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Martin Helmer, Rafael Mohr. A New Algorithm for Whitney Stratification of Varieties. 2024. hal-04663472

HAL Id: hal-04663472

<https://hal.science/hal-04663472v1>

Preprint submitted on 28 Jul 2024

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Faster Computation of Whitney Stratifications and Their Minimization

Martin Helmer and Rafael Mohr

ABSTRACT. We describe two new algorithms for the computation of Whitney stratifications of real and complex algebraic varieties. The first algorithm is a modification of the algorithm of Helmer and Nanda (HN) [7, 9], but is made more efficient by using techniques for equidimensional decomposition rather than computing the set of associated primes of a polynomial ideal at a key step in the HN algorithm. We note that this modified algorithm may fail to produce a minimal Whitney stratification even when the HN algorithm would produce a minimal stratification.

The second algorithm coarsens a given Whitney stratification of a complex variety to the unique minimal Whitney stratification; we refer to this as the *minimization* of a stratification. The theoretical basis for our approach is a classical result of Teissier. To our knowledge this yields the first algorithm for computing a minimal Whitney stratification.

1. Introduction

First introduced by Whitney in [18], the concept of *Whitney stratification* is now an essential tool in the study of singular spaces. This construction provides the basic structure needed to decompose singular spaces into smooth manifolds which join together in a desirable way. In this note we present a new method to compute such stratifications which substantially improves on the state of the art in many cases, building on the recent works of [7, 9] by incorporating ideas used in symbolic algorithms which compute equidimensional decompositions of polynomial ideals, e.g. [2]. We additionally present an algorithm for minimizing a given Whitney stratification.

Our main objects of study will be algebraic varieties defined by systems of polynomial equations f_1, \dots, f_r in the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ over the field of complex numbers \mathbb{C} :

$$\mathbf{V}(f_1, \dots, f_r) := \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid f_1(x) = \dots = f_r(x) = 0\}.$$

To demonstrate the utility, and arguably the necessity, of Whitney stratifications when studying the geometry and topology of singular varieties we consider a concrete example over the real numbers \mathbb{R} for illustrative purposes. Suppose we wish to study the curve in \mathbb{R}^2 defined by the parametric polynomial

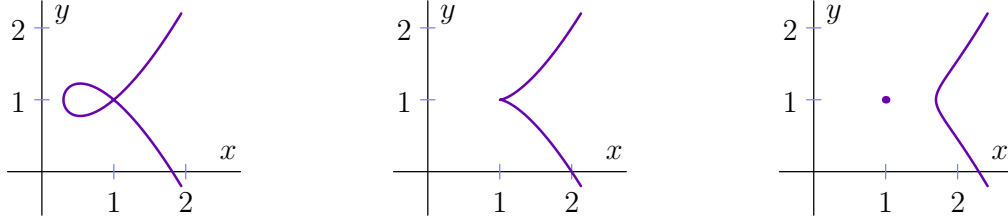
$$f_z(x, y) = (y - 1)^2 - (x - z)(x - 1)^2 \tag{1}$$

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in variables x, y with parameter z . For $z < 1$ the real variety $f_z = 0$ is a (connected) nodal cubic, for $z > 1$ the real variety $f_z = 0$ has two connected components and is smooth, and at $z = 0$ the curve is a cubic cusp, see Figure 1.



(A) When $z < 1$ we obtain a nodal cubic with one loop. (B) When $z = 1$ we obtain a cusp cubic. (C) When $z > 1$, over the reals, we get a curve with two connected components.

FIGURE 1. Plots of the curve defined by (1) for different parameter values z ; the topology of the curve changes at $z = 1$. While the curve in (c) is smooth and has two connected components (of different dimensions) in \mathbb{R}^2 it is connected and singular, with singularity at $(1, 1)$, in \mathbb{C}^2 .

To understand the geometry underlying this change in topology at $z = 1$ for the parametric curve f_z consider the variety $X = \mathbf{V}(f)$ in \mathbb{R}^3 defined by the same polynomial $f(x, y, z) = (y - 1)^2 - (x - z)(x - 1)^2$, plotted in Figure 2. The surface X is singular along the entire line $\text{Sing}(X) = \mathbf{V}(x - 1, y - 1)$ and the projection map $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}$ onto the last coordinate, $(x, y, z) \rightarrow z$, restricted to either the manifold $X - \mathbf{V}(x - 1, y - 1)$ or to the manifold $\mathbf{V}(x - 1, y - 1)$ is a submersion, however the behavior of the fibers of π restricted to $X - \mathbf{V}(x - 1, y - 1)$ depends on whether z is greater than or less than 1, while the behavior of the fibers of the restriction to $\mathbf{V}(x - 1, y - 1)$ is independent of z . *Whitney's Condition (B)* (see Definition 2.8) fails for the pair of manifolds $X - \mathbf{V}(x - 1, y - 1)$ and $\mathbf{V}(x - 1, y - 1)$ at the point $(1, 1, 1)$. The failure of condition (B) at this point captures the fact that a fundamental change in the local geometry of points on $\mathbf{V}(x - 1, y - 1)$ relative to the surface X occurs at the point $(1, 1, 1)$. As it turns out, this in particular allows us to know exactly when the topology of the fibers of the map π change.

More precisely a classical result known as *Thom's Isotopy Lemma* [12, Proposition 11.1] allows us to partition the codomain of (proper) maps into regions of constant topology; Whitney's condition is a critical component of this construction as illustrated by the example above¹.

Effective algorithms to compute Whitney stratifications of real and complex algebraic varieties, and of algebraic maps between them, were developed in [7, 8, 9]. These algorithms compute a Whitney stratification of a variety X by way of computing the set of associated

¹Note that in fact the map π in the example is not proper, so Thom's Isotopy lemma does not hold for the affine map, however we can see that the result still holds for this example by passing to the projective closure in $\mathbb{P}^2 \times \mathbb{C}$, where the corresponding map is proper.

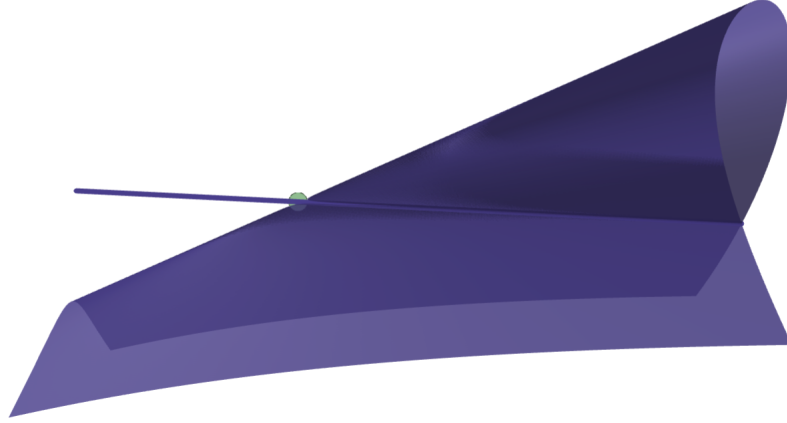


FIGURE 2. The surface $X = \mathbf{V}((y-1)^2 - (x-z)(x-1)^2)$ in \mathbb{R}^3 . Its singular locus is $Y = \mathbf{V}(x-1, y-1)$ and its Whitney stratification arises from the flag $\{(1, 1, 1)\} \subset Y \subset X$. The dimension 2 stratum is $X - Y$, the two dimension 1 strata are the connected components of $Y - \{(1, 1, 1)\}$, and the dimension 0 stratum is $(1, 1, 1)$.

primes of certain (scheme theoretic) fibers in the so-called *conormal space* associated to X (see Definition 2.10). These fibers are often much more complicated than their image, making the computation of the required associated primes difficult even for relatively simple varieties.

