Real almost reducibility of differentiable real quasi-periodic cocycles

Maxime Chatal, Claire Chavaudret, L. Hakan Eliasson

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M.Chatal[*], C.Chavaudret[†], L.H.Eliasson[‡]

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Abstract

We prove that infinitely differentiable almost reducible quasi-periodic cocycles, under a Diophantine condition on the frequency vector, are almost reducible to a real constant cocycle with a real conjugation, up to a period doubling.

1 Introduction

Let $\omega \in \mathbb{R}^d$ a rationally independent vector, i.e a vector satisfying

$$\langle k, \omega \rangle \neq 0, \quad \forall k \in \mathbb{Z}^d \setminus \{0\},$$

and let $A : \mathbb{T}^d \to gl(n, \mathbb{C})$ of class $\mathcal{C}^\infty$, where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. The quasiperiodic cocycle associated to A is the map (of class $\mathcal{C}^\infty$) $X_{\omega,A} : \mathbb{R} \times \mathbb{T}^d \to Gl(n, \mathbb{C})$ which is solution of

$$\begin{cases} \frac{d}{dt} X^t_{\omega,A}(\theta) = A(\theta + t\omega) X^t_{\omega,A}(\theta) \\ X^0_{\omega,A}(\theta) = I. \end{cases}$$

Remark 1.1. In this paper, all functions and mappings will, unless otherwise specified, be of class $\mathcal{C}^\infty$.

We will say that the cocycle $X_{\omega,A}$ is *real* if A is a real valued map, and that it is *constant* if A is a constant map.
A cocycle $X_{\omega,A}$ is *conjugated* to a cocycle $X_{\omega,B}$ if and only if there exists a map $Z : \mathbb{T}^d \to Gl(n, \mathbb{C})$ such that

$$X^t_{\omega,A}(\theta) = Z(\theta + t\omega) X^t_{\omega,B} Z(\theta)^{-1} \quad \forall (t, \theta) \in \mathbb{R} \times \mathbb{T}^d. \tag{1.1}$$

The mapping $Z : \mathbb{T}^d \to Gl(n, \mathbb{C})$ is a *conjugation* between $X_{\omega,A}$ and $X_{\omega,B}$. It satisfies the condition

$$\partial_\omega Z(\theta) = A(\theta) Z(\theta) - Z(\theta) B(\theta) \quad \forall \theta \in \mathbb{T}^d, \tag{1.2}$$

where

$$\partial_\omega Z(\theta) = \frac{d}{dt} Z(\theta + t\omega)_{|t=0},$$

[*]IMJ-PRG, Université Paris-Cité, chatal.maxime@gmail.com,
[†]IMJ-PRG, Université Paris-Cité, chavaudr@math.univ-paris-diderot.fr
[‡]IMJ-PRG, Université Paris-Cité, hakan.eliasson@imj-prg.fr.

which is equivalent to (1.1).
A cocycle is *reducible* if and only if it is conjugated to a constant cocycle. A cocycle is *real reducible* if and only if it is real and conjugated to a constant cocycle by a real conjugation.

A natural question is whether a real and reducible cocycle is real reducible. The answer is yes modulo a "*period-doubling*":

Theorem 1.2. *If $X_{\omega,A}$ is a real and reducible cocycle, then $X_{\frac{\omega}{2},A_2}$ is real reducible, where*

$$A_2(\theta) = A(2\theta) \quad \theta \in \mathbb{T}^d.$$

Hence, there exist a map $Z : \mathbb{T}^d \to Gl(n,\mathbb{R})$ and a constant matrix $B \in gl(n,\mathbb{R})$ such that

$$\partial_{\frac{\omega}{2}} Z(\theta) = A(2\theta)Z(\theta) - Z(\theta)B \quad \forall \theta \in \mathbb{T}^d. \tag{1.3}$$

If we denote

$$W(2\theta) = Z(\theta),$$

then (1.3) says that

$$\partial_\omega W(\theta) = A(\theta)W(\theta) - W(\theta)B \quad \forall \theta \in 2\mathbb{T}^d.$$

which looks very much like real reducibility of $X_{\omega,A}$. But with the difference that W is not defined on $\mathbb{T}^d$, but only on the 2^d-fold covering $\mathbb{R}^d/(2\mathbb{Z})^d$ of $\mathbb{T}^d$ – this "period-doubling" cannot be avoided in general.

Theorem 1.2 was proven in the article [4], which also contains several other results of similar nature.

In this paper we shall discuss a similar result in the framework of almost reducible cocycles, i.e. cocycles that can be conjugated arbitrarily close to constant cocycles. There is no canonical meaning of "arbitrarily close" and we shall use a pretty stringent formulation.

A cocycle $X_{\omega,A}$ is *almost reducible* if and only if there exist sequences of maps $Z_j : \mathbb{T}^d \to Gl(n,\mathbb{C})$, $F_j : \mathbb{T}^d \to gl(n,\mathbb{C})$ and a sequence of matrices $B_j \in gl(n,\mathbb{C})$ such that

$$\partial_\omega Z_j(\theta) = A(\theta)Z_j(\theta) - Z_j(\theta)\Big(B_j + F_j(\theta)\Big) \quad \forall \theta \in \mathbb{T}^d \tag{1.4}$$

with

$$\lim_{j\to+\infty} \|Z_j^{\pm 1}\|_{\mathcal{C}^r}^m \|F_j\|_{\mathcal{C}^r} = 0, \quad \forall r, m \in \mathbb{N}. \ ^{1} \tag{1.5}$$

We will say that a real cocycle $X_{\omega,A}$ is *real almost reducible* if and only if it is almost reducible with a sequence of real-valued maps Z_j and a sequence of real matrices B_j verifying (1.4) and (1.5) (then F_j is automatically real).

In this paper we shall prove:

Theorem 1.3. *If $X_{\omega,A}$ is a real cocycle which is almost reducible and ω is Diophantine, then $X_{\frac{\omega}{2},A_2}$ is real almost reducible.*

This means that there exist a sequences of maps $Z_j : \mathbb{T}^d \to Gl(n,\mathbb{R})$, $F_j : \mathbb{T}^d \to gl(n,\mathbb{R})$ and a sequence of matrices $B_j \in gl(n,\mathbb{R})$ such that

$$\partial_{\frac{\omega}{2}} Z_j(\theta) = A(2\theta)Z_j(\theta) - Z_j(\theta)(B_j + F_j(\theta)) \quad \forall \theta \in \mathbb{T}^d$$

[1] this formulation denotes the condition: $Z_j^{+1} = Z_j$

satisfying (1.5).

An important difference with respect to the reducible case is worth pointing. In the reducible case, no arithmetical conditions on the frequency vector ω is needed, but in the almost reducible case the proof requires some such conditions. Indeed, we need to assure that the primitive of a quasi-periodic function

$$\mathbb{R} \ni t \mapsto f(t\omega)$$

is quasi-periodic and smooth, and this requires arithmetical conditions on the frequency vector ω. We have no idea if the theorem remains true without such conditions.

Reducibility and almost-reducibility of quasiperiodic cocycles are important properties to understand the behaviour of a cocycle. See for example [6] and [11] for applications to one-dimensional quasi-periodic Schrödinger operators, and [7] for applications to quasi-periodic cocycles on $SO(3,\mathbb{R})$.

The notion of almost reducibility is strictly weaker than that of reducibility [2]. For example, in the analytic perturbative case, under arithmetical assumptions on the frequency vector, there are constant cocycles, all of whose perturbations are almost reducible but, generically, not reducible – see for example [6] and [7].

Both reducibility and almost-reducibility are[3] very much perturbative phenomena. Most results are available in analytic or ultradifferentiable category (see [6],[7],[10],[3]...), much fewer in class $\mathcal{C}^\infty$ – for a result in $\mathcal{C}^\infty$ see for example [9].

Perturbative reducibility results require arithmetical conditions on the frequency vector. A Diophantine condition is most often used but it can be relaxed to a Brjuno-Rüssmann condition (see [5],[2]...).

Without arithmetical conditions, weaker notions [4] have been proven using renormalization techniques – see for example [1]. These results are for the moment very much restricted to two frequencies $\omega = (\omega_1, \omega_2)$.

1.1 A word about the proof

It is pretty forward to show that if a real cocycle can be conjugated to a real matrix by a complex conjugation, then it can be conjugated by a real conjugation. Therefore it suffices to prove that a real cocycle can be conjugated to a real matrix.

In the article [4] one uses (complex) invariant subbundles of a real cocycle to construct real invariant subbundles. In this paper we give another proof of this. Indeed here we prove that a complex matrix to which a real reducible cocycle can be conjugated, has the same spectral properties (i.e. Jordan normal form) as a real matrix – it can therefore be conjugated to a real matrix. We prove this in section 2. We generalize then this approach to almost reducible cocycles, but there are several complications.

We would like to prove that if a real cocycle can be almost conjugated to a sequence of real matrices by complex conjugations, then it can thus be conjugated by real conjugations. This may be true, but we have only been able to prove this under a Diophantine condition on the frequency vector ω – see section 3.

We can always conjugate a matrix to Jordan normal form but we have no control on the conjugation. In the reducible case this gives no problem, but in the almost reducible case it does.

[2] except when $n = 1$ or $d = 1$

[3] except when $n = 1$ or $d = 1$

[4] rotational reducibility and almost rotational reducibility

This problem is treated in section 4. Finally an almost reducibility to a Jordan normal form cocycle result is given in section 5.

In section 6 we analyse the spectral properties of a complex matrix to which a real cocycle is almost reduced. We show that, up to a sufficiently small perturbation, it has the same spectral properties as a real matrix.

Finally we put the results from sections 5 and 6 together and show that the estimates obtained are good enough to guarantee almost reducibility to real cocycles.

1.2 Notations

For any set X, we denote by $\#X$ its cardinality.

For any $n \times n$-matrix A we denote by $\sigma(A)$ its spectrum, that is to say the subset of $\mathbb{C}$ consisting of the eigenvalues of A. Clearly $\#\sigma(A) \leq n$.

Since all norms on $gl(n, \mathbb{C})$ are equivalent, the definition of almost reducibility does not depend on the choice of matrix norms. We shall usually, unless otherwise said, use the operator norm, but any other norm on $gl(n, \mathbb{C})$ would do.
For a vector $k = (k_1, \dots k_d) \in \mathbb{R}^d$, denote by $|k|$ its l^1 norm: $|k| = \sum |k_i|$.
As for the function-norms, they are the usual:

$$\|A\|_{\mathcal{C}^0} = \sup_\theta \|A(\theta)\|$$

and for all $r \in \mathbb{N}$,

$$\|A\|_{\mathcal{C}^r} = \max\{\|\partial^\alpha A\|_{\mathcal{C}^0} : \alpha \in \mathbb{N}^d, |\alpha| \leq r\}.$$

The norm $\|.\|_{\mathcal{C}^r}$ is a complete norm on the space of (matrix-valued) $\mathcal{C}^r$-functions.
Let us also recall two inequalities which we shall use frequently:

$$\|AB\|_{\mathcal{C}^r} \leq C_r \|A\|_{\mathcal{C}^r} \|B\|_{\mathcal{C}^r}, \quad \forall A, B \in \mathcal{C}^r(\mathbb{T}^d, gl(n, \mathbb{C})) \tag{1.6}$$

$$\|A^{-1}\|_{\mathcal{C}^r} \leq C_r \|A^{-1}\|_{\mathcal{C}^0}^{r+1} \|A\|_{\mathcal{C}^r}^r, \quad \forall A \in \mathcal{C}^r(\mathbb{T}^d, Gl(n, \mathbb{C})) \tag{1.7}$$

where C_r is a constant which depends on r.

Let us finally recall the Diophantine condition. We say that $\omega \in \mathcal{DC}(\kappa, \tau)$ (for some $\kappa > 0$ and $\tau > d - 1$) if and only if

$$|\langle k, \omega \rangle| \geq \frac{\kappa}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}. \tag{1.8}$$

2 Real reducibility

In this section, we prove a real reducibility proposition. This result has been already proved in [4], but the proof here is different, and will be useful to understand the real almost reducibility result later.

Proposition 2.1. *If two real cocycles $X_{\omega,A}$ and $X_{\omega,B}$ of dimension n are conjugated, then they are conjugated by a real conjugation.*

Proof. Let $Z : \mathbb{T}^d \to Gl(n, \mathbb{C})$ a conjugation between $X_{\omega,A}$ and $X_{\omega,B}$

$$\partial_\omega Z = AZ - ZB.$$

The polynomial

$$\det(\Re Z(0) + \lambda \Im Z(0))$$

is of degree n and is not the zero polynomial because it doesn't vanish when $\lambda = i$ (since $Z(0) = \Re Z(0) + i\Im Z(0)$ is invertible) and thus it has at most n zeros. Choose $\lambda \in \mathbb{R}$ such that $\Re Z(0) + \lambda \Im Z(0)$ is invertible and let

$$W(\theta) = \Re Z(\theta) + \lambda \Im Z(\theta),$$

then

$$X^t_{\omega,A}(\theta)W(\theta) = W(\theta + t\omega)X^t_{\omega,B}(\theta), \quad \theta \in \mathbb{T}^d.$$

Moreover, if there exists $\theta \in \mathbb{T}^d$ such that $\det W(\theta) = 0$, then the previous relations imply $\det W(\theta + t\omega) = 0$ for all $t \in \mathbb{R}$, and since ω is rationally independent, $\{[t\omega]; t \in \mathbb{R}\}$ is a dense set in $\mathbb{T}^d$ and the continuity of $\det W$ implies $\det W = 0$, which is impossible because we chose λ real such that $\Re Z(0) + \lambda \Im Z(0)$ is invertible. Thus $W(\theta)$ is invertible for all $\theta \in \mathbb{T}^d$. □

Remark 2.2. The real conjugation in the proposition 2.1 and the given conjugation have the same period: there is no period doubling.

Proposition 2.3. *Let $n \in \mathbb{N}^*$, $U : \mathbb{T}^d \to Gl(n, \mathbb{C})$ continuous and let $B \in gl(n, \mathbb{C})$ in Jordan normal form. If*

$$\partial_\omega U = BU - U\bar{B} \tag{2.1}$$

then there exist $W : \mathbb{T}^d \to Gl(n, \mathbb{C})$ of class $\mathcal{C}^\infty$ and $B' \in gl(n, \mathbb{R})$ such that, for all $\theta \in \mathbb{T}^d$

$$\partial_{\frac{\omega}{2}} W(\theta) = BW(\theta) - W(\theta)B'.$$

We postpone the proof of Proposition 2.3 after a few lemmas.
Denote

$$\mathcal{M} = \left\{2i\pi\langle k, \omega\rangle, k \in \mathbb{Z}^d\right\}.$$

Let $\sigma(B) = \{\alpha_1, \cdots, \alpha_l\}$ the spectrum of B. The relation (2.1) implies, denoting $B = diag(B_j)_{j=1,\dots,l}$, α_j the eigenvalue of a block B_j and $U = (U^j_i)_{i,j=1,\dots,l}$

$$\partial_\omega U^j_i = B_i U^j_i - U^j_i \bar{B}_j. \tag{2.2}$$

Lemma 2.4. *1. If $\alpha_i - \bar{\alpha}_j \notin \mathcal{M}$, the block U^j_i is zero. In particular, if $\alpha_i - \bar{\alpha}_j \notin \mathcal{M}$ for a given j and for all i, then $\det U = 0$.*

2. Moreover, if $\alpha_i - \bar{\alpha}_j = 2i\pi\langle k_{i,j}, \omega\rangle$ for some $k_{i,j} \in \mathbb{Z}^d \backslash \{0\}$, then the only non zero Fourier mode of the block U^j_i is indexed by $k_{i,j}$.

Proof. The relation (2.1) implies, if $U^j_i = (u_{i',j'})$, denoting $s_i = \text{mult}(\alpha_i)$ and $s_j = \text{mult}(\alpha_j)$, for all $(i', j') \in [\![1, s_i]\!] \times [\![1, s_j]\!]$ (letting $\delta_0 = \delta_{s_i} = 0$),

$$\partial_\omega u_{i',j'} = (\alpha_i - \bar{\alpha}_j)u_{i',j'} + \delta_i u_{i'+1,j'} - \delta_{j'-1} u_{i',j'-1}, \tag{2.3}$$

with the $\delta_{i'}, \delta_{j'} \in \{0,1\}$. Let i,j such that $\alpha_i - \bar{\alpha}_j \notin \mathcal{M}$, then decomposing these equations with Fourier coefficients and solving the equations in the right order, this implies

$$u_{i',j'} = 0$$

whenever $\alpha_{i'} = \alpha_i$ and $\alpha_{j'} = \alpha_j$.

Now, given α_j, if $\alpha_j - \beta \notin \mathcal{M}$ for all β, this implies that there is a zero column in U and then $\det U = 0$ which is impossible by assumption on U.

Now assume that for some i,j, $\alpha_i - \bar{\alpha}_j = 2i\pi\langle k_{i,j}, \omega\rangle$. If $\delta_{i'} = \delta_{j'-1} = 0$, then (2.3) implies that the only non zero Fourier mode of $u_{i',j'}$ is indexed by $k_{i,j}$. Then recursively on i', j' corresponding to the same two eigenvalues of B, one proves that the only non zero Fourier mode of the block U_i^j is indexed by $k_{i,j}$.

□

Definition 2.5. We will say that two complex numbers α_i and α_j are linked if and only if there exists $k_{i,j} \in \mathbb{Z}^d$ such that

$$\alpha_i - \bar{\alpha}_j = 2i\pi\langle k_{ij}, \omega\rangle$$

(that is to say $\alpha_i - \bar{\alpha}_j \in \mathcal{M}$).

Remark 2.6. The relation of being linked is symmetric, but neither reflexive nor transitive.

Definition 2.7. We call *chain of length* $k-1$ a sequence $\alpha_{i_1}, \cdots, \alpha_{i_k}$ such that for all $j \in \{1, \cdots, k-1\}$, α_{i_j} is linked to $\alpha_{i_{j+1}}$. If moreover $\alpha_{i_k} = \alpha_{i_1}$ and $k \geq 2$, we will say it is a *loop of length* $k-1$.

We will say that two numbers α, β are chain-linked if there is a chain between α and β.

Remark 2.8. The relation of being chain-linked is an equivalence relation on any set of complex numbers Γ which satisfy

$$\text{for all } x \in \Gamma, \text{ there exists } y \in \Gamma \text{ such that } x \text{ and } y \text{ are linked} \tag{2.4}$$

(this will be the case when we will consider the spectrum of our matrix B). Considering such a set $\Gamma \subseteq \mathbb{C}$, we shall denote by $[\alpha]$ the equivalence class of $\alpha \in \Gamma$.

