



HAL
open science

Volatility and jump activity estimation in a stable Cox-Ingersoll-Ross model

Emmanuelle Clément, Elise Bayraktar

► **To cite this version:**

Emmanuelle Clément, Elise Bayraktar. Volatility and jump activity estimation in a stable Cox-Ingersoll-Ross model. 2024. hal-04662856

HAL Id: hal-04662856

<https://hal.science/hal-04662856v1>

Preprint submitted on 26 Jul 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Volatility and jump activity estimation in a stable Cox-Ingersoll-Ross model

ELISE BAYRAKTAR¹ and EMMANUELLE CLÉMENT¹

¹LAMA, Univ Gustave Eiffel, Univ Paris Est Creteil, CNRS, F-77447
Marne-la-Vallée, France.

July, 26 2024

Abstract. We consider the parametric estimation of the volatility and jump activity in a stable Cox-Ingersoll-Ross (α -stable CIR) model driven by a standard Brownian Motion and a non-symmetric stable Lévy process with jump activity $\alpha \in (1, 2)$. The main difficulties to obtain rate efficiency in estimating these quantities arise from the superposition of the diffusion component with jumps of infinite variation. Extending the approach proposed in Mies [16], we address the joint estimation of the volatility, scaling and jump activity parameters from high-frequency observations of the process and prove that the proposed estimators are rate optimal up to a logarithmic factor.

MSC 2020. 60G51; 60G52; 60J75; 62F12.

Key words: Cox-Ingersoll-Ross model, Lévy process; Parametric inference; Stable process; Stochastic Differential Equation.

1 Introduction

Jump processes are widely studied in the literature due to their many areas of application, especially in financial modeling where interest rates or asset price evolutions are often described by Itô semimartingales. These processes are characterized by the superposition of a drift term, a diffusion part and a jump component and the estimation of the different components is the subject of specific studies depending on the available observations. In this paper we are interested in the estimation of the volatility and the jump activity from high-frequency observations. The estimation of such quantities has concentrated a lot of activity in the last decades and it is well known that the presence of jumps, and more precisely small jumps, makes the estimation of the volatility more difficult. Jump activity is characterized by the Blumenthal-Gettoor index $\alpha \in (0, 2)$ and when jumps are of infinite variation, i.e. $\alpha > 1$, the usual statistics to estimate the volatility such as the truncated realized quadratic variation have an asymptotic bias. On the other hand, the estimation of the jump activity is disturbed by the presence of a diffusion component since the process is dominated by the Brownian Motion.

In this work, we focus on processes with infinite variation jumps. Many works have attempted to estimate the volatility or the jump activity in this context and we briefly present the most relevant. An efficient estimator of the volatility, with optimal rate of convergence \sqrt{n} , has been proposed in Jacod and Todorov [11] in presence of infinite variation jumps with stable-like behavior. Their method is based on the local estimation of the volatility by using the real part of the empirical characteristic function and the asymptotic bias is eliminated by combining statistics with different arguments in the characteristic function. Their estimator achieves rate and variance optimality for $\alpha \in (1, 2)$ if the jumps of the

driving Lévy process are symmetric. For the truncated realized volatility, some debiasing procedures have been studied in Amorino and Gloter [3] and in Cooper Boniece and al. [6]. By considering different truncation levels and with a two-step debiasing procedure, efficiency for the estimation of the volatility is obtained in [6], with the restriction $\alpha \in (1, 8/5)$. Turning to the estimation of the jump activity, Aït-Sahalia and Jacod in [1] have proposed an estimator of α , in presence of a diffusion component, based on jump counting. For locally stable jumps, their estimator is consistent with rate of convergence $n^{\alpha/10}$ ($n^{\alpha/8}$ in the stable case) but unfortunately it fails to achieve the optimal rate $n^{\alpha/4} \log(n)^{1-\alpha/4}$ identified in Aït-Sahalia and Jacod [2]. This estimation method is improved in Bull [7] by combining increments of the process with different time-scales, achieving the near optimal rate of convergence $n^{\alpha/4-\epsilon}$ for ϵ arbitrarily small. All these methods are non-parametric and apply to semimartingales but are not completely satisfactory. In a parametric framework, rate optimality in estimating the jump activity has been attained by Mies [16] for a Lévy process with characteristic triplet (μ, σ^2, ν) , assuming that the jump part is close to the superposition of M stable processes. Using a method of moments, a joint estimator of the volatility σ^2 , the jump activity indices $\alpha = (\alpha_m)_{1 \leq m \leq M}$ and some scaling coefficients $r = (r_m^+, r_m^-)_{1 \leq m \leq M}$ is proposed. This procedure leads to an efficient estimator of the volatility (without debiasing procedure and without restriction on the jump activity) and to rate optimal estimators of the jump activity and scaling coefficients.

In this paper, our aim is to study the joint parametric estimation of the volatility and jump activity of a stochastic differential equation with infinite variation jumps. We will focus our work on the stable Cox-Ingersoll-Ross model (stable CIR model) described by the stochastic equation

$$dX_t = (a - bX_t)dt + \sigma X_t^{1/2} dB_t + \delta^{1/\alpha} X_{t-}^{1/\alpha} dL_t^\alpha,$$

where (B_t) is a standard Brownian Motion and (L_t^α) a spectrally positive Lévy process with jump activity $\alpha \in (1, 2)$. This model is very popular in financial modeling (see Jiao et al. [12], [13]) and also presents some specific difficulties since the coefficients of the equation are not Lipschitz and not bounded away from zero. When $\sigma = 0$ (pure-jump process), some estimation results have been obtained in Li and Ma [15], Yang [19] and more recently in Bayraktar and Clément [5]. But when $\sigma \neq 0$, there are almost no estimation results in the literature for the stable CIR model and to our knowledge the only existing study concerns the estimation of the drift term b proposed in Barczy et al. [4] from continuous observations of the process. Inspired by the work of Mies [16], we propose here a joint estimation method of the volatility coefficient σ^2 , the scaling parameter δ and the jump activity index α from high-frequency observations of the process that leads to an efficient estimation of the volatility and attains rate optimality up to a factor $\log(n)$ for the estimation of (δ, α) . Our main results are derived in the case $n\Delta_n$ fixed, where n is the number of observations and Δ_n the step between two consecutive observations (we recall that in this framework and in presence of a Brownian component, the drift parameters (a, b) are not identifiable) but also in the long time asymptotics $n\Delta_n \rightarrow +\infty$.

The paper is organized as follows. In Section 2, we present the model and notation. In Section 3, we describe the estimation method and study the asymptotic properties of estimators in both cases $n\Delta_n$ fixed and $n\Delta_n \rightarrow +\infty$. Section 4 contains some intermediate key results that are of intrinsic interest. Proofs are gathered in Section 5.

2 Setting and notation

We consider $(B_t)_{t \geq 0}$ a Brownian motion and $(L_t^\alpha)_{t \geq 0}$ an independent stable Lévy process with exponent $\alpha \in (1, 2)$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where $(\mathcal{F}_t)_{t \geq 0}$ is the natural augmented filtration. We assume that $(L_t^\alpha)_{t \geq 0}$ is a spectrally positive Lévy

process with characteristic function

$$\mathbb{E}\left(e^{izL_1^\alpha}\right) = \exp\left(-|z|^\alpha\left(1 - i \tan\left(\frac{\pi\alpha}{2}\right) \operatorname{sgn}(z)\right)\right). \quad (2.1)$$

This non-symmetric Lévy process is strictly stable (see for example Sato [18]) and admits the representation

$$L_t^\alpha = \int_0^t \int_{\mathbb{R}_+} z \tilde{N}(ds, dz), \quad \tilde{N} = N - \bar{N},$$

where N is a Poisson random measure with compensator \bar{N} given by

$$\bar{N}(dt, dz) = dt F^\alpha(dz) \quad \text{for} \quad F^\alpha(dz) = \frac{c_\alpha}{z^{\alpha+1}} \mathbf{1}_{(0, +\infty)}(z) dz. \quad (2.2)$$

From Lemma 14.11 in Sato [18], we have

$$c_\alpha = -\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}, \quad \text{with} \quad \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \quad \alpha > 0. \quad (2.3)$$

We will study the estimation of the parameter $\theta = (\sigma^2, \delta, \alpha)$ of the process (stable CIR model) given by the stochastic differential equation

$$dX_t = (a - bX_t)dt + \sigma X_t^{1/2} dB_t + \delta^{1/\alpha} X_t^{1/\alpha} dL_t^\alpha, \quad X_0 = x_0 > 0. \quad (2.4)$$

Assuming $a > 0$, $b \in \mathbb{R}$, $\delta > 0$, $\sigma > 0$, $x_0 > 0$ and $2a \geq \sigma^2$, we have according to Proposition 3.4. in Jiao et al [12] that $(X_t)_t$ is positive.

We use the following notation. For any random variable Z , we write $\mathbb{E}_i(Z) = \mathbb{E}(Z|\mathcal{F}_{i\Delta_n})$ and for $p > 0$, $\mathbb{E}^p(Z) = [\mathbb{E}(Z)]^p$ ($\mathbb{E}_i^p(Z)$ is defined accordingly). For any process Y , we denote the increments with step size Δ_n and the symmetrized increments by

$$\Delta_i^n Y = Y_{(i+1)\Delta_n} - Y_{i\Delta_n}, \quad \Delta_i^{s,n} Y = \Delta_{i+1}^n Y - \Delta_i^n Y.$$

We write \implies for the convergence in distribution, and $\xrightarrow{\mathcal{L}-s}$ for the stable convergence in law. Throughout the paper, we denote by C all positive constants (or C_p if it depends on an auxiliary parameter p), which might change from line to line.

3 Main results

From now on, we assume that we observe $(X_{i\Delta_n})_{0 \leq i \leq n}$ that solves (2.4) for the parameter values (a_0, b_0, θ_0) satisfying assumption **H** below.

H. We assume that $\theta_0 = (\sigma_0^2, \delta_0, \alpha_0) \in \Theta = (0, +\infty) \times (0, +\infty) \times (1, 2)$ and $a_0 > \sigma_0^2$.

Under **H**, $(X_t)_{t \geq 0}$ is positive as previously mentioned, but also satisfies that $\mathbb{E}(1/X_t^2) < \infty$ (see Proposition 4.1). To estimate θ_0 , we propose a moment method based on the increments of X , that extends to stochastic equations the estimation method proposed in Mies [16] for Lévy processes. For $f : \mathbb{R} \mapsto \mathbb{R}^3$, we consider the estimating function $F_n(\theta)$ defined by

$$F_n(\theta) = \frac{1}{n} \sum_{i=0}^{\lfloor n/2 \rfloor - 1} \left[f\left(\frac{u_n}{\sqrt{\Delta_n}} \frac{\Delta_{2i}^{s,n} X}{\sqrt{X_{2i\Delta_n}}}\right) - P_n f(X_{2i\Delta_n}, \theta) \right], \quad (3.1)$$

where the centering term $P_n f(x, \theta)$, defined by (3.2), is constructed on the following way. Considering the Euler approximation of $\Delta_{2i}^{s,n} X$ omitting the drift term

$$\Delta_{2i}^{s,n} \bar{X} = \sigma_0 X_{2i\Delta_n}^{1/2} \Delta_{2i}^{s,n} B + \delta_0^{1/\alpha_0} X_{2i\Delta_n}^{1/\alpha_0} \Delta_{2i}^{s,n} L^{\alpha_0},$$

we approximate $\mathbb{E}_{2i}f\left(\frac{u_n}{\sqrt{\Delta_n}}\frac{\Delta_{2i}^{s,n}X}{\sqrt{X_{2i\Delta_n}}}\right)$ by $\mathbb{E}_{2i}f\left(\frac{u_n}{\sqrt{\Delta_n}}\frac{\Delta_{2i}^{s,n}\bar{X}}{\sqrt{X_{2i\Delta_n}}}\right)$. Now observing that $(\Delta_{2i}^{s,n}B, \Delta_{2i}^{s,n}L^\alpha)$ has the distribution of $(\sqrt{2}\sqrt{\Delta_n}B_1, (2\Delta_n)^{1/\alpha}S_1^\alpha)$, where S_1^α is a symmetric stable random variable, independent of B_1 , with characteristic function $\mathbb{E}e^{izS_1^\alpha} = e^{-|z|^\alpha}$, we have

$$P_n f(x, \theta) = \mathbb{E}_{2i}f\left(\frac{u_n}{\sqrt{\Delta_n}}\frac{\Delta_{2i}^{s,n}\bar{X}}{\sqrt{X_{2i\Delta_n}}}\right) = \mathbb{E}\left[f\left(\sqrt{2}u_n\sigma B_1 + \frac{(2\delta x)^{1/\alpha}}{\sqrt{x}}u_n\Delta_n^{1/\alpha-1/2}S_1^\alpha\right)\right]. \quad (3.2)$$

At this stage, we assume that u_n goes to zero as n goes to infinity and $u_n/\sqrt{\Delta_n}$ goes to infinity, we will precise later the rate of convergence of the sequence (u_n) . We estimate the parameter θ_0 by solving the estimating equation $F_n(\theta) = 0$ for F_n defined by (3.1) with $f = (f_k)_{1 \leq k \leq 3}$, $f_1(x) = \cos(x)$, $f_2(x) = 1 - K(x)$, $f_3(x) = f_2(2x)$ and

$$K(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ (1 + \exp(\frac{1}{2-|x|} - \frac{1}{1-|x|}))^{-1} & \text{if } 1 < |x| < 2, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

The choice of f_1 is natural to estimate σ^2 and has been used in Jacod and Todorov [11]. For this choice, the expression of $P_n f_1$ is explicit, it is the real part of the characteristic function of $u_n\Delta_{2i}^{s,n}\bar{X}/\sqrt{\Delta_n X_{2i\Delta_n}}$

$$P_n f_1(x, \theta) = \exp(-u_n^2\sigma^2 - 2\delta x^{1-\alpha/2}u_n^\alpha\Delta_n^{1-\alpha/2}). \quad (3.3)$$

To estimate δ and α , related to the jump component of the process, we consider here smooth truncations of the small jumps, with different truncation levels. $P_n f_2$ and $P_n f_3$ are not explicit (an asymptotic expansion is given in Lemma 4.2) but easy to compute numerically.