Example 1.1. Consider the variety $X := \mathbf{V}(x_1^7 - 2x_1^5x_4 + x_1^3x_4^2 + x_1^2x_2^2 - x_1x_2^2x_3 + x_2^3) \subset \mathbb{C}^4$. The singular locus of X is $X_{\text{Sing}} = \mathbf{V}(x_2, x_1^3 - x_1x_4)$. Inside the singular locus Whitney's condition (B) fails to hold on the following subvarieties of dimension 1:

$$\mathbf{V}(x_4, x_2, x_1), \mathbf{V}(x_2, x_1, 4x_3^3 - 27x_4^2), \mathbf{V}(x_2, x_1 - x_3, x_3^2 - x_4). \quad (2)$$

The three varieties listed in (2) intersect at the origin. The origin along with the varieties (2) and X_{Sing} , define the Whitney stratification of X . As noted above this is computed via studying certain fibers of the conormal space. The conormal space of X , $\text{Con}(X) \subset \mathbb{C}^4 \times \mathbb{P}^3$, is defined by an ideal generated by 22 polynomials in eight variables, most with between 8 and 10 terms each (for a total of 190 terms across all polynomials). The fiber of the singular locus in the conormal space of X , obtained by adding the ideal of $\text{Con}(X)$ to the two equations defining X_{Sing} , is specified by a non-radical ideal generated by 20 polynomials in eight variables. These polynomials contain between 1 and 16 terms each, for a total of 127 terms across all 20 polynomials. For this latter ideal we were unable to compute the associated primes (though the minimal primes can be computed relatively quickly).

In line with this observation, practical experience has shown that the computation of these associated primes is precisely the main bottleneck for these algorithms. We note that, theoretically speaking (to apply the known algebraic criterion for condition (B), see Theorem 2.11), for the ideals involved it is absolutely essential that we obtain both the embedded and the isolated primary components. In particular, algorithms which compute only the isolated

primes of an ideal cannot be used. In this paper we address this bottleneck using a modified version of a well-known Gröbner basis algorithm for *equidimensional* decomposition (see e.g. [2]) which allows us to compute the intersection of all required associated primes of fibers in some conormal space with a single much simpler Gröbner basis computation. While the worst case complexity of the approach is unchanged this, nonetheless, yields quite substantial speedups in practice on a wide variety of examples. However, because we may compute a larger variety, which contains the desired one, this can yield non-minimal Whitney stratifications even when the algorithm of [7, 8, 9] would yield a minimal stratification. While this is not necessarily a problem, in some applications it may be more desirable to obtain the minimal stratification. To this end we also use classical results of Tessier [4, page 751–752] based on the concept of *local polar varieties* to develop a second algorithm which ensures that the minimal stratification will be produced (at least in the case of complex varieties).

This paper is organized as follows: in Section 2 we briefly review the necessary background on Gröbner bases, Whitney stratifications and local polar varieties. The main results, and the resulting algorithm for computing the Whitney stratification of a variety and for minimizing a given Whitney stratification, are presented in Section 3. Finally in Section 4 we give run time comparisons.

2. Background

In this section we briefly recall several important facts regarding Gröbner bases, Whitney stratifications, and the effective version of Whitney’s condition given in [8]. These facts will be needed to construct our new stratification algorithm in the sequel.

2.1. Gröbner Bases. Let us briefly state the necessary definitions and properties of Gröbner bases needed for our algorithm. Just for this section, let \mathbb{K} be any field and let $R := \mathbb{K}[\mathbf{x}]$ be the polynomial ring over \mathbb{K} in a finite set of variables \mathbf{x} . Denote by $\text{Mon}(\mathbf{x})$ the set of monomials in \mathbf{x} .

Definition 2.1 (Monomial Order). A *monomial order* \prec on \mathbf{x} is a total order on $\text{Mon}(\mathbf{x})$ which:

- (1) extends the partial order on $\text{Mon}(\mathbf{x})$ given by divisibility, and
- (2) is compatible with multiplication i.e. we have,

$$u \prec v \Rightarrow wu \prec wv \quad \forall u, v, w \in \text{Mon}(\mathbf{x}).$$

A monomial order on \mathbf{x} yields a notion of *leading monomial* and *leading coefficient* in R .

Definition 2.2 (Leading Monomial). Let \prec be a monomial order on $\text{Mon}(\mathbf{x})$. For a nonzero element $f \in R$ the *leading monomial* of f with respect to \prec , denoted $\text{lm}_\prec(f)$, is the \prec -largest monomial in the support of f . The *leading coefficient*, denoted $\text{lc}_\prec(f)$, is the corresponding coefficient of $\text{lm}_\prec(f)$ in f . For a finite set F in R we define $\text{lm}_\prec(F) := \{\text{lm}_\prec(f) \mid f \in F\}$. For an ideal I in R we define the *leading monomial ideal* of I as $\text{lm}_\prec(I) := \langle \text{lm}_\prec(f) \mid f \in I \rangle$.

We finally define the notion of Gröbner bases.

Definition 2.3 (Gröbner Basis). A *Gröbner basis* of an ideal $I \subset R$ with respect to a monomial order \prec is a finite set $G \subset I$ such that $\langle \text{lm}_\prec(G) \rangle = \text{lm}_\prec(I)$. It is called *minimal* if for any $g \in G$, $\text{lm}_\prec(g)$ is not divisible by any element in $\text{lm}_\prec(G - \{g\})$

Any Gröbner basis can be turned into a minimal one by removing all elements whose leading monomials are divisible by another leading monomial occurring in the Gröbner basis. For our algorithm, we will need to compute Gröbner bases for *block orders*.

Definition 2.4 (Block Order). Let \mathbf{x} and \mathbf{y} be two finite sets of variables. Write each monomial $u \in \text{Mon}(\mathbf{x} \cup \mathbf{y})$ uniquely as a product $u = u_{\mathbf{x}} u_{\mathbf{y}}$ with $u_{\mathbf{x}} \in \text{Mon}(\mathbf{x})$ and $u_{\mathbf{y}} \in \text{Mon}(\mathbf{y})$. Fix a monomial order \prec_1 on $\text{Mon}(\mathbf{x})$ and a monomial order \prec_2 on $\text{Mon}(\mathbf{y})$. The corresponding *block order eliminating \mathbf{x}* is defined as follows: $u \prec v$ if and only if $u_{\mathbf{x}} \prec_1 v_{\mathbf{x}}$ or $u_{\mathbf{x}} = v_{\mathbf{x}}$ and $u_{\mathbf{y}} \prec_2 v_{\mathbf{y}}$ for $u, v \in \text{Mon}(\mathbf{x} \cup \mathbf{y})$.