Notice that if α, β are linked by a chain with even length, then $\alpha - \beta \in \mathcal{M}$.

Also, it is easy to notice that if there is a chain of length 3 then the first and the last numbers are linked (simply write the resonances relations).

Sublemma 2.9. Let $\Gamma \subseteq \mathbb{C}$ satisfying property (2.4). Given $\alpha \in \Gamma$, α is linked to itself if and only if $[\alpha]$ contains a loop of odd length.

Proof. If α is linked to itself, then there is a loop of length 1 between α and itself. Suppose now there is a loop of odd length in $[\alpha]$. Denote $\alpha_1, \ldots \alpha_k$ this loop, and suppose $\alpha = \alpha_1$. Then,

$$\alpha_1 - \bar{\alpha}_1 = \alpha_1 - \bar{\alpha}_2 + \bar{\alpha}_2 - \alpha_3 + \cdots + \alpha_k - \bar{\alpha}_1 \in \mathcal{M}$$

and then α_1 is linked to itself. □

Let us investigate the possible links between the eigenvalues of B.

Lemma 2.10. *Under the assumptions of Proposition 2.3, $\sigma(B)$ satisfies property* (2.4)*. Moreover, for all $\alpha \in \sigma(B)$, if $[\alpha]$ does not contain any loop of odd length, then there is a partition of $[\alpha] = \Sigma_1 \cup \Sigma_2$ where*

- *every element of Σ_1 is linked to every element of Σ_2,*
- *the sum of multiplicities of the eigenvalues in Σ_1 equals the sum of multiplicities of the eigenvalues in Σ_2.*

Proof. By the lemma 2.4, the property (2.4) holds, and the blocks of U relating different equivalence classes of eigenvalues are zero. Therefore, if U has invertible values, the blocks of U corresponding to an equivalence class of $\sigma(B)$ are invertible.

Define the equivalence relation

$$\alpha_i \sim \alpha_j \Leftrightarrow \text{there exists a chain of even length between } \alpha_i \text{ and } \alpha_j \text{ in } [\alpha].$$

From remark 2.8, there are only 2 distinct equivalence classes $\Sigma_1 = \{\beta_1, \dots, \beta_r\}$ and $\Sigma_2 = \{\gamma_1, \cdots, \gamma_s\}$. Moreover, two elements of the same equivalence class cannot be linked, otherwise we would have a loop of odd length. Any element in Σ_1 is linked to all elements of Σ_2. Indeed let $\beta \in \Sigma_1$ and $\gamma \in \Sigma_2$, then β and γ are linked by a chain (because they both are in $[\alpha]$), now this chain has an odd length by definition of Σ_1, Σ_2, therefore they are linked.

Then from lemma 2.4, the block $\tilde{U}$ corresponding to $[\alpha]$ has the form

$$\tilde{U} = \begin{array}{c} \\ \beta_1 \\ \vdots \\ \beta_r \\ \gamma_1 \\ \vdots \\ \gamma_s \end{array} \begin{array}{c} \begin{array}{cccccc} \beta_1 & \cdots & \beta_r & \gamma_1 & \cdots & \gamma_s \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & * & \cdots & * \\ * & \cdots & * & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & 0 & \cdots & 0 \end{pmatrix} \end{array}$$

(because the β_i are not linked to each other, nor are the γ_i), which is invertible only if $\sum_{i=1}^{r} \text{mult}(\beta_i) = \sum_{i=1}^{s} \text{mult}(\gamma_i)$. □

We can now prove proposition 2.3:

Proof. [of the proposition 2.3] **Construction of W.** By lemma 2.10, $\sigma(B)$ satisfies the property (2.4) therefore the relation of being chain linked is an equivalence. Let $[\alpha] \subset \sigma(B)$. There are two cases:

1. If $[\alpha]$ contains a loop of odd length, then by sub-lemma 2.9, for all $\alpha_i \in [\alpha]$,

$$\alpha_i - \bar{\alpha}_i = 2i\pi\langle k_i, \omega \rangle$$

with $k_i \in \mathbb{Z}^d$.

2. Otherwise, $[\alpha]$ does not contain odd loops, and from lemma 2.10 write $[\alpha] = \Sigma_1 \cup \Sigma_2$. Choose arbitrarily $\alpha_j \in \Sigma_1$. Then from remark 2.8, for all $\alpha_i \in \Sigma_1$, there exists $k_i \in \mathbb{Z}^d$ such that

$$\alpha_i - \alpha_j = 2i\pi\langle k_i, \omega\rangle.$$

Also, for all $\alpha_i \in \Sigma_2$, there is $k_i \in \mathbb{Z}^d$ such that

$$\alpha_i - \bar{\alpha}_j = 2i\pi\langle k_i, \omega\rangle.$$

Then we construct $W \in C^0(\mathbb{T}^d, Gl(n,\mathbb{C})), W = \mathrm{diag}(w_k)_{k=1,\cdots,l}$ (where w_k is a sub-matrix associated to the generalized eigenspaces of the eigenvalue α_k) such that for all $\theta \in \mathbb{T}^d$,

- $$w_i(\theta) = e^{2i\pi\langle k_i,\theta\rangle} I$$

  in case 1;
- $$w_i(\theta) = e^{4i\pi\langle k_i,\theta\rangle} I$$

  in case 2.

The relation

$$\partial_{\frac{\omega}{2}} W = BW - WB' \tag{2.5}$$

defines a matrix B' of dimension $n \times n$ (notice that $B' - B$ is a diagonal matrix, since the coefficients for B and B' outside the diagonal are the same), whose diagonal coefficients are either real (in case 1) or come by pairs of complex conjugates with the same multiplicity (in case 2).

Now we will prove that, in case 2, the two blocks of B' corresponding to complex conjugate eigenvalues are algebraically conjugate, which will imply that they have the same Jordan structure.

The relation (2.2) combined with (2.5) implies that for all i, j, if we denote $\tilde{U}_i^j(\theta) = U_i^j(2\theta)$,

$$\partial_{\frac{\omega}{2}}(w_i^{-1}\tilde{U}_i^j\bar{w}_j) = B'_i(w_i^{-1}\tilde{U}_i^j\bar{w}_j) - (w_i^{-1}\tilde{U}_i^j\bar{w}_j)\bar{B}'_j$$

and by construction and the second statement of Lemma 2.4, $w_i^{-1}\tilde{U}_i^j\bar{w}_j$ is constant. Up to a permutation, one can assume that the blocks of B corresponding to the eigenvalues in the same equivalence class are next to each other and can be grouped in a block $B_{[\alpha]}$ (where $[\alpha]$ stands for the equivalence class in question). For any $\alpha \in \sigma(B)$, letting $W_{[\alpha]} = \mathrm{diag}(w_i, i \in [\alpha])$ and $\tilde{U}_{[\alpha]}(\theta)$ be the block of $\tilde{U}(\theta) := U(2\theta)$ corresponding to $[\alpha]$ (so $\tilde{U} = \mathrm{diag}(\tilde{U}_{[\alpha]})_{[\alpha]}$), then $W_{[\alpha]}^{-1}\tilde{U}_{[\alpha]}W_{[\alpha]}$ satisfies

$$\partial_{\frac{\omega}{2}}(W_{[\alpha]}^{-1}\tilde{U}_{[\alpha]}\bar{W}_{[\alpha]}) = B'_{[\alpha]}W_{[\alpha]}^{-1}\tilde{U}_{[\alpha]}\bar{W}_{[\alpha]} - W_{[\alpha]}^{-1}\tilde{U}_{[\alpha]}\bar{W}_{[\alpha]}\bar{B}'_{[\alpha]}.$$

Moreover, $W_{[\alpha]}^{-1}\tilde{U}_{[\alpha]}\bar{W}_{[\alpha]}$ is constant and also invertible since $\tilde{U}$ is invertible. Thus $B'_{[\alpha]}$ and $\bar{B}'_{[\alpha]}$ are algebraically conjugate, which means that two blocks of B' corresponding to complex conjugate eigenvalues have the same Jordan structure.

The matrix B' is not necessary real, but can be conjugated to a real matrix applying lemma 8.9 in appendix. □

We now prove the first main theorem:

Theorem 2.11 (Real reducibility)**.** *Let $A : \mathbb{T}^d \to gl(n,\mathbb{R})$ such that $X_{\omega,A}$ is reducible. Then $X_{\frac{\omega}{2},A_2}$ is real reducible, where*

$$A_2(\theta) := A(2\theta), \quad \forall \theta \in \mathbb{T}^d.$$

Proof. By assumptions, there exist $B \in gl(n,\mathbb{C})$ and $Z : \mathbb{T}^d \to Gl(n,\mathbb{C})$ such that

$$\partial_\omega Z = AZ - ZB$$

and then

$$\partial_\omega \bar{Z} = A\bar{Z} - \bar{Z}\bar{B}$$

and

$$\partial_\omega Z^{-1} = BZ^{-1} - Z^{-1}A.$$

Let P an invertible matrix such that $B = PJP^{-1}$, where J is in normal Jordan form. Let

$$U = P^{-1}Z^{-1}\bar{Z}\bar{P},$$

then

$$\begin{aligned}
\partial_\omega U &= \partial_\omega(P^{-1}Z^{-1}\bar{Z}\bar{P}) \\
&= P^{-1}(\partial_\omega(Z^{-1})\bar{Z} + Z^{-1}\partial_\omega(\bar{Z}))\bar{P} \\
&= P^{-1}((BZ^{-1} - Z^{-1}A)\bar{Z} + Z^{-1}(A\bar{Z} - \bar{Z}\bar{B}))\bar{P} \\
&= JU - U\bar{J}.
\end{aligned}$$

We can apply proposition 2.3 and deduce that there exist $W : \mathbb{T}^d \to Gl(n,\mathbb{C})$ and $J' \in gl(n,\mathbb{R})$ such that

$$\partial_{\frac{\omega}{2}} W = JW - WJ'.$$

Let $Z'(\theta) = Z(2\theta)PW(\theta)$. Therefore

$$\begin{aligned}
\partial_{\frac{\omega}{2}} Z'(\theta) &= \partial_{\frac{\omega}{2}} Z(2\theta)PW(\theta) + Z(2\theta)P\partial_{\frac{\omega}{2}} W(\theta) \\
&= (A(2\theta)Z(2\theta) - Z(2\theta)B)PW(\theta) + Z(2\theta)P(JW(\theta) - W(\theta)J') \\
&= A(2\theta)Z(2\theta)PW(\theta) - Z(2\theta)PW(\theta)J' \quad \text{since } BP = JP \\
&= A_2(\theta)Z'(\theta) - Z'(\theta)J'.
\end{aligned}$$

Remark that Z' is not necessary real, however by proposition 2.1, there exists $\lambda \in \mathbb{R}$ such that

$$\Re Z' + \lambda \Im Z' : \mathbb{T}^d \to Gl(n,\mathbb{R})$$

conjugates the two real cocycles $X_{\frac{\omega}{2},A_2}$ and $X_{\frac{\omega}{2},J'}$.

□

In the remainder of the article, we will prove the second main result, which is that almost reducibility for a real cocycle implies real almost reducibility.

3 Construction of a real change of variables

3.1 Lemmas

In this section, we prove a few lemmas about the trace of a system and the determinant of a conjugation. They will be used to construct real changes of variables for real almost-reducible cocycles. Here, we need an arithmetical condition on ω.

3.1.1 A small divisor lemma

Let $f : \mathbb{T}^d \to \mathbb{C}$ be C^∞ and consider the equation

$$\begin{cases} \partial_\omega g = f - \hat{f}(0) \\ \hat{g}(0) = 0. \end{cases} \tag{3.1}$$

Lemma 3.1. *If $\omega \in \mathcal{DC}(\kappa, \tau)$, then there exists a unique solution $g : \mathbb{T}^d \to \mathbb{C}$ to (3.1) and it satisfies*

$$\|g\|_{\mathcal{C}^r} \leq C_r \frac{1}{\kappa} \|f\|_{\mathcal{C}^{\tau+d+1+r}}, \quad \forall r \geq 0,$$

where $C_{r,d}$ is a constant depending only on r, d.

Proof. Developing g in Fourier series, we get

$$g(\theta) \simeq \sum_{k \neq 0} \frac{\hat{f}(k)}{2i\pi\langle k, \omega\rangle} e^{2i\pi\langle k,\theta\rangle}.$$

Then for all $s \geq 1$, since $\omega \in \mathcal{DC}(\kappa, \tau)$,

$$\begin{aligned} \|g\|_{\mathcal{C}^0} &\leq \sum_{k\neq 0} \frac{|k|^\tau |\hat{f}(k)|}{2\pi\kappa} \\ &\leq C_s \frac{1}{\kappa} \sum_{k\neq 0} |k|^{\tau-s} \sup_{|\alpha|=s} |\widehat{\partial^\alpha f(k)}| \\ &\leq C_s \frac{1}{\kappa} \|f\|_{\mathcal{C}^s} \sum_{k\neq 0} |k|^{\tau-s} \\ &\leq C_{s,d} \frac{1}{\kappa} \|f\|_{\mathcal{C}^s} \sum_{j>0} j^{\tau-s+d-1} \end{aligned}$$

and this converges if $s \geq \tau + d + 1$.

The higher derivatives are obtained by differentiating in the Fourier series. □

3.1.2 Trace and determinant

Lemma 3.2. *Let $A : \mathbb{T}^d \to gl(n, \mathbb{C})$ and $\omega \in \mathcal{DC}(\kappa, \tau)$. There exist $Z : \mathbb{T}^d \to Gl(n, \mathbb{C})$ and $B \in gl(n, \mathbb{C})$ of constant trace and of class $\mathcal{C}^\infty$ such that*

$$\partial_\omega Z = AZ - ZB.$$

Proof. Let $B = A - (\operatorname{Tr} A - \int_{\mathbb{T}^d} \operatorname{Tr} A(\theta) d\theta) I$. Then B has constant trace. If $f = \operatorname{Tr} A - \int_{\mathbb{T}^d} \operatorname{Tr} A(\theta) d\theta$, then $\hat{f}(0) = 0$ and the equation

$$\begin{cases} \partial_\omega g = f \\ \hat{g}(0) = 0 \end{cases}$$

has a unique solution $\mathbb{T}^d \to \mathbb{C}$ (which is of class $\mathcal{C}^\infty$) by Lemma 3.1. Let now $Z = e^g I : \mathbb{T}^d \to Gl(n, \mathbb{C})$, then Z satisfies

$$\partial_\omega Z = AZ - ZB.$$

□

Lemma 3.3. *Let $A, B, F, Z : \mathbb{T}^d \to gl(n, \mathbb{C})$, with Z differentiable. If*

$$\partial_\omega Z = AZ - ZB + F$$

then

$$\partial_\omega \det Z = \operatorname{Tr}(A - B) \det Z + \operatorname{Tr}(F Z^{adj})$$

where Z^{adj} is the transpose of the cofactor matrix of Z.

Proof. We have, for all $\theta \in \mathbb{T}^d$, $Z(\theta) Z^{adj}(\theta) = \det Z(\theta) \cdot I$, so

$$\partial_\omega \det Z(\theta) \cdot I = \partial_\omega Z(\theta) Z^{adj}(\theta) + Z(\theta) \partial_\omega Z^{adj}(\theta),$$

and taking the trace we find

$$n \partial_\omega \det Z = \operatorname{Tr}((AZ - ZB + F) Z^{adj}) + \operatorname{Tr}(Z \partial_\omega Z^{adj}) =$$
$$= \operatorname{Tr}(A - B) \det Z + \operatorname{Tr}(F Z^{adj}) + \operatorname{Tr}(Z \partial_\omega Z^{adj}).$$

We want to show that $(n-1) \partial_\omega \det Z = \operatorname{Tr}(Z \partial_\omega Z^{adj})$. From the formula of the differential of the determinant:

$$D(\det A)[H] = \operatorname{Tr}(A^{adj} H)$$

where A^{adj} is the transpose of the cofactor matrix of A, and from the formula of the derivative of composite functions

$$\frac{\partial}{\partial t}(F \circ f)|_{t=0} = D_{f(t)} F [\frac{\partial f}{\partial t}|]_{t=0}$$

where here

$$F = \det, f = Z(\theta + t\omega) \Rightarrow f(0) = Z(\theta), \quad \frac{\partial f}{\partial t}|_{t=0} = \partial_\omega Z$$

and therefore

$$D_{f(t)} F|_{t=0} = \operatorname{Tr}(Z^{adj}).$$

Hence,

$$\begin{aligned} \partial_\omega \det Z &= Tr(Z^{adj} \partial_\omega Z) = Tr(\partial_\omega Z Z^{adj}) \\ &= Tr(\partial_\omega (Z Z^{adj})) - Tr(Z \partial_\omega Z^{adj}) \\ &= Tr(\partial_\omega (\det Z Id)) - Tr(Z \partial_\omega Z^{adj}) \\ &= n \partial_\omega \det Z - Tr(Z \partial_\omega Z^{adj}). \end{aligned}$$

Finally,

$$\begin{aligned} \partial_\omega \det Z &= n \partial_\omega \det Z - Tr(Z \partial_\omega Z^{adj}) \\ &= \operatorname{Tr}(A - B) \det Z + \operatorname{Tr}(F Z^{adj}). \end{aligned}$$

□

Remark 3.4. This lemma does not require any arithmetical condition on ω.

The following lemma will be used to construct a real invertible conjugation out of a complex one.

Lemma 3.5. *If $Z : \mathbb{T}^d \to Gl(n, \mathbb{C})$ satisfies*

$$|\int_{\mathbb{T}^d} \det Z(\theta) d\theta| = 1,$$

then there exists $\lambda \in [-1, 1]$ such that

$$|\int_{\mathbb{T}^d} \det(\Re Z(\theta) + \lambda \Im Z(\theta)) d\theta| \geq C_n > 0, \tag{3.2}$$

and the constant C_n only depends on n.