The asymptotic properties of estimators depend on the connections between the number of observations n and the discretization step Δ_n and we study separately the two cases $n\Delta_n$ fixed and $n\Delta_n \rightarrow +\infty$.

3.1 Fixed observation window

We consider that $n\Delta_n$ is fixed and without loss of generality, we assume that $n\Delta_n = 1$. We first prove the existence of a joint estimator with non-diagonal rate of convergence Λ_n defined by

$$\Lambda_n(\theta) = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{u_n^{\alpha/2}\Delta_n^{1/2-\alpha/4}} & -\delta \ln\left(\frac{u_n}{\sqrt{\Delta_n}}\right)\frac{1}{u_n^{\alpha/2}\Delta_n^{1/2-\alpha/4}} \\ 0 & 0 & \frac{1}{u_n^{\alpha/2}\Delta_n^{1/2-\alpha/4}} \end{pmatrix}. \quad (3.4)$$

Theorem 3.1. *We assume \mathbf{H} , $n\Delta_n = 1$ and $u_n = 1/[\ln(1/\Delta_n)]^p$ with $p > 1/2$. Then there exists a sequence of random vectors $(\hat{\theta}_n)$ solving $F_n(\hat{\theta}_n) = 0$ with probability tending to one, i.e. $\mathbb{P}(F_n(\hat{\theta}_n) = 0) \rightarrow 1$. This sequence satisfies $\hat{\theta}_n \rightarrow \theta_0$ in probability as $n \rightarrow \infty$, and*

$$\Lambda_n(\theta_0)^{-1}(\hat{\theta}_n - \theta_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}-s} W(\theta_0)^{-1}\Sigma(\theta_0)^{1/2}\mathcal{N},$$

where \mathcal{N} is a standard Gaussian variable independent of \mathcal{F}_1 . Σ is the symmetric definite positive matrix and W is the non-singular matrix respectively defined by

$$\Sigma(\theta) = \frac{1}{2} \begin{pmatrix} 2\sigma^4 & 0 & 0 \\ 0 & c_\alpha\delta I(\alpha, X) \int_{\mathbb{R}} \frac{f_2^2(v)}{|v|^{\alpha+1}} dv & c_\alpha\delta I(\alpha, X) \int_{\mathbb{R}} \frac{(f_2 f_3)(v)}{|v|^{\alpha+1}} dv \\ 0 & c_\alpha\delta I(\alpha, X) \int_{\mathbb{R}} \frac{(f_2 f_3)(v)}{|v|^{\alpha+1}} dv & c_\alpha\delta I(\alpha, X) \int_{\mathbb{R}} \frac{f_3^2(v)}{|v|^{\alpha+1}} dv \end{pmatrix}, \quad (3.5)$$

$$W(\theta) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\psi(\alpha)I(\alpha, X) & -\delta\partial_\alpha[\psi(\alpha)I(\alpha, X)] \\ 0 & -2^\alpha\psi(\alpha)I(\alpha, X) & -\delta\partial_\alpha[2^\alpha\psi(\alpha)I(\alpha, X)] \end{pmatrix}, \quad (3.6)$$

with $I(\alpha, X) = \int_0^1 X_t^{1-\alpha/2} dt$, $\psi(\alpha) = c_\alpha \int \frac{f_2(v)}{|v|^{\alpha+1}} dv$ and c_α given by (2.3).

We deduce from Theorem 3.1 that $\sqrt{n}(\hat{\sigma}_n^2 - \sigma_0^2) \rightarrow \mathcal{N}(0, 4\sigma_0^4)$, so the estimator is rate optimal and achieves the optimal variance up to a factor 2, which is explained by the use of the symmetrized increments $\Delta_{2i}^{s,n} X$. Optimality for the marginal estimation of (δ, α) has been discussed in Aït-Sahalia and Jacod [2]. For the process $X_t = \sigma B_t + \delta L_T^\alpha$ with $\sigma > 0$, they show that the optimal rates of convergence for the estimation of δ and α are respectively $1/((\log(1/\Delta_n)^\alpha \Delta_n^{\alpha/4})$ and $\log(1/\Delta_n)/((\log(1/\Delta_n)^\alpha \Delta_n^{\alpha/4})$. But from Mies [16] a diagonal normalisation with marginal optimality for each parameter leads to a singular information matrix, suggesting that the optimal rate of convergence in the joint estimation of (δ, α) is non-diagonal with slower rate of convergence by a factor $\log(1/\Delta_n)$ and probably achieved by the choice $u_n = 1/(\log(1/\Delta_n))^{1/2}$ in (3.4). This choice is theoretically possible up to a constant depending on the true value of the parameters since the proof of Lemma 4.2 still works with $u_n = 1/(4\sigma_0(\log(1/\Delta_n))^{1/2})$ (see (5.16)). This choice is not possible in practice, this is why we assume the near optimal condition $u_n = 1/(\log(1/\Delta_n))^p$ with $p > 1/2$ in Theorem 3.1.

We now assume that α_0 is known and only consider the estimation of $\beta = (\sigma^2, \delta)$ by solving $F_n(\sigma^2, \delta, \alpha_0) = 0$, where F_n is defined by (3.1) with $f = (f_1, f_2)$. In that case, we can also prove a uniqueness result, which implies that any solution to the estimating equation is consistent.

Theorem 3.2. *We assume that α_0 is known and that (σ_0^2, δ_0) belongs to the interior of a compact subset of $(0, +\infty) \times (0, +\infty)$. Then under the assumptions of Theorem 3.1 any sequence $(\hat{\sigma}_n^2, \hat{\delta}_n)$ that solves $F_n(\sigma^2, \delta, \alpha_0) = 0$ converges in probability to (σ_0^2, δ_0) . Such a sequence exists and is unique in the sense that if $(\tilde{\sigma}_n^2, \tilde{\delta}_n)$ is another sequence that solves $F_n(\sigma^2, \delta, \alpha_0) = 0$ then $\mathbb{P}((\hat{\sigma}_n^2, \hat{\delta}_n) \neq (\tilde{\sigma}_n^2, \tilde{\delta}_n))$ goes to zero as n goes to infinity. This sequence obviously satisfies the asymptotic properties of Theorem 3.1.*

3.2 Ergodic case

We consider in this section the long time asymptotics $n\Delta_n \rightarrow +\infty$ and we restrict our study to the case $b_0 > 0$. With this assumption, the process $(X_t)_{t \geq 0}$ is geometrically ergodic (we refer to Li and Ma [15] Proposition 2.2 and Theorem 2.5) and the stationary distribution π_0 has Laplace transform

$$\int_0^{+\infty} e^{-\lambda y} \pi_0(dy) = \exp\left(-\int_0^\lambda \frac{a_0 x}{\frac{\sigma_0^2}{2} x^2 + \frac{\bar{\delta}_0^{-\alpha_0}}{\alpha_0} + b_0 x} dx\right),$$

with $\bar{\delta}_0^{-\alpha_0} = \delta_0 \alpha_0 / |\cos(\frac{\pi \alpha_0}{2})|$. Since the Laplace transform is \mathcal{C}^1 at zero, we check easily that $\int_0^\infty x \pi_0(dx) < +\infty$, and setting $\bar{I}(\alpha) = \int_0^\infty x^{1-\alpha/2} \pi_0(dx)$ we deduce that $\alpha \mapsto \bar{I}(\alpha)$ is derivable for $\alpha \in (1, 2)$.

Theorem 3.3. *We assume **H** and $b_0 > 0$, $n\Delta_n \rightarrow +\infty$, $u_n = 1/[\ln(1/\Delta_n)]^p$ with $p > 1/2$, and the additional condition $\sqrt{n}\Delta_n^{1/\alpha_0 + \alpha_0/4 - 1/2 - \epsilon_0} \rightarrow 0$ for $\epsilon_0 > 0$. Then the result of Theorem 3.1 holds with*

$$\Lambda_n(\theta_0)^{-1}(\hat{\theta}_n - \theta_0) \xrightarrow[n \rightarrow \infty]{\Longrightarrow} \bar{W}(\theta_0)^{-1} \bar{\Sigma}(\theta_0)^{1/2} \mathcal{N},$$

where \mathcal{N} is a standard Gaussian variable. $\bar{\Sigma}$ is the symmetric definite positive (deterministic) matrix and \bar{W} is the (deterministic) non-singular matrix respectively defined by (3.5) and (3.6) replacing $I(\alpha, X)$ by $\bar{I}(\alpha) = \int_0^{+\infty} x^{1-\alpha/2} \pi_0(dx)$.

The condition $\sqrt{n}\Delta_n^{1/\alpha_0 + \alpha_0/4 - 1/2 - \epsilon_0} \rightarrow 0$ is compatible with $n\Delta_n \rightarrow +\infty$ if $\alpha_0 \in (1, 2)$ but stronger for α_0 near 2. We also mention that we can also prove in the ergodic case a uniqueness result for the estimation of (σ^2, δ) (similar to Theorem 3.2) if we assume that α_0 is known.

4 Auxiliary results

4.1 Moments and approximation

Since X_t is positive but not lower bounded by some strictly positive constant, we first establish conditions to ensure that $\mathbb{E}(1/X_t^p)$ is finite. We also give some bounds for the moments of $(X_t)_t$ and its increments.

Proposition 4.1. *Let $(X_t)_{t \geq 0}$ be the solution of (2.4) with $a > 0$, $b \in \mathbb{R}$, $\delta > 0$, $\sigma > 0$, $x_0 > 0$ and $2a > \sigma^2$. Then we have*

$$(i) \quad \text{for } 1 \leq p < \frac{2a}{\sigma^2}, \quad \sup_{t \geq 0} \mathbb{E}\left(\frac{1}{X_t^p}\right) < +\infty,$$

and for $1 \leq p < \alpha$, $\exists C_p > 0$ such that $\forall t_1, t_2 \in [0, 1]$ and $\forall i \in \llbracket 0; n-1 \rrbracket$

$$\begin{aligned} (ii) \quad & \mathbb{E}\left(\sup_{t_1 \leq u \leq t_2} X_u^p | \mathcal{F}_{t_1}\right) \leq C_p(1 + X_{t_1}^p), \\ (iii) \quad & \mathbb{E}_i\left(\sup_{s \in [i\Delta_n, (i+1)\Delta_n]} \left| \int_{i\Delta_n}^s X_{t-}^{1/\alpha} dL_t^\alpha \right|^p\right) \leq C_p \Delta_n^{p/\alpha} (1 + X_{i\Delta_n}^{p/\alpha}), \\ (iv) \quad & \mathbb{E}_i\left(\sup_{s \in [i\Delta_n, (i+1)\Delta_n]} |X_s - X_{i\Delta_n}|^p\right) \leq C_p \Delta_n^{p/2} (1 + X_{i\Delta_n}^p), \\ (v) \quad & \mathbb{E}_i\left(\left| \int_{i\Delta_n}^{(i+1)\Delta_n} (X_{s-}^{1/\alpha} - X_{i\Delta_n}^{1/\alpha}) dL_s^\alpha \right|^p\right) \leq C_p \Delta_n^{p/\alpha + p/2} \left(1 + \frac{1}{X_{i\Delta_n}^{p(1-1/\alpha)}} + X_{i\Delta_n}^{p/\alpha}\right). \end{aligned}$$

We next give a sharp bound for the Euler approximation (without drift) of the symmetrized normalized increments of X . This result can be seen as an extension of Lemma 11 in Jacod Todorov [11]. We obtain here a better rate of weak approximation and this result is a key step in the proof of Proposition 4.2. To shorten the notation, we set

$$\rho_i^n = \frac{u_n}{\sqrt{\Delta_n}} \frac{\Delta_i^{s,n} X}{\sqrt{X_{i\Delta_n}}} \quad \text{and} \quad \bar{\rho}_i^n = \frac{u_n}{\sqrt{\Delta_n}} \frac{\Delta_i^{s,n} \bar{X}}{\sqrt{X_{i\Delta_n}}}, \quad (4.1)$$

where $\Delta_i^{s,n} \bar{X} = \sigma_0 X_{i\Delta_n}^{1/2} \Delta_i^{s,n} B + \delta_0^{1/\alpha_0} X_{i\Delta_n}^{1/\alpha_0} \Delta_i^{s,n} L^{\alpha_0}$ and $(X_t)_{t \geq 0}$ solves (2.4) with $a_0 > 0$, $b_0 \in \mathbb{R}$, $\delta_0 > 0$, $\sigma_0 > 0$, $x_0 > 0$ and $2a_0 \geq \sigma_0^2$.