The key property of Gröbner bases with respect to block orders that we need is the following (Lemmas 8.91 and 8.93 in [1]):

Proposition 2.5. *Let \mathbf{x} and \mathbf{y} be any two finite sets of variables and let \prec be any block order eliminating \mathbf{x} . Let G be a Gröbner basis of some ideal I with respect to \prec . Then*

- (1) G is also a Gröbner basis of the ideal $\mathbb{K}(\mathbf{y})[\mathbf{x}]$.
- (2) Let H be the corresponding minimal Gröbner basis of G . Let $h \in \mathbb{K}[\mathbf{y}]$ be the least common multiple of the leading coefficients of H , regarded as a subset of $\mathbb{K}(\mathbf{y})[\mathbf{x}]$. Then

$$\mathbb{K}(\mathbf{y})[\mathbf{x}] \cap \mathbb{K}[\mathbf{x}, \mathbf{y}] = (I : h^\infty).$$

We will want to compute Gröbner bases of ideals of the form $\mathbb{K}(\mathbf{y})[\mathbf{x}]$ when \mathbf{y} is chosen such that the map $\mathbb{K}[\mathbf{y}] \rightarrow \mathbb{K}[\mathbf{x}, \mathbf{y}]/I$ is injective. To this end, define:

Definition 2.6 (Maximally Independent Subset). A *maximally independent subset* $\mathbf{u} \subset \mathbf{x}$ of an ideal $I \subset \mathbb{K}[\mathbf{x}]$ is a set such that the map $\mathbb{K}[\mathbf{y}] \rightarrow \mathbb{K}[\mathbf{x}, \mathbf{y}]/I$ is injective and such that the cardinality of \mathbf{u} is equal to $\dim(I)$.

Computationally, we will determine maximally independent subsets from any Gröbner basis of I (Theorem 9.27 in [1]):

Proposition 2.7. *Let G be any Gröbner basis of an ideal $I \subset \mathbb{K}[\mathbf{x}]$. Let $\mathbf{u} \subset \mathbf{x}$ be maximal such that no leading monomial of any $g \in G$ depends only on \mathbf{u} . Then \mathbf{u} is a maximally independent subset.*

2.2. Whitney Stratifications. Throughout the paper, let \mathbb{C}^n be the n -dimensional affine space over the complex numbers \mathbb{C} , \mathbb{P}^n be the n -dimensional projective space over \mathbb{C} with dual projective space $(\mathbb{P}^n)^*$, and let $\mathbf{Gr}(k, n)$ denote the projective variety of k -dimensional vector spaces in \mathbb{C}^n . We assume throughout the rest of the paper that all varieties in \mathbb{C}^n are defined over \mathbb{Q} , i.e. cut out by n -variate polynomials with coefficients in \mathbb{Q} . For any variety $X \subset \mathbb{C}^n$ we will denote by $I(X)$ the radical ideal of all polynomials vanishing on X , whereas I_X will denote any ideal with $\mathbf{V}(I_X) = X$, and in particular if X is non-reduced (i.e. a scheme) I_X will denote the ideal defining the scheme X .

Our goal is to compute a Whitney stratification of a singular variety $X \subset \mathbb{C}^n$, which is a certain flag on X where the successive differences between varieties in the flag are smooth

and where these smooth pieces satisfy an additional regularity condition. This additional regularity condition, called *Whitney's Condition (B)* is designed to enforce that the local structure of each connected component of each smooth piece of the stratification is geometrically identical in a precise way.

Definition 2.8 (Whitney's Condition (B)). Let $Y \subset X$ be algebraic varieties in \mathbb{C}^n with $\dim(Y) < \dim(X)$ and with $X - Y$ smooth. Let Y_{reg} denote the smooth part of Y . The pair (X, Y) satisfies *Whitney's condition (B)* if for any $p \in Y_{\text{reg}}$

- and any sequences $(x_i) \subset X$ and $(y_i) \subset Y_{\text{reg}}$ converging to p ,
- if the secant lines $\ell_i = [x_i, y_i] \in \mathbb{P}^{n-1}$ converge to some $\ell \in \mathbb{P}^{n-1}$,
- and if the tangent spaces $T_{x_i}(X - Y) \in \mathbf{Gr}(\dim(X), n)$ converge to some $T \in \mathbf{Gr}(\dim(X), n)$,

then $\ell \subset T$.

This now allows to formally define a Whitney stratification of a variety $X \subset \mathbb{C}^n$, see e.g. [17, 18]:

Definition 2.9 (Whitney Stratification). A *Whitney stratification* of X is a flag

$$\emptyset = W_{-1} \subset W_0 \subset \cdots \subset W_{k-1} \subset W_k = X$$

where each difference $M_i := W_i - W_{i-1}$ is a smooth variety. The connected components of each M_i are called *strata* and each pair of strata must satisfy Whitney's condition (B). Note that the condition that M_i is smooth in particular means that for each W_i we must have that the singular locus of W_i is contained in W_{i-1} , i.e. $(W_i)_{\text{Sing}} \subset W_{i-1}$.

A *minimal* Whitney stratification of X is a Whitney stratification such that, after removing any stratum, the resulting stratification fails to be a Whitney stratification. The existence of a unique minimal stratification follows from results of [16], see also [4].

In [7] an algebraic criterion based on primary decomposition of polynomial ideals is given that allows to check computationally if a given pair $X, Y \subset \mathbb{C}^n$ satisfies condition (B), and as such, to compute a Whitney stratification of a given variety by succesively computing singular loci and then checking condition (B) for all pairs of resuling varieties. The central contribution of this present work is to still apply the same algebraic criterion but to give a simple way in which having to explicitly compute primary decompositions can nonetheless be avoided, see Section 3.1.

Let us now fix an *equidimensional*, affine variety $X \subset \mathbb{C}^n$, where equidimensional means that all irreducible components of X have the same dimension. We now introduce the aforementioned algebraic criterion, which will use the notion of a *conormal space*.

Definition 2.10 (Conormal Space). Denote by X_{reg} the set of regular points of X . The *conormal space* $\mathbf{Con}(X) \subset \mathbb{C}^n \times (\mathbb{P}^{n-1})^*$ of X is the Zariski closure of the set

$$\{(p, \zeta) \mid p \in X_{\text{reg}} \text{ and } T_p X_{\text{reg}} \subset \zeta^\perp\}.$$

The canonical projection $\kappa_X : \mathbf{Con}(X) \rightarrow X$ is called the *conormal map*.

This notion was used in [7] to give the aforementioned algebraic criterion for condition (B), which we restate below.

THEOREM 2.11 (Theorem 4.3 in [7]). *Let $\emptyset \neq Y \subset X_{\text{sing}}$ be equidimensional and let I_Y be any ideal with $\mathbf{V}(I_Y) = Y$. Let*

$$I_Y + I(\mathbf{Con}(X)) = \bigcap_{i \in I} Q_i$$

be a primary decomposition. Let

$$J := \{i \in I \mid \dim(\kappa_X(\mathbf{V}(Q_i))) < \dim(Y)\}.$$

Further, let

$$A := \left[\bigcup_{j \in J} \kappa_X(\mathbf{V}(Q_j)) \right] \cup Y_{\text{sing}}.$$

Then the pair $(X_{\text{reg}}, Y - A)$ satisfies condition (B).

2.3. Local Polar Varieties and Minimal Whitney Stratifications. While our new algebraic criterion for Whitney's condition (B) avoids the computation of primary decompositions, and thus speeds up the stratification process, it may fail to produce a minimal Whitney stratification even when the algorithm in [7] does. In order to rectify this, we will use the concept of *local polar varieties* which we introduce here.