Proof. Let

$$P(\lambda) = \int_{\mathbb{T}^d} \det(\Re Z(\theta) + \lambda \Im Z(\theta)) d\theta.$$

Then P is a polynomial of degree n which is non zero because $P(i)$ is complex number of unit modulus by assumption. Hence

$$P(\lambda) = \mu(\lambda - \alpha_1) \dots (\lambda - \alpha_n)$$

for some $\mu \neq 0$, $\alpha, \cdots, \alpha_n \in \mathbb{C}$, and we have

$$\mu = \frac{P(i)}{(i - \alpha_1) \dots (i - \alpha_n)}.$$

By the Pigeon hole principle, there exists $k \in \{0, \dots, n\}$ such that

$$(-1 + \frac{2k}{n+1}, -1 + \frac{2(k+1)}{n+1}) \bigcap \{\Re\alpha_1, \dots, \Re\alpha_n\} = \emptyset.$$

If $\lambda_0 = -1 + \frac{2k+1}{n+1}$, then one has

$$|\lambda_0 - \alpha_j| \geq |\lambda_0 - \Re\alpha_j| \geq \frac{1}{n+1}, \quad \forall j.$$

If $|\alpha_j| \leq 3$, then $\frac{|\lambda_0 - \alpha_j|}{|i - \alpha_j|} \geq \frac{1}{4(n+1)}$. If now $|\alpha_j| > 3$, we have $|\frac{\lambda_0}{\alpha_j}| \leq \frac{2}{3}$ therefore $|\frac{\lambda_0}{\alpha_j} - 1| \geq \frac{1}{3}$, and also $|\frac{i}{\alpha_j}| \leq \frac{1}{3}$ which implies $|\frac{i}{\alpha_j} - 1| \leq \frac{4}{3}$. Thus

$$\frac{|\lambda_0 - \alpha_j|}{|i - \alpha_j|} = \frac{|\frac{\lambda_0}{\alpha_j} - 1|}{|\frac{i}{\alpha_j} - 1|} \geq \frac{1}{4}$$

so

$$|P(\lambda_0)| \geq \prod_{|\alpha_j| \leq 3} \frac{|\lambda_0 - \alpha_j|}{|i - \alpha_j|} \prod_{|\alpha_j| > 3} \frac{|\lambda_0 - \alpha_j|}{|i - \alpha_j|} \geq \frac{1}{(4(n+1))^n},$$

so λ_0 satisfies (3.2).

□

3.2 Construction of a sequence of real changes of variables

Proposition 3.6. *Let $X_{\omega,A}$ be a real cocycle which is almost reducible to real matrices, i.e. there exist sequences $Z_j : \mathbb{T}^d \to Gl(n,\mathbb{C})$, $B_j \in gl(n,\mathbb{R})$ and $F_j : \mathbb{T}^d \to gl(n,\mathbb{C})$ such that*

$$\partial_\omega Z_j = AZ_j - Z_j(B_j + F_j)$$

and

$$\lim_{j\to+\infty} \|Z_j^{\pm 1}\|_{\mathcal{C}^r}^m \|F_j\|_{\mathcal{C}^r} = 0, \quad \forall m, r \geq 0.$$

If $\omega \in \mathcal{DC}(\kappa,\tau)$, then $X_{\omega,A}$ is real almost reducible.

Proof. By Lemma 3.2 we can assume $\operatorname{Tr} A$ is constant, and since $\operatorname{Tr}(A)I$ commutes with Z_j,

$$\partial_\omega Z_j = (A - \frac{1}{n}Tr(A)\cdot I)Z_j - Z_j(B_j + F_j - \frac{1}{n}Tr(A)\cdot I)$$

therefore, by replacing B_j with $B_j - \frac{1}{n}Tr(A)\cdot I$, we can assume that

$$\operatorname{Tr} A = 0.$$

We have, for all $\theta \in \mathbb{T}^d$,

$$\|\det Z_j^{\pm 1}(\theta)\| \leq C_n \|Z_j^{\pm 1}\|_{\mathcal{C}^0}^n$$

and, hence for all $\theta \in \mathbb{T}^d$,

$$\|\det Z_j^{\pm 1}(\theta)\| = \|\frac{1}{\det Z_j^{\mp 1}(\theta)}\| \geq \frac{1}{C_n}\|Z_j^{\mp 1}\|_{\mathcal{C}^0}^{-n}.$$

So the quantity

$$a_j = \Big(\int_{\mathbb{T}^d} \det Z_j(\theta)\, d\theta\Big)^{\frac{1}{n}} \in \mathbb{C}$$

satisfies

$$0 < \frac{1}{C_n'\|Z_j^{-1}\|_{\mathcal{C}^0}} \leq |a_j| \leq C_n'\|Z_j\|_{\mathcal{C}^0}. \tag{3.3}$$

Define now

$$\tilde{Z}_j = \frac{1}{a_j} Z_j.$$

Then

$$\partial_\omega \tilde{Z}_j = A\tilde{Z}_j - \tilde{Z}_j(B_j + F_j)$$

and for all $r \in \mathbb{N}$, (3.3) implies

$$\|\tilde{Z}_j\|_{\mathcal{C}^r} \leq C_n'\|Z_j^{-1}\|_{\mathcal{C}^0}\|Z_j\|_{\mathcal{C}^r}, \quad \|\tilde{Z}_j^{-1}\|_{\mathcal{C}^r} \leq C_n'\|Z_j\|_{\mathcal{C}^0}\|Z_j^{-1}\|_{\mathcal{C}^r},$$

which implies that

$$\lim_{j\to+\infty} \|\tilde{Z}_j^{\pm 1}\|_{\mathcal{C}^r}^m \|F_j\|_{\mathcal{C}^r} = 0, \quad \forall m, r \geq 0.$$

Replacing Z_j by $\tilde{Z}_j$, we can therefore simply assume that

$$\int_{\mathbb{T}^d} \det Z_j(\theta)\, d\theta = 1, \quad \forall j. \tag{3.4}$$

Applying Lemma 3.3 with $B = B_j$ and $F = -Z_j F_j$, we have

$$\begin{aligned}\partial_\omega \det Z_j &= \operatorname{Tr}(A - B_j) \det Z_j - \operatorname{Tr}(Z_j F_j Z_j^{adj}) \\ &= -\operatorname{Tr}(B_j + F_j) \det Z_j.\end{aligned}$$

Hence

$$0 = \int_{\mathbb{T}^d} \partial_\omega \det Z_j(\theta) d\theta = -\operatorname{Tr}(B_j) - \int_{\mathbb{T}^d} \operatorname{Tr}(F_j(\theta)) \det Z_j(\theta) d\theta.$$

We can therefore simply assume (replacing B_j by $B_j + \frac{1}{n}\int_{\mathbb{T}^d} \operatorname{Tr}(F_j(\theta)) \det Z_j(\theta) d\theta I$, which is real since B_j is real and from the previous equality), that

$$\operatorname{Tr}(B_j) = 0, \quad \forall j.$$

By Lemma 3.5, which can be applied thanks to (3.4), there exists a $\lambda_j \in [-1, 1]$ and $C_n > 0$ such that

$$W_j = \Re Z_j + \lambda_j \Im Z_j$$

satisfies

$$|\widehat{\det W_j}(0)| = |\int_{\mathbb{T}^d} \det W_j(\theta) d\theta| \geq C_n, \quad \forall j. \tag{3.5}$$

Clearly, since A and B_j are real,

$$\partial_\omega W_j = A W_j - W_j (B_j + G_j)$$

where $G_j = W_j^{-1}(\Re(Z_j F_j) + \lambda_j \Im(Z_j F_j))$. In particular

$$\lim_{j \to +\infty} \|W_j\|_{\mathcal{C}^r}^m \|G_j\|_{\mathcal{C}^r} = 0, \quad \forall m, r \geq 0.$$

There remains to study the inverse of W_j.
By Lemma 3.3, we have

$$\partial_\omega \det W_j = \operatorname{Tr}(A - B_j) \det W_j - \operatorname{Tr}(W_j G_j W_j^{adj}) =$$

$$= -\operatorname{Tr}(W_j G_j W_j^{adj}) = -Tr(G_j \det(W_j)) := H_j.$$

Since

$$\|H_j\|_{\mathcal{C}^r} \leq C_r \|G_j\|_{\mathcal{C}^r} \|W_j\|_{\mathcal{C}^r}^n$$

we have

$$\lim_{j \to \infty} \|H_j\|_{\mathcal{C}^r} = 0, \quad \forall r \geq 0.$$

By Lemma 3.1, if $\omega \in \mathcal{DC}(\kappa, \tau)$, we have

$$\|\det W_j - \widehat{\det W_j}(0)\|_{\mathcal{C}^0} \leq C_{r,d} \frac{1}{\kappa} \|H_j\|_{\mathcal{C}^{\tau+d+1}}.$$

So for j sufficiently large and thanks to (3.5), we have

$$|\det W_j(\theta)| \geq \frac{1}{2} C_n, \quad \forall \theta \in \mathbb{T}^d.$$

This implies that W_j is invertible, and for j sufficiently large,

$$\|W_j^{-1}\|_{\mathcal{C}^0} \leq C'_n \|W_j\|_{\mathcal{C}^0}^{n-1}$$

and, by (1.7),

$$\|W_j^{-1}\|_{\mathcal{C}^r} \leq C_r \|W_j^{-1}\|_{\mathcal{C}^0}^{r+1} \|W_j\|_{\mathcal{C}^r}^r, \quad \forall r.$$

In particular

$$\lim_{j \to +\infty} \|W_j^{-1}\|_{\mathcal{C}^r}^m \|G_j\|_{\mathcal{C}^r} = 0, \quad \forall m, r \geq 0.$$

□

4 Jordan normal form with estimates on the conjugation matrix

This section is devoted to conjugating a matrix to its Jordan normal form with sufficient estimates.

4.1 Column echelon form by an algebraic conjugation

Definition 4.1. We say that a matrix A is in *column echelon form* if it has strictly increasing column lengths, except the first columns which can be zero. Its *pivots* are the last non zero coefficient of each non zero column.

We will now conjugate a nonzero nilpotent matrix $A \in gl(n, \mathbb{C})$ to a nilpotent one in column echelon form. Moreover, the conjugation will be unitary.
Let m the index of A, that is to say the smallest integer m such that $A^m = 0$ (here $m \geq 2$ since A is nonzero). Denote $L : \mathbb{C}^n \to \mathbb{C}^n$ the linear map represented by A in the canonical basis, and $K_j = \ker L^j$ the kernel of the iterates of L, for all $j \in \{0, \dots, m\}$. Then we have

$$\{0\} = K_0 \subset_{\neq} K_1 \subseteq K_2 \subseteq \cdots \subseteq K_{m-1} \subset_{\neq} K_m = \mathbb{C}^n.$$

Let for all $j \in \{1, \dots, m\}$, $U_j = K_j \cap K_{j-1}^{\perp}$ where $K_{j-1}^{\perp}$ is the orthogonal complement of K_{j-1} in $\mathbb{C}^n$ equipped with the standard inner product. This implies the orthogonal direct sum

$$K_j = K_{j-1} \oplus^{\perp} U_j$$

from which we can define $\mathrm{proj}_{U_j} : K_j \to U_j$ the orthogonal projection onto U_j. Finally,

$$K_j = U_1 \oplus^{\perp} U_2 \oplus^{\perp} \cdots \oplus^{\perp} U_j.$$

We will also denote

$$r_j = \dim U_j.$$

Lemma 4.2. *With the above notations,*

(i) $L(K_j) \subseteq K_{j-1}$ *for all* $j \in \{1, \dots, m\}$,

(ii) the restriction $L|_{U_j}$ *of* L *to* U_j *is injective for all* $j \in \{2, \dots, m\}$,

(iii) $L(U_j) \cap K_{j-2} = \{0\}$ *for all* $j \in \{2, \dots, m\}$,

(iv) $(\dim U_i)_{i=1,\dots,m}$ *is a non increasing sequence.*

Proof. (i) For all $j \in \{1,\dots,m\}$,

$$u \in K_j \Leftrightarrow L^j u = 0 \Leftrightarrow L^{j-1}(Lu) = 0 \Leftrightarrow Lu \in K_{j-1}.$$

(ii) Let $j \in \{2,\dots,m\}$, and let $u \in U_j = K_j \cap K_{j-1}^{\perp}$ with $u \neq 0$. Then $u \notin K_{j-1}$ and then $L^{j-1}u \neq 0$, which implies (since $j-1 \geq 1$) $Lu \neq 0$.

(iii) Let $j \in \{2,\dots,m\}$ and let a nonzero $u \in U_j = K_j \cap K_{j-1}^{\perp}$, then $u \notin K_{j-1}$, and from (i), $Lu \in K_{j-1}$. Reasoning by absurdity, suppose that $Lu \in K_{j-2}$, then $L^{j-1}u = L^{j-2}(Lu) = 0$, which implies $u \in K_{j-1}$ leading to a contradiction.

(iv) Let $j \in \{2,\dots,m\}$ and $u_1,\dots,u_r$ linearly independent vectors in $U_j \subset K_j$. From (ii), $Lu_1,\dots,Lu_r$ are linearly independent vectors in $K_{j-1} = K_{j-2} \oplus^{\perp} U_{j-1}$ and from (iii), $\mathrm{proj}_{U_{j-1}}(Lu_1),\dots,\mathrm{proj}_{U_{j-1}}(Lu_r)$ are linearly independent vectors in U_{j-1}, which implies that $\dim U_{j-1} \geq \dim U_j$.

□

Proposition 4.3. *Let A a nilpotent matrix of index $m \geq 2$. With the above notations, for all $j \in \{1,\dots,m\}$, there exists an orthonormal basis $\mathcal{B}^j = \{u_1^j,\dots,u_{r_j}^j\}$ of U_j (where $r_j = \dim U_j$) such that, for all $j \geq 2$ and for all $k \in \{1,\dots,r_j\}$, letting $K_{-1} = \{0\}$,*

$$Lu_k^j \in \mathrm{span}(u_1^{j-1},\dots,u_k^{j-1}) \oplus^{\perp} K_{j-2}$$

and

$$\langle Lu_k^j, u_k^{j-1}\rangle \neq 0.$$

Proof. Let $\mathcal{B}^m = \{u_1^m,\dots,u_{r_m}^m\}$ an orthonormal basis of U_m. From (ii) of lemma 4.2, $\{Lu_1^m,\dots,Lu_{r_m}^m\}$ are linearly independant vectors of $K_{m-1} = K_{m-2} \oplus^{\perp} U_{m-1}$ and from (iii) of lemma 4.2, $\{v_1 = \mathrm{proj}_{U_{m-1}}(Lu_1^m),\dots,v_{r_m} = \mathrm{proj}_{U_{m-1}}(Lu_{r_m}^m)\}$ are linearly independant vectors of U_{m-1}. Apply Gram–Schmidt on $\{v_1,\dots,v_{r_m}\}$. Then there exists an orthonormal basis $\mathcal{B} = \{u_1',\dots,u_{r_{m-1}}'\}$ (taking $\{v_1,\dots,v_{r_m}\}$ and completing into an orthonormal basis if $r_{m-1} - r_m > 0$) of U_{m-1} such that for all $k \in \{1,\dots,r_m\}$ (recall $r_m \leq r_{m-1}$),

$$v_k \in \mathrm{span}(u_1',\dots,u_k')$$

and

$$\langle u_k', v_k\rangle \neq 0.$$

Then for all $k \in \{1,\cdots,r_m\}$,

$$Lu_k^m \in \mathrm{span}(u_1',\dots,u_k') \oplus^{\perp} K_{m-2}$$

and

$$\langle Lu_k^m, u_k'\rangle = \langle u_k', v_k\rangle \neq 0,$$

and let $\mathcal{B}^{m-1} = \mathcal{B}$. Now if $m = 2$ we are done, and if $m \geq 3$, we construct every $\mathcal{B}^j$ the same way from U_{j+1} and $\mathcal{B}^{j+1}$. □

Corollary 4.4. Let L be a nilpotent linear application of index $m \geq 2$. With the above notations, there exists an orthonormal basis $\mathcal{B} = \{u_1, \ldots, u_n\}$ of $\mathbb{C}^n$ such that L is represented in this basis by the matrix

$$A = \begin{pmatrix} 0 & A_1^2 & * & \cdots & \cdots \\ 0 & 0 & A_2^3 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 0 & A_{m-1}^m \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.1}$$

where, for all $j \in \{2, m\}$, A_{j-1}^j is a $r_{j-1} \times r_j$ matrix of the form

$$\begin{pmatrix} \alpha_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & \alpha_{r_j} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{pmatrix}$$

if $r_{j-1} > r_j$ or

$$\begin{pmatrix} \alpha_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & \alpha_{r_j} \end{pmatrix}$$

if $r_{j-1} = r_j$, and with $\alpha_i \neq 0$ for all $i \in \{1, \ldots, r_j\}$.

Proof. Concatenating the basis $\mathcal{B}^i$ obtained for all $i \in \{1, \ldots, m\}$ in proposition 4.3, blocks A_{j-1}^j have coefficients

$$\langle u_k^{j-1}, L u_l^j \rangle$$

for all $k \in \{1, \ldots, r_{j-1}\}$, $l \in \{1, \ldots, r_j\}$. □

Lemma 4.5. *Let A of the form (4.1). Let B with the same block-triangular form as A, that is to say, according to the notation of corollary 4.4, for all $i \geq j$,*

$$A_i^j = 0 \Rightarrow B_i^j = 0.$$

Denote by L_A and L_B the linear maps represented by A and B respectively in the canonical basis. Then

(*i*)
$$\ker L_B^j \supseteq \ker L_A^j, \quad \forall j.$$

(*ii*) *If*
$$\ker L_B^j = \ker L_A^j, \quad \forall j,$$

then the block B_{j-1}^j is of maximal rank r_j for all j.

Proof. (*i*) follows from the assumption on B.

To see (*ii*), notice that

$$K_j^A = \ker L_A^j = \mathbb{R}^{r_1 + \cdots + r_j} \times \{0\}$$

and

$$U_j^A = \{0\} \times \mathbb{R}^{r_j} \times \{0\}.$$

Then

$$L_A : U_j^A \ni \begin{pmatrix} 0 \\ u \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ A_{j-1}^j u \\ 0 \end{pmatrix} \in U_{j-1}^A$$

which is an injective map.

If now $\ker L_B^j = \ker L_A^j$ then, by the second statement of Lemma 4.2,

$$L_B : U_j^A \ni \begin{pmatrix} 0 \\ u \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ B_{j-1}^j u \\ 0 \end{pmatrix} \in U_{j-1}^A$$

is injective. This is the same as B_{j-1}^j being of maximal rank r_j.

□

4.2 Reduced column echelon form

Definition 4.6. A matrix $A \in gl(n, \mathbb{C})$ is on *reduced column echelon form* if and only if it is on column echelon form with all pivots equal to 1.

We shall conjugate a (nonzero) nilpotent matrix on column echelon form to reduced column echelon form modulo a perturbation with control on the conjugation.

