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Maximal Admissible Set for Graph-Dependent Switched Systems with Dwell Time Restriction

Hoai-Nam Nguyen*

[1]SAMOVAR, Télécom SudParis, Institut Polytechnique de Paris, 91120 Palaiseau, France

Correspondence
*Hoai-Nam Nguyen, Email: hoai-nam.nguyen@telecom-sudparis.eu

Summary

In this paper we present an extension of the concept of the maximal admissible set for constrained graph-dependent switched systems with bounded disturbances under dwell time restriction. Termed as the maximal switch-dependently robustly admissible set (MSDRAS), we provide efficient numerical procedures for its determination. For this purpose, we exploit available information about the current active mode, minimum dwell times, and mode transition graph. We employ the MSDRAS for three purposes: i) characterizing the largest set of initial states ensuring constraint satisfaction; ii) establishing necessary and sufficient conditions for stability of switched systems; iii) computing the minimum mode-dependent dwell times. Furthermore, we extend the results to switched systems with parametric uncertainties. Through three numerical examples, we compare our solutions with earlier from the literature to illustrate the effectiveness of our approach.

KEYWORDS:
Switched System, Dwell Time, Mode Transition Graph, Admissible Set, Constrained Control

1 | INTRODUCTION

The concept of the maximal admissible set (MAS) plays an important role in safety-critical applications for dynamical systems[1,2,3,4]. The MAS is the largest set of initial states from which the system can operate while satisfying constraints. The MAS serves as fundamental tools for evaluating the system's capability to maintain states within predefined safe regions. In addition, the MAS offers critical insights into the system's safety margins. Moreover, the MAS has attracted significant interest from the control community due to its close relationship with stability theory.

Different numerical procedures have been proposed in the literature to compute the MAS for different classes of discrete-time systems, e.g.,[5,6,7]. Effort often lies on searching for the maximal invariant set (MIS), which for many classes of systems coincides with the MAS. Recently, it was shown[8] that for a particular class of systems, the MAS can be larger than the MIS.

In this paper, we study the MAS for a linear discrete-time switched system with bounded disturbances. The system is subject to various constraints, including bounds on both the state and the disturbances. Additionally, there are two other types of constraints. The first one is on the mode transitions. The system operates in different subsystems or equivalently modes, each representing a distinct behavior. These modes can be switched between based on certain conditions. The constraints on admissible mode transitions refer to the permissible changes from one mode to another. This can be represented by a directed graph, where nodes correspond to different modes, and edges represent permissible transitions between these modes. The second constraint is on the dwell times, which refer to the minimum duration that the system must remain in a particular mode before transitioning

to another. In the past decade, switched systems with restrictions on the mode transition graph and on the dwell times find applications in various domains such as power systems, communication networks, control theory [9, 10, 11, 12, 13].

Related works. To characterize the MAS for the considered class of switched systems, to the best of the author's knowledge, all of the works in the literature are concerned with the computation of invariant sets. There are three works [14, 15, 16] that studied different concepts of invariance, closely related to the research presented in this paper. Since we will use the results in [14] as the foundation for Algorithm 1, we will detail the work [14] in Section 3.

In [15], the concept of multi-set invariance was considered for switched systems represented on a graph without specific consideration of dwell time restrictions. In parallel, in [17], the graph based modeling framework was extended to encompass switched systems with minimum dwell time. Nevertheless, the graph structures in [15, 17] are different from those used in the current study. Specifically, the nodes in [15, 17] do not correspond to the modes, and the edges do not signify admissible mode transitions, as they do in the current paper. In addition, the lifting technique is required to model a switched system with dwell time. Hence, the complexity of the lifted system can increase significantly compared to the original switched system, especially for large dwell times. This complexity arises from the increased number of nodes and edges needed to represent the system's state transitions accurately. This complexity can be mitigated by using techniques such as the unavoidable set of nodes [17]. Despite this reduction, the number of sets in the invariant multi-set remains substantial, as it mirrors the complexity of the non-reduced lifted system.

In [16], the authors introduced the concept of switch - robustly "control" invariant set, but the definition deviates from the standard, see definition 3 in [16]. This leads different computational procedures and results compared to those presented in the paper.

The paper is concerned with the computation of the MAS for constrained switched systems with bounded disturbances. Using available information on the current active mode, we introduce the notions of the maximal switch-dependent robustly admissible set (MSDRAS), and of the maximal switch-dependent robustly invariant set (MSDRIS). The main contributions are:

- We show that the MSDRAS is equivalent to the MSDRIS.
- We provide two new numerical algorithms to construct the MSDRAS. The algorithms are designed to operate directly within the original, non-lifted system, offering a significant computational advantage by avoiding the complexities associated with system lifting.
- We show that the MSDRAS/MSDRIS is a star-shaped set.
- We extend the results to constrained switched systems with parametric uncertainties.
- We show that the existence of the MSDRAS provides a necessary and sufficient condition for the stability of switched systems.

The paper is organized as follows. Section II is dedicated to the problem formulation and preliminaries. Section III is concerned with earlier works. Section IV presents the main results on the computation of invariant sets for switched systems. Three simulated examples are evaluated in Section V before drawing the conclusions in Section VI.

1.1 | Notation

We denote by $\mathbb{N}$ the set of natural numbers, by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^{n\times m}$ the set of real $n \times m$ matrices. For a given integer $N > 0$, we use $\overline{1, N}$ to denote the set $\{1, 2, \dots, N\}$. For given sets $\Omega_i, \forall i \in \overline{1, N}$, we use $\bigcap_{i=1}^{N} \Omega_i$ and $\bigcup_{i=1}^{N} \Omega_i$, respectively, to denote the intersection, and the union of $\Omega_i \subseteq \mathbb{R}^n, \forall i \in \overline{1, N}$, i.e.,

$$\bigcap_{i=1}^{N} \Omega_i = \{x \in \mathbb{R}^n : x \in \Omega_1 \text{ and } \dots \text{ and } x \in \Omega_N\}$$
$$\bigcup_{i=1}^{N} \Omega_i = \{x \in \mathbb{R}^n : x \in \Omega_1 \text{ or } \dots \text{ or } x \in \Omega_N\}$$

For a given set Ω, we use $\mathrm{Bd}(\Omega)$ to denote its boundary. Given two sets $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^n$, the Minkowski sum Z of X and Y is denoted as $Z = X \oplus Y$, and is defined by

$$Z := \{z \in \mathbb{R}^n : z = x + y, \forall x \in X, \forall y \in Y\}$$

2 | PROBLEM FORMULATION AND PRELIMINARIES

2.1 | Problem Formulation

We consider the following constrained switched linear discrete-time system

$$x(k+1) = A_{\sigma(k)}x(k) + w(k) \tag{1}$$

$$x(k) \in X_{\sigma(k)}, w(k) \in W_{\sigma(k)} \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^n$ is the disturbance. The switching signal $\sigma(k)$ is an exogenous input that takes values in a finite set $\mathcal{I} = \overline{1, N}$ at each time instant. N is the number of subsystems. $\sigma(k)$ indicates the active dynamics $A_i \in \mathbb{R}^{n\times n}$, and the active constraint sets X_i, W_i at time k. In this paper, it is assumed that all eigenvalues of A_i lie strictly inside the unit circle, $\forall i \in \mathcal{I}$.

The set X_i is

$$X_i = \{x \in \mathbb{R}^n \,:\, F_i x \leq g_i\} \tag{3}$$

where F_i, g_i are known matrices such that $g_i > 0, \forall i \in \mathcal{I}$. This implies that X_i contains the origin in its interior. The inequalities are taken element-wise. W_i is a polyhedral set, and contains the origin.

Remark 1: For simplicity, we consider only linear switched systems in this paper. However, our technique can be straightforwardly extended to affine switched systems of the form

$$x(k+1) = A_{\sigma(k)}x(k) + v_{\sigma(k)} + w(k)$$

where $v_i \in \mathbb{R}^n$ is constant, $\forall i \in \mathcal{I}$. □

We consider two constraints on the switching sequence $\sigma(k)$. The first one is a restriction on the admissible mode transitions. They are represented by a directed graph $\mathcal{G} = \{\mathcal{I}, \mathcal{E}\}$, where nodes in $\mathcal{I}$ denotes modes, and edge $(i, l) \in \mathcal{E}$ indicates that a switch from mode i to mode l is possible, $l \neq i$. We use $\sigma(k) \in \mathcal{G}$ to denote the set of $\sigma(k)$ with the restricted mode transition graph $\mathcal{G}$.

