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► **To cite this version:**

Ruben Chenevat, Bruno Chevion, Sébastien Roux, Alain Rapaport. About the Bang-Bang Principle for piecewise affine systems. 63rd IEEE Conference on Decision and Control (CDC 2024), IEEE Control Systems Society, Dec 2024, Milan, Italy. hal-04663130

**HAL Id: hal-04663130**

**<https://hal.science/hal-04663130v1>**

Submitted on 26 Jul 2024

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# About the Bang-Bang Principle for piecewise affine systems\*

Ruben Chenevat<sup>1</sup>, Bruno Cheviron<sup>2</sup>, Sébastien Roux<sup>1</sup> and Alain Rapaport<sup>1</sup>

**Abstract**—We show that for piecewise affine controlled systems, the Bang-Bang Principle does not hold even when the dynamics is continuous with respect to the state variable. However, we show that there exist minimal time trajectories with extreme controls at the loci where the dynamics is differentiable. We give two examples which exhibit singular arcs exactly at the loci of non-differentiability.

## I. INTRODUCTION

The Bang-Bang Principle is a pillar of the theory of linear controlled systems (see [10], [15], [8], [11]). Consider a linear dynamical system

$$\dot{X} = A(t)X + B(t)u, \quad X \in \mathbb{R}^n, \quad u \in U \subset \mathbb{R}^p \quad (1)$$

where  $A(\cdot)$ ,  $B(\cdot)$  are time-measurable matrices of adequate dimensions, and  $U$  is a convex compact set. Denote  $E(U)$  the set of extreme points of the set  $U$ . For any initial condition  $(t_0, X_0)$  in  $\mathbb{R}_+ \times \mathbb{R}^n$ , the attainability set  $\mathcal{A}(t)$ , resp.  $\mathcal{A}_{BB}(t)$ , is defined as the set of  $X(t)$  where  $X(\cdot)$  is the solution of (1) for the initial condition  $(t_0, X_0)$  and any time-measurable controls  $u(\cdot)$  that takes values in  $U$ , resp. in  $E(U)$ . The Bang-Bang Principle states that  $\mathcal{A}(t)$  and  $\mathcal{A}_{BB}(t)$  coincides. This result has consequences in terms of controllability and the nature of trajectories that join two points of the state space in minimal time.

We shall consider in this paper piecewise affine controlled systems, defined by a family of controlled affine systems

$$\dot{X} = f_i(t, X) := A_i(t)X + B(t)u + C_i(t), \quad i \in \mathcal{I}$$

over a partition of  $\mathbb{R}^n$ , and whose dynamics is given by one of these linear systems on each subset of the partition. Such a dynamics is in general discontinuous and the question of well-posedness of solutions raises. In particular sliding modes [17] or Filippov solutions [7] may be considered. Controllability of such systems has been well investigated in the literature (see [4], [6], [14], [18]). In the present work, we shall assume that the piecewise affine systems are continuous with respect to  $X$  (but not necessarily differentiable), which does not pose any problem to define unique solutions as for any Lipschitz continuous differential equations. There is much less works dedicated to this precise setting [16].

\*This work was supported by Labex Numev (ANR-10-LABX-0020) and by the French National Research Agency under the Investments for the Future Program, referred as ANR-16-CONV-0004.

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One may wonder if the Bang-Bang Principle still holds in this simple setting, and if not where the control has to be an non extreme point. It seems that this question has not addressed specifically in the literature, although the problem of regional optimal control has get recent interests in the literature (see [3], [1], [2]). For instance in [2], a minimal time problem with piecewise affine dynamics is presented, but whose dynamics is discontinuous. Note that considering a family of linear systems with the continuity assumption is very restrictive (as it imposes the common boundary of two domains  $D_i$ ,  $D_j$  to belongs to the kernel of  $A_i(t) - A_j(t)$  for any  $t$ ). This is why we consider families of affine rather than linear systems. This allows in particular to consider dynamics with saturation. Let us underline that applying directly the Bang-Bang Principle for trajectories joining two points in a same subset  $D_i$  does not guarantee that the trajectory with extreme controls remains in the same set  $D_i$ . This is precisely the matter investigated in the present work.

The paper is organized as follows. In the next section, we make precise our notations and assumptions, making distinctions between linear and affine Bang-Bang Principle for the minimal time control problem. In Section III, we give our main result. Section IV is devoted to examples.

## II. PRELIMINARIES

In this section we present the framework of piecewise affine systems, and highlight their similarities and differences with classical linear systems.

We consider an non-autonomous system that is linear with respect to  $u$ :

$$\dot{X} = F(t, X) + B(t)u \quad (2)$$

where  $X \in \mathbb{R}^n$  and  $u \in U \subset \mathbb{R}^r$  under the following assumptions.