Lemma 4.7. *Let B be a matrix on column echelon form with pivots $(\alpha_j)_j$. If*

$$|\alpha_j| \geq \delta > 0, \quad \forall j,$$

then there exists a diagonal matrix S,

$$\|S^{\pm}\| \leq C(\frac{\|B\|}{\delta})^{\frac{n}{2}},$$

such that $S^{-1}BS$ is on reduced column echelon form.
The constant C only depends on n.

Proof. Let B in column echelon form, then up to a change of orthonormal basis, it has the form of Corollary 4.4; we will look for a block-diagonal S where the diagonal blocks S_j are diagonal and their dimensions are r_j (the number of columns in B_{j-1}^j). Then for all j, we want to find S_j such that

$$S_{j-1}^{-1} B_{j-1}^j S_j = \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \end{pmatrix}$$

where S_{j-1} is given. These equations can be solved uniquely, one by one, starting with $S_1 = I$. Thus,

$$\|S\| \leq \frac{1}{\delta^n}, \quad \|S^{-1}\| \leq \|B\|^n.$$

One can then just multiply S by $\sqrt{\|B\|^n \delta^n}$ to get the estimate.

□

Proposition 4.8. *Let A a nilpotent block diagonal matrix. Let $\varepsilon > 0$ and $(\delta_k)_{k\geq 0}$ a positive decreasing sequence such that*

$$\varepsilon^{\delta_1} + \cdots + \varepsilon^{\delta_k} \leq 2\varepsilon^{\delta_k}, \quad \forall k \geq 1.$$

Then there exists $1 \leq k \leq \frac{n^2}{2}$ and $S \in Gl(n, \mathbb{C})$ with

$$\|S^{\pm}\| \leq C(\|A\| + 2\varepsilon^{\delta_{k-1}})^{\frac{n}{2}} \varepsilon^{-\frac{n}{2}\delta_k} \tag{4.2}$$

such that

$$S^{-1}AS = A' + F,$$

with

$$\|F\| \leq C(\|A\| + 2\varepsilon^{\delta_{k-1}})^n \varepsilon^{-n\delta_k + \delta_{k-1}}, \tag{4.3}$$

where the constant C only depends on n, and S and A' are block diagonal with the same block decomposition as A, and each block of A' is on reduced column echelon form.

Proof. For any nilpotent matrix A, define

$$\sigma ker(A) = (\dim\ker L_A, \dim\ker L_A^2, \ldots, \dim\ker L_A^{n-1})$$

where L_A is associated to A in the canonical basis. This is an increasing sequence of integers $\in [\![1, n]\!]$ and

$$\sigma\ker(A) = (n, n, \ldots, n) \quad \Longleftrightarrow \quad A = 0. \tag{4.4}$$

If B is another nilpotent matrix we say that $\sigma\ker(A) > \sigma\ker(B)$ if and only if

$$\dim\ker L_A^j \quad \begin{cases} \geq \dim\ker L_B^j & \text{for all } j \\ > \dim\ker L_B^j & \text{for some } \ j \end{cases}$$

(this is of course not a total ordering).

By applying Corollary 4.4 to each block of A we can assume, without restriction, that each block of $A_1 = A$ is on column echelon form. By induction:

Base case: If no pivot is $\leq \varepsilon^{\delta_1}$, applying Lemma 4.7 to each block of A_1, there exists a diagonal matrix S_1 with $\|S_1^{\pm 1}\| \leq C(\frac{\|A_1\|}{\varepsilon^{\delta_1}})^{\frac{n}{2}}$ such that $S_1^{-1}A_1S_1$ is on reduced column echelon form, so we are done with $F = 0$.

If there are pivots which are $\leq \varepsilon^{\delta_1}$, let $F_1 \in gl(n, \mathbb{C})$ whose non zero coefficients are those pivots (then $\|F_1\| \leq \varepsilon^{\delta_1}$, and $A_1 - F_1$ has a block not of maximal rank). Apply Corollary 4.4 to $A_1 - F_1$: there exists a unitary matrix $U_1 \in Gl(n, \mathbb{C})$ such that $A_2 = U_1^{-1}(A_1 - F_1)U_1$ is block diagonal, with blocks on column echelon form and by Lemma 4.5, $\sigma\ker(A_2) > \sigma\ker(A_1)$. We have estimates

$$\|F_1\| \leq \varepsilon^{\delta_1}$$

$$\|A_2\| = \|U_1^{-1}(A_1 - F_1)U_1\| \leq \|A_1\| + \varepsilon^{\delta_1}.$$

Induction step: Assume that for some $k > 1$ there exists a unitary $U'_{k-1} \in Gl(n, \mathbb{C})$ with ${U'}_{k-1}^{-1} A U'_{k-1} = A_k + F'_k$ such that A_k is block diagonal and has blocks on column echelon form, with $\|F'_k\| \leq \varepsilon^{\delta_{k-1}} + \cdots + \varepsilon^{\delta_1}$ and $\|A_k\| \leq \|A\| + \varepsilon^{\delta_1} + \cdots + \varepsilon^{\delta_{k-1}}$.

If no pivot of A_k is $\leq \varepsilon^{\delta_k}$, applying Lemma 4.7 to each block of A_k, we get $S_k \in Gl(n,\mathbb{C})$ diagonal, with $\|S_k^{\pm 1}\| \leq C(\frac{\|A_k\|}{\varepsilon^{\delta_k}})^{\frac{n}{2}}$, and $A' = S_k^{-1} A_k S_k$ is block diagonal with blocks on reduced column echelon form. Let then $S = U'_{k-1} S_k$, then

$$S^{-1}AS = (U'_{k-1}S_k)^{-1} A (U'_{k-1}S_k) = S_k^{-1}(A_k + F'_k)S_k$$

$$= A' + F'$$

with $F' \in gl(n,\mathbb{C})$ of norm

$$\|F'\| \leq \|S_k\| \, \|S_k^{-1}\| (\varepsilon^{\delta_{k-1}} + \cdots + \varepsilon^{\delta_1}) \leq 2C^2 (\frac{\|A_k\|}{\varepsilon^{\delta_k}})^n \varepsilon^{\delta_{k-1}} \leq 2C^2 (\|A\| + 2\varepsilon^{\delta_{k-1}})^n \varepsilon^{-n\delta_k + \delta_{k-1}}$$

where the constant C depends only on n, and the proposition is proved.

If some pivots of A_k are $\leq \varepsilon^{\delta_k}$, let $F_k \in gl(n,\mathbb{C})$ whose non zero coefficients are those pivots (then $\|F_k\| \leq \varepsilon^{\delta_k}$, and $A_k - F_k$ is block diagonal and has an block not of maximal rank). Apply Corollary 4.4 to $A_k - F_k$: there exists a unitary matrix $U_k \in Gl(n,\mathbb{C})$ such that $A_{k+1} = U_k^{-1}(A_k - F_k)U_k$ is block diagonal and has blocks in column echelon form and by Lemma 4.5, $\sigma \ker(A_{k+1}) > \sigma \ker(A_k)$ with estimates

$$\|F_k\| \leq \varepsilon^{\delta_k},$$

$$\|A_{k+1}\| \leq \|A_k\| + \varepsilon^{\delta_k}$$

therefore

$$U_k^{-1} U_{k-1}'^{-1} A U'_{k-1} U_k = U_k^{-1}(A_k + F'_k)U_k = A_{k+1} + U_k^{-1}(F_k + F'_k)U_k$$

and then, letting $U'_k = U'_{k-1}U_k$ and $F'_{k+1} = F_k + F'_k$, the induction step is established. By construction, the constructed matrices have the same block decomposition as A.

After at most $\frac{n^2}{2}$ steps, the algorithm stops according to 4.4, since the matrix is zero and is then trivially in normal Jordan form. □

4.3 From reduced echelon to Jordan

We shall conjugate a (nonzero) nilpotent matrix $A \in gl(n,\mathbb{C})$ on reduced column echelon form to Jordan normal form with control on the conjugation. If A is on reduced column echelon form, iterating the following lemma will remove the non zero coefficients above the pivots of A. We denote in the following $A_{i,\cdot}$ the i:th row of A, and $A_{\cdot,l}$ the l:th column of A.

Fix $1 \leq k_0 < i_0 < j_0$ and define $M := M_{k_0,i_0} \in gl(n,\mathbb{C})$ by

$$M_{i,j} = \begin{cases} 0 & (i,j) \neq (k_0,i_0) \\ 1 & (i,j) = (k_0,i_0) \end{cases}$$

Multiplying a matrix A to the left by $I + aM$ amounts to replacing the k_0:th row $A_{k_0,\cdot}$ by $A_{k_0,\cdot} + aA_{i_0,\cdot}$.

Multiplying a matrix A to the right by

$$(I + aM)^{-1} = I - aM$$

amounts to replacing the i_0:th column $A_{\cdot,i_0}$ by $A_{\cdot,i_0} - aA_{\cdot,k_0}$.

Let A be a nilpotent matrix on reduced column echelon form with a pivot

$$A_{i_0,j_0}, \quad i_0 < j_0.$$

Let $k_0 \in \{1, \dots, i_0 - 1\}$ the row index of the last non zero coefficient before the pivot in the j_0:th column of A, let M as above, and

$$B = (I - aM)A(I + aM), \quad a = A_{k_0,j_0}.$$

The following lemma will be used to remove that coefficient from the matrix A.

Lemma 4.9. *The matrix B is on reduced column echelon form with the same pivots as A.*

If the only non-zero coefficients in $A_{\cdot,j}$, $j > j_0$ are the pivots, then

$$B_{\cdot,j} = A_{\cdot,j}, \quad j > j_0$$

and

$$B_{i,j_0} = \begin{cases} A_{i,j_0} & i \neq k_0 \\ 0 & i = k_0 \end{cases}$$

Proof. We have

$$\forall j \neq i_0,\ B_{k_0,j} = A_{k_0,j} - aA_{i_0,j} \quad \& \quad \forall i \neq k_0,\ B_{i,i_0} = A_{i,i_0} + aA_{i,k_0}$$

and $B_{k_0,i_0} = A_{k_0,i_0}$, and for $i \neq k_0$ and $j \neq j_0$,

$$B_{i,j} = (A - aMA + aAM - a^2MAM)_{i,j} = A_{i,j}.$$

Since $k_0 < i_0$, the column $A_{\cdot,k_0}$ is strictly shorter than $A_{\cdot,i_0}$. The pivot in $B_{\cdot,i_0}$ is therefore the same as in $A_{\cdot,i_0}$.

Since A_{i_0,j_0} is a pivot, we have $A_{i_0,j} = 0$ for all $j < j_0$. Hence

$$B_{k_0,j} = A_{k_0,j} - aA_{i_0,j} = A_{k_0,j}, \quad \forall j < j_0.$$

So if there is a pivot in $B_{k_0,\cdot}$, it is the same as that in $A_{k_0,\cdot}$.

If the only non-zero coefficients in $A_{\cdot,j}$, $j > j_0$, are the pivot, then, since $k_0 < j_0$ and $i_0 < j_0$,

$$B_{k_0,j} = A_{k_0,j} - aA_{i_0,j} = 0 \quad \forall j > j_0.$$

Hence $B_{\cdot,j} = 0$ for all $j > j_0$. Moreover,

$$B_{i,j_0} = \begin{cases} A_{i,j_0} & i \neq k_0 \\ B_{k_0,j_0} = A_{k_0,j_0} - aA_{i_0,j_0} = 0 & i = k_0 \end{cases}.$$

□

Now we can conjugate to the Jordan normal form:

Proposition 4.10. *Let A be a non zero block diagonal nilpotent matrix, with each block on reduced column echelon form. There exists $S \in Gl(n, \mathbb{C})$,*

$$\|S^{\pm 1}\| \leq C(1 + \|A\|)^{n!},$$

such that $S^{-1}AS$ is on Jordan normal form and S are also block diagonal with the same block decomposition as A. The constant C only depends on n.

Proof. We start with the column A^n and apply the lemma 4.9 to each coefficient above the pivot. This gives a $S_1 \in Gl(n, \mathbb{C})$,

$$\|S_1^{\pm 1}\| \leq C_1(1 + \|A\|)^{n-1},$$

such that

$$A_1 = S_1^{-1}AS_1$$

diagonal on reduced column echelon form with the same pivots as A, and whose only non-zero coefficient in the last column is the pivot.

Then we do the same with the next to last column in A_1, and so on and so forth. This stops after at most n steps producing a $S \in Gl(n, \mathbb{C})$,

$$\|S^{\pm 1}\| \leq C_n(1 + \|A\|)^{n!},$$

such that

$$B = S^{-1}AS$$

is block diagonal on reduced column echelon form with the same pivots as A, and whose only non-zero coefficient are the pivots. By construction the block decomposition is the same as A.

Since there are only finitely many such matrices, they can be conjugated to Jordan normal form with (uniform) bound on the conjugation and its inverse.

□

4.4 Jordan normal form with estimates

Proposition 4.11. *Let N be a non zero nilpotent block diagonal matrix. For any $\varepsilon \in (0, 1)$ and $m \in \mathbb{N}^*$, there exists $S \in Gl(n, \mathbb{C})$ block diagonal, a constant $C > 0$ only depending on n and constants $c > 0, c' \in (0, \frac{1}{2}]$ only depending on n and m (in particular, they do not depend on ε) such that*

$$\|S^{\pm 1}\| \leq C(\|N\| + 1)^c \varepsilon^{-\frac{1}{2(m+2)}} \tag{4.5}$$

$$S^{-1}NS = A' + F',$$

with A' on Jordan normal form and

$$\|S^{\pm 1}\|^m \|F'\| \leq (C(\|N\| + 1)^c)^{m+1} \varepsilon^{c'}. \tag{4.6}$$

Moreover, S and A' have the same block diagonal decomposition as N.

Proof. Let $\varepsilon \in (0,1)$ and $m \geq 1$. Let $\delta_0 = 1$, $c_n = 2(n+1)(2n)! + n$, and for all $k \geq 1$,

$$\delta_{k-1} = 2(m+2)c_n\delta_k.$$

Apply Proposition 4.8 to N with this choice of (δ_k). This gives a $1 \leq k \leq n^2$ and a block diagonal matrix $S_1 \in Gl(n,\mathbb{C})$,

$$\|S_1^{\pm 1}\| \leq C(\|N\|+1)^{\frac{n}{2}}\varepsilon^{-\frac{n}{2}\delta_k}$$

such that

$$S_1^{-1}NS_1 = N' + G,$$

with N' on reduced column echelon form and

$$\|G\| \leq C(\|N\|+1)^n\varepsilon^{-n\delta_k+\delta_{k-1}}.$$

Moreover, S_1 and N' are block diagonal with the same block structure as N.
Apply now Proposition 4.10 to each block of N' to find a block diagonal $S_2 \in Gl(n,\mathbb{C})$ satisfying

$$\|S_2^{\pm 1}\| \leq C(1+\|N'\|)^{n!}$$

such that the block diagonal matrix $A' = S_2^{-1}N'S_2$ is on Jordan normal form. Then we have, with $S = S_1S_2$,

$$S^{-1}NS = S_2^{-1}(N'+G)S_2 = A' + S_2^{-1}GS_2.$$

Moreover, S and A' are block diagonal with the same block structure as N.

The estimates:

$$\begin{aligned}\|N'\| = \|S_1^{-1}NS_1 - G\| \leq \\ \leq C(\|N\|+1)^n\varepsilon^{-n\delta_k}\|N\| + C(\|N\|+1)^n\varepsilon^{-n\delta_k+\delta_{k-1}} \\ \leq C(\|N\|+1)^{n+1}\varepsilon^{-n\delta_k}\end{aligned}$$

Hence

$$\|S_2^{\pm 1}\| \leq C(\|N\|+1)^{(n+1)n!}\varepsilon^{-n\cdot n!\delta_k}.$$

In particular

$$\|S^{\pm 1}\| \leq C(\|N\|+1)^{(n+1)!+\frac{n}{2}}\varepsilon^{-n\cdot n!\delta_k-\frac{n}{2}\delta_k} \leq C(\|N\|+1)^{c_n}\varepsilon^{-c_n\delta_k}$$

(recall $c_n = 2(n+1)(2n)!+n$) and by the definition of (δ_k), $c_n\delta_k \leq \frac{1}{2(m+2)}$ so (4.5) holds. Moreover,

$$\begin{aligned}\|S_2^{-1}GS_2\| \leq C(\|N\|+1)^{2(n+1)!+n}\varepsilon^{-2n\cdot n!\delta_k-n\delta_k}\varepsilon^{\delta_{k-1}} \\ \leq C(\|N\|+1)^{c_n}\varepsilon^{-c_n\delta_k+\delta_{k-1}}.\end{aligned}$$

Hence

$$\|S^{\pm 1}\|^m\|S_2^{-1}GS_2\| \leq C^{m+1}(\|N\|+1)^{(m+1)c_n}\varepsilon^{-(m+1)c_n\delta_k+\delta_{k-1}}$$

$$\leq (C(\|N\|+1)^{c_n})^{m+1}\varepsilon^{(m+1)c_n\delta_k},$$

so (4.6) holds with $c = c_n$ and $c' = (m+1)c_n\delta_k$. ∎

5 An almost reducibility result to a Jordan normal form

In this section, we construct an almost conjugation to a cocycle which is in Jordan normal form. A control of the estimates requires to do it in two steps.

Proposition 5.1. *Let $X_{\omega,A}$ an almost reducible cocycle in $\mathcal{C}^\infty$. If ω is Diophantine, then there exist sequences $(Z_j), (F_j), (B_j)$ such that every B_j is block diagonal, each block being upper triangular with only one eigenvalue,*

$$\partial_\omega Z_j = AZ_j - Z_j(B_j + F_j)$$

and for all $m \in \mathbb{N}$,

$$\|Z_j^{\pm 1}\|_{\mathcal{C}^m}^m \|F_j\|_{\mathcal{C}^m} \to 0. \tag{5.1}$$

Proof. We will construct such a sequence for a fixed m, and then the lemma 8.10 will imply the conclusion.