Let $\{k_s\}$ be the sequence of switching times with $k_0 = 0$, $k_s < k_{s+1}$, and $\sigma(k_{s+1}) \neq \sigma(k_s)$. This implies that $\sigma(k_s) = \sigma(k_s + 1) = \ldots = \sigma(k_{s+1} - 1)$. The second constraint for $\sigma(k)$ is on k_s. Define τ_i as the dwell time of mode $i \in \mathcal{I}$, i.e.,

$$\tau_i := \min\{k_{s+1} - k_s \,:\, \sigma(k_s) = i, s \in \mathbb{N}\}$$

In the paper we consider only switching sequences $\sigma(k)$ with dwell times of at least τ_i time steps for each mode $i \in \mathcal{I}$. No future information of σ is available, but we assume that $\sigma(k)$ is known at time k.

For proceeding further, we use Σ to denote the set of all admissible switching sequences $\sigma(k)$, i.e., $\forall\sigma(k) \in \mathcal{G}$ that satisfy the constraints on the dwell time.

The objective of this paper is to provide a numerical procedure to compute the largest set of all the states that if the initial state $x(0)$ belongs to this set, then the constraints are satisfied all the time, i.e., $x(k+1) \in X_{\sigma(k+1)}$, $\forall w(k) \in W_{\sigma(k)}$, $\forall\sigma(k) \in \Sigma$, $\forall k \geq 0$. We will give a precise definition of the set at the beginning of the next sub-section.

2.2 | Preliminaries

Definition 1: Switch-Dependently Robustly Admissible Set. Given the sets $\Phi_i \subseteq X_i, \forall i \in \mathcal{I}$. The union $\bigcup_{i=1}^{N} \Phi_i$ is a switch-dependently robustly constraint-admissible set (SDRAS) for system (1) with constraints (2) if and only if $\forall x(0) \in \Phi_{\sigma(0)} \subseteq X_{\sigma(0)}$, one has $x(k+1) \in X_{\sigma(k+1)}$, $\forall w(k) \in W_{\sigma(k)}$, $\forall\sigma(k) \in \Sigma$, $\forall k \geq 0$. Furthermore, if every SDRAS is contained in $\bigcup_{i=1}^{N} \Phi_i$, then $\bigcup_{i=1}^{N} \Phi_i$ is the maximal switch-dependently robustly constraint-admissible set (MSDRAS).

Definition 2: Switch-Dependently Robustly Invariant Set. Given the sets $\Phi_i \subseteq X_i, \forall i \in \mathcal{I}$. The union $\bigcup_{i=1}^{N} \Omega_i$ is a switch-dependently robustly invariant set (SDRIS) for system (1) with constraints (2) if and only if $\forall x(0) \in \Omega_{\sigma(0)}$, one has $x(k+1) \in \Omega_{\sigma(k+1)}$, $\forall w(k) \in W_{\sigma(k)}$, $\forall\sigma(k) \in \Sigma$, $\forall k \geq 0$. Furthermore, if every SDRIS is contained in $\bigcup_{i=1}^{N} \Omega_i$, then $\bigcup_{i=1}^{N} \Omega_i$ is the maximal switch-dependently robustly constraint-admissible set (MSDRIS).

The following result holds.

Proposition 1: The set $\bigcup_{i=1}^{N} \Omega_i$ with $\Omega_i \subseteq X_i, \forall i \in \mathcal{I}$ is the MSDRIS for (1), (2) if and only if $\bigcup_{i=1}^{N} \Omega_i$ is the MSDRAS.

Proof: If $\bigcup_{i=1}^{N} \Omega_i$ is the MSDRIS, then $\forall x(0) \in \Omega_{\sigma(0)} \in X_{\sigma(0)}$, one has $x(k+1) \in \Omega_{\sigma(k+1)} \in X_{\sigma(k+1)}, \forall w(k) \in W_{\sigma(k)}, \forall \sigma(k) \in \Sigma$, $\forall k \geq 0$. Hence, $\bigcup_{i=1}^{N} \Omega_i$ is a SDRAS. Therefore, it is contained in the MSDRAS.

Conversely, if $\bigcup_{i=1}^{N} \Omega_i$ is the MSDRAS, then $\forall x(0) \in \Omega_{\sigma(0)}$, one has $x(k+1) \in X_{\sigma(k+1)}, \forall w(k) \in W_{\sigma(k)}, \forall \sigma(k) \in \Sigma, \forall k \geq 0$. Because $\Omega_{\sigma(k+1)}$ is the largest SDRAS in $X_{\sigma(k+1)}$, one should have $x(k+1) \in \Omega_{\sigma(k+1)}$. Hence, $\bigcup_{i=1}^{N} \Omega_i$ is a SDRIS. Consequently, it is contained in the MSDRIS. The proof is complete. □

Using Proposition 1, we conclude that the problem of computing the MSDRAS is equivalent to the problem of finding MSDRIS. In the rest of the paper, we will characterize the MSDRAS mostly via the MSDRIS.

Given an integer $t \geq 1$. In the context of set invariance theory, the t-step set plays an important role [1,18]. We recall this concept in the following. Consider the linear discrete-time system subject to the bounded disturbance $\delta(k) \in \Delta$

$$x(k+1) = \mathcal{A}x(k) + \delta(k) \tag{4}$$

Using (4), the $t-$step ahead state $x(k+t)$ is given as, $t \geq 1$

$$x(k+t) = \mathcal{A}^t x(k) + \sum_{m=0}^{t-1} \mathcal{A}^m \delta(k+t-1-m) \tag{5}$$

Definition 3: t-Step Set. Given a set $\Psi \subset \mathbb{R}^n$, the t-step set $\mathcal{Q}_t(\Psi)$ for system (4) is the set of all $x(0)$ such that $x(t) \in \Psi$, i.e.,

$$\mathcal{Q}_t(\Psi) := \{x(0) \in \mathbb{R}^n \,:\, x(t) \in \Psi\}$$

in spite of disturbances $\forall \delta(0) \in \Delta, \ldots, \forall \delta(t-1) \in \Delta$.

Using (5), if Ψ is a polyhedral set, i.e.,

$$\Psi = \{x \in \mathbb{R}^n \,:\, F_\psi x \leq g_\psi\}$$

then $\mathcal{Q}_t(\Psi)$ is a polyhedral set, and is computed by

$$\mathcal{Q}_t(\Psi) = \left\{x \in \mathbb{R}^n \,:\, F_\psi \mathcal{A}^t x \leq g_\psi - g_\delta\right\} \tag{6}$$

where $g_\delta = \max_{\delta \in \Delta} F_\psi \delta + \max_{\delta \in \Delta} F_\psi \mathcal{A}\delta + \ldots + \max_{\delta \in \Delta} F_\psi \mathcal{A}^{t-1}\delta$.

Definition 4: Robustly Invariant Set. $\Psi \subset \mathbb{R}^n$ is a robustly invariant set (RIS) for system (4) if and only if $x(k) \in \Psi$, one has $x(k+1) \in \Psi, \forall \delta(k) \in \Delta, \forall k \geq 0$.

If Ψ contains every RIS, then Ψ is the maximal robustly invariant set (MRIS). If Ψ is contained in any RIS, then Ψ is the minimal robustly invariant set (mRIS).

It is well known [1] that Ψ is a RIS if and only if $\Psi \subseteq \mathcal{Q}_1(\Psi)$. It is also well known [19] that if Ψ is the mRIS for system (4), then Ψ can be computed by $\Psi = \lim_{t \to \infty} \bigoplus_{k=0}^{t} \mathcal{A}^k \Delta$.

3 | EARLIER WORKS ON INVARIANT SETS

To the best of the author's knowledge, [14] is the first work that considers the problem of calculating the so-called dwell time robustly invariant set for system (1). In the following, we will recall key definitions, and extend key results of [14]. We will use them later to construct our new algorithms.

Definition 5: Dwell Time Robustly Invariant Set. $\Omega \subset \mathbb{R}^n$ is a dwell time robustly invariant set (DTRIS) for system (1) if and only if $\forall x(0) \in \Omega$, one has $x(k_s) \in \Omega, \forall w(k) \in W_{\sigma(k)}, \forall \sigma(k) \in \Sigma$, for any sequence of switching times k_s, and $\forall k \geq 0$.