*Assumption 2.1:* There exists a partition of the state space

$$\mathbb{R}^n = \bigsqcup_{i \in \mathcal{I}} D_i$$

where  $\mathring{D}_i$  is a non-empty connected set for any  $i \in \mathcal{I}$  (where  $\mathcal{I}$  is countable), such that

$$F(t, X) = A_i(t)X + C_i(t), \quad X \in D_i, \quad t \geq 0$$

where  $A_i$  and  $C_i$  are  $n \times n$ - and  $n \times 1$ - time measurable matrix-valued functions.  $B$  is a  $n \times r$ - time measurable matrix-valued function. Moreover  $F$  is continuous with respect to  $X$  for any  $t$ .

Let us stress that we do not impose any regularity of the boundaries of the sets  $D_i$ .

*Assumption 2.2:*  $U$  is a convex compact set.

We consider the minimum time problem to reach a compact target set  $\Omega \subset \mathbb{R}^n$  from an initial state  $X(0) = X_0 \in \mathbb{R}^n$  that belongs to the attainability set of  $\Omega$ . For this problem, the Bang-Bang Principle is often stated for linear dynamics [9], as follows.

In the case where  $\mathcal{S} = \{\alpha\}$  and  $C_\alpha = 0$ , we have in fact a classical linear system

$$\dot{X}(t) = A_\alpha(t)X(t) + B(t)u(t) \quad (3)$$

*Theorem 2.1:* If  $X_0$  belongs to the attainability set of  $\Omega$ , then there exists a control function  $u(\cdot)$  that takes values in  $E(U)$  for which the associated trajectory of (3) reaches  $\Omega$  in minimal time from the initial condition  $X(0) = X_0$ .

We recall that this statement holds true also for affine systems.

*Lemma 2.2:* Consider the affine system in  $\mathbb{R}^n$

$$\dot{Y}(t) = L(t)Y(t) + M(t)u(t) + N(t) \quad (4)$$

where  $u(t) \in U$ , a convex compact set of  $\mathbb{R}^p$ , and  $L, M$  and  $N$  are  $n \times n$ -,  $n \times r$ - and  $n \times 1$ - time measurable matrix-valued functions. Then, the conclusion of Theorem 2.1 holds for dynamics (4).

*Proof:* The proof only consists in rewriting the affine system into an equivalent linear one in higher dimension, and we explicitly construct it as follows :

$$\dot{\tilde{Y}}(t) = \tilde{L}(t)\tilde{Y}(t) + \tilde{M}(t)u(t) \quad (5)$$

where  $\tilde{L}$  and  $\tilde{M}$  are  $(n+1) \times (n+1)$ - and  $(n+1) \times r$ -matrix-valued functions. Let us set the additional state variable :

$$\dot{y}_{n+1}(t) = 0 \quad , y_{n+1}(0) = 1$$

Then we define  $\tilde{Y} \in \mathbb{R}^{n+1}$  such that the  $j$ -th coefficient of  $\tilde{Y}$  is  $\tilde{Y}_j = Y_j$  for all  $1 \leq j \leq n$ , and  $\tilde{Y}_{n+1} = y_{n+1}$ . We posit  $\tilde{L} \in \mathcal{M}_{n+1, n+1}(\mathbb{R})$  such that the coefficients  $(k, j)$  are  $\tilde{L}_{k,j} = L_{k,j}$  for all  $1 \leq k, j \leq n$ , the coefficients  $(k, n+1)$  are  $\tilde{L}_{k, n+1} = N_k$  for all  $1 \leq k \leq n$ , and the coefficients  $(n+1, j)$  are  $\tilde{L}_{n+1, j} = 0$  for all  $1 \leq j \leq n+1$ . We similarly define  $\tilde{M} \in \mathcal{M}_{n+1, r}(\mathbb{R})$  such that the coefficients  $(k, j)$  are  $\tilde{M}_{k,j} = M_{k,j}$  for all  $1 \leq k \leq n$  and  $1 \leq j \leq r$ , and the coefficients  $(n+1, j)$  are  $\tilde{M}_{n+1, j} = 0$  for all  $1 \leq j \leq r$ . With such a construction, we easily check that the system (5) describes the same dynamics as the system (4). ■