Lemma 4.1. *Let f be an even bounded \mathcal{C}^2 function with bounded derivatives. Then we have (with ρ_i^n and $\bar{\rho}_i^n$ defined by (4.1)) for $\epsilon > 0$, $\exists C_\epsilon$ such that $\forall i \in \llbracket 0; n-1 \rrbracket$*

$$|\mathbb{E}_i(f(\rho_i^n) - f(\bar{\rho}_i^n))| \leq C_\epsilon u_n \Delta_n^{1/\alpha_0 - \epsilon} \left(1 + X_{i\Delta_n}^{1/2} \frac{1}{X_{i\Delta_n}} \left[1 + \frac{1}{\Delta_n} \mathbb{E}_i \int_{i\Delta_n}^{(i+2)\Delta_n} \frac{dv}{X_v}\right]\right), \quad (4.2)$$

and for $1 \leq p < \alpha_0$, $\exists C_p$ such that $\forall i \in \llbracket 0; n-1 \rrbracket$

$$\mathbb{E}_i(|f(\rho_i^n) - f(\bar{\rho}_i^n)|^p) \leq C_p u_n^p \Delta_n^{p/2} (1 + \frac{1}{X_{i\Delta_n}^p} + X_{i\Delta_n}^{p/2}). \quad (4.3)$$

Our estimating equations are based on the computation of the expectation $P_n f(x, \theta)$ given by (3.2). For $f(x) = \cos(x)$ this expectation has an explicit expression but not for $f(x) = 1 - K(x)$. The next lemma gives an asymptotic expansion of $P_n f(x, \theta)$ for functions vanishing in a neighborhood of zero. It is similar to Lemma 6.1 (i) in [16] derived from Itô's Formula. Here we propose a different proof, which easily allows higher order expansion, by using the explicit form of the conditional distribution of the Euler approximation $\Delta_i^{s,n} \bar{X}$ as a convolution between a Gaussian distribution and a Stable distribution.

Lemma 4.2. *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a bounded even function such that $f = 0$ on $[-\eta, +\eta]$ for some $\eta > 0$. We assume that $u_n [\ln(1/\Delta_n)]^{1/2} \rightarrow 0$. Then we have $\forall x > 0$*

$$P_n f(x, \theta) = u_n^\alpha \Delta_n^{1-\alpha/2} c_\alpha \delta x^{1-\alpha/2} \int \frac{f(v)}{|v|^{\alpha+1}} dv + u_n^\alpha \Delta_n^{1-\alpha/2} \varepsilon_n(\theta, x), \quad \text{with } c_\alpha \text{ given by (2.3),}$$

where for any compact subset A of Θ , $\sup_{\theta \in A} |\varepsilon_n(\theta, x)| \leq (1+x)\varepsilon_n$, with $\varepsilon_n \rightarrow 0$.

4.2 Convergences

The asymptotic behavior of the estimator $\hat{\theta}_n$ is derived using the theory of estimating equations (see [10]) and is based on the convergence of $F_n(\theta_0)$ defined by (3.1) and its gradient $\nabla_\theta F_n(\theta)$. We set

$$A_n(\theta) = \sqrt{n} \operatorname{diag} \left(\frac{1}{u_n^2}, \frac{1}{u_n^{\alpha/2} \Delta_n^{1/2-\alpha/4}}, \frac{1}{u_n^{\alpha/2} \Delta_n^{1/2-\alpha/4}} \right). \quad (4.4)$$

We first state a Central Limit Theorem for $F_n(\theta_0)$ with rate $A_n(\theta_0)$.

Proposition 4.2. *We assume \mathbf{H} and $u_n = 1/[\ln(1/\Delta_n)]^p$ with $p > 1/2$.*

1. *If $n\Delta_n = 1$, then we have the stable convergence in law*

$$A_n(\theta_0)F_n(\theta_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}\text{-}s} \Sigma(\theta_0)^{1/2}\mathcal{N},$$

where \mathcal{N} is a standard Gaussian variable independent of \mathcal{F}_1 and $\Sigma(\theta_0)$ is the symmetric definite positive matrix defined by (3.5).

2. *If $n\Delta_n \rightarrow +\infty$, $b_0 > 0$ and $\sqrt{n}\Delta_n^{1/\alpha_0 + \alpha_0/4 - 1/2 - \epsilon_0} \rightarrow 0$ for some $\epsilon_0 > 0$, then we have the convergence in distribution*

$$A_n(\theta_0)F_n(\theta_0) \xrightarrow[n \rightarrow \infty]{} \bar{\Sigma}(\theta_0)^{1/2}\mathcal{N},$$

where \mathcal{N} is a standard Gaussian variable and $\bar{\Sigma}(\theta_0)$ is the symmetric definite positive matrix defined by (3.5) replacing $I(\alpha, X)$ by $\bar{I}(\alpha) = \int_0^{+\infty} x^{1-\alpha/2} \pi_0(dx)$.

Turning to the convergence of $\nabla_\theta F_n$ and recalling that $\Lambda_n(\theta)$ is defined by (3.4), we obtain the following convergence.

Proposition 4.3. *Let A be a compact subset of Θ . We assume \mathbf{H} and $u_n = 1/[\ln(1/\Delta_n)]^p$ with $p > 1/2$.*

1. *If $n\Delta_n = 1$, then we have the convergence in probability*

$$\sup_{\theta \in A} |A_n(\theta) \nabla_\theta F_n(\theta) \Lambda_n(\theta) - W(\theta)| \rightarrow 0,$$

where $W(\theta)$ is the non-singular matrix defined by (3.6).

2. *If $n\Delta_n \rightarrow +\infty$ and $b_0 > 0$, then we have the convergence in probability*

$$\sup_{\theta \in A} |A_n(\theta) \nabla_\theta F_n(\theta) \Lambda_n(\theta) - \bar{W}(\theta)| \rightarrow 0,$$

where $\bar{W}(\theta)$ is defined by (3.6) replacing $I(\alpha, X)$ by $\bar{I}(\alpha) = \int_0^{+\infty} x^{1-\alpha/2} \pi_0(dx)$.

5 Proofs

5.1 Auxiliary results

5.1.1 Proof of Proposition 4.1

We only prove (i) since the other items can be deduced with similar arguments as the ones given for the proof of Proposition 2.2 in [5]. The Laplace transform of X_t has the explicit expression (we refer to Li [14], see also Section 3.1 in [12] and Proposition 2.1. in [4])

$$\mathbb{E}(e^{-uX_t}) = \exp \left(-x_0 v_t(u) - \int_{v_t(u)}^u \frac{F(z)}{R(z)} dz \right), \quad (5.1)$$

where $t \rightarrow v_t(u)$ is the unique locally bounded solution of

$$\frac{\partial}{\partial t} v_t(u) = -R(v_t(u)), \quad v_0(u) = u. \quad (5.2)$$

For the α -CIR model, the branching mechanism and the immigration rate are given by

$$R(z) = \frac{\sigma^2}{2} z^2 + \frac{\bar{\delta}^\alpha}{\alpha} z^\alpha + bz, \quad F(z) = az, \quad z \in [0, +\infty),$$

with $\bar{\delta} = \delta^{1/\alpha} (\alpha / |\cos(\frac{\pi\alpha}{2})|)^{1/\alpha}$. Setting $z_0 = \inf\{z > 0, R(z) \geq 0\}$, $z_0 = 0$ when $b \geq 0$ and $z_0 > 0$ when $b < 0$ with $z_0 < +\infty$ as $\lim_{z \rightarrow +\infty} R(z) = +\infty$. From Corollary 3.2 of Li [14], $\lim_{t \rightarrow \infty} v_t(u) = z_0$ and $\forall u > z_0$, $t \rightarrow v_t(u)$ is decreasing. Now we have $\forall t \geq 0$

$$\mathbb{E}\left(\frac{1}{X_t^p}\right) = C_p \int_{\mathbb{R}_+} u^{p-1} \mathbb{E}(e^{-uX_t}) du \leq C_p(z_0) + C_p \int_{z_0+1}^{+\infty} u^{p-1} \mathbb{E}(e^{-uX_t}) du.$$

We fix $\epsilon \in (0, 1)$ and for $u \geq z_0 + 1$ we set $t_u^\epsilon = \inf\{t \geq 0, v_t(u) \leq u^\epsilon\}$. With $u_{0,\epsilon} = \max(z_0 + 1, z_0^{1/\epsilon})$

$$\sup_{t \geq 0} \int_{u_{0,\epsilon}}^{+\infty} u^{p-1} \mathbb{E}(e^{-uX_t}) du \leq \int_{u_{0,\epsilon}}^{+\infty} u^{p-1} \left[\sup_{t \in [0, t_u^\epsilon]} \mathbb{E}(e^{-uX_t}) + \sup_{t \geq t_u^\epsilon} \mathbb{E}(e^{-uX_t}) \right] du. \quad (5.3)$$

Using that $t \rightarrow v_t(u)$ is non-increasing for $u > z_0$, we obtain from (5.1) and observing that $v_{t_u^\epsilon}(u) = u^\epsilon$

$$\begin{aligned} \sup_{t \in [0, t_u^\epsilon]} \mathbb{E}(e^{-uX_t}) &\leq \sup_{t \in [0, t_u^\epsilon]} \exp(-x_0 v_t(u)) \leq \exp(-x_0 u^\epsilon), \\ \sup_{t \geq t_u^\epsilon} \mathbb{E}(e^{-uX_t}) &\leq \sup_{t \geq t_u^\epsilon} \exp\left(-\int_{v_t(u)}^u \frac{F(z)}{R(z)} dz\right) \leq \exp\left(-\int_{u^\epsilon}^u \frac{F(z)}{R(z)} dz\right). \end{aligned}$$

For $z \in [u^\epsilon, u]$

$$\frac{F(z)}{R(z)} \geq \frac{2a}{\sigma^2} \frac{1}{z(1 + \frac{\bar{\delta}^\alpha 2}{\alpha \sigma^2 z^{2-\alpha}}) + \frac{2b}{\sigma^2}} \geq \frac{2a}{\sigma^2} \frac{1}{zg(u) + \frac{2b}{\sigma^2}}, \quad \text{with } g(u) = 1 + \frac{\bar{\delta}^\alpha 2}{\alpha \sigma^2 u^{\epsilon(2-\alpha)}}.$$

Hence integrating the previous inequality

$$\int_{u^\epsilon}^u \frac{F(z)}{R(z)} dz \geq \frac{2a}{\sigma^2 g(u)} \ln \left(\frac{ug(u) + \frac{2b}{\sigma^2}}{u^\epsilon g(u) + \frac{2b}{\sigma^2}} \right) = \frac{2a}{\sigma^2 g(u)} [\ln(u^{1-\epsilon}) + h(u)],$$

with

$$h(u) = \ln \left(\frac{g(u) + \frac{2b}{u\sigma^2}}{g(u) + \frac{2b}{u^\epsilon \sigma^2}} \right).$$

We have $g(u) \geq 1$, $\lim_{u \rightarrow +\infty} g(u) = 1$ and $\lim_{u \rightarrow +\infty} h(u) = 0$, then

$$\exp\left(-\int_{u^\epsilon}^u \frac{F(z)}{R(z)} dz\right) \leq C_\epsilon / u^{(1-\epsilon) \frac{2a}{\sigma^2 g(u)}}.$$

If $1 \leq p < \frac{2a}{\sigma^2}$, we choose $\epsilon > 0$ such that $p - \frac{2a}{\sigma^2} + \epsilon \frac{2a}{\sigma^2} < 0$, then we deduce

$$\int_{u_{0,\epsilon}}^{+\infty} u^{p-1} \exp\left(-\int_{u^\epsilon}^u \frac{F(z)}{R(z)} dz\right) du < +\infty,$$

and this allows us to conclude.

5.1.2 Proof of Lemma 4.1

We introduce a little more notation. We set

$$a_t = \mathbf{1}_{\{(i+1)\Delta_n \leq t \leq (i+2)\Delta_n\}} - \mathbf{1}_{\{i\Delta_n \leq t < (i+1)\Delta_n\}}, \quad t \in [i\Delta_n, (i+2)\Delta_n].$$

With this notation, we have

$$\Delta_i^{s,n} X = \int_{i\Delta_n}^{(i+2)\Delta_n} a_t dX_t \quad \text{and} \quad \Delta_i^{s,n} \bar{X} = \sigma_0 X_{i\Delta_n}^{1/2} \int_{i\Delta_n}^{(i+2)\Delta_n} a_t dB_t + \delta_0^{1/\alpha_0} X_{i\Delta_n}^{1/\alpha_0} \int_{i\Delta_n}^{(i+2)\Delta_n} a_t dL_t^{\alpha_0}.$$

We then deduce the decomposition $\rho_i^n = \bar{\rho}_i^n + \theta(1)_i^n + \theta(2)_i^n + \theta(3)_i^n$ with

$$\begin{aligned} \theta(1)_i^n &= \sigma_0 \frac{u_n}{\sqrt{\Delta_n} \sqrt{X_{i\Delta_n}}} \int_{i\Delta_n}^{(i+2)\Delta_n} a_t (X_t^{1/2} - X_{i\Delta_n}^{1/2}) dB_t, \\ \theta(2)_i^n &= \delta_0^{1/\alpha_0} \frac{u_n}{\sqrt{\Delta_n} \sqrt{X_{i\Delta_n}}} \int_{i\Delta_n}^{(i+2)\Delta_n} a_t (X_t^{1/\alpha_0} - X_{i\Delta_n}^{1/\alpha_0}) dL_t^{\alpha_0}, \\ \theta(3)_i^n &= -b_0 \frac{u_n}{\sqrt{\Delta_n} \sqrt{X_{i\Delta_n}}} \int_{i\Delta_n}^{(i+2)\Delta_n} a_t (X_t - X_{i\Delta_n}) dt. \end{aligned}$$

We have the following estimates

$$\forall p \in [1, \alpha_0), \mathbb{E}_i(|\theta(1)_i^n|^p) \leq C_p u_n^p \Delta_n^{p/2} \left(1 + \frac{1}{X_{i\Delta_n}^p}\right), \quad (5.4)$$

$$\forall p \in [1, \alpha_0), \mathbb{E}_i(|\theta(2)_i^n|^p) \leq C_p u_n^p \Delta_n^{p/\alpha_0} \left(1 + \frac{1}{X_{i\Delta_n}^{p/2+p(1-1/\alpha_0)}} + X_{i\Delta_n}^{p/\alpha_0-p/2}\right), \quad (5.5)$$

$$\forall p \in [1, \alpha_0), \mathbb{E}_i(|\theta(3)_i^n|^p) \leq C_p u_n^p \Delta_n^p \left(1 + \frac{1}{X_{i\Delta_n}^{p/2}} + X_{i\Delta_n}^{p/2}\right). \quad (5.6)$$

We deduce easily (5.5) and (5.6) from Proposition 4.1, (5.4) follows from Burkholder's inequality, Proposition 4.1 and the bound

$$\forall x, y \in (0, +\infty) |y^{1/2} - x^{1/2}| \leq C|y - x|/x^{1/2}.$$

Combining the estimates (5.4), (5.5), (5.6) and $|f(x+y) - f(x)| \leq C|y|$ as f' is bounded, we deduce immediately (4.3).