Define

$$\mathcal{T}_i := \{\tau_i, \tau_i + 1, \ldots, 2\tau_i - 1\}, \forall i \in \mathcal{I} \tag{7}$$

Consider the following associated system of (1)

$$\hat{x}(k+1) = A_i^t \hat{x}(k) + \hat{\omega}_{i,t}(k) \tag{8}$$

where $t \in \mathcal{T}_i$, and $\hat{w}_{i,t} \in W_{i,t}$ with

$$W_{i,t} = W_i \oplus A_i W_i \oplus \ldots \oplus A_i^{t-1} W_i \tag{9}$$

In the following, we will provide a generalization of the results in [14]. They were presented only for the case $\tau_i = \tau$, $W_i = W$, $\forall i \in \mathcal{I}$. In this paper, there is no additional requirement on $\tau_i, W_i, \forall i \in \mathcal{I}$, except the ones in Section 2.1.

Lemma 1: Under the assumption that $\mathcal{G}$ is the complete graph, $\Omega \subset \mathbb{R}^n$ is a DTRIS for system (1) if and only if it is a RIS for (8).

Proof: The proof follows the same steps as Theorem 1 in [14]. Hence, it is omitted here. □

An interesting feature of Ω is that if $x(k) \in \Omega$ and $x(k+1) \notin \Omega$ for system (1) with $\sigma(k) = i$, then $x(k+\tau_i) \in \Omega$ for any $i \in \mathcal{I}$. In other words, $x(k)$ is allowed to leave temporally Ω, but will come back to Ω in no more than τ_i time steps under the dynamics

$$x(k+1) = A_i x(k) + w(k) \tag{10}$$

Define $X := \bigcap_{i=1}^{N} X_i$. Since X_i contains the origin in its interior $\forall i \in \mathcal{I}$, the set X is non-empty. Using (3), one gets

$$X := \left\{ x \in \mathbb{R}^n : \begin{bmatrix} F_1 \\ \vdots \\ F_N \end{bmatrix} x \leq \begin{bmatrix} g_1 \\ \vdots \\ g_N \end{bmatrix} \right\} \tag{11}$$

Clearly, a necessary condition for constraint admissibility is that $\Omega \subseteq X$. If $x(0) \in \Omega$ and $\Omega \subseteq X$ is a DTRIS, then by imposing $x(k) \in X, \forall k \in \mathcal{S}_i$ with

$$\mathcal{S}_i := \{1, 2, \ldots, \tau_i - 1\} \tag{12}$$

under the dynamics (10), $\forall i \in \mathcal{I}$, we will have $x(k) \in X \subseteq X_i, \forall i \in \mathcal{I}, \forall k \geq 0$.

For a given set $\Psi \subset \mathbb{R}^n$, define $\mathcal{Q}_{i,\mathcal{S}_i}(\Psi) := \bigcap_{t \in \mathcal{S}_i} \mathcal{Q}_{i,t}(\Psi)$ and $\mathcal{Q}_{i,\mathcal{T}_i}(\Psi) := \bigcap_{t \in \mathcal{T}_i} \mathcal{Q}_{i,t}(\Psi)$, where $\mathcal{Q}_{i,t}(\Psi)$ as the $t-$step set of Ψ under the dynamics (10), $\forall i \in \mathcal{I}$

The following algorithm computes a DTRIS for system (1). Step (1) of algorithm imposes the constraints for the first $\tau_i - 1$

Algorithm 1: Computation of DTRIS

1: Set $q \leftarrow 0$ and let $\Omega^{(q)} \leftarrow X \cap \left(\bigcap_{i=1}^{N} \mathcal{Q}_{i,\mathcal{S}_i}(X) \right)$.

2: Let $\Omega^{(q+1)} \leftarrow \Omega^{(q)} \cap \left(\bigcap_{i=1}^{N} \mathcal{Q}_{i,\mathcal{T}_i}(\Omega^{(q)}) \right)$.

3: If $\Omega^{(q+1)} \equiv \Omega^{(q)}$ set $\Omega \leftarrow \Omega^{(q)}$ then stop, else set $q \leftarrow q+1$ and go to step 2.

steps to assure constraint satisfaction of Ω under the dynamics (10). Step (2) imposes the condition of Lemma 1. It ensures that Ω is robustly invariant for system (8). Since X is a polyhedral set, so are $\Omega^{(q)}$, $\Omega^{(q+1)}$ and Ω.

Lemma 2: Suppose that the disturbance-free system (1) is asymptotically stable with dwell times τ_i under the dynamics (10) and with the complete mode transition graph $\mathcal{G}$. Then: i) $\Omega^{(q+1)} \subseteq \Omega^{(q)} \subseteq X, \forall q \geq 0$; ii) Algorithm 1 terminates after a finite number of steps; iii) If Ω is non-empty, then Ω is the maximal dwell time robustly invariant set (MDTRIS) for (1), (2); iv) If Ω is empty, then so is the MSDRAS, i.e., there is no $x(0) \in X_{\sigma(0)}$ such that $x(k) \in X_{\sigma(k)}, \forall k \geq 1$.

Proof: It is omitted here, since it follows the same steps of Theorem 3 in [14]. □

4 | COMPUTATION OF MSDRIS

4.1 | Complete Mode Transition Graph Case

In this section it is assumed that: i) the graph $\mathcal{G}$ is complete; ii) the MDTRIS Ω is already computed, and is non-empty. Our aim is to provide a numerical algorithm to construct the MSDRPI.

It is worth noticing that Ω is a SDRAS for (1), (2), as $\forall x(0) \in \Omega$, one has $x(k) \in X \subseteq X_i, \forall i \in \mathcal{I}, \forall k \geq 0$. However, Ω is not the MSDRAS. $\forall x(0) \in \Omega$ we have the guarantee that $x(\tau_i) \in \Omega$ under the dynamics (10), $\forall i \in \mathcal{I}$. Nonetheless $x(k)$ is allowed to leave Ω, $\forall k \in \overline{1, \tau_i - 1}$. It is clear that $\Omega \cup \left(\bigcup_{k=1}^{\tau_i - 1} x(k) \right)$ is a SDRAS, and that $\Omega \subset \Omega \cup \left(\bigcup_{k=1}^{\tau_i - 1} x(k) \right)$. Hence Ω is not the MSDRAS.

Define, $\forall i \in \mathcal{I}$

$$Z_i := X_i \cap \mathcal{Q}_{i,\tau_i}(\Omega) \tag{13}$$

i.e., Z_i is the set of all states $x \in X_i$ that can be brought into $\mathcal{Q}_{i,\tau_i}(\Omega)$ in no more than τ_i time steps under the dynamics (10). Clearly, Z_i is a polyhedral set.

For any $i \in \mathcal{I}$, the following algorithm is used to construct the MSDRIS or equivalently the MSDRAS for system (1), (2).

Algorithm 2: Computation of MSDRIS - Complete Mode Transition Graph

1: Set $q \leftarrow 0$ and let $\Omega_i^{(q)} \leftarrow Z_i$.
2: Let $\Omega_i^{(q+1)} \leftarrow \Omega_i^{(q)} \cap \mathcal{Q}_{i,1}\left(\Omega_i^{(q)}\right)$.
3: If $\Omega_i^{(q+1)} \equiv \Omega_i^{(q)}$ set $\Omega_i \leftarrow \Omega_i^{(q)}$ then stop, else set $q \leftarrow q + 1$ and go to step 2.

By executing N times Algorithm 2 for all $i \in \mathcal{I}$, one obtains N sets Ω_i. Because Z_i is a polyhedral set, so is Ω_i.

It is worth noticing that Algorithm 2 is a standard procedure[2] to compute the MRIS for system (10) with the constraints

$$x(k) \in Z_i, \ w(k) \in W_i \tag{14}$$

Step (1) initializes the construction of $\Omega_i^{(q)}$ with the outer approximation $\Omega_i^{(q)} = Z_i$. Step (2) removes states $x \in \Omega_i^{(q)}$ that cannot by kept in $\Omega_i^{(q)}$ under the dynamics (10). Algorithm 2 stops when $\Omega_i^{(q+1)} = \Omega_i^{(q)}$.

The following result holds

Proposition 2: Suppose the MDTRIS Ω produced by Algorithm 1 is non-empty. Then Ω_i is non-empty and is finitely determined for any $i \in \mathcal{I}$.

Proof: Denote the mRIS of (10) as Φ_i. It is well known[20] that for proving that Ω_i is non-empty, and is finitely determined by Algorithm 2, it suffices to show $\Phi_i \subseteq Z_i$.

Since Ω is a DTRIS, one gets $x(k_s) \in \Omega$, $\forall x(0) \in \Omega$. It follows that

$$A_i^{k_s}\Omega \oplus \bigoplus_{t=0}^{k_s-1} A_i^t W_i \subseteq \Omega \tag{15}$$

Because all eigenvalues of A_i lie strictly inside the unit circle, on has $\lim_{k_s \to \infty} A_i^{k_s}\Omega = 0$. Hence, $\lim_{k_s \to \infty} \bigoplus_{t=0}^{k_s-1} A_i^t W_i \subseteq \Omega$, or equivalently $\Phi_i \subseteq \Omega$.