We may now introduce the main issue of this work. Considering a time-optimal problem subject to the dynamics (2), we will discuss about a revisit of the Bang-Bang principle for such settings. It is well known that for hybrid dynamical systems or switched systems the optimal time trajectory may lie on the locus of discontinuity of the dynamics, and it is called a sliding mode. In this work, the systems we consider might be seen as spatially hybrid systems, but with an additional condition of continuity at the boundaries. One may wonder if such sliding mode behaviours is still possible with continuous piecewise affine dynamics. The Lemma 2.2 suggests that the Theorem 2.1 should remain relevant where

the dynamics of (2) is affine. However a direct application of the Bang-Bang principle would fail, since we can not ensure that the obtained Bang-Bang trajectory doesn't cross any boundary, which would thus modify the dynamics driving the system. Therefore we have to be a bit more cautious and judiciously examine the evolution of the system in order to locally apply the Bang-Bang principle, and then build a candidate Bang-Bang control step by step. In the end, we still show that there exists an optimal solution composed of a succession of Bang-Bang arcs over time intervals where the state trajectory belongs to a region  $\mathring{D}_i$ , but an optimal trajectory may contain a portion located in a boundary  $\partial D_i$ , thus consisting in singular arcs. We shall present in the Section IV examples where there is indeed singular arcs, showing this way that the Bang-Bang principle is not true globally for piecewise affine problems, even when the dynamics is continuous with respect to the state.

### III. MAIN RESULTS

*Proposition 3.1:* For a minimum-time control problem subject to the piecewise affine settings (2), there exists an optimal solution composed of a succession of Bang-Bang arcs over time interval where the state trajectory belongs to a region  $\mathring{D}_i$ .

Let  $u^*(\cdot) \in \mathcal{U}$  be an optimal solution of the time-optimal problem (2). Denote  $X^*(\cdot)$  the associated optimal trajectory and  $T^* \in \mathbb{R}_+^*$  the corresponding optimal time. Assume that there exists a time interval  $[a, b] \subset [0, T^*]$  such that  $X^*(b) \in \partial D_i$  where  $D_i$  is a fixed region of the partition, and  $X^*([a, b]) \subset \mathring{D}_i$ . For any  $[t_d, t_f] \subset (a, b)$ , we study the evolution of the system over  $[t_d, t_f]$ , thus following the dynamics of the region  $D_i$  :

$$\dot{X}(t) = A_i(t)X(t) + B(t)u(t) + C_i(t) \quad (6)$$

We aim to apply the Bang-Bang principle for affine systems, but we need to ensure that all the trajectories undergo the same dynamics. To this aim we will have to study the local behaviour of trajectories and we will define an appropriate neighbourhood of  $X^*([t_d, t_f])$  where the principle may be applied.

*Lemma 3.2:* With the notations above, for any  $[t_d, t_f] \subset [a, b]$ , there exists a Bang-Bang control such that its associated trajectory  $Z(\cdot)$  verifies :

$$Z(t_d) = X^*(t_d) \quad , \quad Z(t_f) = X^*(t_f) \quad , \quad Z([t_d, t_f]) \subset \mathring{D}_i$$

*Proof:* The distance from this portion of trajectory to the boundary is  $r = d(X^*([t_d, t_f]), \partial D_i) > 0$ . Let us fix  $\hat{t} \in [t_d, t_f]$ . Consider the problem defined by the dynamics of (6) with the initial condition  $X(\hat{t}) = X^*(\hat{t})$ , and select an admissible control  $u(\cdot)$ . We want to evaluate the difference between the associated trajectory  $X(\cdot)$  and the reference trajectory  $X^*(\cdot)$ . In particular, we are looking for a time  $t^+$  such that for all  $t \in [\hat{t}, t^+]$  we have  $\|X(t) - X^*(t)\| \leq r$ . For  $t \geq \hat{t}$  we have :

$$\begin{aligned} X(t) - X^*(t) &= X(t) - X(\hat{t}) - (X^*(t) - X^*(\hat{t})) \\ &= \int_{\hat{t}}^t A_i(s)(X(s) - X^*(s)) + B(s)(u(s) - u^*(s)) ds \end{aligned}$$

Then :

$$\begin{aligned} \|X(t) - X^*(t)\| &\leq 2M(t - \hat{t}) \sup_{s \in [a, b]} \|B(s)\| \\ &\quad + \int_{\hat{t}}^t \|A_i(s)\| \|X(s) - X^*(s)\| ds \end{aligned}$$