We now prove (4.2). With the previous notation, we have the decomposition

$$\begin{aligned} f(\rho_i^n) - f(\bar{\rho}_i^n) &= f(\rho_i^n) - f(\bar{\rho}_i^n + \theta(1)_i^n + \theta(2)_i^n) + f(\bar{\rho}_i^n + \theta(1)_i^n + \theta(2)_i^n) - f(\bar{\rho}_i^n + \theta(1)_i^n) \\ &\quad + f(\bar{\rho}_i^n + \theta(1)_i^n) - f(\bar{\rho}_i^n), \end{aligned}$$

and since f' is bounded it yields

$$|\mathbb{E}_i(f(\rho_i^n) - f(\bar{\rho}_i^n))| \leq C(\mathbb{E}_i|\theta(3)_i^n| + \mathbb{E}_i|\theta(2)_i^n| + |\mathbb{E}_i(f(\bar{\rho}_i^n + \theta(1)_i^n) - f(\bar{\rho}_i^n))|).$$

Using (5.5) and (5.6) with $p = 1$

$$\mathbb{E}_i|\theta(3)_i^n| + \mathbb{E}_i|\theta(2)_i^n| \leq C u_n \Delta_n^{1/\alpha_0} \left(1 + \frac{1}{X_{i\Delta_n}^{3/2-1/\alpha_0}} + X_{i\Delta_n}^{1/2}\right). \quad (5.7)$$

It remains to bound $|\mathbb{E}_i(f(\bar{\rho}_i^n + \theta(1)_i^n) - f(\bar{\rho}_i^n))|$. Since f'' is bounded, we have

$$|\mathbb{E}_i(f(\bar{\rho}_i^n + \theta(1)_i^n) - f(\bar{\rho}_i^n))| \leq |\mathbb{E}_i(\theta(1)_i^n f'(\bar{\rho}_i^n))| + C \mathbb{E}_i|\theta(1)_i^n|^2.$$

To finish the proof, we show the following estimates

$$\mathbb{E}_i |\theta(1)_i^n|^2 \leq C u_n^2 \Delta_n \left(1 + \frac{1}{X_{i\Delta_n}} \left[1 + \frac{1}{\Delta_n} \mathbb{E}_i \int_{i\Delta_n}^{(i+2)\Delta_n} \frac{dv}{X_v} \right] \right), \quad (5.8)$$

and for $\epsilon > 0$

$$|\mathbb{E}_i(\theta(1)_i^n f'(\bar{\rho}_i^n))| \leq C_\epsilon u_n \Delta_n^{1/\alpha_0 - \epsilon} \left(1 + \frac{1}{X_{i\Delta_n}^{1/2}} \left[1 + \frac{1}{\Delta_n} \mathbb{E}_i \int_{i\Delta_n}^{(i+2)\Delta_n} \frac{dv}{X_v^{1/2}} \right] \right). \quad (5.9)$$

Collecting (5.7), (5.8) and (5.9), we deduce (4.2).

Proof of (5.8) We write Itô's formula for the function $x \rightarrow \sqrt{x}$ which is C^2 on $(0, +\infty)$

$$\sqrt{X_t} - \sqrt{X_s} = \int_s^t \bar{b}_v dv + \int_s^t \frac{\sigma_0}{2} dB_v + \int_s^t \int \bar{c}_{v-} \tilde{N}(dv, dz),$$

where

$$\bar{b}_v = \frac{(a_0 - b_0 X_v)}{2X_v^{1/2}} - \frac{\sigma_0^2}{8X_v^{1/2}} + \int (\sqrt{X_{v-} + \delta_0^{1/\alpha_0} X_{v-}^{1/\alpha_0} z} - \sqrt{X_{v-}} - \frac{1}{2X_{v-}^{1/2}} \delta_0^{1/\alpha_0} X_{v-}^{1/\alpha_0} z) F^{\alpha_0}(z) dz,$$

$$\bar{c}_{v-} = (X_{v-} + \delta_0^{1/\alpha_0} X_{v-}^{1/\alpha_0} z)^{1/2} - (X_{v-})^{1/2}.$$

Moreover, we check that $|\bar{b}_v| \leq C(1 + \frac{1}{\sqrt{X_v}} + \sqrt{X_v})$ and that $|\bar{c}_v| \leq C(1 + \sqrt{X_v})(\sqrt{z}\mathbf{1}_{\{z>1\}} + z\mathbf{1}_{\{0<z\leq 1\}})$. We obtain the decomposition

$$\theta(1)_i^n = \mu(1)_i^n + \mu(2)_i^n + \mu(3)_i^n, \quad (5.10)$$

with

$$\begin{aligned} \mu(1)_i^n &= \frac{u_n \sigma_0}{\sqrt{\Delta_n X_{i\Delta_n}}} \int_{i\Delta_n}^{(i+2)\Delta_n} a_t \left(\int_{i\Delta_n}^t \bar{b}_v dv \right) dB_t, \\ \mu(2)_i^n &= \frac{u_n \sigma_0}{\sqrt{\Delta_n X_{i\Delta_n}}} \frac{\sigma_0}{2} \int_{i\Delta_n}^{(i+2)\Delta_n} a_t (B_t - B_{i\Delta_n}) dB_t, \\ \mu(3)_i^n &= \frac{u_n \sigma_0}{\sqrt{\Delta_n X_{i\Delta_n}}} \int_{i\Delta_n}^{(i+2)\Delta_n} a_t \left(\int_{i\Delta_n}^t \int \bar{c}_{v-} \tilde{N}(dv, dz) \right) dB_t. \end{aligned}$$

From Burkholder's inequality and Proposition 4.1 (ii), we have

$$\forall p \in [1, 2], \mathbb{E}_i |\mu(1)_i^n|^p \leq C_p u_n^p \Delta_n^p \left(1 + \frac{1}{X_{i\Delta_n}^{p/2}} \left[1 + \frac{1}{\Delta_n} \mathbb{E}_i \int_{i\Delta_n}^{(i+2)\Delta_n} \frac{dv}{X_v^{p/2}} \right] \right), \quad (5.11)$$

$$\forall p \geq 1, \mathbb{E}_i |\mu(2)_i^n|^p \leq C_p \frac{u_n^p}{X_{i\Delta_n}^{p/2}} \Delta_n^{p/2}, \quad (5.12)$$

$$\forall p \in (\alpha_0, 2], \mathbb{E}_i |\mu(3)_i^n|^p \leq C_p u_n^p \Delta_n \left(1 + \frac{1}{X_{i\Delta_n}^{p/2}} \right). \quad (5.13)$$

Note that to obtain (5.13), we have used

$$\mathbb{E}_i \sup_{i\Delta_n \leq t \leq (i+2)\Delta_n} \left| \int_{i\Delta_n}^t \int \bar{c}_{v-} \tilde{N}(dv, dz) \right|^p \leq C_p \mathbb{E}_i \int_{i\Delta_n}^{(i+2)\Delta_n} \int |\bar{c}_{v-}|^p \bar{N}(dv, dz),$$

combined with $|\bar{c}_v| \leq C(1 + \sqrt{X_v})(\sqrt{z}\mathbf{1}_{\{z>1\}} + z\mathbf{1}_{\{0<z\leq 1\}})$. We then deduce (5.8) from the previous bounds with $p = 2$.

Proof of (5.9) Since f' is bounded, we have using (5.10) and Hölder's inequality with $\alpha_0 < p \leq 2$

$$|\mathbb{E}_i(\theta(1)_i^n f'(\bar{\rho}_i^n))| \leq C(\mathbb{E}_i|\mu(1)_i^n| + |\mathbb{E}_i(\mu(2)_i^n f'(\bar{\rho}_i^n))| + \mathbb{E}_i^{1/p}|\mu(3)_i^n|^p). \quad (5.14)$$

It remains to consider $\mathbb{E}_i(\mu(2)_i^n f'(\bar{\rho}_i^n))$. We split $\bar{\rho}_i^n$ in two parts (Brownian part and pure-jump part)

$$\bar{\rho}_i^n = \frac{\sigma_0 u_n}{\sqrt{\Delta_n}} \Delta_i^{s,n} B + \frac{\delta_0^{1/\alpha_0} X_{i\Delta_n}^{1/\alpha_0} u_n}{\sqrt{\Delta_n} \sqrt{X_{i\Delta_n}}} \Delta_i^{s,n} L^{\alpha_0} := \bar{\rho}_i^{1,n} + \bar{\rho}_i^{2,n},$$

and write

$$\mathbb{E}_i(\mu(2)_i^n f'(\bar{\rho}_i^n)) = \mathbb{E}_i(\mu(2)_i^n f'(\bar{\rho}_i^{1,n})) + \mathbb{E}_i(\mu(2)_i^n (f'(\bar{\rho}_i^n) - f'(\bar{\rho}_i^{1,n}))).$$

Since f' is odd, an explicit computation gives $\mathbb{E}_i(\mu(2)_i^n f'(\bar{\rho}_i^{1,n})) = 0$. For the second term, we use successively that f'' is bounded, Hölder's inequality with $1 < q < \alpha_0$ and the scaling property of L^{α_0} to obtain

$$\begin{aligned} \mathbb{E}_i|\mu(2)_i^n (f'(\bar{\rho}_i^n) - f'(\bar{\rho}_i^{1,n}))| &\leq C \mathbb{E}_i|\mu(2)_i^n \bar{\rho}_i^{2,n}| \leq C_q \frac{u_n}{\sqrt{\Delta_n}} (1 + X_{i\Delta_n}^{1/2}) \mathbb{E}_i^{\frac{1}{q}} |\mu(2)_i^n|^{q'} \mathbb{E}_i^{\frac{1}{q}} |\Delta_i^{s,n} L^{\alpha_0}|^q \\ &\leq C_q u_n^2 (1 + \frac{1}{X_{i\Delta_n}^{1/2}}) \Delta_n^{1/\alpha_0}. \end{aligned} \quad (5.15)$$

We finally deduce (5.9) from (5.14) (using (5.11) with $p = 1$ and (5.13) with $p = \alpha_0 + \epsilon$) and (5.15).

5.1.3 Proof of Lemma 4.2

We assume that $\theta = (\sigma^2, \delta, \alpha) \in A$ a compact subset of Θ . We denote by g_α the density of S_1^α , by Φ the density of the standard Gaussian distribution and we set $a_n(x, \theta) = (2\delta)^{1/\alpha} u_n (x \Delta_n)^{1/\alpha - 1/2}$. Then

$$P_n f(x, \theta) = 2 \int_{\mathbb{R}_+} f(v) dv \int_{\mathbb{R}} g_\alpha \left(\frac{v-y}{a_n(x, \theta)} \right) \frac{1}{a_n(x, \theta)} \Phi \left(\frac{y}{\sqrt{2} u_n \sigma} \right) \frac{1}{\sqrt{2} u_n \sigma} dy.$$

Using successively a change of variables and Fubini's Theorem yields

$$P_n f(x, \theta) = 2 \int_{\mathbb{R}} \Phi(z) dz \int_{\mathbb{R}_+} f(v) g_\alpha \left(\frac{v - \sqrt{2} u_n \sigma z}{a_n(x, \theta)} \right) \frac{1}{a_n(x, \theta)} dv.$$

Recalling that $f = 0$ on $[-\eta, \eta]$, we split the previous integral in two parts $P_n f(x, \theta) = I_n^1 + I_n^2$ where

$$\begin{aligned} I_n^1 &= 2 \int_{\{|z| > \frac{\eta}{2\sqrt{2} u_n \sigma}\}} \Phi(z) dz \int_{\mathbb{R}_+} f(v) g_\alpha \left(\frac{v - \sqrt{2} u_n \sigma z}{a_n(x, \theta)} \right) \frac{1}{a_n(x, \theta)} dv, \\ I_n^2 &= 2 \int_{\{|z| \leq \frac{\eta}{2\sqrt{2} u_n \sigma}\}} \Phi(z) dz \int_{\{v > \eta\}} f(v) g_\alpha \left(\frac{v - \sqrt{2} u_n \sigma z}{a_n(x, \theta)} \right) \frac{1}{a_n(x, \theta)} dv. \end{aligned}$$

By assumption, $u_n^2 = \bar{u}_n / \ln(1/\Delta_n)$ with $\bar{u}_n \rightarrow 0$, so we deduce immediately, since f is bounded, that

$$|I_n^1| \leq C \exp\left(-\frac{\eta^2}{16 u_n^2 \sigma^2}\right) \leq C \Delta_n^{C'/\bar{u}_n}, \quad (5.16)$$

where C and C' do not depend on θ . Turning to I_n^2 , we observe that for $|z| \leq \frac{\eta}{2\sqrt{2} u_n \sigma}$ and $v > \eta$,

$$\frac{v - \sqrt{2} u_n \sigma z}{a_n(x, \theta)} \geq \frac{\eta}{2 a_n(x, \theta)} > 0.$$