Let $X \subset \mathbb{C}^n$ be a variety of dimension d and consider the conormal space $\mathbf{Con}(X) \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$ with associated conormal map $\kappa_X : \mathbf{Con}(X) \rightarrow X$. For fixed i with $1 \leq i \leq d-1$, consider a dimension $d+i-1$ linear space in \mathbb{C}^n with dual $L_i \subset (\mathbb{P}^{n-1})^*$. We use the convention $L_0 = \mathbb{C}^n \times (\mathbb{P}^{n-1})^*$.

Definition 2.12 (Polar Variety). *A polar variety is a variety of the form*

$$\delta_i(X) := \kappa_X(\mathbf{Con}(X) \cap L_i).$$

Consider now a point $y \in X$. Picking our linear spaces as general linear spaces through y with dual $\tilde{L}_i \subset \mathbb{P}^{n-1}$ we define

Definition 2.13 (Local Polar Variety). *A codimension i local polar variety through y is a variety of the form*

$$\delta_i(X, y) := \kappa_X(\mathbf{Con}(X) \cap \tilde{L}_i).$$

Since the conormal space is reduced (as a scheme), i.e. is a variety [4, Proposition 2.9], and since the L_i (and the \tilde{L}_i) are general it follows that the local polar varieties are also reduced [4, Remark 3.10 (b)] and are hence each defined by a radical ideal.

If our linear space \tilde{L}_i are chosen sufficiently generic, then, using a dimension count, some of the local polar varieties will contain y and some will not (see [4, Remark 3.1]). We will want to compute the *multiplicity* of points y in local polar varieties $\delta_i(X, y)$. Let us define the notion of multiplicity used here:

Definition 2.14 (Multiplicity). Let $Z \subset \mathbb{C}^n$ be a subvariety and let $z \in \mathbb{C}^n$ and write $\mathfrak{m}_z := I(z)$. If $z \in Z$, the *multiplicity* of Z at z , denoted $m_z(Z)$, is defined as the Hilbert-Samuel multiplicity (see e.g. Chapter 12 in [3]) of the local ring $(\mathbb{C}[\mathbf{x}]/I(Z))_{\mathfrak{m}_z}$ at \mathfrak{m}_z . If $z \notin Z$ we define $m_z(Z) := 0$.

For a subvariety $Y \subset X$ and a given point $y \in Y$, computing multiplicities of the form $m_y(\delta_i(X, y))$ can be used to check if Whitney's condition (B) holds for X and Y at y . More precisely (see p. 69 and Proposition 3.6 in [4]):

THEOREM 2.15. *Let $Y \subset X$ be a subvariety and $y \in Y$. Then the sequence*

$$m_\bullet(X, x) := (m_x(X), m_x(\delta_1(X, x)), \dots, m_x(\delta_{\dim(X)-1}(X, x)))$$

is independent of the linear subspaces chosen to construct the local polar varieties if they are sufficiently general. In addition, Whitney's condition (B) is satisfied at y for X and Y if the sequence takes the same value for every x in a euclidean neighborhood of y in Y .

Making this theorem computationally profitable will require us to compute sequences $m_\bullet(X, y)$ where y lies in a subvariety $Y \subset X$. We will show how to do this in Section 3.3.

3. New Whitney Stratification Algorithms

In this section we present the main results of this work, namely a new algebraic condition to identify subvarieties where condition (B) fails, new algorithms to compute a Whitney stratification of an algebraic variety and an algorithm to produce the unique minimal Whitney stratification for complex varieties along with theoretical results which guarantee the correctness of our algorithms. Our new formulation of Condition B is given in Section 3.1, the resulting stratification is given in Section 3.2, and finally an algorithm to coarsen a given Whitney stratification to the unique minimal stratification is given in Section 3.3.

3.1. A New Algebraic Criterion for Checking Whitney's Condition. Throughout this section we denote by $\mathbb{C}[\mathbf{x}]$ the underlying polynomial ring of \mathbb{C}^n . We will study the problem of effectively computing a Whitney stratification of a complex affine algebraic variety $X \subset \mathbb{C}^n$. We first show how to, in the setting of Theorem 2.11, compute a superset of A without having to compute a primary decomposition of $I_Y + I(\mathbf{Con}(X))$. For this we denote by $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ the underlying polynomial ring of $\mathbb{C}^n \times \mathbb{P}^{n-1}$, where \mathbb{P}^{n-1} denotes the $(n-1)$ -dimensional projective space over \mathbb{C} .

First recall the following elementary proposition from commutative algebra:

Proposition 3.1. *Let $P \subset \mathbb{C}[\mathbf{x}]$ be a prime ideal and let $\mathbf{u} \subset \mathbf{x}$ be of cardinality d . If $P \cap \mathbb{C}[\mathbf{u}] = 0$, then $\dim(P) \geq d$.*

PROOF. This follows from the definition of Krull dimension of an ideal: Any maximal chain of prime ideals in $\mathbb{C}[\mathbf{u}]$ extends to a chain of prime ideals in $\mathbb{C}[\mathbf{x}]/P$ since $\mathbb{C}[\mathbf{u}] \rightarrow \mathbb{C}[\mathbf{x}]/P$ is injective. \square

Using Theorem 2.11 we obtain an alternative algebraic criterion which will also allow us to construct a Whitney stratification. When reading the result below keep in mind that \mathbf{x}, \mathbf{u} , and \mathbf{y} (and hence $\mathbf{x} - \mathbf{u}$) denote sets of variables.

THEOREM 3.2. *Let $X \subset \mathbb{C}^n$ be an algebraic variety and let Y be any (not necessarily equidimensional) subvariety of the singular locus X_{sing} of X . Choose any maximal independent subset of*

variables $\mathbf{u} \subset \mathbf{x}$ of any defining ideal I_Y of Y . Let $J := I(\mathbf{Con}(X)) + I_Y$. Let G be a minimal Gröbner basis of $J\mathbf{C}(\mathbf{u})[\mathbf{x} - \mathbf{u}, \mathbf{y}]$ with respect to any monomial order \prec with $G \subset \mathbf{C}[\mathbf{x}, \mathbf{y}]$. Let

$$h = \text{lcm}\{\text{lc}_\prec(g) \mid g \in G\} \in \mathbf{C}[\mathbf{u}].$$

Then

- (1) $Y - \mathbf{V}(h)$ is equidimensional of dimension $\dim(Y)$.
- (2) The pair $(X, Y - (\mathbf{V}(h) \cup Y_{\text{sing}}))$ satisfies Whitney's condition (B).

PROOF. Throughout this proof, we denote by $\text{Assoc}(I)$ the set of associated primes of any ideal I .