Fix $m \in \mathbb{N}\backslash\{0\}$ and define the parameters

$$\beta = \frac{1}{4n^3}, \gamma_1 = \frac{1}{4(16mn^3)^n},$$

and for $i \geq 1$, the increasing sequence

$$\gamma_{i+1} = 16mn^3\gamma_i.$$

Estimate of B_j The conjugation relation can be written as

$$B_j = Z_j^{-1}(AZ_j - \partial_\omega Z_j) - F_j$$

and then

$$\|B_j\| \leq C_{A,\omega,d}\|Z_j\|_{\mathcal{C}^1}\|Z_j^{-1}\|_{\mathcal{C}^0} + \|F_j\|_{\mathcal{C}^r} \leq C'_{A,\omega,d}\|Z_j\|_{\mathcal{C}^1}\|Z_j^{-1}\|_{\mathcal{C}^0},$$

with $C_{A,\omega,d}$ and $C'_{A,\omega,d}$ depending only on A, ω and d.
The assumption

$$\lim_{j\to+\infty} \|Z_j^{\pm 1}\|_{\mathcal{C}^r}^m \|F_j\|_{\mathcal{C}^r} = 0, \quad \forall r, m \in \mathbb{N}$$

implies that for J_1 large enough, for all $j \geq J_1$,

$$(\|Z_j\|_{\mathcal{C}^m}\|Z_j^{-1}\|_{\mathcal{C}^m})^{\frac{1}{\beta}}\|F_j\|_{\mathcal{C}^m} \leq {C'_{A,\omega,d}}^{-\frac{1}{\beta}}$$

therefore for all $j \geq J_1$,

$$\|B_j\| \leq \|F_j\|_{\mathcal{C}^0}^{-\beta}. \tag{5.2}$$

From now on, we will work with $j \geq J_1$.

Construction of a matrix similar to B_j, block diagonal with separated spectrum Let $\varepsilon_j = \|F_j\|_{\mathcal{C}^m}$. From Corollary 8.5 given in appendix and applied to B_j and $\Gamma_i = \varepsilon_j^{\gamma_i}$, there exists $d_0 \in \{1, \dots, n\}$ such that we can conjugate B_j to a matrix D_j which is block diagonal, with each block being upper triangular with $\varepsilon_j^{\gamma_{d_0}}$-connected spectrum (as defined in 8.2), and if we denote by $M_j \in GL(n, \mathbb{C})$ the conjugation (so that $D_j = M_j^{-1} B_j M_j$ and $G_j = M_j^{-1} F_j M_j$), then

$$\|M_j\|, \|M_j^{-1}\| \leq n^{3n} (\frac{\|B_j\|}{\varepsilon_j^{2\gamma_{d_0-1}}})^{n^3}. \tag{5.3}$$

Writing $D_j + G_j = \hat{B}_j + \hat{F}_j$, where $\hat{F}_j$ is obtained from D_j by adding a diagonal with coefficients smaller than $n\|F_j\|_{\mathcal{C}^m}^{\gamma_{d_0}}$, and $\hat{B}_j$ is block diagonal, with blocks upper triangular having only one eigenvalue, we obtain a conjugation

$$M_j^{-1}(B_j + F_j)M_j = \hat{B}_j + \hat{F}_j$$

with the estimate

$$\|\hat{F}_j\|_{\mathcal{C}^m} \leq \|G_j\|_{\mathcal{C}^m} + n\|F_j\|_{\mathcal{C}^m}^{\gamma_{d_0}} \leq \|M_j\|\|M_j^{-1}\|\varepsilon_j + n\varepsilon_j^{\gamma_{d_0}} \leq C_n \varepsilon_j^{1-2\beta n^3 - 4n^3\gamma_{d_0-1}} + n\varepsilon_j^{\gamma_{d_0}} \leq C_n \varepsilon_j^{\gamma_{d_0}}$$

(where C_n only depends on n). Then

$$\|Z_j^{\pm 1}\|_{\mathcal{C}^m}^m \|M_j\|^m \|M_j^{-1}\|^m \|\hat{F}_j\|_{\mathcal{C}^m} \leq \|Z_j^{\pm 1}\|_{\mathcal{C}^m}^m n^{6mn^3} (\frac{\|B_j\|}{\varepsilon_j^{2\gamma_{d_0-1}}})^{2mn^3} \varepsilon_j^{\gamma_{d_0}}$$

$$\leq C(m,n)\|Z_j^{\pm 1}\|_{\mathcal{C}^m}^m \varepsilon_j^{-4mn^3\gamma_{d_0-1} - 2mn^3\beta + \gamma_{d_0}}$$

By the choice of β and the sequence γ_i, the exponent on ε_j is positive, therefore the almost reducibility assumption implies that this quantity tends to 0. □

Proposition 5.2. *Let $X_{\omega,A}$ an almost reducible cocycle in C^∞. If ω is Diophantine, then it is almost reducible to a sequence (B_j) of matrices that are in Jordan normal form.*

Proof. By Proposition 5.1, there exist sequences $(Z_j), (F_j), (B_j)$ such that

$$\partial_\omega Z_j = AZ_j - Z_j(B_j + F_j)$$

where the convergence condition (5.1) holds, and the matrices B_j are block diagonal, every block being upper triangular with only one eigenvalue.

Fix $m \in \mathbb{N}$. Define $\beta = \frac{c'}{2c(m+2)}$ where $c > 0, c' \in (0, \frac{1}{2}]$ are given by the proposition 4.11 and depend only on n, m.
Reasoning similarly as in the proof of Proposition 5.1, there exists an index J_2 such that if $j \geq J_2$, then

$$\|B_j\| \leq \|F_j\|_{\mathcal{C}^m}^{-\beta}.$$

We shall apply Proposition 4.11 with N being the matrix B_j without its diagonal coefficients, and $\varepsilon = \varepsilon_j = \|F_j\|_{\mathcal{C}^m}$.
Let $A' = A'_j, F' = F'_j, S = S_j$ given by proposition 4.11 (in particular S_j block diagonal with the same block structure as B_j). Therefore, since the diagonal part of B_j commutes with S_j (recall

that B_j has only one eigenvalue for each diagonal block), $B_j + F_j$ is conjugate via S_j to $\tilde{B}_j + \tilde{F}_j$, such that $\tilde{B}_j$ is in Jordan normal form and $\tilde{F}_j = F'_j + S_j^{-1} F_j S$ with

$$\|S_j^{\pm 1}\| \leq C(\|B_j\| + 1)^c \varepsilon_j^{-\frac{1}{2(m+2)}} \leq C(m,n)\varepsilon_j^{-\beta c - \frac{1}{2(m+2)}} \tag{5.4}$$

and

$$\|S_j^{\pm 1}\|^m \|F'_j\| \leq (C(\|B_j\| + 1)^c)^{m+1} \varepsilon_j^{c'} \leq C(m,n)\varepsilon_j^{-c\beta(m+1)+c'}. \tag{5.5}$$

Therefore

$$\begin{aligned} \|S_j^{\pm 1}\|^m \|\tilde{F}_j\|_{\mathcal{C}^m} &\leq C(m,n)\varepsilon_j^{-c\beta(m+1)+c'} + \|S_j^{\pm 1}\|^{m+2} \|F_j\|_{\mathcal{C}^m} \\ &\leq C(m,n)\varepsilon_j^{-c\beta(m+1)+c'} + C(m,n)\varepsilon_j^{-(m+2)(\beta c + \frac{1}{2(m+2)})+1} \end{aligned} \tag{5.6}$$

Thus

$$\begin{aligned} &\|Z_j^{\pm 1}\|_{\mathcal{C}^m}^m \|S_j^{\pm 1}\|^m \|\tilde{F}_j\|_{\mathcal{C}^m} \\ &\leq C(m,n)\|Z_j^{\pm 1}\|_{\mathcal{C}^m}^m (\varepsilon_j^{-c\beta(m+1)+c'} + \varepsilon_j^{-(m+2)(\beta c + \frac{1}{2(m+2)})+1}) \end{aligned}$$

The choice of the parameter β implies that the exponent on ε_j on the right hand side will be positive, thus the convergence condition holds for fixed m. Applying Lemma **??**, there is almost reducibility to the sequence $(\tilde{B}_j)$. □

6 Construction of a conjugation to a real matrix

In this section, $B = \mathrm{diag}(B_j)_{j=1}^l$ will be a block diagonal matrix where each block B_j is on Jordan normal form with only one eigenvalue α_j. The spectrum of B is

$$\sigma(B) = \{\alpha_1, \ldots, \alpha_l\} \quad (\#\sigma(B) = l).$$

We shall study the equation

$$F = \partial_\omega V - BV + V\bar{B}. \tag{6.1}$$

where $V : \mathbb{T}^d \to Gl(n, \mathbb{C})$ and $\omega \in \mathcal{DC}(\kappa, \tau)$, and where F is supposed to be "small".
This equation decomposes into its block-components

$$F_i^j = \partial_\omega V_i^j - B_i V_i^j + V_i^j \bar{B}_j \tag{6.2}$$

for each i, j.

6.1 (N, ρ)-linkedness

In this section, we will study resonances between the eigenvalues of the matrix B. This way, we want to create sets of eigenvalues with same cardinality, linked by resonances. Fix $N \in \mathbb{N}, \rho > 0$.

Definition 6.1. Two complex numbers α and β are $(N, \rho) -$ linked if and only if

$$|i2\pi\langle k, \omega\rangle - (\alpha_i - \bar{\alpha}_j)| < \rho$$

for some $|k| \leq N$.

Let Γ be a finite set of complex numbers with $0 < \#\Gamma \leq n$.

Definition 6.2. An $(N, \rho) -$ chain in Γ of length $r-1$ is a sequence

$$\alpha_1, \alpha_2, \ldots, \alpha_{r-1}, \alpha_r$$

in Γ such that α_j and α_{j+1} are (N, ρ)- linked for all j. The numbers α_1 and α_r are then said to be $(N, \rho) -$ chain $-$ linked.
An $(N, \rho) -$ loop in Γ is an (N, ρ)- *chain*

$$\alpha_1, \alpha_2, \ldots, \alpha_{r-1}, \alpha_r$$

such that $\alpha_1 = \alpha_r$. An $(N, \rho) -$ loop is odd if it is of odd length.

It is easy to verify that if α and β are (N, ρ)-chain-linked, then they are (N, ρ)-chain-linked by a chain of length $\leq n$.

Lemma 6.3. *Let*

$$\alpha_1, \alpha_2, \ldots, \alpha_{r-1}, \alpha_r$$

be an $(N, \rho) -$ *chain in* Γ.

- *If* $r-1$ *is odd, then* α_1 *and* α_r *are* $(nN, n\rho) -$ *linked.*
- *If* $r-1$ *is even, then* α_1 *and* $\bar{\alpha}_r$ *are* $(nN, n\rho) -$ *linked.*

Proof. We can assume without restriction that $r - 1 \leq n$.
We have for all j and for some k_j,

$$|i2\pi\langle k_j, \omega\rangle - (\alpha_j - \bar{\alpha}_{j+1})| \leq \rho, \quad |k_j| \leq N,$$

Let $k = \sum_{j \text{ odd}} k_j - \sum_{j \text{ even}} k_j$, then

$$|k| \leq nN.$$

If $r-1$ is odd, then

$$\alpha_1 - \bar{\alpha}_r = (\alpha_1 - \bar{\alpha}_2) + (\bar{\alpha}_2 - \alpha_3) + (\alpha_3 - \bar{\alpha}_4) + \cdots + (\alpha_{r-1} - \bar{\alpha}_r)$$

so

$$i2\pi\langle k, \omega\rangle - (\alpha_1 - \bar{\alpha}_r) = \sum_{j \text{ odd}} \Big(i2\pi\langle k_j, \omega\rangle - (\alpha_j - \bar{\alpha}_{j+1})\Big) + \sum_{j \text{ even}} \Big(i2\pi\langle -k_j, \omega\rangle - (\bar{\alpha}_j - \alpha_{j+1})\Big).$$

This implies that

$$|i2\pi\langle k, \omega\rangle - (\alpha_1 - \bar{\alpha}_r)| \leq n\rho.$$

If $r-1$ is even, then

$$\alpha_1 - \alpha_r = (\alpha_1 - \bar{\alpha}_2) + (\bar{\alpha}_2 - \alpha_3) + (\alpha_3 - \bar{\alpha}_4) + \cdots + (\bar{\alpha}_{r-1} - \alpha_r)$$

so

$$i2\pi\langle k, \omega\rangle - (\alpha_1 - \alpha_r) = \sum_{j \text{ odd}} \Big(i2\pi\langle k_j, \omega\rangle - (\alpha_j - \bar{\alpha}_{j+1})\Big) + \sum_{j \text{ even}} \Big(i2\pi\langle -k_j, \omega\rangle - (\bar{\alpha}_j - \alpha_{j+1})\Big).$$

This implies that

$$|i2\pi\langle k, \omega\rangle - (\alpha_1 - \alpha_r)| \leq n\rho.$$

□

We shall assume that Γ is such that

$$\text{for any } \alpha \in \Gamma, \text{ there is a } \beta \text{ such that } \alpha \text{ and } \beta \text{ are } (\mathrm{N}, \rho) - \text{linked.} \tag{6.3}$$

Then "being (N, ρ)-chain-linked" is an equivalence relation and we denote by

$$[\alpha]$$

the equivalence class of $\alpha \in \Gamma$ – it depends on (N, ρ).

Lemma 6.4. *If $[\alpha]$ contains an odd (N, ρ)-loop, then any $\beta \in [\alpha]$ is $((2n-1)N, (2n-1)\rho)$-linked to itself.*

Proof. Consider an odd (N, ρ)-loop

$$\alpha_1, \alpha_2, \ldots, \alpha_{r-1}, \alpha_r = \alpha_1, \quad r \geq 2.$$

If $r - 1 \geq n + 1$, then there exist $1 \leq i < j \leq r - 1$ such that $\alpha_i = \alpha_j$. Then

$$\alpha_i, \alpha_{i+1}, \ldots, \alpha_j = \alpha_i$$

is an (N, ρ)-loop of length $j - i$. Moreover

$$\alpha_j, \alpha_{j+1}, \ldots, \alpha_{r-1}, \alpha_r = \alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i = \alpha_j$$

is an (N, ρ)-loop of length $(r - j) + (i - 1)$.
Since

$$(j - i) + (r - j) + (i - 1) = r - 1$$

is odd, one of these two "sub-loops" must be odd. So there exists a shorter odd (N, ρ)-loop, and we conclude that there exists an odd (N, ρ)-loop of length $< r - 1$, i.e. we can assume that $r - 1 \leq n$.
Let now $\beta \in [\alpha]$. Then there is an (N, ρ)-chain connecting β to the odd (N, ρ)-loop

$$\alpha_1, \alpha_2, \ldots, \alpha_{r-1}, \alpha_r = \alpha_1,$$

i.e.

$$\beta = \beta_1, \beta_2, \ldots, \beta_{s-1}, \beta_s = \alpha_j, \quad s - 1 \leq n - (r - 1)$$

We can without restriction assume that $j = 1$. Then

$$\beta = \beta_1, \ldots, \beta_{s-1}, \beta_s = \alpha_1, \alpha_2, \ldots, \alpha_r = \alpha_1 = \beta_s, \beta_{s-1}, \ldots, \beta_2, \beta_1 = \beta$$

is an (N, ρ)-loop of length

$$(s - 1) + (r - 1) + (s - 1) \leq 2(n - r + 1) + (r - 1) = 2n - r + 1 \leq 2n - 1.$$

This length is odd, thus by Lemma 6.3, β is $(nN, n\rho)$-linked to itself. □

Lemma 6.5. *If $[\alpha]$ contains no odd (N, ρ)-loops, then there exists a partition $[\alpha] = \Sigma_1 \cup \Sigma_2$ such that*

(*i*) *β and γ are not (N,ρ)-linked if $\beta,\gamma\in\Sigma_1$, and the same holds for Σ_2,*
(*ii*) *β and $\bar\gamma$ are $(nN,n\rho)$-linked if $\beta,\gamma\in\Sigma_1$, and the same holds for Σ_2,*
(*iii*) *β and γ are $(nN,n\rho)$-linked if $\beta\in\Sigma_1$ and $\gamma\in\Sigma_2$.*

Proof. Fix $\alpha\in\Gamma$ and define

$$\Sigma_1=\{\beta\in[\alpha]:\text{there exists a }(N,\rho)-\text{chain of even length between }\alpha\text{ and }\beta\}\cup\{\alpha\}$$

and

$$\Sigma_2=\{\beta\in[\alpha]:\text{there exists a }(N,\rho)-\text{chain of odd length between }\alpha\text{ and }\beta\}.$$

Notice that Σ_2 contains all elements that are (N,ρ)-linked to α. In particular, $\Sigma_2\neq\emptyset$, by assumption (6.3) on Γ.
Notice also that $\Sigma_1\cap\Sigma_2=\emptyset$, because if not, then $[\alpha]$ would contain an odd (N,ρ)-loop.

