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Hybrid high-order approximations of div-curl systems on domains with general topology

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Abstract

We devise and analyze hybrid polyhedral methods of arbitrary order for the approximation of div-curl systems on three-dimensional domains featuring non-trivial topology. The systems we focus on stem from magnetostatics models, and can either be first-order (field formulation) or second-order (vector potential formulation). The well-posedness of our methods essentially relies on topologically general discrete (hybrid) versions of the first and second Weber inequalities. In turn, our error analysis is performed under low regularity assumptions on the solutions. Finally, we provide a comprehensive numerical validation of our methodology.

Keywords: Div-curl systems; Polyhedral meshes; Hybrid methods; de Rham cohomology; Computational topology.

AMS Subject Classification 2020: 65N12, 14F40, 35Q60.

1 Introduction

Let Ω be a domain in \mathbb{R}^3 , i.e. a bounded and connected Lipschitz open set of \mathbb{R}^3 . In the L^2 framework, the standard (primal) de Rham complex reads:

$$\{0\} \xrightarrow{0} H^1(\Omega) \xrightarrow{\mathbf{grad}} \mathbf{H}(\mathbf{curl}; \Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}(\mathbf{div}; \Omega) \xrightarrow{\mathbf{div}} L^2(\Omega) \xrightarrow{0} \{0\}, \quad (1)$$

with cohomology spaces defined by

$$\begin{aligned} \mathfrak{H}^0 &:= \text{Ker}(\mathbf{grad}) / \text{Im}(0), & \mathfrak{H}^1 &:= \text{Ker}(\mathbf{curl}) / \text{Im}(\mathbf{grad}), \\ \mathfrak{H}^2 &:= \text{Ker}(\mathbf{div}) / \text{Im}(\mathbf{curl}), & \mathfrak{H}^3 &:= (\text{Ker}(0) =) L^2(\Omega) / \text{Im}(\mathbf{div}). \end{aligned} \quad (2)$$

The spaces \mathfrak{H}^n have respective dimensions equal to the Betti numbers $\beta_0 = 1$ (number of connected components of Ω), $\beta_1 \in \mathbb{N}$ (number of tunnels crossing through Ω), $\beta_2 \in \mathbb{N}$ (number of voids encapsulated into Ω), and $\beta_3 = 0$ (since $\Omega \subset \mathbb{R}^3$). In turn, the dual de Rham complex reads (modulo missing minus signs in front of the operators \mathbf{div}_0 and \mathbf{grad}_0):

$$\{0\} \xleftarrow{0} L^2(\Omega) \xleftarrow{\mathbf{div}_0} \mathbf{H}_0(\mathbf{div}; \Omega) \xleftarrow{\mathbf{curl}_0} \mathbf{H}_0(\mathbf{curl}; \Omega) \xleftarrow{\mathbf{grad}_0} H_0^1(\Omega) \xleftarrow{0} \{0\}, \quad (3)$$

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where the operators div_0 , \mathbf{curl}_0 and \mathbf{grad}_0 are respectively L^2 -adjoint to the operators $-\mathbf{grad}$, \mathbf{curl} and $-\operatorname{div}$. The zero subscripts are here meant to recall that the domains of the adjoint differential operators embed zero boundary conditions. The corresponding homology spaces are defined by

$$\begin{aligned}\mathfrak{H}_0^3 &:= (\operatorname{Ker}(0) =) L^2(\Omega) / \operatorname{Im}(\operatorname{div}_0), & \mathfrak{H}_0^2 &:= \operatorname{Ker}(\operatorname{div}_0) / \operatorname{Im}(\mathbf{curl}_0), \\ \mathfrak{H}_0^1 &:= \operatorname{Ker}(\mathbf{curl}_0) / \operatorname{Im}(\mathbf{grad}_0), & \mathfrak{H}_0^0 &:= \operatorname{Ker}(\mathbf{grad}_0) / \operatorname{Im}(0).\end{aligned}\quad (4)$$

Importantly, there holds $\mathfrak{H}_0^n = \mathfrak{H}_0^{3-n}$ (this is the so-called *Poincaré–Lefschetz duality*).

Let μ be a possibly variable coefficient, assumed to be bounded by above and by below by positive constants. In what follows, the symbols \perp and \perp_μ respectively stand for standard and μ -weighted L^2 -orthogonality. We are interested in solving problems of the following form: Given some datum $\mathbf{j} \in \operatorname{Ker}(\operatorname{Div})$, find $\mathbf{h} \in \operatorname{Dom}(\operatorname{Curl}) \cap \operatorname{Im}(\operatorname{Grad})^{\perp_\mu}$ such that

$$\mathfrak{C}(\mathbf{h}) = \mathbf{j}, \quad (5)$$

where the bounded linear operator $\mathfrak{C} : \operatorname{Dom}(\operatorname{Curl}) \cap \operatorname{Im}(\operatorname{Grad})^{\perp_\mu} \rightarrow \operatorname{Ker}(\operatorname{Div})$ is such that $\mathfrak{C}(\mathbf{v}) = \operatorname{Curl}(\mathbf{v})$ for all $\mathbf{v} \in \operatorname{Dom}(\mathfrak{C})$. Here, the sequence of operators $\{\operatorname{Grad}, \operatorname{Curl}, \operatorname{Div}\}$ can either be $\{\mathbf{grad}, \mathbf{curl}, \operatorname{div}\}$ from the primal de Rham complex (1), or $\{\mathbf{grad}_0, \mathbf{curl}_0, \operatorname{div}_0\}$ from the dual de Rham complex (3). Accordingly, in the first case we let $\mathfrak{h}^n := \mathfrak{H}^n$, whereas in the second case we let $\mathfrak{h}^n := \mathfrak{H}_0^n$. It is an easy matter to verify that the operator \mathfrak{C} is Fredholm, with null space $\operatorname{Ker}(\mathfrak{C}) = \operatorname{Ker}(\operatorname{Curl}) \cap \operatorname{Im}(\operatorname{Grad})^{\perp_\mu} \cong \mathfrak{h}^1$, and defect space $\operatorname{Codom}(\mathfrak{C}) / \operatorname{Im}(\mathfrak{C}) = \operatorname{Ker}(\operatorname{Div}) / \operatorname{Im}(\operatorname{Curl}) = \mathfrak{h}^2$. The following Fredholm alternative thus holds true for Problem (5):

- either $\dim(\mathfrak{h}^1) = \dim(\mathfrak{h}^2) = 0$, then Problem (5) admits a unique solution;
- in the opposite case, for Problem (5) to admit a solution, it is necessary that $\mathbf{j} \in \operatorname{Im}(\mathfrak{C}) = \operatorname{Im}(\operatorname{Curl})$ (which amounts, for $\mathbf{j} \in \operatorname{Ker}(\operatorname{Div})$, to additionally satisfying $\mathbf{j} \perp \mathfrak{h}^2$); the solution $\mathbf{h} \perp_\mu \operatorname{Im}(\operatorname{Grad})$ is then unique up to an element of \mathfrak{h}^1 (retrieving uniqueness thus amounts to imposing $\dim(\mathfrak{h}^1)$ additional constraints on \mathbf{h}).

In practice, $\mathbf{j} : \Omega \rightarrow \mathbb{R}^3$ is a given current density, $\mu : \Omega \rightarrow \mathbb{R}_+^*$ is the magnetic permeability of the medium, and $\mathbf{h} : \Omega \rightarrow \mathbb{R}^3$ is the sought magnetic field. Problem (5) is thus the prototype of a first-order magnetostatics model (field formulation), which can be endowed with either tangential (based on (3)) or normal (based on (1)) boundary conditions. We refer the reader to Section 4.1.1 for a somewhat more conventional rewriting of the latter problem. Letting $\operatorname{Curl}^\star$ denote the L^2 -adjoint of the operator Curl , since $\operatorname{Ker}(\operatorname{Curl})^\perp = \operatorname{Im}(\operatorname{Curl}^\star)$, if the solution $\mathbf{h} \perp_\mu \operatorname{Im}(\operatorname{Grad})$ to Problem (5) (assuming that it exists) additionally satisfies $\mathbf{h} \perp_\mu \mathfrak{h}^1$, then it can be written as $\mathbf{h} = \mu^{-1} \operatorname{Curl}^\star(\mathbf{a})$ for some vector potential $\mathbf{a} \in \operatorname{Dom}(\operatorname{Curl}^\star)$. Obviously, the vector potential \mathbf{a} is non-unique, and is only defined up to an element of $\operatorname{Ker}(\operatorname{Curl}^\star)$ (gauge). Enforcing (among other possible choices) zero gauge, Problem (5) then equivalently rewrites: Find $\mathbf{a} \in \operatorname{Ker}(\operatorname{Curl}^\star)^\perp$ such that

$$\operatorname{Curl}(\mu^{-1} \operatorname{Curl}^\star(\mathbf{a})) = \mathbf{j}, \quad (6)$$

and the (unique) solution to this problem does exist as soon as $\mathbf{j} \in \operatorname{Ker}(\operatorname{Div})$ additionally satisfies $\mathbf{j} \perp \mathfrak{h}^2$. In practice, $\mathbf{a} : \Omega \rightarrow \mathbb{R}^3$ represents the magnetic vector potential. Problem (6) is thus the prototype of a second-order magnetostatics model (vector potential formulation). We refer the reader to Section 4.1.2 for a more conventional rewriting of the latter model.

In this work, we aim at devising arbitrary-order *hybrid* polyhedral discretizations of the first- and second-order model problems (5) and (6). By definition, hybrid methods only attach unknowns to the faces and to the cells of the partition. Examples of such approaches are the Hybridizable Discontinuous Galerkin (HDG) [16], the Weak Galerkin (WG) [36], the Hybrid High-Order (HHO) [21, 22], or

the non-conforming Virtual Element (ncVE) [32, 5] methods. As may be expected, and as first documented in [14, 15], all these technologies share very tight links. The devising of hybrid polyhedral methods for div-curl systems has already been addressed to some extent in the HDG and WG literatures. For second-order problems, these contributions include [34, 12, 11] (cf. also [23]) for HDG and [33] for WG, all of them restricted to trivial topologies. Regarding first-order problems, the only contributions we are aware of are [31], in which a primal WG method is introduced, and [35, 8, 9], in which alternative (primal-dual) WG approaches are studied. In all these contributions, full face polynomial spaces are employed to discretize the vectorial variable. Also, either the case of non-trivial topologies is not covered (like in [31]), or it is but in a non-systematic way. Recently, the discretization of both first- and second-order magnetostatics models has been studied in the HHO context [10]. Therein, the vectorial variable is sought into trimmed face polynomial spaces, reusing and extending ideas introduced in [12]. The contribution [10], however, only covers the case of trivial topologies. Our objective in the present article is to fill this gap. We aim at devising, analyzing, and numerically assessing trimmed HHO methods for (first- and second-order) problems posed on domains with arbitrary topology. With respect to [35, 8, 9], apart from using trimmed face spaces, our approach for first-order models is also based on a different, more suitable variational formulation (inspired from [29]), which enables a leaner construction and a systematic handling of harmonic fields. In turn, our analysis hinges on the systematic use of the (topologically general) hybrid Weber inequalities recently established by the two last authors in [30]. Let us finally stress that, from an application point of view, non-trivial topologies are ubiquitous (think, e.g., of a toroidal coil). Their proper handling is thus of paramount importance in practice.

The material is organized as follows. In Section 2, we introduce both the topological and functional frameworks, and we recall the first and second Helmholtz–Hodge decompositions. In Section 3, we introduce the discrete setting, including polyhedral discretizations, polynomial decompositions, and hybrid spaces. In Section 4, we devise and analyze HHO methods for (first- and second-order) model problems of the form (5) and (6). Finally, in Section 5, we provide a comprehensive set of numerical experiments on non-trivial domains, assessing the relevance of our methodology.

2 Preliminaries

2.1 Topological framework

We recall that Ω denotes a domain in \mathbb{R}^3 , that is a bounded and connected Lipschitz open set of \mathbb{R}^3 . Let $\Gamma := \partial\Omega$ denote its boundary, and $\mathbf{n} : \Gamma \rightarrow \mathbb{R}^3$ be the (almost everywhere defined) unit vector field normal to Γ pointing outward from Ω . We recall that the Betti numbers β_1 and β_2 of Ω respectively denote the number of tunnels crossing through Ω ($\beta_1 \in \mathbb{N}$), and the number of voids encapsulated into Ω ($\beta_2 \in \mathbb{N}$). For Ω simply-connected, $\beta_1 = 0$. Likewise, when the boundary Γ of Ω is connected, $\beta_2 = 0$. In what follows, when both β_1, β_2 are equal to zero, we shall say that the topology of Ω is trivial.

When $\beta_1 > 0$, we make the following standard assumption: there exist β_1 (non-intersecting) orientable two-dimensional manifolds with boundary $\Sigma_1, \dots, \Sigma_{\beta_1}$, called *cutting surfaces*, satisfying $\partial\Sigma_i \subset \Gamma$ for all $i \in \{1, \dots, \beta_1\}$, and such that the open set $\hat{\Omega} := \Omega \setminus \cup_{i \in \{1, \dots, \beta_1\}} \Sigma_i$ is not crossed by any tunnel (its first Betti number is zero). Remark that $\hat{\Gamma} := \partial\hat{\Omega} = \Gamma \cup \cup_{i \in \{1, \dots, \beta_1\}} \Sigma_i$. We will assume in what follows that $\hat{\Omega}$ is connected (which is generally the case starting from a connected domain Ω), and that the cutting surfaces are sufficiently regular so that the set $\hat{\Omega}$ is pseudo-Lipschitz (cf. [4, Def. 3.2.2]). For any $i \in \{1, \dots, \beta_1\}$, we let $\mathbf{n}_{\Sigma_i} : \Sigma_i \rightarrow \mathbb{R}^3$ be the (almost everywhere defined) unit vector field normal to Σ_i , the orientation of which we arbitrarily prescribe (recall that the cutting surfaces Σ_i are supposed to be orientable). The orientation of \mathbf{n}_{Σ_i} being prescribed, we associate

the tag “+” to the side of $\hat{\Omega}$ (with respect to Σ_i) for which \mathbf{n}_{Σ_i} is outward, whereas the tag “−” is associated to the side of $\hat{\Omega}$ for which \mathbf{n}_{Σ_i} is inward.

When $\beta_2 > 0$, letting Γ_0 be the (connected) boundary of the only unbounded component of the exterior open set $\mathbb{R}^3 \setminus \overline{\Omega}$, there exist β_2 (maximally) connected components $\Gamma_1, \dots, \Gamma_{\beta_2}$ of Γ such that $\Gamma = \cup_{j \in \{0, \dots, \beta_2\}} \Gamma_j$. When $\beta_2 = 0$, there holds $\Gamma = \Gamma_0$.

2.2 Functional framework

Let $\mu : \Omega \rightarrow \mathbb{R}$ be a given function satisfying, for real numbers $0 < \mu_b \leq \mu_{\sharp} < \infty$,

$$\mu_b \leq \mu(\mathbf{x}) \leq \mu_{\sharp} \quad \text{for a.e. } \mathbf{x} \in \Omega. \quad (7)$$

For $m \in \{2, 3\}$, and for X an m -dimensional, (relatively) open pseudo-Lipschitz subset of $\overline{\Omega}$, we let $L^2(X)$ (respectively, $\mathbf{L}^2(X)$) denote the Lebesgue space of square-integrable functions (respectively, \mathbb{R}^m -valued vector fields) over X . The standard inner products (and norms) in $L^2(X)$ and $\mathbf{L}^2(X)$ are irrespectively denoted by $(\mathbf{f}, \mathbf{g})_X := \int_X \mathbf{f} \cdot \mathbf{g}$ (and $\|\cdot\|_{0,X} := \sqrt{(\cdot, \cdot)_X}$). We also define $L_0^2(X) := \{v \in L^2(X) \mid \int_X v = 0\}$ and $\mathbf{L}_0^2(X) := \{v \in \mathbf{L}^2(X) \mid \int_X v = \mathbf{0}\}$. For $s > 0$, we let $H^s(X)$ (respectively, $\mathbf{H}^s(X)$) denote the Sobolev space of functions in $L^2(X)$ (respectively, \mathbb{R}^m -valued vector fields in $\mathbf{L}^2(X)$) possessing square-integrable partial weak derivatives up to order s over X (for fractional s , we follow the classical Sobolev–Slobodeckij construction). The standard norms (and semi-norms) in $H^s(X)$ and $\mathbf{H}^s(X)$ are irrespectively denoted by $\|\cdot\|_{s,X}$ (and $|\cdot|_{s,X}$). Given a Lipschitz partition $\mathcal{K} := \{K\}$ of X (in the sense that $\bigcup_{K \in \mathcal{K}} \overline{K} = \overline{X}$), we also define the following broken Sobolev space:

$$\mathbf{H}^s(X) := \{v \in \mathbf{L}^2(X) : v|_K \in \mathbf{H}^s(K) \forall K \in \mathcal{K}\}.$$

Let Y be a three-dimensional, open Lipschitz subset of Ω . Classically, we let

$$\begin{aligned} \mathbf{H}(\mathbf{curl}; Y) &:= \{v \in \mathbf{L}^2(Y) \mid \mathbf{curl} v \in \mathbf{L}^2(Y)\}, \\ \mathbf{H}(\text{div}_{\mu}; Y) &:= \{v \in \mathbf{L}^2(Y) \mid \text{div}(\mu v) \in L^2(Y)\}, \end{aligned}$$

with $\mathbf{H}(\text{div}; Y) := \mathbf{H}(\text{div}_1; Y)$, as well as their subspaces

$$\begin{aligned} \mathbf{H}(\mathbf{curl}^0; Y) &:= \{v \in \mathbf{H}(\mathbf{curl}; Y) \mid \mathbf{curl} v \equiv \mathbf{0}\}, \\ \mathbf{H}(\text{div}_{\mu}^0; Y) &:= \{v \in \mathbf{H}(\text{div}_{\mu}; Y) \mid \text{div}(\mu v) \equiv 0\}. \end{aligned}$$

Let $\mathbf{n}_{\partial Y} : \partial Y \rightarrow \mathbb{R}^3$ denote the (almost everywhere defined) outward, unit normal vector field to ∂Y . For $v \in \mathbf{H}(\text{div}_{\mu}; Y)$, the normal trace of μv on ∂Y can be defined as an element of $H^{-\frac{1}{2}}(\partial Y)$ (space of bounded linear forms on $H^{\frac{1}{2}}(\partial Y)$), denoted $(\mu v)|_{\partial Y} \cdot \mathbf{n}_{\partial Y}$, with continuous mapping $v \mapsto (\mu v)|_{\partial Y} \cdot \mathbf{n}_{\partial Y}$ from $\mathbf{H}(\text{div}_{\mu}; Y)$ to $H^{-\frac{1}{2}}(\partial Y)$. Likewise, for $v \in \mathbf{H}(\mathbf{curl}; Y)$, one can give a sense to the rotated tangential trace of v on ∂Y as an element of $H^{-\frac{1}{2}}(\partial Y)^3$ (space of bounded linear forms on $H^{\frac{1}{2}}(\partial Y)^3$), denoted $v|_{\partial Y} \times \mathbf{n}_{\partial Y}$, with continuous mapping $v \mapsto v|_{\partial Y} \times \mathbf{n}_{\partial Y}$ from $\mathbf{H}(\mathbf{curl}; Y)$ to $H^{-\frac{1}{2}}(\partial Y)^3$. Following [30, Rmk. 1], from now on, tangential vector fields are identified to two-dimensional vector fields. The relevant boundary functional spaces then become $\mathbf{L}^2(\partial Y) = L^2(\partial Y)^2$, $\mathbf{H}^{\frac{1}{2}}(\partial Y) := H^{\frac{1}{2}}(\partial Y)^2$, and $\mathbf{H}^{-\frac{1}{2}}(\partial Y) := H^{-\frac{1}{2}}(\partial Y)^2$, and one may abuse the notation and write $v|_{\partial Y} \times \mathbf{n}_{\partial Y} \in \mathbf{H}^{-\frac{1}{2}}(\partial Y)$. In what follows, the duality pairings between $H^{-\frac{1}{2}}(\partial Y)$ and $H^{\frac{1}{2}}(\partial Y)$ on the one side, and between $\mathbf{H}^{-\frac{1}{2}}(\partial Y)$ and $\mathbf{H}^{\frac{1}{2}}(\partial Y)$ on the other side, are irrespectively denoted by $\langle \cdot, \cdot \rangle_{\partial Y}$. If $v \in \mathbf{H}(\mathbf{curl}; Y) \cap \mathbf{H}(\text{div}; Y) \cap \mathbf{H}^s(Y)$ for some $s > \frac{1}{2}$, then for almost every $\mathbf{x} \in \partial Y$,

$$v|_{\partial Y}(\mathbf{x}) = (v|_{\partial Y} \cdot \mathbf{n}_{\partial Y})(\mathbf{x}) \mathbf{n}_{\partial Y}(\mathbf{x}) + \mathbf{n}_{\partial Y}(\mathbf{x}) \times (v|_{\partial Y} \times \mathbf{n}_{\partial Y})(\mathbf{x}).$$

In this case, $\mathbf{v}|_{\partial Y} \cdot \mathbf{n}_{\partial Y} \in L^2(\partial Y)$ and $\mathbf{v}|_{\partial Y} \times \mathbf{n}_{\partial Y} \in L^2(\partial Y)$.