Using (15) with $k_s = \tau_i$, one obtains

$$A_i^{\tau_i}\Omega \oplus \bigoplus_{t=0}^{\tau_i-1} A_i^t W_i \subseteq \Omega$$

Hence $\Omega \subseteq \mathcal{Q}_{i,\tau_i}(\Omega)$. Recall that $\Omega \subseteq X \subseteq X_i$. Therefore $\Omega \subseteq X_i \cap \mathcal{Q}_{i,\tau_i}(\Omega)$, or equivalently, $\Omega \subseteq Z_i$. It follows that $\Phi_i \subseteq Z_i$. The proof is complete. □

We will use the following two propositions to show that $\bigcup_{i=1}^{N} \Omega_i$ is the MSDRIS for system (1) with constraints (2).

Proposition 3: Consider Ω_i constructed by Algorithm 2. The following relation holds $\forall t \geq \tau_i$, $\forall i \in \mathcal{I}$

$$A_i^t\Omega_i \oplus \bigoplus_{k=0}^{t-1} A_i^t W_i \subseteq \bigcap_{l=1}^{N} \Omega_l \tag{16}$$

Proof: We prove (16) by showing,

$$A_i^t\Omega_i \oplus \bigoplus_{k=0}^{t-1} A_i^k W_i \subseteq \Omega, \forall t \geq \tau_i, \forall i \in \mathcal{I} \tag{17}$$

$$\Omega \subseteq \bigcap_{l=1}^{N} \Omega_l \tag{18}$$

We show (17) by induction $\forall t \geq \tau_i$, $\forall i \in \mathcal{I}$. From the construction of Ω_i, one has $A_i^{\tau_i}\Omega_i \oplus \bigoplus_{t=0}^{\tau_i-1} A_i^t W_i \subseteq \Omega$. Hence, (17) holds for $t = \tau_i$. Assume now that (17) holds for $t = t_0$, i.e.,

$$A_i^{t_0}\Omega_i \oplus \bigoplus_{k=0}^{t_0-1} A_i^k W_i \subseteq \Omega$$

We need to show that (17) holds for $t = t_0 + 1$. Because Ω_i is a RIS for (10), (14), one has $A_i\Omega_i + W_i \subseteq \Omega_i$. It follows that

$$\begin{aligned} A_i^{t_0+1}\Omega_i \oplus \bigoplus_{k=0}^{t_0} A_i^k W_i &= A_i^{t_0}\left(A_i\Omega_i + W_i\right) \oplus \bigoplus_{k=0}^{t_0-1} A_i^k W_i \\ &\subseteq A_i^{t_0}\Omega_i \oplus \bigoplus_{k=0}^{t_0-1} A_i^k W_i \subseteq \Omega \end{aligned}$$

Hence (17) holds $\forall t \geq \tau_i$, $\forall i \in \mathcal{I}$.

Now we will prove (18) by showing that $\Omega \subseteq \Omega_i, \forall i \in \mathcal{I}$. Recall that Ω_i is the MRPIS for (10), (14), $\forall i \in \mathcal{I}$. Consequently, it is well known[20] that Ω_i is also the maximal robustly constraint-admissible set for (10), (14).

Due to the construction of Z_i, and the fact that Ω is a dwell time robustly invariant set for (1), (2), one has $\Omega \subseteq Z_i$ and $x(k) \in Z_i$, $\forall x(0) \in \Omega$ under the dynamics (10), $\forall k \geq 0$. It follows that Ω is a robustly constraint-admissible set for (10), (14). Consequently, $\Omega \subseteq \Omega_i, \forall i \in \mathcal{I}$. This completes the proof. □

Proposition 4: Consider Ω, $\Omega_i, \forall i \in \mathcal{I}$, constructed by Algorithm 1 and Algorithm 2, respectively. The following relation holds

$$\Omega = \bigcap_{i=1}^{N} \Omega_i \tag{19}$$

Proof: Define $\hat{\Omega} := \bigcap_{i=1}^{N} \Omega_i$. Using the proof of Proposition 3, one has $\Omega \subseteq \hat{\Omega}$. It remains to show that $\hat{\Omega} \subseteq \Omega$. Because $\Omega_i \subseteq X_i$ and $\bigcap_{i=1}^{N} X_i = X$, one gets $\hat{\Omega} \subseteq X$. Using (16), and since $\hat{\Omega} \subseteq \Omega_i$, one has $A_i^t\hat{\Omega} \oplus \bigoplus_{k=0}^{t-1} A_i^k W_i \subseteq \hat{\Omega}, \forall t \in \mathcal{T}_i, \forall i \in \mathcal{I}$. It follows that $\hat{\Omega}$ is a RIS for system (8) and is constraint-admissible for $x(k) \in X$. Recall that Ω is the MRIS for (8) and for the constraint $x(k) \in X$. It follows that $\hat{\Omega} \subseteq \Omega$. The proof is complete. □

We are ready to state the main theorem of this section.

Theorem 1: Under the assumption that the mode transition graph $\mathcal{G}$ is complete, $\bigcup_{i=1}^{N} \Omega_i$ is the MSDRIS for (1), (2).

Proof: We decompose the proof of Theorem 1 into two parts. First we will prove that $\bigcup_{i=1}^{N} \Omega_i$ is a SDRIS for (1), (2). Then we will show that $\bigcup_{i=1}^{N} \Omega_i$ is the largest SDRIS.

To prove $\bigcup_{i=1}^{N} \Omega_i$ is a SDRIS, we need to show that $\forall x(0) \in \Omega_{\sigma(0)}$, one has $x(k) \in \Omega_{\sigma(k)}, \forall k \geq 0$. Without loss of generality, assume that $x(0) \in \Omega_1$ and $\sigma(0) = 1$. Since Ω_1 is a RIS for (10), (14), one has $x(k) \in \Omega_1, \forall k \in \overline{0, \tau_1 - 1}$. Using Proposition 2, one obtains $x(k) \in \bigcap_{i=1}^{N} \Omega_i, \forall k \geq \tau_1$ if (10) with $i = 1$ is the active dynamics. Note that $\bigcap_{i=1}^{N} \Omega_i \subseteq \Omega_l$ for any $l \in \mathcal{I}$. If for the next switching time k_s, $\sigma(k_s) = l$, one has $x(k_s + k) \in \Omega_l$, $\forall k \in \overline{0, \tau_l - 1}$ because Ω_l is a RIS. Using Proposition 2, it follows that $x(k_s + k) \in \bigcap_{i=1}^{N} \Omega_i, \forall k \geq \tau_l$, e.t.c. We conclude that $x(k) \in \Omega_{\sigma(k)}, \forall k \geq 0$.

The proof that $\bigcup_{i=1}^{N} \Omega_i$ is the MSDRIS comes from three facts: (i) Proposition 4; (ii) Ω_i is the MRIS for (10), (14); (iii) Ω is the MDTRIS for (1), (2). □

4.2 | Arbitrary Mode Transition Graph Case

Simplicity is the main advantage of Algorithm 2. Once Ω is available, the set Ω_i is constructed separately, $\forall i \in \mathcal{I}$. Nevertheless, it is not trivial to extend Algorithm 2, or more precisely Algorithm 1 for an arbitrary mode transition graph $\mathcal{G}$. This is because Lemma 1 heavily relies on the assumption that $\mathcal{G}$ is the complete graph. We will show later that if $\mathcal{G}$ is not complete, then Ω could be empty. The aim of this section is to provide a new procedure for computing the MSDRIS without any requirement on $\mathcal{G}$.

The following two remarks can be made concerning Ω_i.

- $\Omega_i \subseteq X_i$ is the MRIS for system (10), $\forall i \in \mathcal{I}$.
- $A_i^{\tau_i}\Omega_i \oplus \bigoplus_{k=0}^{\tau_i-1} A_i^k W_i \subseteq \bigcap_{l=1}^{N} \Omega_l, \forall i \in \mathcal{I}$.

Using the two remarks, the following procedure is used to build the MSDRIS for (1), (2) in the arbitrary mode transition graph case.