(Proof of 1) By Proposition 2.5, we have

$$(J : h^\infty) = J\mathbf{C}(\mathbf{u})[\mathbf{x} - \mathbf{u}, \mathbf{y}] \cap \mathbf{C}[\mathbf{x}, \mathbf{y}]. \quad (3)$$

The ring $\mathbf{C}(\mathbf{u})[\mathbf{x} - \mathbf{u}, \mathbf{y}]$ is the localization of $\mathbf{C}[\mathbf{x}, \mathbf{y}]$ at the multiplicative set $\mathbf{C}[\mathbf{u}] - \{0\}$. Hence, using Theorem 3.1 in [3],

$$\text{Assoc}((J : h^\infty)) = \{P \in \text{Assoc}(J) \mid P \cap \mathbf{C}[\mathbf{u}] = 0\}.$$

Further, clearly, $(J : h^\infty) \cap \mathbf{C}[\mathbf{x}] = (I_Y : h^\infty)$. For P a minimal prime over $(I_Y : h^\infty)$, choose a minimal prime Q over $(J : h^\infty)$ with $Q \cap \mathbf{C}[\mathbf{x}] = P$. Then we have

$$0 = Q \cap \mathbf{C}[\mathbf{u}] = P \cap \mathbf{C}[\mathbf{u}]$$

and therefore $\dim(P) \geq \dim(Y)$ by Proposition 3.1. On the other hand, P is also minimal over I_Y . Therefore $\dim(P) \leq \dim(Y)$ and finally $\dim(P) = \dim(Y)$. This shows that $Y - \mathbf{V}(h)$ is equidimensional of dimension equal to $\dim(Y)$.

(Proof of 2): Let $P \in \text{Assoc}(J)$ such that $\dim(\kappa_X(\mathbf{V}(P))) < \dim(Y)$. This implies $P \cap \mathbf{C}[\mathbf{u}] \neq 0$ again by Proposition 3.1, therefore $P \notin \text{Assoc}(J\mathbf{C}(\mathbf{u})[\mathbf{x} - \mathbf{u}, \mathbf{y}])$ and therefore also $P \notin \text{Assoc}((J : h^\infty))$. Hence necessarily $h \in P$ or, since $h \in \mathbf{C}[\mathbf{u}]$, $\kappa_X(\mathbf{V}(P)) \subset \mathbf{V}(h)$. If we then define

$$A := \bigcup_{\substack{P \in \text{Assoc}(J) \\ \dim(\kappa_X(\mathbf{V}(P))) < \dim(Y)}} \kappa_X(\mathbf{V}(P))$$

then we have $A \subset \mathbf{V}(h)$. By Theorem 2.11, the pair $(X, Y - (A \cup Y_{\text{sing}}))$ satisfies Whitney's condition (B) and since $Y - \mathbf{V}(h) \subset Y - A$, so does $(X, Y - (\mathbf{V}(h) \cup Y_{\text{sing}}))$. \square

3.2. The Whitney Stratification Algorithm. To start, let us briefly recall how to compute an ideal defining the conormal space of a given $X \subset \mathbb{C}^n$ (see also Section 4.1 in [7]). Suppose that $X = \mathbf{V}(f_1, \dots, f_r)$ for certain $f_1, \dots, f_r \in \mathbf{C}[\mathbf{x}]$. Let $F := (f_1, \dots, f_r)$. We then build the augmented Jacobian matrix

$$\mathbf{J}_y(F) := \begin{bmatrix} y_0 & y_1 & \dots & y_{n-1} \\ \partial_1 f_1 & \partial_2 f_1 & \dots & \partial_n f_1 \\ \vdots & & & \vdots \\ \partial_1 f_r & \partial_2 f_r & \dots & \partial_n f_r \end{bmatrix}$$

with new variables $\mathbf{y} := \{y_0, \dots, y_{n-1}\}$. Here, ∂_i denotes the partial derivative by the i th variable in \mathbf{x} . Denote by $\mathbf{J}(F)$ the usual Jacobian matrix of F . Now (see again [7]) we have the following fact.

Proposition 3.3. *With the notation above let $c := \text{codim}(X)$, let M be the set of $(c \times c)$ -minors of $\mathbf{J}(F)$, and let $M_{\mathbf{y}}$ be the set of $(c+1) \times (c+1)$ -minors of $\mathbf{J}_{\mathbf{y}}(F)$. Then*

$$\mathbf{Con}(X) = \mathbf{V}(\langle F \rangle + \langle M_{\mathbf{y}} \rangle : M^{\infty}).$$

Remark 3.4. Recall that the necessary saturation to compute conormal spaces can be performed using Gröbner basis computations with block orders and *Rabinowitsch's trick*, see e.g. Exercise 15.41 in [3].

Now we can give the following algorithm subroutine **WhitPoints**, which given any pair of varieties X and Y with $Y \subset X_{\text{sing}}$, returns a polynomial h as in Theorem 3.2. This subroutine is the centerpiece of our algorithm for computing Whitney stratifications. As previously mentioned, we avoid the computation of associated primes in the conormal space of X which an immediate application of Theorem 2.11 would require.

	WhitPoints (X, Y)
	Input: An equidimensional affine variety X , any closed $Y \subset X_{\text{sing}}$. Output: An element $h \in \mathbb{C}[\mathbf{u}]$ as in Theorem 3.2.
1	$I_X \leftarrow$ any ideal defining X , $I_Y \leftarrow$ any ideal defining Y .
2	Let \mathbf{u} be any maximally independent subset of variables of I_Y .
3	Let \prec be any monomial ordering eliminating $(\mathbf{x} \cup \mathbf{y}) - \mathbf{u}$.
4	Set G to be a \prec -Gröbner basis of $I(\mathbf{Con}(X)) + I_Y$.
5	Minimize G over $\mathbb{C}(\mathbf{u})[\mathbf{x} - \mathbf{u}, \mathbf{y}]$.
6	Return $h := \text{lcm}\{\text{lc}(g) \mid g \in G\}$.

Remark 3.5. In line 2, we use a Gröbner basis of I_Y and Proposition 2.7 to compute the desired maximally independent subset of variables of I_Y .

THEOREM 3.6. *For a given pair of varieties X and $Y \subset X_{\text{sing}}$, the output polynomial h of **WhitPoints**(X, Y) is such that the pair $(X, Y - (\mathbf{V}(h) \cup Y_{\text{sing}}))$ satisfies Whitney's condition (B).*

PROOF. Note that, thanks to Proposition 2.5, a Gröbner basis of an ideal $I \subset \mathbb{C}[\mathbf{x}]$ with respect to a monomial order eliminating $\mathbf{x} - \mathbf{u}$, where $\mathbf{u} \subset \mathbf{x}$, is also a Gröbner basis of $I\mathbb{C}(\mathbf{u})[\mathbf{x} - \mathbf{u}]$. The stated result then follows immediately from Theorem 3.2. \square

Finally, we can use this subroutine to give the new algorithm **Whitney** below to compute a Whitney stratification of an affine equidimensional variety X . In this algorithm, each occurring variety Z is represented as a union of its \mathbb{Q} -irreducible components $Z = \bigcup_{i=1}^r Z_i$. Let us define three subroutines used by this algorithm:

- We define **Components**(Z) to return the set $\{Z_1, \dots, Z_r\}$.
- For two varieties Z, W we define the routine **Add**(Z, W) to return $Z \cup W$ represented again by its \mathbb{Q} -irreducible components.

- For a flag W_\bullet and an irreducible variety Z we then define **Update**(W_\bullet, Z) to change $W_{\dim(Z)}$ to **Add**($W_{\dim(Z)}, Z$).