To account for essential boundary conditions, we introduce the following subspaces of $\mathbf{H}(\mathbf{curl}; \Omega)$ and $\mathbf{H}(\text{div}_\mu; \Omega)$:

$$\mathbf{H}_0(\mathbf{curl}; \Omega) := \{ \mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega) \mid \mathbf{v}|_{\Gamma} \times \mathbf{n} \equiv \mathbf{0} \}$$

and $\mathbf{H}_0(\mathbf{curl}^0; \Omega) := \mathbf{H}_0(\mathbf{curl}; \Omega) \cap \mathbf{H}(\mathbf{curl}^0; \Omega)$, as well as

$$\mathbf{H}_0(\text{div}_\mu; \Omega) := \{ \mathbf{v} \in \mathbf{H}(\text{div}_\mu; \Omega) \mid (\mu \mathbf{v})|_{\Gamma} \cdot \mathbf{n} \equiv 0 \}$$

and $\mathbf{H}_0(\text{div}_\mu^0; \Omega) := \mathbf{H}_0(\text{div}_\mu; \Omega) \cap \mathbf{H}(\text{div}_\mu^0; \Omega)$. We also define $H_0^1(\Omega) := \{ v \in H^1(\Omega) \mid v|_{\Gamma} \equiv 0 \}$. Last, assume that the first Betti number β_1 of Ω is positive. Then, for $v \in L^2(\hat{\Omega})$ (resp. $\mathbf{v} \in \mathbf{L}^2(\hat{\Omega})$), we denote by \check{v} (resp. $\check{\mathbf{v}}$) its continuation to $L^2(\Omega)$ (resp. $\mathbf{L}^2(\Omega)$). Also, for any $v : \hat{\Omega} \rightarrow \mathbb{R}$, and $i \in \{1, \dots, \beta_1\}$, denoting $v|_{\Sigma_i}^+$ and $v|_{\Sigma_i}^-$ the traces of v on Σ_i defined (if need be, in a weak sense) from both sides of $\hat{\Omega}$ (respectively tagged by “+” and “-”), we define the jump of v on Σ_i by

$$\llbracket v \rrbracket_{\Sigma_i} := v|_{\Sigma_i}^+ - v|_{\Sigma_i}^-. \quad (8)$$

2.3 Helmholtz–Hodge decompositions

We here collect classical results about Helmholtz–Hodge decompositions; for further details, we refer the reader to [26, 19, 2] (cf. also [27, 4], as well as [30, Sec. 2.3]). Before proceeding, we define $L_\mu^2(\Omega) := (L^2(\Omega), (\mu \cdot, \cdot)_\Omega)$.

2.3.1 1st Helmholtz–Hodge decomposition

Let us first consider the harmonic space

$$\mathbf{H}_\tau(\Omega) := \mathbf{H}_0(\mathbf{curl}^0; \Omega) \cap \mathbf{H}(\text{div}_\mu^0; \Omega), \quad (9)$$

which, by (4), satisfies $\mathbf{H}_\tau(\Omega) \cong \mathfrak{H}_0^1$. In particular, $\mathbf{H}_\tau(\Omega)$ has dimension β_2 , and it can be proved that vector fields $\mathbf{w} \in \mathbf{H}_\tau(\Omega)$ are entirely characterized by the data of $(\langle (\mu \mathbf{w})|_{\Gamma_j} \cdot \mathbf{n}, \mathbf{1} \rangle_{\Gamma_j} \in \mathbb{R})_{j \in \{1, \dots, \beta_2\}}$. The following $L_\mu^2(\Omega)$ -orthogonal Helmholtz–Hodge decomposition holds true:

$$L^2(\Omega) = \mathbf{grad}(H_0^1(\Omega)) \overset{\perp_\mu}{\oplus} \frac{\mu \sharp}{\mu} \mathbf{curl}(H^1(\Omega) \cap L_0^2(\Omega)) \overset{\perp_\mu}{\oplus} \mathbf{H}_\tau(\Omega). \quad (10)$$

Furthermore, letting

$$H_\Gamma^1(\Omega) := \{ v \in H^1(\Omega), v|_{\Gamma_0} \equiv 0 \mid \exists (\gamma_j) \in \mathbb{R}^{\beta_2}, v|_{\Gamma_j} \equiv \gamma_j \forall j \in \{1, \dots, \beta_2\} \}, \quad (11)$$

it can be noticed that

$$\mathbf{grad}(H_\Gamma^1(\Omega)) = \mathbf{grad}(H_0^1(\Omega)) \overset{\perp_\mu}{\oplus} \mathbf{H}_\tau(\Omega). \quad (12)$$

Remark, also, that $\mathbf{grad}(H_\Gamma^1(\Omega)) \subset \mathbf{H}_0(\mathbf{curl}^0; \Omega)$.

2.3.2 2nd Helmholtz–Hodge decomposition

Let us now consider the harmonic space

$$\mathbf{H}_n(\Omega) := \mathbf{H}(\mathbf{curl}^0; \Omega) \cap \mathbf{H}_0(\text{div}_\mu^0; \Omega), \quad (13)$$

which, by (4), satisfies $\mathbf{H}_n(\Omega) \cong \mathfrak{H}_0^2$. In particular, $\mathbf{H}_n(\Omega)$ has dimension β_1 , and it can be proved that vector fields $\mathbf{w} \in \mathbf{H}_n(\Omega)$ are entirely characterized by the data of $(\langle (\mu \mathbf{w})|_{\Sigma_i} \cdot \mathbf{n}_{\Sigma_i}, 1 \rangle_{\Sigma_i} \in \mathbb{R})_{i \in \{1, \dots, \beta_1\}}$. The following $L_\mu^2(\Omega)$ -orthogonal Helmholtz–Hodge decomposition holds true:

$$L^2(\Omega) = \mathbf{grad} (H^1(\Omega) \cap L_0^2(\Omega)) \overset{\perp_\mu}{\oplus} \frac{\mu_\#}{\mu} \mathbf{curl} (H^1(\Omega) \cap \mathbf{H}_0(\mathbf{curl}; \Omega)) \overset{\perp_\mu}{\oplus} \mathbf{H}_n(\Omega). \quad (14)$$

Furthermore, letting

$$H_\Sigma^1(\hat{\Omega}) := \{v \in H^1(\hat{\Omega}) \cap L_0^2(\hat{\Omega}) \mid \exists (\sigma_i) \in \mathbb{R}^{\beta_1}, \llbracket v \rrbracket_{\Sigma_i} \equiv \sigma_i \forall i \in \{1, \dots, \beta_1\}\}, \quad (15)$$

it can be noticed that

$$\check{\mathbf{grad}}(H_\Sigma^1(\hat{\Omega})) = \mathbf{grad} (H^1(\Omega) \cap L_0^2(\Omega)) \overset{\perp_\mu}{\oplus} \mathbf{H}_n(\Omega). \quad (16)$$

We remind the reader that, for $v \in H^1(\hat{\Omega})$, $\check{\mathbf{grad}} v$ is the continuation to $L^2(\Omega)$ of $\mathbf{grad} v \in L^2(\hat{\Omega})$, and that the jump $\llbracket \cdot \rrbracket_{\Sigma_i}$ is defined in (8). Note also that $\check{\mathbf{grad}}(H_\Sigma^1(\hat{\Omega})) \subset \mathbf{H}(\mathbf{curl}^0; \Omega)$.

3 Discrete setting

From now on, we assume that the domain $\Omega \subset \mathbb{R}^3$ is a (Lipschitz) polyhedron.

3.1 Polyhedral discretizations

We consider discretizations $\mathcal{D} := (\mathcal{T}, \mathcal{F})$ of $\Omega \subset \mathbb{R}^3$ in the sense of [20, Def. 1.4].

The set \mathcal{T} is a finite collection of (disjoint) open Lipschitz polyhedra T (the mesh cells), which is assumed to form a partition of the domain, that is $\bar{\Omega} = \bigcup_{T \in \mathcal{T}} \bar{T}$. For all $T \in \mathcal{T}$, we let $h_T := \max_{\mathbf{x}, \mathbf{y} \in \bar{T}} |\mathbf{x} - \mathbf{y}|$ denote the diameter of the cell T . We also let $h_{\mathcal{T}}$ be s.t. $h_{\mathcal{T}|T} := h_T$ for all $T \in \mathcal{T}$, and we define the mesh size by $h_{\mathcal{D}} := \max_{T \in \mathcal{T}} h_T$. In turn, the set \mathcal{F} is a finite collection of (disjoint) connected subsets of $\bar{\Omega}$ (the mesh faces) such that, for all $F \in \mathcal{F}$,

- (i) F is a relatively open, Lipschitz polygonal subset of an affine hyperplane, and
- (ii) either there are two distinct mesh cells $T^+, T^- \in \mathcal{T}$ s.t. $\bar{F} \subseteq \partial T^+ \cap \partial T^-$ (F is then an interface), or there is a mesh cell $T \in \mathcal{T}$ s.t. $\bar{F} \subseteq \partial T \cap \Gamma$ (F is then a boundary face).

The set of mesh faces is further assumed to form a partition of the mesh skeleton, that is to satisfy $\bigcup_{T \in \mathcal{T}} \partial T = \bigcup_{F \in \mathcal{F}} \bar{F}$. For all $F \in \mathcal{F}$, we let $h_F := \max_{\mathbf{x}, \mathbf{y} \in \bar{F}} |\mathbf{x} - \mathbf{y}|$ denote the diameter of the face F . Interfaces are collected in the set \mathcal{F}° , whereas boundary faces are collected in the set \mathcal{F}^∂ . For all $T \in \mathcal{T}$, we denote by \mathcal{F}_T the subset of \mathcal{F} which collects the mesh faces lying on the boundary of T , so that $\partial T = \bigcup_{F \in \mathcal{F}_T} \bar{F}$. For all $T \in \mathcal{T}$, consistently with our notation so far, we let $\mathbf{n}_{\partial T} : \partial T \rightarrow \mathbb{R}^3$ denote the (almost everywhere defined) unit vector field normal to ∂T pointing outward from T . For all $F \in \mathcal{F}_T$, we also let $\mathbf{n}_{T,F} := \mathbf{n}_{\partial T|F}$ be the (constant) unit vector normal to the hyperplane containing F and pointing outward from T . For all $F \in \mathcal{F}$, we define \mathbf{n}_F as the (constant) unit vector normal to F such that either $\mathbf{n}_F := \mathbf{n}_{T^+,F}$ if $F \subset \partial T^+ \cap \partial T^- \in \mathcal{F}^\circ$, or $\mathbf{n}_F := \mathbf{n}_{T,F} (= \mathbf{n}_{|F})$ if $F \subset \partial T \cap \Gamma \in \mathcal{F}^\partial$. For further use, we also let, for all $T \in \mathcal{T}$ and all $F \in \mathcal{F}_T$, $\varepsilon_{T,F} \in \{-1, 1\}$ be such that $\varepsilon_{T,F} := \mathbf{n}_{T,F} \cdot \mathbf{n}_F$. Finally, for any $X \in \mathcal{T} \cup \mathcal{F}$, we let $\mathbf{x}_X \in \mathbb{R}^3$ be some point inside X .

When $\beta_1 > 0$, for all $i \in \{1, \dots, \beta_1\}$, we assume that there exists a subset $\mathcal{F}_{\Sigma_i}^\circ$ of \mathcal{F}° such that $\bar{\Sigma}_i = \bigcup_{F \in \mathcal{F}_{\Sigma_i}^\circ} \bar{F}$, and for which $\mathbf{n}_F = \mathbf{n}_{\Sigma_i|F}$ for all $F \in \mathcal{F}_{\Sigma_i}^\circ$. This requirement ensures that we do associate the tag “+” to the side of $\hat{\Omega}$ (with respect to Σ_i) for which \mathbf{n}_{Σ_i} is outward, consistently with

our assumption from Section 2.1. Notice that, since the cutting surfaces are piecewise planar, the set $\hat{\Omega}$ is indeed pseudo-Lipschitz. We also let $\mathcal{F}_\Sigma^\circ := \bigcup_{i=1}^{\beta_1} \mathcal{F}_{\Sigma_i}^\circ$. When $\beta_2 > 0$, for all $j \in \{0, \dots, \beta_2\}$, we let $\mathcal{F}_{\Gamma_j}^\partial$ denote the subset of \mathcal{F}^∂ such that $\Gamma_j = \bigcup_{F \in \mathcal{F}_{\Gamma_j}^\partial} \overline{F}$.

At the discrete level, the parameter $\mu : \Omega \rightarrow [\mu_b, \mu_\#]$ introduced in (7) is assumed to be piecewise constant over the partition \mathcal{T} of the domain Ω , and we let

$$\mu_b \leq \mu_T := \mu|_T \leq \mu_\# \quad \forall T \in \mathcal{T}. \quad (17)$$

When studying asymptotic properties with respect to the mesh size, one has to adopt a measure of regularity for refined sequences of discretizations. Let us thus consider $(\mathcal{D}_n)_{n \in \mathbb{N}}$, a sequence of polyhedral discretizations $\mathcal{D}_n := (\mathcal{T}_n, \mathcal{F}_n)$ such that $h_{\mathcal{D}_n}$ tends to zero as n goes to infinity. We classically follow [20, Def. 1.9], in which regularity for refined mesh sequences is quantified through a uniform-in- n parameter $\varrho \in (0, 1)$ (the mesh regularity parameter). In a nutshell, it is assumed that, for all $n \in \mathbb{N}$, there exists a matching tetrahedral subtessellation of \mathcal{T}_n , (i) which is uniformly-in- n shape-regular, and (ii) whose elements have a diameter that is uniformly-in- n comparable to the diameter of the mesh cell in \mathcal{T}_n they belong to. In what follows, we write $a \lesssim b$ (resp. $a \gtrsim b$) in place of $a \leq Cb$ (resp. $a \geq Cb$), if $C > 0$ only depends on Ω , on the mesh regularity parameter ϱ , and (if need be) on the underlying polynomial degree, but is independent of both n (and thus $h_{\mathcal{D}_n}$) and μ . When $a \lesssim b \lesssim a$, we simply write $a \approx b$. In particular, for regular sequences $(\mathcal{D}_n)_{n \in \mathbb{N}}$ of discretizations, for all $n \in \mathbb{N}$ and $T \in \mathcal{T}_n$, there holds $\text{card}(\mathcal{F}_T) \lesssim 1$, as well as $h_T \lesssim h_F \leq h_T$ for all $F \in \mathcal{F}_T$ (cf. [20, Lem. 1.12]). Also, we assume that, for all $n \in \mathbb{N}$ and $X \in \mathcal{T}_n \cup \mathcal{F}_n$, the point \mathbf{x}_X is the center of an X -inscribed ball/disk of radius $h_X \lesssim r_X \leq h_X$. This last assumption can always be satisfied for regular sequences of discretizations.

3.2 Polynomial spaces

For $\ell \in \mathbb{N}$ and $m \in \{2, 3\}$, we let \mathcal{P}_m^ℓ denote the linear space of m -variate polynomials of total degree at most ℓ , with the convention that \mathcal{P}_m^0 is identified to \mathbb{R} , and that $\mathcal{P}_m^{-1} := \{0\}$. For any $X \in \mathcal{T} \cup \mathcal{F}$, we let $\mathcal{P}^\ell(X)$ be the linear space spanned by the restrictions to X of the polynomials in \mathcal{P}_3^ℓ . For X of dimension $m \in \{2, 3\}$, $\mathcal{P}^\ell(X)$ is isomorphic to \mathcal{P}_m^ℓ (cf. [20, Prop. 1.23]). We let $\pi_{\mathcal{P}, X}^\ell$ denote the $L^2(X)$ -orthogonal projector onto $\mathcal{P}^\ell(X)$. For convenience, we also let $\mathcal{P}^\ell(X) := \mathcal{P}^\ell(X)^m$, that is, for all $T \in \mathcal{T}$, $\mathcal{P}^\ell(T) = \mathcal{P}^\ell(T)^3$, and for all $F \in \mathcal{F}$, $\mathcal{P}^\ell(F) = \mathcal{P}^\ell(F)^2$. We finally define $\pi_{\mathcal{P}, X}^\ell$ as the $L^2(X)$ -orthogonal projector onto $\mathcal{P}^\ell(X)$.

For any $T \in \mathcal{T}$, we introduce the following subspaces of $\mathcal{P}^\ell(T)$, $\ell \in \mathbb{N}$:

$$\mathcal{G}^\ell(T) := \mathbf{grad}(\mathcal{P}^{\ell+1}(T)), \quad \mathcal{G}^{c, \ell}(T) := \mathcal{P}^{\ell-1}(T) \times (\mathbf{x} - \mathbf{x}_T).$$

The space $\mathcal{G}^{c, \ell}(T)$ is the Koszul complement of $\mathcal{G}^\ell(T)$. The (non $L^2(T)$ -orthogonal) polynomial decomposition below is a by-product of the *homotopy formula* (cf. [3, Thm. 7.1]):

$$\mathcal{P}^\ell(T) = \mathcal{G}^\ell(T) \oplus \mathcal{G}^{c, \ell}(T). \quad (18)$$

In addition, letting $\mathcal{R}^\ell(T) := \mathbf{curl}(\mathcal{P}^{\ell+1}(T))$, by exactness of the polynomial de Rham complex, the differential mapping $\mathbf{curl} : \mathcal{G}^{c, \ell}(T) \rightarrow \mathcal{R}^{\ell-1}(T)$ is an isomorphism.

For any $F \in \mathcal{F}$ now, we let H_F be the hyperplane containing F , that we orient according to the normal \mathbf{n}_F . For any function $w : F \rightarrow \mathbb{R}$, we let $\mathbf{grad}_F w : F \rightarrow \mathbb{R}^2$ denote the (tangential) gradient of w . We also let $\mathbf{rot}_F w : F \rightarrow \mathbb{R}^2$ be such that $\mathbf{rot}_F w := (\mathbf{grad}_F w)^\perp$, where \mathbf{z}^\perp is defined as the rotation of angle $-\pi/2$ of \mathbf{z} in the oriented hyperplane H_F . Let us introduce the following subspaces of $\mathcal{P}^\ell(F)$, $\ell \in \mathbb{N}$:

$$\mathcal{R}^\ell(F) := \mathbf{rot}_F(\mathcal{P}^{\ell+1}(F)), \quad \mathcal{R}^{c, \ell}(F) := \mathcal{P}^{\ell-1}(F) \times (\mathbf{x} - \mathbf{x}_F),$$

where, for $\mathbf{x} \in F$, $(\mathbf{x} - \mathbf{x}_F) \subset H_F$ is identified to its (two-dimensional) tangential counterpart. The space $\mathcal{R}^{c,\ell}(F)$ is the Koszul complement of $\mathcal{R}^\ell(F)$. By the homotopy formula, in that case again, the following (non $L^2(F)$ -orthogonal) polynomial decomposition holds true:

$$\mathcal{P}^\ell(F) = \mathcal{R}^\ell(F) \oplus \mathcal{R}^{c,\ell}(F). \quad (19)$$

For all $T \in \mathcal{T}$ and $F \in \mathcal{F}_T$, since for any $v : T \rightarrow \mathbb{R}$ we have $\mathbf{n}_F \times ((\mathbf{grad} v)|_F \times \mathbf{n}_F) = \mathbf{grad}_F(v|_F)$ (identifying $\mathbf{n}_F \times ((\mathbf{grad} v)|_F \times \mathbf{n}_F)$ to its two-dimensional proxy), there holds

$$\mathcal{G}^\ell(T)|_F \times \mathbf{n}_F = \mathcal{R}^\ell(F), \quad (20)$$

where vectors in $\mathcal{G}^\ell(T)|_F \times \mathbf{n}_F$ are also identified to their tangential counterparts. The identity (20) provides a characterization for the rotated tangential traces of curl-free vector polynomials. This characterization is instrumental in the design of stable and optimally consistent trimmed HHO methods (cf. [30, Rmk. 10] for further insight).

