relation of this theorem to Theorem 2.1, and consequently its proof, is the same as that of the Main Theorem (Theorem 3.6) in [13]. Nevertheless, we reproduce it here with minor changes for the sake of clarity and completeness.

Proof. The constraint $n \leq N$ is a consequence of (a) in Theorem 2.1 and Bott–Borel–Weil theorem.

It follows from (c) in Theorem 2.1 that $\text{GS}(\mathfrak{m}V_0, V_0)^\perp$ is a parametrization of the space of full holomorphic isometric embeddings $f : \text{Gr}_p \mathbf{C}^{p+q} \rightarrow \text{Gr}_N(\mathbf{R}^{N+2})$ of degree k . Since the standard map into $\text{Gr}_N(\mathbf{R}^{N+2})$ is the composite of the Kodaira embedding $\text{Gr}_m(\mathbf{C}^{m+2}) \rightarrow \mathbf{C}P^{\frac{N}{2}}$ and the totally geodesic embedding $\mathbf{C}P^{\frac{N}{2}} \hookrightarrow \text{Gr}_N(\mathbf{R}^{N+2})$, we can apply Theorem 2.2 and Corollary 3.5 to conclude that \mathcal{M}_k is a bounded connected open convex body in V_k with the topology induced by the L^2 scalar product.

Under the natural compactification in the L^2 -topology, the boundary points correspond to endomorphisms T which are not positive definite, but positive semi-definite. It follows from Theorem 2.1 that each of these endomorphisms defines a full holomorphic isometric embedding $\text{Gr}_p(\mathbf{C}^{p+q}) \rightarrow \text{Gr}_h(\mathbf{R}^{h+2})$, of degree k with $h = 2k - \dim \text{Ker } T$, whose target embeds in $\text{Gr}_N(\mathbf{R}^{N+2})$ as a totally geodesic submanifold. The image of the embedding $\text{Gr}_h(\mathbf{R}^{h+2}) \hookrightarrow \text{Gr}_N(\mathbf{R}^{N+2})$ is determined by the common zero set of sections in $\text{Ker } T$. (See also the Remark after Proposition 5.14 in [14] for the geometric meaning of the compactification of the moduli space.) \square

Remark 3. It follows from Corollary 5.18 in [15] that the first condition in (c) is automatically satisfied. Alternatively, using the same techniques as in Lemma 3.4 it can be shown that

$$\text{GS}(V_0, V_0) \cap \text{H}(W)^\perp = \mathbf{V}(2k, 2k, 0, \dots, 0).$$

The centraliser $S^1 \cong \text{U}(1)$ of the holonomy subgroup $\text{K} = \text{S}(\text{U}(m) \times \text{U}(2))$ of the structure group of the line bundle acts on \mathcal{M}_k . For the general theory, see [15]. Therefore, the same argument and proof as the one of Theorem 8.1 in [11] applies leading to

Corollary 3.7. *The moduli space \mathbf{M}_k of image equivalence classes of holomorphic isometric embeddings $\text{Gr}_m(\mathbf{C}^{m+2}) \rightarrow \text{Gr}_N(\mathbf{R}^{N+2})$, $N + 2 = \dim V_k$, of degree k , is*

$$\mathbf{M}_k = \mathcal{M}_k / S^1.$$

Remark 4. There is a natural, induced complex structure defined on \mathcal{M}_k from its embedding in V_k . It is also equipped with a compatible metric induced from the inner

product, so \mathcal{M}_k is a Kähler manifold. The aforesaid S^1 -action preserves the Kähler structure on \mathcal{M}_k . The moment map $\mu : \mathcal{M}_k \rightarrow \mathbf{R} : |Id - T^2|^2$ induces the Kähler quotient, and \mathbf{M}_k has a foliation whose general leaves are the complex projective spaces.

4. A WORKED-OUT EXAMPLE: EMBEDDING OF $Gr_q(\mathbf{C}^{2q})$

This example generalises the case of the holomorphic isometric embedding of the complexified, compactified Minkowski Space $Gr_2(\mathbf{C}^4)$ into quadrics, discussed in [13]. For that end, assume $p = q$ in the general discussion above. Then, we have that

$$\begin{aligned} \otimes^2 \mathbf{F}(k\varpi_q) &= \bigoplus_{i_1, i_2, \dots, i_q=0}^{k \geq \sum_{\alpha=1}^q i_\alpha} \mathbf{V}(2k - i_q, 2k - (i_{q-1} + i_q), \dots, 2k - \sum_{\alpha=1}^q i_\alpha, \\ &\quad \sum_{\alpha=1}^q i_\alpha, \sum_{\alpha=2}^q i_\alpha, \dots, i_{q-1} + i_q, i_q) \\ &= \bigoplus_{i_\alpha=0}^{k \geq \sum_{\alpha=1}^q i_\alpha} \mathbf{F}(\underbrace{i_{q-1}, i_{q-2}, \dots, i_2, i_1}_{q-1}, 2k - 2 \sum_{\alpha=1}^q i_\alpha, \underbrace{i_1, i_2, \dots, i_{q-1}}_{q-1}) \\ &= \bigoplus_{j_\alpha=0}^{k \geq \sum_{\alpha=0}^{q-1} j_\alpha} F \left(\sum_{i=1}^{q-1} j_i \varpi_i + \left(2k - 2 \sum_{\alpha=0}^{q-1} j_\alpha \right) \varpi_q + \sum_{i=1}^{q-1} j_{q-i} \varpi_{q+i} \right). \end{aligned}$$