Proof of (i): Two elements in Σ_1 cannot be (N,ρ)-linked to each other, because then there would be an odd (N,ρ)-loop in $[\alpha]$. Idem for Σ_2.
Proof of (ii): Any two elements in Σ_1 are linked by an even (N,ρ)-chain (by transversality). Idem for Σ_2. Therefore (ii) follows from Lemma 6.3.
Proof of (iii): Any element in Σ_1 are linked to any element of Σ_2 by a chain of odd length (by transversality). Therefore (iii) follows from Lemma 6.3. □

6.2 Analysis of resonances

Lemma 6.6. *If for some $0<\rho\leq 1$,*

$$|i2\pi\langle k,\omega\rangle-(\alpha_i-\bar\alpha_j)|\geq\rho,$$

then

$$\|\hat V_i^j(k)\|\leq(\frac{2}{\rho})^{2n-1}\|F\|_{\mathcal{C}^0}.$$

Proof. From (6.2) we have

$$(i2\pi\langle k,\omega\rangle-(\alpha_i-\bar\alpha_j))\hat V_i^j(k)=N_i\hat V_i^j(k)-\hat V_i^j(k)N_j+\hat F_i^j(k),$$

where

$$N_i=B_i-\alpha_iI\quad\text{and}\quad N_j=B_j-\alpha_jI.$$

Changing notations, we can write this as

$$\gamma X=N_iX-XN_j+Y=\mathcal{L}X+Y.$$

The operator $\mathcal{L}$ verifies for all matrix X in the domain of $\mathcal{L}$,

$$\|\mathcal{L}X\|\leq 2\|X\|\quad\&\quad\mathcal{L}^{2n-1}=0.$$

Then

$$X=\frac{1}{\gamma}\big(Y+\mathcal{L}X\big)=\frac{1}{\gamma}Y+\frac{1}{\gamma^2}\mathcal{L}Y+\frac{1}{\gamma^3}\mathcal{L}^2X+\dots$$

$$= \sum_{0\le j\le 2n-2} \frac{1}{\gamma^{j+1}} \mathcal{L}^j Y.$$

Hence, since $\gamma \ge \rho$,

$$\|X\| \le \sum_{0\le j\le 2n-2} \frac{1}{\rho^{j+1}} 2^j \|Y\| \le \frac{1}{2-\rho} (\frac{2}{\rho})^{2n-1} \|Y\|.$$

□

Lemma 6.7. *Assume*

1.
$$\|V\|_{\mathcal{C}^0} + \|V^{-1}\|_{\mathcal{C}^0} \le \xi$$

2. *V is a trigonometric polynomial of degree $\le N$*

3.
$$\rho < (2N)^{-\tau}\kappa$$

4.
$$\|F\|_{\mathcal{C}^0} \le (4^n n!(3N)^d \xi^n)^{-1} \rho^{2n-1}.$$

Then, for any i, there exists a j and a unique $|k_{i,j}| \le N$ such that

$$|i2\pi\langle k_{i,j}, \omega\rangle - (\alpha_i - \bar{\alpha}_j)| < \rho.$$

Proof. Given i, suppose that

$$|i2\pi\langle k, \omega\rangle - (\alpha_i - \bar{\alpha}_j)| \ge \rho$$

for all j and all $|k| \le N$. By Lemma 6.6,

$$\|\hat{V}_i^j(k)\| \le (\frac{2}{\rho})^{2n-1} \|F\|_{\mathcal{C}^0}$$

so

$$\|V_i^j\|_{\mathcal{C}^0} \le (3N)^d (\frac{2}{\rho})^{2n-1} \|F\|_{\mathcal{C}^0} = \varepsilon.$$

This implies that

$$\|\det V\|_{\mathcal{C}^0} \le n!\varepsilon\xi^{n-1}.$$

Since

$$\begin{aligned} \partial_\omega V^{-1} - \bar{B}V^{-1} + V^{-1}B &= V^{-1}\partial_\omega V V^{-1} - \bar{B}V^{-1} + V^{-1}B \\ &= V^{-1}(BV - V\bar{B} + F)V^{-1} - \bar{B}V^{-1} + V^{-1}B \\ &= V^{-1}FV^{-1} \end{aligned} \tag{6.4}$$

we find, by a similar reasoning, that

$$\|\det(V^{-1})\|_{\mathcal{C}^0} \le n!\varepsilon\xi^2\xi^{n-1}.$$

Hence

$$1 = \| \det(VV^{-1})\|_{\mathcal{C}^0} \leq (n!\varepsilon\xi^n)^2$$

which is forbidden by assumption 4.

Uniqueness: Suppose there exists $k \neq l$ such that $|k|,\ |l| \leq N$ and

$$|i2\pi\langle k, \omega\rangle - (\alpha_i - \bar{\alpha}_j)| < \rho \quad \& \quad |i2\pi\langle l, \omega\rangle - (\alpha_i - \bar{\alpha}_j)| < \rho.$$

Then, since $\omega \in DC(\kappa, \tau)$,

$$2\pi\frac{\kappa}{(2N)^\tau} \leq |i2\pi\langle k - l, \omega\rangle| \leq 2\rho$$

which is forbidden by assumption 3. □

Corollary 6.8. Under assumptions $1-4$ of Lemma 6.7, being (N, ρ)-chain-linked is an equivalence relation on $\sigma(B)$.

Indeed these assumptions imply that the condition (6.3) holds for $\sigma(B)$.

6.3 The Jordan structure of B

Assumption: From now on, in this section, we assume properties $1 - 4$.

Let

$$[\alpha] \subseteq \sigma(B)$$

be an equivalence class that contains no odd (N, ρ)-loop. By Lemma 6.5 there exists a partition $[\alpha] = \Sigma_1 \cup \Sigma_2$ such that

(*i*) β and γ are not (N, ρ)-linked if $\beta, \gamma \in \Sigma_1$, and the same holds for Σ_2,
(*ii*) β and $\bar{\gamma}$ are $(nN, n\rho)$-linked if $\beta, \gamma \in \Sigma_1$, and the same holds for Σ_2,
(*iii*) β and γ are $(nN, n\rho)$-linked if $\beta \in \Sigma_1$ and $\gamma \in \Sigma_2$.

Let $\Sigma_3 = \sigma(B) \setminus [\alpha]$ and define

$$X^{\Sigma_v}_{\Sigma_u} = \left(X^j_i\right)^{\alpha_j \in \Sigma_v}_{\alpha_i \in \Sigma_u}, \quad u, v = 1, 2, 3,$$

for any matrix X – we often write X_{Σ_u} for the diagonal block $X^{\Sigma_u}_{\Sigma_u}$.

Lemma 6.9. *If*

$$\|F\|_{\mathcal{C}^0} \leq \frac{1}{4^n 3n(3N)^d\xi}\rho^{2n-1}$$

then $V^{\Sigma_2}_{\Sigma_1}$ is an invertible square-matrix and

$$\|(V^{\Sigma_2}_{\Sigma_1})^{-1} - (V^{-1})^{\Sigma_1}_{\Sigma_2}\|_{\mathcal{C}^0} \leq 6n(3N)^d\xi^2(\frac{1}{\rho})^{2n-1}\|F\|_{\mathcal{C}^0}.$$

Proof. We can assume without restriction that

$$\Sigma_1 = \{\alpha_1, \dots, \alpha_r\}, \quad \Sigma_2 = \{\alpha_{r+1}, \dots, \alpha_{r+s}\}$$

so

$$V = \left(U_u^v\right)_{u,v=1,2,3}, \quad U_u^v = V_{\Sigma_u}^{\Sigma_v}.$$

By Lemma 6.6, for any $|k| \leq N$,

$$\|\hat{V}_i^j(k)\| \leq (\frac{2}{\rho})^{2n-1}\|F\|_{\mathcal{C}^0} = \varepsilon \quad \text{if } \begin{cases} \alpha_i \in \Sigma_1, \alpha_j \in \Sigma_1 \cup \Sigma_3 \\ \text{or } \alpha_i \in \Sigma_2, \alpha_j \in \Sigma_2 \cup \Sigma_3 \\ \text{or } \alpha_i \in \Sigma_3, \alpha_j \in \Sigma_1 \cup \Sigma_2 \end{cases}$$

which implies that

$$\|\hat{U}_u^v(k)\| \leq n\varepsilon \quad \text{if } \begin{cases} u = 1 & v = 1, 3 \\ u = 2 & v = 2, 3 \\ u = 3 & v = 1, 2. \end{cases}$$

Hence

$$\|U_u^v\|_{\mathcal{C}^0} \leq n(3N)^d\varepsilon \quad \text{if } \begin{cases} u = 1 & v = 1, 3 \\ u = 2 & v = 2, 3 \\ u = 3 & v = 1, 2. \end{cases}$$

Let now

$$W = \begin{pmatrix} 0 & U_1^2 & 0 \\ U_2^1 & 0 & 0 \\ 0 & 0 & U_3^3 \end{pmatrix}.$$

Since

$$\|W - V\|_{\mathcal{C}^0} \leq 3n(3N)^d\varepsilon = \delta$$

we get that W is invertible and

$$\|W^{-1} - V^{-1}\|_{\mathcal{C}^0} \leq \sum_{j\geq 1} \|V^{-1}(W - V)\|_{\mathcal{C}^0}^j \|V^{-1}\|_{\mathcal{C}^0} \leq \sum_{j\geq 1} (\delta\xi)^j \xi$$

Now if

$$\delta\xi \leq \frac{1}{2} \quad \iff \quad \varepsilon \leq \frac{1}{6n(3N)^d\xi} \quad \iff \quad \|F\|_{\mathcal{C}^0} \leq \frac{1}{4^n 3n(3N)^d\xi}\rho^{2n-1}$$

(which holds by assumption), then

$$\|W^{-1} - V^{-1}\|_{\mathcal{C}^0} \leq \frac{1}{1 - \delta\xi}\delta\xi^2 \leq 2\delta\xi^2 = 6n(3N)^d\varepsilon\xi^2 = 6n(3N)^d(\frac{2}{\rho})^{2n-1}\xi^2\|F\|_{\mathcal{C}^0}$$

Finally, by a computation,

$$W^{-1} = \begin{pmatrix} 0 & (U_2^1)^{-1} & 0 \\ (U_1^2)^{-1} & 0 & 0 \\ 0 & 0 & (U_3^3)^{-1} \end{pmatrix},$$

thus the estimate on $(V_{\Sigma_1}^{\Sigma_2})^{-1} - (V^{-1})_{\Sigma_2}^{\Sigma_1}$ holds. □

Lemma 6.10. *There exists a constant C – only depending on n and d – such that if*

$$\begin{cases} \|F\|_{\mathcal{C}^0} < \frac{1}{CN^d\xi^n}\rho^{2n-1} \\ \rho < \frac{1}{CN^d\xi^{n+1}}, \end{cases} \tag{6.5}$$

then B_{Σ_1} and B_{Σ_2} have the same Jordan structure, i.e. for any $k \geq 1$, they have the same number of Jordan blocks of dimension k.

Proof. From (6.2) we have

$$(i2\pi\langle k,\omega\rangle - (\alpha_i - \bar{\alpha}_j))\hat{V}_i^j(k) = N_i\hat{V}_i^j(k) - \hat{V}_i^j(k)N_j + \hat{F}_i^j(k), \quad \forall 1 \leq i,j \leq l,\ \forall k \in \mathbb{Z}^d,$$

where

$$N_i = B_i - \alpha_i I \quad \text{and} \quad N_j = B_j - \alpha_j I.$$

If

$$|i2\pi\langle k,\omega\rangle - (\alpha_i - \bar{\alpha}_j)| \geq \rho,$$

then, by Lemma 6.6,

$$\|\hat{V}_i^j(k)\| \leq (\frac{2}{\rho})^{2n-1}\|F\|_{\mathcal{C}^0}.$$

If

$$|i2\pi\langle k,\omega\rangle - (\alpha_i - \bar{\alpha}_j)| < \rho,$$

then

$$\|N_i\hat{V}_i^j(k) - \hat{V}_i^j(k)N_j\| \leq \rho\xi + \|F\|_{\mathcal{C}^0}.$$

Hence

$$\|N_i\hat{V}_i^j(k) - \hat{V}_i^j(k)N_j\| \leq \max(2(\frac{2}{\rho})^{2n-1}\|F\|_{\mathcal{C}^0}, \rho\xi + \|F\|_{\mathcal{C}^0}) = \varepsilon$$

and

$$\|N_iV_i^j - V_i^jN_j\|_{\mathcal{C}^0} \leq (3N)^d\varepsilon.$$

This implies that

$$\|N_{\Sigma_1}V_{\Sigma_1}^{\Sigma_2} - V_{\Sigma_1}^{\Sigma_2}N_{\Sigma_2}\|_{\mathcal{C}^0} \leq n(3N)^d\varepsilon.$$

By Lemma 6.9, the bound on $\|F\|_{\mathcal{C}^0}$ implies

$$\|V_{\Sigma_1}^{\Sigma_2}\|_{\mathcal{C}^0}, \quad \|(V_{\Sigma_1}^{\Sigma_2})^{-1}\|_{\mathcal{C}^0} \ \leq \ 2\xi,$$

so, by Proposition 8.8, N_{Σ_1} and N_{Σ_2}, hence B_{Σ_1} and B_{Σ_2}, have the same Jordan structure if

$$n(3N)^d\varepsilon < \frac{1}{n\cdot n!2^n\xi^n} \quad \iff \quad \varepsilon < \frac{1}{n^2\cdot n!2^n(3N)^d\xi^n}.$$

This holds if

$$\|F\|_{\mathcal{C}^0} < \frac{1}{n^2\cdot n!8^n(3N)^d\xi^n}\rho^{2n-1}$$

and

$$\rho < \frac{1}{n^2\cdot n!2^{n+1}(3N)^d\xi^{n+1}}.$$

□

6.4 Construction of W and conjugation to a real matrix.

Lemma 6.11. *Assume that V is a trigonometric polynomial of degree N; let $\xi \geq 0$ such that*

$$\|V\|_{\mathcal{C}^0} + \|V^{-1}\|_{\mathcal{C}^0} \leq \xi. \tag{6.6}$$

There exists a constant C which only depends on n, d such that if

$$\|F\|_{\mathcal{C}^0} \leq \frac{\rho^{2n-1}}{CN^d\xi^n}, \tag{6.7}$$

and

$$\rho \leq \frac{1}{C}\min(\frac{1}{N^d\xi^{n+1}}, \frac{\kappa}{N^\tau}),$$

then there exist $W : \mathbb{T}^d \to Gl(n, \mathbb{C})$ of class $\mathcal{C}^\infty$, $B' \in gl(n, \mathbb{R})$ and $B'' \in gl(n, \mathbb{C})$ such that

1. $$\partial_{\frac{\omega}{2}} W = BW - W(B' + B''),$$

2. *W commutes with B, B' and B'', and B' has the same diagonal block structure as B,*

3. $$\|B''\| \leq 2n\rho,$$

4. *for all $r \in \mathbb{R}$,*
$$\|W^{\pm 1}\|_{\mathcal{C}^r} \leq (4n\pi N)^r,$$

5. *there is the estimate*
$$\|B'\| \leq C\|B\|.$$

Proof. By Corollary 6.8, $\sigma(B)$ satisfies the property (6.3). Let $[\alpha] \subset \sigma(B)$ be an equivalence class.

- Case 1: if $[\alpha]$ contains an odd (N, ρ)-loop, then by Lemma 6.3, for all $\alpha_i \in [\alpha]$, there exists $k_i \in \mathbb{Z}^d$ with $|k_i| \leq (2n-1)N$ such that

$$|\alpha_i - \bar{\alpha}_i - 2i\pi\langle k_i, \omega\rangle| \leq (2n-1)n\rho.$$

- Case 2: if $[\alpha]$ does not contain any odd (N, ρ)-loop, let $[\alpha] = \Sigma_1 \cup \Sigma_2$ be the partition given by the Lemma 6.5. Choose arbitrarily $\alpha_0 \in \Sigma_1 \cap [\alpha]$. Then for all $\alpha_i \in \Sigma_2 \cap [\alpha]$, by Lemma 6.3, there exists $k_i \in \mathbb{Z}^d$, $|k_i| \leq nN$ such that

$$|\alpha_i - \bar{\alpha}_0 - 2i\pi\langle k_i, \omega\rangle| \leq n\rho.$$

  And for all $\alpha_i \in \Sigma_1 \cap [\alpha]$, there exists $k_i \in \mathbb{Z}^d$, $|k_i| \leq 2nN$ such that

$$|\alpha_i - \alpha_0 - 2i\pi\langle k_i, \omega\rangle| \leq 2n\rho.$$

  and if $\alpha_i = \alpha_0$ then $k_i = 0$;

Then we construct $W \in \mathcal{C}^0(\mathbb{T}^d, Gl(n,\mathbb{C}))$ the diagonal matrix whose diagonal coefficients are (w_j) defined as follows: given an eigenvalue $\alpha_i \in \sigma(B)$ associated with k_i as defined before, if α_i appears on line j in B, then

$$w_j(\theta) = e^{2i\pi\langle k_i,\theta\rangle} I$$

in case (1) and

$$w_j(\theta) = e^{4i\pi\langle k_i,\theta\rangle} I$$

in case (2).

Then it holds that

$$\partial_{\frac{\omega}{2}} W = BW - W(B' + B'')$$

where the coefficients of $B' + B''$ will be defined as follows:

- in case (1): Since

$$\alpha_i - i\pi\langle k_i,\omega\rangle = \frac{1}{2}(\alpha_i + \bar{\alpha}_i) + \frac{1}{2}(\alpha_i - \bar{\alpha}_i - 2i\pi\langle k_i,\omega\rangle),$$

  one can define $\Re\alpha_i$ as the coefficient of B' and $\frac{1}{2}(\alpha_i - \bar{\alpha}_i - 2i\pi\langle k_i,\omega\rangle)$ as the coefficient of B'';
- in case (2), if $\alpha_i \in \Sigma_2 \cap [\alpha]$,

$$\alpha_i - 2i\pi\langle k_i,\omega\rangle = \bar{\alpha} + (\alpha_i - \bar{\alpha} - 2i\pi\langle k_i,\omega\rangle)$$

  Then $\bar{\alpha}$ is the coefficient of B' and $\alpha_i - \bar{\alpha} - 2i\pi\langle k_i,\omega\rangle$ is the coefficient of B''. If $\alpha_i \in \Sigma_1 \cap [\alpha]$,

$$\alpha_i - 2i\pi\langle k_i,\omega\rangle = \alpha + (\alpha_i - \alpha - 2i\pi\langle k_i,\omega\rangle)$$

  Then α is the coefficient of B' and $\alpha_i - \alpha - 2i\pi\langle k_i,\omega\rangle$ is the coefficient of B''.

Therefore, if α is an eigenvalue of B', then $\bar{\alpha}$ is also an eigenvalue with the same multiplicity. Moreover, by Lemma 6.10, the blocks with eigenvalues α and $\bar{\alpha}$ have the same Jordan structure. Thus, by Lemma 8.9, one can assume up to a unitary transformation that B' is in real Jordan normal form.
With our choices of the k_i, the values of W commute with B, B' and B''. The norm of W and the norm of W^{-1} follows from the fact that $|k_i| \leq (2n-1)N$.

The matrix B'' is bounded by $(2n-1)\rho$. Moreover, the coefficients of the matrix B' are the same as those of B outside the diagonal. The diagonal coefficients of B' are in $\sigma(B) \cup \overline{\sigma(B)} \cup \Re\sigma(B)$, which implies

$$\|B'\| \leq C(n)\|B\|.$$

□

6.5 Application of the main lemma 6.11

To apply the result of the previous section, it will be necessary to have an estimate of the truncation of an application U satisfying $U^{-1} = \bar{U}$, which will be obtained in the following lemma.