Last, we introduce broken versions of the 3-variate polynomial spaces \mathcal{P}^ℓ and \mathcal{P}^ℓ :

$$\begin{aligned} \mathcal{P}^\ell(\mathcal{T}) &:= \{v \in L^2(\Omega) \mid v|_T \in \mathcal{P}^\ell(T) \forall T \in \mathcal{T}\}, \\ \mathcal{P}^\ell(\mathcal{T}) &:= \{v \in L^2(\Omega) \mid v|_T \in \mathcal{P}^\ell(T) \forall T \in \mathcal{T}\}. \end{aligned}$$

We classically define on $\mathcal{P}^\ell(\mathcal{T})$ the broken gradient operator $\mathbf{grad}_{\mathcal{T}}$, and on $\mathcal{P}^\ell(\mathcal{T})$ the broken rotational operator $\mathbf{curl}_{\mathcal{T}}$. We finally let $\pi_{\mathcal{P},\mathcal{T}}^\ell$ (resp. $\pi_{\mathcal{P},\mathcal{T}}^\ell$) be the $L^2(\Omega)$ -orthogonal (resp. $L^2(\Omega)$ -orthogonal) projector onto $\mathcal{P}^\ell(\mathcal{T})$ (resp. $\mathcal{P}^\ell(\mathcal{T})$).

3.3 Hybrid spaces

We introduce hybrid counterparts of the spaces $\mathbf{H}(\mathbf{curl}; \Omega)$ and $H^1(\hat{\Omega})$ (as well as of relevant subspaces of the latter). The first discrete spaces will serve for the discretization of the magnetic variable, whereas the second will be related to the pressure-like variable (Lagrange multiplier). Henceforth, let $k \in \mathbb{N}^*$ denote a given polynomial degree.

3.3.1 $\mathbf{H}(\mathbf{curl})$ -like hybrid spaces

Global spaces: We define the following discrete counterpart of the space $\mathbf{H}(\mathbf{curl}; \Omega)$:

$$\underline{\mathbf{U}}_{\mathcal{D}}^k := \left\{ \underline{\mathbf{v}}_{\mathcal{D}} := \left((\mathbf{v}_T)_{T \in \mathcal{T}}, (\mathbf{v}_{F,\tau})_{F \in \mathcal{F}} \right) : \begin{array}{ll} \mathbf{v}_T \in \mathcal{P}^k(T) & \forall T \in \mathcal{T} \\ \mathbf{v}_{F,\tau} \in \mathbf{Q}^k(F) & \forall F \in \mathcal{F} \end{array} \right\}, \quad (21)$$

where the (possibly trimmed) polynomial space $\mathbf{Q}^k(F)$ shall satisfy

$$\mathcal{R}^k(F) \subseteq \mathbf{Q}^k(F) \subseteq \mathcal{P}^k(F). \quad (22)$$

We let $\pi_{\mathbf{Q},F}^k$ denote the $L^2(F)$ -orthogonal projector onto $\mathbf{Q}^k(F)$. In (21), $\mathbf{v}_{F,\tau}$ stands for the rotated tangential trace of the (magnetic) variable. We also introduce the subspace

$$\underline{\mathbf{U}}_{\mathcal{D},\mathbf{0}}^k := \{ \underline{\mathbf{v}}_{\mathcal{D}} \in \underline{\mathbf{U}}_{\mathcal{D}}^k \mid \mathbf{v}_{F,\tau} \equiv \mathbf{0} \forall F \in \mathcal{F}^\partial \}, \quad (23)$$

discrete counterpart of $\mathbf{H}_0(\mathbf{curl}; \Omega)$.

Local traits: Given a cell $T \in \mathcal{T}$, we denote by \underline{U}_T^k the restriction of $\underline{U}_{\mathcal{D}}^k$ to T , and by $\underline{v}_T := (v_T, (v_{F,\tau})_{F \in \mathcal{F}_T}) \in \underline{U}_T^k$ the restriction of the generic element $\underline{v}_{\mathcal{D}} \in \underline{U}_{\mathcal{D}}^k$ to the cell T . For $\underline{v}_{\mathcal{D}} \in \underline{U}_{\mathcal{D}}^k$, we also let $v_{\mathcal{T}}$ be the broken polynomial vector field in $\mathcal{P}^k(\mathcal{T})$ such that $v_{\mathcal{T}|T} := v_T$ for all $T \in \mathcal{T}$. Let us now define an $\mathbf{H}(\mathbf{curl})$ -like hybrid semi-norm on \underline{U}_T^k . We let, for all $\underline{v}_T \in \underline{U}_T^k$,

$$|\underline{v}_T|_{\mathbf{curl},T}^2 := \|\mathbf{curl} v_T\|_{0,T}^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|\pi_{\mathbf{Q},F}^k(v_T|_F \times \mathbf{n}_F) - v_{F,\tau}\|_{0,F}^2. \quad (24)$$

In turn, at the global level, we classically set, for all $\underline{v}_{\mathcal{D}} \in \underline{U}_{\mathcal{D}}^k$, $|\underline{v}_{\mathcal{D}}|_{\mathbf{curl},\mathcal{D}}^2 := \sum_{T \in \mathcal{T}} |\underline{v}_T|_{\mathbf{curl},T}^2$. Remark that, whenever $|\underline{v}_{\mathcal{D}}|_{\mathbf{curl},\mathcal{D}} = 0$ for some $\underline{v}_{\mathcal{D}} \in \underline{U}_{\mathcal{D}}^k$, then by (20) one has $v_{\mathcal{T}} \in \mathbf{H}(\mathbf{curl}^0; \Omega) \cap \mathcal{G}^k(\mathcal{T})$. We then define the rotational reconstruction operator. For any $T \in \mathcal{T}$, we let $\mathbf{C}_T^{k-1} : \underline{U}_T^k \rightarrow \mathcal{P}^{k-1}(T)$ be the operator such that, for all $\underline{v}_T \in \underline{U}_T^k$, $\mathbf{C}_T^{k-1}(\underline{v}_T) \in \mathcal{P}^{k-1}(T)$ is the unique solution to

$$(\mathbf{C}_T^{k-1}(\underline{v}_T), z)_T = (v_T, \mathbf{curl} z)_T - \sum_{F \in \mathcal{F}_T} \varepsilon_{T,F} (v_{F,\tau}, \mathbf{n}_F \times (z|_F \times \mathbf{n}_F))_F \quad \forall z \in \mathcal{P}^{k-1}(T), \quad (25)$$

where $\mathbf{n}_F \times (z|_F \times \mathbf{n}_F)$ is identified to its tangential proxy. At the global level, we let $\mathbf{C}_{\mathcal{T}}^{k-1} : \underline{U}_{\mathcal{D}}^k \rightarrow \mathcal{P}^{k-1}(\mathcal{T})$ be such that, for all $\underline{v}_{\mathcal{D}} \in \underline{U}_{\mathcal{D}}^k$, $\mathbf{C}_{\mathcal{T}}^{k-1}(\underline{v}_{\mathcal{D}})|_T := \mathbf{C}_T^{k-1}(\underline{v}_T)$ for all $T \in \mathcal{T}$. It now remains to define an $\mathbf{H}(\mathbf{curl})$ -like hybrid stabilizer. For any $T \in \mathcal{T}$, we introduce the following symmetric, positive semi-definite bilinear form: for all $\underline{v}_T, \underline{w}_T \in \underline{U}_T^k$,

$$S_{\mathbf{curl},T}(\underline{w}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1} (\pi_{\mathbf{Q},F}^k(w_T|_F \times \mathbf{n}_F) - w_{F,\tau}, \pi_{\mathbf{Q},F}^k(v_T|_F \times \mathbf{n}_F) - v_{F,\tau})_F, \quad (26)$$

where $w_T|_F \times \mathbf{n}_F$ and $v_T|_F \times \mathbf{n}_F$ are identified to their tangential proxies. At the global level, we then classically let $S_{\mathbf{curl},\mathcal{D}}(\underline{w}_{\mathcal{D}}, \underline{v}_{\mathcal{D}}) := \sum_{T \in \mathcal{T}} S_{\mathbf{curl},T}(\underline{w}_T, \underline{v}_T)$ for all $\underline{v}_{\mathcal{D}}, \underline{w}_{\mathcal{D}} \in \underline{U}_{\mathcal{D}}^k$. Given $T \in \mathcal{T}$, under the assumption that $\mathbf{Q}^k(F)$ (already satisfying (22)) additionally satisfies $\mathcal{P}^{k-1}(F) \subset \mathbf{Q}^k(F)$ for all $F \in \mathcal{F}_T$, it is an easy matter to prove that, for all $\underline{v}_T \in \underline{U}_T^k$,

$$\|\mathbf{C}_T^{k-1}(\underline{v}_T)\|_{0,T}^2 + S_{\mathbf{curl},T}(\underline{v}_T, \underline{v}_T) \approx |\underline{v}_T|_{\mathbf{curl},T}^2. \quad (27)$$

Approximation: Let $q > 2$. Given $T \in \mathcal{T}$, we introduce the local reduction operator $\underline{\mathbf{I}}_T^k : \mathbf{H}(\mathbf{curl}; T) \cap \mathbf{L}^q(T) \rightarrow \underline{U}_T^k$ such that, for any $\mathbf{v} \in \mathbf{H}(\mathbf{curl}; T) \cap \mathbf{L}^q(T)$,

$$\underline{\mathbf{I}}_T^k(\mathbf{v}) := \left(\pi_{\mathcal{P},T}^k(\mathbf{v}), (\pi_{\mathbf{Q},F}^k(\mathbf{v}|_F \times \mathbf{n}_F))_{F \in \mathcal{F}_T} \right), \quad (28)$$

where, as now standard, $\mathbf{v}|_F \times \mathbf{n}_F$ is identified to its (two-dimensional) tangential counterpart. Following [24, Sec. 5.1], note that the regularity $\mathbf{v} \in \mathbf{H}(\mathbf{curl}; T) \cap \mathbf{L}^q(T)$ is sufficient to give a (weak) meaning to the face polynomial projections in (28) (see also [30, Rmk. 11]). At the domain level, now, we define the global reduction operator $\underline{\mathbf{I}}_{\mathcal{D}}^k : \mathbf{H}(\mathbf{curl}; \Omega) \cap \mathbf{L}^q(\Omega) \rightarrow \underline{U}_{\mathcal{D}}^k$ such that, for any $\mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega) \cap \mathbf{L}^q(\Omega)$,

$$\underline{\mathbf{I}}_{\mathcal{D}}^k(\mathbf{v}) := \left((\pi_{\mathcal{P},T}^k(\mathbf{v}|_T))_{T \in \mathcal{T}}, (\pi_{\mathbf{Q},F}^k(\mathbf{v}|_F \times \mathbf{n}_F))_{F \in \mathcal{F}} \right).$$

Remark that, since $\mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega)$, the quantity $\mathbf{v}|_F \times \mathbf{n}_F$ is single-valued at interfaces $F \in \mathcal{F}^\circ$. Notice also that $\underline{\mathbf{I}}_{\mathcal{D}}^k(\mathbf{H}_0(\mathbf{curl}; \Omega) \cap \mathbf{L}^q(\Omega)) \subset \underline{U}_{\mathcal{D},0}^k$. Given $T \in \mathcal{T}$, under the assumption that $\mathbf{Q}^k(F)$ (already satisfying (22)) additionally satisfies $\mathcal{P}^{k-1}(F) \subset \mathbf{Q}^k(F)$ for all $F \in \mathcal{F}_T$, it is an easy matter to prove the following commutation property: for all $\mathbf{v} \in \mathbf{H}(\mathbf{curl}; T) \cap \mathbf{L}^q(T)$,

$$(\mathbf{C}_T^{k-1} \circ \underline{\mathbf{I}}_T^k)(\mathbf{v}) = \pi_{\mathcal{P},T}^{k-1}(\mathbf{curl} \mathbf{v}). \quad (29)$$

Finally, without this time the need for additional assumptions on $\mathbf{Q}^k(F)$, the following (optimal) polynomial consistency result holds true: for all $\mathbf{p} \in \mathcal{P}^k(T)$,

$$S_{\mathbf{curl},T}(\underline{\mathbf{I}}_T^k(\mathbf{p}), \underline{v}_T) = 0 \quad \forall \underline{v}_T \in \underline{U}_T^k.$$

3.3.2 H^1 -like hybrid spaces

Global spaces: We define the following discrete counterpart of the space $H^1(\hat{\Omega})$:

$$\hat{\underline{P}}_{\mathcal{D}}^k := \left\{ \underline{q}_{\mathcal{D}} := \left((q_T)_{T \in \mathcal{T}}, (q_F)_{F \in \mathcal{F} \setminus \mathcal{F}_{\Sigma}^{\circ}}, (q_F^+, q_F^-)_{F \in \mathcal{F}_{\Sigma}^{\circ}} \right) : \begin{array}{ll} q_T \in \mathcal{P}^{k-1}(T) & \forall T \in \mathcal{T} \\ q_F \in \mathcal{P}^k(F) & \forall F \in \mathcal{F} \setminus \mathcal{F}_{\Sigma}^{\circ} \\ q_F^+, q_F^- \in \mathcal{P}^k(F) & \forall F \in \mathcal{F}_{\Sigma}^{\circ} \end{array} \right\}. \quad (30)$$

Here, for any $i \in \{1, \dots, \beta_1\}$ and $F \in \mathcal{F}_{\Sigma_i}^{\circ}$, q_F^+ and q_F^- stand for the traces on F of the pressure-like variable defined, respectively, from the “+” and “-” sides of $\hat{\Omega}$ with respect to the cutting surface Σ_i . Mirroring (8), for $\underline{q}_{\mathcal{D}} \in \hat{\underline{P}}_{\mathcal{D}}^k$, we define its jump through $F \in \mathcal{F}_{\Sigma_i}^{\circ}$ by

$$\llbracket \underline{q}_{\mathcal{D}} \rrbracket_F := q_F^+ - q_F^-. \quad (31)$$

We then introduce the following subspace of $\hat{\underline{P}}_{\mathcal{D}}^k$:

$$\hat{\underline{P}}_{\mathcal{D}, \Sigma}^{b,k} := \left\{ \underline{q}_{\mathcal{D}} \in \hat{\underline{P}}_{\mathcal{D}}^k : \exists (\sigma_i) \in \mathbb{R}^{\beta_1}, \llbracket \underline{q}_{\mathcal{D}} \rrbracket_F \equiv \sigma_i \forall F \in \mathcal{F}_{\Sigma_i}^{\circ}, i \in \{1, \dots, \beta_1\} \right\}, \quad (32)$$

which is, modulo the zero-mean condition over $\hat{\Omega}$, an hybrid counterpart of the space $H_{\Sigma}^1(\hat{\Omega})$ defined in (15). Next, we define the following discrete counterpart of the space $H^1(\Omega)$:

$$\underline{P}_{\mathcal{D}}^k := \left\{ \underline{q}_{\mathcal{D}} := \left((q_T)_{T \in \mathcal{T}}, (q_F)_{F \in \mathcal{F}} \right) : \begin{array}{ll} q_T \in \mathcal{P}^{k-1}(T) & \forall T \in \mathcal{T} \\ q_F \in \mathcal{P}^k(F) & \forall F \in \mathcal{F} \end{array} \right\}. \quad (33)$$

Interestingly, remark that whenever $\beta_1 = 0$, then $\hat{\underline{P}}_{\mathcal{D}, \Sigma}^{b,k} = \hat{\underline{P}}_{\mathcal{D}}^k = \underline{P}_{\mathcal{D}}^k$. For $\beta_1 > 0$, the space $\underline{P}_{\mathcal{D}}^k$ is isomorphic to the subspace of $\hat{\underline{P}}_{\mathcal{D}, \Sigma}^{b,k}$ such that $\llbracket \underline{q}_{\mathcal{D}} \rrbracket_F \equiv 0$ for all $F \in \mathcal{F}_{\Sigma}^{\circ}$. We may now introduce the following subspaces of $\underline{P}_{\mathcal{D}}^k$:

$$\underline{P}_{\mathcal{D}, 0}^k := \left\{ \underline{q}_{\mathcal{D}} \in \underline{P}_{\mathcal{D}}^k : q_F \equiv 0 \forall F \in \mathcal{F}_{\Gamma_0}^{\partial} \right\}, \quad (34)$$

as well as

$$\underline{P}_{\mathcal{D}, \Gamma}^k := \left\{ \underline{q}_{\mathcal{D}} \in \underline{P}_{\mathcal{D}, 0}^k : \exists (\gamma_j) \in \mathbb{R}^{\beta_2}, q_F \equiv \gamma_j \forall F \in \mathcal{F}_{\Gamma_j}^{\partial}, j \in \{1, \dots, \beta_2\} \right\}. \quad (35)$$

The space $\underline{P}_{\mathcal{D}, \Gamma}^k$ is the hybrid counterpart of the space $H_{\Gamma}^1(\Omega)$ in (11). Remark that, whenever $\beta_2 = 0$, then $\underline{P}_{\mathcal{D}, \Gamma}^k = \underline{P}_{\mathcal{D}, 0}^k$ with $\underline{P}_{\mathcal{D}, 0}^k$ being, in that case, the hybrid counterpart of $H_0^1(\Omega)$.

Local traits: Given $T \in \mathcal{T}$, we denote by \underline{P}_T^k the restriction of $\hat{\underline{P}}_{\mathcal{D}}^k$ or $\underline{P}_{\mathcal{D}}^k$ to T , and by $\underline{q}_T := (q_T, (q_{F,T})_{F \in \mathcal{F}_T}) \in \underline{P}_T^k$ the restriction of the generic element $\underline{q}_{\mathcal{D}}$ in $\hat{\underline{P}}_{\mathcal{D}}^k$ or $\underline{P}_{\mathcal{D}}^k$ to the cell T . For $\underline{q}_{\mathcal{D}} \in \underline{P}_{\mathcal{D}}^k$, $q_{F,T} := q_F$ for all $F \in \mathcal{F}_T$, whereas for $\underline{q}_{\mathcal{D}} \in \hat{\underline{P}}_{\mathcal{D}}^k$,

- if $F \in \mathcal{F}_T \setminus \mathcal{F}_{\Sigma}^{\circ}$, then $q_{F,T} := q_F$;
- if $F \in \mathcal{F}_T \cap \mathcal{F}_{\Sigma}^{\circ}$ with $F \subset \partial T^+ \cap \partial T^-$, then $q_{F,T} := q_F^+$ if $T = T^+$ whereas $q_{F,T} := q_F^-$ if $T = T^-$.