Remark 3.7. Note that, as Theorem 3.2 does not rely on Y being irreducible, it is not strictly needed for the correctness of our algorithm to represent the occurring varieties by their \mathbb{Q} -irreducible components. We chose to present the algorithm like this here for ease of reading. It would not be difficult to adapt Algorithm 5 in [2] to our situation, then we would have to perform only *equidimensional decomposition* instead of finding the minimal primes over a given ideal which is generally much easier.

However, as remarked in the introduction, the key contribution of this work is to avoid the computation of associated primes *in conormal spaces*. Even with our new method, we observed that the required computations in conormal spaces are still the bottleneck of our algorithm, compared to representing varieties by their irreducible components as above.

	Whitney (X)
	Input: An equidimensional affine variety X . Output: A Whitney stratification of X .
1	$d \leftarrow \dim(X)$
2	$W_0 \leftarrow \emptyset, \dots, W_{d-1} \leftarrow \emptyset, W_d \leftarrow X$
3	For i from d to 0
4	$Z_1, \dots, Z_r \leftarrow \mathbf{Components}(W_i)$
5	For j, k from 1 to r with $j < k$
6	Update ($W_\bullet, Z_j \cap Z_k$)
7	For Z in Components (W_i)
8	Update ($W_\bullet, Z_{\text{sing}}$)
9	For j from $d-1$ to 0
10	For Y in Components (W_j) if $Y \subset Z$
11	$h \leftarrow \mathbf{WhitPoints}(Z, Y)$
12	$Y' \leftarrow Y \cap \mathbf{V}(h)$
13	Update (W_\bullet, Y')
14	Return W_\bullet .

THEOREM 3.8. For a given equidimensional affine variety $X \subset \mathbb{C}^n$, **Whitney**(X) terminates and outputs a Whitney stratification of X .

PROOF. The termination of the algorithm is clear. Note that the singular locus of any variety consists of the the union of the singular loci of its \mathbb{Q} -irreducible components together with the intersections of all its \mathbb{Q} -irreducible components. Thus lines 4 and 6 in **Whitney** guarantee the smoothness of all strata of the output of **Whitney**(X). Further, the fact that Whitney's condition (B) is satisfied for all pair of the strata of the output of **Whitney**(X) follows immediately from Theorem 3.6. \square

Remark 3.9. While we have focused on the case of varieties over \mathbb{C} in this note, applying the results of [9], it follows that the algorithm **Whitney** presented above would also give a valid stratification for real algebraic varieties. In particular, this follows by [9, Theorem 3.3] and

the fact that ideal addition and Gröbner basis computation leave the coefficient field of the polynomials unchanged. Hence a Whitney stratification of a real algebraic variety may also be computed using these techniques. Note however, that the resulting stratification may fail to be minimal (see [9] for more discussions) and additionally the algorithm in Section 3.3 below to minimize a given Whitney stratification **does not** necessarily return the minimal stratification of a real variety.

3.3. The Minimization Algorithm. Our first goal here is to show how to compute, for a given $X \subset \mathbb{C}^n$ and a random point y in a subvariety $Y \subset X$, the sequence $m_\bullet(X, y)$ without being given y explicitly. Next, we will show that the sequence $m_\bullet(X, \bullet)$ is constant on a Zariski-open subset of Y , if Y is \mathbb{Q} -irreducible. These two things combined allow us to compute, probabilistically, the “generic” multiplicities $m_\bullet(X, y)$, $y \in Y$. Combining this with Theorem 2.15 we obtain a procedure to produce the unique minimal Whitney stratification of a complex algebraic variety given a Whitney stratification computed by **Whitney**.

Suppose for the moment that $Y = \mathbf{V}(f_1, \dots, f_c)$ where $c \leq n$ and f_1, \dots, f_c is such that $F := (f_1, \dots, f_c, \ell_{c+1}, \dots, \ell_n)$ is a reduced regular sequence, where the ℓ_i are generic degree one polynomials in \mathbf{x} , i.e. the determinant of $\mathbf{J}(F)$ is non-zero at every point of Y . Let now

$$\begin{aligned} \phi : \mathbb{C}^n &\rightarrow \mathbb{C}^n \\ p := (p_1, \dots, p_n) &\mapsto (f_1(p), \dots, f_c(p), \ell_{c+1}(p), \dots, \ell_n(p)), \end{aligned}$$

and let \tilde{X} and \tilde{Y} be the closures of the images under ϕ of X and Y respectively. Denote by z_1, \dots, z_n the coordinates on the codomain of ϕ . Note that the origin $0 \in \mathbb{C}^n$ lies in \tilde{Y} . Now we can compute the polar varieties $\delta_i(\tilde{X}, 0) = \delta_i(\tilde{X})$ and their multiplicities at 0 by choosing linear spaces through the origin. This gives us the correct multiplicities because of the following fact.

Proposition 3.10. *Using the notation above, for any $y \in \phi^{-1}(0)$ we have*

$$m_\bullet(X, y) = m_\bullet(\tilde{X}, 0).$$

PROOF. Since F is a reduced regular sequence, ϕ defines a local isomorphism near any $y \in Y$ by the inverse function theorem. This means that for any $y \in Y$, ϕ induces an isomorphism of completions

$$\phi_y^* : \hat{\mathcal{O}}_{\mathbb{C}^n, \phi(y)} \rightarrow \hat{\mathcal{O}}_{\mathbb{C}^n, y}.$$

Now, since the Hilbert-Samuel multiplicity of a local ring (R, \mathfrak{m}) depends only on $\text{gr}_{\mathfrak{m}}(R)$ and since $\text{gr}_{\mathfrak{m}}(R) = \text{gr}_{\mathfrak{m}}(\hat{R})$, the result follows. \square

Next suppose that $Y \subset X$ is equidimensional of codimension c but no longer necessarily cut out by a reduced regular sequence. Choose, at random, a sequence $f_1, \dots, f_c \in I(Y)$, for example as random linear combinations of given generators of $I(Y)$. Defining, $Z := \mathbf{V}(f_1, \dots, f_c)$ we then have the following

Proposition 3.11. *Use the notations above. If f_1, \dots, f_c are sufficiently generic, there exists a Zariski-open subset $U \subset \mathbb{C}^n$ such that*

$$Y \cap U = Z \cap U$$

and $Y \cap U$ is dense in Y . Moreover, for sufficiently generic degree one polynomials $\ell_{c+1}, \dots, \ell_n$, the sequence

$$F := (f_1, \dots, f_c, \ell_{c+1}, \dots, \ell_n)$$

is a reduced regular sequence on Y .

PROOF. Since f_1, \dots, f_c is sufficiently generic, f_1, \dots, f_c defines a maximal regular sequence in $I(Y)$. Hence there exists some polynomial h such that $hI(Y) \subset \langle f_1, \dots, f_c \rangle$. Choosing $U := \mathbb{C}^n - \mathbf{V}(h)$ proves the first part of the statement. Now, at every point $p \in U$, we have $I(Y)\mathbb{C}[\mathbf{x}]_{I(p)} = \langle f_1, \dots, f_c \rangle \mathbb{C}[\mathbf{x}]_{I(p)}$. This shows that f_1, \dots, f_c defines a radical ideal at every point $p \in U$. By Exercise 12.11 in [3], the same remains true for the sequence $f_1, \dots, f_c, \ell_{c+1}, \dots, \ell_n$ where the ℓ_i are sufficiently generic degree one polynomials in \mathbf{x} , proving the proposition. \square

From this we obtain the following corollary.

