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Volatility and jump activity estimation in a stable Cox-Ingersoll-Ross model

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abstract We consider the parametric estimation of the volatility and jump activity in a stable Cox-Ingersoll-Ross (α -stable CIR) model driven by a standard Brownian Motion and a non-symmetric stable Lévy process with jump activity $\alpha \in (1, 2)$. The main difficulties to obtain rate efficiency in estimating these quantities arise from the superposition of the diffusion component with jumps of infinite variation. Extending the approach proposed in Mies [30], we address the joint estimation of the volatility, scaling and jump activity parameters from high-frequency observations of the process and prove that the proposed estimators are rate optimal up to a logarithmic factor.

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Key words: Cox-Ingersoll-Ross process, Lévy process; parametric inference; stable process; stochastic differential equation.

1 Introduction

Jump processes are widely studied in the literature due to their many areas of application, especially in financial modeling where interest rates or asset price evolutions are often described by Itô semimartingales. These processes are characterized by the superposition of a drift term, a diffusion part and a jump component and the estimation of the different components is the subject of specific studies depending on the available observations. In this paper we are interested in the estimation of the volatility and the jump activity from high-frequency observations.

The estimation of the volatility or integrated volatility (quantities of interest in financial modeling) has concentrated a lot of activity in the last decades. For diffusion processes, it is well known that the realized quadratic variation is a rate and variance efficient estimator of the integrated volatility, but models with jumps are more appropriate to financial markets as financial price series exhibit jumps (see Huang and Tauchen [14]). Jump activity is characterized by the Blumenthal-Gettoor index $\alpha \in [0, 2)$. When jumps are of finite variation ($\alpha < 1$) the main statistics to estimate the volatility are the truncated realized quadratic variation, which is rate and variance efficient, and the bi-power or multipower variations, which are rate efficient but not variance efficient. We refer to Mancini [27] [28] [29], Barndorff-Nielsen and Shephard [6], Barndorff-Nielsen et al. [7] for the properties of these methods. When jumps are of infinite variation, i.e. $\alpha > 1$, the previous statistics are still consistent but with a rate of convergence slower than the usual one \sqrt{n} (see Jacod and Reiss [16]) and also exhibit an asymptotic bias. The first efficient estimator of the volatility, with optimal rate of convergence \sqrt{n} , has been proposed in Jacod and Todorov [19] and [20] in presence of infinite variation jumps with stable-like behavior. Their method is based on the local estimation of the volatility by using the real part of the empirical characteristic function and the asymptotic bias is eliminated by combining

statistics with different arguments in the characteristic function. Their estimator achieves rate and variance optimality for $\alpha \in (1, 2)$ if the jumps of the driving Lévy process are symmetric, and for $\alpha < 3/2$ for non-symmetric jumps (the estimator is only rate efficient for larger values of α). For the truncated realized volatility, some debiasing procedures have been studied in Amorino and Gloter [4] and in Cooper Boniece et al. [9]. By considering different truncation levels and with a two-step debiasing procedure, efficiency for the estimation of the volatility is obtained in [9], with the restriction $\alpha \in (1, 8/5)$.

Turning to the estimation of the jump activity, the previous discussion makes clear the importance of the parameter α , in addition to its intrinsic interest. In presence of a diffusion component, Aït-Sahalia and Jacod in [1] have proposed an estimator of α based on jump counting. For locally stable jumps, their estimator is consistent with rate of convergence $n^{\alpha/10}$ ($n^{\alpha/8}$ in the stable case) but unfortunately it fails to achieve the optimal rate $n^{\alpha/4} \log(n)^{1-\alpha/4}$ identified in Aït-Sahalia and Jacod [2]. This estimation method is improved in Bull [10] by combining increments of the process with different time-scales, achieving the near optimal rate of convergence $n^{\alpha/4-\epsilon}$ for ϵ arbitrarily small. All these methods are non-parametric and apply to semimartingales but are not completely satisfactory. In a parametric framework, rate optimality in estimating the jump activity has been attained by Mies [30] for a Lévy process with characteristic triplet (μ, σ^2, ν) , assuming that the jump part is close to the superposition of M stable processes. Using a method of moments, a joint estimator of the volatility σ^2 , the jump activity indices $\alpha = (\alpha_m)_{1 \leq m \leq M}$ and some scaling coefficients $r = (r_m^+, r_m^-)_{1 \leq m \leq M}$ is proposed. This procedure leads to an efficient estimator of the volatility (without debiasing procedure and without restriction on the jump activity) and to rate optimal estimators of the jump activity and scaling coefficients.

In this paper, our aim is to study the joint parametric estimation of the volatility and jump activity of a stochastic differential equation with infinite variation jumps. We will focus our work on the stable Cox-Ingersoll-Ross model (stable CIR model) mainly for the following reasons. This parametric model is very popular in financial modeling (see Jiao et al. [21], [22]) and it also presents some specific difficulties since the coefficients of the equation are not Lipschitz and not bounded away from zero. However we mention that the estimation method proposed in this paper could be extended to a larger class of stochastic equations with jumps.

The stable CIR model belongs to the class of affine processes (or Continuous Branching processes with Immigration, we refer to Dawson and Li [11] and Li [25] for a general presentation of these processes) and is described by the stochastic equation

$$dX_t = (a - bX_t)dt + \sigma X_t^{1/2} dB_t + \delta^{1/\alpha} X_t^{1/\alpha} dL_t^\alpha,$$

where (B_t) is a standard Brownian Motion and (L_t^α) a spectrally positive α -stable Lévy process with jump activity $\alpha \in (1, 2)$. When $\sigma = 0$ (pure-jump process), some estimation results have been obtained in Li and Ma [26], Yang [33] and more recently in Bayraktar and Clément [8]. But when $\sigma \neq 0$, there are almost no estimation results in the literature for the stable CIR model and to our knowledge the only existing study concerns the estimation of the drift term b proposed in Barczy et al. [5] from continuous observations of the process. Inspired by the work of Mies [30], we propose here a joint estimation method of the volatility coefficient σ^2 , the scaling parameter δ and the jump activity index α from high-frequency observations of the process that leads to an efficient estimation of the volatility, for any $\alpha \in (1, 2)$, and attains rate optimality up to a factor $\log(n)$ for the estimation of (δ, α) . Our main results are derived in the case $n\Delta_n$ fixed, where n is the number of observations and Δ_n the step between two consecutive observations (we recall that in this framework and in presence of a Brownian component, the drift parameters (a, b) are not identifiable) but also in the long time asymptotics $n\Delta_n \rightarrow +\infty$.

The paper is organized as follows. In Section 2, we present the model and notation. In Section 3, we describe the estimation method and study the asymptotic properties of estimators in both cases $n\Delta_n$ fixed and $n\Delta_n \rightarrow +\infty$. Section 4 contains some intermediate key results

that are of intrinsic interest. The theoretical results are illustrated in the case $n\Delta_n$ fixed by numerical simulations in Section 5. Proofs are gathered in Section 6.

2 Setting and notation

We consider $(B_t)_{t \geq 0}$ a Brownian motion and $(L_t^\alpha)_{t \geq 0}$ an independent stable Lévy process with exponent $\alpha \in (1, 2)$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where $(\mathcal{F}_t)_{t \geq 0}$ is the natural augmented filtration. We assume that $(L_t^\alpha)_{t \geq 0}$ is a spectrally positive Lévy process with characteristic function

$$\mathbb{E} \left(e^{izL_t^\alpha} \right) = \exp \left(-|z|^\alpha \left(1 - i \tan \left(\frac{\pi\alpha}{2} \right) \operatorname{sgn}(z) \right) \right). \quad (2.1)$$

This non-symmetric Lévy process is strictly stable (see for example Theorem 14.15 in Sato [32]) and admits the representation

$$L_t^\alpha = \int_0^t \int_{\mathbb{R}_+} z \tilde{N}(ds, dz), \quad \tilde{N} = N - \bar{N},$$

where N is a Poisson random measure with compensator \bar{N} given by

$$\bar{N}(dt, dz) = dt F^\alpha(dz) \quad \text{for} \quad F^\alpha(dz) = \frac{c_\alpha}{z^{\alpha+1}} \mathbf{1}_{(0, +\infty)}(z) dz. \quad (2.2)$$

From Lemma 14.11 in Sato [32], we have

$$c_\alpha = -\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha) \cos(\pi\alpha/2)}, \quad \text{with} \quad \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \quad \alpha > 0. \quad (2.3)$$

We will study the estimation of the parameter $\theta = (\sigma^2, \delta, \alpha)$ of the process (stable CIR model) given by the stochastic differential equation

$$dX_t = (a - bX_t)dt + \sigma X_t^{1/2} dB_t + \delta^{1/\alpha} X_t^{1/\alpha} dL_t^\alpha, \quad X_0 = x_0 > 0. \quad (2.4)$$

Assuming $a > 0$, $b \in \mathbb{R}$, $\delta > 0$, $\sigma > 0$, $x_0 > 0$, equation (2.4) admits a pathwise unique nonnegative solution (see Fu and Li [12]), which has the Markov property. Since the coefficients of (2.4) are not Lipschitz, the condition $\alpha > 1$ is required to prove this result. If additionally $2a \geq \sigma^2$, we have according to Proposition 3.4. in Jiao et al [21] that $(X_t)_t$ is positive.

We use the following notation. For any random variable Z , we write $\mathbb{E}_i(Z) = \mathbb{E}(Z | \mathcal{F}_{i\Delta_n})$ and for $p > 0$, $\mathbb{E}^p(Z) = [\mathbb{E}(Z)]^p$ ($\mathbb{E}_i^p(Z)$ is defined accordingly). For any process Y , we denote the increments with step size Δ_n by

$$\Delta_i^n Y = Y_{(i+1)\Delta_n} - Y_{i\Delta_n}.$$

We write \implies for the convergence in distribution, and $\xrightarrow{\mathcal{L}-s}$ for the stable convergence in law. Throughout the paper, we denote by C all positive constants (or C_p if it depends on an auxiliary parameter p), which might change from line to line.

3 Main results

From now on, we assume that we observe $(X_{i\Delta_n})_{0 \leq i \leq n}$ that solves (2.4) for the parameter values (a_0, b_0, θ_0) satisfying assumption **H** below.

H. We assume that $\theta_0 = (\sigma_0^2, \delta_0, \alpha_0) \in \Theta = (0, +\infty) \times (0, +\infty) \times (1, 2)$ and $a_0 > \sigma_0^2$.

Under **H**, $(X_t)_{t \geq 0}$ is positive as previously mentioned, but also satisfies that $\mathbb{E}(1/X_t^2) < \infty$ (see Proposition 4.1). To estimate θ_0 , we propose a moment method based on the increments of

X , that extends to stochastic equations the estimation method proposed in Mies [30] for Lévy processes. For $f : \mathbb{R} \mapsto \mathbb{R}^3$, we consider the estimating function $F_n(\theta)$ defined by

$$F_n(\theta) = \frac{1}{n} \sum_{i=0}^{n-1} \left[f \left(\frac{u_n}{\sqrt{\Delta_n}} \frac{\Delta_i^n X}{\sqrt{X_{i\Delta_n}}} \right) - P_n f(X_{i\Delta_n}, \theta) \right], \quad (3.1)$$

where the centering term $P_n f(x, \theta)$, defined by (3.2), is constructed on the following way. Considering the Euler approximation of $\Delta_i^n X$ omitting the drift term

$$\Delta_i^n \bar{X} = \sigma_0 X_{i\Delta_n}^{1/2} \Delta_i^n B + \delta_0^{1/\alpha_0} X_{i\Delta_n}^{1/\alpha_0} \Delta_i^n L^{\alpha_0},$$

we approximate $\mathbb{E}_i f \left(\frac{u_n}{\sqrt{\Delta_n}} \frac{\Delta_i^n X}{\sqrt{X_{i\Delta_n}}} \right)$ by $\mathbb{E}_i f \left(\frac{u_n}{\sqrt{\Delta_n}} \frac{\Delta_i^n \bar{X}}{\sqrt{X_{i\Delta_n}}} \right)$. Now observing that $(\Delta_i^n B, \Delta_i^n L^\alpha)$ has the distribution of $(\sqrt{\Delta_n} B_1, \Delta_n^{1/\alpha} L_1^\alpha)$, we set

$$P_n f(x, \theta) = \mathbb{E} \left[f \left(u_n \sigma B_1 + \frac{(\delta x)^{1/\alpha}}{\sqrt{x}} u_n \Delta_n^{1/\alpha-1/2} L_1^\alpha \right) \right], \quad (3.2)$$

and in particular we have $P_n f(X_{i\Delta_n}, \theta_0) = \mathbb{E}_i f \left(\frac{u_n}{\sqrt{\Delta_n}} \frac{\Delta_i^n \bar{X}}{\sqrt{X_{i\Delta_n}}} \right)$.

At this stage, we assume that u_n goes to zero as n goes to infinity and $u_n/\sqrt{\Delta_n}$ goes to infinity, we will precise later the rate of convergence of the sequence (u_n) . We estimate the parameter θ_0 by solving the estimating equation $F_n(\theta) = 0$ for F_n defined by (3.1) with $f = (f_k)_{1 \leq k \leq 3}$, $f_1(x) = \cos(x)$, $f_2(x) = 1 - K(x)$, $f_3(x) = f_2(2x)$ and

$$K(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ (1 + \exp(\frac{1}{2-|x|} - \frac{1}{1-|x|}))^{-1} & \text{if } 1 < |x| < 2, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

Let us comment this choice of estimating functions. To estimate the volatility σ^2 , the idea is to consider even functions, quadratic near zero. The choice $f_1(x) = \cos(x)$ (also used in Jacod and Todorov [19]) has the additional property that the expression of $P_n f_1$ is explicit, it is the real part of the characteristic function of $u_n \sigma B_1 + \frac{(\delta x)^{1/\alpha}}{\sqrt{x}} u_n \Delta_n^{1/\alpha-1/2} L_1^\alpha$ given by

$$P_n f_1(x, \theta) = \exp \left(-u_n^2 \frac{\sigma^2}{2} - \delta x^{1-\alpha/2} u_n^\alpha \Delta_n^{1-\alpha/2} \right) \cos(\delta x^{1-\alpha/2} u_n^\alpha \Delta_n^{1-\alpha/2} \tan(\pi\alpha/2)). \quad (3.3)$$

For the jump parameters α and δ , we follow the idea of Aït-Sahalia and Jacod [1], where to neglect the diffusion contribution in estimating α , the following truncation of the small jumps is considered

$$\sum_{i=1}^n \mathbf{1}_{|\Delta_i^n X| > \Delta_n^\omega}, \quad \text{with } 0 < \omega < 1/2.$$

We also refer to Jing et al. [23], who replace the indicator function by smooth functions not vanishing near zero to take into account the small jumps. Here we replace the indicator function (truncation of small jumps) by the smooth function $1 - K$, with different truncation levels. $P_n f_2$ and $P_n f_3$ are not explicit (an asymptotic expansion is given in Lemma 4.2) but can be computed numerically.

The asymptotic properties of estimators depend on the connections between the number of observations n and the discretization step Δ_n and we study separately the two cases $n\Delta_n$ fixed and $n\Delta_n \rightarrow +\infty$.