For $\underline{q}_{\mathcal{D}}$ in $\hat{\underline{P}}_{\mathcal{D}}^k$ or $\underline{P}_{\mathcal{D}}^k$, we also let $q_{\mathcal{T}}$ be the broken polynomial function in $\mathcal{P}^{k-1}(\mathcal{T})$ such that $q_{\mathcal{T}|T} := q_T$ for all $T \in \mathcal{T}$. Let us now define an (L^2 -scaled) H^1 -like hybrid semi-norm on $\underline{P}_{\mathcal{T}}^k$. We let, for all $\underline{q}_{\mathcal{T}} \in \underline{P}_{\mathcal{T}}^k$,

$$|\underline{q}_{\mathcal{T}}|_{\mathbf{grad},\mathcal{T}}^2 := h_T^2 \|\mathbf{grad} q_T\|_{0,T}^2 + \sum_{F \in \mathcal{F}_T} h_F \|q_{T|F} - q_{F,T}\|_{0,F}^2. \quad (36)$$

In turn, at the global level, for all $\underline{q}_{\mathcal{D}}$ in $\hat{\underline{P}}_{\mathcal{D}}^k$ or $\underline{P}_{\mathcal{D}}^k$, we set $|\underline{q}_{\mathcal{D}}|_{\mathbf{grad},\mathcal{D}}^2 := \sum_{T \in \mathcal{T}} |\underline{q}_{\mathcal{T}}|_{\mathbf{grad},\mathcal{T}}^2$. Remark that, whenever $|\underline{q}_{\mathcal{D}}|_{\mathbf{grad},\mathcal{D}} = 0$ for some $\underline{q}_{\mathcal{D}}$ in $\hat{\underline{P}}_{\mathcal{D}}^k$ or $\underline{P}_{\mathcal{D}}^k$, then there exists $c \in \mathbb{R}$ such that $\underline{q}_{\mathcal{D}} \cong c \underline{1}_{\mathcal{D}}$, with $\underline{1}_{\mathcal{D}}$ the vector of $\underline{P}_{\mathcal{D}}^k$ whose components are all equal to one. We now define the gradient reconstruction operator. For any $T \in \mathcal{T}$, we let $\mathbf{G}_T^k : \underline{P}_T^k \rightarrow \mathcal{P}^k(T)$ be the operator such that, for all $\underline{q}_{\mathcal{T}} \in \underline{P}_{\mathcal{T}}^k$, $\mathbf{G}_T^k(\underline{q}_{\mathcal{T}}) \in \mathcal{P}^k(T)$ is the unique solution to

$$(\mathbf{G}_T^k(\underline{q}_{\mathcal{T}}), \mathbf{z})_T = -(q_T, \operatorname{div} \mathbf{z})_T + \sum_{F \in \mathcal{F}_T} (q_{F,T}, \mathbf{z}|_F \cdot \mathbf{n}_{T,F})_F \quad \forall \mathbf{z} \in \mathcal{P}^k(T). \quad (37)$$

At the global level, we let $\mathbf{G}_{\mathcal{T}}^k : \{\hat{\underline{P}}_{\mathcal{D}}^k, \underline{P}_{\mathcal{D}}^k\} \rightarrow \mathcal{P}^k(\mathcal{T})$ be such that, for all $\underline{q}_{\mathcal{D}}$ in $\hat{\underline{P}}_{\mathcal{D}}^k$ or $\underline{P}_{\mathcal{D}}^k$, $\mathbf{G}_{\mathcal{T}}^k(\underline{q}_{\mathcal{D}})|_T := \mathbf{G}_T^k(\underline{q}_{\mathcal{T}})$ for all $T \in \mathcal{T}$. It now remains to define an (L^2 -scaled) H^1 -like hybrid stabilizer. For any $\underline{r}_{\mathcal{T}} \in \underline{P}_{\mathcal{T}}^k$, we introduce the following symmetric, positive semi-definite bilinear form: for all $\underline{q}_{\mathcal{T}}, \underline{r}_{\mathcal{T}} \in \underline{P}_{\mathcal{T}}^k$,

$$S_{\mathbf{grad},\mathcal{T}}(\underline{r}_{\mathcal{T}}, \underline{q}_{\mathcal{T}}) := \sum_{F \in \mathcal{F}_T} h_F (r_{T|F} - r_{F,T}, q_{T|F} - q_{F,T})_F. \quad (38)$$

At the global level, we then set $S_{\mathbf{grad},\mathcal{D}}(\underline{r}_{\mathcal{D}}, \underline{q}_{\mathcal{D}}) := \sum_{T \in \mathcal{T}} S_{\mathbf{grad},\mathcal{T}}(\underline{r}_{\mathcal{T}}, \underline{q}_{\mathcal{T}})$ for all $\underline{q}_{\mathcal{D}}, \underline{r}_{\mathcal{D}}$ in $\hat{\underline{P}}_{\mathcal{D}}^k$ or $\underline{P}_{\mathcal{D}}^k$. Given $T \in \mathcal{T}$, it is an easy matter to prove that, for all $\underline{q}_{\mathcal{T}} \in \underline{P}_{\mathcal{T}}^k$,

$$h_T^2 \|\mathbf{G}_T^k(\underline{q}_{\mathcal{T}})\|_{0,T}^2 + S_{\mathbf{grad},\mathcal{T}}(\underline{q}_{\mathcal{T}}, \underline{q}_{\mathcal{T}}) \approx |\underline{q}_{\mathcal{T}}|_{\mathbf{grad},\mathcal{T}}^2. \quad (39)$$

Besides, under the assumption that T belongs to a regular tetrahedral mesh sequence, by [10, Lem. 1] (and adapting the arguments from the proof of [13, Lem. 3.2]), the following important equivalence holds true: for all $\underline{q}_{\mathcal{T}} \in \underline{P}_{\mathcal{T}}^k$,

$$h_T^2 \|\mathbf{G}_T^k(\underline{q}_{\mathcal{T}})\|_{0,T}^2 \approx |\underline{q}_{\mathcal{T}}|_{\mathbf{grad},\mathcal{T}}^2. \quad (40)$$

Approximation: Given $T \in \mathcal{T}$, we introduce the local reduction operator $\underline{I}_T^k : H^1(T) \rightarrow \underline{P}_T^k$ such that, for any $q \in H^1(T)$,

$$\underline{I}_T^k(q) := \left(\pi_{\mathcal{P},T}^{k-1}(q), (\pi_{\mathcal{P},F}^k(q|_F))_{F \in \mathcal{F}_T} \right). \quad (41)$$

At the global level, now, we define the reduction operator $\underline{I}_{\mathcal{D}}^k : \{H^1(\hat{\Omega}), H^1(\Omega)\} \rightarrow \{\hat{\underline{P}}_{\mathcal{D}}^k, \underline{P}_{\mathcal{D}}^k\}$ in the following way:

- for any $q \in H^1(\hat{\Omega})$,

$$\underline{I}_{\mathcal{D}}^k(q) := \left((\pi_{\mathcal{P},T}^{k-1}(q|_T))_{T \in \mathcal{T}}, (\pi_{\mathcal{P},F}^k(q|_F))_{F \in \mathcal{F} \setminus \mathcal{F}_{\Sigma}^{\circ}}, (\pi_{\mathcal{P},F}^k(q|_F^+), \pi_{\mathcal{P},F}^k(q|_F^-))_{F \in \mathcal{F}_{\Sigma}^{\circ}} \right),$$

where, for $F \in \mathcal{F}_{\Sigma}^{\circ}$ such that $F \subset \partial T^+ \cap \partial T^-$, we have let $q^+ := q|_{T^+}$ and $q^- := q|_{T^-}$;

- for any $q \in H^1(\Omega)$,

$$\underline{I}_{\mathcal{D}}^k(q) := \left((\pi_{\mathcal{P},T}^{k-1}(q|_T))_{T \in \mathcal{T}}, (\pi_{\mathcal{P},F}^k(q|_F))_{F \in \mathcal{F}} \right).$$

Remark that, for $F \in \mathcal{F}^\circ$ with $F \subset \partial T^+ \cap \partial T^-$, whenever $q \in H^1(\overline{T^+} \cup \overline{T^-})$, then the quantity $q|_F$ is single-valued. Notice also that $\underline{I}_{\mathcal{D}}^k(H_\Sigma^1(\hat{\Omega})) \subset \hat{P}_{\mathcal{D},\Sigma}^{b,k}$ and that $\underline{I}_{\mathcal{D}}^k(H_\Gamma^1(\Omega)) \subset \underline{P}_{\mathcal{D},\Gamma}^k$. Given $T \in \mathcal{T}$, the following commutation property is valid: for all $q \in H^1(T)$,

$$(\mathbf{G}_T^k \circ \underline{I}_T^k)(q) = \pi_{\mathcal{P},T}^k(\mathbf{grad} q). \quad (42)$$

Also, the following polynomial consistency result holds true: for all $p \in \mathcal{P}^{k-1}(T)$,

$$S_{\mathbf{grad},T}(\underline{I}_T^k(p), \underline{q}_T) = 0 \quad \forall \underline{q}_T \in \underline{P}_T^k.$$

4 HHO methods

We consider two magnetostatics models. The first one is the (first-order) field formulation of magnetostatics, endowed with normal boundary conditions, that is, Problem (5) based on the primal de Rham complex (1). The second model is the (second-order) vector potential formulation of magnetostatics, endowed with tangential boundary conditions, that is, Problem (6) (still) based on the primal de Rham complex. This last model was already considered in [10, Sec. 3.2], but therein within a trivial topological setting. These two problems are actually equivalent.

4.1 Magnetostatics models

Recall the definitions (11) and (15) of the spaces $H_\Gamma^1(\Omega)$ and $H_\Sigma^1(\hat{\Omega})$. Henceforth, the (variable) coefficient μ satisfying (17) denotes the magnetic permeability of the medium.

4.1.1 Field formulation

Let $\mathbf{j} : \Omega \rightarrow \mathbb{R}^3$ be a given current density satisfying the following compatibility conditions:

$$\operatorname{div} \mathbf{j} = 0 \text{ in } \Omega, \quad \langle \mathbf{j}|_{\Gamma_j} \cdot \mathbf{n}, 1 \rangle_{\Gamma_j} = 0 \text{ for all } j \in \{1, \dots, \beta_2\}. \quad (43)$$

We seek the magnetic field $\mathbf{h} : \Omega \rightarrow \mathbb{R}^3$ such that

$$\mathbf{curl} \mathbf{h} = \mathbf{j} \quad \text{in } \Omega, \quad (44a)$$

$$\operatorname{div} \mathbf{b} = 0 \quad \text{in } \Omega, \quad (44b)$$

$$\mathbf{b}|_{\Gamma} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (44c)$$

$$\langle \mathbf{b}|_{\Sigma_i} \cdot \mathbf{n}_{\Sigma_i}, 1 \rangle_{\Sigma_i} = 0 \quad \text{for all } i \in \{1, \dots, \beta_1\}, \quad (44d)$$

with constitutive law $\mathbf{b} = \mu \mathbf{h}$, where $\mathbf{b} : \Omega \rightarrow \mathbb{R}^3$ represents the magnetic induction.

In what follows, we assume that $\mathbf{j} \in \mathbf{L}^2(\Omega)$. By (12), remark that (43) is equivalent to

$$(\mathbf{j}, \mathbf{z})_\Omega = 0 \quad \forall \mathbf{z} \in \mathbf{grad} (H_\Gamma^1(\Omega)). \quad (45)$$

In the spirit of Kikuchi [29], we consider the following equivalent (cf. Remark 1 below) weak form for Problem (44): Find $(\mathbf{h}, p) \in \mathbf{H}(\mathbf{curl}; \Omega) \times H_\Sigma^1(\hat{\Omega})$ s.t.

$$A(\mathbf{h}, \mathbf{v}) + B(\mathbf{v}, p) = (\mathbf{j}, \mathbf{curl} \mathbf{v})_\Omega \quad \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega), \quad (46a)$$

$$-B(\mathbf{h}, q) = 0 \quad \forall q \in H_\Sigma^1(\hat{\Omega}), \quad (46b)$$

with bilinear forms $A : \mathbf{H}(\mathbf{curl}; \Omega) \times \mathbf{H}(\mathbf{curl}; \Omega) \rightarrow \mathbb{R}$, $B : \mathbf{H}(\mathbf{curl}; \Omega) \times H^1(\hat{\Omega}) \rightarrow \mathbb{R}$ set to

$$A(\mathbf{w}, \mathbf{v}) := (\mathbf{curl} \mathbf{w}, \mathbf{curl} \mathbf{v})_{\Omega}, \quad B(\mathbf{w}, q) := (\mu \mathbf{w}, \mathbf{grad} q)_{\Omega}. \quad (47)$$

The pressure-like variable p is the Lagrange multiplier of the constraints (44b)–(44d) on the magnetic induction \mathbf{b} . Testing (46a) with $\mathbf{v} = \mathbf{grad} p \in \mathbf{grad}(H_{\Sigma}^1(\hat{\Omega})) \subset \mathbf{H}(\mathbf{curl}^0; \Omega)$, and recalling that $p \in L_0^2(\hat{\Omega})$, one can actually infer that $p = 0$ in $\hat{\Omega}$ (recall that $\hat{\Omega}$ is assumed to be connected). The well-posedness of Problem (46) is a direct consequence of the second Weber inequality, more precisely of [30, Rmk. 6] with $\eta := \mu$ (combined with (46b) and (16)).

Remark 1 (Weak-strong equivalence). *Whereas it is clear that any solution to Problem (44) also solves Problem (46), the converse is less straightforward, in particular when it comes to retrieving (44a). To do so, one has to use the first Helmholtz–Hodge decomposition (10) (with $\mu \leftarrow 1$ therein), combine it to (12), then recall the compatibility condition (45), so as to infer that $(\mathbf{curl} \mathbf{h} - \mathbf{j}, \mathbf{z})_{\Omega} = 0$ for all $\mathbf{z} \in \mathbf{L}^2(\Omega)$, thereby yielding (44a).*

Remark 2 (Regularity theory). *Recall that Ω is a Lipschitz polyhedron. When the coefficient μ is globally smooth (which, under (17), amounts to assuming that μ is constant in Ω), it is known (cf. [17, Thm. 2] and [2, Prop. 3.7]) that the solution to Problem (44) (and more generally to Problem (5)) satisfies $\mathbf{h} \in \mathbf{H}^s(\Omega)$ for some $s > \frac{1}{2}$, with $s \geq 1$ if Ω is convex. When the coefficient μ is piecewise smooth (here, under (17), piecewise constant), however, the solution \mathbf{h} to Problem (44) (and more generally to Problem (5)) is only known (cf. [18, 28, 7]) to belong to $\mathbf{H}^s(\Omega)$ for some $s > 0$.*

4.1.2 Vector potential formulation

Let $\mathbf{j} : \Omega \rightarrow \mathbb{R}^3$ be a given current density complying with (43). We seek the magnetic vector potential $\mathbf{a} : \Omega \rightarrow \mathbb{R}^3$ such that

$$\mathbf{curl}(\mu^{-1} \mathbf{curl} \mathbf{a}) = \mathbf{j} \quad \text{in } \Omega, \quad (48a)$$

$$\operatorname{div} \mathbf{a} = 0 \quad \text{in } \Omega, \quad (48b)$$

$$\mathbf{a}|_{\Gamma} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma, \quad (48c)$$

$$\langle \mathbf{a}|_{\Gamma_j} \cdot \mathbf{n}, 1 \rangle_{\Gamma_j} = 0 \quad \text{for all } j \in \{1, \dots, \beta_2\}. \quad (48d)$$

In what follows, we assume that $\mathbf{j} \in \mathbf{L}^2(\Omega)$. Recall that (43) is equivalent to (45). We consider the following weak form for Problem (48): Find $(\mathbf{a}, p) \in \mathbf{H}_0(\mathbf{curl}; \Omega) \times H_{\Gamma}^1(\Omega)$ s.t.

$$A(\mathbf{a}, \mathbf{v}) + B(\mathbf{v}, p) = (\mathbf{j}, \mathbf{v})_{\Omega} \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega), \quad (49a)$$

$$-B(\mathbf{a}, q) = 0 \quad \forall q \in H_{\Gamma}^1(\Omega), \quad (49b)$$

with bilinear forms $A : \mathbf{H}(\mathbf{curl}; \Omega) \times \mathbf{H}(\mathbf{curl}; \Omega) \rightarrow \mathbb{R}$, $B : \mathbf{H}(\mathbf{curl}; \Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$ set to

$$A(\mathbf{w}, \mathbf{v}) := (\mu^{-1} \mathbf{curl} \mathbf{w}, \mathbf{curl} \mathbf{v})_{\Omega}, \quad B(\mathbf{w}, q) := (\mathbf{w}, \mathbf{grad} q)_{\Omega}. \quad (50)$$

The pressure-like variable p is the Lagrange multiplier of the constraints (48b) and (48d) on the magnetic vector potential \mathbf{a} . Testing (49a) with $\mathbf{v} = \mathbf{grad} p \in \mathbf{grad}(H_{\Gamma}^1(\Omega)) \subset \mathbf{H}_0(\mathbf{curl}^0; \Omega)$, using (45), and recalling the definition (11), one can actually infer that $p = 0$ in Ω . Consequently, Problems (48) and (49) are equivalent (that is, weak-strong equivalence holds true). The well-posedness of Problem (49) is a direct consequence of the first Weber inequality, more precisely of [30, Rmk. 3] with $\eta := 1$ (combined with (49b) and (12)).

Remark 3 (Equivalence with Problem (44)). *Whereas it is clear that, for any solution \mathbf{a} to Problem (48), letting $\mathbf{b} := \mathbf{curl} \mathbf{a}$, then $\mathbf{h} = \mu^{-1} \mathbf{b}$ is solution to Problem (44), the converse is less obvious. It is a consequence of the fact that the equations (44b)–(44d) imply, by the second Helmholtz–Hodge decomposition (14), the existence of $\mathbf{a} \in \mathbf{H}_0(\mathbf{curl}; \Omega)$ such that $\mathbf{h} = \mu^{-1} \mathbf{curl} \mathbf{a}$. The latter vector potential is non-unique, which is the reason why one has to further impose a gauge condition. In our case, we impose the so-called Coulomb gauge, which amounts to enforcing (48b) and (48d).*

Remark 4 (Regularity theory). *Since $\mathbf{h} = \mu^{-1} \mathbf{curl} \mathbf{a}$ is solution to Problem (44), we infer from Remark 2 that $\mu^{-1} \mathbf{curl} \mathbf{a} \in \mathbf{H}^r(\Omega)$, with $r > \frac{1}{2}$ if μ is globally smooth, or $r > 0$ if μ is piecewise smooth. In turn, since $\mathbf{a} \in \mathbf{H}_0(\mathbf{curl}; \Omega) \cap \mathbf{H}(\text{div}^0; \Omega)$, there holds $\mathbf{a} \in \mathbf{H}^s(\Omega)$ for some $s > \frac{1}{2}$. In particular, if μ is globally constant, both $\mu^{-1} \mathbf{curl} \mathbf{a}$ and \mathbf{a} belong to $\mathbf{H}^t(\Omega)$ for some $t > \frac{1}{2}$, with $t \geq 1$ if Ω is convex. In the general case, the solution to Problem (48) (and more generally to Problem (6)) satisfies $\mu^{-1} \mathbf{curl} \mathbf{a} \in \mathbf{H}^r(\Omega)$ for some $r > 0$, and $\mathbf{a} \in \mathbf{H}^s(\Omega)$ for some $s > \frac{1}{2}$.*

4.2 HHO schemes

The HHO methods we devise are directly built upon the weak formulations (46) and (49). At the discrete level, we heavily leverage the fact that the continuous Lagrange multipliers are equal to zero. In what follows, to avoid any confusion with the notation adopted for the mesh size, we will denote the magnetic field \mathbf{h} by \mathbf{u} . For the sake of consistency, we will also extend this notation to the magnetic vector potential \mathbf{a} .

4.2.1 Field formulation

Recall the definition (21) of the space $\underline{U}_{\mathcal{D}}^k$, discrete counterpart of $\mathbf{H}(\mathbf{curl}; \Omega)$, and assume that, for all $F \in \mathcal{F}$, the space $\mathbf{Q}^k(F)$ is given by

$$\mathbf{Q}^k(F) := \mathcal{R}^k(F) \oplus \mathcal{R}^{c,k-1}(F), \quad (51)$$

in such a way that (22) is indeed fulfilled, and that the inclusion $\mathcal{P}^{k-1}(F) \subset \mathbf{Q}^k(F)$ additionally holds true. Recall, in addition, the definition (32) of the space $\hat{\underline{P}}_{\mathcal{D},\Sigma}^{b,k}$. We introduce the following hybrid counterpart of the space $H_{\Sigma}^1(\hat{\Omega})$:

$$\hat{\underline{P}}_{\mathcal{D},\Sigma}^k := \left\{ \underline{q}_{\mathcal{D}} \in \hat{\underline{P}}_{\mathcal{D},\Sigma}^{b,k} : \int_{\hat{\Omega}} q_{\mathcal{T}} = 0 \right\}. \quad (52)$$

Remark that $\underline{I}_{\mathcal{D}}^k(H_{\Sigma}^1(\hat{\Omega})) \subset \hat{\underline{P}}_{\mathcal{D},\Sigma}^k$, since functions in $H_{\Sigma}^1(\hat{\Omega})$ also belong to $L_0^2(\hat{\Omega})$.