Using the asymptotic expansion of g_α (see Sato [18]), we have

$$\forall y > 0 \quad |g_\alpha(y) - \frac{c_\alpha}{2y^{\alpha+1}}| \leq \frac{C}{y^{2\alpha+1}}, \text{ with } c_\alpha \text{ given by (2.3)}$$

and we deduce

$$g_\alpha \left(\frac{v - \sqrt{2}u_n\sigma z}{a_n(x, \theta)} \right) = \frac{c_\alpha a_n(x, \theta)^{\alpha+1}}{2v^{\alpha+1}} \frac{1}{(1 - \sqrt{2}\sigma u_n z/v)^{\alpha+1}} (1 + R_n^0(z, v)),$$

with

$$|R_n^0(z, v)| \leq C \frac{a_n(x, \theta)^\alpha}{v^\alpha} \frac{1}{(1 - \sqrt{2}\sigma u_n z/v)^\alpha} \leq C \frac{a_n(x, \theta)^\alpha}{v^\alpha},$$

where we used $|\sqrt{2}\sigma u_n z/v| \leq 1/2$ for the last inequality. Moreover, we have the first order expansion

$$\frac{1}{(1 - \sqrt{2}\sigma u_n z/v)^{\alpha+1}} = 1 + R_n^1(z, v), \text{ with } |R_n^1(z, v)| \leq C u_n z/v.$$

Reporting these results in I_n^2 , we obtain

$$\begin{aligned} I_n^2 &= \int_{\{|z| \leq \frac{\eta}{2\sqrt{2}u_n\sigma}\}} \Phi(z) dz \int_{\{v > \eta\}} f(v) \frac{c_\alpha a_n(x, \theta)^\alpha}{v^{\alpha+1}} (1 + R_n^1(z, v))(1 + R_n^0(z, v)) dv \\ &= c_\alpha a_n(x, \theta)^\alpha \left(\int_{\{v > \eta\}} \frac{f(v)}{v^{\alpha+1}} dv + \int_{\{|z| > \frac{\eta}{2\sqrt{2}u_n\sigma}\}} \Phi(z) dz \int_{\{v > \eta\}} \frac{f(v)}{v^{\alpha+1}} dv \right. \\ &\quad \left. + \int_{\{|z| \leq \frac{\eta}{2\sqrt{2}u_n\sigma}\}} \Phi(z) dz \int_{\{v > \eta\}} \frac{f(v)}{v^{\alpha+1}} [R_n^1(z, v)(1 + R_n^0(z, v)) + R_n^0(z, v)] dv \right). \end{aligned}$$

This finally gives

$$I_n^2 = c_\alpha a_n(x, \theta)^\alpha \left(\int_{\{v > \eta\}} \frac{f(v)}{v^{\alpha+1}} dv + R_n \right), \quad (5.17)$$

with

$$|R_n| \leq C(\Delta_n^{C'/\bar{u}_n} + u_n + u_n^\alpha (x\Delta_n)^{1-\alpha/2}) \leq C(1 + \sqrt{x})(\Delta_n^{C'/\bar{u}_n} + u_n).$$

From (5.16) and (5.17), we deduce the result of Lemma 4.2.

5.1.4 Proof of Proposition 4.2

The proof consists first to replace, in the expression of $F_n(\theta_0)$, ρ_{2i}^n by $\bar{\rho}_{2i}^n$ (defined by (4.1)) by using Lemma 4.1 and next to study the convergence in distribution of

$$A_n(\theta_0) \frac{1}{n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} [f(\bar{\rho}_{2i}^n) - P_n f(X_{2i\Delta_n}, \theta_0)].$$

We set

$$\zeta_{2i}^n = A_n(\theta_0) \frac{1}{n} [f(\bar{\rho}_{2i}^n) - P_n f(X_{2i\Delta_n}, \theta_0)], \quad (5.18)$$

hence $\zeta_{2i}^n = (\zeta_{2i}^{k,n})_{1 \leq k \leq 3}$ with

$$\begin{aligned} \zeta_{2i}^{1,n} &= \frac{1}{u_n^2 \sqrt{n}} [f_1(\bar{\rho}_{2i}^n) - P_n f_1(X_{2i\Delta_n}, \theta_0)], \\ \zeta_{2i}^{k,n} &= \frac{1}{u_n^{\alpha_0/2} \Delta_n^{1/2 - \alpha_0/4} \sqrt{n}} [f_k(\bar{\rho}_{2i}^n) - P_n f_k(X_{2i\Delta_n}, \theta_0)], \text{ for } k \in \{2, 3\}. \end{aligned}$$

1. Case $n\Delta_n = 1$. We first remark that for any $0 < p \leq 1$ we have the convergence in probability

$$\frac{1}{n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} X_{2i\Delta_n}^p \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \frac{1}{2} \int_0^1 X_t^p dt.$$

Moreover from Proposition 4.1 and recalling $a_0 > \sigma_0^2$, we have for $1 \leq p \leq 2$

$$\mathbb{E} \left(\frac{1}{n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{1}{X_{i\Delta_n}^p} \right) \leq \sup_{t \geq 0} \mathbb{E} \left(\frac{1}{X_t^p} \right) < +\infty, \quad (5.19)$$

and using $1/(X_{i\Delta_n} X_v) \leq 1/(2X_{i\Delta_n}^2) + 1/(2X_v^2)$

$$\mathbb{E} \left(\frac{1}{n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{1}{\Delta_n} \mathbb{E}_i \int_{i\Delta_n}^{(i+2)\Delta_n} \frac{dv}{X_{i\Delta_n} X_v} \right) \leq \sup_{t \geq 0} \mathbb{E} \left(\frac{1}{X_t^2} \right) < +\infty. \quad (5.20)$$

Next from Lemma 4.1 and choosing $\epsilon > 0$ such that $1/\alpha_0 - \epsilon > 1/2$ and $1/\alpha_0 - \epsilon > 1 - \alpha_0/4$, which is possible since $1 < \alpha_0 < 2$, we have the convergence in probability

$$A_n(\theta_0) \frac{1}{n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} |\mathbb{E}_{2i}(f(\rho_{2i}^n) - f(\bar{\rho}_{2i}^n))| \rightarrow 0.$$

Observing that for f bounded $|\mathbb{E}_{2i}(f(\bar{\rho}_{2i}^n) - P_n f(X_{2i\Delta_n}, \theta_0))|^2 \leq C \mathbb{E}_{2i}(f(\bar{\rho}_{2i}^n) - P_n f(X_{2i\Delta_n}, \theta_0))^2$ with $p < \alpha_0$, we check with Lemma 4.1

$$\begin{aligned} \frac{n}{u_n^4} \frac{1}{n^2} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i} |f_1(\rho_{2i}^n) - f_1(\bar{\rho}_{2i}^n)|^2 &\rightarrow 0, \\ \frac{n}{u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2}} \frac{1}{n^2} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i} |f_k(\rho_{2i}^n) - f_k(\bar{\rho}_{2i}^n)|^2 &\rightarrow 0 \quad k \in \{2, 3\}. \end{aligned}$$

Consequently, we have the convergence in probability

$$A_n(\theta_0) \frac{1}{n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} [f(\rho_{2i}^n) - f(\bar{\rho}_{2i}^n)] \rightarrow 0, \quad (5.21)$$

and the proof of Proposition 4.2 reduces to show the stable convergence

$$A_n(\theta_0) \frac{1}{n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} [f(\bar{\rho}_{2i}^n) - P_n f(X_{2i\Delta_n}, \theta_0)] \xrightarrow[n \rightarrow \infty]{\mathcal{L}-s} \Sigma(\theta_0)^{1/2} \mathcal{N},$$

for \mathcal{N} a standard Gaussian variable independent of $\Sigma(\theta_0)$ defined by (3.5). According to Theorem IX.7.26 in Jacod and Shiryaev [9], it is sufficient to show the following convergences

in probability (with ζ_{2i}^n defined by (5.18))

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i}(\zeta_{2i}^n) \rightarrow 0, \quad (5.22)$$

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i}(|\zeta_{2i}^{k,n}|^p) \rightarrow 0 \quad \text{for some } p > 2, \quad k = 1, 2, 3, \quad (5.23)$$

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i}(\zeta_{2i}^{k,n} \zeta_{2i}^{l,n}) \rightarrow \Sigma_{k,l}(\theta_0) \quad \text{for } k, l = 1, 2, 3, \quad (5.24)$$

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i}(\zeta_{2i}^{k,n} (M_{2(i+1)\Delta_n} - M_{2i\Delta_n})) \rightarrow 0 \quad \text{for } k = 1, 2, 3, \quad (5.25)$$

where the last convergence holds for any square integrable martingale M . The convergence (5.22) is immediate since $\mathbb{E}_{2i}f(\bar{\rho}_{2i}^n) = P_n f(X_{2i\Delta_n}, \theta_0)$ and (5.25) is addressed in Lemma 5.1 below.

We deduce (5.23) from Lemma 4.2 for $k = 2$ and $k = 3$. For $k = 1$, we recall that using the notation (4.1) with $f_1(x) = \cos(x)$

$$P_n f_1(X_{2i\Delta_n}, \theta_0) = \mathbb{E}_{2i} f_1(\bar{\rho}_{2i}^n) = \exp(-u_n^2 \sigma_0^2 - 2\delta_0 X_{2i\Delta_n}^{1-1/\alpha_0} u_n^{\alpha_0} \Delta_n^{1-1/\alpha_0}), \quad (5.26)$$

and

$$\mathbb{E}_{2i} f_1(2\bar{\rho}_{2i}^n) = \exp(-4u_n^2 \sigma_0^2 - 2^{1+\alpha_0} \delta_0 X_{2i\Delta_n}^{1-1/\alpha_0} u_n^{\alpha_0} \Delta_n^{1-1/\alpha_0}).$$

Using $\cos^2(a) = (1 + \cos(2a))/2$, we deduce

$$\mathbb{E}_{2i}(f_1(\bar{\rho}_{2i}^n)^2) = (1 + \mathbb{E}_{2i} f_1(2\bar{\rho}_{2i}^n))/2,$$

and for $q \geq 2$, since f_1 is bounded

$$\mathbb{E}_{2i}(|f_1(\bar{\rho}_{2i}^n) - P_n f_1(X_{2i\Delta_n}, \theta_0)|^q) \leq \mathbb{E}_{2i}(f_1(\bar{\rho}_{2i}^n) - \mathbb{E}_{2i} f_1(\bar{\rho}_{2i}^n))^2 = \frac{\mathbb{E}_{2i} f_1(2\bar{\rho}_{2i}^n) - 2\mathbb{E}_{2i}^2 f_1(\bar{\rho}_{2i}^n) + 1}{2}. \quad (5.27)$$

We now expand $\mathbb{E}_{2i} f_1(\bar{\rho}_{2i}^n)$ and $\mathbb{E}_{2i} f_1(2\bar{\rho}_{2i}^n)$ around 0 and get that

$$\mathbb{E}_{2i} f_1(2\bar{\rho}_{2i}^n) - 2\mathbb{E}_{2i}^2 f_1(\bar{\rho}_{2i}^n) + 1 = 4u_n^4 \sigma_0^4 + R_n(x, \theta_0), \quad \text{with } |R_n(x, \theta_0)| \leq C(u_n^6 + u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2})(1+x). \quad (5.28)$$

This gives (5.23) for $k = 1$.

We now prove (5.24). From (5.27) and (5.28) we have

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i}(\zeta_{2i}^{1,n})^2 = \frac{1}{n} \frac{1}{u_n^4} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{\mathbb{E}_{2i} f_1(2\bar{\rho}_{2i}^n) - 2\mathbb{E}_{2i}^2 f_1(\bar{\rho}_{2i}^n) + 1}{2} \rightarrow \sigma_0^4.$$

We then apply Lemma 4.2 to f_k , f_l and $f_k f_l$ with $k, l \in \{2, 3\}$. It yields

$$\begin{aligned} & \mathbb{E}_{2i}[f_k(\bar{\rho}_{2i}^n) - P_n f_k(X_{2i\Delta_n}, \theta_0)][f_l(\bar{\rho}_{2i}^n) - P_n f_l(X_{2i\Delta_n}, \theta_0)] \\ &= P_n(f_k f_l)(X_{2i\Delta_n}, \theta_0) - P_n f_k(X_{2i\Delta_n}, \theta_0) P_n f_l(X_{2i\Delta_n}, \theta_0) \\ &= u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2} c_{\alpha_0} \delta_0 X_{2i\Delta_n}^{1-\alpha_0/2} \int \frac{f_k f_l(v)}{|v|^{\alpha_0+1}} dv + u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2} \varepsilon_n(\theta_0, X_{2i\Delta_n}), \end{aligned}$$

where

$$|\varepsilon_n(\theta_0, X_{2i\Delta_n})| \leq (1 + X_{2i\Delta_n}) \varepsilon_n, \quad \text{with } \varepsilon_n \rightarrow 0.$$

Consequently, we deduce for $k, l \in \{2, 3\}$

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i}(\zeta_{2i}^{k,n} \zeta_{2i}^{l,n}) \rightarrow \frac{1}{2} c_{\alpha_0} \delta_0 \int_0^1 X_s^{1-\alpha_0/2} ds \int \frac{f_k f_l(v)}{|v|^{\alpha_0+1}} dv.$$

We finally address the cross terms $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i}(\zeta_{2i}^{1,n} \zeta_{2i}^{k,n})$ for $k \in \{2, 3\}$. We have

$$\begin{aligned} & \mathbb{E}_{2i}[(f_1(\bar{\rho}_{2i}^n) - P_n f_1(X_{2i\Delta_n}, \theta_0))[f_k(\bar{\rho}_{2i}^n) - P_n f_k(X_{2i\Delta_n}, \theta_0)]) \\ &= P_n(f_1 f_k)(X_{2i\Delta_n}, \theta_0) - P_n f_1(X_{2i\Delta_n}, \theta_0) P_n f_k(X_{2i\Delta_n}, \theta_0). \end{aligned}$$

Using again Lemma 4.2 for $f_1 f_k$ and f_k with $k \in \{2, 3\}$ and observing that $P_n f_1(X_{2i\Delta_n}, \theta_0)$ is bounded we have

$$|P_n(f_1 f_k)(X_{2i\Delta_n}, \theta_0) - P_n f_1(X_{2i\Delta_n}, \theta_0) P_n f_k(X_{2i\Delta_n}, \theta_0)| \leq C u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2} (1 + X_{2i\Delta_n}),$$

and we obtain for $k \in \{2, 3\}$,

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i}(\zeta_{2i}^{1,n} \zeta_{2i}^{k,n}) \rightarrow 0.$$

We finally conclude that

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i}(\zeta_{2i}^n (\zeta_{2i}^n)^T) \rightarrow \Sigma(\theta_0).$$

We have proved equation (5.24).