3.1 Fixed observation window

We consider that $n\Delta_n$ is fixed and without loss of generality, we assume that $n\Delta_n = 1$. We first prove the existence of a joint estimator with non-diagonal rate of convergence Λ_n defined by

$$\Lambda_n(\theta) = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{u_n^{\alpha/2} \Delta_n^{1/2-\alpha/4}} & -\delta \ln\left(\frac{u_n}{\sqrt{\Delta_n}}\right) \frac{1}{u_n^{\alpha/2} \Delta_n^{1/2-\alpha/4}} \\ 0 & 0 & \frac{1}{u_n^{\alpha/2} \Delta_n^{1/2-\alpha/4}} \end{pmatrix}. \quad (3.4)$$

Theorem 3.1. *We assume \mathbf{H} , $n\Delta_n = 1$ and $u_n = 1/[\ln(1/\Delta_n)]^p$ with $p > 1/2$. Then there exists a sequence of random vectors $(\hat{\theta}_n)$ solving $F_n(\hat{\theta}_n) = 0$ with probability tending to one, i.e. $\mathbb{P}(F_n(\hat{\theta}_n) = 0) \rightarrow 1$. This sequence satisfies $\hat{\theta}_n \rightarrow \theta_0$ in probability as $n \rightarrow \infty$, and*

$$\Lambda_n(\theta_0)^{-1}(\hat{\theta}_n - \theta_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}-s} W(\theta_0)^{-1} \Sigma(\theta_0)^{1/2} \mathcal{N},$$

where \mathcal{N} is a standard Gaussian variable independent of \mathcal{F}_1 . Σ is the symmetric positive definite matrix and W is the non-singular matrix respectively defined by

$$\Sigma(\theta) = \frac{1}{2} \begin{pmatrix} \sigma^4 & 0 & 0 \\ 0 & c_\alpha \delta I(\alpha, X) \int_{\mathbb{R}} \frac{f_2^2(v)}{|v|^{\alpha+1}} dv & c_\alpha \delta I(\alpha, X) \int_{\mathbb{R}} \frac{(f_2 f_3)(v)}{|v|^{\alpha+1}} dv \\ 0 & c_\alpha \delta I(\alpha, X) \int_{\mathbb{R}} \frac{(f_2 f_3)(v)}{|v|^{\alpha+1}} dv & c_\alpha \delta I(\alpha, X) \int_{\mathbb{R}} \frac{f_3^2(v)}{|v|^{\alpha+1}} dv \end{pmatrix}, \quad (3.5)$$

$$W(\theta) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\psi(\alpha) I(\alpha, X) & -\delta \partial_\alpha [\psi(\alpha) I(\alpha, X)] \\ 0 & -2^\alpha \psi(\alpha) I(\alpha, X) & -\delta \partial_\alpha [2^\alpha \psi(\alpha) I(\alpha, X)] \end{pmatrix}, \quad (3.6)$$

with $I(\alpha, X) = \int_0^1 X_t^{1-\alpha/2} dt$, $\psi(\alpha) = c_\alpha \int \frac{f_2(v)}{|v|^{\alpha+1}} dv$ and c_α given by (2.3).

We deduce from Theorem 3.1 that $\sqrt{n}(\hat{\sigma}_n^2 - \sigma_0^2) \rightarrow \mathcal{N}(0, 2\sigma_0^4)$, so the estimator is rate and variance efficient, for any $\alpha_0 \in (1, 2)$. Turning to the jump parameters, with $u_n = (\log(1/\Delta_n))^p$, the marginal estimation rates are respectively $1/((\log(1/\Delta_n))^{p\alpha_0/2} \Delta_n^{\alpha_0/4} \log(u_n/\sqrt{\Delta_n}))$ for δ and $1/((\log(1/\Delta_n))^{p\alpha_0/2} \Delta_n^{\alpha_0/4})$ for α . Since the asymptotic variance does not depend on the parameter p and the estimation rate increases as p decreases, the theoretical asymptotic accuracy of the estimation method is improved by choosing p as small as possible, with the restriction $p > 1/2$. The restriction $p > 1/2$ comes mainly from Lemma 4.2 and ensures that the contribution of the term given in (6.16) is negligible. However from the bound (6.16), this assumption could be weakened up to the choice $u_n = C/(\log(1/\Delta_n))^{1/2}$ with $0 < C \leq 1/4\sigma_0$ (similar to condition (U) in [30]). As the choice of the constant C depends on the unknown value σ_0 , we preferred to assume in the statement of Theorem 3.1 the stronger condition $u_n = (\log(1/\Delta_n))^p$ with $p > 1/2$. From the asymptotic theoretical point of view, the optimal choice would be $p = 1/2$ but in practice for finite sample, it is less clear and we refer to the empirical study carried out in Section 5 where the choice $p = 1$ gives satisfactory results.

Efficiency for the joint estimation of (δ, α) in presence of a Brownian motion is still an open problem even for the Lévy process $X_t = \sigma B_t + \delta L_t^\alpha$ with $\sigma > 0$ (the Local Asymptotic Normality property with non-singular information matrix is not established). However marginal optimality has been discussed in Aït-Sahalia and Jacod [2] in the Lévy case showing that the marginal optimal rates of convergence for the estimation of δ and α are respectively $1/((\log(1/\Delta_n))^{\alpha/4} \Delta_n^{\alpha/4})$ and $\log(1/\Delta_n)/((\log(1/\Delta_n))^{\alpha/4} \Delta_n^{\alpha/4})$. These results have been completed in Mies [30] by showing that a diagonal normalisation with marginal optimality for each parameter leads to a singular information matrix, suggesting that the optimal rate of convergence in the joint estimation of (δ, α) is non-diagonal with a loss of rate of convergence of order $\log(1/\Delta_n)$. These partial results

allow us to conclude that our estimator is near rate optimal up to a factor $\log(1/\Delta_n)^\epsilon$, with ϵ arbitrarily small.

We now assume that α_0 is known and only consider the estimation of $\beta = (\sigma^2, \delta)$ by solving $F_n(\sigma^2, \delta, \alpha_0) = 0$, where F_n is defined by (3.1) with $f = (f_1, f_2)$. In that case, we can also prove a uniqueness result, which implies that any solution to the estimating equation is consistent.

Theorem 3.2. *We assume that α_0 is known and that (σ_0^2, δ_0) belongs to the interior of a compact subset of $(0, +\infty) \times (0, +\infty)$. Then under the assumptions of Theorem 3.1 any sequence $(\hat{\sigma}_n^2, \hat{\delta}_n)$ that solves $F_n(\sigma^2, \delta, \alpha_0) = 0$ converges in probability to (σ_0^2, δ_0) . Such a sequence exists and is unique in the sense that if $(\tilde{\sigma}_n^2, \tilde{\delta}_n)$ is another sequence that solves $F_n(\sigma^2, \delta, \alpha_0) = 0$ then $\mathbb{P}((\hat{\sigma}_n^2, \hat{\delta}_n) \neq (\tilde{\sigma}_n^2, \tilde{\delta}_n))$ goes to zero as n goes to infinity. This sequence obviously satisfies the asymptotic properties of Theorem 3.1.*

3.2 Ergodic case

We consider in this section the long time asymptotics $n\Delta_n \rightarrow +\infty$ and we restrict our study to the case $b_0 > 0$. With this assumption, the process $(X_t)_{t \geq 0}$ is geometrically ergodic (we refer to Li and Ma [26] Proposition 2.2 and Theorem 2.5) and the stationary distribution π_0 has Laplace transform

$$\int_0^{+\infty} e^{-\lambda y} \pi_0(dy) = \exp \left(- \int_0^\lambda \frac{a_0 x}{\frac{\sigma_0^2}{2} x^2 + \frac{\bar{\delta}_0^{\alpha_0}}{\alpha_0} x^{\alpha_0} + b_0 x} dx \right),$$

with $\bar{\delta}_0^{\alpha_0} = \delta_0 \alpha_0 / |\cos(\frac{\pi \alpha_0}{2})|$. Since the Laplace transform is \mathcal{C}^1 at zero, we check easily that $\int_0^\infty x \pi_0(dx) < +\infty$, and setting $\bar{I}(\alpha) = \int_0^\infty x^{1-\alpha/2} \pi_0(dx)$ we deduce that $\alpha \mapsto \bar{I}(\alpha)$ is derivable for $\alpha \in (1, 2)$.

Theorem 3.3. *We assume **H** and $b_0 > 0$, $n\Delta_n \rightarrow +\infty$, $u_n = 1/[\ln(1/\Delta_n)]^p$ with $p > 1/2$, and the additional condition $\sqrt{n}\Delta_n^{1/\alpha_0 + \alpha_0/4 - 1/2 - \epsilon_0} \rightarrow 0$ for $\epsilon_0 > 0$. Then the result of Theorem 3.1 holds with*

$$\Lambda_n(\theta_0)^{-1}(\hat{\theta}_n - \theta_0) \xrightarrow[n \rightarrow \infty]{} \bar{W}(\theta_0)^{-1} \bar{\Sigma}(\theta_0)^{1/2} \mathcal{N},$$

where \mathcal{N} is a standard Gaussian variable. $\bar{\Sigma}$ is the symmetric positive definite (deterministic) matrix and \bar{W} is the (deterministic) non-singular matrix respectively defined by (3.5) and (3.6) replacing $I(\alpha, X)$ by $\bar{I}(\alpha) = \int_0^{+\infty} x^{1-\alpha/2} \pi_0(dx)$.

The condition $\sqrt{n}\Delta_n^{1/\alpha_0 + \alpha_0/4 - 1/2 - \epsilon_0} \rightarrow 0$ is compatible with $n\Delta_n \rightarrow +\infty$ if $\alpha_0 \in (1, 2)$ but stronger for α_0 near 2. We also mention that we can also prove in the ergodic case a uniqueness result for the estimation of (σ^2, δ) (similar to Theorem 3.2) if we assume that α_0 is known.

Remark 3.1. *Let us remark that the estimation of the drift parameters (a, b) is possible in the long time asymptotic setting $n\Delta_n \rightarrow +\infty$. We refer to Amorino and Gloter [3] (that extends Gloter et al. [13] to jumps with infinite variation), who propose an efficient contrast method to estimate the drift parameters of a stochastic equation with jumps. Their method refines the quasi-likelihood method based on the Euler approximation of the continuous part of the stochastic equation by replacing the drift contribution in the Euler approximation by the conditional expectation $\mathbb{E}_i X_{(i+1)\Delta_n}$. The sufficient conditions given in [3] are not satisfied by the stable CIR model, in particular the coefficients of equation (2.4) are non-Lipschitz and not bounded from below and the process (X_t) has only finite moments of order p for $p < \alpha_0$. Consequently the results of [13] or [3] cannot be directly applied. But using the results of Proposition 4.1, it should be possible to extend them to the stable CIR model. We do not investigate the drift estimation in this paper and we leave it for future research.*

4 Auxiliary results

4.1 Moments and approximation

Since X_t is positive but not lower bounded by some strictly positive constant, we first establish conditions to ensure that $\mathbb{E}(1/X_t^p)$ is finite. We also give some bounds for the moments of $(X_t)_t$ and its increments.

Proposition 4.1. *Let $(X_t)_{t \geq 0}$ be the solution of (2.4) with $a > 0$, $b \in \mathbb{R}$, $\delta > 0$, $\sigma > 0$, $x_0 > 0$ and $2a > \sigma^2$. Then we have*

$$(i) \quad \text{for } 1 \leq p < \frac{2a}{\sigma^2}, \quad \sup_{t \geq 0} \mathbb{E}\left(\frac{1}{X_t^p}\right) < +\infty,$$

and for $1 \leq p < \alpha$, $\exists C_p > 0$ such that $\forall t_1, t_2 \in [0, 1]$ and $\forall 0 \leq i \leq n-1$

$$\begin{aligned} (ii) \quad & \mathbb{E}\left(\sup_{t_1 \leq u \leq t_2} X_u^p | \mathcal{F}_{t_1}\right) \leq C_p(1 + X_{t_1}^p), \\ (iii) \quad & \mathbb{E}_i\left(\sup_{s \in [i\Delta_n, (i+1)\Delta_n]} \left| \int_{i\Delta_n}^s X_{t-}^{1/\alpha} dL_t^\alpha \right|^p\right) \leq C_p \Delta_n^{p/\alpha} \left(1 + X_{i\Delta_n}^{p/\alpha}\right), \\ (iv) \quad & \mathbb{E}_i\left(\sup_{s \in [i\Delta_n, (i+1)\Delta_n]} |X_s - X_{i\Delta_n}|^p\right) \leq C_p \Delta_n^{p/2} (1 + X_{i\Delta_n}^p), \\ (v) \quad & \mathbb{E}_i\left(\left| \int_{i\Delta_n}^{(i+1)\Delta_n} (X_{s-}^{1/\alpha} - X_{i\Delta_n}^{1/\alpha}) dL_s^\alpha \right|^p\right) \leq C_p \Delta_n^{p/\alpha + p/2} \left(1 + \frac{1}{X_{i\Delta_n}^{p(1-1/\alpha)}} + X_{i\Delta_n}^{p/\alpha}\right). \end{aligned}$$

We next give a sharp bound for the Euler approximation (without drift) of the normalized increments of X . This result can be seen as an extension of Lemma 11 in Jacod and Todorov [19]. We obtain here a better rate of weak approximation and this result is a key step in the proof of Proposition 4.2. To shorten the notation, we set

$$\rho_i^n = \frac{u_n}{\sqrt{\Delta_n}} \frac{\Delta_i^n X}{\sqrt{X_{i\Delta_n}}} \quad \text{and} \quad \bar{\rho}_i^n = \frac{u_n}{\sqrt{\Delta_n}} \frac{\Delta_i^n \bar{X}}{\sqrt{X_{i\Delta_n}}}, \quad (4.1)$$

where $\Delta_i^n \bar{X} = \sigma_0 X_{i\Delta_n}^{1/2} \Delta_i^n B + \delta_0^{1/\alpha_0} X_{i\Delta_n}^{1/\alpha_0} \Delta_i^n L^{\alpha_0}$ and $(X_t)_{t \geq 0}$ solves (2.4) with $a_0 > 0$, $b_0 \in \mathbb{R}$, $\delta_0 > 0$, $\sigma_0 > 0$, $x_0 > 0$ and $2a_0 \geq \sigma_0^2$.

Lemma 4.1. *Let f be an even bounded \mathcal{C}^2 function with bounded derivatives. Then we have (with ρ_i^n and $\bar{\rho}_i^n$ defined by (4.1)) for $\epsilon > 0$, $\exists C_\epsilon$ such that $\forall 0 \leq i \leq n-1$*

$$|\mathbb{E}_i(f(\rho_i^n) - f(\bar{\rho}_i^n))| \leq C_\epsilon u_n \Delta_n^{1/\alpha_0 - \epsilon} \left(1 + X_{i\Delta_n} + \frac{1}{X_{i\Delta_n}} \left[1 + \frac{1}{\Delta_n} \mathbb{E}_i \int_{i\Delta_n}^{(i+1)\Delta_n} \frac{dv}{X_v}\right]\right), \quad (4.2)$$

and for $1 \leq p < \alpha_0$, $\exists C_p$ such that $\forall 0 \leq i \leq n-1$

$$\mathbb{E}_i(|f(\rho_i^n) - f(\bar{\rho}_i^n)|^p) \leq C_p u_n^p \Delta_n^{p/2} \left(1 + \frac{1}{X_{i\Delta_n}^p} + X_{i\Delta_n}^{p/2}\right). \quad (4.3)$$

Our estimating equations are based on the computation of the expectation $P_n f(x, \theta)$ given by (3.2). For $f(x) = \cos(x)$ this expectation has an explicit expression but not for $f(x) = 1 - K(x)$. The next lemma gives an asymptotic expansion of $P_n f(x, \theta)$ for functions vanishing in a neighborhood of zero. It is similar to Lemma 6.1 (i) in [30] derived from Itô's Formula. Here we propose a different proof, which easily allows higher order expansion, by using the explicit form of the conditional distribution of the Euler approximation $\Delta_i^n \bar{X}$ as a convolution between a Gaussian distribution and a Stable distribution.

Lemma 4.2. *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a bounded even function such that $f = 0$ on $[-\eta, +\eta]$ for some $\eta > 0$. We assume that $u_n [\ln(1/\Delta_n)]^{1/2} \rightarrow 0$. Then we have $\forall x > 0$*

$$P_n f(x, \theta) = \frac{1}{2} u_n^\alpha \Delta_n^{1-\alpha/2} c_\alpha \delta x^{1-\alpha/2} \int \frac{f(v)}{|v|^{\alpha+1}} dv + u_n^\alpha \Delta_n^{1-\alpha/2} \varepsilon_n(\theta, x), \quad \text{with } c_\alpha \text{ given by (2.3),}$$

where for any compact subset A of Θ , $\sup_{\theta \in A} |\varepsilon_n(\theta, x)| \leq (1+x)\varepsilon_n$, with $\varepsilon_n \rightarrow 0$.

4.2 Convergences

The asymptotic behavior of the estimator $\hat{\theta}_n$ is derived using the theory of estimating equations (see [18]) and is based on the convergence of $F_n(\theta_0)$ defined by (3.1) and its gradient $\nabla_{\theta}F_n(\theta)$. We set

$$A_n(\theta) = \sqrt{n} \operatorname{diag} \left(\frac{1}{u_n^2}, \frac{1}{u_n^{\alpha/2} \Delta_n^{1/2-\alpha/4}}, \frac{1}{u_n^{\alpha/2} \Delta_n^{1/2-\alpha/4}} \right). \quad (4.4)$$

We first state a Central Limit Theorem for $F_n(\theta_0)$ with rate $A_n(\theta_0)$.