Given $T \in \mathcal{T}$, we introduce the local bilinear forms $A_T : \underline{U}_T^k \times \underline{U}_T^k \rightarrow \mathbb{R}$, $B_T : \underline{U}_T^k \times \underline{P}_T^k \rightarrow \mathbb{R}$, and $N_T : \underline{P}_T^k \times \underline{P}_T^k \rightarrow \mathbb{R}$ defined by: for all $\underline{\mathbf{v}}_T, \underline{\mathbf{w}}_T \in \underline{U}_T^k$ and $\underline{q}_T, \underline{r}_T \in \underline{P}_T^k$,

$$A_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) := (\mathbf{C}_T^{k-1}(\underline{\mathbf{w}}_T), \mathbf{C}_T^{k-1}(\underline{\mathbf{v}}_T))_T + S_{\mathbf{curl},T}(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T), \quad (53a)$$

$$B_T(\underline{\mathbf{w}}_T, \underline{q}_T) := \mu_T(\underline{\mathbf{w}}_T, \mathbf{G}_T^k(\underline{q}_T))_T, \quad (53b)$$

$$N_T(\underline{r}_T, \underline{q}_T) := \mu_T^2 h_T^2 (\mathbf{grad} \underline{r}_T, \mathbf{grad} \underline{q}_T)_T + \mu_T^2 S_{\mathbf{grad},T}(\underline{r}_T, \underline{q}_T), \quad (53c)$$

where the local stabilizers $S_{\mathbf{curl},T}$ and $S_{\mathbf{grad},T}$ are respectively defined by (26) and (38). The scaling μ_T^2 (also h_T^2) in (53c) is chosen so as to restore consistency between the physical units. In turn, and as standard, the global bilinear forms $A_{\mathcal{D}} : \underline{U}_{\mathcal{D}}^k \times \underline{U}_{\mathcal{D}}^k \rightarrow \mathbb{R}$, $B_{\mathcal{D}} : \underline{U}_{\mathcal{D}}^k \times \hat{\underline{P}}_{\mathcal{D}}^k \rightarrow \mathbb{R}$, and

$N_{\mathcal{D}} : \hat{\underline{P}}_{\mathcal{D}}^k \times \hat{\underline{P}}_{\mathcal{D}}^k \rightarrow \mathbb{R}$ are assembled by simply summing the local contributions. The discrete HHO problem reads: Find $(\underline{\mathbf{u}}_{\mathcal{D}}, \underline{\mathbf{p}}_{\mathcal{D}}) \in \underline{\mathbf{U}}_{\mathcal{D}}^k \times \hat{\underline{P}}_{\mathcal{D},\Sigma}^k$ s.t.

$$A_{\mathcal{D}}(\underline{\mathbf{u}}_{\mathcal{D}}, \underline{\mathbf{v}}_{\mathcal{D}}) + B_{\mathcal{D}}(\underline{\mathbf{v}}_{\mathcal{D}}, \underline{\mathbf{p}}_{\mathcal{D}}) = (\mathbf{j}, \mathbf{C}_{\mathcal{T}}^{k-1}(\underline{\mathbf{v}}_{\mathcal{D}}))_{\Omega} \quad \forall \underline{\mathbf{v}}_{\mathcal{D}} \in \underline{\mathbf{U}}_{\mathcal{D}}^k, \quad (54a)$$

$$-B_{\mathcal{D}}(\underline{\mathbf{u}}_{\mathcal{D}}, \underline{\mathbf{q}}_{\mathcal{D}}) + N_{\mathcal{D}}(\underline{\mathbf{p}}_{\mathcal{D}}, \underline{\mathbf{q}}_{\mathcal{D}}) = 0 \quad \forall \underline{\mathbf{q}}_{\mathcal{D}} \in \hat{\underline{P}}_{\mathcal{D},\Sigma}^k. \quad (54b)$$

We now establish well-posedness for Problem (54).

Lemma 5 (Well-posedness). *For all $\underline{\mathbf{v}}_{\mathcal{D}} \in \underline{\mathbf{U}}_{\mathcal{D}}^k$ and $\underline{\mathbf{q}}_{\mathcal{D}} \in \hat{\underline{P}}_{\mathcal{D},\Sigma}^k$, there holds:*

$$|\underline{\mathbf{v}}_{\mathcal{D}}|_{\mathbf{curl},\mathcal{D}}^2 + \sum_{T \in \mathcal{T}} \mu_T^2 |q_T|_{\mathbf{grad},T}^2 \lesssim A_{\mathcal{D}}(\underline{\mathbf{v}}_{\mathcal{D}}, \underline{\mathbf{v}}_{\mathcal{D}}) + N_{\mathcal{D}}(\underline{\mathbf{q}}_{\mathcal{D}}, \underline{\mathbf{q}}_{\mathcal{D}}). \quad (55)$$

Consequently, Problem (54) is well-posed.

Proof. Let $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$ and $q_T \in \hat{\underline{P}}_T^k$. Starting from definition (53a), and invoking (27) (which is valid under the assumption (51)), there holds

$$A_T(\underline{\mathbf{v}}_T, \underline{\mathbf{v}}_T) = \|\mathbf{C}_T^{k-1}(\underline{\mathbf{v}}_T)\|_{0,T}^2 + S_{\mathbf{curl},T}(\underline{\mathbf{v}}_T, \underline{\mathbf{v}}_T) \approx |\underline{\mathbf{v}}_T|_{\mathbf{curl},T}^2. \quad (56)$$

In the same vein, by definition (53c), there holds $N_T(q_T, q_T) = \mu_T^2 |q_T|_{\mathbf{grad},T}^2$. From the above, summing over all $T \in \mathcal{T}$, we readily obtain (55).

Now, since the linear system corresponding to Problem (54) is square, proving well-posedness is actually equivalent to proving the uniqueness of its solution. Hence, let us show that, if $\mathbf{j} \equiv \mathbf{0}$, then necessarily $(\underline{\mathbf{u}}_{\mathcal{D}}, \underline{\mathbf{p}}_{\mathcal{D}}) = (\mathbf{0}_{\mathcal{D}}, \mathbf{0}_{\mathcal{D}})$. First, remark that the map $|\cdot|_{\mathbf{grad},\mathcal{D}}$ (cf. (36)) defines a norm on $\hat{\underline{P}}_{\mathcal{D},\Sigma}^k$ (cf. (52)), as a by-product of the built-in discrete zero-mean condition over $\hat{\Omega}$. Second, let us test Problem (54) (for $\mathbf{j} \equiv \mathbf{0}$) with $\underline{\mathbf{v}}_{\mathcal{D}} = \underline{\mathbf{u}}_{\mathcal{D}}$ and $\underline{\mathbf{q}}_{\mathcal{D}} = \underline{\mathbf{p}}_{\mathcal{D}}$. Summing (54a) and (54b), and leveraging (55), we directly infer that $|\underline{\mathbf{u}}_{\mathcal{D}}|_{\mathbf{curl},\mathcal{D}} = 0$ and $|\underline{\mathbf{p}}_{\mathcal{D}}|_{\mathbf{grad},\mathcal{D}} \leq \mu_b^{-2} \sum_{T \in \mathcal{T}} \mu_T^2 |p_T|_{\mathbf{grad},T}^2 = 0$. From the second relation, since $\underline{\mathbf{p}}_{\mathcal{D}} \in \hat{\underline{P}}_{\mathcal{D},\Sigma}^k$, we deduce that $\underline{\mathbf{p}}_{\mathcal{D}} = \mathbf{0}_{\mathcal{D}}$. Plugging this value into (54b), we then infer that

$$B_{\mathcal{D}}(\underline{\mathbf{u}}_{\mathcal{D}}, \underline{\mathbf{q}}_{\mathcal{D}}) = (\mu \mathbf{u}_{\mathcal{T}}, \mathbf{G}_{\mathcal{T}}^k(q_{\mathcal{D}}))_{\Omega} = 0 \quad \forall \underline{\mathbf{q}}_{\mathcal{D}} \in \hat{\underline{P}}_{\mathcal{D},\Sigma}^k.$$

Since, for any $q \in H_{\Sigma}^1(\hat{\Omega})$ there holds $I_{\mathcal{D}}^k(q) \in \hat{\underline{P}}_{\mathcal{D},\Sigma}^k$, letting $\underline{\mathbf{q}}_{\mathcal{D}} = I_{\mathcal{D}}^k(q)$ in the relation above, and leveraging the local commutation property (42) together with the fact that $\mu \mathbf{u}_{\mathcal{T}} \in \mathcal{P}^k(\mathcal{T})$, we infer

$$(\mu \mathbf{u}_{\mathcal{T}}, z)_{\Omega} = 0 \quad \forall z \in \check{\mathbf{grad}}(H_{\Sigma}^1(\hat{\Omega})).$$

The conclusion then follows from the combination of the discrete second Weber inequality of [30, Rmk. 16] (with $\eta := \mu$) along with (16), with the fact that $|\underline{\mathbf{u}}_{\mathcal{D}}|_{\mathbf{curl},\mathcal{D}} = 0$. This yields $\underline{\mathbf{u}}_{\mathcal{D}} = \mathbf{0}_{\mathcal{D}}$. \square

Last, we prove an error estimate in energy-norm for the solution to Problem (54).

Theorem 6 (Energy-error estimate). *Assume that the solution $\mathbf{u} := \mathbf{h} \in \mathbf{H}(\mathbf{curl}; \Omega)$ to Problem (46) further satisfies $\mathbf{curl} \mathbf{u} \in \mathbf{H}^r(\mathcal{T})$ for some $r \in [0, k]$ and $\mathbf{u} \in \mathbf{H}^s(\mathcal{T})$ for some $s \in (\frac{1}{2}, k + 1]$. Then, the following estimate holds true:*

$$|\underline{\mathbf{u}}_{\mathcal{D}} - \underline{\mathbf{I}}_{\mathcal{D}}^k(\mathbf{u})|_{\mathbf{curl},\mathcal{D}}^2 + \sum_{T \in \mathcal{T}} \mu_T^2 |p_T|_{\mathbf{grad},T}^2 \lesssim \sum_{T \in \mathcal{T}} (h_T^{2r} |\mathbf{curl} \mathbf{u}|_{r,T}^2 + h_T^{2(s-1)} |\mathbf{u}|_{s,T}^2), \quad (57)$$

where $(\underline{\mathbf{u}}_{\mathcal{D}}, \underline{\mathbf{p}}_{\mathcal{D}}) \in \underline{\mathbf{U}}_{\mathcal{D}}^k \times \hat{\underline{P}}_{\mathcal{D},\Sigma}^k$ is the unique solution to Problem (54).

Proof. First remark that, by Sobolev embedding, since $\mathbf{u} \in \mathbf{H}^s(\mathcal{T})$ for some $s > \frac{1}{2}$ ($s > 0$ would be enough), then $\mathbf{u} \in \mathbf{L}^q(\Omega)$ for some $q > 2$, and one can thus give a meaning to $\underline{\mathbf{I}}_{\mathcal{D}}^k(\mathbf{u}) \in \underline{\mathbf{U}}_{\mathcal{D}}^k$. The starting point of the proof is the following simple inequality: for all $(\underline{\mathbf{v}}_{\mathcal{D}}, \underline{\mathbf{q}}_{\mathcal{D}}) \in \underline{\mathbf{U}}_{\mathcal{D}}^k \times \hat{\underline{\mathbf{P}}}_{\mathcal{D},\Sigma}^k$,

$$|(\underline{\mathbf{v}}_{\mathcal{D}}, \underline{\mathbf{q}}_{\mathcal{D}})|_{\mathcal{D}} \leq \max_{\substack{(\underline{\mathbf{w}}_{\mathcal{D}}, \underline{\mathbf{r}}_{\mathcal{D}}) \in \underline{\mathbf{U}}_{\mathcal{D}}^k \times \hat{\underline{\mathbf{P}}}_{\mathcal{D},\Sigma}^k \\ |(\underline{\mathbf{w}}_{\mathcal{D}}, \underline{\mathbf{r}}_{\mathcal{D}})|_{\mathcal{D}}=1}} A_{\mathcal{D}}(\underline{\mathbf{v}}_{\mathcal{D}}, \underline{\mathbf{w}}_{\mathcal{D}}) + B_{\mathcal{D}}(\underline{\mathbf{w}}_{\mathcal{D}}, \underline{\mathbf{q}}_{\mathcal{D}}) - B_{\mathcal{D}}(\underline{\mathbf{v}}_{\mathcal{D}}, \underline{\mathbf{r}}_{\mathcal{D}}) + N_{\mathcal{D}}(\underline{\mathbf{q}}_{\mathcal{D}}, \underline{\mathbf{r}}_{\mathcal{D}}),$$

where we set $|(\underline{\mathbf{v}}_{\mathcal{D}}, \underline{\mathbf{q}}_{\mathcal{D}})|_{\mathcal{D}}^2 := A_{\mathcal{D}}(\underline{\mathbf{v}}_{\mathcal{D}}, \underline{\mathbf{v}}_{\mathcal{D}}) + N_{\mathcal{D}}(\underline{\mathbf{q}}_{\mathcal{D}}, \underline{\mathbf{q}}_{\mathcal{D}})$. Plugging $\underline{\mathbf{v}}_{\mathcal{D}} = (\underline{\mathbf{u}}_{\mathcal{D}} - \underline{\mathbf{I}}_{\mathcal{D}}^k(\mathbf{u})) \in \underline{\mathbf{U}}_{\mathcal{D}}^k$ and $\underline{\mathbf{q}}_{\mathcal{D}} = (\underline{\mathbf{p}}_{\mathcal{D}} - \underline{\mathbf{I}}_{\mathcal{D}}^k(p)) = \underline{\mathbf{p}}_{\mathcal{D}} \in \hat{\underline{\mathbf{P}}}_{\mathcal{D},\Sigma}^k$ (where we have used that $p \equiv 0$), leveraging Problem (54) to linearly develop the expression into the max, and invoking (55), we infer

$$\left(|\underline{\mathbf{u}}_{\mathcal{D}} - \underline{\mathbf{I}}_{\mathcal{D}}^k(\mathbf{u})|_{\text{curl}, \mathcal{D}}^2 + \sum_{T \in \mathcal{T}} \mu_T^2 |\underline{\mathbf{p}}_T|_{\text{grad}, T}^2 \right)^{1/2} \lesssim \max_{\substack{(\underline{\mathbf{w}}_{\mathcal{D}}, \underline{\mathbf{r}}_{\mathcal{D}}) \in \underline{\mathbf{U}}_{\mathcal{D}}^k \times \hat{\underline{\mathbf{P}}}_{\mathcal{D},\Sigma}^k \\ |(\underline{\mathbf{w}}_{\mathcal{D}}, \underline{\mathbf{r}}_{\mathcal{D}})|_{\mathcal{D}}=1}} \mathcal{E}_{\mathcal{D}}((\underline{\mathbf{w}}_{\mathcal{D}}, \underline{\mathbf{r}}_{\mathcal{D}})),$$

where $\mathcal{E}_{\mathcal{D}}((\underline{\mathbf{w}}_{\mathcal{D}}, \underline{\mathbf{r}}_{\mathcal{D}})) := (\mathbf{j}, \mathbf{C}_{\mathcal{T}}^{k-1}(\underline{\mathbf{w}}_{\mathcal{D}}))_{\Omega} - A_{\mathcal{D}}(\underline{\mathbf{I}}_{\mathcal{D}}^k(\mathbf{u}), \underline{\mathbf{w}}_{\mathcal{D}}) + B_{\mathcal{D}}(\underline{\mathbf{I}}_{\mathcal{D}}^k(\mathbf{u}), \underline{\mathbf{r}}_{\mathcal{D}})$. Let us set

$$\begin{aligned} \mathfrak{I}_1 &:= (\mathbf{j}, \mathbf{C}_{\mathcal{T}}^{k-1}(\underline{\mathbf{w}}_{\mathcal{D}}))_{\Omega} - A_{\mathcal{D}}(\underline{\mathbf{I}}_{\mathcal{D}}^k(\mathbf{u}), \underline{\mathbf{w}}_{\mathcal{D}}) \\ &= (\mathbf{curl} \mathbf{u} - (\mathbf{C}_{\mathcal{T}}^{k-1} \circ \underline{\mathbf{I}}_{\mathcal{D}}^k)(\mathbf{u}), \mathbf{C}_{\mathcal{T}}^{k-1}(\underline{\mathbf{w}}_{\mathcal{D}}))_{\Omega} - S_{\text{curl}, \mathcal{D}}(\underline{\mathbf{I}}_{\mathcal{D}}^k(\mathbf{u}), \underline{\mathbf{w}}_{\mathcal{D}}), \end{aligned}$$

and

$$\mathfrak{I}_2 := B_{\mathcal{D}}(\underline{\mathbf{I}}_{\mathcal{D}}^k(\mathbf{u}), \underline{\mathbf{r}}_{\mathcal{D}}) = (\boldsymbol{\pi}_{\mathcal{P}, \mathcal{T}}^k(\boldsymbol{\mu}\mathbf{u}), \mathbf{G}_{\mathcal{T}}^k(\underline{\mathbf{r}}_{\mathcal{D}}))_{\Omega},$$

where we have invoked (44a) to replace \mathbf{j} by $\mathbf{curl} \mathbf{u}$ in \mathfrak{I}_1 , and used the fact that $\boldsymbol{\mu}$ is piecewise constant in \mathfrak{I}_2 . For \mathfrak{I}_1 , using (i) the local commutation property (29) along with the fact that $\mathbf{curl} \mathbf{u} \in \mathbf{H}^r(\mathcal{T})$, and (ii) the fact that $\mathbf{u} \in \mathbf{H}^s(\mathcal{T})$ for some $s > \frac{1}{2}$ (so that the full trace of \mathbf{u} can be given a meaning in $L^2(F)^3$ on both sides of each interface, and on each boundary face), we infer from the fractional approximation results of [13, Lem. 2.5] that

$$\mathfrak{I}_1 \lesssim \left(\sum_{T \in \mathcal{T}} (h_T^{2r} |\mathbf{curl} \mathbf{u}|_{r, T}^2 + h_T^{2(s-1)} |\mathbf{u}|_{s, T}^2) \right)^{1/2} A_{\mathcal{D}}(\underline{\mathbf{w}}_{\mathcal{D}}, \underline{\mathbf{w}}_{\mathcal{D}})^{1/2}.$$

For \mathfrak{I}_2 now, using the definition (37) of the (local) gradient reconstruction, and integrating by parts the volume term therein, we infer that

$$\mathfrak{I}_2 = (\boldsymbol{\mu}\mathbf{u}, \mathbf{grad}_{\mathcal{T}} r_{\mathcal{T}})_{\Omega} + \sum_{T \in \mathcal{T}} \sum_{F \in \mathcal{F}_T} (\boldsymbol{\pi}_{\mathcal{P}, T}^k(\boldsymbol{\mu}_T \mathbf{u}|_T)|_F \cdot \mathbf{n}_{T, F}, r_{F, T} - r_T|_F)_F,$$

where we have invoked the fact that $\mathbf{grad}_{\mathcal{T}} r_{\mathcal{T}} \in \mathcal{P}^{k-2}(\mathcal{T})$ to remove the projector in the first term. A (cell-by-cell) integration by parts of the first term in the right-hand side, and the combination of $\mathbf{u} \in \mathbf{H}_0(\text{div}_{\boldsymbol{\mu}}^0; \Omega)$ (according to (44b) and (44c)) and $\underline{\mathbf{r}}_{\mathcal{D}} \in \hat{\underline{\mathbf{P}}}_{\mathcal{D},\Sigma}^k$ (cf. (52) and (32)), yields

$$\mathfrak{I}_2 = \sum_{T \in \mathcal{T}} \sum_{F \in \mathcal{F}_T} \varepsilon_{T, F} ((\boldsymbol{\pi}_{\mathcal{P}, T}^k(\boldsymbol{\mu}_T \mathbf{u}|_T)|_F - (\boldsymbol{\mu}\mathbf{u})|_F) \cdot \mathbf{n}_F, r_{F, T} - r_T|_F)_F + \sum_{i=1}^{\beta_1} \sum_{F \in \mathcal{F}_{\Sigma_i}^{\circ}} \sigma_i((\boldsymbol{\mu}\mathbf{u})|_F \cdot \mathbf{n}_F, 1)_F,$$

where we have additionally leveraged the fact that $\boldsymbol{\mu}\mathbf{u} \in \mathbf{H}^s(\mathcal{T})$ for some $s > \frac{1}{2}$. Now, since for any $i \in \{1, \dots, \beta_1\}$, there holds $\sum_{F \in \mathcal{F}_{\Sigma_i}^{\circ}} \sigma_i((\boldsymbol{\mu}\mathbf{u})|_F \cdot \mathbf{n}_F, 1)_F = \sigma_i((\boldsymbol{\mu}\mathbf{u})|_{\Sigma_i} \cdot \mathbf{n}_{\Sigma_i}, 1)_{\Sigma_i}$, as a consequence of (44d), we infer that

$$\mathfrak{I}_2 = \sum_{T \in \mathcal{T}} \sum_{F \in \mathcal{F}_T} \varepsilon_{T, F} ((\boldsymbol{\pi}_{\mathcal{P}, T}^k(\boldsymbol{\mu}_T \mathbf{u}|_T)|_F - (\boldsymbol{\mu}\mathbf{u})|_F) \cdot \mathbf{n}_F, r_{F, T} - r_T|_F)_F.$$

The approximation results of [13, Lem. 2.5] along with standard arguments then show that

$$\mathfrak{I}_2 \lesssim \left(\sum_{T \in \mathcal{T}} h_T^{2(s-1)} |\mathbf{u}|_{s,T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}} \mu_T^2 S_{\text{grad},T}(\underline{r}_T, \underline{r}_T) \right)^{1/2}.$$

Since $\sum_{T \in \mathcal{T}} \mu_T^2 S_{\text{grad},T}(\underline{r}_T, \underline{r}_T) \leq N_{\mathcal{D}}(\underline{r}_{\mathcal{D}}, \underline{r}_{\mathcal{D}})$, gathering the estimates on \mathfrak{I}_1 and \mathfrak{I}_2 , and using that $|(\underline{\mathbf{w}}_{\mathcal{D}}, \underline{r}_{\mathcal{D}})|_{\mathcal{D}} = 1$, we conclude the proof. \square

Before concluding this section, a few remarks are in order.