Corollary 3.12. *Let $m_\bullet(\tilde{X}, 0)$ be the multiplicity sequence constructed as above, with Y replaced by Z . Then $m_\bullet(\tilde{X}, 0) = m_\bullet(X, y)$ for every $y \in \phi^{-1}(0)$.*

PROOF. Choose $U \subset \mathbb{C}^n$ as in Proposition 3.11. By the same proposition, with probability 1, the intersection $Y \cap \mathbf{V}(\ell_{c+1}, \dots, \ell_n) \cap U \subset \phi^{-1}(0)$ is not empty. For any choice of $y \in Y \cap \mathbf{V}(\ell_{c+1}, \dots, \ell_n) \cap U$ we then have $m_\bullet(\tilde{X}, 0) = m_\bullet(X, y)$ by Proposition 3.10. \square

Now, given X and Y we can sketch an algorithm that computes the sequence $m_\bullet(X, y)$ for a random point $y \in Y$.

	Mult (X, Y)
	Input: An affine variety X , any closed $Y \subset X$. Output: The sequence $m_\bullet(X, y)$ for a randomly chosen $y \in Y$.
1	$I_Y \leftarrow \langle g_1, \dots, g_r \rangle$, the radical ideal defining Y .
2	$c \leftarrow \text{codim}(Y)$.
3	Choose f_1, \dots, f_c as random linear combinations of the g_i .
4	Choose $\ell_{c+1}, \dots, \ell_n$ as random degree one polynomials.
5	$\phi \leftarrow$ the map given by $\mathbf{x} \mapsto (f_1, \dots, f_c, \ell_{c+1}, \dots, \ell_n)$
6	Compute ideals defining $\phi(X) = \delta_0(\phi(X), 0), \dots, \delta_{\dim(X)-1}(\phi(X), 0)$
7	Using these ideals, Return $(m_0(\delta_0(\phi(X), 0)), \dots, m_0(\delta_{\dim(X)-1}(\phi(X), 0)))$

Remark 3.13. In line 6, an ideal defining $\phi(X)$ is again computed using Gröbner basis computations with block orders, see e.g. Proposition 15.30 in [3]. Using the notation of Proposition 3.3, ideals defining the necessary local polar varieties can then be computed as ideals of the form

$$(I(\phi(X)) + \langle M_y \rangle + \tilde{L} : M^\infty) \cap \mathbb{C}[\mathbf{x}]$$

where M_y are the suitable minors of the augmented Jacobian matrix associated to generators of $I(\phi(X))$, M are the suitable minors of the Jacobian matrix of generators of $I(\phi(X))$ and \tilde{L} is a collection of degree one forms, defining a linear space of suitable dimension in \mathbb{P}^{n-1} .

In line 7, to compute the required multiplicities we use degree computations via Gröbner basis as described in [6] and implemented in the SegreClasses Macaulay2 package [5], see in particular [6, Theorem 5.3]. Alternatively one can use standard basis computations, see e.g. [15] for details.

Our next goal is the following: Suppose that $Y \subset X$ is \mathbb{Q} -irreducible. Then we want to show that the sequence $m_\bullet(X, \bullet)$ is constant on a Zariski-dense open subset of Y . This is concluded as follows from Theorem 2.15:

Corollary 3.14. *Given a variety X and \mathbb{Q} -irreducible $Y \subset X$, there exists a Zariski-dense open subset $U \subset Y$ such that $m_\bullet(X, \bullet)$ is constant on U .*

PROOF. Suppose $\dim(Y) = e$. Using Proposition 3.11 and the same argument as in the proof of Proposition 3.10 we may suppose that $Y = \mathbf{V}(x_{e+1}, \dots, x_n)$ since we are computing local multiplicities and they are preserved by the map ϕ used in the proof of Proposition 3.10. Now, the set of points $W \subset Y$ at which the pair (X, Y) fails to satisfy Whitney's condition (B) is Zariski-closed in Y and defined over \mathbb{Q} (this can be seen for example by using Remark 4.2 in [4]). Let $U := Y - W$. By Theorem 2.15, every point $y \in U$ has a euclidean neighborhood on which the sequence $m_\bullet(X, \bullet)$ is equal to $m_\bullet(X, y)$. But U is connected, therefore $m_\bullet(X, \bullet)$ is constant on U , proving the theorem. \square

Now we have the ability to (probabilistically) minimize a given Whitney stratification. For a given variety $X \subset \mathbb{C}^n$, the output of $\mathbf{Whitney}(X)$ consists of a flag

$$W_0 \subset W_1 \subset \dots \subset W_d = X$$

on X . Each W_i is given by its \mathbb{Q} -irreducible components W_{i1}, \dots, W_{ir_i} . Let us define $W_{00} := X$. These W_{ij} may now be organized in a tree as follows: Each node of this tree contains one of the W_{ij} . The children of a node containing W_{ij} are given by those $W_{i+1,k}$ with $W_{i+1,k} \subset W_{ij}$. Let us call such a data structure a *Whitney tree* of X . Note that such a tree may be extracted from the output of $\mathbf{Whitney}(X)$ by simple containment checks using Gröbner bases. From such a tree, we may now minimize a given Whitney stratification as follows:

WhitneyMinimize (\mathcal{T}_X)	
	Input: A Whitney tree \mathcal{T}_X on X . Output: A minimal Whitney stratification of X .
1	For each node W in \mathcal{T}_X
2	$P \leftarrow \{(W, Z, Y) \mid Z, Y \text{ are nodes of } \mathcal{T}_X \text{ with } W \subset Z \subset Y\}$
3	If $\mathbf{Mult}(Y, W) = \mathbf{Mult}(Y, Z)$ for all $(W, Z, Y) \in P$
4	Delete the node W from \mathcal{T}_X
5	For each level i of \mathcal{T}_X
6	$W_{d-i} \leftarrow$ union of all components in level i
7	Return W_\bullet

Lemma 3.15. *WhitneyMinimize is correct and terminates.*

PROOF. The termination is clear. Suppose that we have a triple of varieties $W \subset Z \subset Y$ with W and Z irreducible and that for generic $w \in W$ and $z \in Z$ we have $m_\bullet(Y, w) =$

$m_\bullet(Y, z)$. Then there exists a Zariski-dense subset U of Z such that $U \cap W \neq \emptyset$ and such that $m_\bullet(Y, \bullet)$ is constant on U . Hence the points in W at which the pair (Y, Z) does not satisfy Whitney's condition (B) have at least codimension one in W by Theorem 2.15. In the situation of **WhitneyMinimize**, W can thus be removed without destroying the property of the output flag W_\bullet being a Whitney stratification. This proves the correctness. \square

Remark 3.16. Note that we may also combine **WhitneyMinimize** directly with **WhitPoints**: For given X and irreducible Y let h be the output of **WhitPoints**(X, Y) and let $A := Y \cap \mathbf{V}(h)$. We know that the set of points in Y where Whitney's condition (B) fails to hold with respect to X and Y is contained in A . We may then compute the irreducible components A_1, \dots, A_r of A and compare for each A_i the output of **Mult**(X, Y) with **Mult**(X, A_i). If these sequences are equal we recursively go through the same procedure with X and A_i . If they are not equal we append A_i to a list of output components which we return in the end. This is potentially more optimal than minimizing after the stratification is finished because it prevents a build up of unneeded varieties.