By the Grönwall lemma, we get :

$$\begin{aligned} \|X(t) - X^*(t)\| &\leq 2M(t - \hat{t}) K_{[a, b]}^B \exp\left(\int_{\hat{t}}^t \|A_i(s)\| ds\right) \\ &\leq 2M(t - \hat{t}) K_{[a, b]}^B \exp((t - \hat{t}) K_{[a, b]}^A) \end{aligned}$$

where  $K_{[a, b]}^A = \sup_{s \in [a, b]} \|A_i(s)\| < +\infty$  and  $K_{[a, b]}^B = \sup_{s \in [a, b]} \|B(s)\| < +\infty$ . Then there exists  $t^+ \in (\hat{t}, t_f]$  such that :

$$\|X(t) - X^*(t)\| \leq r, \quad \forall t \in [\hat{t}, t^+]$$

Moreover, the distance  $\delta = |t^+ - \hat{t}|$  only depends on  $r$ . Then there exists  $N \in \mathbb{N}^*$  such that  $t_d + (N - 1)\delta < t_f \leq t_d + N\delta$ , and we define a finite sequence  $(\theta_k)_{0 \leq k \leq N}$  by :

$$\begin{aligned} \theta_k &= t_d + k\delta, \quad \text{for } 0 \leq k \leq N - 1 \\ \theta_N &= t_f \end{aligned}$$

Therefore from above we deduce that for any  $0 \leq k \leq N - 1$ , all the trajectories  $Y_k(\cdot)$  following the dynamics (6) with initial condition  $Y_k(\theta_k) = X^*(\theta_k)$  satisfy :

$$Y_k(t) \in B(X^*(t), r), \quad \forall t \in [\theta_k, \theta_{k+1}]$$

In particular, we have :

$$Y_k([\theta_k, \theta_{k+1}]) \subset \bigcup_{t \in [\theta_k, \theta_{k+1}]} B(X^*(t), r) \subset \mathring{D}_i$$

As a consequence, we now may apply the Bang-Bang principle as follows. Consider the dynamics of (6) over the whole space  $\mathbb{R}^n$ , with the initial condition  $Y_k(\theta_k) = X^*(\theta_k)$ . By assumption, the target  $X^*(\theta_{k+1})$  is attainable at the time  $\theta_{k+1}$ , thus the Bang-Bang principle for linear time-optimal problems indicates that there exists a Bang-Bang control  $\bar{u}_k(\cdot)$  such that the corresponding trajectory  $\bar{Y}_k(\cdot)$  reaches  $X^*(\theta_{k+1})$  at the time  $\theta_{k+1}$ . Furthermore, we know from the well-chosen construction detailed above that this trajectory verifies :

$$\bar{Y}_k([\theta_k, \theta_{k+1}]) \subset \mathring{D}_i$$

Let us now define  $v(\cdot)$  as the concatenation of the  $\bar{u}_k(\cdot)$  for  $0 \leq k \leq N - 1$ , and denote  $Z(\cdot)$  the corresponding trajectory as we can see in Fig. 1.

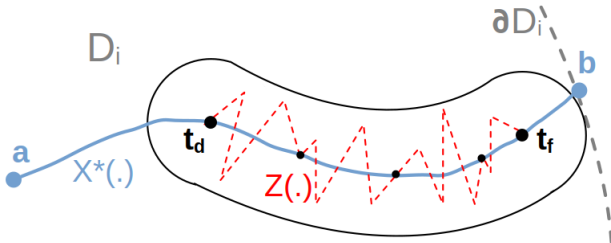


Fig. 1. Optimal Bang-Bang trajectory constructed from  $X^*$  on  $[t_d, t_f]$

Finally we showed that for any  $[t_d, t_f] \subset [a, b)$  the obtained control  $v(\cdot)$  is a Bang-Bang control defined over  $[t_d, t_f]$ , and the associated trajectory  $Z(\cdot)$  satisfies  $Z(t_d) = X^*(t_d)$ ,  $Z(t_f) = X^*(t_f)$  and :

$$Z([t_d, t_f]) \subset \bigcup_{0 \leq k \leq N-1} \left( \bigcup_{t \in [\theta_k, \theta_{k+1}]} B(X^*(t), r) \right) \subset \mathring{D}_i$$

*Proof:* (of Proposition 3.1) To conclude for the main statement, it then suffices to apply the previous Lemma 3.2 on well-chosen intervals. We set a sequence of times  $(\sigma_f^j)_{j \geq 0}$  by :

$$\sigma_f^j = b - (b - a)/2^j, \quad \text{for } j \in \mathbb{N}$$

We get a corresponding family of radiuses  $r_j = d(X^*([\sigma_f^j, \sigma_f^{j+1}]), \partial D_i) > 0$ , and the previous reasoning provides a family of Bang-Bang controls  $(v^j)_{j \geq 0}$ , and their associated trajectories  $(Z^j)_{j \geq 0}$ , defined on each portion of time interval by :

$$\begin{aligned} v^0 : [a, \sigma_f^1] &\longrightarrow U \quad \text{given by Lemma 3.2 over } [a, \sigma_f^1] \\ &\text{with } Z^0(a) = X^*(a), Z^0(\sigma_f^1) = X^*(\sigma_f^1) \\ &\text{and } Z^0([a, \sigma_f^1]) \subset \mathring{D}_i \end{aligned}$$

$$\begin{aligned} v^j : [\sigma_f^j, \sigma_f^{j+1}] &\longrightarrow U \quad \text{given by Lemma 3.2 over } [\sigma_f^j, \sigma_f^{j+1}] \\ &\text{with } Z^j(\sigma_f^j) = X^*(\sigma_f^j), Z^j(\sigma_f^{j+1}) = X^*(\sigma_f^{j+1}) \\ &\text{and } Z^j([\sigma_f^j, \sigma_f^{j+1}]) \subset \mathring{D}_i \end{aligned}$$