Remark 7 (Solution regularity and convergence rate). *Let us comment on the error estimate (57) from Theorem 6. First, notice that when $r = k$ and $s = k + 1$, the HHO scheme converges with optimal rate $h_{\mathcal{D}}^k$. Second, remark that our analysis requires that $\mathbf{u} \in \mathbf{H}^s(\mathcal{T})$ for some $s > \frac{1}{2}$ (so that one can give a classical sense to the trace of \mathbf{u} on mesh faces) which, by Remark 2, is only met for a globally constant coefficient μ . However, adapting the arguments from [24, Sec. 5], it would be possible to relax this assumption to $s > 0$, which would cover the case of a piecewise constant coefficient. We do not pursue herein further in this direction. Finally, remark that, for our choice of error measure, our HHO scheme is convergent only if $s > 1$ and $r > 0$. To recover convergence for any $s > 0$ and $r \geq 0$, one has to consider a weaker error-norm, formally on $h_{\mathcal{T}} \mathbf{C}_{\mathcal{T}}^{k-1}$ instead of $\mathbf{C}_{\mathcal{T}}^{k-1}$. This amounts to multiplying A_T in (53a) by h_T^2 (and the rhs in (54a) by h_T^2), and N_T in (53c) by h_T^{-2} . This is precisely the approach pursued in [25] in the context of DG methods (with scaling $h_{\mathcal{T}}^{1/2}$ therein for $s > \frac{1}{2}$).*

Remark 8 (Variant of the scheme). *Since the model is first-order, following [10, Sec. 3.1], it is also possible to define the discrete rotational operator as the broken rotational $\mathbf{curl}_{\mathcal{T}}$, instead of $\mathbf{C}_{\mathcal{T}}^{k-1}$. In practice, the new scheme is obtained replacing in Problem (54) every occurrence of $\mathbf{C}_{\mathcal{T}}^{k-1}$ by $\mathbf{curl}_{\mathcal{T}}$. The advantage of doing so is that, in this case, one can relax the assumption that $\mathcal{P}^{k-1}(F) \subset \mathbf{Q}^k(F)$ for all $F \in \mathcal{F}$, and consider in place of (51) the leaner space $\mathbf{Q}^k(F) := \mathcal{R}^k(F)$. This choice is actually sufficient (and necessary, thus optimal) for ensuring stability (cf. [30, Rmk. 10]). Redefining $A_T(\underline{\mathbf{v}}_T, \underline{\mathbf{v}}_T)$, there now holds (in place of (56)) that $A_T(\underline{\mathbf{v}}_T, \underline{\mathbf{v}}_T) = |\underline{\mathbf{v}}_T|_{\mathbf{curl},T}^2$. Importantly, this identity holds true without having to assume that $\mathcal{P}^{k-1}(F) \subset \mathbf{Q}^k(F)$ for all $F \in \mathcal{F}_T$. There is, of course, a price to pay for this complexity reduction. The commutation property (29) is no longer valid in that case, and the proxy for $\mathbf{curl} \mathbf{u}$ is then given by $\mathbf{curl}_{\mathcal{T}} \pi_{\mathcal{P},\mathcal{T}}^k(\mathbf{u})$ in place of $\pi_{\mathcal{P},\mathcal{T}}^{k-1}(\mathbf{curl} \mathbf{u})$. Unfortunately, and as supported by our numerical experiments (cf. Section 5.4), for singular solutions, this alternative proxy usually retains insufficient approximability properties.*

Remark 9 (Tetrahedra and stability). *Suppose \mathcal{T} is a member of a regular tetrahedral mesh sequence. Then, as already pointed out in [10, Rmk. 3.1], the contribution $N_{\mathcal{D}}(\underline{p}_{\mathcal{D}}, \underline{q}_{\mathcal{D}})$ can be removed from Problem (54) without compromising its stability. As a matter of fact, by (40), there holds in that case that $\sum_{T \in \mathcal{T}} \mu_T^2 |\underline{q}_T|_{\text{grad},T}^2 \lesssim \sum_{T \in \mathcal{T}} \mu_T^2 h_T^2 \|\mathbf{G}_T^k(\underline{q}_T)\|_{0,T}^2$ for all $\underline{q}_{\mathcal{D}} \in \hat{\underline{P}}_{\mathcal{D}}^k$. In turn, proceeding as in [10, Lem. 3.6], one can control the right-hand side of the above estimate, thus yielding stability for this alternative scheme. When \mathcal{T} belongs to a (regular) general mesh sequence, Equation (39) suggests that defining $N_T(\underline{r}_T, \underline{q}_T) := \mu_T^2 S_{\text{grad},T}(\underline{r}_T, \underline{q}_T)$ (only part remaining to be controlled) is sufficient to ensure the stability of the scheme. This said, remark that using $N_{\mathcal{D}}(\underline{p}_{\mathcal{D}}, \underline{q}_{\mathcal{D}})$ as in Problem (54) (built from (53c)) does not add any consistency error to the scheme, owing to the fact that the continuous Lagrange multiplier is always zero. This strategy might even yield slightly more accurate results, as a consequence of the improved stability of the discrete problem (cf. [10, Rmk. 3.2]). Other choices of the same nature can be made for $N_{\mathcal{D}}$, see e.g. [10, Eq. (3.4c)]. The present remark also applies to Problem (59) (vector potential case).*

4.2.2 Vector potential formulation

Recall the definition (23) of $\underline{U}_{\mathcal{D},\mathbf{0}}^k$, discrete counterpart of $\mathbf{H}_0(\mathbf{curl}; \Omega)$, and assume that the space $\mathbf{Q}^k(F)$ is still given by (51) for all $F \in \mathcal{F}$. Recall, in addition, the definition (35) of $\underline{P}_{\mathcal{D},\Gamma}^k$, hybrid counterpart of $H_\Gamma^1(\Omega)$.

Given $T \in \mathcal{T}$, we introduce the local bilinear forms $A_T : \underline{U}_T^k \times \underline{U}_T^k \rightarrow \mathbb{R}$, $B_T : \underline{U}_T^k \times \underline{P}_T^k \rightarrow \mathbb{R}$, and $N_T : \underline{P}_T^k \times \underline{P}_T^k \rightarrow \mathbb{R}$ defined by: for all $\underline{v}_T, \underline{w}_T \in \underline{U}_T^k$ and $\underline{q}_T, \underline{r}_T \in \underline{P}_T^k$,

$$A_T(\underline{w}_T, \underline{v}_T) := \mu_T^{-1} (\mathbf{C}_T^{k-1}(\underline{w}_T), \mathbf{C}_T^{k-1}(\underline{v}_T))_T + \mu_T^{-1} S_{\mathbf{curl},T}(\underline{w}_T, \underline{v}_T), \quad (58a)$$

$$B_T(\underline{w}_T, \underline{q}_T) := (\underline{w}_T, \mathbf{G}_T^k(\underline{q}_T))_T, \quad (58b)$$

$$N_T(\underline{r}_T, \underline{q}_T) := \mu_T h_T^2 (\mathbf{grad} r_T, \mathbf{grad} q_T)_T + \mu_T S_{\mathbf{grad},T}(\underline{r}_T, \underline{q}_T), \quad (58c)$$

where the local stabilizers $S_{\mathbf{curl},T}$ and $S_{\mathbf{grad},T}$ are respectively defined by (26) and (38). Here as well, the scaling μ_T (also h_T^2) in (58c) is chosen so as to restore consistency between the physical units. In turn, the global bilinear forms $A_{\mathcal{D}} : \underline{U}_{\mathcal{D}}^k \times \underline{U}_{\mathcal{D}}^k \rightarrow \mathbb{R}$, $B_{\mathcal{D}} : \underline{U}_{\mathcal{D}}^k \times \underline{P}_{\mathcal{D}}^k \rightarrow \mathbb{R}$, and $N_{\mathcal{D}} : \underline{P}_{\mathcal{D}}^k \times \underline{P}_{\mathcal{D}}^k \rightarrow \mathbb{R}$ are assembled by summing the local contributions. The discrete HHO problem reads: Find $(\underline{u}_{\mathcal{D}}, \underline{p}_{\mathcal{D}}) \in \underline{U}_{\mathcal{D},\mathbf{0}}^k \times \underline{P}_{\mathcal{D},\Gamma}^k$ s.t.

$$A_{\mathcal{D}}(\underline{u}_{\mathcal{D}}, \underline{v}_{\mathcal{D}}) + B_{\mathcal{D}}(\underline{v}_{\mathcal{D}}, \underline{p}_{\mathcal{D}}) = (\mathbf{j}, \mathbf{v}_{\mathcal{T}})_{\Omega} \quad \forall \underline{v}_{\mathcal{D}} \in \underline{U}_{\mathcal{D},\mathbf{0}}^k, \quad (59a)$$

$$-B_{\mathcal{D}}(\underline{u}_{\mathcal{D}}, \underline{q}_{\mathcal{D}}) + N_{\mathcal{D}}(\underline{p}_{\mathcal{D}}, \underline{q}_{\mathcal{D}}) = 0 \quad \forall \underline{q}_{\mathcal{D}} \in \underline{P}_{\mathcal{D},\Gamma}^k. \quad (59b)$$

Let us establish well-posedness for Problem (59).

Lemma 10 (Well-posedness). *For all $\underline{v}_{\mathcal{D}} \in \underline{U}_{\mathcal{D}}^k$ and $\underline{q}_{\mathcal{D}} \in \underline{P}_{\mathcal{D}}^k$, there holds:*

$$\sum_{T \in \mathcal{T}} \mu_T^{-1} |\underline{v}_T|_{\mathbf{curl},T}^2 + \sum_{T \in \mathcal{T}} \mu_T |q_T|_{\mathbf{grad},T}^2 \lesssim A_{\mathcal{D}}(\underline{v}_{\mathcal{D}}, \underline{v}_{\mathcal{D}}) + N_{\mathcal{D}}(\underline{q}_{\mathcal{D}}, \underline{q}_{\mathcal{D}}). \quad (60)$$

Consequently, Problem (59) is well-posed.

Proof. Let $\underline{v}_T \in \underline{U}_T^k$ and $\underline{q}_T \in \underline{P}_T^k$. Starting from definition (58a), and invoking (27) (which is valid under the assumption (51)), there holds

$$A_T(\underline{v}_T, \underline{v}_T) = \mu_T^{-1} \|\mathbf{C}_T^{k-1}(\underline{v}_T)\|_{0,T}^2 + \mu_T^{-1} S_{\mathbf{curl},T}(\underline{v}_T, \underline{v}_T) \approx \mu_T^{-1} |\underline{v}_T|_{\mathbf{curl},T}^2. \quad (61)$$

Similarly, by definition (58c), there holds $N_T(\underline{q}_T, \underline{q}_T) = \mu_T |q_T|_{\mathbf{grad},T}^2$. From the above, summing over all $T \in \mathcal{T}$, we immediately obtain (60).

Now, since the linear system corresponding to Problem (59) is square, proving well-posedness is actually equivalent to proving the uniqueness of its solution. Hence, let us show that, if $\mathbf{j} \equiv \mathbf{0}$, then necessarily $(\underline{u}_{\mathcal{D}}, \underline{p}_{\mathcal{D}}) = (\mathbf{0}_{\mathcal{D}}, \mathbf{0}_{\mathcal{D}})$. First, remark that the map $|\cdot|_{\mathbf{grad},\mathcal{D}}$ (cf. (36)) defines a norm on $\underline{P}_{\mathcal{D},\Gamma}^k$ (cf. (34) and (35)), as a by-product of the essential zero discrete boundary condition over Γ_0 . Now, let us test Problem (59) (for $\mathbf{j} \equiv \mathbf{0}$) with $\underline{v}_{\mathcal{D}} = \underline{u}_{\mathcal{D}}$ and $\underline{q}_{\mathcal{D}} = \underline{p}_{\mathcal{D}}$. Summing (59a) and (59b), and leveraging (60), we directly infer that $|\underline{u}_{\mathcal{D}}|_{\mathbf{curl},\mathcal{D}} \leq \mu_{\sharp}^{-1} \sum_{T \in \mathcal{T}} \mu_T^{-1} |\underline{u}_T|_{\mathbf{curl},T}^2 = 0$ and $|\underline{p}_{\mathcal{D}}|_{\mathbf{grad},\mathcal{D}} \leq \mu_{\flat}^{-1} \sum_{T \in \mathcal{T}} \mu_T |p_T|_{\mathbf{grad},T}^2 = 0$. From the second relation, since $\underline{p}_{\mathcal{D}} \in \underline{P}_{\mathcal{D},\Gamma}^k$, we deduce that $\underline{p}_{\mathcal{D}} = \mathbf{0}_{\mathcal{D}}$. Plugging this value into (59b), we then infer that

$$B_{\mathcal{D}}(\underline{u}_{\mathcal{D}}, \underline{q}_{\mathcal{D}}) = (\underline{u}_{\mathcal{T}}, \mathbf{G}_{\mathcal{T}}^k(\underline{q}_{\mathcal{D}}))_{\Omega} = 0 \quad \forall \underline{q}_{\mathcal{D}} \in \underline{P}_{\mathcal{D},\Gamma}^k.$$

Since, for any $q \in H^1_\Gamma(\Omega)$ there holds $\underline{I}_D^k(q) \in \underline{P}_{D,\Gamma}^k$, letting $\underline{q}_D = \underline{I}_D^k(q)$ in the relation above, and leveraging the local commutation property (42) along with the fact that $\mathbf{u}_\mathcal{T} \in \mathcal{P}^k(\mathcal{T})$, we infer that

$$(\mathbf{u}_\mathcal{T}, \mathbf{z})_\Omega = 0 \quad \forall \mathbf{z} \in \mathbf{grad}(H^1_\Gamma(\Omega)).$$

The conclusion then follows from the combination of the discrete first Weber inequality of [30, Rmk. 14] (with $\eta := 1$) along with (12), with the fact that $\underline{\mathbf{u}}_D \in \underline{U}_{D,0}^k$ (so that [30, Rmk. 14] is applicable) and $|\underline{\mathbf{u}}_D|_{\mathbf{curl},D} = 0$. This yields $\underline{\mathbf{u}}_D = \mathbf{0}_D$. \square

Let us now prove an error estimate in energy-norm for the solution to Problem (59).

Theorem 11 (Energy-error estimate). *Assume that the solution $\mathbf{u} := \mathbf{a} \in \mathbf{H}_0(\mathbf{curl}; \Omega)$ to Problem (49) further satisfies $\mathbf{curl} \mathbf{u} \in \mathbf{H}^r(\mathcal{T})$ for some $r \in (\frac{1}{2}, k]$ and $\mathbf{u} \in \mathbf{H}^s(\mathcal{T})$ for some $s \in (\frac{1}{2}, k + 1]$. Then, the following estimate holds true:*

$$\sum_{T \in \mathcal{T}} \mu_T^{-1} |\underline{\mathbf{u}}_T - \underline{\mathbf{I}}_T^k(\mathbf{u}|_T)|_{\mathbf{curl},T}^2 + \sum_{T \in \mathcal{T}} \mu_T |p_T|_{\mathbf{grad},T}^2 \lesssim \sum_{T \in \mathcal{T}} \mu_T^{-1} (h_T^{2r} |\mathbf{curl} \mathbf{u}|_{r,T}^2 + h_T^{2(s-1)} |\mathbf{u}|_{s,T}^2), \quad (62)$$

where $(\underline{\mathbf{v}}_D, \underline{p}_D) \in \underline{U}_{D,0}^k \times \underline{P}_{D,\Gamma}^k$ is the unique solution to Problem (59).

Proof. First remark that, by Sobolev embedding, since $\mathbf{u} \in \mathbf{H}^s(\mathcal{T})$ for some $s > \frac{1}{2}$ ($s > 0$ would be enough), then $\mathbf{u} \in \mathbf{L}^q(\Omega)$ for some $q > 2$, and one can thus give a meaning to $\underline{\mathbf{I}}_D^k(\mathbf{u}) \in \underline{U}_{D,0}^k$. The starting point of the proof is the following simple inequality: for all $(\underline{\mathbf{v}}_D, \underline{q}_D) \in \underline{U}_{D,0}^k \times \underline{P}_{D,\Gamma}^k$,

$$|(\underline{\mathbf{v}}_D, \underline{q}_D)|_D \leq \max_{\substack{(\underline{\mathbf{w}}_D, \underline{r}_D) \in \underline{U}_{D,0}^k \times \underline{P}_{D,\Gamma}^k \\ |(\underline{\mathbf{w}}_D, \underline{r}_D)|_D = 1}} A_D(\underline{\mathbf{v}}_D, \underline{\mathbf{w}}_D) + B_D(\underline{\mathbf{w}}_D, \underline{q}_D) - B_D(\underline{\mathbf{v}}_D, \underline{r}_D) + N_D(\underline{q}_D, \underline{r}_D),$$

where we set $|(\underline{\mathbf{v}}_D, \underline{q}_D)|_D^2 := A_D(\underline{\mathbf{v}}_D, \underline{\mathbf{v}}_D) + N_D(\underline{q}_D, \underline{q}_D)$. Plugging $\underline{\mathbf{v}}_D = (\underline{\mathbf{u}}_D - \underline{\mathbf{I}}_D^k(\mathbf{u})) \in \underline{U}_{D,0}^k$ and $\underline{q}_D = (\underline{p}_D - \underline{\mathbf{I}}_D^k(p)) = \underline{p}_D \in \underline{P}_{D,\Gamma}^k$ (where we have used that $p \equiv 0$), leveraging Problem (59) to linearly develop the expression into the max, and invoking (60), we infer

$$\left(\sum_{T \in \mathcal{T}} \mu_T^{-1} |\underline{\mathbf{u}}_T - \underline{\mathbf{I}}_T^k(\mathbf{u}|_T)|_{\mathbf{curl},T}^2 + \sum_{T \in \mathcal{T}} \mu_T |p_T|_{\mathbf{grad},T}^2 \right)^{1/2} \lesssim \max_{\substack{(\underline{\mathbf{w}}_D, \underline{r}_D) \in \underline{U}_{D,0}^k \times \underline{P}_{D,\Gamma}^k \\ |(\underline{\mathbf{w}}_D, \underline{r}_D)|_D = 1}} \mathcal{E}_D((\underline{\mathbf{w}}_D, \underline{r}_D)),$$

where $\mathcal{E}_D((\underline{\mathbf{w}}_D, \underline{r}_D)) := (\mathbf{j}, \mathbf{w}_\mathcal{T})_\Omega - A_D(\underline{\mathbf{I}}_D^k(\mathbf{u}), \underline{\mathbf{w}}_D) + B_D(\underline{\mathbf{I}}_D^k(\mathbf{u}), \underline{r}_D)$. Let us set

$$\begin{aligned} \mathfrak{I}_1 &:= (\mathbf{j}, \mathbf{w}_\mathcal{T})_\Omega - A_D(\underline{\mathbf{I}}_D^k(\mathbf{u}), \underline{\mathbf{w}}_D), \\ \mathfrak{I}_2 &:= B_D(\underline{\mathbf{I}}_D^k(\mathbf{u}), \underline{r}_D) = (\boldsymbol{\pi}_{\mathcal{P},\mathcal{T}}^k(\mathbf{u}), \mathbf{G}_\mathcal{T}^k(\underline{r}_D))_\Omega. \end{aligned}$$