Comparing this last expression with the general one for the case $p > q$, the difference is the vanishing of the term $j_0 \varpi_{2q}$. Hence,

$$Y \cdot \check{\omega} = y^{-2k + \frac{p+q}{p}} \sum_{i=1}^{q-1} (q-i) j_i + 2j_0 \check{\omega}.$$

The term $2j_0$ is the same as $(1 + \frac{q}{p})j_0$ when $p = q$. Hence, if $p = q$ the content of Lemma 3.2 would be restated as

Lemma 4.1. *The lowest weight of*

$$F \left(\sum_{i=1}^{q-1} j_i \varpi_i + \left(2k - 2 \sum_{\alpha=0}^{q-1} j_\alpha \right) \varpi_q + \sum_{i=1}^{q-1} j_{q-i} \varpi_{q+i} \right)$$

is

$$-2k + \frac{2}{q} \sum_{i=1}^{q-1} (q-i) j_i + 2j_0.$$

Now, from Lemma 3.3,

$$\mathfrak{m}V_0 = \mathbf{C}^q \otimes (\mathbf{C}^q)^* \otimes \mathbf{C}_{-k + \frac{2}{q}}.$$

Therefore, $\text{GS}(\mathfrak{m}V_0, V_0)$ has the weight

$$-k - k + \frac{2}{q} = -2k + \frac{2}{q}.$$

The lowest weight of

$$F \left(\sum_{i=1}^{q-1} j_i \varpi_i + \left(2k - 2 \sum_{\alpha=0}^{q-1} j_\alpha \right) \varpi_q + \sum_{i=1}^{q-1} j_{q-i} \varpi_{q+i} \right)$$

is lower than the one of $\text{GS}(\mathfrak{m}V_0, V_0)$ in $S^2(\mathbf{F}(k\varpi_q))$ if and only if

$$-2k + \frac{2}{q} \sum_{i=1}^{q-1} (q-i)j_i + 2j_0 \leq -2k + \frac{2}{q},$$

equivalently

$$\frac{2}{q} \left(-1 + \sum_{i=1}^{q-1} (q-i)j_i \right) + 2j_0 \leq 0$$

if and only if

$$\sum_{i=1}^{q-1} (q-i)j_i - 1 \leq 0, \quad j_0 = 0.$$

Thus,

$$(q-1)j_1 + (q-2)j_2 + \cdots + 2j_{q-2} + j_{q-1} \leq 1, \quad j_0 = 0.$$

It follows that $j_{q-1} \in \{0, 1\}$ and the others terms must vanish. As a consequence,

$$\text{GS}(\mathfrak{m}V_0, V_0) \subset \underbrace{\mathbf{F}(2k\varpi_q)}_{j_{q-1}=0} \oplus \underbrace{\mathbf{F}((2k-2)\varpi_q + \varpi_{q+1})}_{j_{q-1}=1}$$

and then again, since $\mathbf{F}(2k\varpi_q) \in S^2(\mathbf{F}(k\varpi_q))$ but $\mathbf{F}((2k-2)\varpi_q + \varpi_{q+1}) \subset \Lambda^2(\mathbf{F}(k\varpi_q))$ we have that

$$\text{GS}(\mathfrak{m}V_0, V_0) = \mathbf{F}(2k\varpi_q).$$

The real dimension of V_k is therefore computed as in the general case Eqn. (3.19) where now $n = 2q$, that is, using

$$\mathbf{d}_{2q}(k\varpi_q) = \prod_{i=1}^q \prod_{j=1}^k \frac{2q-i+j}{1+(q-i)+(k-j)}, \quad \mathbf{d}_{2q}(2k\varpi_q) = \prod_{i=1}^q \prod_{j=1}^{2k} \frac{2q-i+j}{1+(q-i)+(2k-j)}.$$

Moreover, if q is even, say $q = 2m$, then $\Lambda^q \mathbf{C}^{2q} = \Lambda^{2m} \mathbf{C}^{4m}$ has an invariant real structure, which will be denoted by σ , determining a real subspace $U = (\Lambda^{2m} \mathbf{C}^{4m})^{\mathbf{R}} \subset \Lambda^{2m} \mathbf{C}^{4m}$. Therefore, if $k = 1$ we obtain a one-parameter family of holomorphic isometric embeddings generalising Corollary 3.7 of [13]

Corollary 4.2. *There exists a one-parameter family $\{f_t\}$ with $t \in [0, 1]$ of $\text{SU}(4m)$ -equivariant non-congruent holomorphic isometric embeddings of degree 1 of $\text{Gr}_{2m}(\mathbf{C}^{4m})$ into complex quadrics. The mapping f_0 corresponds to the Kodaira embedding, while f_1 is the real standard map defined by the pair $(\mathcal{O}(1), U)$.*

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