Lemma 6.12. *Let $N \in \mathbb{N}$. Let $U : \mathbb{T}^d \to Gl(n, \mathbb{C})$ of class $\mathcal{C}^\infty$ such that*

$$U^{-1} = \bar{U},$$

and let

$$V(\theta) = \mathcal{T}_N U(\theta) := \sum_{|k| \leq N} \hat{U}(k) e^{2i\pi\langle k, \theta\rangle}.$$

There exists a constant $C_d > 0$ depending only on d such that if N satisfies

$$C_d \|U\|_{\mathcal{C}^{d+1}} \|U\|_{\mathcal{C}^0} \leq N, \tag{6.8}$$

then

$$\|V^{-1} - \bar{V}\|_{\mathcal{C}^0} \leq \frac{1}{4} \|V\|_{\mathcal{C}^0}. \tag{6.9}$$

Proof. We have

$$\|\hat{U}(k)\| \leq (\frac{1}{2\pi|k|})^{d+1} \|U\|_{\mathcal{C}^{d+1}}, \quad k \in \mathbb{Z}^d,$$

which implies

$$\|V - U\|_{\mathcal{C}^0} \leq \sum_{|k|>N} \|\hat{U}(k)\| \leq \frac{C}{2\pi N} \|U\|_{\mathcal{C}^{d+1}} =: \sigma_N. \tag{6.10}$$

where C depends only on d. Therefore

$$\begin{aligned}
\|\bar{V}U - I\|_{\mathcal{C}^0} &\leq \|\bar{V} - U^{-1}\|_{\mathcal{C}^0} \|U\|_{\mathcal{C}^0} \\
&\leq \|\bar{V} - \bar{U}\|_{\mathcal{C}^0} \|U\|_{\mathcal{C}^0} \quad \text{from } U^{-1} = \bar{U} \\
&\leq \|V - U\|_{\mathcal{C}^0} \|U\|_{\mathcal{C}^0} \\
&\leq \sigma_N \|U\|_{\mathcal{C}^0} \\
&\leq \frac{1}{2} \qquad (*_1)
\end{aligned}$$

The last line of the equation is satisfied by hypothesis (6.8) with $C_d \geq \frac{C}{\pi}$. This implies that, for all θ, $X(\theta) = \bar{V}(\theta)U(\theta)$ is invertible and

$$X^{-1} = (I + (\bar{V}U - I))^{-1} = \sum_{k \geq 0} (-1)^k (\bar{V}U - I)^k$$

hence

$$\|X^{-1} - I\|_{\mathcal{C}^0} \leq \sum_{k \geq 1} \|(\bar{V}U - I)\|_{\mathcal{C}^0}^k \leq 2\|V - U\|_{\mathcal{C}^0} \|U\|_{\mathcal{C}^0} \leq 2\sigma_N \|U\|_{\mathcal{C}^0}.$$

Therefore

$$\begin{aligned}
\bar{V}^{-1} &= UU^{-1}\bar{V}^{-1} = UX^{-1} = U + U(X^{-1} - I) \\
&= \bar{U}^{-1} + U(X^{-1} - I) \quad \text{since } \bar{U} = U^{-1} \\
&= \bar{X}^{-1}V + U(X^{-1} - I) \\
&= V + (\bar{X}^{-1} - I)V + U(X^{-1} - I),
\end{aligned}$$

and then

$$V^{-1} - \bar{V} = (X^{-1} - I)\bar{V} + \bar{U}(\bar{X}^{-1} - I)$$

which gives

$$\begin{aligned}\|V^{-1}-\bar{V}\|_{\mathcal{C}^0} &= \|(X^{-1}-I)\bar{V}+\bar{U}(\bar{X}^{-1}-I)\|_{\mathcal{C}^0}\\ &\leq \|X^{-1}-I\|_{\mathcal{C}^0}(\|V\|_{\mathcal{C}^0}+\|U\|_{\mathcal{C}^0})\\ &\leq 2\sigma_N\|U\|_{\mathcal{C}^0}(\|V\|_{\mathcal{C}^0}+\|U\|_{\mathcal{C}^0})\\ &\leq 4\sigma_N\|U\|_{\mathcal{C}^0}(\|V\|_{\mathcal{C}^0}+\frac{1}{2}\sigma_N)\end{aligned}$$

(by definition of σ_N), therefore, using (6.8) with $C_d \geq \frac{16C}{\pi}$,

$$\|V^{-1}-\bar{V}\|_{\mathcal{C}^0} \leq \frac{1}{8}(\|V\|_{\mathcal{C}^0}+\frac{\sigma_N}{2}) \leq \frac{1}{8}\|V\|_{\mathcal{C}^0}+\frac{1}{16}\sigma_N. \tag{6.11}$$

Now (6.8) also implies

$$\sigma_N \leq \frac{1}{32\|U\|_{\mathcal{C}^0}}$$

and the property that $U^{-1}=\bar{U}$ implies that $\|U\|_{\mathcal{C}^0} \geq 1$, so $3\sigma_N \leq 2\|U\|_{\mathcal{C}^0}$ also holds. This, together with (6.10), implies that

$$\sigma_N \leq 2(\|U\|_{\mathcal{C}^0}-\sigma_N) \leq 2(\|U\|_{\mathcal{C}^0}-\|V-U\|_{\mathcal{C}^0}) \leq 2\|V\|_{\mathcal{C}^0}.$$

Thus (6.11) implies that (6.9) holds. □

The following proposition allows us to apply the previous lemma to the change of variables given by the almost reducibility, in order to conjugate the almost reducible cocycle to a cocycle arbitrarily close to a real one.

Proposition 6.13. *Let $N \in \mathbb{N}$, $\rho > 0$. Let $U : \mathbb{T}^d \to Gl(n,\mathbb{C})$ of class $\mathcal{C}^{d+1}$ such that*

$$U^{-1}=\bar{U}.$$

Let $B \in gl(n,\mathbb{C})$ in Jordan normal form, and let

$$G = \partial_\omega U - BU + U\bar{B}.$$

There exists a constant $C_1 \geq 0$ (depending only on n,d) such that if

1. $$C_1\|U\|_{\mathcal{C}^{d+1}}\|U\|_{\mathcal{C}^0} \leq N, \tag{6.12}$$

2. $$\|G\|_{\mathcal{C}^{d+1}} \leq C_1^{-1}\rho^{2n-1}N^{-(d-1)}\|U\|_{\mathcal{C}^0}^{-n}, \tag{6.13}$$

3. $$\rho \leq C_1^{-1}\min(\frac{1}{N^d\|U\|_{\mathcal{C}^0}^{n+1}}, \frac{\kappa}{N^\tau}), \tag{6.14}$$

then there exist $W \in \mathcal{C}^\infty(\mathbb{T}^d, Gl(n,\mathbb{C}))$, $B' \in gl(n,\mathbb{C})$, *and* $B'' \in gl(n,\mathbb{C})$ *such that*

$$\partial_{\frac{\omega}{2}} W = BW - W(B' + B'')$$

where B' *is as in Lemma 6.11. Moreover,*

$$\|B''\| \leq 2n^2\rho,$$

and

$$\forall s \in \mathbb{N},\ \|W^{\pm 1}\|_{\mathcal{C}^s} \leq (2\pi n N)^{2s}.$$

Proof. Denote $\xi = \|U\|_{\mathcal{C}^0} \geq 1$ (the lower bound comes from $U^{-1} = \bar{U}$). Let

$$V = \mathcal{R}_N U = \sum_{|k| \leq N} \hat{U}(k) e^{2i\pi\langle k,\theta\rangle}.$$

We have

$$\partial_\omega V = BV - V\bar{B} + F, \quad F = \mathcal{R}_N G.$$

We will now apply lemma 6.11 and for this, we need to find an upper bound of $\|V\|_{\mathcal{C}^0}$ and an upper bound of $\|V^{-1}\|_{\mathcal{C}^0}$. We have

$$\|V - U\|_{\mathcal{C}^0} \leq \sum_{|k|>N} \|\hat{U}(k)\| \leq C_2 \frac{1}{N} \|U\|_{\mathcal{C}^{d+1}}$$

where $C_2 \geq 1$ only depends on d. Suppose that $C_1 \geq 32C_2$ and that $N \geq C_1\|U\|_{\mathcal{C}^{d+1}}\|U\|_{\mathcal{C}^0}$, then

$$\|V - U\|_{\mathcal{C}^0} \leq C_2 \frac{1}{N} \frac{N}{C_1\|U\|_{\mathcal{C}^0}} \leq \frac{1}{2\xi}.$$

Therefore since $\xi \geq 1$,

$$\|V\|_{\mathcal{C}^0} \leq \|U\|_{\mathcal{C}^0} + \|V - U\|_{\mathcal{C}^0} \leq \xi + \frac{1}{2\xi} \leq 2\xi.$$

Let C_d the constant given by lemma 6.12. If moreover $C_1 \geq C_d$, we have

$$N \geq C_d \|U\|_{\mathcal{C}^{d+1}} \|U\|_{\mathcal{C}^0},$$

and we can apply lemma 6.12 which gives

$$\|V^{-1}\|_{\mathcal{C}^0} \leq \|V^{-1} - \bar{V}\|_{\mathcal{C}^0} + \|\bar{V}\|_{\mathcal{C}^0} \leq \frac{1}{4}\|V\|_{\mathcal{C}^0} + \|V\|_{\mathcal{C}^0} \leq 3\xi,$$

and the assumption (6.6) of lemma 6.11 is satisfied with 4ξ instead of ξ. So,

$$\|F\|_{\mathcal{C}^0} \leq \|G\|_{\mathcal{C}^0} + \|F - G\|_{\mathcal{C}^0} \leq \|G\|_{\mathcal{C}^0} + C_2\left(\frac{1}{2\pi N}\right)\|G\|_{\mathcal{C}^{d+1}}$$

$$\leq 2C_2\|G\|_{\mathcal{C}^{d+1}} \leq 2C_2C_1^{-1}\rho^{2n-1}\Big(N\|U\|_{\mathcal{C}^0}\Big)^{-dn^2}$$

so, if C_1 is large enough depending on n, d, the assumption (6.7) of Lemma 6.11 holds since

$$\partial_\omega V = BV - V\bar{B} + F, \quad F = \mathcal{R}_N G.$$

Therefore one can apply the Lemma 6.11 which directly gives the conclusions. ∎

7 Proof of the main result

Theorem 7.1. *Let $X_{\omega,A}$ an almost reducible cocycle in $\mathcal{C}^\infty$. If $X_{\omega,A}$ is real and if ω is Diophantine, then $X_{\frac{\omega}{2},A_2}$ is real almost reducible in $\mathcal{C}^\infty$, where $A_2(\theta) = A(2\theta)$ for all $\theta \in \mathbb{T}^d$.*

Proof. By the proposition 5.2, there exist $Z_j : \mathbb{T}^d \to Gl(n,\mathbb{C})$ of class $\mathcal{C}^\infty$, $B_j \in gl(n,\mathbb{C})$ in Jordan normal form and $F_j \in C^\infty(\mathbb{T}^d, gl(n,\mathbb{C}))$ such that

$$\forall j,\ \partial_\omega Z_j = AZ_j - Z_j(B_j + F_j)$$

$$\|Z_j^{\pm 1}\|_{\mathcal{C}^r}^m \|F_j\|_{\mathcal{C}^r} \to_{j\to\infty} 0, \quad \forall r, m \in \mathbb{N}. \tag{7.1}$$

Fix $r \geq d+1$ and denote $\|F_j\|_{\mathcal{C}^r} = \varepsilon_j$. By a similar reasoning as when proving (5.2), and from the convergence condition (7.1), given $\beta' < \min(\frac{1}{2+8n^3\tau}, \frac{1}{16n^6})$, there exists J_3 such that, for all $j \geq J_3$,

$$\|Z_j^{\pm 1}\|_{\mathcal{C}^r} \leq \varepsilon_j^{-\beta'}. \tag{7.2}$$

Let $U_j = {Z_j}^{-1}\bar{Z}_j$. Then $U_j^{-1} = \bar{U}_j$ and

$$\xi_j = \|U_j\|_{\mathcal{C}^0} = \|U_j^{-1}\|_{\mathcal{C}^0} \geq 1,$$

and

$$\|U_j\|_{\mathcal{C}^r} \leq \varepsilon_j^{-2\beta'}$$

Moreover

$$\begin{aligned}
\partial_\omega U_j &= \partial_\omega({Z_j}^{-1}\bar{Z}_j) \\
&= \partial_\omega({Z_j}^{-1})\bar{Z}_j + {Z_j}^{-1}\partial_\omega\bar{Z}_j \\
&= ((B_j + F_j){Z_j}^{-1} - {Z_j}^{-1}A)\bar{Z}_j + {Z_j}^{-1}(\overline{AZ_j - Z_j(B_j + F_j)}) \\
&= B_j{Z_j}^{-1}\bar{Z}_j + F_j{Z_j}^{-1}\bar{Z}_j - {Z_j}^{-1}\bar{Z}_j\bar{B}_j - {Z_j}^{-1}\bar{Z}_j\bar{F}_j + {Z_j}^{-1}(\bar{A} - A)\bar{Z}_j
\end{aligned}$$

and since A is real,

$$\partial_\omega U_j = B_jU_j - U_j\bar{B}_j + G_j$$

where

$$G_j = F_jU_j - U_j\bar{F}_j$$

with, from inequality (1.6) given in introduction,

$$\|G_j\|_{\mathcal{C}^r} \leq 2C_r\|U_j\|_{\mathcal{C}^r}\|F_j\|_{\mathcal{C}^r}.$$

Let

$$\begin{aligned}
\varepsilon_j' &= 2C_r\|U_j\|_{\mathcal{C}^r}\|F_j\|_{\mathcal{C}_r} \\
N_j &= C_1\|U_j\|_{\mathcal{C}^{d+1}}\|U_j\|_{\mathcal{C}^0}
\end{aligned}$$

with C_1 the constant of proposition 6.13. We will apply Proposition 6.13 with $N = N_j, \rho = \rho_j = {\varepsilon_j'}^{\frac{1}{2n^3}}$, $U = U_j, B = B_j, G = G_j$. There are three assumptions to check. The assumption (6.12) holds by definition of N.
By definition of β', we have

$$\rho_j\|U_j\|_{\mathcal{C}^0}^{n+1}(2N_j)^\tau \to_{j\to+\infty} 0,$$

and there exists J_4 (which we can choose $\geq J_3$) such that for all $j \geq J_4$,

$$\kappa > C_1 \rho_j \|U_j\|_{\mathcal{C}^0}^{n+1} (2N_j)^\tau,$$

therefore the assumption (6.14) of Proposition 6.13 holds. Moreover, by definition of N_j and ε'_j,

$$\begin{aligned}\rho^{-(2n-1)} (N_j \|U_j\|_{\mathcal{C}^0})^{dn^2} \varepsilon'_j &\leq (\varepsilon'_j)^{1-\frac{2n-1}{2n^3}} (C_1 \varepsilon_j^{-6\beta'})^{dn^2} \\ &\leq C (\varepsilon_j^{-2\beta'})^{1-\frac{2n-1}{2n^3}+3dn^2} \varepsilon_j^{1-\frac{2n-1}{2n^3}}\end{aligned}$$

where C is a constant depending only on d, n, r, and the right hand side tends to 0 as j tends to infinity. Therefore there exists J_5 ($\geq J_4$) such that, for $j \geq J_5$,

$$\|G_j\|_{\mathcal{C}^r} \leq \varepsilon'_j \leq C_1^{-1} \rho^{2n-1} (N_j \|U_j\|_{\mathcal{C}^0})^{-dn^2}$$

therefore the assumption (6.13) of Proposition 6.13 holds. Apply the proposition 6.13: there exist $W_j : \mathbb{T}^d \to Gl(n, \mathbb{C})$ of class $\mathcal{C}^\infty$, $B'_j \in gl(n, \mathbb{R})$ and $B''_j \in gl(n, \mathbb{C})$ such that B'_j has the same block diagonal structure as B_j, with

$$\partial_{\frac{\omega}{2}} W_j = B_j W_j - W_j (B'_j + B''_j),$$

$$\begin{aligned}\|B''_j\| \leq 2n\rho = 2n{\varepsilon'_j}^{\frac{1}{2n^3}} \leq C_{n,r} \|U_j\|_{\mathcal{C}^r}^{\frac{1}{2n^3}} \varepsilon_j^{\frac{1}{2n^3}}, \\ \|W_j^{\pm 1}\|_{\mathcal{C}^s} \leq (2\pi n N_j)^{2s}, \quad \forall s \in \mathbb{N},\end{aligned} \tag{7.3}$$

Moreover, denoting $G_j = B''_j + W_j^{-1} F_j W_j$, we get

$$\begin{aligned}\|G_j\|_{\mathcal{C}_r} &\leq C_{n,r} \|U_j\|_{\mathcal{C}^r}^{\frac{1}{2n^3}} \varepsilon_j^{\frac{1}{2n^3}} + (2\pi n N_j)^{4r} \varepsilon_j \\ &\leq C_{n,r} \|U_j\|_{\mathcal{C}^r}^{\frac{1}{2n^3}} \varepsilon_j^{\frac{1}{2n^3}} + (32 C_1 n)^{4r} \|U_j\|_{\mathcal{C}^{d+1}}^{4r} \|U_j\|_{\mathcal{C}^0}^{4r} \varepsilon_j \\ &\leq C'_{r,n} \|U_j\|_{\mathcal{C}^r}^{8r} \varepsilon_j^{\frac{1}{2n^3}}.\end{aligned}$$

Finally, for our fixed m and $r \geq d+1$,

$$\begin{aligned}\|(Z_j W_j)^{\pm 1}\|_{\mathcal{C}^r}^m \|G_j\|_{\mathcal{C}^r} &\leq C''_{r,n} \|W_j^{\pm 1}\|_{\mathcal{C}^r}^m \|Z_j^{\pm 1}\|_{\mathcal{C}^r}^m \|U_j\|_{\mathcal{C}^r}^{8r} \varepsilon^{\frac{1}{2n^3}} \\ &\leq C'''_{r,n} N_j^{4rm} \|Z_j^{\pm 1}\|_{\mathcal{C}^r}^m \|U_j\|_{\mathcal{C}^r}^{8r} \varepsilon^{\frac{1}{2n^3}} \\ &\leq C''''_{r,n} \|U_j\|_{\mathcal{C}^r}^{8rm+8r} \|Z_j^{\pm 1}\|_{\mathcal{C}^r}^m \varepsilon^{\frac{1}{2n^3}} \\ &\leq C'''''_{r,n} \|Z_j^{\pm 1}\|_{\mathcal{C}^r}^{16rm+16r+m} \varepsilon^{\frac{1}{2n^3}}\end{aligned}$$

which tends to 0 as $j \to \infty$ by assumption (7.1).

This construction depends on m, r; however, applying Lemma 8.10, one gets almost reducibility to a sequence of real matrices.