Proposition 4.2. *We assume \mathbf{H} and $u_n = 1/[\ln(1/\Delta_n)]^p$ with $p > 1/2$.*

1. *If $n\Delta_n = 1$, then we have the stable convergence in law*

$$A_n(\theta_0)F_n(\theta_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}\text{-}s} \Sigma(\theta_0)^{1/2}\mathcal{N},$$

where \mathcal{N} is a standard Gaussian variable independent of \mathcal{F}_1 and $\Sigma(\theta_0)$ is the symmetric positive definite matrix defined by (3.5).

2. *If $n\Delta_n \rightarrow +\infty$, $b_0 > 0$ and $\sqrt{n}\Delta_n^{1/\alpha_0 + \alpha_0/4 - 1/2 - \epsilon_0} \rightarrow 0$ for some $\epsilon_0 > 0$, then we have the convergence in distribution*

$$A_n(\theta_0)F_n(\theta_0) \xrightarrow[n \rightarrow \infty]{} \bar{\Sigma}(\theta_0)^{1/2}\mathcal{N},$$

where \mathcal{N} is a standard Gaussian variable and $\bar{\Sigma}(\theta_0)$ is the symmetric positive definite matrix defined by (3.5) replacing $I(\alpha, X)$ by $\bar{I}(\alpha) = \int_0^{+\infty} x^{1-\alpha/2} \pi_0(dx)$.

Turning to the convergence of $\nabla_{\theta}F_n$ and recalling that $\Lambda_n(\theta)$ is defined by (3.4), we obtain the following convergence.

Proposition 4.3. *Let A be a compact subset of Θ . We assume \mathbf{H} and $u_n = 1/[\ln(1/\Delta_n)]^p$ with $p > 1/2$.*

1. *If $n\Delta_n = 1$, then we have the convergence in probability*

$$\sup_{\theta \in A} |A_n(\theta)\nabla_{\theta}F_n(\theta)\Lambda_n(\theta) - W(\theta)| \rightarrow 0,$$

where $W(\theta)$ is the non-singular matrix defined by (3.6).

2. *If $n\Delta_n \rightarrow +\infty$ and $b_0 > 0$, then we have the convergence in probability*

$$\sup_{\theta \in A} |A_n(\theta)\nabla_{\theta}F_n(\theta)\Lambda_n(\theta) - \bar{W}(\theta)| \rightarrow 0,$$

where $\bar{W}(\theta)$ is defined by (3.6) replacing $I(\alpha, X)$ by $\bar{I}(\alpha) = \int_0^{+\infty} x^{1-\alpha/2} \pi_0(dx)$.

5 Monte-Carlo experiments

In this section, we investigate the finite sample performances of our estimator with a Monte-Carlo study and compare with the estimators proposed respectively by Bull [10] for the Blumenthal-Gettoor index α and by Jacod and Todorov [19] for the volatility σ^2 . We assume that $(X_{i\Delta_n})_{0 \leq i \leq n}$ solves (2.4) with parameters (satisfying assumption \mathbf{H}) $x_0 = 1, a_0 = 10, b_0 = 3, \sigma_0 = 1.2, \delta_0 = 1.5$ and for $\alpha_0 \in \{1.2, 1.4, 1.7\}$. The observations are simulated with the implicit Euler scheme (with step $1/100n$) proposed by Li and Taguchi [24] to preserve the positivity of the simulated process.

Instead of solving $F_n(\hat{\theta}_n) = 0$, we minimize the weighted function given by

$$\left(\frac{\sum_{i=0}^{n-1} \left[f_j \left(\frac{u_n}{\sqrt{\Delta_n}} \frac{\Delta_i^n X}{\sqrt{X_{i\Delta_n}}} \right) - P_n f_j(X_{i\Delta_n}, \theta) \right]}{\sum_{i=0}^{n-1} f_j \left(\frac{u_n}{\sqrt{\Delta_n}} \frac{\Delta_i^n X}{\sqrt{X_{i\Delta_n}}} \right)} \right)^2, \quad j = 1, 2, 3 \quad (5.1)$$

as this equivalent problem leads to better numerical results. The minimization is done in Python using the `scipy.optimize.newton` method from the Scipy package, which uses the secant method. We ensure that the initial value is intentionally distinct from the unknown true parameter value.

The computation of $F_n(\theta)$ requires the computation of the expectations $(P_n f(X_{i\Delta_n}, \theta))_{0 \leq i \leq n-1}$. $P_n f_1$ is explicitly given in equation (3.3), but $P_n f_2$ and $P_n f_3$ require a Monte-Carlo method. We calibrate $P_n f_2$ and $P_n f_3$ with $n_{exp} = 100$ replications. Even using such a small number of replications and after parallelization of the computation of these expectations, the implementation of the estimation method still remains computationally expensive. This is the main limitation of this method for non-constant coefficients, and explains why we do less simulations than in Mies [30].

We conduct our simulation study with the choice of the threshold parameter $u_n = 1/\log(n)$. Our asymptotic results hold for $u_n = 1/(\log(n))^p$ with $p > 1/2$ and we already discussed that the asymptotic properties of our estimators are improved by decreasing the parameter p . However when experimenting smaller values of p , the failure rate in our minimization problem increases (this is also observed in [30] Table 1). This poses a larger problem in our setting, as our estimator is already very costly to compute due to the non-constant coefficients. The choice $p = 1$ leads to a reasonable failure rate when minimizing $F_n(\theta)$ (see Table 1 below) and shows good finite sample properties of our estimation method.

According to Theorem 3.1, we have the convergences in law :

$$\sqrt{n}(\hat{\sigma}_n^2 - \sigma_0^2) \xrightarrow[n \rightarrow \infty]{\mathcal{L}-s} \sqrt{2}\sigma_0^2 \mathcal{N}, \quad (5.2)$$

$$\frac{u_n^{\alpha_0/2} n^{\alpha_0/4}}{\log(u_n \sqrt{n})} (\hat{\delta}_n - \delta_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}-s} \delta_0 \sqrt{V_{33}} \mathcal{N}, \quad (5.3)$$

$$u_n^{\alpha_0/2} n^{\alpha_0/4} (\hat{\alpha}_n - \alpha) \xrightarrow[n \rightarrow \infty]{\mathcal{L}-s} \sqrt{V_{33}} \mathcal{N}, \quad (5.4)$$

where \mathcal{N} is a standard Gaussian variable independent of \mathcal{F}_1 and $V(\theta_0) = W(\theta_0)^{-1} \Sigma(\theta_0) (W(\theta_0)^T)^{-1}$, with $W(\theta_0)$ and $\Sigma(\theta_0)$ defined in equations (3.6) and (3.5). We calibrate the (theoretical) asymptotic distribution of the estimators from the above convergences replacing, in the expression of $V(\theta_0)$, $I(\alpha_0, X)$ by the Riemann sum $\frac{1}{n} \sum_i X_{i/n}^{1-\alpha_0/2}$.

The method used in Bull [10] has a tuning parameter $m \in \mathbb{N}$, where larger values of m increase the rate of convergence, although they do not perform as well in practice. We set the tuning parameters to $C = 0.05$, $\rho = 2$, $m = 3$ as suggested in Bull [10]. The estimator of Jacod and Todorov [19] is implemented as in equation (5.3) therein. To enhance the stability of the estimator, we split the estimated bias term into two components as recommended. We use the values $\zeta = 1.5$ and $u_n^{JT} = \log(n)^{-1/30}$ as suggested by the authors. The values for all of these tuning parameters are also the ones used in Mies [30]. For the last tuning parameter in Jacod and Todorov [19], which is necessary for a stochastic equation, we set the number of observations in each block to $k_n = 300$. In practice, all these estimation methods are very sensitive to the tuning parameters. For our estimation method, as previously mentioned, the choice of u_n can affect the performance of the estimators, even for the estimation of the volatility parameter whose rate of convergence does not depend on u_n . Similarly, the estimation method is sensitive to the choice of the truncation functions f_2 and f_3 . We also note that although the theoretical results are valid for any value of σ_0^2 (providing assumption **H**), some values perform

better numerically. The sensitivity to the true value of the volatility is also observed in Boniece et al. [9] for the debiased truncated realized variation method.

In Table 1, we compare the finite sample performance of our moment estimator for the estimation of the volatility σ^2 (comparison with the estimator of Jacod and Todorov [19]) and for the estimation of the jump activity α (comparison with the estimator of Bull [10]). We use the median absolute deviation to compare estimators, following what was done in Mies [30] and Jacod and Todorov [19]. We choose $n = 5000$ (respectively $n = 25000$) to be compared to $h = 1/n = 5/23400$ (respectively $h = 1/23400$) in Mies [30]. Table 1 also presents the percentage of simulations for which the minimisation did not converge. For the volatility σ^2 , our estimator performs better (for $n = 25000$) if $\alpha_0 \in \{1.2; 1.4\}$ and is comparable to the one of Jacod and Todorov in the case $\alpha_0 = 1.7$. While the asymptotic theoretical variance of $\hat{\sigma}_n^2$ does not depend on α_0 , we observe that the finite sample performance of our estimator decreases when α_0 is close to 2. This is also the case in Mies [30] (Table 1), and in Boniece et al. [9] (Figure 1 and Table 12), reflecting the difficulty to separate the contribution of the Brownian Motion from the contribution of the small jumps, when α_0 is close to 2. To complete the estimation results for the volatility parameter, we also presents in Figure 1 the distribution of $\hat{\sigma}_n^2$ for $\alpha_0 \in \{1.2, 1.4\}$, and the distribution of Jacod and Todorov estimator. This confirms that our estimator performs a bit better, but in both cases the finite sample distributions are far from the theoretical asymptotic distribution (not represented here).

For the estimation of the jump activity α , the results of Table 1 show that our method performs better than the one of Bull, in all cases, even for small sample size $n = 5000$. The good finite sample performance of our estimator is confirmed in Figure 2 representing the distribution of $\hat{\alpha}_n$ compared with its asymptotic theoretical distribution. There is nevertheless a slight bias when α_0 is large ($\alpha_0 = 1.7$). These results are comparable to the one of Mies [30] (Figures 1 and 2) despite our simulations are more expensive since we estimate a stochastic equation. Figure 3 shows the distribution of $\hat{\delta}_n$ (not presented in [30]). As for the estimation of α , we see that the estimation method works well and that the finite sample distribution of the estimator is more concentrated than the asymptotic theoretical distribution, especially for large values of α_0 .

To summarize the results obtained from these Monte-Carlo experiments, we conclude that our estimation method leads to mixed results for the estimation of σ^2 as it is computationally expensive and does not really improve the finite sample properties of the estimator proposed by Jacod and Todorov. In contrast, this method shows good finite sample performance for the estimation of the jump parameters, where the comparison can only be made with the estimator of α proposed by Bull as the estimation of the scaling parameter has not be studied before for a stochastic equation. We also emphasize that our method is the first near rate efficient method for the joint estimation of (δ, α) for a stochastic equation in presence of a Brownian component.

Table 1: Median absolute errors for the estimation of σ^2 and α for different estimators. All values are based on 500 Monte Carlo replications.

α	n	$\hat{\sigma}_n^2$	$\hat{\alpha}_n$	Infeasible	JT σ^2	Bull α
1.2	5 000	0.10	0.12	0.46	0.11	0.41
1.2	25 000	0.030	0.089	0.42	0.054	0.37
1.4	5 000	0.12	0.10	0.38	0.14	0.46
1.4	25 000	0.044	0.086	0.35	0.11	0.37
1.7	5 000	0.42	0.19	0.49	0.22	0.58
1.7	25 000	0.23	0.098	0.48	0.22	0.43

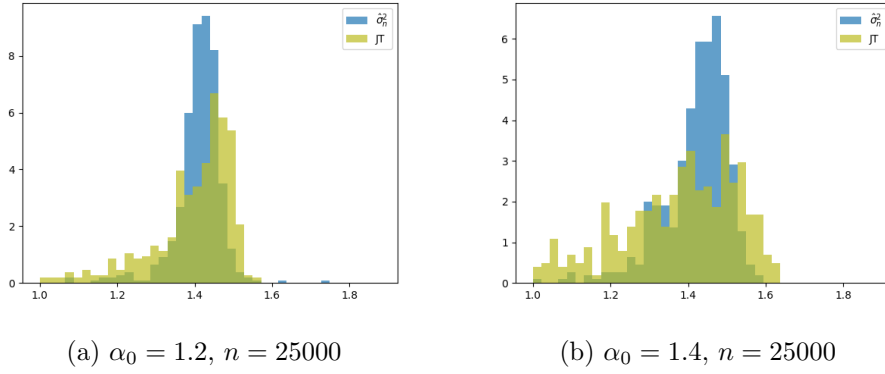


Figure 1: Simulated distribution of $\hat{\sigma}_n^2$ ($\sigma_0^2 = 1.44$) and comparison with Jacod and Todorov (based on 500 Monte Carlo replications)

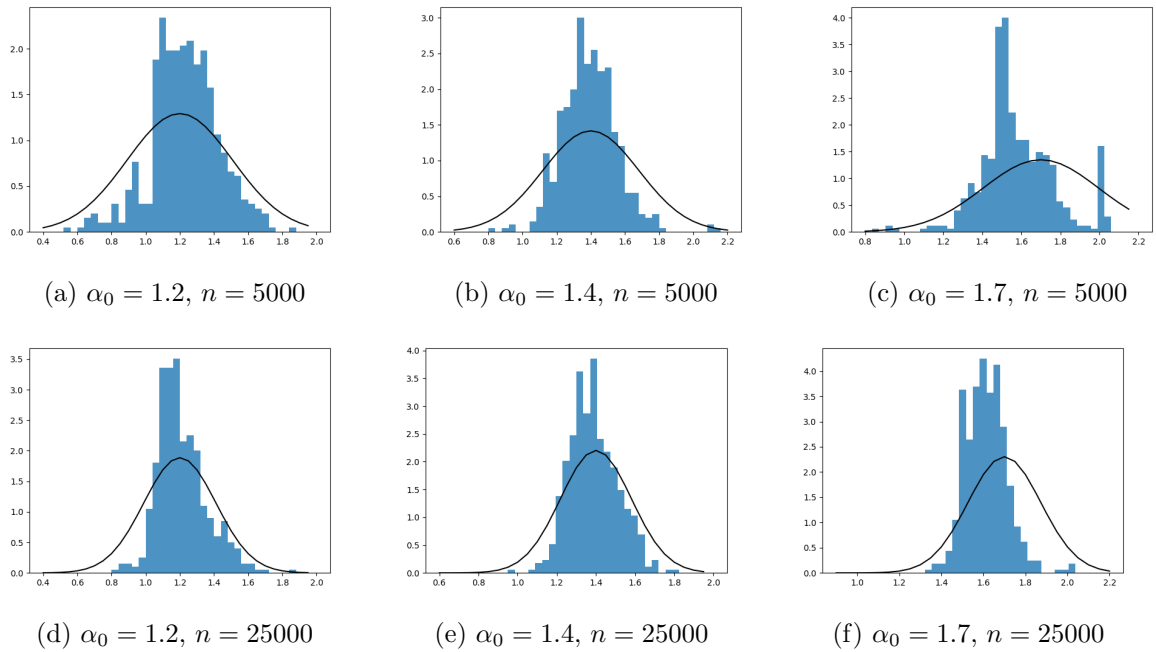


Figure 2: Simulated distribution of $\hat{\alpha}_n$ and comparison with its asymptotic distribution (based on 500 Monte Carlo replications)

6 Proofs

6.1 Auxiliary results

6.1.1 Proof of Proposition 4.1

We only prove (i) since the other items can be deduced with similar arguments as the ones given for the proof of Proposition 2.2 in [8]. The Laplace transform of X_t has the explicit expression (we refer to Li [25], see also Section 3.1 in [21] and Proposition 2.1. in [5])

$$\mathbb{E}(e^{-uX_t}) = \exp\left(-x_0v_t(u) - \int_{v_t(u)}^u \frac{F(z)}{R(z)} dz\right), \quad (6.1)$$

where $t \rightarrow v_t(u)$ is the unique locally bounded solution of

$$\frac{\partial}{\partial t}v_t(u) = -R(v_t(u)), \quad v_0(u) = u. \quad (6.2)$$