We check easily that $\Sigma(\theta_0)$ is definite positive and we finish the proof of Proposition 4.2 in the case $n\Delta_n = 1$ with Lemma 5.1 below.

Lemma 5.1. *Let M be a square-integrable martingale. Let f_1 and f_2 be even bounded functions. We assume moreover that f_1 is \mathcal{C}^1 with bounded derivative, $f_2 = 0$ on $[-\eta, \eta]$, for $\eta > 0$ and that $\Delta_n = 1/n$ and $u_n = 1/[\ln(1/\Delta_n)]^p$ with $p > 1/2$. Then we have the convergence in probability (where $\bar{\rho}_{2i}^n$ is defined by (4.1) and $\Delta_{2i}^{2,n} M = M_{2(i+1)\Delta_n} - M_{2i\Delta_n}$)*

$$(i) \frac{1}{u_n^2 \sqrt{n}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i}[(f_1(\bar{\rho}_{2i}^n) - P_n f_1(X_{2i\Delta_n}, \theta_0)) \Delta_{2i}^{2,n} M] \rightarrow 0,$$

$$(ii) \frac{1}{u_n^{\alpha_0/2} \Delta_n^{1/2-\alpha_0/4} \sqrt{n}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i}[(f_2(\bar{\rho}_{2i}^n) - P_n f_2(X_{2i\Delta_n}, \theta_0)) \Delta_{2i}^{2,n} M] \rightarrow 0.$$

Proof. From Theorem 2.2.15 in [8], we only consider the cases where M is the Brownian Motion B or M is orthogonal to B .

(i) Proceeding as in the proof of (5.9), we separate the Brownian component and the pure-jump component $\bar{\rho}_{2i}^n = \bar{\rho}_{2i}^{1,n} + \bar{\rho}_{2i}^{2,n}$ and since f_1 and f_1' are bounded, for $1 \leq p < \alpha_0$

$$\mathbb{E}_{2i} |f_1(\bar{\rho}_{2i}^n) - f_1(\bar{\rho}_{2i}^{1,n})|^2 \leq C_p u_n^p \Delta_n^{p(1/\alpha_0-1/2)} X_{2i\Delta_n}^{p(1/\alpha_0-1/2)}.$$

Choosing p close to α_0 and using Proposition 4.1 (ii), it yields for $\epsilon > 0$

$$\mathbb{E} |f_1(\bar{\rho}_{2i}^n) - f_1(\bar{\rho}_{2i}^{1,n})|^2 \leq C_\epsilon u_n^{\alpha_0-\epsilon} \Delta_n^{1-\alpha_0/2-\epsilon}.$$

Setting

$$S_n^1 = \frac{1}{u_n^2 \sqrt{n}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i}[(f_1(\bar{\rho}_{2i}^n) - f_1(\bar{\rho}_{2i}^{1,n})) \Delta_{2i}^{2,n} M],$$

we deduce from Cauchy-Schwarz inequality and the previous bound with ϵ such that $1 - \alpha_0/2 - \epsilon > 0$

$$\mathbb{E}|S_n^1| \leq \frac{C_\epsilon}{u_n^2 \sqrt{n}} \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} u_n^{\alpha_0 - \epsilon} \Delta_n^{1 - \alpha_0/2 - \epsilon} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}(\Delta_{2i}^{2,n} M)^2 \right)^{1/2} \rightarrow 0,$$

observing that the second sum is bounded since M is a square-integrable martingale. So the proof of (i) reduces to $S_n^2 \rightarrow 0$ where

$$S_n^2 = \frac{1}{u_n^2 \sqrt{n}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i}[f_1(\bar{\rho}_{2i}^{1,n}) \Delta_{2i}^{2,n} M].$$

If $M = B$, using that f_1 is even and recalling that $\bar{\rho}_{2i}^{1,n} = \frac{\sigma_0 u_n}{\sqrt{\Delta_n}} \Delta_{2i}^{s,n} B$, an explicit calculus gives $\mathbb{E}_{2i}[f_1(\bar{\rho}_{2i}^{1,n}) \Delta_{2i}^{2,n} M] = 0$. If M is a square integrable martingale orthogonal to B , we also have $\mathbb{E}_{2i}[f_1(\bar{\rho}_{2i}^{1,n}) \Delta_{2i}^{2,n} M] = 0$ from the representation Theorem and Itô's Formula.

(ii) We follow the proof of Lemma 8 in [7]. We first assume that $M = B$. From Hölder's inequality with $p > 1$, we have

$$|\mathbb{E}_{2i}[f_2(\bar{\rho}_{2i}^n) \Delta_{2i}^{2,n} B]| \leq C \mathbb{E}_{2i}^{1/p}(f_2(\bar{\rho}_{2i}^n))^p \sqrt{\Delta_n} \leq C \Delta_n^{(1 - \alpha_0/2)/p} \sqrt{\Delta_n} (1 + X_{2i\Delta_n}),$$

where for the second inequality we applied Lemma 4.2 to f_2^p . We deduce then (ii) in that case choosing $p < 2$.

We now assume that M is orthogonal to B . From the representation Theorem (we refer to Chapter III.4 of [9]), we have for some predictable process (G_t)

$$M_t = \int_0^t \int G_s \tilde{N}(ds, dz) \text{ with } \mathbb{E} \int_0^1 \int G_s^2 \bar{N}(ds, dz) < \infty,$$

and also with (h_t) and (H_t) predictable and (H_t) bounded (since f_2 is bounded)

$$f_2(\bar{\rho}_{2i}^n) - P_n f_2(X_{2i\Delta_n}, \theta_0) = \int_{2i\Delta_n}^{2(i+1)\Delta_n} h_s dB_s + \int_{2i\Delta_n}^{2(i+1)\Delta_n} \int H_s \tilde{N}(ds, dz).$$

We then deduce from Itô's Formula

$$\mathbb{E}_{2i}[(f_2(\bar{\rho}_{2i}^n) - P_n f_2(X_{2i\Delta_n}, \theta_0)) \Delta_{2i}^{2,n} M] = \mathbb{E}_{2i} \int_{2i\Delta_n}^{2(i+1)\Delta_n} \int H_s G_s \bar{N}(ds, dz).$$

With the decomposition $G_s = G_s \mathbf{1}_{|G_s| > \epsilon_n} + G_s \mathbf{1}_{|G_s| \leq \epsilon_n}$, the proof reduces to study the convergence of

$$\begin{aligned} T_n^1 &= \frac{1}{u_n^{\alpha_0/2} \Delta_n^{1/2 - \alpha_0/4} \sqrt{n}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i} \int_{2i\Delta_n}^{2(i+1)\Delta_n} \int H_s G_s \mathbf{1}_{|G_s| > \epsilon_n} \bar{N}(ds, dz), \\ T_n^2 &= \frac{1}{u_n^{\alpha_0/2} \Delta_n^{1/2 - \alpha_0/4} \sqrt{n}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E}_{2i} \int_{2i\Delta_n}^{2(i+1)\Delta_n} \int H_s G_s \mathbf{1}_{|G_s| \leq \epsilon_n} \bar{N}(ds, dz). \end{aligned}$$

Since H is bounded, we have immediately

$$\begin{aligned} \mathbb{E}|T_n^1| &\leq \frac{C}{u_n^{\alpha_0/2} \Delta_n^{1/2 - \alpha_0/4} \sqrt{n}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathbb{E} \int_{2i\Delta_n}^{2(i+1)\Delta_n} \int \frac{G_s^2}{\epsilon_n} \bar{N}(ds, dz), \\ &\leq \frac{C}{\epsilon_n u_n^{\alpha_0/2} \Delta_n^{1/2 - \alpha_0/4} \sqrt{n}} \mathbb{E} \int_0^1 \int G_s^2 \bar{N}(ds, dz) \leq \frac{C}{\epsilon_n u_n^{\alpha_0/2} \Delta_n^{1/2 - \alpha_0/4} \sqrt{n}}. \end{aligned} \quad (5.29)$$

With $\epsilon_n = \Delta_n^{\alpha_0/8}$, we deduce $\mathbb{E}|T_n^1| \rightarrow 0$.

We now turn to T_n^2 . We first apply Lemma 4.2 to f_2 to obtain

$$\mathbb{E}_{2i} \int_{2i\Delta_n}^{2(i+1)\Delta_n} \int H_s^2 \bar{N}(ds, dz) \leq \mathbb{E}_{2i} (f_2(\bar{\rho}_{2i}^n) - P_n f_2(X_{2i\Delta_n}, \theta_0))^2 \leq C u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2} (1 + X_{2i\Delta_n}).$$

From Cauchy-Schwarz inequality it yields

$$\begin{aligned} & |\mathbb{E}_{2i} \int_{2i\Delta_n}^{2(i+1)\Delta_n} \int H_s G_s \mathbf{1}_{|G_s| \leq \epsilon_n} \bar{N}(ds, dz)| \leq \\ & C u_n^{\alpha_0/2} \Delta_n^{1/2-\alpha_0/4} (1 + \sqrt{X_{2i\Delta_n}}) \mathbb{E}_{2i}^{1/2} \int_{2i\Delta_n}^{2(i+1)\Delta_n} \int G_s^2 \mathbf{1}_{|G_s| \leq \epsilon_n} \bar{N}(ds, dz), \end{aligned}$$

and applying once again Cauchy-Schwarz inequality

$$\mathbb{E}|T_n^2| \leq C \left(\mathbb{E} \frac{1}{n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} (1 + X_{2i\Delta_n}) \times \mathbb{E} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \int_{2i\Delta_n}^{2(i+1)\Delta_n} \int G_s^2 \mathbf{1}_{|G_s| \leq \epsilon_n} \bar{N}(ds, dz) \right)^{1/2}.$$

The first expectation is bounded from Proposition 4.1, consequently

$$\mathbb{E}|T_n^2| \leq C \mathbb{E}^{1/2} \int_0^1 \int G_s^2 \mathbf{1}_{|G_s| \leq \epsilon_n} \bar{N}(ds, dz). \quad (5.30)$$

As ϵ_n goes to zero, we conclude $\mathbb{E}|T_n^2| \rightarrow 0$ by dominated convergence and the proof of (ii) is finished. \square

2. Case $n\Delta_n \rightarrow +\infty$. The proof follows essentially the same lines as in the previous case and we only indicate the main changes. The bounds (5.19) and (5.20) still hold. From the ergodic Theorem (if $b_0 > 0$, the process (X_t) is geometrically ergodic, see [15]), we have the convergence in probability for $0 < p \leq 1$

$$\frac{1}{n\Delta_n} \int_0^{n\Delta_n} X_t^p dt \xrightarrow[n \rightarrow \infty]{} \int_0^{+\infty} x^p \pi_0(dx),$$

since $\int_0^{+\infty} x \pi_0(dx) < +\infty$. Moreover we know that $\mathbb{E}(X_t) \rightarrow a_0/b_0$ (see [4]), then $\sup_t \mathbb{E}(X_t) < +\infty$ and combining this with Proposition 4.1 (iv) we can prove

$$\mathbb{E} \left| \frac{1}{n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} X_{2i\Delta_n}^p - \frac{1}{2n\Delta_n} \int_0^{n\Delta_n} X_t^p dt \right| \xrightarrow[n \rightarrow \infty]{} 0.$$

We deduce the convergence in probability

$$\frac{1}{n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} X_{2i\Delta_n}^p \xrightarrow[n \rightarrow \infty]{} \frac{1}{2} \int_0^{+\infty} x^p \pi_0(dx). \quad (5.31)$$

Now replacing ρ_{2i}^n by $\bar{\rho}_{2i}^n$, we obtain (5.21) under the additional assumption $\sqrt{n}\Delta_n^{1/\alpha + \alpha/4 - 1/2 - \epsilon_0} \rightarrow 0$. It remains to prove the convergence in distribution of

$$A_n(\theta_0) \frac{1}{n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} [f(\bar{\rho}_{2i}^n) - P_n f(X_{2i\Delta_n}, \theta_0)].$$

We check (5.22), (5.23), (5.24) (with $\bar{\Sigma}$) by proceeding as in the case $n\Delta_n$ fixed but using (5.31) and $\sup_t \mathbb{E}(X_t) < +\infty$ to conclude.