Algorithm 3: Computation of MSDRIS - Arbitrary Mode Transition Graph

1: Set $q \leftarrow 0$ and let $\tilde{\Omega}_i^{(q)} \leftarrow X_i, \forall i \in \mathcal{I}$.
2: **for** each $i \in \mathcal{I}$ **do**
3: $\quad \tilde{\Omega}_i^{(q+1)} \leftarrow \tilde{\Omega}_i^{(q)} \cap \mathcal{Q}_{i,1}\left(\tilde{\Omega}_i^{(q)}\right) \cap \bigcap_{(i,l)\in\mathcal{E}} \mathcal{Q}_{i,\tau_i}\left(\tilde{\Omega}_l^{(q)}\right)$.
4: **end for**
5: If $\tilde{\Omega}_i^{(q+1)} \equiv \tilde{\Omega}_i^{(q)}, \forall i \in \mathcal{I}$ set $\tilde{\Omega}_i \leftarrow \tilde{\Omega}_i^{(q)}, \forall i \in \mathcal{I}$ then stop, else set $q \leftarrow q+1$ and go to step 2.

Algorithm 3 initializes the construction of $\tilde{\Omega}_i, \forall i \in \mathcal{I}$ in Step (1) with the outer approximation $\tilde{\Omega}_i = X_i, \forall i \in \mathcal{I}$. At each iteration and for each mode $i \in \mathcal{I}$, Step (3) of Algorithm 3 removes states $x \in \tilde{\Omega}_i^{(q)}$ that cannot be kept in $\tilde{\Omega}_i^{(q)}$ under the dynamics (10) and cannot reach $\tilde{\Omega}_l^{(q)}$ for all possible mode transitions $(i,l) \in \mathcal{E}$ in τ_i time steps under the dynamics (10). Algorithm 3 stops when $\tilde{\Omega}_i^{(q+1)} = \tilde{\Omega}_i^{(q)}, \forall i \in \mathcal{I}$.

Remark 2: It is clear that if $\tilde{\Omega}_i^{(q)}$ is empty at any iteration q and for any $i \in \mathcal{I}$, then Algorithm 3 will stop and all the sets $\tilde{\Omega}_i, \forall i \in \mathcal{I}$ are empty. □

We will prove the finite termination of Algorithm 3 later. For the moment, let us assume that Algorithm 3 terminates in finite time, and that $\tilde{\Omega}_i$ is non-empty, $\forall i \in \mathcal{I}$.

The following result holds.

Proposition 5: Assume that $\tilde{\Omega}_i$ produced by Algorithm 3 is non-empty, $\forall i \in \mathcal{I}$. The following relation holds, $\forall i \in \mathcal{I}$

$$A_i^t \tilde{\Omega}_i \oplus \bigoplus_{k=0}^{t-1} A_i^k W_i \subseteq \bigcap_{(i,l)\in\mathcal{E}} \tilde{\Omega}_l, \forall t \geq \tau_i \tag{20}$$

Proof: The proof of Proposition 5 follows closely the one of Proposition 3, and is done by induction. By the construction of $\tilde{\Omega}_i$, it is clear that (20) holds for $t = \tau_i$. Assume now that (20) holds for $t = t_0$, i.e.,

$$A_i^{t_0} \tilde{\Omega}_i \oplus \bigoplus_{k=0}^{t_0-1} A_i^k W_i \subseteq \bigcap_{(i,l)\in\mathcal{E}} \tilde{\Omega}_l \tag{21}$$

One needs to show that (20) holds for $t = t_0 + 1$. One has

$$\begin{aligned} A_i^{t_0+1} \tilde{\Omega}_i \oplus \bigoplus_{k=0}^{t_0} A_i^k W_i &= A_i^{t_0}\left(A_i \tilde{\Omega}_i + W_i\right) \oplus \bigoplus_{k=0}^{t_0-1} A_i^k W_i \\ &\subseteq A_i^{t_0} \tilde{\Omega}_i \oplus \bigoplus_{k=0}^{t_0-1} A_i^k W_i \subseteq \bigcap_{(i,l)\in\mathcal{E}} \tilde{\Omega}_l \end{aligned} \tag{22}$$

For the last equation of (22), we used (21) and the fact that $A_i\tilde{\Omega}_i + W_i \subseteq \tilde{\Omega}_i$. It follows that (20) holds $\forall t \geq \tau_i$, $\forall i \in \mathcal{I}$. This completes the proof. □

The following result shows that $\bigcup_{i=1}^{N} \tilde{\Omega}_i$ is the MSDRIS.

Theorem 2: Assume that $\tilde{\Omega}_i$ is non-empty, $\forall i \in \mathcal{I}$. Then, $\bigcup_{i=1}^{N} \tilde{\Omega}_i$ is the MSDRIS for (1), (2).

Proof: As for Theorem 1, we decompose the proof of Theorem 2 into two parts. First we will prove that $\bigcup_{i=1}^{N} \tilde{\Omega}_i$ is a SDRIS. Then we will show that $\bigcup_{i=1}^{N} \tilde{\Omega}_i$ is the MSDRIS.

We omit here the proof that $\bigcup_{i=1}^{N} \tilde{\Omega}_i$ is a SDRIS, as with Proposition 5, it follows the same as the one of Theorem 1.

We prove that $\bigcup_{i=1}^{N} \tilde{\Omega}_i$ is the MSDRIS or equivalently the MSDRAS for (1), (2). It is clear that by the construction $\tilde{\Omega}_i \in X_i$ is the largest set such that

$$A_i\tilde{\Omega}_i \oplus W_i \subseteq \tilde{\Omega}_i, \forall i \in \mathcal{I} \tag{23}$$

$$A_i^{\tau_i}\tilde{\Omega}_i \oplus \bigoplus_{k=0}^{\tau_i-1} A_i^k W_i \subseteq \bigcap_{(i,l)\in\mathcal{E}} \tilde{\Omega}_l \tag{24}$$

Since $\sigma(k) = i, \forall k \geq 0$ is an admissible switching sequence, it is clear that (23) is a necessary condition for invariance of $\bigcup_{i=1}^{N} \tilde{\Omega}_i$. We will prove now that (24) is also a necessary condition by contradiction. Assume there exist $(i, l) \in \mathcal{E}$ such that

$$A_i^{\tau_i}\tilde{\Omega}_i \oplus \bigoplus_{k=0}^{\tau_i-1} A_i^k W_i \not\subset \tilde{\Omega}_l$$

That is $\exists x(0) \in \tilde{\Omega}_i$ such that $\exists w(0) \in W_i, \dots, \exists w(\tau_i - 1) \in W_i$ such that

$$x(\tau_i) = A_i^{\tau_i} x_0 + \sum_{k=0}^{\tau_i-1} A_i^k w(k) \notin \tilde{\Omega}_l$$

Consider the following admissible switching sequence $\sigma(k) = i, \forall k = \overline{0, \tau_i - 1}$, and $\sigma(k) = l, \forall k \geq \tau_i$. If $x(\tau_i) \notin X_l$, then the constraint (2) is violated. Consider now the case when $x(\tau_i) \in X_l$. Recall that $\tilde{\Omega}_i \in X_i$ is the largest set for conditions (23), (24). Consequently, $x(\tau_i)$ cannot belong to any RIS for the mode l. This implies that $x(\tau_i)$ does not belong to the maximal constraint-admissible set for the mode l. It follows that the constraint $x(k) \in X_l$ will be eventually violated. We conclude that (24) is a necessary condition for invariance of $\bigcup_{i=1}^{N} \tilde{\Omega}_i$. □

Using Theorem 2, it is clear that the outputs of Algorithm 3 is the same as that of Algorithm 2, but for a more general case. From this point on for simplicity, we use $\Omega_i, \forall i \in \mathcal{I}$ as the outputs of Algorithm 3.

It is worth noticing that unlike the complete mode transition graph case, the MDTRIS Ω is generally empty in the case where $\mathcal{G}$ is not complete. For example, consider a case where $\mathcal{G}$ is a disconnected graph, i.e., at least two modes of $\mathcal{G}$ are not connected by an edge. Without loss of generality, suppose that mode 1 and mode 2 in $\mathcal{G}$ are not connected. Using Algorithm 3, one obtains Ω_1 and Ω_2 for these two modes. For any $x(0) \in \Omega_1$, if $\sigma(k) = 2$, then there is no guarantee that $x(k) \in \Omega_1 \cap \Omega_2$, for any $k = k_s$. Hence, Ω is empty.

The following result is a direct consequence of Theorem 2.