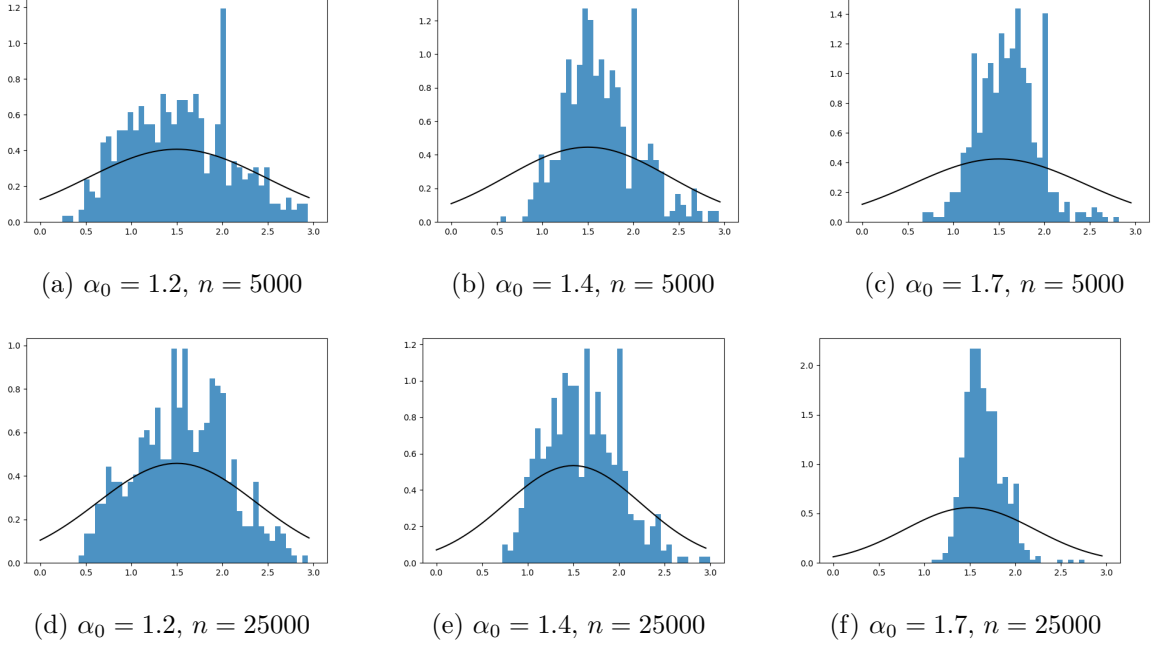


Figure 3: Simulated distribution of $\hat{\delta}_n$ ($\delta_0 = 1.5$) and comparison with its asymptotic distribution (based on 500 Monte Carlo replications)

For the α -CIR model, the branching mechanism and the immigration rate are given by

$$R(z) = \frac{\sigma^2}{2}z^2 + \frac{\bar{\delta}^\alpha}{\alpha}z^\alpha + bz, \quad F(z) = az, \quad z \in [0, +\infty),$$

with $\bar{\delta} = \delta^{1/\alpha}(\alpha/|\cos(\frac{\pi\alpha}{2})|)^{1/\alpha}$. Setting $z_0 = \inf\{z > 0, R(z) \geq 0\}$, $z_0 = 0$ when $b \geq 0$ and $z_0 > 0$ when $b < 0$ with $z_0 < +\infty$ as $\lim_{z \rightarrow +\infty} R(z) = +\infty$. From Corollary 3.2 of Li [25], $\lim_{t \rightarrow \infty} v_t(u) = z_0$ and $\forall u > z_0, t \rightarrow v_t(u)$ is decreasing. Now we have $\forall t \geq 0$

$$\mathbb{E}\left(\frac{1}{X_t^p}\right) = C_p \int_{\mathbb{R}_+} u^{p-1} \mathbb{E}(e^{-uX_t}) du \leq C_p(z_0) + C_p \int_{z_0+1}^{+\infty} u^{p-1} \mathbb{E}(e^{-uX_t}) du.$$

We fix $\epsilon \in (0, 1)$ and for $u \geq z_0 + 1$ we set $t_u^\epsilon = \inf\{t \geq 0, v_t(u) \leq u^\epsilon\}$. With $u_{0,\epsilon} = \max(z_0 + 1, z_0^{1/\epsilon})$

$$\sup_{t \geq 0} \int_{u_{0,\epsilon}}^{+\infty} u^{p-1} \mathbb{E}(e^{-uX_t}) du \leq \int_{u_{0,\epsilon}}^{+\infty} u^{p-1} \left[\sup_{t \in [0, t_u^\epsilon]} \mathbb{E}(e^{-uX_t}) + \sup_{t \geq t_u^\epsilon} \mathbb{E}(e^{-uX_t}) \right] du. \quad (6.3)$$

Using that $t \rightarrow v_t(u)$ is non-increasing for $u > z_0$, we obtain from (6.1) and observing that $v_{t_u^\epsilon}(u) = u^\epsilon$

$$\begin{aligned} \sup_{t \in [0, t_u^\epsilon]} \mathbb{E}(e^{-uX_t}) &\leq \sup_{t \in [0, t_u^\epsilon]} \exp(-x_0 v_t(u)) \leq \exp(-x_0 u^\epsilon), \\ \sup_{t \geq t_u^\epsilon} \mathbb{E}(e^{-uX_t}) &\leq \sup_{t \geq t_u^\epsilon} \exp\left(-\int_{v_t(u)}^u \frac{F(z)}{R(z)} dz\right) \leq \exp\left(-\int_{u^\epsilon}^u \frac{F(z)}{R(z)} dz\right). \end{aligned}$$

For $z \in [u^\epsilon, u]$

$$\frac{F(z)}{R(z)} \geq \frac{2a}{\sigma^2} \frac{1}{z(1 + \frac{\bar{\delta}^\alpha 2}{\alpha \sigma^2 z^{2-\alpha}}) + \frac{2b}{\sigma^2}} \geq \frac{2a}{\sigma^2} \frac{1}{zg(u) + \frac{2b}{\sigma^2}}, \quad \text{with } g(u) = 1 + \frac{\bar{\delta}^\alpha 2}{\alpha \sigma^2 u^{\epsilon(2-\alpha)}}.$$

Hence integrating the previous inequality

$$\int_{u^\epsilon}^u \frac{F(z)}{R(z)} dz \geq \frac{2a}{\sigma^2 g(u)} \ln \left(\frac{ug(u) + \frac{2b}{\sigma^2}}{u^\epsilon g(u) + \frac{2b}{\sigma^2}} \right) = \frac{2a}{\sigma^2 g(u)} [\ln(u^{1-\epsilon}) + h(u)],$$

with

$$h(u) = \ln \left(\frac{g(u) + \frac{2b}{u\sigma^2}}{g(u) + \frac{2b}{u^\epsilon \sigma^2}} \right).$$

We have $g(u) \geq 1$, $\lim_{u \rightarrow +\infty} g(u) = 1$ and $\lim_{u \rightarrow +\infty} h(u) = 0$, then

$$\exp \left(- \int_{u^\epsilon}^u \frac{F(z)}{R(z)} dz \right) \leq C_\epsilon / u^{(1-\epsilon) \frac{2a}{\sigma^2 g(u)}}.$$

If $1 \leq p < \frac{2a}{\sigma^2}$, we choose $\epsilon > 0$ such that $p - \frac{2a}{\sigma^2} + \epsilon \frac{2a}{\sigma^2} < 0$, then we deduce

$$\int_{u_{0,\epsilon}}^{+\infty} u^{p-1} \exp \left(- \int_{u^\epsilon}^u \frac{F(z)}{R(z)} dz \right) du < +\infty,$$

and this allows us to conclude.

6.1.2 Proof of Lemma 4.1

From the expressions of $\Delta_i^n X$ and $\Delta_i^n \bar{X}$, we deduce the decomposition $\rho_i^n = \bar{\rho}_i^n + \theta(1)_i^n + \theta(2)_i^n + \theta(3)_i^n$ with

$$\begin{aligned} \theta(1)_i^n &= \sigma_0 \frac{u_n}{\sqrt{\Delta_n} \sqrt{X_{i\Delta_n}}} \int_{i\Delta_n}^{(i+1)\Delta_n} (X_t^{1/2} - X_{i\Delta_n}^{1/2}) dB_t + u_n \sqrt{\Delta_n} (a_0 X_{i\Delta_n}^{-1/2} - b_0 X_{i\Delta_n}^{1/2}), \\ \theta(2)_i^n &= \delta_0^{1/\alpha_0} \frac{u_n}{\sqrt{\Delta_n} \sqrt{X_{i\Delta_n}}} \int_{i\Delta_n}^{(i+1)\Delta_n} (X_t^{1/\alpha_0} - X_{i\Delta_n}^{1/\alpha_0}) dL_t^{\alpha_0}, \\ \theta(3)_i^n &= -b_0 \frac{u_n}{\sqrt{\Delta_n} \sqrt{X_{i\Delta_n}}} \int_{i\Delta_n}^{(i+1)\Delta_n} (X_t - X_{i\Delta_n}) dt. \end{aligned}$$

We have the following estimates

$$\forall p \in [1, \alpha_0), \mathbb{E}_i(|\theta(1)_i^n|^p) \leq C_p u_n^p \Delta_n^{p/2} \left(1 + \frac{1}{X_{i\Delta_n}^p} + X_{i\Delta_n}^{p/2} \right), \quad (6.4)$$

$$\forall p \in [1, \alpha_0), \mathbb{E}_i(|\theta(2)_i^n|^p) \leq C_p u_n^p \Delta_n^{p/\alpha_0} \left(1 + \frac{1}{X_{i\Delta_n}^{p/2 + p(1-1/\alpha_0)}} + X_{i\Delta_n}^{p/\alpha_0 - p/2} \right), \quad (6.5)$$

$$\forall p \in [1, \alpha_0), \mathbb{E}_i(|\theta(3)_i^n|^p) \leq C_p u_n^p \Delta_n^p \left(1 + \frac{1}{X_{i\Delta_n}^{p/2}} + X_{i\Delta_n}^{p/2} \right). \quad (6.6)$$

We deduce easily (6.5) and (6.6) from Proposition 4.1, (6.4) follows from Burkholder's inequality, Proposition 4.1 and the bound

$$\forall x, y \in (0, +\infty) |y^{1/2} - x^{1/2}| \leq C|y - x|/x^{1/2}.$$

Combining the estimates (6.4), (6.5), (6.6) and $|f(x+y) - f(x)| \leq C|y|$ as f' is bounded, we deduce immediately (4.3).

We now prove (4.2). With the previous notation, we have the decomposition

$$\begin{aligned} f(\rho_i^n) - f(\bar{\rho}_i^n) &= f(\rho_i^n) - f(\bar{\rho}_i^n + \theta(1)_i^n + \theta(2)_i^n) + f(\bar{\rho}_i^n + \theta(1)_i^n + \theta(2)_i^n) - f(\bar{\rho}_i^n + \theta(1)_i^n) \\ &\quad + f(\bar{\rho}_i^n + \theta(1)_i^n) - f(\bar{\rho}_i^n), \end{aligned}$$

and since f' is bounded it yields

$$|\mathbb{E}_i(f(\rho_i^n) - f(\bar{\rho}_i^n))| \leq C(\mathbb{E}_i|\theta(3)_i^n| + \mathbb{E}_i|\theta(2)_i^n| + |\mathbb{E}_i(f(\bar{\rho}_i^n + \theta(1)_i^n) - f(\bar{\rho}_i^n))|).$$

Using (6.5) and (6.6) with $p = 1$

$$\mathbb{E}_i|\theta(3)_i^n| + \mathbb{E}_i|\theta(2)_i^n| \leq C u_n \Delta_n^{1/\alpha_0} \left(1 + \frac{1}{X_{i\Delta_n}^{3/2-1/\alpha_0}} + X_{i\Delta_n}^{1/2} \right). \quad (6.7)$$

It remains to bound $|\mathbb{E}_i(f(\bar{\rho}_i^n + \theta(1)_i^n) - f(\bar{\rho}_i^n))|$. Since f'' is bounded, we have

$$|\mathbb{E}_i(f(\bar{\rho}_i^n + \theta(1)_i^n) - f(\bar{\rho}_i^n))| \leq |\mathbb{E}_i(\theta(1)_i^n f'(\bar{\rho}_i^n))| + C \mathbb{E}_i|\theta(1)_i^n|^2.$$

To finish the proof, we show the following estimates

$$\mathbb{E}_i|\theta(1)_i^n|^2 \leq C u_n^2 \Delta_n \left(1 + X_{i\Delta_n} + \frac{1}{X_{i\Delta_n}} \left[1 + \frac{1}{\Delta_n} \mathbb{E}_i \int_{i\Delta_n}^{(i+1)\Delta_n} \frac{dv}{X_v} \right] \right), \quad (6.8)$$

and for $\epsilon > 0$

$$|\mathbb{E}_i(\theta(1)_i^n f'(\bar{\rho}_i^n))| \leq C_\epsilon u_n \Delta_n^{1/\alpha_0 - \epsilon} \left(1 + X_{i\Delta_n} + \frac{1}{X_{i\Delta_n}^{1/2}} \left[1 + \frac{1}{\Delta_n} \mathbb{E}_i \int_{i\Delta_n}^{(i+1)\Delta_n} \frac{dv}{X_v^{1/2}} \right] \right). \quad (6.9)$$

Collecting (6.7), (6.8) and (6.9), we deduce (4.2).

Proof of (6.8) We write Itô's formula for the function $x \rightarrow \sqrt{x}$ which is \mathcal{C}^2 on $(0, +\infty)$

$$\sqrt{X_t} - \sqrt{X_s} = \int_s^t \bar{b}_v dv + \int_s^t \frac{\sigma_0}{2} dB_v + \int_s^t \int \bar{c}_{v-}(z) \tilde{N}(dv, dz),$$

where

$$\bar{b}_v = \frac{(a_0 - b_0 X_v)}{2X_v^{1/2}} - \frac{\sigma_0^2}{8X_v^{1/2}} + \int (\sqrt{X_{v-} + \delta_0^{1/\alpha_0} X_{v-}^{1/\alpha_0} z} - \sqrt{X_{v-}} - \frac{1}{2X_{v-}^{1/2}} \delta_0^{1/\alpha_0} X_{v-}^{1/\alpha_0} z) F^{\alpha_0}(z) dz,$$

$$\bar{c}_{v-}(z) = (X_{v-} + \delta_0^{1/\alpha_0} X_{v-}^{1/\alpha_0} z)^{1/2} - (X_{v-})^{1/2}.$$

Moreover, we check that $|\bar{b}_v| \leq C(1 + \frac{1}{\sqrt{X_v}} + \sqrt{X_v})$ and that $|\bar{c}_v(z)| \leq C(1 + \sqrt{X_v})(\sqrt{z} \mathbf{1}_{\{z>1\}} + z \mathbf{1}_{\{0<z\leq 1\}})$. We obtain the decomposition

$$\theta(1)_i^n = \mu(1)_i^n + \mu(2)_i^n + \mu(3)_i^n, \quad (6.10)$$

with

$$\begin{aligned} \mu(1)_i^n &= \frac{u_n \sigma_0}{\sqrt{\Delta_n X_{i\Delta_n}}} \int_{i\Delta_n}^{(i+1)\Delta_n} \left(\int_{i\Delta_n}^t \bar{b}_v dv \right) dB_t, \\ \mu(2)_i^n &= \frac{u_n \sigma_0}{\sqrt{\Delta_n X_{i\Delta_n}}} \frac{\sigma_0}{2} \int_{i\Delta_n}^{(i+1)\Delta_n} (B_t - B_{i\Delta_n}) dB_t + u_n \sqrt{\Delta_n} (a_0 X_{i\Delta_n}^{-1/2} - b_0 X_{i\Delta_n}^{1/2}), \\ \mu(3)_i^n &= \frac{u_n \sigma_0}{\sqrt{\Delta_n X_{i\Delta_n}}} \int_{i\Delta_n}^{(i+1)\Delta_n} \left(\int_{i\Delta_n}^t \int \bar{c}_{v-}(z) \tilde{N}(dv, dz) \right) dB_t. \end{aligned}$$