5.1.5 Proof of Proposition 4.3

By definition of $F_n(\theta)$, we have

$$A_n(\theta)\nabla_\theta F_n(\theta)\Lambda_n(\theta) = -\frac{1}{n} \sum_{i=0}^{\lfloor n/2 \rfloor - 1} A_n(\theta)\nabla_\theta P_n f(X_{2i\Delta_n}, \theta)\Lambda_n(\theta).$$

We start by computing

$$\nabla_\theta P_n f(x, \theta) = \begin{pmatrix} \partial_{\sigma^2} P_n f_1(x, \theta) & \partial_\delta P_n f_1(x, \theta) & \partial_\alpha P_n f_1(x, \theta) \\ \partial_{\sigma^2} P_n f_2(x, \theta) & \partial_\delta P_n f_2(x, \theta) & \partial_\alpha P_n f_2(x, \theta) \\ \partial_{\sigma^2} P_n f_3(x, \theta) & \partial_\delta P_n f_3(x, \theta) & \partial_\alpha P_n f_3(x, \theta) \end{pmatrix}.$$

As $P_n f_1(x, \theta) = \exp(-u_n^2 \sigma^2 - 2\delta x^{1-\alpha/2} u_n^\alpha \Delta_n^{1-\alpha/2})$, we have

$$\begin{aligned} \partial_{\sigma^2} P_n f_1(x, \theta) &= -u_n^2 P_n f_1(x, \theta), \\ \partial_\delta P_n f_1(x, \theta) &= -2x^{1-\alpha/2} u_n^\alpha \Delta_n^{1-\alpha/2} P_n f_1(x, \theta), \\ \partial_\alpha P_n f_1(x, \theta) &= -2\delta x^{1-\alpha/2} u_n^\alpha \Delta_n^{1-\alpha/2} \ln(u_n/\sqrt{x\Delta_n}) P_n f_1(x, \theta). \end{aligned} \tag{5.32}$$

Next observing that $\nabla_\theta P_n f_2(x, \theta) = -\nabla_\theta P_n K(x, \theta)$ (the same connection holds for f_3 replacing $K(x)$ by $K(2x)$), we evaluate $\nabla_\theta P_n K(x, \theta)$. We recall that from (3.2)

$$P_n K(x, \theta) = \mathbb{E}K(\sqrt{2}u_n\sigma B_1 + \frac{(2\delta x)^{1/\alpha}}{\sqrt{x}} u_n \Delta_n^{1/\alpha-1/2} S_1^\alpha),$$

and we denote by $g_{n,\theta,x}$ the density of $\sqrt{2}u_n\sigma B_1 + \frac{(2\delta x)^{1/\alpha}}{\sqrt{x}} u_n \Delta_n^{1/\alpha-1/2} S_1^\alpha$. We obtain the following result where $\mathcal{F}f$ is the Fourier transform of f , defined by $\mathcal{F}f(y) = \int_{\mathbb{R}} e^{-iyz} f(z) dz$.

Lemma 5.2. *Let K be defined by (3). Then we have*

$$\begin{aligned} \partial_{\sigma^2} P_n K(x, \theta) &= u_n^2 P_n K''(x, \theta), \\ \partial_\delta P_n K(x, \theta) &= -x^{1-\alpha/2} u_n^\alpha \Delta_n^{1-\alpha/2} \frac{2}{2\pi} \int \mathcal{F}K(y) |y|^\alpha \mathcal{F}g_{n,\theta,x}(y) dy, \\ \partial_\alpha P_n K(x, \theta) &= -\delta x^{1-\alpha/2} u_n^\alpha \Delta_n^{1-\alpha/2} \left(\frac{2}{2\pi} \int \mathcal{F}K(y) |y|^\alpha \ln(|y|) \mathcal{F}g_{n,\theta,x}(y) dy \right. \\ &\quad \left. + (\ln(\frac{u_n}{\sqrt{\Delta_n}}) - \ln(\sqrt{x})) \frac{2}{2\pi} \int \mathcal{F}K(y) |y|^\alpha \mathcal{F}g_{n,\theta,x}(y) dy \right). \end{aligned}$$

Proof. Since K is compactly supported and even, its Fourier transform is a Schwartz function and $\mathcal{F}\mathcal{F}K = 2\pi K$. We also have $-y^2 \mathcal{F}K(y) = \mathcal{F}K''(y)$. With the previous notation it yields

$$\begin{aligned} \nabla_\theta P_n K(x, \theta) &= \nabla_\theta \int_{\mathbb{R}} K(y) g_{n,\theta,x}(y) dy = \frac{1}{2\pi} \nabla_\theta \int_{\mathbb{R}} (\mathcal{F}\mathcal{F}K)(y) g_{n,\theta,x}(y) dy \\ &= \frac{1}{2\pi} \nabla_\theta \int_{\mathbb{R}} \mathcal{F}K(y) \mathcal{F}g_{n,\theta,x}(y) dy = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}K(y) \nabla_\theta \mathcal{F}g_{n,\theta,x}(y) dy, \end{aligned}$$

where we successively used duality and Lebesgue's Theorem. By definition of $g_{n,\theta,x}$, we have

$$\mathcal{F}g_{n,\theta,x}(y) = \exp(-u_n^2 \sigma^2 y^2 - 2|y|^\alpha \delta x^{1-\alpha/2} u_n^\alpha \Delta_n^{1-\alpha/2}). \tag{5.33}$$

The computation of $\nabla_\theta \mathcal{F}g_{n,\theta,x}$ is straightforward and the expression of $\nabla_\theta P_n K$ follows. \square

Coming back to $\nabla_\theta P_n f_2(x, \theta)$ we obtain the following result.

Lemma 5.3. *We assume that $u_n[\ln(1/\Delta_n)]^{1/2} \rightarrow 0$. Then we have $\forall x > 0$*

$$\begin{aligned}
(i) \quad & \left| \frac{1}{u_n^{\alpha/2} \Delta_n^{1/2-\alpha/4}} \partial_{\sigma^2} P_n f_2(x, \theta) \right| \leq \varepsilon_n^1(x, \theta), \\
(ii) \quad & \frac{1}{u_n^\alpha \Delta_n^{1-\alpha/2}} \partial_\delta P_n f_2(x, \theta) = x^{1-\alpha/2} \psi(\alpha) + x^{1-\alpha/2} \varepsilon_n^2(x, \theta), \\
(iii) \quad & \frac{1}{u_n^\alpha \Delta_n^{1-\alpha/2}} \left(\partial_\alpha P_n f_2(x, \theta) - \delta \ln\left(\frac{u_n}{\sqrt{\Delta_n}}\right) \partial_\delta P_n f_2(x, \theta) \right) = \delta x^{1-\alpha/2} [\partial_\alpha \psi(\alpha) - \ln(\sqrt{x}) \psi(\alpha)] \\
& \quad + x^{1-\alpha/2} \varepsilon_n^3(x, \theta) - x^{1-\alpha/2} \ln(\sqrt{x}) \varepsilon_n^2(x, \theta),
\end{aligned}$$

where $\psi(\alpha) = c_\alpha \int \frac{f_2(y)}{|y|^{\alpha+1}} dy = c_\alpha \int \frac{1-K(y)}{|y|^{\alpha+1}} dy$ and for any compact subset $A \subset \Theta$

$$\sup_{\theta \in A} (|\varepsilon_n^1(x, \theta)| + |\varepsilon_n^2(x, \theta)| + |\varepsilon_n^3(x, \theta)|) \leq C u_n (1 + \sqrt{x}).$$

The same result holds for f_3 replacing $\psi(\alpha)$ by $2^\alpha \psi(\alpha)$.

Proof. We first prove that

$$\frac{2}{2\pi} \int \mathcal{F}K(y) |y|^\alpha dy = c_\alpha \int \frac{(1-K(y))}{|y|^{\alpha+1}} dy = \psi(\alpha). \quad (5.34)$$

By definition of c_α we have $2|y|^\alpha = c_\alpha \int_{\mathbb{R}} \frac{1-\cos(uy)}{|u|^{\alpha+1}} du$. Then for $\epsilon > 0$

$$\begin{aligned}
\frac{2}{2\pi} \int \mathcal{F}K(y) |y|^\alpha dy &= \frac{c_\alpha}{2\pi} \int \mathcal{F}K(y) \left(\int_{\mathbb{R}} \frac{1-\cos(uy)}{|u|^{\alpha+1}} du \right) dy \\
&= \frac{c_\alpha}{2\pi} \int \mathcal{F}K(y) \left(\int_{\mathbb{R}} \frac{1-\cos(uy)}{|u|^{\alpha+1}} (\mathbf{1}_{|u| \leq \epsilon} + \mathbf{1}_{|u| > \epsilon}) du \right) dy.
\end{aligned}$$

Since $1 - \cos(uy) = h(uy)u^2y^2$ with h bounded, we can apply Fubini's Theorem

$$\begin{aligned}
\frac{2}{2\pi} \int \mathcal{F}K(y) |y|^\alpha dy &= \frac{c_\alpha}{2\pi} \int_{\mathbb{R}} \frac{u^2}{|u|^{\alpha+1}} \mathbf{1}_{|u| \leq \epsilon} \left(\int \mathcal{F}K(y) h(uy) y^2 dy \right) du \\
&\quad + \frac{c_\alpha}{2\pi} \int_{\mathbb{R}} \frac{1}{|u|^{\alpha+1}} \mathbf{1}_{|u| > \epsilon} \left(\int \mathcal{F}K(y) (1 - \cos(uy)) dy \right) du.
\end{aligned}$$

Moreover, we have

$$\int \mathcal{F}K(y) dy = 2\pi K(0) = 2\pi, \quad \int \mathcal{F}K(y) \cos(uy) dy = \mathcal{F}(\mathcal{F}K)(y) = 2\pi K(u).$$

Letting ϵ go to zero, we obtain (5.34).

Next from (5.33), we have

$$\sup_{\theta \in A} |\mathcal{F}g_{n,\theta,x}(y) - 1| \leq C u_n (1 + y^2) (1 + \sqrt{x}),$$

and consequently

$$\begin{aligned}
\sup_{\theta \in A} \left| \frac{2}{2\pi} \int \mathcal{F}K(y) |y|^\alpha \mathcal{F}g_{n,\theta,x}(y) dy - \psi(\alpha) \right| &\leq C u_n (1 + \sqrt{x}), \\
\sup_{\theta \in A} \left| \frac{2}{2\pi} \int \mathcal{F}K(y) |y|^\alpha \ln(|y|) \mathcal{F}g_{n,\theta,x}(y) dy - \partial_\alpha \psi(\alpha) \right| &\leq C u_n (1 + \sqrt{x}).
\end{aligned}$$

Since $f_2 = 1 - K$, we obtain (i) combining Lemma 5.2 with Lemma 4.2 (applied to K''), (ii) and (iii) follow from Lemma 5.2, (5.34) and the previous uniform bounds. \square

With this background, we can finish the proof of Proposition 4.3. An explicit calculus leads to $A_n(\theta)\nabla_\theta P_n f(x, \theta)\Lambda_n(\theta) = W_n(x, \theta)$ with

$$W_n = \begin{pmatrix} \frac{1}{v_n^2}\partial_{\sigma^2}P_n f_1 & \frac{v_n}{u_n^2}\partial_\delta P_n f_1 & \frac{v_n}{u_n^2}[\partial_\alpha P_n f_1 - \delta \ln(\frac{u_n}{\sqrt{\Delta_n}})\partial_\delta P_n f_1] \\ v_n\partial_{\sigma^2}P_n f_2 & v_n^2\partial_\delta P_n f_2 & v_n^2[\partial_\alpha P_n f_2 - \delta \ln(\frac{u_n}{\sqrt{\Delta_n}})\partial_\delta P_n f_2] \\ v_n\partial_{\sigma^2}P_n f_3 & v_n^2\partial_\delta P_n f_3 & v_n^2[\partial_\alpha P_n f_3 - \delta \ln(\frac{u_n}{\sqrt{\Delta_n}})\partial_\delta P_n f_3] \end{pmatrix}, \quad v_n = \frac{1}{u_n^{\alpha/2}\Delta_n^{1/2-\alpha/4}}.$$

1. Case $n\Delta_n = 1$. We check the uniform convergences in probability for $\alpha \in [\underline{a}, \bar{a}]$, with $1 < \underline{a} < \bar{a} < 2$

$$\sup_{\alpha \in [\underline{a}, \bar{a}]} \left| \frac{1}{n} \sum_{i=0}^{\lfloor n/2 \rfloor - 1} X_{2i\Delta_n}^{1-\alpha/2} - \frac{1}{2} \int_0^1 X_t^{1-\alpha/2} dt \right| \rightarrow 0, \quad (5.35)$$

$$\sup_{\alpha \in [\underline{a}, \bar{a}]} \left| \frac{1}{n} \sum_{i=0}^{\lfloor n/2 \rfloor - 1} X_{2i\Delta_n}^{1-\alpha/2} \ln \sqrt{X_{2i\Delta_n}} - \frac{1}{2} \int_0^1 X_t^{1-\alpha/2} \ln \sqrt{X_t} dt \right| \rightarrow 0. \quad (5.36)$$

We just detail (5.36). Let $h(\alpha, x) = x^{1-\alpha/2} \ln(\sqrt{x})$. We have for $\epsilon > 0$,

$$\forall x > 0, |h'(\alpha, x)| \leq \frac{1}{x^{\alpha/2}}(1 + |\ln(x)|) \leq C(1 + \frac{1}{x^{\alpha/2+\epsilon}}),$$

and from Taylor's formula for $x, y > 0$

$$|h(\alpha, y) - h(\alpha, x)| \leq C|y - x| \int_0^1 (1 + \frac{1}{((1-c)x + cy)^{\alpha/2+\epsilon}}) dc.$$

Choosing ϵ such that $\bar{a}/2 + \epsilon < 1$, we deduce

$$\sup_{\alpha \in [\underline{a}, \bar{a}]} |h(\alpha, y) - h(\alpha, x)| \leq C|y - x|(1 + \frac{1}{x}).$$

From Proposition 4.1 it yields

$$\mathbb{E}_{2i} \sup_{t \in [2i\Delta_n, 2(i+1)\Delta_n]} \sup_{\alpha \in [\underline{a}, \bar{a}]} |h(\alpha, X_t) - h(\alpha, X_{2i\Delta_n})| \leq C\sqrt{\Delta_n}(1 + X_{2i\Delta_n} + \frac{1}{X_{2i\Delta_n}}),$$

and we obtain (5.36).