Corollary 1: Suppose that the disturbance-free system (1) is asymptotically stable with dwell times τ_i for the dynamics A_i and with the mode transition graph $\mathcal{G}$. Then i) Algorithm 3 terminates after a finite number of steps; ii) If Ω_i exists, then $\bigcup_{i=1}^{N} \Omega_i$ is the MSDRIS for (1), (2); iii) If Ω_i is empty, $\forall i \in \mathcal{I}$, then there is no $x(0) \in X_{\sigma(0)}$ such that $x(k) \in X_{\sigma(k)}, \forall k \geq 0$ for any admissible switching sequence $\sigma(k)$, i.e., the MSDRAS is empty.

Proof: The proof comes directly from Theorem 2 and the fact that the sequences of sets $\Omega_i^{(q)}$ are non-increasing, i.e., $\Omega_i^{(q+1)} \subseteq \Omega_i^{(q)}, \forall q \geq 0$, and are bounded from below $\emptyset \subseteq \Omega_i^{(q)}, \forall q \geq 0, \forall i \in \mathcal{I}$. □

Remark 3: Algorithm 3 could also terminate in finite time if the disturbance-free system (1) is only stable. For example consider (1) with two nodes

$$A_1 = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix}; A_2 = \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix}$$

The mode transition graph $\mathcal{G}$ is complete. The dwell times are $\tau_1 = \tau_2 = 1$. Note that (1) is only stable. The constraint sets are $X_1 = \{x \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}$, $X_2 = X_1$, $W_1 = W_2 = 0$. Algorithm 3 terminates in two steps. The sets Ω_1, Ω_2 are $\Omega_1 = \{x \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1, |x_1 + x_2| \leq \sqrt{2}, |x_1 - x_2| \leq \sqrt{2}\}$. □

Note that Ω_i in Algorithm 3 is a polyhedral set containing the origin in its interior, $\forall i \in \mathcal{I}$. In this case $\bigcup_{i=1}^{N} \Omega_i$ is a star-shaped or radially convex set. It means that $\forall x \in \bigcup_{i=1}^{N} \Omega_i$, $\forall 0 \leq \lambda \leq 1$, one has $\lambda x \in \bigcup_{i=1}^{N} \Omega_i$.

We can use Algorithm 3 as a tool to compute the minimal dwell time $\tau_{i,min}, \forall i \in \mathcal{I}$ that ensures stability of the origin of the disturbance-free system (1), (2). For simplicity, in the rest of this section, when we write system (1) we refer to (1) without the disturbance $w(k)$.

The following result holds

Proposition 6: Given polyhedral sets Ω_i containing the origin in the interior, $\forall i \in \mathcal{I}$. $\bigcup_{i=1}^{N} \Omega_i$ is a SDRIS for (1), (2) if and only if $\bigcup_{i=1}^{N} \Omega_{i,\lambda}$ with $\Omega_{i,\lambda} = \lambda\Omega_i$ for any $\lambda \in (0, 1]$ is a SDRIS for (1), (2).

Proof: Note that $\bigcup_{i=1}^{N} \Omega_{i,\lambda} \subseteq \bigcup_{i=1}^{N} \Omega_i$, $\forall \lambda \in (0, 1]$. We can represent $\Omega_{i,\lambda}$ by, $\forall i \in \mathcal{I}$

$$\Omega_{i,\lambda} = \lambda\Omega_i = \left\{x \in \mathbb{R}^n : F_{i,w}x \leq \lambda g_{i,w}\right\}$$

where $(F_{i,w}, g_{i,w})$ is a half-space representation of Ω_i, i.e.,

$$\Omega_i = \left\{x \in \mathbb{R}^n : F_{i,w}x \leq g_{i,w}\right\}$$

Therefore, $\forall x(0) \in \lambda\Omega_{\sigma(0)}$, one has $\frac{x(0)}{\lambda} \in \Omega_{\sigma(0)}$. Since $\bigcup_{i=1}^{N} \Omega_i$ is a SDRIS for (1), one obtains $A_{\sigma(k)}\frac{x(0)}{\lambda} \in \Omega_{\sigma(k)}$ for all admissible switching sequence $\sigma(k)$. It follows that $A_{\sigma(k)}x(k) \in \lambda\Omega_{\sigma(k)}$. Hence, $\bigcup_{i=1}^{N} \Omega_{i,\lambda}$ is a SDRIS for (1), (2). □

Remark 4: In Proposition 6 for simplicity, we consider only polyhedral sets $\Omega_i, \forall i \in \mathcal{I}$. However, Proposition 6 holds for any convex sets $\Omega_i, \forall i \in \mathcal{I}$ containing the origin in the interior. □

Theorem 3: System (1) is stable under admissible switching sequence $\sigma(k) \in \mathcal{G}$ with dwell time $\tau_i, \forall i \in \mathcal{I}$ if and only if Algorithm 3 produces a non-empty set Ω_i, $\forall i \in \mathcal{I}$. In addition, if for any $i \in \mathcal{I}$ there is no $x(0) \in \Omega_i$ such that $x(k) \in \text{Bd}\left(\bigcap_{(i,l)\in\mathcal{E}} \Omega_l\right), \forall k \geq 0$ under any admissible switching sequence $\sigma(k)$, then (1) is asymptotically stable.

Proof: ($\Rightarrow$) comes directly from Corollary 1. ($\Leftarrow$) If Algorithm 3 produces a non-empty set $\Omega_i \subseteq X_i$, $\forall i \in \mathcal{I}$. In this case $\bigcup_{i=1}^{N} \Omega_i$ is a SDRIS for (1). Hence, $\forall x(0) \in \Omega_{\sigma(0)}$, one has $x(k) \in \Omega_{\sigma(k)}, \forall k \geq 0$. Consequently, $x(k), \forall k \geq 0$ is bounded. In other words (1) is stable.

For the asymptotic stability proof, for simplicity, we consider only the complete mode transition graph case. If there is no mode i and no state $x \in \Omega_i$ such that the state trajectory stays forever on $\text{Bd}\left(\bigcap_{l=1}^{N} \Omega_l\right)$ any admissible switching sequence $\sigma(k)$. In this case, $\exists 0 < \lambda < 1$ and a finite index k_1 such that $x(k_1) \in \text{Bd}\left(\bigcap_{l=1}^{N} \Omega_{l,\lambda}\right)$. Using Proposition 6, $\bigcup_{l=1}^{N} \Omega_{l,\lambda}$ is a SDRIS for (1). Since $\Omega_{l,\lambda}$ is a scaled version of Ω_l, one has $\exists k_2$ such that $x(k_2) \in \text{Bd}\left(\bigcap_{l=1}^{N} \Omega_{l,\lambda^2}\right)$. As such $\exists k_m$ such that $x(k_m) \in \text{Bd}\left(\bigcap_{l=1}^{N} \Omega_{l,\lambda^m}\right)$. Since $\lambda < 1$, one has $\lim_{m\to\infty} x(k_m) \to 0$. Hence (1) is asymptotically stable. □

4.3 | Extension to Switched Systems with Parametric Uncertainties

We extend the results in Section 4.2 to switched systems with parametric uncertainties. Consider the uncertain and/or time-varying switched linear discrete-time systems

$$x(k+1) = A_{\sigma(k)}(\theta_{\sigma(k)}(k))x(k) + w(k) \tag{25}$$

The matrix $A_i(\theta_i(k))$ satisfies, $\forall i \in \mathcal{I}$

$$A_i(\theta_i(k)) = \sum_{j=1}^{r_i} \theta_{i,j}(k) A_{i,j} \tag{26}$$

where $A_{i,j} \in \mathbb{R}^{n\times n}, \forall j = \overline{1, r_i}$ are known matrices. $\theta_i(k) = [\theta_{i,1}(k)\ \theta_{i,2}(k)\ \dots\ \theta_{i,r_i}(k)]^T \in \Theta_i$ is a vector of unknown and/or time-varying parameters, with

$$\Theta_i = \left\{ \theta_i \in \mathbb{R}^{r_i} : \sum_{j=1}^{r_i} \theta_{i,j} = 1, \theta_{i,j} \geq 0 \right\} \tag{27}$$

$x(k)$ and $w(k)$ are subject to constraints (2), i.e., $x(k) \in X_{\sigma(k)}, w(k) \in W_{\sigma(k)}, \forall k \geq 0$. The switching sequence $\sigma(k)$ satisfies $\sigma(k) \in \Sigma, \forall k \geq 0$.

Our objective is to provide a numerical procedure to construct the MSDRAS for (25), (2). Using similar arguments as that of Proposition 1, we can conclude that for (25), (2), the MSDRAS is the RSDRIS.