From Burkholder's inequality and Proposition 4.1 (ii), we have

$$\forall p \in [1, 2], \mathbb{E}_i |\mu(1)_i^n|^p \leq C_p u_n^p \Delta_n^p \left(1 + \frac{1}{X_{i\Delta_n}^{p/2}} \left[1 + \frac{1}{\Delta_n} \mathbb{E}_i \int_{i\Delta_n}^{(i+1)\Delta_n} \frac{dv}{X_v^{p/2}} \right] \right), \quad (6.11)$$

$$\forall p \geq 1, \mathbb{E}_i |\mu(2)_i^n|^p \leq C_p u_n^p \Delta_n^{p/2} \left(\frac{1}{X_{i\Delta_n}^{p/2}} + X_{i\Delta_n}^{p/2} \right), \quad (6.12)$$

$$\forall p \in (\alpha_0, 2], \mathbb{E}_i |\mu(3)_i^n|^p \leq C_p u_n^p \Delta_n \left(1 + \frac{1}{X_{i\Delta_n}^{p/2}} \right). \quad (6.13)$$

Let us give some details for (6.11). We have

$$\begin{aligned}
\mathbb{E}_i |\mu(1)_i^n|^p &\leq C_p \frac{u_n^p}{\Delta_n^{p/2} X_{i\Delta_n}^{p/2}} \mathbb{E}_i \left(\int_{i\Delta_n}^{(i+1)\Delta_n} \left(\int_{i\Delta_n}^t |\bar{b}_v| dv \right)^2 dt \right)^{p/2} \\
&\leq C_p \frac{u_n^p}{X_{i\Delta_n}^{p/2}} \mathbb{E}_i \left(\int_{i\Delta_n}^{(i+1)\Delta_n} |\bar{b}_v| dv \right)^p \leq C_p \frac{u_n^p \Delta_n^p}{X_{i\Delta_n}^{p/2}} \frac{1}{\Delta_n} \int_{i\Delta_n}^{(i+1)\Delta_n} |\bar{b}_v|^p dv \\
&\leq C_p \frac{u_n^p \Delta_n^p}{X_{i\Delta_n}^{p/2}} \frac{1}{\Delta_n} \int_{i\Delta_n}^{(i+1)\Delta_n} \left(1 + \frac{1}{X_v^{p/2}} + X_v^{p/2} \right) dv,
\end{aligned}$$

and we conclude with $\mathbb{E}_i \sup_{i\Delta_n \leq v \leq (i+1)\Delta_n} X_v^{p/2} \leq C(1 + X_{i\Delta_n}^{p/2})$ (Proposition 4.1 (ii)). The bound (6.12) is immediate and to obtain (6.13), we use for $p \in [1, 2]$ (see Lemma 2.1.5 in [15])

$$\mathbb{E}_i \sup_{i\Delta_n \leq t \leq (i+1)\Delta_n} \left| \int_{i\Delta_n}^t \int \bar{c}_{v-}(z) \tilde{N}(dv, dz) \right|^p \leq C_p \mathbb{E}_i \int_{i\Delta_n}^{(i+1)\Delta_n} \int |\bar{c}_{v-}(z)|^p \bar{N}(dv, dz),$$

combined with $|\bar{c}_v(z)| \leq C(1 + \sqrt{X_v})(\sqrt{z} \mathbf{1}_{\{z>1\}} + z \mathbf{1}_{\{0<z\leq 1\}})$. We then deduce (6.8) from the previous bounds with $p = 2$.

Proof of (6.9) Since f' is bounded, we have using (6.10) and Hölder's inequality with $\alpha_0 < p \leq 2$

$$|\mathbb{E}_i(\theta(1)_i^n f'(\bar{\rho}_i^n))| \leq C(\mathbb{E}_i |\mu(1)_i^n| + |\mathbb{E}_i(\mu(2)_i^n f'(\bar{\rho}_i^n))| + \mathbb{E}_i^{1/p} |\mu(3)_i^n|^p). \quad (6.14)$$

It remains to consider $\mathbb{E}_i(\mu(2)_i^n f'(\bar{\rho}_i^n))$. We split $\bar{\rho}_i^n$ in two parts (Brownian part and pure-jump part)

$$\bar{\rho}_i^n = \frac{\sigma_0 u_n}{\sqrt{\Delta_n}} \Delta_i^n B + \frac{\delta_0^{1/\alpha_0} X_{i\Delta_n}^{1/\alpha_0} u_n}{\sqrt{\Delta_n} \sqrt{X_{i\Delta_n}}} \Delta_i^n L^{\alpha_0} := \bar{\rho}_i^{1,n} + \bar{\rho}_i^{2,n},$$

and write

$$\mathbb{E}_i(\mu(2)_i^n f'(\bar{\rho}_i^n)) = \mathbb{E}_i(\mu(2)_i^n f'(\bar{\rho}_i^{1,n})) + \mathbb{E}_i(\mu(2)_i^n (f'(\bar{\rho}_i^n) - f'(\bar{\rho}_i^{1,n}))).$$

Since f' is odd, an explicit computation gives $\mathbb{E}_i(\mu(2)_i^n f'(\bar{\rho}_i^{1,n})) = 0$. For the second term, we use successively that f'' is bounded, Hölder's inequality with $1 < q < \alpha_0$ and the scaling property of L^{α_0} to obtain

$$\begin{aligned}
\mathbb{E}_i |\mu(2)_i^n (f'(\bar{\rho}_i^n) - f'(\bar{\rho}_i^{1,n}))| &\leq C \mathbb{E}_i |\mu(2)_i^n \bar{\rho}_i^{2,n}| \leq C_q \frac{u_n}{\sqrt{\Delta_n}} (1 + X_{i\Delta_n}^{1/2}) \mathbb{E}_i^{\frac{1}{q}} |\mu(2)_i^n|^q \mathbb{E}_i^{\frac{1}{q}} |\Delta_i^n L^{\alpha_0}|^q \\
&\leq C_q u_n^2 \left(1 + \frac{1}{X_{i\Delta_n}^{1/2}} + X_{i\Delta_n} \right) \Delta_n^{1/\alpha_0}. \quad (6.15)
\end{aligned}$$

We finally deduce (6.9) from (6.14) (using (6.11) with $p = 1$ and (6.13) with $p = \alpha_0 + \epsilon$) and (6.15).

6.1.3 Proof of Lemma 4.2

We assume that $\theta = (\sigma^2, \delta, \alpha) \in A$ a compact subset of Θ . We denote by g_α the density of L_1^α , by Φ the density of the standard Gaussian distribution and we set $a_n(x, \theta) = \delta^{1/\alpha} u_n (x \Delta_n)^{1/\alpha - 1/2}$. Then

$$P_n f(x, \theta) = \int_{\mathbb{R}} f(v) dv \int_{\mathbb{R}} g_\alpha \left(\frac{v - y}{a_n(x, \theta)} \right) \frac{1}{a_n(x, \theta)} \Phi \left(\frac{y}{u_n \sigma} \right) \frac{1}{u_n \sigma} dy.$$

Using successively a change of variables and Fubini's Theorem yields

$$P_n f(x, \theta) = \int_{\mathbb{R}} \Phi(z) dz \int_{\mathbb{R}} f(v) g_\alpha \left(\frac{v - u_n \sigma z}{a_n(x, \theta)} \right) \frac{1}{a_n(x, \theta)} dv.$$

Recalling that $f = 0$ on $[-\eta, \eta]$, we split the previous integral in three parts $P_n f(x, \theta) = I_n^1 + I_n^2 + I_n^3$ where

$$\begin{aligned} I_n^1 &= \int_{\{|z| > \frac{\eta}{2u_n\sigma}\}} \Phi(z) dz \int_{\mathbb{R}} f(v) g_\alpha \left(\frac{v - u_n\sigma z}{a_n(x, \theta)} \right) \frac{1}{a_n(x, \theta)} dv, \\ I_n^2 &= \int_{\{|z| \leq \frac{\eta}{2u_n\sigma}\}} \Phi(z) dz \int_{\{v > \eta\}} f(v) g_\alpha \left(\frac{v - u_n\sigma z}{a_n(x, \theta)} \right) \frac{1}{a_n(x, \theta)} dv, \\ I_n^3 &= \int_{\{|z| \leq \frac{\eta}{2u_n\sigma}\}} \Phi(z) dz \int_{\{v < -\eta\}} f(v) g_\alpha \left(\frac{v - u_n\sigma z}{a_n(x, \theta)} \right) \frac{1}{a_n(x, \theta)} dv. \end{aligned}$$

By assumption, $u_n^2 = \bar{u}_n / \ln(1/\Delta_n)$ with $\bar{u}_n \rightarrow 0$, so we deduce immediately, since f is bounded, that

$$|I_n^1| \leq C \exp\left(-\frac{\eta^2}{8u_n^2\sigma^2}\right) \leq C \Delta_n^{C'/\bar{u}_n}, \quad (6.16)$$

where C and C' do not depend on θ . Turning to I_n^2 , we observe that for $|z| \leq \frac{\eta}{2u_n\sigma}$ and $v > \eta$,

$$\frac{v - u_n\sigma z}{a_n(x, \theta)} \geq \frac{\eta}{2a_n(x, \theta)} > 0.$$

Using the asymptotic expansion of g_α (see Sato [32] equation 14.34 p88), we have

$$\forall y > 0 \quad |g_\alpha(y) - \frac{c_\alpha}{y^{\alpha+1}}| \leq \frac{C}{y^{2\alpha+1}}, \text{ with } c_\alpha \text{ given by (2.3),}$$

and we deduce

$$g_\alpha \left(\frac{v - u_n\sigma z}{a_n(x, \theta)} \right) = \frac{c_\alpha a_n(x, \theta)^{\alpha+1}}{v^{\alpha+1}} \frac{1}{(1 - \sigma u_n z/v)^{\alpha+1}} (1 + R_n^0(z, v)),$$

with

$$|R_n^0(z, v)| \leq C \frac{a_n(x, \theta)^\alpha}{v^\alpha} \frac{1}{(1 - \sigma u_n z/v)^\alpha} \leq C \frac{a_n(x, \theta)^\alpha}{v^\alpha},$$

where we used $|\sigma u_n z/v| \leq 1/2$ for the last inequality. Moreover, we have the first order expansion

$$\frac{1}{(1 - \sigma u_n z/v)^{\alpha+1}} = 1 + R_n^1(z, v), \text{ with } |R_n^1(z, v)| \leq C u_n z/v.$$

Reporting these results in I_n^2 , we obtain

$$\begin{aligned} I_n^2 &= \int_{\{|z| \leq \frac{\eta}{2u_n\sigma}\}} \Phi(z) dz \int_{\{v > \eta\}} f(v) \frac{c_\alpha a_n(x, \theta)^\alpha}{v^{\alpha+1}} (1 + R_n^1(z, v))(1 + R_n^0(z, v)) dv \\ &= c_\alpha a_n(x, \theta)^\alpha \left(\int_{\{v > \eta\}} \frac{f(v)}{v^{\alpha+1}} dv + \int_{\{|z| > \frac{\eta}{2u_n\sigma}\}} \Phi(z) dz \int_{\{v > \eta\}} \frac{f(v)}{v^{\alpha+1}} dv \right. \\ &\quad \left. + \int_{\{|z| \leq \frac{\eta}{2u_n\sigma}\}} \Phi(z) dz \int_{\{v > \eta\}} \frac{f(v)}{v^{\alpha+1}} [R_n^1(z, v)(1 + R_n^0(z, v)) + R_n^0(z, v)] dv \right). \end{aligned}$$

This finally gives

$$I_n^2 = c_\alpha a_n(x, \theta)^\alpha \left(\int_{\{v > \eta\}} \frac{f(v)}{v^{\alpha+1}} dv + R_n \right), \quad (6.17)$$

with

$$|R_n| \leq C(\Delta_n^{C'/\bar{u}_n} + u_n + u_n^\alpha (x\Delta_n)^{1-\alpha/2}) \leq C(1 + \sqrt{x})(\Delta_n^{C'/\bar{u}_n} + u_n).$$

It remains to bound I_n^3 . Using that f is bounded and a change of variables we have

$$I_n^3 \leq C \int_{\{|z| \leq \frac{\eta}{2u_n\sigma}\}} \Phi(z) dz \int_{\{y < \frac{-\eta - u_n\sigma z}{a_n(x, \theta)}\}} g_\alpha(y) dy \leq C \int_{\{y < -\frac{\eta}{2a_n(x, \theta)}\}} g_\alpha(y) dy.$$

From Sato [32] equation 14.35 p88, we have the bound

$$\forall y < 0 \quad 0 \leq g_\alpha(y) \leq C \xi^{\frac{2-\alpha}{2\alpha}} e^{-\xi} \left(1 + \frac{1}{\xi}\right), \text{ with } \xi = (\alpha - 1) \left| \frac{y}{\alpha} \right|^{\frac{\alpha}{\alpha-1}},$$

it yields changing y in $-y$ and next setting $\xi = (\alpha - 1) \left(\frac{y}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}$ and $\xi_{n,\theta,x} = (\alpha - 1) \left(\frac{\eta}{2\alpha a_n(x,\theta)}\right)^{\frac{\alpha}{\alpha-1}}$

$$\begin{aligned} I_n^3 &\leq C \int_{\{\xi > \xi_{n,\theta,x}\}} \xi^{\frac{2-\alpha}{2\alpha}} e^{-\xi} (1 + 1/\xi) \left(\frac{\xi}{\alpha - 1}\right)^{-1/\alpha} d\xi \\ &\leq C \int_{\{\xi > \xi_{n,\theta,x}\}} e^{-\xi} \xi^{-1/2} (1 + 1/\xi) d\xi \leq C e^{-\xi_{n,\theta,x}/2} (1 + 1/\xi_{n,\theta,x}) \int_{\mathbb{R}_+} e^{-\xi/2} \xi^{-1/2} d\xi \\ &\leq C e^{-\xi_{n,\theta,x}/2} (1 + 1/\xi_{n,\theta,x}). \end{aligned}$$

Since we have for $x > 0$, $e^{-1/2x}(1+x) \leq Cx^{2\alpha(\alpha-1)/\alpha}$, we finally obtain

$$I_n^3 \leq C a_n(x, \theta)^{2\alpha}. \quad (6.18)$$

Collecting (6.16), (6.17) and (6.18), we deduce the result of Lemma 4.2.