Finally, the control defined by a forward concatenation of the controls  $v^j$  above is :

$$V(t) = v^j(t) \quad \text{for } t \in [\sigma_f^j, \sigma_f^{j+1}]$$

Thus  $V$  is well-defined over  $[a, b)$ , is Bang-Bang, and its corresponding trajectory  $\xi(\cdot)$  satisfies :

$$\xi(a) = X^*(a), \quad \xi(t) \xrightarrow[t \rightarrow b]{} X^*(b), \quad \xi([a, b]) \subset \mathring{D}_i$$

Hence the existence of an optimal solution composed of Bang-Bang arcs in the regions where the dynamics is differentiable. ■

*Remark 3.3:* Since the classical Bang-Bang principle does not give information about the number of commutations, we may have constructed an optimal Bang-Bang solution that has an infinite (countable) number of commutations. However, if the boundaries fulfill the interior sphere condition (see [12]), then the radiuses  $r_j$  in our construction may not converge to zero, and we could be able to have a finite number of concatenations with an appropriate sequence of time intervals.

#### IV. EXAMPLES

In this section we are going to present two examples. The first one is a very simple autonomous system driving an optimal time problem where the geometry of the partition of the space gives the intuition of the existence of a singular arc. The second one is from a crop irrigation model, and is written as a non-autonomous system driving an optimization

problem with a fixed terminal time, a terminal cost, and constraints over the input and the state. Therefore it is much more difficult to study the optimality of solutions, but we will show that for some settings a singular arc is strictly better than Bang-Bang controls over a certain time interval.

#### A. Turnpike-like Example

Let us consider the following time-optimal control problem :

$$(P1) : \begin{cases} \dot{x}(t) = u(t) \\ \dot{y}(t) = F(x(t)) \quad \text{where } F(x) = \begin{cases} x & , x \leq 1 \\ 2-x & , x \geq 1 \end{cases} \end{cases}$$

This systems (P1) respects the framework of piecewise affine dynamics and continuous with respect to the state, the states space being partitioned in two pieces separated by the line  $x = 1$ . We set the initial conditions  $x(0) = y(0) = 0$ , and the target is the singleton  $(x_f, y_f) = (2, 3)$ . Moreover the control  $u(\cdot)$  is a measurable function with values constrained in the compact  $U = [-1, 1]$ . We seek to minimize the time to reach the target from the initial point.

By definition, the magnitude of  $\dot{y}$  is maximum when  $x = 1$ , then the intuition would say that the trajectory will tend to use this line as much as possible, in order to move as quickly as possible. We are going to show that this strategy (as illustrated in Fig. 2) is indeed optimal.

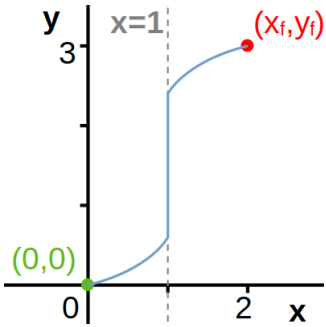


Fig. 2. Minimum time "Bang-Singular-Bang" trajectory for (P1)

Consider an optimal control  $u^*(\cdot)$ , with its associated trajectory  $(x^*(\cdot), y^*(\cdot))$  and the corresponding optimal time  $T^*$ . By continuity, the trajectory has to cross the boundary  $x = 1$ . Define  $W = \{t \geq 0, \text{ st } : x^*(t) = 1\}$ , which is non-empty. Denote  $t^- = \inf W$  and  $t^+ = \sup W$ . By definition of the dynamics of  $x$ , we have  $t^- \geq 1$  and  $t^+ \leq T^* - 1$ .