4. Runtime Tests

In this section we collect some runtime comparisons of the new Whitney stratification algorithm described in this note with the algorithm of [7, 9] for a variety of examples. We also make note of if the resulting stratification is minimal and, if the stratification is not minimal, the time it takes to compute the coarsening to the minimal one.

Note that in Table 1 we do not compare our implementation to the algorithms of [11] or of [13, 14] as all these algorithms are unable to successfully compute the Whitney stratification of the Whitney umbrella, the simplest non-trivial example, which is a surface in \mathbb{C}^3 defined by the binomial equation $x^2 - y^2z = 0$. In both cases we have let these algorithms run for greater than 24 hours on the Whitney umbrella on our test machine and neither finished. Additionally, in the case of the quantifier elimination based method of [13, 14], the Maple implementation of quantifier elimination eventually gives a stack limit exceeded error even if we allow it to make the stack arbitrarily large on our test machine (which has an Intel Xeon W-3365 CPU with 1032 GB of RAM). For comparison the Whitney umbrella example takes 0.1s to run with the algorithm of [7, 9] and 0.08s with the algorithm of Whitney algorithm of §3; both produce a minimal stratification.

The implementation of our Algorithm used to create these benchmarks is available in version 2.11, and above, of the `WhitneyStratifications` Macaulay2 package, which is available on the first author's website at:

<http://martin-helmer.com/Software/WhitStrat/WhitneyStratifications.m2>.

With this Macaulay2 package loaded and given a polynomial ideal I our algorithm in Section 3 is called with `whitneyStratify(I, AssocPrimes=>false)`, while the algorithm of [7, 9] is called with `whitneyStratify(I, AssocPrimes=>true)`. Given a stratification W output by one of these methods the **WhitneyMinimize** procedure is run using the command `minCoarsenWS(W)`.

	Associated Primes Alg. of [7, 9]		Whitney Alg. of §3		WhitneyMinimize
	Run time	Minimal	Run time	Minimal	Run time
Input					
Whitney Cusp	0.12s	Yes	0.09s	Yes	0.3s
Example 1.1	–	–	7.3s	No	140.5s
X_1 , see (4)	85.3s	Yes	82.8s	Yes	1.1s
X_2 , see (5)	4.8s	No	4.6s	No	68.1s
X_3 , see (6)	182.3s	–	192.7s	No	–
X_4 , see (7)	2.3s	Yes	1.5s	Yes	0.5s
X_5 , see (8)	1.3s	Yes	0.9s	Yes	2.4s
X_6 , see (9)	–	–	6080.8s	–	–
X_7 , see (10)	–	–	15.4s	No	20.2s
X_8 , see (11)	–	–	1177.6s	No	1934.6s

TABLE 1. Run times of several examples, – denotes examples for which execution was stopped after 24 hours. These computations were run in version 1.24.5 of Macaulay2 on a workstation with an Intel Xeon W-3365 CPU and 1TB of RAM.

The Whitney cusp is defined by the equation $x_2^2 + x_1^3 - x_1^2 x_3^2 = 0$. Below we list the defining equations of the remaining examples presented in Table 1. Examples X_3 and X_6 are computations which arise when using map stratification to identify the singularities of Feynman integrals in quantum field theory in physics [10]. While most examples which did not finish were stopped after 8 hours, given the longer run time of the example X_6 with the new algorithm we left the associated primes algorithm running for over 7 days, without it finishing. We note that the optimization described in Remark 3.16 is not part of our implementation. With this optimization we hope to improve our runtimes further: On the example X_3 the stratification produced by our algorithm includes a substantial amount of additional strata compared to running the HN algorithm [7, 9] on X_3 , likely causing a significant computational overhead due to a build up of redundant varieties.

$$X_1 = \mathbf{V} \left(x_1^2 x_3 - x_2^2, x_2^4 - x_1 x_2^2 - x_3 x_4^2, x_1^2 x_2^2 - x_1^3 - x_4^2 \right) \subset \mathbb{C}^4 \quad (4)$$

$$X_2 = \mathbf{V} \left(x_1^6 + x_2^6 + x_1^4 x_3 x_4 + x_3^3 \right) \subset \mathbb{C}^4 \quad (5)$$

$$X_3 = \mathbf{V} \left(-p_1 y_1 y_3 y_4 - p_1 y_2 y_3 y_4 - s y_1 y_3 y_5 - p_2 y_1 y_4 y_5 - t y_2 y_4 y_5 \right) \subset \mathbb{C}^9 \quad (6)$$

$$X_4 = \mathbf{V} \left(x_1^3 - 2x_1^2 x_4 + x_1^3 + x_5^2 x_4 + x_2^2 x_1 - x_2 x_1 x_3 + x_5^3 \right) \subset \mathbb{C}^5 \quad (7)$$

$$X_5 = \mathbf{V} \left(x_2 x_5^2 - x_1^2, x_2 x_4 x_5 - x_3^2 - x_2, x_1^2 x_4 - x_3^2 x_5 - x_2 x_5 \right) \subset \mathbb{C}^5 \quad (8)$$

$$X_6 = \mathbf{V}(-p_1x_1x_3x_4 - p_1x_2x_3x_4 - s x_1x_3x_5 - p_2x_1x_4x_5 - t x_2x_4x_5 - p_1x_3x_4x_5) \subset \mathbb{C}^9 \quad (9)$$

$$X_7 = \mathbf{V}(x_1x_2^3 - 2x_1x_2^2x_4 + x_1x_2x_3x_5 + x_1x_2x_4^2 + x_1x_2x_5 - x_1x_3x_4x_5 - x_1x_4x_5 - x_2^2x_3x_5 - x_2^2x_4 - x_2^2x_5 + x_2x_3x_4x_5 - x_2x_3x_5 + 2x_2x_4^2 + x_2x_4x_5 - x_2x_5 - x_3^2x_5^2 + x_3x_4x_5 - 2x_3x_5^2 - x_4^3 + x_4x_5 - x_5^2) \quad (10)$$

$$X_8 = \mathbf{V}(x_1^2x_2^2x_3 - 2x_1^2x_2x_3x_4 - 2x_1^2x_2x_3x_5 + x_1^2x_3x_4^2 + 2x_1^2x_3x_4x_5 + x_1^2x_3x_5^2 - 2x_1x_2^3x_3 + 4x_1x_2^2x_3x_4 + 4x_1x_2^2x_3x_5 - x_1x_2^2x_4 - 2x_1x_2x_3x_4^2 - 6x_1x_2x_3x_4x_5 - 2x_1x_2x_3x_5^2 + x_1x_2x_4^2 + x_1x_2x_4x_5 + 2x_1x_3x_4^2x_5 + 2x_1x_3x_4x_5^2 - x_1x_4^2x_5 + x_2^4x_3 - 2x_2^3x_3x_4 - 2x_2^3x_3x_5 + x_2^3x_4 + x_2^2x_3x_4^2 + 4x_2^2x_3x_4x_5 + x_2^2x_3x_5^2 - 2x_2^2x_4^2 - x_2^2x_4x_5 - 2x_2x_3x_4^2x_5 - 2x_2x_3x_4x_5^2 + x_2x_4^3 + 2x_2x_4^2x_5 + x_3x_4^2x_5^2 - x_4^3x_5) \quad (11)$$

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