6.1.4 Proof of Proposition 4.2

The proof consists first to replace, in the expression of $F_n(\theta_0)$, ρ_i^n by $\bar{\rho}_i^n$ (defined by (4.1)) by using Lemma 4.1 and next to study the convergence in distribution of

$$A_n(\theta_0) \frac{1}{n} \sum_{i=0}^{n-1} [f(\bar{\rho}_i^n) - P_n f(X_{i\Delta_n}, \theta_0)].$$

1. Case $n\Delta_n = 1$. We first remark that for any $0 < p \leq 1$ we have the convergence in probability

$$\frac{1}{n} \sum_{i=0}^{n-1} X_{i\Delta_n}^p \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^1 X_t^p dt.$$

Moreover from Proposition 4.1 and recalling $a_0 > \sigma_0^2$, we have for $1 \leq p \leq 2$

$$\mathbb{E} \left(\frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{X_{i\Delta_n}^p} \right) \leq \sup_{t \geq 0} \mathbb{E} \left(\frac{1}{X_t^p} \right) < +\infty, \quad (6.19)$$

and using $1/(X_{i\Delta_n} X_v) \leq 1/(2X_{i\Delta_n}^2) + 1/(2X_v^2)$

$$\mathbb{E} \left(\frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{\Delta_n} \mathbb{E}_i \int_{i\Delta_n}^{(i+1)\Delta_n} \frac{dv}{X_{i\Delta_n} X_v} \right) \leq \sup_{t \geq 0} \mathbb{E} \left(\frac{1}{X_t^2} \right) < +\infty. \quad (6.20)$$

Next from Lemma 4.1 and choosing $\epsilon > 0$ such that $1/\alpha_0 - \epsilon > 1/2$ and $1/\alpha_0 - \epsilon > 1 - \alpha_0/4$, which is possible since $1 < \alpha_0 < 2$, we have the convergence in probability

$$A_n(\theta_0) \frac{1}{n} \sum_{i=0}^{n-1} |\mathbb{E}_i(f(\rho_i^n) - f(\bar{\rho}_i^n))| \rightarrow 0.$$

Observing that for f bounded $\mathbb{E}_i |f(\bar{\rho}_i^n) - P_n f(X_{i\Delta_n}, \theta_0)|^2 \leq C \mathbb{E}_i |f(\bar{\rho}_i^n) - P_n f(X_{i\Delta_n}, \theta_0)|^p$ with $p < \alpha_0$, we check with Lemma 4.1

$$\begin{aligned} \frac{n}{u_n^4} \frac{1}{n^2} \sum_{i=0}^{n-1} \mathbb{E}_i |f_1(\rho_i^n) - f_1(\bar{\rho}_i^n)|^2 &\rightarrow 0, \\ \frac{n}{u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2}} \frac{1}{n^2} \sum_{i=0}^{n-1} \mathbb{E}_i |f_k(\rho_i^n) - f_k(\bar{\rho}_i^n)|^2 &\rightarrow 0 \quad k \in \{2, 3\}. \end{aligned}$$

Consequently, we have the convergence in probability

$$A_n(\theta_0) \frac{1}{n} \sum_{i=0}^{n-1} [f(\rho_i^n) - f(\bar{\rho}_i^n)] \rightarrow 0, \quad (6.21)$$

and the proof of Proposition 4.2 reduces to show the stable convergence

$$A_n(\theta_0) \frac{1}{n} \sum_{i=0}^{n-1} [f(\bar{\rho}_i^n) - P_n f(X_{i\Delta_n}, \theta_0)] \xrightarrow[n \rightarrow \infty]{\mathcal{L}\text{-}s} \Sigma(\theta_0)^{1/2} \mathcal{N},$$

for \mathcal{N} a standard Gaussian variable independent of $\Sigma(\theta_0)$ defined by (3.5). We set

$$\zeta_i^n = A_n(\theta_0) \frac{1}{n} [f(\bar{\rho}_i^n) - P_n f(X_{i\Delta_n}, \theta_0)], \quad (6.22)$$

hence $\zeta_i^n = (\zeta_i^{k,n})_{1 \leq k \leq 3}$ with

$$\begin{aligned} \zeta_i^{1,n} &= \frac{1}{u_n^2 \sqrt{n}} [f_1(\bar{\rho}_i^n) - P_n f_1(X_{i\Delta_n}, \theta_0)], \\ \zeta_i^{k,n} &= \frac{1}{u_n^{\alpha_0/2} \Delta_n^{1/2 - \alpha_0/4} \sqrt{n}} [f_k(\bar{\rho}_i^n) - P_n f_k(X_{i\Delta_n}, \theta_0)], \quad \text{for } k \in \{2, 3\}. \end{aligned}$$

According to Theorem IX.7.26 in Jacod and Shiryaev [17], it is sufficient to show the following convergences in probability (with ζ_i^n defined by (6.22))

$$\sum_{i=0}^{n-1} \mathbb{E}_i(\zeta_i^n) \rightarrow 0, \quad (6.23)$$

$$\sum_{i=0}^{n-1} \mathbb{E}_i(|\zeta_i^{k,n}|^p) \rightarrow 0 \quad \text{for some } p > 2, \quad k = 1, 2, 3, \quad (6.24)$$

$$\sum_{i=0}^{n-1} \mathbb{E}_i(\zeta_i^{k,n} \zeta_i^{l,n}) \rightarrow \Sigma_{k,l}(\theta_0) \quad \text{for } k, l = 1, 2, 3, \quad (6.25)$$

$$\sum_{i=0}^{n-1} \mathbb{E}_i(\zeta_i^{k,n} (M_{(i+1)\Delta_n} - M_{i\Delta_n})) \rightarrow 0 \quad \text{for } k = 1, 2, 3, \quad (6.26)$$

where the last convergence holds for any square integrable martingale M . The convergence (6.23) is immediate since $\mathbb{E}_i f(\bar{\rho}_i^n) = P_n f(X_{i\Delta_n}, \theta_0)$ and (6.26) is addressed in Lemma 6.1 below.

We deduce (6.24) from Lemma 4.2 for $k = 2$ and $k = 3$. For $k = 1$, we recall that using the notation (4.1) with $f_1(x) = \cos(x)$, we have $\mathbb{E}_i f_1(\bar{\rho}_i^n) = P_n f_1(X_{i\Delta_n}, \theta_0)$ with

$$P_n f_1(X_{i\Delta_n}, \theta_0) = \exp\left(-u_n^2 \frac{\sigma_0^2}{2} - \delta_0(X_{i\Delta_n}, \Delta_n)^{1-\alpha_0/2} u_n^{\alpha_0}\right) \cos(\delta_0(X_{i\Delta_n}, \Delta_n)^{1-\alpha_0/2} u_n^{\alpha_0} \tan(\pi\alpha_0/2)), \quad (6.27)$$

and

$$\mathbb{E}_i f_1(2\bar{\rho}_i^n) = \exp\left(-2u_n^2 \sigma_0^2 - 2^{\alpha_0} \delta_0(X_{i\Delta_n}, \Delta_n)^{1-\alpha_0/2} u_n^{\alpha_0}\right) \cos(2^{\alpha_0} \delta_0(X_{i\Delta_n}, \Delta_n)^{1-\alpha_0/2} u_n^{\alpha_0} \tan(\pi\alpha_0/2)).$$

Using $\cos^2(a) = (1 + \cos(2a))/2$, we deduce

$$\mathbb{E}_i(f_1(\bar{\rho}_i^n)^2) = (1 + \mathbb{E}_i f_1(2\bar{\rho}_i^n))/2,$$

and for $q \geq 2$, since f_1 is bounded

$$\mathbb{E}_i(|f_1(\bar{\rho}_i^n) - P_n f_1(X_{i\Delta_n}, \theta_0)|^q) \leq \mathbb{E}_i(f_1(\bar{\rho}_i^n) - \mathbb{E}_i f_1(\bar{\rho}_i^n))^2 = \frac{\mathbb{E}_i f_1(2\bar{\rho}_i^n) - 2\mathbb{E}_i^2 f_1(\bar{\rho}_i^n) + 1}{2}. \quad (6.28)$$

We now expand $\mathbb{E}_i f_1(\bar{\rho}_i^n)$ and $\mathbb{E}_i f_1(2\bar{\rho}_i^n)$ around 0 and get that

$$\mathbb{E}_i f_1(2\bar{\rho}_i^n) - 2\mathbb{E}_i^2 f_1(\bar{\rho}_i^n) + 1 = u_n^4 \sigma_0^4 + R_n(X_{i\Delta_n}, \theta_0), \text{ with } |R_n(x, \theta_0)| \leq C(u_n^6 + u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2})(1+x). \quad (6.29)$$

This gives (6.24) for $k = 1$.

We now prove (6.25). From (6.28) and (6.29) we have

$$\sum_{i=0}^{n-1} \mathbb{E}_i(\zeta_i^{1,n})^2 = \frac{1}{n} \frac{1}{u_n^4} \sum_{i=0}^{n-1} \frac{\mathbb{E}_i f_1(2\bar{\rho}_i^n) - 2\mathbb{E}_i^2 f_1(\bar{\rho}_i^n) + 1}{2} \rightarrow \frac{\sigma_0^4}{2}.$$

We then apply Lemma 4.2 to f_k, f_l and $f_k f_l$ with $k, l \in \{2, 3\}$. It yields

$$\begin{aligned} & \mathbb{E}_i[f_k(\bar{\rho}_i^n) - P_n f_k(X_{i\Delta_n}, \theta_0)][f_l(\bar{\rho}_i^n) - P_n f_l(X_{i\Delta_n}, \theta_0)] \\ &= P_n(f_k f_l)(X_{i\Delta_n}, \theta_0) - P_n f_k(X_{i\Delta_n}, \theta_0) P_n f_l(X_{i\Delta_n}, \theta_0) \\ &= u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2} c_{\alpha_0} \delta_0 X_{i\Delta_n}^{1-\alpha_0/2} \frac{1}{2} \int \frac{f_k f_l(v)}{|v|^{\alpha_0+1}} dv + u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2} \varepsilon_n(\theta_0, X_{i\Delta_n}), \end{aligned}$$

where

$$|\varepsilon_n(\theta_0, X_{i\Delta_n})| \leq (1 + X_{i\Delta_n}) \varepsilon_n, \quad \text{with } \varepsilon_n \rightarrow 0.$$

Consequently, we deduce for $k, l \in \{2, 3\}$

$$\sum_{i=0}^{n-1} \mathbb{E}_i(\zeta_i^{k,n} \zeta_i^{l,n}) \rightarrow \frac{1}{2} c_{\alpha_0} \delta_0 \int_0^1 X_s^{1-\alpha_0/2} ds \int \frac{f_k f_l(v)}{|v|^{\alpha_0+1}} dv.$$

We finally address the cross terms $\sum_{i=0}^{n-1} \mathbb{E}_i(\zeta_i^{1,n} \zeta_i^{k,n})$ for $k \in \{2, 3\}$. We have

$$\begin{aligned} & \mathbb{E}_i[(f_1(\bar{\rho}_i^n) - P_n f_1(X_{i\Delta_n}, \theta_0)][f_k(\bar{\rho}_i^n) - P_n f_k(X_{i\Delta_n}, \theta_0)]) \\ &= P_n(f_1 f_k)(X_{i\Delta_n}, \theta_0) - P_n f_1(X_{i\Delta_n}, \theta_0) P_n f_k(X_{i\Delta_n}, \theta_0). \end{aligned}$$

Using again Lemma 4.2 for $f_1 f_k$ and f_k with $k \in \{2, 3\}$ and observing that $P_n f_1(X_{i\Delta_n}, \theta_0)$ is bounded we have

$$|P_n(f_1 f_k)(X_{i\Delta_n}, \theta_0) - P_n f_1(X_{i\Delta_n}, \theta_0) P_n f_k(X_{i\Delta_n}, \theta_0)| \leq C u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2} (1 + X_{i\Delta_n}),$$

and we obtain for $k \in \{2, 3\}$,

$$\sum_{i=0}^{n-1} \mathbb{E}_i(\zeta_i^{1,n} \zeta_i^{k,n}) \rightarrow 0.$$

We finally conclude that

$$\sum_{i=0}^{n-1} \mathbb{E}_i(\zeta_i^n (\zeta_i^n)^T) \rightarrow \Sigma(\theta_0).$$

We have proved equation (6.25).

We check easily that $\Sigma(\theta_0)$ is positive definite and we finish the proof of Proposition 4.2 in the case $n\Delta_n = 1$ with Lemma 6.1 below.

Lemma 6.1. *Let M be a square-integrable martingale. Let f_1 and f_2 be even bounded functions. We assume moreover that f_1 is \mathcal{C}^1 with bounded derivative, $f_2 = 0$ on $[-\eta, \eta]$, for $\eta > 0$ and that $\Delta_n = 1/n$ and $u_n = 1/[\ln(1/\Delta_n)]^p$ with $p > 1/2$. Then we have the convergence in probability (where $\bar{\rho}_i^n$ is defined by (4.1))*

$$\begin{aligned} (i) \quad & \frac{1}{u_n^2 \sqrt{n}} \sum_{i=0}^{n-1} \mathbb{E}_i[(f_1(\bar{\rho}_i^n) - P_n f_1(X_{i\Delta_n}, \theta_0)) \Delta_i^n M] \rightarrow 0, \\ (ii) \quad & \frac{1}{u_n^{\alpha_0/2} \Delta_n^{1/2-\alpha_0/4} \sqrt{n}} \sum_{i=0}^{n-1} \mathbb{E}_i[(f_2(\bar{\rho}_i^n) - P_n f_2(X_{i\Delta_n}, \theta_0)) \Delta_i^n M] \rightarrow 0. \end{aligned}$$

Proof. From Theorem 2.2.15 in [15], we only consider the cases where M is the Brownian Motion B or M is orthogonal to B .

(i) Proceeding as in the proof of (6.9), we separate the Brownian component and the pure-jump component $\bar{\rho}_i^n = \bar{\rho}_i^{1,n} + \bar{\rho}_i^{2,n}$ and since f_1 and f_1' are bounded, for $1 \leq p < \alpha_0$

$$\mathbb{E}_i |f_1(\bar{\rho}_i^n) - f_1(\bar{\rho}_i^{1,n})|^2 \leq C_p u_n^p \Delta_n^{p(1/\alpha_0 - 1/2)} X_{i\Delta_n}^{p(1/\alpha_0 - 1/2)}.$$

Choosing p close to α_0 and using Proposition 4.1 (ii), it yields for $\epsilon > 0$

$$\mathbb{E} |f_1(\bar{\rho}_i^n) - f_1(\bar{\rho}_i^{1,n})|^2 \leq C_\epsilon u_n^{\alpha_0 - \epsilon} \Delta_n^{1 - \alpha_0/2 - \epsilon}.$$

Setting

$$S_n^1 = \frac{1}{u_n^2 \sqrt{n}} \sum_{i=0}^{n-1} \mathbb{E}_i [(f_1(\bar{\rho}_i^n) - f_1(\bar{\rho}_i^{1,n})) \Delta_i^n M],$$

we deduce from Cauchy-Schwarz inequality and the previous bound with ϵ such that $1 - \alpha_0/2 - \epsilon > 0$

$$\mathbb{E} |S_n^1| \leq \frac{C_\epsilon}{u_n^2 \sqrt{n}} \left(\sum_{i=0}^{n-1} u_n^{\alpha_0 - \epsilon} \Delta_n^{1 - \alpha_0/2 - \epsilon} \sum_{i=0}^{n-1} \mathbb{E} (\Delta_i^n M)^2 \right)^{1/2} \rightarrow 0,$$

observing that the second sum is bounded since M is a square-integrable martingale. So the proof of (i) reduces to $S_n^2 \rightarrow 0$ where

$$S_n^2 = \frac{1}{u_n^2 \sqrt{n}} \sum_{i=0}^{n-1} \mathbb{E}_i [f_1(\bar{\rho}_i^{1,n}) \Delta_i^n M].$$

If $M = B$, using that f_1 is even and recalling that $\bar{\rho}_i^{1,n} = \frac{\sigma_0 u_n}{\sqrt{\Delta_n}} \Delta_i^n B$, an explicit calculus gives $\mathbb{E}_i [f_1(\bar{\rho}_i^{1,n}) \Delta_i^n M] = 0$. If M is a square integrable martingale orthogonal to B , we have from the representation Theorem (Theorem III 4.34 in [17]) using that $f_1(\bar{\rho}_i^{1,n})$ is adapted to the Brownian filtration

$$f_1(\bar{\rho}_i^{1,n}) = \mathbb{E}_i f_1(\bar{\rho}_i^{1,n}) + \int_{i\Delta_n}^{(i+1)\Delta_n} h_s dB_s,$$

and since M is orthogonal to B

$$M_t = \int_0^t \int G_s \tilde{N}(ds, dz) \text{ with } \mathbb{E} \int_0^1 \int G_s^2 \bar{N}(ds, dz) < \infty,$$

where (h_t) and (G_t) are predictable processes. We conclude with Itô's formula for the product of stochastic integrals that $\mathbb{E}_i [f_1(\bar{\rho}_i^{1,n}) \Delta_i^n M] = 0$

(ii) We follow the proof of Lemma 8 in [10]. We first assume that $M = B$. From Hölder's inequality with $p > 1$, we have

$$|\mathbb{E}_i [f_2(\bar{\rho}_i^n) \Delta_i^n B]| \leq C \sqrt{\Delta_n} \mathbb{E}_i^{1/p} f_2(\bar{\rho}_i^n)^p \leq C \Delta_n^{(1 - \alpha_0/2)/p} \sqrt{\Delta_n} (1 + X_{i\Delta_n}),$$

where for the second inequality we applied Lemma 4.2 to f_2^p . We deduce then (ii) in that case choosing $p < 2$.