It is easy to observe that Algorithm 3 can be straightforwardly extended to the case (25), (2). The main difficulty here is the computation of the set $\mathcal{Q}_{i,\tau_i}\left(\Omega_j^{(q)}\right)$. Recall that $\mathcal{Q}_{i,\tau_i}\left(\Omega_l^{(q)}\right)$ is the set of all states x that can reach to $\Omega_l^{(q)}$ in no more than τ_i steps under the dynamics

$$x(k+1) = A_i(\theta_i(k))x(k) + w(k) \tag{28}$$

$\forall w(k) \in W_i$. Applying directly (6) to system (28) may lead to a computationally prohibitive condition.

Our idea to overcome the computational issue is to calculate $\mathcal{Q}_{i,\tau_i}\left(\Omega_l^{(q)}\right)$ recursively as

$$\mathcal{Q}_{i,t}\left(\Omega_l^{(q)}\right) = \mathcal{Q}_{i,1}\left(\mathcal{Q}_{i,t-1}\left(\Omega_l^{(q)}\right)\right), \forall t \in \overline{1, \tau_i} \tag{29}$$

starting from $t = 1$. It is well known[18] that for a given polyhedral set $\Psi = \left\{x \in \mathbb{R}^n : F_\psi x \leq g_\psi\right\}$, the set $\mathcal{Q}_{i,1}(\Psi)$ for system (28) can be computed as

$$\mathcal{Q}_{i,1}(\Psi) = \left\{ x \in \mathbb{R}^n : \begin{bmatrix} F_\psi A_{i,1} \\ \vdots \\ F_\psi A_{i,r_i} \end{bmatrix} x \leq \begin{bmatrix} g_w \\ \vdots \\ g_w \end{bmatrix} \right\} \tag{30}$$

where $g_w = g_\psi - \max_{w\in W_i}\{F_\psi w\}$. In general, $\mathcal{Q}_{i,1}(\Psi)$ in (30) contains many redundant constraints. It is well known[18] that these constraints can be eliminated by using linear program.

Combining (29), (30), we obtain the following algorithm to compute $\mathcal{Q}_{i,\tau_i}\left(\Omega_l^{(q)}\right)$ for system (28).

Algorithm 4: Computation of $\mathcal{Q}_{i,\tau_i}\left(\Omega_l^{(q)}\right)$

1: Let $\Psi \leftarrow \Omega_l^{(q)}$.
2: **for** $t \leftarrow 1$ to τ_i **do**
3: $\quad \Psi \leftarrow \mathcal{Q}_{i,1}(\Psi)$.
4: **end for**
5: Set $\mathcal{Q}_{i,\tau_i}\left(\Omega_l^{(q)}\right) \leftarrow \Psi$ and stop.

The following Algorithm, which is an extension of Algorithm 3, can be used to compute the MSDRIS for (25), (2).

Corollary 2: Suppose that the disturbance-free switched system (25) is robustly asymptotically stable. Then i) Algorithm 5 terminates after a finite number of steps; ii) If Ω_i exists, then $\bigcup_{i=1}^{N} \Omega_i$ is the MSDRIS for (25), (2); iii) If Ω_i is empty, $\forall i \in \mathcal{I}$, then so is the MSDRAS.

Algorithm 5: Computation of MSDRIS - Switched Systems with Parametric Uncertainties

1: Set $q \leftarrow 0$ and let $\Omega_i^{(q)} \leftarrow X_i, \forall i \in \mathcal{I}$.
2: **for** each $i \in \mathcal{I}$ **do**
3: **for** each $l \in \mathcal{I}$ such that $(i,l) \in \mathcal{E}$ **do**
4: Using Algorithm 4 to obtain $\mathcal{Q}_{i,\tau_i}\left(\Omega_l^{(q)}\right)$.
5: **end for**
6: $\Omega_i^{(q+1)} \leftarrow \Omega_i^{(q)} \cap \mathcal{Q}_{i,1}\left(\Omega_i^{(q)}\right) \cap \bigcap\limits_{(i,l)\in\mathcal{E}} \mathcal{Q}_{i,\tau_i}\left(\Omega_l^{(q)}\right)$.
7: **end for**
8: If $\Omega_i^{(q+1)} \equiv \Omega_i^{(q)}, \forall i \in \mathcal{I}$ set $\Omega_i \leftarrow \Omega_i^{(q)}, \forall i \in \mathcal{I}$ then stop, else set $q \leftarrow q+1$ and go to step 2.

Proof: It is omitted here, since it follows the same steps as that of Theorem 2, and of Corollary 1. □

As for Algorithm 3, we can use Algorithm 5 to compute the minimal dwell time $\tau_{i,min}, \forall i \in \mathcal{I}$ that assures robust stability of the origin of system (25), (2) without the disturbance $w(k)$. The following result holds.

Corollary 3: The disturbance-free system (25) is robustly stable under admissible switching sequence $\sigma(k) \in \mathcal{G}$ with dwell time $\tau_i, \forall i \in \mathcal{I}$ if and only if Algorithm 5 produces a non-empty set $\Omega_i, \forall i \in \mathcal{I}$. In addition, if for any $i \in \mathcal{I}$ there is no $x(0) \in \Omega_i$ such that $x(k) \in \text{Bd}\left(\bigcap\limits_{(i,l)\in\mathcal{E}} \Omega_l\right), \forall k \geq 0$ under any admissible switching sequence $\sigma(k)$, then (25) is robustly asymptotically stable.

Proof: It is omitted here. □

5 | EXAMPLES

We demonstrate the obtained results via three examples in this section. In all three examples, a numerical description of the obtained sets is not reported, but will be sent to reader upon request.

5.1 | Example 1

This example is taken from [14]. Consider system (1) with

$$A_1 = \begin{bmatrix} 0.1321 & 0.2494 \\ -2.4940 & -0.1173 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.9885 & 0.4406 \\ -0.0441 & 0.7682 \end{bmatrix}$$

The constraint sets are

$$X_1 = \{x \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}, X_2 = X_1,$$
$$W_1 = \{w \in \mathbb{R}^2 : |w_1| \leq 0.001, |w_2| \leq 0.001\}, W_2 = W_1$$

The graph $\mathcal{G}$ is complete. The dwell times are $\tau_1 = \tau_2 = 6$.

Since $\mathcal{G}$ is complete, both Algorithms 2 and 3 are applicable, and both produce the same result. Using Algorithm 2, Fig. 1 presents the sets Ω_1 (red), Ω_2 (blue). Recall that $\Omega_1 \cup \Omega_2$ is the MSDRAS/MSDRIS. Algorithm 2 requires 7 and 6 iterations to calculate Ω_1, Ω_2, respectively. Fig. 1 also presents the MDTRIS $\Omega = \Omega_1 \cap \Omega_2$ (green). Note that Ω is $\mathbf{O}_\infty$ in the example in [14]. Fig. 1 also presents a phase trajectory (magenta) starting from the initial condition $x(0) = [-1 \quad 0.2481]^T$. It can be observed that the phase trajectory is always confined in $\Omega_1 \cup \Omega_2$.

For this example using Algorithm 3, we found that the minimum dwell times are $\tau_1 = 6, \tau_2 = 1$. The number of iterations is $q = 11$. Fig. 2 shows Ω_1 (red), Ω_2 (blue), Ω (green). Fig. 2 also shows a phase trajectory (magenta) starting from the initial condition $x(0) = [-0.6831 \quad 0.5755]^T$.

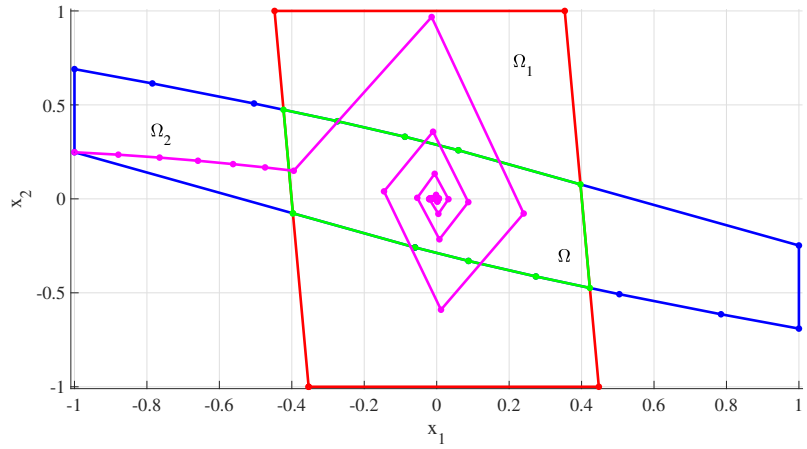

FIGURE 1 Invariant sets $\Omega_1 \cup \Omega_2$, Ω, and phase trajectory for $\tau_1 = \tau_2 = 6$ for example 1.