Suppose that  $y^*(t^-) > 1/2$ . Denoting  $\mathcal{C} = \{(x, y), y = x^2/2\}$ , we define  $\Gamma = \{t \in [0, t^-], \text{ st } : (x^*(t), y^*(t)) \in \mathcal{C}\}$ . The set  $\Gamma$  is non-empty since  $y^*(0) = (0, 0)$ , and we define  $\hat{t} = \sup \Gamma \in [0, t^-)$ . Then, for all  $t \in (\hat{t}, t^-)$ , we have either  $y^*(t) \geq 1/2$  and  $x^*(t) < 1$ , or  $y^*(t) < 1/2$  and  $y^*(t) - x^*(t)^2/2 > 0$ . In the first case, we directly have  $\dot{y}^*(t) < 1$ , and in the second case, we have  $x^*(t) < \sqrt{2y^*(t)}$ , and the point  $(\sqrt{2y^*(t)}, y^*(t)) \in \mathcal{C}$  is attainable from the point  $(x^*(\hat{t}), y^*(\hat{t}))$ . As a consequence, we have  $\dot{y}^*(t) < \sqrt{2y^*(t)}$ . Starting from the point  $(x^*(\hat{t}), y^*(\hat{t}))$  at time  $\hat{t}$ , we define a trajectory  $(\tilde{x}, \tilde{y})$  driven by the control  $\tilde{u}(t) = 1$  for  $t \in$

$[\hat{t}, \hat{t} + 1 - x^*(\hat{t})]$ , and  $\tilde{u}(t) = 0$  for  $t \in [\hat{t} + 1 - x^*(\hat{t}), \hat{t} + 1 - x^*(\hat{t}) + y^*(t^-) - 1/2]$ . This trajectory follows the curb  $\mathcal{C}$  until it reaches the point  $(1, 1/2)$ , then it goes straight up on the boundary  $x = 1$  until the point  $(1, y^*(t^-))$ . According to what was said above, we then have for all  $t \in (\hat{t}, \hat{t} + 1 - x^*(\hat{t}) + y^*(t^-) - 1/2)$ ,  $\dot{y}^*(t) < \dot{\tilde{y}}(t)$ . We deduce that the trajectory  $(\tilde{x}, \tilde{y})$  reaches the point  $(x^*(t^-), y^*(t^-))$  at a time  $\tilde{t} < t^-$ , which contradicts the optimality of the trajectory  $(x^*(\cdot), y^*(\cdot))$ . Hence  $y^*(t^-) \leq 1/2$ .

Moreover if  $y^*(t^-) < 1/2$ , a brief study of reachability of  $(1, y^*(t^-))$  shows that the trajectory of  $y$  has to decrease, and hence the trajectory of  $x$  has to go in the negative values. This way, it would exist a time  $s \in (0, t^-)$  such that  $x^*(s) = 0$ ,  $y^*(s) < 0$ , and since  $s > 1/2 - y^*(t^-)$  (since the speed of movement of  $y$  is smaller around  $x = 0$ , and getting  $|\dot{y}| = 1$  in the area  $x < 0$  would require to waste more than  $s = 1/2$  of time). As a consequence, the time  $\sigma$  to reach  $(1, y^*(t^-))$  has to verify  $\sigma \geq 1 + s > 3/2 - y^*(t^-)$ . Thus either the trajectory stays on the boundary  $x = 1$  until the point  $(1, 1/2)$ , or the quantity  $x$  grows. In the first case, the trajectory starting from  $(0, 0)$  and following  $\mathcal{C}$  would be strictly faster to reach  $(1, 1/2)$ , and in the second case, we could make a similar argument as before by defining  $\mathcal{D} = \{(x, y), y = -x^2/2 + 2x + 1\}$  and considering  $\check{t} = \inf \Lambda$  where  $\Lambda = \{t \in (\sigma, T^*), \text{ st } : (x^*(t), y^*(t)) \in \mathcal{D}\}$ . In any case, we conclude that it would contradict the optimality of the trajectory  $(x^*(\cdot), y^*(\cdot))$ . Hence  $y^*(t^-) = 1/2$ .

By mirroring these arguments for the end of the trajectory, we can show that  $y^*(t^+) = 5/2$ . Finally, any trajectory  $(x(\cdot), y(\cdot))$  that does not stay on  $x(t) = 1$  for all  $t \in [t^-, t^+]$  will necessarily present an interval  $I$  of non-null measure where  $\dot{y}(t) < 1$  for all  $t \in I$ . Therefore it will not reach the point  $(1, 5/2)$  as fast as the trajectory remaining on  $x = 1$ . Between  $(1, 1/2)$  and  $(1, 5/2)$ , the singular control  $u^* = 0$  is then optimal.

To conclude, we necessarily have  $t^- = 1$ ,  $u^*(t) = 1$  for all  $t \in [0, 1)$ ,  $t^+ = 3$ ,  $u^*(t) = 0$  for all  $t \in [1, 3)$ , and  $u^*(t) = 1$  for all  $t \in [3, T^*]$ . Furthermore, the optimal time is then  $T^* = 4$ .