For \mathfrak{I}_1 , (i) replacing \mathbf{j} by $\mathbf{curl}(\mu^{-1} \mathbf{curl} \mathbf{u})$ (leveraging (48a)) then performing a (cell-by-cell) integration by parts in $(\mathbf{j}, \mathbf{w}_\mathcal{T})_\Omega$, and (ii) using the definition (25) of the (local) rotational reconstruction operator for $A_D(\underline{\mathbf{I}}_D^k(\mathbf{u}), \underline{\mathbf{w}}_D)$, we infer that

$$\begin{aligned} \mathfrak{I}_1 &= (\mu^{-1}(\mathbf{curl} \mathbf{u} - (\mathbf{C}_\mathcal{T}^{k-1} \circ \underline{\mathbf{I}}_D^k)(\mathbf{u})), \mathbf{curl}_\mathcal{T} \mathbf{w}_\mathcal{T})_\Omega - \sum_{T \in \mathcal{T}} \mu_T^{-1} S_{\mathbf{curl},T}(\underline{\mathbf{I}}_T^k(\mathbf{u}|_T), \underline{\mathbf{w}}_T) \\ &\quad + \sum_{T \in \mathcal{T}} \sum_{F \in \mathcal{F}_T} \mathcal{E}_{T,F}(\mathbf{w}_T|_F \times \mathbf{n}_F - \mathbf{w}_{F,T}, \mathbf{n}_F \times (\mu^{-1} \mathbf{curl} \mathbf{u} - \mu_T^{-1} (\mathbf{C}_T^{k-1} \circ \underline{\mathbf{I}}_T^k)(\mathbf{u}|_T))|_F \times \mathbf{n}_F)_F, \end{aligned}$$

where we have also leveraged the fact that $\mu^{-1} \mathbf{curl} \mathbf{u} \in \mathbf{H}(\mathbf{curl}; \Omega)$ with $\mu^{-1} \mathbf{curl} \mathbf{u} \in \mathbf{H}^r(\mathcal{T})$ for some $r > \frac{1}{2}$, together with the fact that $\underline{\mathbf{w}}_D \in \underline{U}_{D,0}^k$. Let us then estimate the two first addends in

\mathfrak{I}_1 , that we denote by $\mathfrak{I}_{1,1}$ ($\mathfrak{I}_{1,2}$ will denote the last addend of \mathfrak{I}_1). By (i) the local commutation property (29) along with the fact that $\mathbf{curl} \mathbf{u} \in \mathbf{H}^r(\mathcal{T})$, and (ii) the fact that $\mathbf{u} \in \mathbf{H}^s(\mathcal{T})$ for some $s > \frac{1}{2}$, we get from the fractional approximation results of [13, Lem. 2.5] that

$$\mathfrak{I}_{1,1} \lesssim \left(\sum_{T \in \mathcal{T}} \mu_T^{-1} (h_T^{2r} |\mathbf{curl} \mathbf{u}|_{r,T}^2 + h_T^{2(s-1)} |\mathbf{u}|_{s,T}^2) \right)^{1/2} \left(\sum_{T \in \mathcal{T}} \mu_T^{-1} |\underline{\mathbf{w}}_T|_{\mathbf{curl},T}^2 \right)^{1/2}.$$

In turn, using [30, Lem. 9] together with similar arguments, we infer that

$$\mathfrak{I}_{1,2} \lesssim \left(\sum_{T \in \mathcal{T}} \mu_T^{-1} h_T^{2r} |\mathbf{curl} \mathbf{u}|_{r,T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}} \mu_T^{-1} |\underline{\mathbf{w}}_T|_{\mathbf{curl},T}^2 \right)^{1/2}.$$

By (61), we finally get

$$\mathfrak{I}_1 \lesssim \left(\sum_{T \in \mathcal{T}} \mu_T^{-1} (h_T^{2r} |\mathbf{curl} \mathbf{u}|_{r,T}^2 + h_T^{2(s-1)} |\mathbf{u}|_{s,T}^2) \right)^{1/2} A_{\mathcal{D}}(\underline{\mathbf{w}}_{\mathcal{D}}, \underline{\mathbf{w}}_{\mathcal{D}})^{1/2}.$$

Let us now estimate \mathfrak{I}_2 . Using the definition (37) of the (local) gradient reconstruction (with the fact that $\underline{r}_{\mathcal{D}} \in \underline{P}_{\mathcal{D},\Gamma}^k$), and integrating by parts the volume term therein, we infer that

$$\mathfrak{I}_2 = (\mathbf{u}, \mathbf{grad}_{\mathcal{T}} r_{\mathcal{T}})_{\Omega} + \sum_{T \in \mathcal{T}} \sum_{F \in \mathcal{F}_T} (\boldsymbol{\pi}_{\mathcal{P},T}^k(\mathbf{u}|_T)|_F \cdot \mathbf{n}_{T,F}, r_F - r_T|_F)_F.$$

A (cell-by-cell) integration by parts of the first term in the right-hand side, and the combination of $\mathbf{u} \in \mathbf{H}(\text{div}^0; \Omega)$ (according to (48b)) and $\underline{r}_{\mathcal{D}} \in \underline{P}_{\mathcal{D},\Gamma}^k$ (cf. (34) and (35)), yields

$$\mathfrak{I}_2 = \sum_{T \in \mathcal{T}} \sum_{F \in \mathcal{F}_T} \varepsilon_{T,F} ((\boldsymbol{\pi}_{\mathcal{P},T}^k(\mathbf{u}|_T)|_F - \mathbf{u}|_F) \cdot \mathbf{n}_{F,r_F} - r_T|_F)_F + \sum_{j=1}^{\beta_2} \sum_{F \in \mathcal{F}_{\Gamma_j}^{\partial}} \gamma_j (\mathbf{u}|_F \cdot \mathbf{n}_F, 1)_F,$$

where we have additionally leveraged the fact that $\mathbf{u} \in \mathbf{H}^s(\mathcal{T})$ for some $s > \frac{1}{2}$. Now, since for any $j \in \{1, \dots, \beta_2\}$, there holds $\sum_{F \in \mathcal{F}_{\Gamma_j}^{\partial}} \gamma_j (\mathbf{u}|_F \cdot \mathbf{n}_F, 1)_F = \gamma_j (\mathbf{u}|_{\Gamma_j} \cdot \mathbf{n}, 1)_{\Gamma_j}$, as a consequence of (48d), we infer that

$$\mathfrak{I}_2 = \sum_{T \in \mathcal{T}} \sum_{F \in \mathcal{F}_T} \varepsilon_{T,F} ((\boldsymbol{\pi}_{\mathcal{P},T}^k(\mathbf{u}|_T)|_F - \mathbf{u}|_F) \cdot \mathbf{n}_{F,r_F} - r_T|_F)_F.$$

The approximation results of [13, Lem. 2.5] along with standard arguments then show that

$$\mathfrak{I}_2 \lesssim \left(\sum_{T \in \mathcal{T}} \mu_T^{-1} h_T^{2(s-1)} |\mathbf{u}|_{s,T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}} \mu_T S_{\mathbf{grad},T}(\underline{r}_T, \underline{r}_T) \right)^{1/2}.$$

Since $\sum_{T \in \mathcal{T}} \mu_T S_{\mathbf{grad},T}(\underline{r}_T, \underline{r}_T) \leq N_{\mathcal{D}}(\underline{r}_{\mathcal{D}}, \underline{r}_{\mathcal{D}})$, gathering the estimates on \mathfrak{I}_1 and \mathfrak{I}_2 , and using that $|(\underline{\mathbf{w}}_{\mathcal{D}}, \underline{r}_{\mathcal{D}})|_{\mathcal{D}} = 1$, we conclude the proof. \square

A few remarks are necessary before concluding this section.

Remark 12 (Solution regularity and convergence rate). *Let us comment on the error estimate (62) from Theorem 11. First, notice that when $r = k$ and $s = k + 1$, the HHO scheme converges with optimal rate $h_{\mathcal{D}}^k$. Second, remark that our analysis requires that $\mathbf{curl} \mathbf{u} \in \mathbf{H}^r(\mathcal{T})$ for some $r > \frac{1}{2}$ which, by Remark 4, is only met for a globally constant coefficient μ (that $\mathbf{u} \in \mathbf{H}^s(\mathcal{T})$ for some $s > \frac{1}{2}$*

is always true). However, as in the first-order case, adapting the arguments from [24, Sec. 5], it would be possible to relax this assumption to $r > 0$, which would cover the case of a piecewise constant coefficient. Again, we do not pursue further in this direction. Finally, note that, for the error measure we consider, our scheme is convergent only if $s > 1$ ($r > 0$ is always true). However, adapting Remark 7, one could recover convergence for any $s > \frac{1}{2}$ in a weaker norm (scaled as in [25]).

Remark 13 (Variant of the scheme). *It is possible, instead of (locally) reconstructing the rotational in $\mathcal{P}^{k-1}(T)$ as in (25), to reconstruct it in the smaller space $\mathcal{R}^{k-1}(T) = \mathbf{curl}(\mathcal{P}^k(T))$. This was the strategy advocated in [10, Sec. 3.2]. In this case, the proxy for $\mathbf{curl} \mathbf{u}$ is given by $\pi_{\mathcal{R}, \mathcal{T}}^{k-1}(\mathbf{curl} \mathbf{u})$ (with obvious notation for the projector) in place of $\pi_{\mathcal{P}, \mathcal{T}}^{k-1}(\mathbf{curl} \mathbf{u})$. This proxy has approximability properties no better than $\mathbf{curl}_{\mathcal{T}} \pi_{\mathcal{P}, \mathcal{T}}^k(\mathbf{u})$ (see Remark 8). Therefore, since this alternative construction does not allow to relax (51), and given that the complexity gain in using $\mathcal{R}^{k-1}(T)$ instead of $\mathcal{P}^{k-1}(T)$ to reconstruct the rotational is marginal (local systems are small and local computations embarrassingly parallel), it is most probably a better option to stick to the choice $\mathcal{P}^{k-1}(T)$ as in (25).*

Remark 14 (Robustness to heterogeneity). *Let us introduce $\kappa_{\mu} := \mu_{\sharp} / \mu_{\flat} \geq 1$ the (global) heterogeneity ratio of the magnetic permeability μ . Given that $\mathbf{u} = \mathbf{a}$ here, one can infer from (62) that $|\underline{\mathbf{a}}_{\mathcal{D}} - \underline{\mathbf{I}}_{\mathcal{D}}^k(\mathbf{a})|_{\mathbf{curl}, \mathcal{D}}$ is bounded by $\kappa_{\mu}^{1/2}$. Since $\mathbf{b} = \mathbf{curl} \mathbf{a}$ (recall that \mathbf{a} is the magnetic vector potential), notice that $|\underline{\mathbf{a}}_{\mathcal{D}} - \underline{\mathbf{I}}_{\mathcal{D}}^k(\mathbf{a})|_{\mathbf{curl}, \mathcal{D}}$ is actually an L^2 -like measure of the scheme error on \mathbf{b} . The discrete magnetic induction computed by the scheme is thus only mildly sensitive to heterogeneity.*

5 Numerical experiments

We perform in this section a comprehensive numerical validation of the HHO schemes devised and analyzed in Section 4.2. Our main goal is to numerically assess the convergence rates of the methods, so our focus is on relatively simple geometries and test-cases for which an analytical solution is available. The computational domains under study include geometries with non-trivial topology, for which we consider suitable boundary conditions so that the harmonic spaces $\mathbf{H}_{\mathcal{T}}(\Omega)$ and $\mathbf{H}_n(\Omega)$ given by (9) and (13) are non-empty, as well as a non-convex domain with reentrant edge. Accordingly, we consider exact solutions \mathbf{u} with various regularities, ranging from smooth to singular.

We focus on two families of refined (regular) mesh sequences, of respectively *tetrahedral* and *polyhedral* meshes. The family of refined polyhedral meshes is constructed from the tetrahedral one by merging tetrahedral elements into polyhedral conglomerates in a random fashion. The agglomeration procedure concerns all tetrahedra surrounding randomly selected mesh vertices, in such a way that one can achieve a mesh sequence whose regularity parameter does not increase upon refinement. We solve the discrete Problems (54) and (59) (respectively referred to as first- and second-order Problems), modulo the following adaptations: (i) building on Remark 8, for the first-order Problem, when the considered exact solution \mathbf{u} is regular, we replace the discrete rotational operator $\mathbf{C}_{\mathcal{T}}^{k-1}$ by its broken version $\mathbf{curl}_{\mathcal{T}}$, and the magnetic face space is then set to $\mathbf{Q}^k(F) := \mathcal{R}^k(F)$ for all $F \in \mathcal{F}$; building on Remark 9, (ii) when \mathcal{T} is a tetrahedral mesh, we remove from both the first- and second-order Problems the contribution $N_{\mathcal{D}}(\underline{p}_{\mathcal{D}}, \underline{q}_{\mathcal{D}})$, whereas (iii) when \mathcal{T} is a polyhedral mesh, we remove from both the first- and second-order Problems, in the definition of $N_{\mathcal{D}}(\underline{p}_{\mathcal{D}}, \underline{q}_{\mathcal{D}})$, the contribution stemming from the local bilinear form $h_T^2(\mathbf{grad} p_T, \mathbf{grad} q_T)_T$. Doing so we achieve, without compromising stability, a somewhat leaner implementation, and we bear consistency with the numerical results from [10] (therein for $k' := k - 1$) where these variants were actually implemented (in the case of topologically trivial domains only). Crucially, and as already noticed in [10], the numerical tests that we performed (not reported here) show that these variants achieve virtually the same accuracy as the original schemes (and, interestingly, for examples with non-trivial topology,

even slightly improve it). As standard with skeletal methods, for each discrete Problem, both the magnetic and pressure-like cell unknowns are locally eliminated (without additional fill-in) from the global linear system using a Schur complement technique. The resulting (condensed) global linear system is solved using the SparseLU direct solver of the Eigen library, on a laptop Dell Precision 5570 equipped with an Intel Core i7-12800H processor clocked at 2.80GHz and with 64Gb of RAM. The implementation uses the open-source (under license GNU LGPL v3.0) and publicly available C++ library ParaSkel++¹ [6]. For each computed HHO discrete solution $\underline{\mathbf{u}}_{\mathcal{D}}$ corresponding to the exact solution \mathbf{u} , we evaluate the relative *energy-error* and *L²-error* by

$$\text{En_err} := \frac{\left(\sum_{T \in \mathcal{T}} \eta_T^{-1} |\underline{\mathbf{u}}_T - \underline{\mathbf{I}}_T^k(\mathbf{u}|_T)|_{\text{curl},T}^2\right)^{1/2}}{\left(\sum_{T \in \mathcal{T}} \eta_T^{-1} |\underline{\mathbf{I}}_T^k(\mathbf{u}|_T)|_{\text{curl},T}^2\right)^{1/2}} \quad \text{and} \quad \text{L2_err} := \frac{\|\mathbf{u}_{\mathcal{T}} - \boldsymbol{\pi}_{\mathcal{P},\mathcal{T}}^k(\mathbf{u})\|_{0,\Omega}}{\|\boldsymbol{\pi}_{\mathcal{P},\mathcal{T}}^k(\mathbf{u})\|_{0,\Omega}},$$

with $\eta := 1$ in the first-order case, and $\eta := \mu$ in the second-order case. In all the numerical examples below, we set $\mu \equiv 1$.

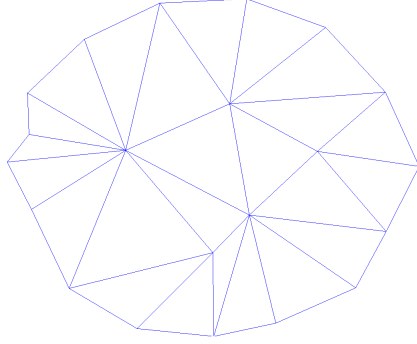


Figure 1: **Test-case 5.1 (torus)**. Example of computed discrete cutting surface for the torus. The surface is piecewise planar, and the shown triangles are faces of a tetrahedral mesh discretizing Ω .

5.1 Torus

Our first example consists in a toroidal domain Ω , with midline radius $R = 2$ and internal radius $r = 1$. For this geometry, $\beta_1 = 1$ (so that there is one cutting surface Σ) and $\beta_2 = 0$. We consider the first-order Problem (44), with the following exact solution, expressed in cylindrical coordinates:

$$\mathbf{u}(\rho, \varphi, z) := \left(\cos\left(\frac{\pi}{r} \sqrt{(\rho - R)^2 + z^2}\right) - c \right) \hat{\boldsymbol{\varphi}}, \quad (63)$$

where the constant $c \in \mathbb{R}$ is set so that the condition $\langle \mathbf{u}|_{\Sigma} \cdot \mathbf{n}_{\Sigma}, 1 \rangle_{\Sigma} = 0$ is satisfied. In our numerical experiments, the current density \mathbf{j} is set according to (44a), whereas the zero normal boundary condition (44c) and zero flux condition (44d) (valid, respectively, on the curved domain boundary Γ and on the Γ -supported cutting surface Σ) are replaced by their non-homogeneous versions computed from (63) on the given polygonal approximations of Γ and Σ . To compute a discrete representative of the cutting surface Σ (cf. Figure 1), which is involved in the definition (30) of the discrete space $\hat{\mathcal{P}}_{\mathcal{D}}^k$, we implemented a simplified version of the algorithm proposed in [1]. For $k \in \{1, 2, 3\}$, we solve Problem (54) (with $\text{curl}_{\mathcal{T}}$), and depict on Figure 2, for both the tetrahedral and (agglomerated)

¹cf. <https://gitlab.inria.fr/simlemai/paraskel/-/tree/electromagnetics>.

polyhedral mesh families, the relative energy-error and L^2 -error as functions of the mesh size ($h_{\mathcal{D}}$) and of the number of degrees of freedom (#DoF) after static condensation. For the two mesh families, as predicted by Theorem 6, we obtain a convergence rate for the energy-error of order k . We also observe a convergence rate of order $k + 1$ for the L^2 -error on the magnetic field.

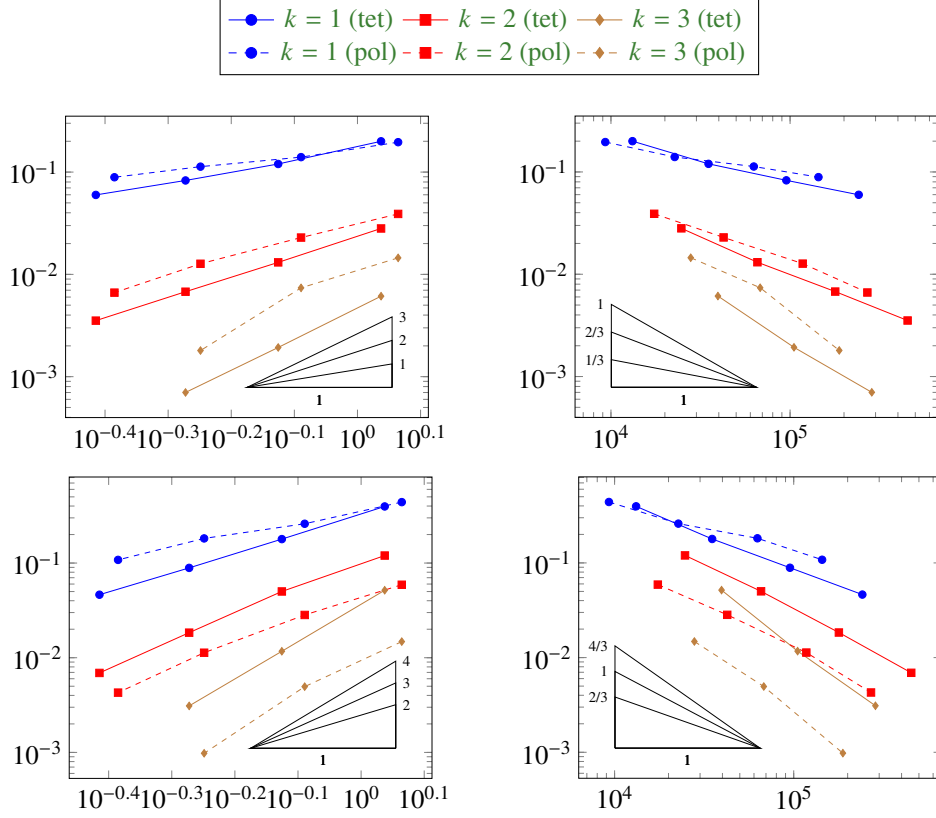


Figure 2: **Test-case 5.1 (torus)**. Relative energy-error (*top row*) and L^2 -error (*bottom row*) vs. $h_{\mathcal{D}}$ (*left column*) and #DoF (*right column*) on tetrahedral (*solid*) and polyhedral (*dashed*) meshes.