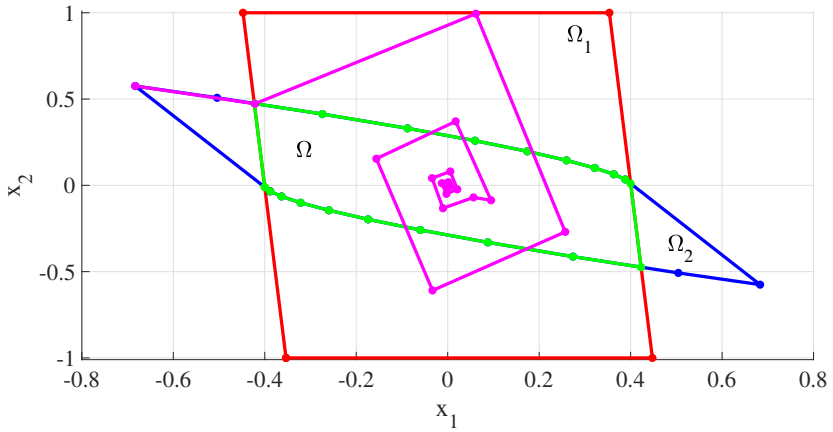

FIGURE 2 Invariant sets $\Omega_1 \cup \Omega_2$, Ω, and phase trajectory for $\tau_1 = 6, \tau_2 = 1$ for example 1.

5.2 | Example 2

The second example system is inspired by a triple integrator. Consider system (1) with

$$A_1 = \begin{bmatrix} 1.1600 & 0.4000 & -1.2200 \\ -0.5000 & 0.3400 & 0.8600 \\ 0.3400 & -0.0600 & 0.0800 \end{bmatrix}, A_2 = \begin{bmatrix} 2.0000 & 2.0000 & 0.0000 \\ -0.8600 & -0.4600 & -0.2600 \\ -0.0800 & 0.0400 & 0.4800 \end{bmatrix}, A_3 = \begin{bmatrix} 1.3510 & -1.3126 & -0.8112 \\ 0.6636 & -0.2480 & -0.6254 \\ -0.6490 & 0.6874 & 1.1888 \end{bmatrix}$$

The constraint sets are

$$\begin{aligned} X_1 &= \{x \in \mathbb{R}^3 : |x_j| \leq 20, \forall j \in \overline{1,3}\}, \\ X_2 &= \{x \in \mathbb{R}^3 : |x_j| \leq 40, \forall j \in \overline{1,3}\}, \\ X_3 &= \{x \in \mathbb{R}^3 : |x_j| \leq 30, \forall j \in \overline{1,3}\}, \\ W_1 &= \{w \in \mathbb{R}^3 : |w_j| \leq 0.01, \forall j \in \overline{1,3}\}, \\ W_2 &= W_1, W_3 = W_1 \end{aligned}$$

Fig. 3 presents the mode transition graph $\mathcal{G}$. The dwell times are $\tau_1 = 4$, $\tau_2 = 8$, $\tau_3 = 1$. The number of iterations is $q = 13$ for Algorithm 3. Fig. 4 illustrates the sets Ω_1 (red), Ω_2 (blue), Ω_3 (green).

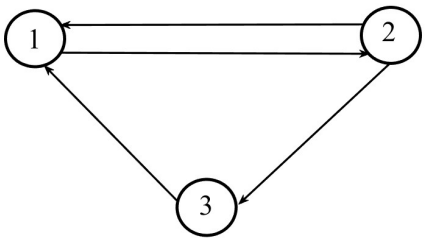

FIGURE 3 Mode transition graph $\mathcal{G}$ for example 2.

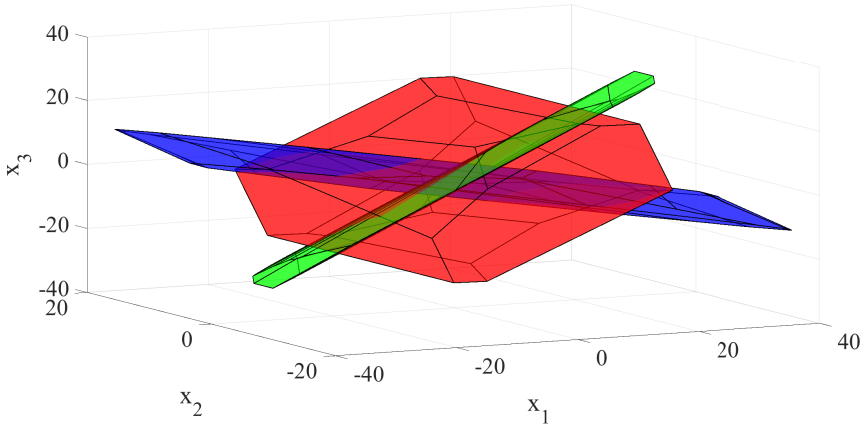

FIGURE 4 Illustration of Ω_1(red), Ω_2(blue), Ω_3(green) for example 2.

5.3 | Example 3

For the third example, we consider the disturbance-free system (25) with

$$A_{11} = \begin{bmatrix} 0.75 & 0 \\ 1.00 & 0.75 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.75 & 0.025 \\ 1.00 & 0.750 \end{bmatrix}, \\ A_{21} = \begin{bmatrix} 0.75 & -2.50 \\ 0 & 0.75 \end{bmatrix}, A_{22} = \begin{bmatrix} 0.750 & -2.50 \\ 0.025 & 0.75 \end{bmatrix} \tag{31}$$

This system is obtained by discretizing the uncertain switched continuous-time system in Example 1 from[21] using Euler's method with the sampling time $T_s = 0.25(sec)$. The constraints are $X_1 = \{x \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}$, $X_2 = X_1$. The graph $\mathcal{G}$ is complete.

By using a method based on homogeneous polynomial Lyapunov function, it was found in[21] that the minimum dwell time for the switched continuous-time system is $3.5(sec)$. This is equivalent to the dwell times $\tau_1 = \tau_2 = \frac{3.5}{T_s} = 14$ for the switched discrete-time system (31). Using Algorithm 5, we found that the minimum dwell times are $\tau_1 = \tau_2 = 13$. The number of iteration is $q = 6$. Fig. 5 presents Ω_1 (red), Ω_2 (blue). This figure also presents a phase space trajectory (magenta) starting from the initial condition $x(0) = [-0.384 \;\; 1]^T$.

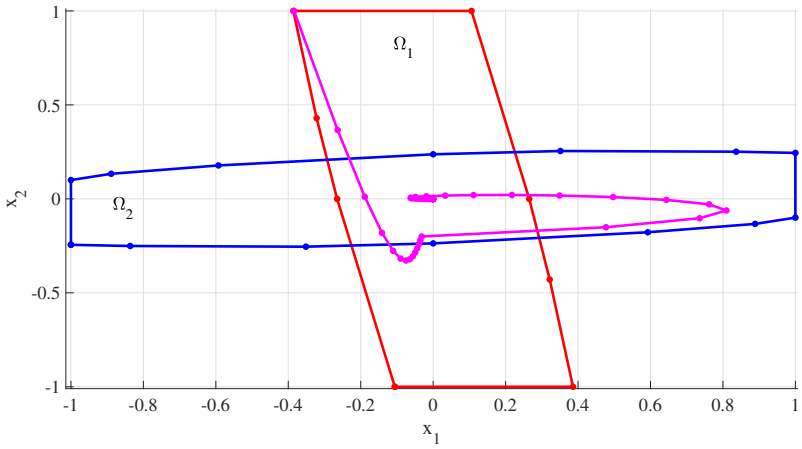

FIGURE 5 Invariant sets $\Omega_1 \cup \Omega_2$, and phase trajectory for example 3.

6 | CONCLUSION

We introduced two new notions of the maximal switch-dependently robustly admissible set (MSDRAS), and of the maximal switch-dependently robustly invariant set (MSDRIS) for constrained graph-dependent switched systems with bounded disturbances under dwell time restriction. We showed that the MSDRAS coincides with the MSDRIS. We presented two new numerical procedures to construct the MSDRAS, and showed that the MSDRAS is a star-shaped set. We extended the obtained results to constrained switched systems with parametric uncertainties. We proved that the existence of the MSDRAS provides a necessary and sufficient condition for the stability of switched systems. Using the proposed algorithms as the foundation to calculate the minimal dwell times needed for stability of the origin of the switched system, we found that the obtained dwell times are smaller, in terms of their sum, than that of earlier solutions in recent literature for examples considered in this paper.

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