#### B. Example from an Irrigation model

Let us define the following optimal control problem, denoted (P2), taken from a crop irrigation model (see [5]), and evolving over a fixed time interval  $[0, T]$ .

$$\begin{cases} \dot{S}(t) = k_1(-\varphi(t)K_S(S(t)) - (1 - \varphi(t))K_R(S(t)) + k_2u(t)) \\ \dot{B}(t) = \varphi(t)K_S(S(t)) \\ \dot{V}(t) = u(t) \end{cases}$$

with initial conditions  $S(0) = S_0 \in (S^*, 1]$ ,  $B(0) = 0$  and  $V(0) = 0$ , and where the functions  $K_S$  and  $K_R$  are continuous piecewise affine, as follows :

$$K_S(S) = \begin{cases} 0 & , S \in [0, S_w] \\ \frac{S - S_w}{S^* - S_w} & , S \in [S_w, S^*] \\ 1 & , S \geq S^* \end{cases}$$

$$K_R(S) = \begin{cases} 0 & , S \in [0, S_h] \\ \frac{S - S_h}{1 - S_h} & , S \geq S_h \end{cases}$$

with  $0 < S_h < S_w < S^* < 1$ . The control  $u(\cdot)$  takes values in  $[0, 1]$ , and the constants verify  $k_1 > 0$  and  $k_2 > 1$  (local controllability condition). The function  $\varphi$  is supposed to be  $C^1$ , increasing, and taking values in  $[0, 1]$ . The optimization problem consists in fixing a terminal time  $T > 0$  and aiming at a maximization of the terminal cost  $B(T)$  under some target constraint  $V(T) = V_{init}$ , and some state inequality constraints  $S_{tol} \leq S(t) \leq 1$  for all  $t \in [0, T]$ .

This system (P2) is a quite complex constrained optimization problem, and does not fulfill the framework of piecewise affine time-optimal problems yet. In order to solve this issue, we first cope with the state constraints. Following an idea in [13], we replace the constraint by two additional state equations :

$$\dot{Y}(t) = \min(0, S(t) - S_{tol}) \quad , \quad \dot{Z}(t) = \max(0, S(t) - 1)$$

with complementary initial and final conditions :

$$Y(0) = Z(0) = 0 \quad , \quad Y(T) = Z(T) = 0$$

We obtain a system of five equations and additional initial and final conditions. The functions  $S \mapsto \min(0, S - S_{tol})$  and  $S \mapsto \max(0, S - 1)$  are piecewise affine and continuous with respect to the state. Then, in a similar way as in the Section II.B, we add a new state in order to measure the time :

$$\dot{\theta}(t) = 1$$

with initial condition  $\theta(0) = 0$  and terminal condition  $\theta(T) = T$ . Finally, let us assume that there exists an optimal solution of the problem (P2). In this case, there is a corresponding optimal terminal cost  $B^* > 0$  and a final state  $S(T) = \tilde{S}$ . We then consider that it gives in fact an additional terminal condition for the state  $B(\cdot)$ , ie.  $B(T) = B^*$ . Now we may replace all the terminal conditions by a target :

$$(S_f, B_f, V_f, Y_f, Z_f, \theta_f) = (\tilde{S}, B^*, V_{init}, 0, 0, T)$$

With all these transformations, we now have a problem written as a piecewise affine system with continuous dynamics with respect to the state, and the previous optimization is thus by definition equivalent to a minimum time problem to reach the target.

If  $k_2$  and  $V_{init}$  are sufficiently high, then the application of the Bang control  $u(t) = 1$  over a time interval will allow the trajectory of  $S(\cdot)$  to reach the threshold  $S^*$ , at a certain time  $t_1$ . Then, if  $V_{init}$  is not too high, we can show (by using the expression of the switching function and the terminal necessary conditions given by the non-smooth PMP, see [5]) that it is optimal to have a Bang arc  $u(t) = 0$  over the final time interval. Under these circumstances, there exists a time  $t_2$  such that  $S(t_2) = S^*$ , and  $S(t) < S^*$  for all  $t > t_2$ . Over the time interval  $[t_1, t_2]$ , we are going to show that it is then optimal to keep the trajectory at  $S = S^*$  rather than making Bang arcs, as we illustrate in Fig. 3.

Let us denote  $\tilde{u}(\cdot)$  an optimal control for (P2), and  $(\tilde{S}(\cdot), \tilde{B}(\cdot), \tilde{V}(\cdot))$  its associated trajectory. Suppose that there exists  $t_1 < t_2$  such that  $S(t_1) = S(t_2) = S^*$ . For any control  $u$  and corresponding trajectory  $S$ , define  $B_{[t_1, t_2]}(u) =$

$\int_{t_1}^{t_2} \varphi(t) K_S(S(t)) dt$  and  $V_{[t_1, t_2]}(u) = \int_{t_1}^{t_2} u(t) dt$ . According to the initial problem, the objective on this interval  $[t_1, t_2]$  is to maximize the quantity  $B_{[t_1, t_2]}(u)$  with the target  $V_{[t_1, t_2]}(u) = V_{[t_1, t_2]}(\tilde{u})$ .