5.2 Hollow ball

Our second example consists in a ball Ω of radius 2, encapsulating a void concentric spherical cavity of radius 1, so that the boundary Γ consists of two connected components Γ_0 and Γ_1 . For such a geometry, $\beta_1 = 0$ and $\beta_2 = 1$. We consider the second-order Problem (48), with the following exact solution, expressed in spherical coordinates:

$$\mathbf{u}(r, \theta, \varphi) := \frac{1}{r^2} \cos(\theta) \hat{\mathbf{r}}. \quad (64)$$

Remark that \mathbf{u} satisfies $\langle \mathbf{u}|_{\Gamma_1} \cdot \mathbf{n}, 1 \rangle_{\Gamma_1} = 0$ (as well as $\langle \mathbf{u}|_{\Gamma_0} \cdot \mathbf{n}, 1 \rangle_{\Gamma_0} = 0$, since $\operatorname{div} \mathbf{u} = 0$ in Ω). In our numerical experiments, the current density \mathbf{j} is set according to (48a), whereas the zero tangential boundary condition (48c) and zero flux condition (48d) (valid on the curved domain boundary Γ or inner boundary Γ_1) are replaced by their non-homogeneous versions computed from (64) on the given polygonal approximation of Γ or Γ_1 . For $k \in \{1, 2, 3\}$, we solve Problem (59), and plot on Figure 3 the relative energy-error and L^2 -error as functions of the mesh size ($h_{\mathcal{D}}$) and of the number of degrees of freedom (#DoF) after static condensation, for both the tetrahedral and polyhedral mesh families. For both mesh families, we obtain, as predicted by Theorem 11, a convergence rate for the

energy-error of order k . We also observe a convergence rate of order $k + 1$ for the L^2 -error on the vector potential.

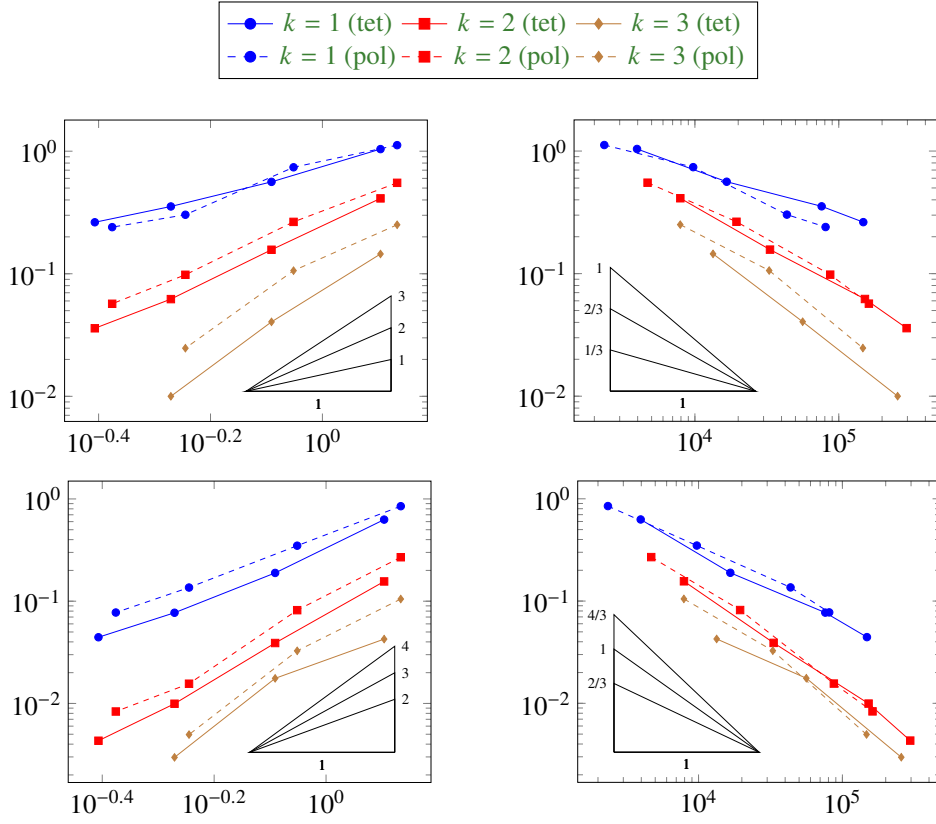


Figure 3: **Test-case 5.2 (hollow ball)**. Relative energy-error (*top row*) and L^2 -error (*bottom row*) vs. $h_{\mathcal{D}}$ (*left column*) and #DoF (*right column*) on tetrahedral (*solid*) and polyhedral (*dashed*) meshes.

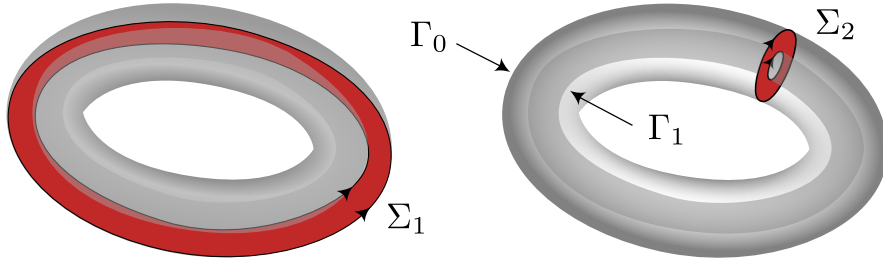


Figure 4: **Test-case 5.3 (hollow torus)**. The toroidal domain Ω , with coaxial toric cavity. The two cutting surfaces Σ_1 and Σ_2 are depicted, as well as the two connected components Γ_0 and Γ_1 of Γ .

5.3 Hollow torus

This test-case is defined on the same toroidal domain as in Section 5.1, except that Ω here additionally encapsulates a void coaxial (i.e. with midline radius 2) toric cavity of internal radius $1/2$ (that is, the cavity has half the internal radius of the torus). For this geometry, $\beta_1 = 2$ (so that there are two cutting surfaces Σ_1 and Σ_2) and $\beta_2 = 1$ (the boundary Γ consists of two connected components Γ_0 and

Γ_1); see Figure 4. We solve the field formulation, with the same exact solution (63) (also same value for the constant c) as in the torus test-case of Section 5.1. The vector field \mathbf{u} given by (63) is actually solution to a non-homogeneous version of the first-order Problem (44): it satisfies (44b)–(44c) and $\langle \mathbf{u}|_{\Sigma_1} \cdot \mathbf{n}_{\Sigma_1}, 1 \rangle_{\Sigma_1} = 0$, but $\langle \mathbf{u}|_{\Sigma_2} \cdot \mathbf{n}_{\Sigma_2}, 1 \rangle_{\Sigma_2} \neq 0$, so that \mathbf{u} “embeds” an harmonic field in $\mathbf{H}_n(\Omega)$. In order to compute discrete representatives of the two cutting surfaces Σ_1 and Σ_2 , we leveraged again in our implementation our adapted version of the algorithm in [1]. Here again, for $k \in \{1, 2, 3\}$, we solve Problem (54) (with \mathbf{curl}_T), and plot on Figure 5 the relative energy-error and L^2 -error as functions of $h_{\mathcal{D}}$ and #DoF, for both the tetrahedral and polyhedral mesh families. The obtained energy-error convergence rates are again consistent with our theory as established in Theorem 6. The observed L^2 -error convergence rates on the magnetic field are again of order $k + 1$.

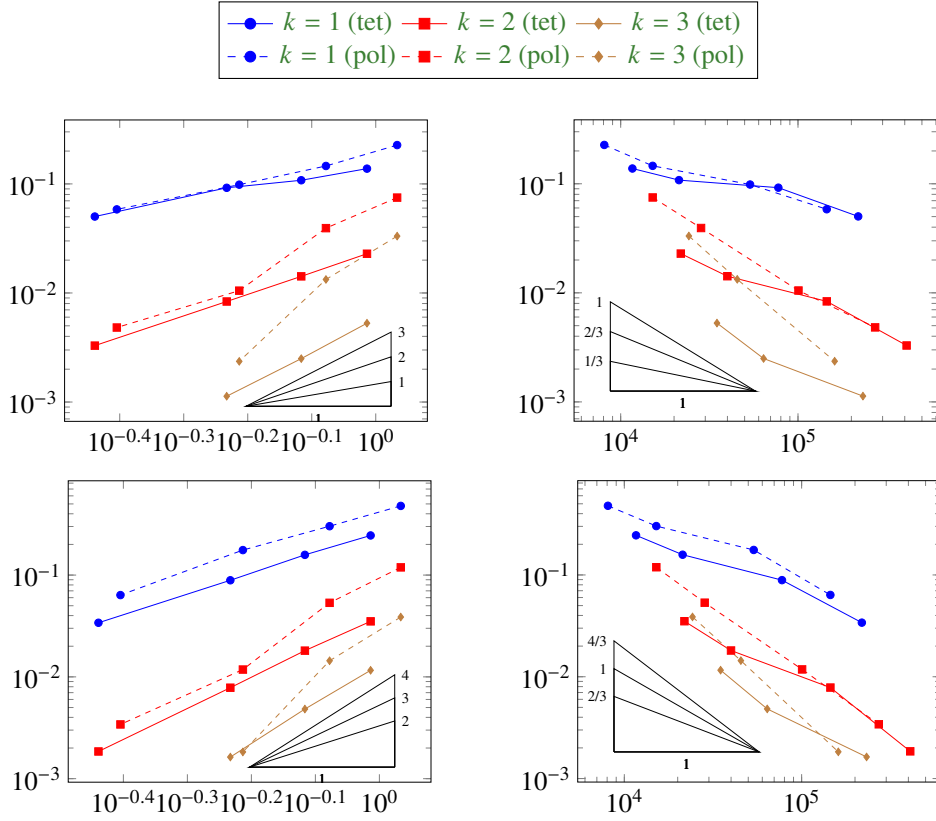


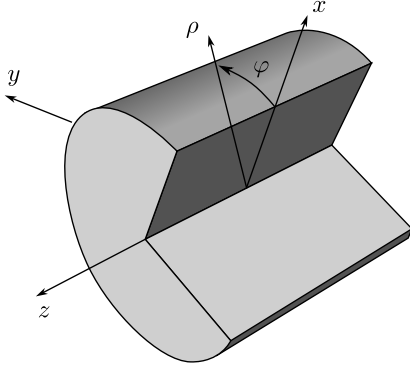
Figure 5: **Test-case 5.3 (hollow torus)**. Relative energy-error (*top row*) and L^2 -error (*bottom row*) vs. $h_{\mathcal{D}}$ (*left column*) and #DoF (*right column*) on tetrahedral (*solid*) and polyhedral (*dashed*) meshes.

5.4 Singular solution

In this last test-case, we consider a singular solution over a non-convex domain Ω . The domain Ω is a cylinder of unit radius and height, to which a quarter has been removed, so that it presents a reentrant edge (cf. the left panel of Figure 6). We solve the field formulation, for the following exact solution, expressed in cylindrical coordinates:

$$\mathbf{u}(\rho, \varphi, z) := \mathbf{grad} \left(\rho^{\frac{2}{3}} \cos \left(\frac{2}{3} \varphi \right) \right). \quad (65)$$

The vector field \mathbf{u} given by (65) is solution to a non-homogeneous version of the first-order Problem (44): it satisfies (44b) (remark that (44d) is here trivial), but fulfills in place of (44c) a non-



$h_{\mathcal{D}}$	L^2 -error	Rate
4.20E-01	3.62E-02	–
2.37E-01	2.72E-02	0.50
1.44E-01	1.99E-02	0.63
7.00E-02	1.24E-02	0.66

Figure 6: **Test-case 5.4 (singular solution)**. The non-convex domain Ω (left panel). L^2 -error and convergence rate on tetrahedral meshes for $k = 1$ (right panel).

homogeneous normal boundary condition on the curved section of Γ . In our numerical experiments, the current density \mathbf{j} is set to zero (accordingly to (44a)), and the boundary data is set in accordance with (65) on the given polygonal approximation of the curved section of Γ . It can be easily verified that $\mathbf{u} \in \mathbf{H}^{\frac{2}{3}-\varepsilon}(\Omega)$ for all $\varepsilon > 0$, and that \mathbf{u} is singular along the reentrant edge. We solve Problem (54) (with $\mathbf{C}_{\mathcal{T}}^{k-1}$), and we collect on the right panel of Figure 6 the results for the relative L^2 -error on the magnetic field for $k = 1$ on the tetrahedral mesh sequence. We observe that the L^2 -error attains its maximum possible rate of convergence of $2/3$. We verified for each $k \in \{2, 3\}$ that this convergence rate does not depend on the polynomial degree. In agreement with Theorem 6, we observed no convergence in energy-norm. Finally, we performed the same test, but with discrete rotational operator given by $\mathbf{curl}_{\mathcal{T}}$ instead of $\mathbf{C}_{\mathcal{T}}^{k-1}$. We observed no convergence whatsoever, even in L^2 -norm. These observations corroborate the discussion in Remark 8.

Acknowledgments

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References

- [1] A. Alonso Rodr ıguez, E. Bertolazzi, R. Ghiloni, and R. Specogna. Efficient construction of 2-chains representing a basis of $H_2(\overline{\Omega}, \partial\Omega; \mathbb{Z})$. *Adv. Comput. Math.*, 44(5):1411–1440, 2018.
- [2] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault. Vector potentials in three-dimensional non-smooth domains. *Math. Methods Appl. Sci.*, 21(9):823–864, 1998.
- [3] D. N. Arnold. *Finite element exterior calculus*, volume 93 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2018.
- [4] F. Assous, P. Ciarlet Jr., and S. Labrunie. *Mathematical foundations of computational electromagnetism*, volume 198 of *Applied Mathematical Sciences*. Springer, Cham, 2018.
- [5] B. Ayuso de Dios, K. Lipnikov, and G. Manzini. The nonconforming virtual element method. *ESAIM Math. Model. Numer. Anal.*, 50(3):879–904, 2016.
- [6] L. Beaud e and S. Lemaire. ParaSkel++: a C++ platform for the high-performance, arbitrary-order, 2/3D numerical approximation of PDEs on general polytopal meshes using skeletal Galerkin methods, 2021.

- [7] A. Bonito, J.-L. Guermond, and F. Luddens. Regularity of the Maxwell equations in heterogeneous media and Lipschitz domains. *J. Math. Anal. Appl.*, 408(2):498–512, 2013.
- [8] S. Cao, C. Wang, and J. Wang. A new numerical method for div-curl systems with low regularity assumptions. *Comput. Math. Appl.*, 114:47–59, 2022.
- [9] W. Cao, C. Wang, and J. Wang. An L^p -primal-dual weak Galerkin method for div-curl systems. *J. Comput. Appl. Math.*, 422:Paper No. 114881, 17, 2023.
- [10] F. Chave, D. A. Di Pietro, and S. Lemaire. A discrete Weber inequality on three-dimensional hybrid spaces with application to the HHO approximation of magnetostatics. *Math. Models Methods Appl. Sci.*, 32(1):175–207, 2022.
- [11] G. Chen, J. Cui, and L. Xu. Analysis of a Hybridizable Discontinuous Galerkin method for the Maxwell operator. *ESAIM Math. Model. Numer. Anal.*, 53(1):301–324, 2019.
- [12] H. Chen, W. Qiu, K. Shi, and M. Solano. A superconvergent HDG method for the Maxwell equations. *J. Sci. Comput.*, 70(3):1010–1029, 2017.
- [13] M. Ciccuttin, A. Ern, and N. Pignet. *Hybrid high-order methods – A primer with applications to solid mechanics*. SpringerBriefs in Mathematics. Springer, Cham, 2021.
- [14] B. Cockburn. Static condensation, hybridization, and the devising of the HDG methods. In *Building bridges: connections and challenges in modern approaches to numerical partial differential equations*, volume 114 of *Lect. Notes Comput. Sci. Eng.*, pages 129–177. Springer, [Cham], 2016.
- [15] B. Cockburn, D. A. Di Pietro, and A. Ern. Bridging the Hybrid High-Order and Hybridizable Discontinuous Galerkin methods. *ESAIM Math. Model. Numer. Anal.*, 50(3):635–650, 2016.
- [16] B. Cockburn, J. Gopalakrishnan, and R. Lazarov. Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems. *SIAM J. Numer. Anal.*, 47(2):1319–1365, 2009.
- [17] M. Costabel. A remark on the regularity of solutions of Maxwell’s equations on Lipschitz domains. *Math. Methods Appl. Sci.*, 12(4):365–368, 1990.
- [18] M. Costabel, M. Dauge, and S. Nicaise. Singularities of Maxwell interface problems. *M2AN Math. Model. Numer. Anal.*, 33(3):627–649, 1999.
- [19] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology – Vol. 3: Spectral theory and applications*. Springer-Verlag, Berlin, 1990. With the collaboration of Michel Artola and Michel Cessenat. Translated from the French by John C. Amson.
- [20] D. A. Di Pietro and J. Droniou. *The Hybrid High-Order method for polytopal meshes – Design, analysis, and applications*, volume 19 of *Modeling, Simulation and Applications*. Springer, Cham, 2020.
- [21] D. A. Di Pietro and A. Ern. A hybrid high-order locking-free method for linear elasticity on general meshes. *Comput. Methods Appl. Mech. Engrg.*, 283:1–21, 2015.
- [22] D. A. Di Pietro, A. Ern, and S. Lemaire. An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators. *Comput. Methods Appl. Math.*, 14(4):461–472, 2014.
- [23] S. Du and F.-J. Sayas. A unified error analysis of Hybridizable Discontinuous Galerkin methods for the static Maxwell equations. *SIAM J. Numer. Anal.*, 58(2):1367–1391, 2020.
- [24] A. Ern and J.-L. Guermond. Quasi-optimal nonconforming approximation of elliptic PDEs with contrasted coefficients and H^{1+r} , $r > 0$, regularity. *Found. Comput. Math.*, 22(5):1273–1308, 2022.
- [25] A. Ern and J.-L. Guermond. The discontinuous Galerkin approximation of the grad-div and curl-curl operators in first-order form is involution-preserving and spectrally correct. *SIAM J. Numer. Anal.*, 61(6):2940–2966, 2023.
- [26] V. Girault and P.-A. Raviart. *Finite element methods for Navier–Stokes equations – Theory and algorithms*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986.

- [27] P. W. Gross and P. R. Kotiuga. *Electromagnetic theory and computation: a topological approach*, volume 48 of *Mathematical Sciences Research Institute Publications*. Cambridge University Press, Cambridge, 2004.
- [28] F. Jochmann. Regularity of weak solutions of Maxwell's equations with mixed boundary conditions. *Math. Methods Appl. Sci.*, 22(14):1255–1274, 1999.
- [29] F. Kikuchi. Mixed formulations for finite element analysis of magnetostatic and electrostatic problems. *Japan J. Appl. Math.*, 6(2):209–221, 1989.
- [30] S. Lemaire and S. Pitassi. Discrete Weber inequalities and related Maxwell compactness for hybrid spaces over polyhedral partitions of domains with general topology. *Found. Comput. Math.*, 2024.
- [31] J. Li, X. Ye, and S. Zhang. A weak Galerkin least-squares finite element method for div-curl systems. *J. Comput. Phys.*, 363:79–86, 2018.
- [32] K. Lipnikov and G. Manzini. A high-order mimetic method on unstructured polyhedral meshes for the diffusion equation. *J. Comput. Phys.*, 272:360–385, 2014.
- [33] L. Mu, J. Wang, X. Ye, and S. Zhang. A weak Galerkin finite element method for the Maxwell equations. *J. Sci. Comput.*, 65(1):363–386, 2015.
- [34] N. C. Nguyen, J. Peraire, and B. Cockburn. Hybridizable Discontinuous Galerkin methods for the time-harmonic Maxwell's equations. *J. Comput. Phys.*, 230(19):7151–7175, 2011.
- [35] C. Wang and J. Wang. Discretization of div-curl systems by weak Galerkin finite element methods on polyhedral partitions. *J. Sci. Comput.*, 68(3):1144–1171, 2016.
- [36] J. Wang and X. Ye. A weak Galerkin finite element method for second-order elliptic problems. *J. Comput. Appl. Math.*, 241:103–115, 2013.