The transformations $Z_j W_j$ are not real, but we can now apply Proposition 3.6. □

8 Appendix

8.1 A small divisors lemma

The following classical lemma is useful to control the small divisors which might occur:

Lemma 8.1. *Let $f : \mathbb{T}^d \to \mathbb{C}$ of class $\mathcal{C}^\infty$ and $N \in \mathbb{N}^*$ such that*

$$\hat{f}(k) = 0, \quad |k| > N.$$

Let $\rho > 0$. If $\alpha \in \mathbb{C}$ is such that

$$|\alpha - 2i\pi\langle k, \omega\rangle| \geq \rho, \quad \forall |k| \leq N,$$

then the equation

$$\begin{cases} \partial_\omega u(\theta) = \alpha u(\theta) + f(\theta), & \theta \in \mathbb{T}^d, \\ \hat{u}(k) = 0, & \text{if } |k| > N \end{cases} \tag{$*$}$$

has a unique solution $u : \mathbb{T}^d \to \mathbb{C}$ of class $\mathcal{C}^\infty$ and it satisfies

$$\|u\|_{\mathcal{C}^0} \leq C\rho^{-1} N^{\frac{d+1}{2}} \|f\|_{\mathcal{C}^0}$$

where C is a constant depending on d.

Proof. Decomposing the first equation $(*)$ into Fourier coefficients, we have

$$2i\pi\langle k, \omega\rangle \hat{u}(k) = \alpha \hat{u}(k) + \hat{f}(k), \quad k \in \mathbb{Z}^d$$

and the unique solution of $(*)$ is defined by

$$\hat{u}(k) = \begin{cases} \frac{1}{2i\pi\langle k,\omega\rangle - \alpha} \hat{f}(k), & \forall |k| \leq N \\ 0, & \forall |k| > N. \end{cases}$$

By Hölder's inequality,

$$\|u\|_{\mathcal{C}^0} \leq \sum_k |\hat{u}(k)| \leq \frac{1}{\rho} \sum_{|k| \leq N} |\hat{f}(k)|$$

$$\leq \frac{1}{\rho} \sqrt{\sum_{|k|\leq N} 1} \sqrt{\sum_{|k|\leq N} |\hat{f}(k)|^2} \leq C \frac{1}{\rho} N^{\frac{d+1}{2}} \|f\|_{L^2} \leq C \frac{1}{\rho} N^{\frac{d+1}{2}} \|f\|_{\mathcal{C}^0},$$

where C depends only on d. □

8.2 Separating the spectrum of a matrix

Definition 8.2 (Γ-separation, Γ-connection)**.** Let E_1, E_2 two finite sets of complex numbers and let $\Gamma > 0$. We say that E_1 and E_2 are Γ-separated if for all $\alpha \in E_1$ and for all $\beta \in E_2$, $|\alpha - \beta| > \Gamma$.

We say that $E_1 = \{\alpha_1, \ldots, \alpha_n\}$ is Γ-connected if there is no decomposition $E_1 = E_1' \cup E_1''$ with E_1' and E_1'' Γ-separated.

Remark 8.3. If E_1 is Γ-connected, then for all $\alpha, \beta \in E_1$, $|\alpha - \beta| \leq \#E_1 \cdot \Gamma$.

Lemma 8.4. *Let $B \in gl(n,\mathbb{C})$ and $\Gamma > 0$. There exists $\tilde{M} \in Gl(n,\mathbb{C})$ such that $\tilde{M}^{-1}B\tilde{M}$ is block diagonal, with each block being upper triangular with Γ-connected spectrum, and such that the spectrum of two distinct blocks are Γ-separated. Moreover $\tilde{M}$ satisfies*

$$\|\tilde{M}\| \leq n^{3n}(\frac{\|B\|}{\Gamma^2})^{n^3}$$

$$\|\tilde{M}^{-1}\| \leq n^{3n}(\frac{\|B\|}{\Gamma^2})^{n^3},$$

and

$$\|\tilde{M}^{-1}B\tilde{M}\| \leq C(n)B$$

where $C(n)$ is a constant depending on n.

Proof. **Step 1:** By Schur decomposition, there exist an unitary matrix $Q \in Gl(n,\mathbb{C})$ and an upper triangular matrix $T \in gl(n,\mathbb{C})$ such that

$$B = QTQ^{-1}$$

Therefore $\|B\| = \|T\|$.

Step 2: Let $E_1, \ldots, E_l$ be Γ-separated sets of eigenvalues of B, each of them being Γ-connected, and write (up to a permutation, which doesn't change the estimates)

$$B = \begin{pmatrix} T_1 & T_2 \\ 0 & T_4 \end{pmatrix}, \quad D = \begin{pmatrix} T_1 & 0 \\ 0 & T_4 \end{pmatrix}$$

where T_1 is the block corresponding to the eigenvalues of E_1 and T_4 is the block corresponding to the eigenvalues of $E_2 \cup \cdots \cup E_l$. Write

$$M = \begin{pmatrix} I & R \\ 0 & I \end{pmatrix}, M^{-1} = \begin{pmatrix} I & -R \\ 0 & I \end{pmatrix},$$

then the conjugation $MBM^{-1} = D$ is equivalent to

$$T_1R + T_2 = RT_4$$

where R is the unknown. Let n_1 be the dimension of T_1 and n_2 the dimension of T_4. Decomposing to coefficients, the previous matrix equation can be written

$$-(T_2)_{i,j} = \sum_{k=i}^{n_1}(T_1)_{i,k}R_{k,j} - \sum_{l=1}^{j} R_{i,l}(T_4)_{l,j}$$

Solve these equations in the following order:

$$(i,j) = (n_1,1),(n_1,2),\ldots,(n_1,n_2),(n_1-1,1),\ldots,(n_1-1,n_2),\ldots,(1,1),(1,2),\ldots,(1,n_2).$$

Therefore, the coefficients of R have upper bound $(n_1n_2)(\frac{\|B\|}{\Gamma^2})^{n_1n_2}$ which implies that

$$\|M\| \leq n(n_1n_2)(\frac{\|B\|}{\Gamma^2})^{n_1n_2}.$$

Iterate step 2 replacing B by T_4 and T_4 by the blocks corresponding to $E_{l-k} \cup \cdots \cup E_l$. The algorithm stops when T_4 only has the eigenvalues of E_l, that is to say after at most $n-1$ steps.

Finally there exist an invertible matrix $\tilde{M}$ of dimension n, a block diagonal matrix $\tilde{D} = (\tilde{D}_1, \ldots, \tilde{D}_l)$ of dimension n, with each block T_i corresponding to the group of eigenvalues E_i (which remains Γ-separated).

$$\tilde{D} = \tilde{M} B \tilde{M}^{-1}$$

with

$$\|\tilde{M}\|, \|\bar{M}^{-1}\| \leq n^{3n} (\frac{\|B\|}{\Gamma^2})^{n^3}.$$

Since $\tilde{D}$ is constructed from B by removing coefficients, its norm is $\leq C(n)\|B\|$ where $C(n)$ is a constant depending on n. □

Corollary 8.5. Let $A \in gl(n, \mathbb{C})$. Given a positive decreasing sequence (Γ_i), there exists $d_0 \in \mathbb{N}$, $d_0 \leq n$, depending on A, and there exists $B \in gl(n, \mathbb{C})$ a block diagonal matrix where each block is upper-triangular with Γ_{d_0}-connected spectrum, and S invertible such that

$$A = SBS^{-1}$$

$$\|S\|, \|S^{-1}\| \leq n^{3n} (\frac{\|B\|}{\Gamma_{d_0-1}^2})^{n^3},$$

and

$$\|A\| \leq C(n)\|B\|$$

where $C(n)$ is a constant depending on n.

Proof. By induction on Γ_i: **Base case:** first apply lemma 8.4 on A with $\Gamma = \Gamma_0$. This gives a conjugation S from A to a matrix B where each block is upper triangular with Γ_0-connected spectrum. If the spectrum of every block of B is also Γ_1-connected, then we are done, if not, apply lemma 8.4 to A with Γ_1.
Induction step: given $d \in \mathbb{N}^*$, apply lemma 8.4 on A with Γ_d. This gives a conjugation S from A to a block diagonal matrix B where each block is upper triangular with Γ_d-connected spectrum. If the spectrum of every block of B is also Γ_{d+1}-connected, then we are done, if not, apply lemma (8.4) to A with Γ_{d+1}.

After each step, the number of eigenvalues in each block decreases (because you split a block at each step). Then there exists an index d_0 such that applying lemma (8.4) on A with Γ_{d_0-1}, this gives B and S where the spectrum of every block of B is Γ_{d_0}-connected and the matrix S satisfies the estimate with Γ_{d_0-1}. □

Remark 8.6. If $\#\sigma(A) = n_0$, then if the induction is applied n_0 times, each block contains only one eigenvalue. Also notice that the induction does not change A, whence the exponents in the estimate.

8.3 Stability of the Jordan structure

Let $A = (a_i^j)_{1\le i,j\le n}$ and $B = (b_i^j)_{1\le i,j\le n}$ be two matrices with entries 0 and 1 such that for some $r \in \{1,\dots,n-1\}$,

$$a_i^j = b_i^j = 0 \quad \text{if } j - i \neq r.$$

Lemma 8.7. *If there exists an invertible matrix C,*

$$\|C\|, \|C^{-1}\| \le \xi$$

such that

$$\|AC - CB\| = \varepsilon < \frac{1}{n!\xi^n},$$

then $\mathrm{rank}A = \mathrm{rank}B$.

Proof. Let

$$I = \{i \in [[1, n-r]] : a_i^{i+r} = 1\} \quad \& \quad J = \{j \in [[r+1, n]] : b_{j-r}^j = 1\}.$$

Then

$$(AC - CB)_i^j = a_i^{i+r}c_{i+r}^j - c_i^{j-r}b_{j-r}^j = \begin{cases} c_{i+r}^j & \text{if } i \in I \ \&\ j \notin J \\ -c_i^{j-r} & \text{if } i \notin I \ \&\ j \in J. \end{cases}$$

thus

$$\begin{cases} |c_{i+r}^j| \le \varepsilon & \text{if } i \in I \ \&\ j \notin J \\ |c_i^{j-r}| \le \varepsilon & \text{if } i \notin I \ \&\ j \in J. \end{cases}$$

It follows that if $\#I \neq \#J$,

$$|\det C| \le n!\varepsilon\xi^{n-1}.$$

Now

$$\|C^{-1}A - BC^{-1}\| = \|C^{-1}\Big(AC - CB\Big)C^{-1}\| \le \varepsilon\xi^2$$

so, in the same way as for C,

$$|\det C^{-1}| \le n!\varepsilon\xi^2\xi^{n-1}$$

if $\#I$ is different from $\#J$.
Therefore

$$1 = |\det(CC^{-1})| \le (n!\xi^n\varepsilon)^2$$

which is impossible by assumption. Hence $\#I = \#J$. □

Let now A and B be two nilpotent $n \times n$-matrices on Jordan normal form.

Proposition 8.8. *If there exists an invertible matrix C,*

$$\|C\|, \|C^{-1}\| \le \xi$$

such that

$$\|AC - CB\| = \varepsilon < \frac{1}{n \cdot n!\xi^n},$$

then A and B have the same Jordan structure, i.e. they have the same number of Jordan blocks of dimension k, for all $k = 1, 2, \dots$.

Proof. Let

$$X_k = A^k C - C B^k.$$

Since $\|X_1\| \leq \varepsilon$, it follows by an easy induction that

$$\|X_k\| \leq k\varepsilon \leq (n-1)\varepsilon :$$

- For $k = 1$, this is the assumption;
- For $k \geq 1$, notice

$$X_{k+1} = A^{k+1}C - CB^{k+1} = A^k AC - CB^{k+1} = A^k(CB + X_1) - CB^k B = X_k B + A^k X_1$$

and the estimate follows by induction since A, B have norm ≤ 1.

Then, by the lemma 8.7,

$$\mathrm{rank} A^k = \mathrm{rank} B^k, \quad \forall k \geq 1$$

This implies the statement.

□

8.4 Conjugation to a real matrix whose spectrum is stable by complex conjugation

Lemma 8.9. *Let A a matrix in Jordan normal form, such that if α is an eigenvalue, then $\bar{\alpha}$ is also an eigenvalue and the Jordan blocks of α and $\bar{\alpha}$ are identical. There exists a unitary matrix P such that P^*AP is on real Jordan normal form.*

Proof. It suffices to prove this for $A = \begin{pmatrix} \alpha I + N & 0 \\ 0 & \bar{\alpha} I + N \end{pmatrix}$, where N is a nilpotent Jordan block (of dimension $n \times n$). After the permutation $(e_1, e_2, \ldots, e_{2n}) \mapsto (e_1, e_{n+1}, e_2, e_{n+2}, \ldots, e_{2n-1}, e_{2n})$, the matrix A takes the block triangular form $A = (A_i^j)_{i,j=1,\ldots,n}$ with

$$A_i^j = \begin{cases} U \text{ if } j = i \\ I \text{ if } j - i = 1 \\ 0 \text{ otherwise} \end{cases}$$

and $U = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}$. Let $C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ (which is unitary) and let $P = \mathrm{diag}(P_j)_{j=1,\ldots,n}$, $P_j = C$. Then $P^*AP = (B_i^j)_{i,j=1,\ldots,n}$ with $B_i^j = \begin{pmatrix} \mathrm{Re}\,\alpha & \mathrm{Im}\,\alpha \\ -\mathrm{Im}\,\alpha & \mathrm{Re}\,\alpha \end{pmatrix}$ if $i = j$, $B_i^j = I$ if $j - i = 1$, and $B_i^j = 0$ otherwise. □

8.5 A lemma about almost reducibility

Lemma 8.10. *Assume for all $m' \in \mathbb{N}$, there exist $(Z_{j,m'})_j$ a sequence of $\mathcal{C}^\infty$ maps defined on $\mathbb{T}^d$ and $(F_{j,m'})_j$ a sequence of $\mathcal{C}^\infty$ maps defined on $\mathbb{T}^d$ such that*

$$\|Z_{j,m'}^{\pm 1}\|_{\mathcal{C}^{m'}}^{m'} \|F_{j,m'}\|_{\mathcal{C}^{m'}} \to_{j \to +\infty} 0,$$

then there exists $(\tilde{Z}_j)$ and $(\tilde{F}_j)$, subsequences of $(Z_{j,m'})_j$ and $(F_{j,m'})_j$ extracted from the same indices, such that for all $m, r \in \mathbb{N}$,

$$\|\tilde{Z}_j^{\pm 1}\|_{\mathcal{C}^r}^m \|\tilde{F}_j\|_{\mathcal{C}^r} \to_{j \to +\infty} 0.$$

Proof. For all $m \in \mathbb{N}$, let $j_m \in \mathbb{N}$ such that, for all $j \geq j_m$,

$$\|Z^{\pm 1}_{j_m,m}\|^m_{\mathcal{C}^m} \|F_{j_m,m}\|_{\mathcal{C}^m} \leq \frac{1}{m}$$

and such that the sequence $(j_m)_{m\in\mathbb{N}}$ is strictly increasing. Then, for all j large enough, there exists $m := m(j) \in \mathbb{N}$ such that $j_m \leq j < j_{m+1}$. Denote then, for all j large enough

$$\tilde{F}_j = F_{j,m(j)}, \quad \tilde{Z}_j = Z_{j,m(j)}.$$

Let $\tilde{m} \in \mathbb{N}$. Then for all j large enough such that $m(j) > \tilde{m}$,

$$\|\tilde{Z}^{\pm 1}_j\|^{\tilde{m}}_{\mathcal{C}^{\tilde{m}}} \|\tilde{F}_j\|_{\mathcal{C}^{\tilde{m}}} \stackrel{def}{=} \|Z^{\pm 1}_{j,m(j)}\|^{\tilde{m}}_{\mathcal{C}^{\tilde{m}}} \|F_{j,m(j)}\|_{\mathcal{C}^{\tilde{m}}} \leq \|Z^{\pm 1}_{j,m(j)}\|^{m(j)}_{\mathcal{C}^{\tilde{m}}} \|F_{j,m(j)}\|_{\mathcal{C}^{\tilde{m}}} \leq \frac{1}{m(j)}.$$

Since the constructed sequence $(j_m)_m$ is increasing, the sequence $(m(j))_j$ is also increasing, and then

$$\|\tilde{Z}_j^{\pm 1}\|^{\tilde{m}}_{\mathcal{C}^{\tilde{m}}} \|\tilde{F}_j\|_{\mathcal{C}^{\tilde{m}}} \to 0, \quad j \to \infty. \tag{8.1}$$

Now let $r, m \in \mathbb{N}$. Since (8.1) holds for all $\tilde{m}$, then in particular for $\tilde{m} = \max(r, m)$,

$$\|\tilde{Z}^{\pm 1}_j\|^{\tilde{m}}_{\mathcal{C}^{\tilde{m}}} \|\tilde{F}_j\|_{\mathcal{C}^{\tilde{m}}} \to 0$$

which implies the convergence condition is satisfied.

□

References

[1] A.Avila, R.Krikorian, *Reducibility or nonuniform hyperbolicity for quasiperiodic Schrödinger cocycles*, Annals of Mathematics, 164 (2006), 911-940

[2] A.Bounemoura, C.Chavaudret, S.Liang, *Reducibility of ultra-differentiable quasi-periodic cocycles under an adapted arithmetic condition*, Proceedings of the American Mathematical Society, vol. 151 (2021), n°4, p. 2999-3012

[3] M.Chatal, C.Chavaudret, *Almost reducibility of quasiperiodic SL(2,R)-cocycles in ultradifferentiable classes*, Journal of Differential Equations 356 n.7 (2023):243-288

[4] C. Chavaudret, *Reducibility of Quasi-Periodic Cocycles in Linear Lie Groups*, Ergodic Theory Dynam. Systems, 31 (2010), 741-769

[5] C.Chavaudret, S.Marmi, *Reducibility of cocycles under a Brjuno-Rüssmann arithmetical condition*, Journal of Modern Dynamics 6, n.1, 2012, 59-78

[6] L. H. Eliasson, *Floquet solutions for the* 1*-dimensional quasi-periodic Schrödinger equation*, Comm. Math. Phys, 146 (1992), 447-482

[7] L. H. Eliasson, *Almost reducibility of linear quasi-periodic systems*, Proc. Sympos. Pure Math., 69, Amer. Math. Soc., Providence, RI, (2001), 679-705

[8] L. H. Eliasson, *Ergodic skew-systems on* $\mathbb{T}^d \times SO(3,\mathbb{R})$, in Ergodic Theory Dynam. Systems, 22 (2002), 1429-1449

[9] B. Fayad, R.Krikorian, *Regidity results for quasiperiodic* $SL(2,\mathbb{R})$*-cocycles*, j. Mod. Dyn. , 3 (2009), 497-510

[10] Hou, X., You, J. Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems. Invent. math. 190, 209–260 (2012)

[11] J.You, *Quantitative almost reducibility and its applications*, Proc. int. cong. of math., vol.1, 2018, 2231-2154