We now assume that M is orthogonal to B . As previously, we apply Theorem III 4.34 in [17] to write for some predictable process (G_t)

$$M_t = \int_0^t \int G_s \tilde{N}(ds, dz) \text{ with } \mathbb{E} \int_0^1 \int G_s^2 \bar{N}(ds, dz) < \infty,$$

and also for (h_t) and (H_t) predictable and (H_t) bounded (since f_2 is bounded)

$$f_2(\bar{\rho}_i^n) - P_n f_2(X_{i\Delta_n}, \theta_0) = \int_{i\Delta_n}^{(i+1)\Delta_n} h_s dB_s + \int_{i\Delta_n}^{(i+1)\Delta_n} \int H_s \tilde{N}(ds, dz).$$

We then deduce from Itô's Formula for the product of stochastic integrals

$$\mathbb{E}_i[(f_2(\bar{\rho}_i^n) - P_n f_2(X_{i\Delta_n}, \theta_0))\Delta_i^n M] = \mathbb{E}_i \int_{i\Delta_n}^{(i+1)\Delta_n} \int H_s G_s \bar{N}(ds, dz).$$

With the decomposition $G_s = G_s \mathbf{1}_{|G_s| > \epsilon_n} + G_s \mathbf{1}_{|G_s| \leq \epsilon_n}$, the proof reduces to study the convergence of

$$\begin{aligned} T_n^1 &= \frac{1}{u_n^{\alpha_0/2} \Delta_n^{1/2-\alpha_0/4} \sqrt{n}} \sum_{i=0}^{n-1} \mathbb{E}_i \int_{i\Delta_n}^{(i+1)\Delta_n} \int H_s G_s \mathbf{1}_{|G_s| > \epsilon_n} \bar{N}(ds, dz), \\ T_n^2 &= \frac{1}{u_n^{\alpha_0/2} \Delta_n^{1/2-\alpha_0/4} \sqrt{n}} \sum_{i=0}^{n-1} \mathbb{E}_i \int_{i\Delta_n}^{(i+1)\Delta_n} \int H_s G_s \mathbf{1}_{|G_s| \leq \epsilon_n} \bar{N}(ds, dz). \end{aligned}$$

Since H is bounded, we have immediately

$$\begin{aligned} \mathbb{E}|T_n^1| &\leq \frac{C}{u_n^{\alpha_0/2} \Delta_n^{1/2-\alpha_0/4} \sqrt{n}} \sum_{i=0}^{n-1} \mathbb{E} \int_{i\Delta_n}^{(i+1)\Delta_n} \int \frac{G_s^2}{\epsilon_n} \bar{N}(ds, dz) \\ &\leq \frac{C}{\epsilon_n u_n^{\alpha_0/2} \Delta_n^{1/2-\alpha_0/4} \sqrt{n}} \mathbb{E} \int_0^1 \int G_s^2 \bar{N}(ds, dz) \leq \frac{C}{\epsilon_n u_n^{\alpha_0/2} \Delta_n^{1/2-\alpha_0/4} \sqrt{n}}. \end{aligned} \quad (6.30)$$

With $\epsilon_n = \Delta_n^{\alpha_0/8}$, we deduce $\mathbb{E}|T_n^1| \rightarrow 0$.

We now turn to T_n^2 . We first apply Lemma 4.2 to f_2 to obtain

$$\mathbb{E}_i \int_{i\Delta_n}^{(i+1)\Delta_n} \int H_s^2 \bar{N}(ds, dz) \leq \mathbb{E}_i (f_2(\bar{\rho}_i^n) - P_n f_2(X_{i\Delta_n}, \theta_0))^2 \leq C u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2} (1 + X_{i\Delta_n}).$$

From Cauchy-Schwarz inequality it yields

$$\begin{aligned} |\mathbb{E}_i \int_{i\Delta_n}^{(i+1)\Delta_n} \int H_s G_s \mathbf{1}_{|G_s| \leq \epsilon_n} \bar{N}(ds, dz)| &\leq \\ &\leq C u_n^{\alpha_0/2} \Delta_n^{1/2-\alpha_0/4} (1 + \sqrt{X_{i\Delta_n}}) \mathbb{E}_i^{1/2} \int_{i\Delta_n}^{(i+1)\Delta_n} \int G_s^2 \mathbf{1}_{|G_s| \leq \epsilon_n} \bar{N}(ds, dz), \end{aligned}$$

and applying once again Cauchy-Schwarz inequality

$$\mathbb{E}|T_n^2| \leq C \left(\mathbb{E} \frac{1}{n} \sum_{i=0}^{n-1} (1 + X_{i\Delta_n}) \times \mathbb{E} \sum_{i=0}^{n-1} \int_{i\Delta_n}^{(i+1)\Delta_n} \int G_s^2 \mathbf{1}_{|G_s| \leq \epsilon_n} \bar{N}(ds, dz) \right)^{1/2}.$$

The first expectation is bounded from Proposition 4.1, consequently

$$\mathbb{E}|T_n^2| \leq C \mathbb{E}^{1/2} \int_0^1 \int G_s^2 \mathbf{1}_{|G_s| \leq \epsilon_n} \bar{N}(ds, dz). \quad (6.31)$$

As ϵ_n goes to zero, we conclude $\mathbb{E}|T_n^2| \rightarrow 0$ by dominated convergence and the proof of (ii) is finished. \square

2. Case $n\Delta_n \rightarrow +\infty$. The proof follows essentially the same lines as in the previous case and we only indicate the main changes. The bounds (6.19) and (6.20) still hold. From the ergodic Theorem (if $b_0 > 0$, the process (X_t) is geometrically ergodic, see [26]), we have the convergence in probability for $0 < p \leq 1$

$$\frac{1}{n\Delta_n} \int_0^{n\Delta_n} X_t^p dt \xrightarrow[n \rightarrow \infty]{} \int_0^{+\infty} x^p \pi_0(dx),$$

since $\int_0^{+\infty} x \pi_0(dx) < +\infty$. Moreover we know that $\mathbb{E}(X_t) \rightarrow a_0/b_0$ (see [5]), then $\sup_t \mathbb{E}(X_t) < +\infty$ and combining this with Proposition 4.1 (iv) we can prove

$$\mathbb{E} \left| \frac{1}{n} \sum_{i=0}^{n-1} X_{i\Delta_n}^p - \frac{1}{n\Delta_n} \int_0^{n\Delta_n} X_t^p dt \right| \xrightarrow[n \rightarrow \infty]{} 0.$$

We deduce the convergence in probability

$$\frac{1}{n} \sum_{i=0}^{n-1} X_{i\Delta_n}^p \xrightarrow[n \rightarrow \infty]{} \int_0^{+\infty} x^p \pi_0(dx). \quad (6.32)$$

Now replacing ρ_i^n by $\bar{\rho}_i^n$, we obtain (6.21) under the additional assumption $\sqrt{n}\Delta_n^{1/\alpha + \alpha/4 - 1/2 - \epsilon_0} \rightarrow 0$. It remains to prove the convergence in distribution of

$$A_n(\theta_0) \frac{1}{n} \sum_{i=0}^{n-1} [f(\bar{\rho}_i^n) - P_n f(X_{i\Delta_n}, \theta_0)].$$

We check (6.23), (6.24), (6.25) (with $\bar{\Sigma}$) by proceeding as in the case $n\Delta_n$ fixed but using (6.32) and $\sup_t \mathbb{E}(X_t) < +\infty$ to conclude.

6.1.5 Proof of Proposition 4.3

By definition of $F_n(\theta)$, we have

$$A_n(\theta) \nabla_{\theta} F_n(\theta) \Lambda_n(\theta) = -\frac{1}{n} \sum_{i=0}^{n-1} A_n(\theta) \nabla_{\theta} P_n f(X_{i\Delta_n}, \theta) \Lambda_n(\theta).$$

We start by computing

$$\nabla_{\theta} P_n f(x, \theta) = \begin{pmatrix} \partial_{\sigma^2} P_n f_1(x, \theta) & \partial_{\delta} P_n f_1(x, \theta) & \partial_{\alpha} P_n f_1(x, \theta) \\ \partial_{\sigma^2} P_n f_2(x, \theta) & \partial_{\delta} P_n f_2(x, \theta) & \partial_{\alpha} P_n f_2(x, \theta) \\ \partial_{\sigma^2} P_n f_3(x, \theta) & \partial_{\delta} P_n f_3(x, \theta) & \partial_{\alpha} P_n f_3(x, \theta) \end{pmatrix}.$$

As $P_n f_1(x, \theta) = \exp(-u_n^2 \frac{\sigma^2}{2} - \delta x^{1-\alpha/2} u_n^{\alpha} \Delta_n^{1-\alpha/2}) \cos(\delta x^{1-\alpha/2} u_n^{\alpha} \Delta_n^{1-\alpha/2} \tan(\pi\alpha/2))$, an explicit calculus gives (where the terms coming from the derivative of the cosinus are negligible)

$$\begin{aligned} \partial_{\sigma^2} P_n f_1(x, \theta) &= -\frac{u_n^2}{2} P_n f_1(x, \theta), \\ \partial_{\delta} P_n f_1(x, \theta) &= -x^{1-\alpha/2} u_n^{\alpha} \Delta_n^{1-\alpha/2} (P_n f_1(x, \theta) + \varepsilon_n^0(x, \theta)), \\ \partial_{\alpha} P_n f_1(x, \theta) &= -\delta x^{1-\alpha/2} u_n^{\alpha} \Delta_n^{1-\alpha/2} \ln(u_n / \sqrt{x \Delta_n}) (P_n f_1(x, \theta) + \bar{\varepsilon}_n^0(x, \theta)), \end{aligned} \quad (6.33)$$

with $|\varepsilon_n^0(x, \theta)| + |\bar{\varepsilon}_n^0(x, \theta)| \leq C x^{1-\alpha/2} u_n^{\alpha} \Delta_n^{1-\alpha/2}$. Next observing that $\nabla_{\theta} P_n f_2(x, \theta) = -\nabla_{\theta} P_n K(x, \theta)$ (the same connection holds for f_3 replacing $K(x)$ by $K(2x)$), we evaluate $\nabla_{\theta} P_n K(x, \theta)$. We recall that from (3.2)

$$P_n K(x, \theta) = \mathbb{E} K \left(u_n \sigma B_1 + \frac{(\delta x)^{1/\alpha}}{\sqrt{x}} u_n \Delta_n^{1/\alpha - 1/2} L_1^{\alpha} \right),$$

and we denote by $g_{n,\theta,x}$ the density of $u_n\sigma B_1 + \frac{(\delta x)^{1/\alpha}}{\sqrt{x}}u_n\Delta_n^{1/\alpha-1/2}L_1^\alpha$. We obtain the following result where $\mathcal{F}f$ is the Fourier transform of f , defined by $\mathcal{F}f(y) = \int e^{-iyz}f(z)dz$ and \mathcal{RF} denotes the real part of the Fourier transform.

Lemma 6.2. *Let K be defined by (3). Then we have*

$$\begin{aligned}\partial_{\sigma^2}P_nK(x, \theta) &= \frac{u_n^2}{2}P_nK''(x, \theta), \\ \partial_\delta P_nK(x, \theta) &= -x^{1-\alpha/2}u_n^\alpha\Delta_n^{1-\alpha/2} \left(\frac{1}{2\pi} \int \mathcal{F}K(y)|y|^\alpha \mathcal{RF}g_{n,\theta,x}(y)dy + \varepsilon_n^1(x, \theta) \right), \\ \partial_\alpha P_nK(x, \theta) &= -\delta x^{1-\alpha/2}u_n^\alpha\Delta_n^{1-\alpha/2} \left(\frac{1}{2\pi} \int \mathcal{F}K(y)|y|^\alpha \ln(|y|) \mathcal{RF}g_{n,\theta,x}(y)dy \right. \\ &\quad \left. + \left(\ln\left(\frac{u_n}{\sqrt{\Delta_n}}\right) - \ln(\sqrt{x}) \right) \frac{1}{2\pi} \int \mathcal{F}K(y)|y|^\alpha \mathcal{RF}g_{n,\theta,x}(y)dy + \varepsilon_n^2(x, \theta) \right),\end{aligned}$$

with $|\varepsilon_n^1(x, \theta)| \leq Cx^{1-\alpha/2}u_n^\alpha\Delta_n^{1-\alpha/2}$ and $|\varepsilon_n^2(x, \theta)| \leq Cx^{1-\alpha/2}u_n^\alpha\Delta_n^{1-\alpha/2}(1 + |\ln(\frac{u_n}{\sqrt{\Delta_n x}})|)$.

Proof. Since K is compactly supported and even, its Fourier transform is a Schwartz function and $\mathcal{F}\mathcal{F}K = 2\pi K$. We also have $-y^2\mathcal{F}K(y) = \mathcal{F}K''(y)$. With the previous notation it yields

$$\begin{aligned}\nabla_\theta P_nK(x, \theta) &= \nabla_\theta \int K(y)g_{n,\theta,x}(y)dy = \frac{1}{2\pi} \nabla_\theta \int (\mathcal{F}\mathcal{F}K)(y)g_{n,\theta,x}(y)dy \\ &= \frac{1}{2\pi} \nabla_\theta \int \mathcal{F}K(y)\mathcal{F}g_{n,\theta,x}(y)dy = \frac{1}{2\pi} \nabla_\theta \int \mathcal{F}K(y)\mathcal{RF}g_{n,\theta,x}(y)dy \\ &= \frac{1}{2\pi} \int \mathcal{F}K(y)\nabla_\theta \mathcal{RF}g_{n,\theta,x}(y)dy,\end{aligned}$$

where we successively used duality and Lebesgue's Theorem. By definition of $g_{n,\theta,x}$, we have

$$\mathcal{RF}g_{n,\theta,x}(y) = \exp\left(-\frac{u_n^2}{2}\sigma^2y^2 - |y|^\alpha\delta x^{1-\alpha/2}u_n^\alpha\Delta_n^{1-\alpha/2}\right) \cos(|y|^\alpha\delta x^{1-\alpha/2}u_n^\alpha\Delta_n^{1-\alpha/2} \tan(\pi\alpha/2)). \quad (6.34)$$

We compute explicitly $\nabla_\theta \mathcal{RF}g_{n,\theta,x}$. As previously the terms coming from the derivative of the cosine are negligible and the expression of $\nabla_\theta P_nK$ follows. \square

Coming back to $\nabla_\theta P_n f_2(x, \theta)$ we obtain the following result.

Lemma 6.3. *We assume that $u_n[\ln(1/\Delta_n)]^{1/2} \rightarrow 0$. Then we have $\forall x > 0$*

$$\begin{aligned}(i) \quad & \left| \frac{1}{u_n^{\alpha/2} \Delta_n^{1/2-\alpha/4}} \partial_{\sigma^2} P_n f_2(x, \theta) \right| \leq \bar{\varepsilon}_n^1(x, \theta), \\ (ii) \quad & \frac{1}{u_n^\alpha \Delta_n^{1-\alpha/2}} \partial_\delta P_n f_2(x, \theta) = \frac{1}{2} x^{1-\alpha/2} \psi(\alpha) + \bar{\varepsilon}_n^2(x, \theta), \\ (iii) \quad & \frac{1}{u_n^\alpha \Delta_n^{1-\alpha/2}} \left(\partial_\alpha P_n f_2(x, \theta) - \delta \ln\left(\frac{u_n}{\sqrt{\Delta_n}}\right) \partial_\delta P_n f_2(x, \theta) \right) = \frac{1}{2} \delta x^{1-\alpha/2} [\partial_\alpha \psi(\alpha) - \ln(\sqrt{x}) \psi(\alpha)] \\ & \quad + \bar{\varepsilon}_n^3(x, \theta),\end{aligned}$$

where $\psi(\alpha) = c_\alpha \int \frac{f_2(y)}{|y|^{\alpha+1}} dy = c_\alpha \int \frac{1-K(y)}{|y|^{\alpha+1}} dy$ and for any compact subset $A \subset \Theta$

$$\sup_{\theta \in A} (|\bar{\varepsilon}_n^1(x, \theta)| + |\bar{\varepsilon}_n^2(x, \theta)| + |\bar{\varepsilon}_n^3(x, \theta)|) \leq C u_n (1+x).$$

The same result holds for f_3 replacing $\psi(\alpha)$ by $2^\alpha \psi(\alpha)$.