Let us assume that there exists a time  $\hat{t} \in (t_1, t_2)$  such that  $\tilde{S}(\hat{t}) < S^*$ . By continuity there exists a neighbourhood  $I = [t^-, t^+] \subset [t_1, t_2]$  where  $\tilde{S}(t) < S^*$  for all  $t \in I$  and  $\tilde{S}(t^-) = \tilde{S}(t^+) = S^*$ . Denote  $u_{sing}$  the control keeping  $S$  constant, defined by :

$$u_{sing}(t, S) = \frac{\varphi(t) K_S(S(t)) + (1 - \varphi(t)) K_R(S(t))}{k_2} \in (0, 1)$$

Thus the trajectory of  $S_s$  over  $I$  with initial condition  $S_s(t^-) = S^*$  and the control  $u_{sing}(\cdot, S^*)$  is constant equal to  $S^*$  over  $I$ . For any  $t \in I$ , using the dynamics and the fact that  $K_S$  and  $K_R$  are non-decreasing in  $S$ , we then have :

$$0 < S_s(t) - \tilde{S}(t) \leq k_1 k_2 \int_{t^-}^t u_{sing}(\tau, S^*) - \tilde{u}(\tau) d\tau$$

We deduce that  $\int_I u_{sing}(t, S^*) dt > \int_I \tilde{u}(t) dt$ . As a consequence, since we must have  $V_{[t_1, t_2]}(u_{sing}) = V_{[t_1, t_2]}(\tilde{u})$ , it is not possible to have  $\tilde{S}(t) \leq S^*$  for all  $t \in [t_1, t_2]$ , using a similar argument. Then there exists an interval  $J \subset [t_1, t_2]$  such that  $\tilde{S} > S^*$  for all  $t \in J$ . We define the following sets :

$$G^< = \{t \in [t_1, t_2], \text{ st : } \tilde{S}(t) < S^*\}$$

$$G^> = \{t \in [t_1, t_2], \text{ st : } \tilde{S}(t) > S^*\}$$

By the reasoning above, these sets have non-null measure, and by using the dynamics of  $B$  and the definition of  $K_S$ , we finally have :

$$\int_{G^<} \varphi(t) (K_S(S_s(t)) - K_S(\tilde{S}(t))) dt > 0$$

and :

$$\int_{G^>} \varphi(t) (K_S(S_s(t)) - K_S(\tilde{S}(t))) dt = 0$$

In other words, the quantity  $B_{[t_1, t_2]}(u_{sing})$  is thus strictly bigger than  $B_{[t_1, t_2]}(\tilde{u})$ . To conclude, it is not possible to have  $\hat{t} \in (t_1, t_2)$  such that  $\tilde{S}(\hat{t}) < S^*$  for the optimal solution. A mirror argument holds for  $\tilde{S}(\hat{t}) > S^*$ , and we finally deduce that the trajectory  $S(\cdot) = S^*$  is optimal on such an interval  $[t_1, t_2]$ .

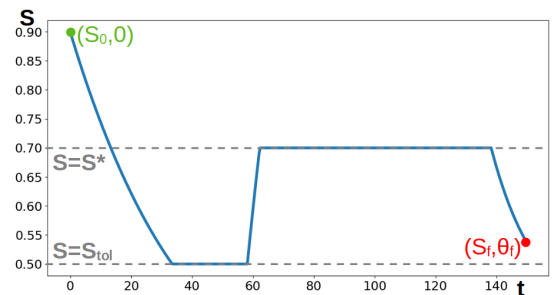


Fig. 3. Optimal mixed Bang and singular trajectory for (P2)

## V. CONCLUSIONS

We focused our study here on a discussion about the validity of the Bang-Bang principle in a piecewise affine framework. As we saw, an optimal solution may in general be composed of Bang arcs inside the areas of the partition, and eventual singular arcs on the boundaries. For future perspectives, we may wonder about deriving the non-smooth Pontryagin principle for a continuous piecewise affine system, and especially we may interest in the conditions of occurrence of such singular arcs. In fact, generalized gradients may arise from the derivation of the Hamiltonian, and the behaviour of the adjoints over time intervals where the state runs through a non-differentiable locus of the dynamics is therefore non-trivial. A more precise description of this situation may be considered as the next step for the study of continuous piecewise affine systems.

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