Proof. We first prove that

$$\frac{1}{2\pi} \int \mathcal{F}K(y)|y|^\alpha dy = \frac{c_\alpha}{2} \int \frac{(1-K(y))}{|y|^{\alpha+1}} dy = \frac{\psi(\alpha)}{2}. \quad (6.35)$$

By definition of c_α we have $|y|^\alpha = \frac{c_\alpha}{2} \int \frac{1-\cos(uy)}{|u|^{\alpha+1}} du$. Then for $\epsilon > 0$

$$\begin{aligned} \frac{1}{2\pi} \int \mathcal{F}K(y)|y|^\alpha dy &= \frac{c_\alpha}{4\pi} \int \mathcal{F}K(y) \left(\int \frac{1-\cos(uy)}{|u|^{\alpha+1}} du \right) dy \\ &= \frac{c_\alpha}{4\pi} \int \mathcal{F}K(y) \left(\int \frac{1-\cos(uy)}{|u|^{\alpha+1}} (\mathbf{1}_{|u|\leq\epsilon} + \mathbf{1}_{|u|>\epsilon}) du \right) dy. \end{aligned}$$

Since $1 - \cos(uy) = h(uy)u^2y^2$ with h bounded, we can apply Fubini's Theorem

$$\begin{aligned} \frac{1}{2\pi} \int \mathcal{F}K(y)|y|^\alpha dy &= \frac{c_\alpha}{4\pi} \int \frac{u^2}{|u|^{\alpha+1}} \mathbf{1}_{|u|\leq\epsilon} \left(\int \mathcal{F}K(y)h(uy)y^2 dy \right) du \\ &\quad + \frac{c_\alpha}{4\pi} \int \frac{1}{|u|^{\alpha+1}} \mathbf{1}_{|u|>\epsilon} \left(\int \mathcal{F}K(y)(1-\cos(uy)) dy \right) du. \end{aligned}$$

Moreover, we have

$$\int \mathcal{F}K(y) dy = 2\pi K(0) = 2\pi, \quad \int \mathcal{F}K(y) \cos(uy) dy = \mathcal{F}(\mathcal{F}K)(u) = 2\pi K(u).$$

Letting ϵ go to zero, we obtain (6.35).

Next from (6.34), we have

$$\sup_{\theta \in A} |\mathcal{R}\mathcal{F}g_{n,\theta,x}(y) - 1| \leq C u_n (1+y^2)(1+\sqrt{x}),$$

and consequently

$$\begin{aligned} \sup_{\theta \in A} \left| \frac{1}{2\pi} \int \mathcal{F}K(y)|y|^\alpha \mathcal{R}\mathcal{F}g_{n,\theta,x}(y) dy - \frac{\psi(\alpha)}{2} \right| &\leq C u_n (1+\sqrt{x}), \\ \sup_{\theta \in A} \left| \frac{1}{2\pi} \int \mathcal{F}K(y)|y|^\alpha \ln(|y|) \mathcal{R}\mathcal{F}g_{n,\theta,x}(y) dy - \frac{\partial_\alpha \psi(\alpha)}{2} \right| &\leq C u_n (1+\sqrt{x}). \end{aligned}$$

Since $f_2 = 1 - K$, we obtain (i) combining Lemma 6.2 with Lemma 4.2 (applied to K''), (ii) and (iii) follow from Lemma 6.2, (6.35) and the previous uniform bounds. \square

With this background, we can finish the proof of Proposition 4.3. An explicit calculus leads to $A_n(\theta) \nabla_\theta P_n f(x, \theta) \Lambda_n(\theta) = W_n(x, \theta)$ with

$$W_n = \begin{pmatrix} u_n^{-2} \partial_{\sigma^2} P_n f_1 & u_n^{-2} v_n \partial_\delta P_n f_1 & u_n^{-2} v_n [\partial_\alpha P_n f_1 - \delta \ln(\frac{u_n}{\sqrt{\Delta_n}}) \partial_\delta P_n f_1] \\ v_n \partial_{\sigma^2} P_n f_2 & v_n^2 \partial_\delta P_n f_2 & v_n^2 [\partial_\alpha P_n f_2 - \delta \ln(\frac{u_n}{\sqrt{\Delta_n}}) \partial_\delta P_n f_2] \\ v_n \partial_{\sigma^2} P_n f_3 & v_n^2 \partial_\delta P_n f_3 & v_n^2 [\partial_\alpha P_n f_3 - \delta \ln(\frac{u_n}{\sqrt{\Delta_n}}) \partial_\delta P_n f_3] \end{pmatrix}, \quad v_n = \frac{1}{u_n^{\alpha/2} \Delta_n^{1/2-\alpha/4}}.$$

1. Case $n\Delta_n = 1$. We check the uniform convergences in probability for $\alpha \in [\underline{a}, \bar{a}]$, with $1 < \underline{a} < \bar{a} < 2$

$$\sup_{\alpha \in [\underline{a}, \bar{a}]} \left| \frac{1}{n} \sum_{i=0}^{n-1} X_{i\Delta_n}^{1-\alpha/2} - \int_0^1 X_t^{1-\alpha/2} dt \right| \rightarrow 0, \quad (6.36)$$

$$\sup_{\alpha \in [\underline{a}, \bar{a}]} \left| \frac{1}{n} \sum_{i=0}^{n-1} X_{i\Delta_n}^{1-\alpha/2} \ln \sqrt{X_{i\Delta_n}} - \int_0^1 X_t^{1-\alpha/2} \ln \sqrt{X_t} dt \right| \rightarrow 0. \quad (6.37)$$

We just detail (6.37). Let $h(\alpha, x) = x^{1-\alpha/2} \ln(\sqrt{x})$. We have for $\epsilon > 0$,

$$\forall x > 0, |h'(\alpha, x)| \leq \frac{1}{x^{\alpha/2}}(1 + |\ln(x)|) \leq C(1 + \frac{1}{x^{\alpha/2+\epsilon}}),$$

and from Taylor's formula for $x, y > 0$

$$|h(\alpha, y) - h(\alpha, x)| \leq C|y - x| \int_0^1 (1 + \frac{1}{((1-c)x + cy)^{\alpha/2+\epsilon}}) dc.$$

Choosing ϵ such that $\bar{a}/2 + \epsilon < 1$, we deduce

$$\sup_{\alpha \in [\underline{a}, \bar{a}]} |h(\alpha, y) - h(\alpha, x)| \leq C|y - x|(1 + \frac{1}{x}).$$

From Proposition 4.1 it yields

$$\mathbb{E}_i \sup_{t \in [i\Delta_n, (i+1)\Delta_n]} \sup_{\alpha \in [\underline{a}, \bar{a}]} |h(\alpha, X_t) - h(\alpha, X_{i\Delta_n})| \leq C\sqrt{\Delta_n}(1 + X_{i\Delta_n} + \frac{1}{X_{i\Delta_n}}),$$

and we obtain (6.37).

Furthermore, from Lebesgue's Theorem $\alpha \mapsto I(\alpha, X) = \int_0^1 X_t^{1-\alpha/2} dt$ admits a derivative and $\partial_\alpha I(\alpha, X) = -\int_0^1 X_t^{1-\alpha/2} \ln(\sqrt{X_t}) dt$, so in the expression of $W(\theta)$ (3.6), we have

$$\partial_\alpha(\psi(\alpha)I(\alpha, X)) = \partial_\alpha \psi(\alpha) \int_0^1 X_t^{1-\alpha/2} dt - \psi(\alpha) \int_0^1 X_t^{1-\alpha/2} \ln(\sqrt{X_t}) dt.$$

Consequently, combining (6.33) and Lemma 6.3 with (6.36) and (6.37), we obtain

$$\sup_{\theta \in A} |\frac{1}{n} \sum_{i=0}^{n-1} W_n(X_{i\Delta_n}, \theta) + W(\theta)| \rightarrow 0,$$

and we finally conclude from $A_n(\theta) \nabla_\theta F_n(\theta) \Lambda_n(\theta) = -\frac{1}{n} \sum_{i=0}^{n-1} W_n(X_{i\Delta_n}, \theta)$. We see easily that W is non-singular since

$$\det W(\theta) = \frac{1}{8} \delta \psi(\alpha)^2 I(\alpha, X)^2 2^\alpha \ln(2) \neq 0.$$

2. Case $n\Delta_n \rightarrow \infty$. When $n\Delta_n \rightarrow +\infty$, we first prove the convergences in probability

$$\sup_{\alpha \in [\underline{a}, \bar{a}]} |\frac{1}{n} \sum_{i=0}^{n-1} X_{i\Delta_n}^{1-\alpha/2} - \int_0^{+\infty} x^{1-\alpha/2} \pi_0(dx)| \rightarrow 0, \quad (6.38)$$

$$\sup_{\alpha \in [\underline{a}, \bar{a}]} |\frac{1}{n} \sum_{i=0}^{n-1} X_{i\Delta_n}^{1-\alpha/2} \ln \sqrt{X_{i\Delta_n}} - \int_0^{+\infty} x^{1-\alpha/2} \ln \sqrt{x} \pi_0(dx)| \rightarrow 0. \quad (6.39)$$

We give the details for (6.39) and study separately the uniform convergences in probability

$$\sup_{\alpha \in [\underline{a}, \bar{a}]} |\frac{1}{n} \sum_{i=0}^{n-1} X_{i\Delta_n}^{1-\alpha/2} \ln \sqrt{X_{i\Delta_n}} - \frac{1}{n\Delta_n} \int_0^{n\Delta_n} X_t^{1-\alpha/2} \ln \sqrt{X_t} dt| \rightarrow 0, \quad (6.40)$$

$$\sup_{\alpha \in [\underline{a}, \bar{a}]} |\frac{1}{n\Delta_n} \int_0^{n\Delta_n} X_t^{1-\alpha/2} \ln \sqrt{X_t} dt - \int_0^{+\infty} x^{1-\alpha/2} \ln \sqrt{x} \pi_0(dx)| \rightarrow 0. \quad (6.41)$$

The proof of (6.40) is similar to the one of (6.37) and we omit it. It remains to check (6.41). We set $H_n(\alpha) = \frac{1}{n\Delta_n} \int_0^{n\Delta_n} X_t^{1-\alpha/2} \ln \sqrt{X_t} dt$ and $H(\alpha) = \int_0^{+\infty} x^{1-\alpha/2} \ln \sqrt{x} \pi_0(dx)$. The map

$\alpha \rightarrow H_n(\alpha)$ is continuous on $[\underline{a}, \bar{a}]$ and $H_n(\alpha) \rightarrow H(\alpha)$, $\forall \alpha \in [\underline{a}, \bar{a}]$. Moreover for $\alpha_1, \alpha_2 \in [\underline{a}, \bar{a}]$ and $x > 0$, $\exists c \in (\alpha_1, \alpha_2)$ such that

$$|x^{1-\alpha_1/2} \ln(\sqrt{x}) - x^{1-\alpha_2/2} \ln(\sqrt{x})| \leq |\alpha_1 - \alpha_2| (\ln(x))^2 x^{1-c/2} \leq C |\alpha_1 - \alpha_2| (1+x).$$

With $\sup_t \mathbb{E} X_t < +\infty$, it follows that $\forall \delta > 0$, $\sup_n \mathbb{E} \sup_{|\alpha_1 - \alpha_2| \leq \delta} |H_n(\alpha_1) - H_n(\alpha_2)| \leq C\delta$, so the sequence $(H_n(\alpha)_{\alpha \in [\underline{a}, \bar{a}]})$ is tight and we obtain the uniform convergence (6.41).

Observing that $\partial_\alpha \bar{I}(\alpha) = -\int_0^\infty x^{1-\alpha/2} \ln(\sqrt{x}) \pi_0(dx)$ and using the same arguments as in the case $n\Delta_n$ fixed, we obtain from (6.33), Lemma 6.3, (6.38) and (6.39)

$$\sup_{\theta \in A} \left| \frac{1}{n} \sum_{i=0}^{n-1} W_n(X_{i\Delta_n}, \theta) + \bar{W}(\theta) \right| \rightarrow 0.$$

This achieves the proof of Proposition 4.3.

6.2 Main results

6.2.1 Proof of Theorem 3.1 and Theorem 3.3

In both cases, $n\Delta_n$ fixed and $n\Delta_n \rightarrow +\infty$, we apply Theorem A.2. in Mies and Podolskij [31], that extends the results of Jacod and Sørensen [18]. We just detail the first case, where we have strengthened condition (E.1) in [31] by assuming the stable convergence in law. With our notation, the result of Theorem 3.1 is a consequence of the following conditions:

$$(E.1) \quad A_n(\theta_0) F_n(\theta_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}-s} \Sigma(\theta_0)^{1/2} \mathcal{N},$$

(E.2) $\theta \rightarrow F_n(\theta)$ is \mathcal{C}^1 and for r_n a sequence of real numbers we have the convergences in probability

$$\sup_{\theta \in \mathcal{B}_{r_n}(\theta_0)} \|A_n(\theta) \nabla_\theta F_n(\theta) \Lambda_n(\theta) - W(\theta)\| \xrightarrow[n \rightarrow \infty]{} 0, \quad \sup_{\theta \in \mathcal{B}_{r_n}(\theta_0)} \frac{\|\Lambda_n(\theta)\| \|A_n(\theta) A_n(\theta_0)^{-1}\|}{r_n} \xrightarrow[n \rightarrow \infty]{} 0,$$

$$(E.3) \quad \sup_{\theta \in \mathcal{B}_{r_n}(\theta_0)} (\|A_n(\theta) A_n(\theta_0)^{-1} - Id\| + \|\Lambda_n(\theta) \Lambda_n(\theta_0)^{-1} - Id\| + \|W(\theta) - W(\theta_0)\|) \xrightarrow[n \rightarrow \infty]{} 0,$$

where $\mathcal{B}_{r_n}(\theta_0)$ is the ball with center θ_0 and radius r_n , $\|\cdot\|$ a matrix norm and Id the identity matrix. Condition (E.1) is proven in Proposition 4.2 and the first condition of (E.2) is proven in Proposition 4.3. The second condition in (E.2) and condition (E.3) are easily checked with $r_n = 1/\ln(\Delta_n)^2$.

6.2.2 Proof of Theorem 3.2

The existence is a consequence of Theorem 3.1. To prove the global uniqueness result we consider the renormalized estimating function (restricted to $\beta = (\sigma^2, \delta)$) with $f = (f_k)_{1 \leq k \leq 2}$

$$\tilde{F}_n(\sigma^2, \delta) = \tilde{A}_n(\alpha_0) F_n(\sigma^2, \delta, \alpha_0), \text{ with } \tilde{A}_n(\alpha_0) = \text{diag} \left(\frac{1}{u_n^2}, \frac{1}{u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2}} \right).$$

From Proposition 4.2, we deduce $\tilde{F}_n(\sigma_0^2, \delta_0) \rightarrow 0$. Next, an explicit calculus gives

$$\nabla_\beta \tilde{F}_n(\sigma^2, \delta) = -\frac{1}{n} \sum_{i=0}^{n-1} \begin{pmatrix} \frac{1}{u_n^2} \partial_{\sigma^2} P_n f_1(X_{i\Delta_n}, \sigma^2, \delta) & \frac{1}{u_n^2} \partial_\delta P_n f_1(X_{i\Delta_n}, \sigma^2, \delta) \\ \frac{1}{u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2}} \partial_{\sigma^2} P_n f_2(X_{i\Delta_n}, \sigma^2, \delta) & \frac{1}{u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2}} \partial_\delta P_n f_2(X_{i\Delta_n}, \sigma^2, \delta) \end{pmatrix},$$

and from Proposition 4.3, we have the convergence in probability for any compact subset A of $(0, +\infty) \times (0, +\infty)$

$$\sup_{(\sigma^2, \delta) \in A} \left\| \nabla_{\beta} \tilde{F}_n(\sigma^2, \delta) - \nabla_{\beta} \tilde{F}(\sigma^2, \delta) \right\| \rightarrow 0, \text{ where } \tilde{F}(\sigma^2, \delta) = \left(\begin{array}{c} \frac{1}{2}(\sigma^2 - \sigma_0^2) \\ \frac{1}{2}\psi(\alpha_0) \int_0^1 X_t^{1-\alpha_0/2} dt (\delta_0 - \delta) \end{array} \right).$$

The global uniqueness result follows then from Theorem 2.7.a) of Jacod and Sørensen [18], since (σ_0^2, δ_0) is the unique root of $\tilde{F}(\sigma^2, \delta) = 0$.

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