Furthermore, from Lebesgue's Theorem $\alpha \mapsto I(\alpha, X) = \int_0^1 X_t^{1-\alpha/2} dt$ admits a derivative and $\partial_\alpha I(\alpha, X) = -\int_0^1 X_t^{1-\alpha/2} \ln(\sqrt{X_t}) dt$, so in the expression of $W(\theta)$ (3.6), we have

$$\partial_\alpha(\psi(\alpha)I(\alpha, X)) = \partial_\alpha \psi(\alpha) \int_0^1 X_t^{1-\alpha/2} dt - \psi(\alpha) \int_0^1 X_t^{1-\alpha/2} \ln(\sqrt{X_t}) dt.$$

Consequently, combining (5.32) and Lemma 5.3 with (5.35) and (5.36), we obtain

$$\sup_{\theta \in A} \left| \frac{1}{n} \sum_{i=0}^{\lfloor n/2 \rfloor - 1} W_n(X_{2i\Delta_n}, \theta) + W(\theta) \right| \rightarrow 0,$$

and we finally conclude from $A_n(\theta)\nabla_\theta F_n(\theta)\Lambda_n(\theta) = -\frac{1}{n} \sum_{i=0}^{\lfloor n/2 \rfloor - 1} W_n(X_{2i\Delta_n}, \theta)$. We see easily that W is non-singular since

$$\det W(\theta) = \frac{1}{8} \delta \psi(\alpha)^2 I(\alpha, X)^2 2^\alpha \ln(2) \neq 0.$$

2. Case $n\Delta_n \rightarrow \infty$. When $n\Delta_n \rightarrow +\infty$, we first prove the convergences

$$\sup_{\alpha \in [\underline{a}, \bar{a}]} \left| \frac{1}{n} \sum_{i=0}^{\lfloor n/2 \rfloor - 1} X_{2i\Delta_n}^{1-\alpha/2} - \frac{1}{2} \int_0^{+\infty} x^{1-\alpha/2} \pi_0(dx) \right| \rightarrow 0, \quad (5.37)$$

$$\sup_{\alpha \in [\underline{a}, \bar{a}]} \left| \frac{1}{n} \sum_{i=0}^{\lfloor n/2 \rfloor - 1} X_{2i\Delta_n}^{1-\alpha/2} \ln \sqrt{X_{2i\Delta_n}} - \frac{1}{2} \int_0^{+\infty} x^{1-\alpha/2} \ln \sqrt{x} \pi_0(dx) \right| \rightarrow 0. \quad (5.38)$$

We give the details for (5.38) and study separately the uniform convergences in probability

$$\sup_{\alpha \in [\underline{a}, \bar{a}]} \left| \frac{1}{n} \sum_{i=0}^{\lfloor n/2 \rfloor - 1} X_{2i\Delta_n}^{1-\alpha/2} \ln \sqrt{X_{2i\Delta_n}} - \frac{1}{2n\Delta_n} \int_0^{n\Delta_n} X_t^{1-\alpha/2} \ln \sqrt{X_t} dt \right| \rightarrow 0, \quad (5.39)$$

$$\sup_{\alpha \in [\underline{a}, \bar{a}]} \left| \frac{1}{n\Delta_n} \int_0^{n\Delta_n} X_t^{1-\alpha/2} \ln \sqrt{X_t} dt - \int_0^{+\infty} x^{1-\alpha/2} \ln \sqrt{x} \pi_0(dx) \right| \rightarrow 0. \quad (5.40)$$

The proof of (5.39) is similar to the one of (5.36) and we omit it. It remains to check (5.40). We set $H_n(\alpha) = \frac{1}{n\Delta_n} \int_0^{n\Delta_n} X_t^{1-\alpha/2} \ln \sqrt{X_t} dt$ and $H(\alpha) = \int_0^{+\infty} x^{1-\alpha/2} \ln \sqrt{x} \pi_0(dx)$. The map $\alpha \rightarrow H_n(\alpha)$ is continuous on $[\underline{a}, \bar{a}]$ and $H_n(\alpha) \rightarrow H(\alpha)$, $\forall \alpha \in [\underline{a}, \bar{a}]$. Moreover for $\alpha_1, \alpha_2 \in [\underline{a}, \bar{a}]$ and $x > 0$, $\exists c \in (\alpha_1, \alpha_2)$ such that

$$|x^{1-\alpha_1/2} \ln(\sqrt{x}) - x^{1-\alpha_2/2} \ln(\sqrt{x})| \leq |\alpha_1 - \alpha_2| (\ln(x))^2 x^{1-c/2} \leq C |\alpha_1 - \alpha_2| (1+x).$$

With $\sup_t \mathbb{E} X_t < +\infty$, it follows that $\forall \delta > 0$, $\sup_n \mathbb{E} \sup_{|\alpha_1 - \alpha_2| \leq \delta} |H_n(\alpha_1) - H_n(\alpha_2)| \leq C\delta$, so the sequence $(H_n(\alpha)_{\alpha \in [\underline{a}, \bar{a}]})$ is tight and we obtain the uniform convergence (5.40).

Observing that $\partial_\alpha \bar{I}(\alpha) = - \int_0^{+\infty} x^{1-\alpha/2} \ln(\sqrt{x}) \pi_0(dx)$ and using the same arguments as in the case $n\Delta_n$ fixed, we obtain from (5.32), Lemma 5.3, (5.37) and (5.38)

$$\sup_{\theta \in A} \left| \frac{1}{n} \sum_{i=0}^{\lfloor n/2 \rfloor - 1} W_n(X_{2i\Delta_n}, \theta) + \bar{W}(\theta) \right| \rightarrow 0.$$

This achieves the proof of Proposition 4.3.

5.2 Main results

5.2.1 Proof of Theorem 3.1 and Theorem 3.3

In both cases, $n\Delta_n$ fixed and $n\Delta_n \rightarrow +\infty$, we apply Theorem A.2. in Mies and Podolskij [17], that extends the results of Jacod and Sørensen [10]. We just detail the first case, where we have strengthened condition (E.1) in [17] by assuming the stable convergence in law. With our notation, the result of Theorem 3.1 is a consequence of the following conditions:

$$(E.1) \quad A_n(\theta_0) F_n(\theta_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}\text{-}s} \Sigma(\theta_0)^{1/2} \mathcal{N},$$

(E.2) $\theta \rightarrow F_n(\theta)$ is \mathcal{C}^1 and for r_n a sequence of real numbers we have the convergences in probability

$$\sup_{\theta \in \mathcal{B}_{r_n}(\theta_0)} \|A_n(\theta) \nabla_\theta F_n(\theta) \Lambda_n(\theta) - W(\theta)\| \xrightarrow[n \rightarrow \infty]{} 0, \quad \sup_{\theta \in \mathcal{B}_{r_n}(\theta_0)} \frac{\|\Lambda_n(\theta)\| \|A_n(\theta) A_n(\theta_0)^{-1}\|}{r_n} \xrightarrow[n \rightarrow \infty]{} 0,$$

$$(E.3) \quad \sup_{\theta \in \mathcal{B}_{r_n}(\theta_0)} (\|A_n(\theta) A_n(\theta_0)^{-1} - Id\| + \|\Lambda_n(\theta) \Lambda_n(\theta_0)^{-1} - Id\| + \|W(\theta) - W(\theta_0)\|) \xrightarrow[n \rightarrow \infty]{} 0,$$

where $\mathcal{B}_{r_n}(\theta_0)$ is the ball with center θ_0 and radius r_n , $\|\cdot\|$ a matrix norm and Id the identity matrix. Condition (E.1) is proven in Proposition 4.2 and the first condition of (E.2) is proven in Proposition 4.3. The second condition in (E.2) and condition (E.3) are easily checked with $r_n = 1/\ln(\Delta_n)^2$.

5.2.2 Proof of Theorem 3.2

The existence is a consequence of Theorem 3.1. To prove the global uniqueness result we consider the renormalized estimating function (restricted to $\beta = (\sigma^2, \delta)$ with $f = (f_k)_{1 \leq k \leq 2}$)

$$\tilde{F}_n(\sigma^2, \delta) = \tilde{A}_n(\alpha_0)F_n(\sigma^2, \delta, \alpha_0), \text{ with } \tilde{A}_n(\alpha_0) = \text{diag} \left(\frac{1}{u_n^2}, \frac{1}{u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2}} \right).$$

From Proposition 4.2, we deduce $\tilde{F}_n(\sigma_0^2, \delta_0) \rightarrow 0$. Next, an explicit calculus gives

$$\nabla_{\beta} \tilde{F}_n(\sigma^2, \delta) = -\frac{1}{n} \sum_{i=0}^{\lfloor n/2 \rfloor - 1} \begin{pmatrix} \frac{1}{u_n^2} \partial_{\sigma^2} P_n f_1 & \frac{1}{u_n^2} \partial_{\delta} P_n f_1 \\ \frac{1}{u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2}} \partial_{\sigma^2} P_n f_2 & \frac{1}{u_n^{\alpha_0} \Delta_n^{1-\alpha_0/2}} \partial_{\delta} P_n f_2 \end{pmatrix},$$

and from Proposition 4.3, we have the convergence in probability for any compact subset A of $(0, +\infty) \times (0, +\infty)$

$$\sup_{(\sigma^2, \delta) \in A} \left\| \nabla \tilde{F}_n(\sigma^2, \delta) - \nabla \tilde{F}(\sigma^2, \delta) \right\| \rightarrow 0, \text{ where } \tilde{F}(\sigma^2, \delta) = \begin{pmatrix} \frac{1}{2}(\sigma^2 - \sigma_0^2) \\ \frac{1}{2} \psi(\alpha_0) \int_0^1 X_t^{1-\alpha_0/2} dt (\delta_0 - \delta) \end{pmatrix}.$$

The global uniqueness result follows then from Theorem 2.7.a) of Jacod and Sørensen [10], since (σ_0^2, δ_0) is the unique root of $\tilde{F}(\sigma^2, \delta) = 0$.

Funding. This work was supported by french ANR reference ANR-21-CE40-0021.

References

- [1] Yacine Aït-Sahalia and Jean Jacod. Estimating the degree of activity of jumps in high frequency data. *Ann. Statist.*, 37(5A):2202–2244, 2009.
- [2] Yacine Aït-Sahalia and Jean Jacod. Identifying the successive Blumenthal-Gettoor indices of a discretely observed process. *Ann. Statist.*, 40(3):1430–1464, 2012.
- [3] Chiara Amorino and Arnaud Gloter. Unbiased truncated quadratic variation for volatility estimation in jump diffusion processes. *Stochastic Process. Appl.*, 130(10):5888–5939, 2020.
- [4] Mátyás Barczy, Mohamed Ben Alaya, Ahmed Kebaier, and Gyula Pap. Asymptotic properties of maximum likelihood estimator for the growth rate of a stable CIR process based on continuous time observations. *Statistics*, 53(3):533–568, 2019.
- [5] Elise Bayraktar and Emmanuelle Clément. Estimation of a pure-jump stable cox-ingersoll-ross process. *Bernoulli in press*, 2024.
- [6] B. Cooper Boniece, José E. Figueroa-López, and Yuchen Han. Efficient integrated volatility estimation in the presence of infinite variation jumps via debiased truncated realized variations. *Preprint arXiv 2209.10128v3*, 2024.
- [7] Adam D. Bull. Near-optimal estimation of jump activity in semimartingales. *Ann. Statist.*, 44(1):58–86, 2016.
- [8] Jean Jacod and Philip Protter. *Discretization of processes*, volume 67 of *Stochastic Modelling and Applied Probability*. Springer, Heidelberg, 2012.
- [9] Jean Jacod and Albert N. Shiryaev. *Limit theorems for stochastic processes*, volume 288. Springer-Verlag, Berlin, second edition, 2003.

- [10] Jean Jacod and Michael Sørensen. A review of asymptotic theory of estimating functions. *Stat. Inference Stoch. Process.*, 21(2):415–434, 2018.
- [11] Jean Jacod and Viktor Todorov. Efficient estimation of integrated volatility in presence of infinite variation jumps. *Ann. Statist.*, 42(3):1029–1069, 2014.
- [12] Ying Jiao, Chunhua Ma, and Simone Scotti. Alpha-CIR model with branching processes in sovereign interest rate modeling. *Finance Stoch.*, 21(3):789–813, 2017.
- [13] Ying Jiao, Chunhua Ma, Simone Scotti, and Chao Zhou. The alpha-Heston stochastic volatility model. *Math. Finance*, 31(3):943–978, 2021.
- [14] Zenghu Li. Continuous-state branching processes with immigration. In *From probability to finance—lecture notes of BICMR Summer School on Financial Mathematics*, Math. Lect. Peking Univ., pages 1–69. Springer, Singapore, 2020.
- [15] Zenghu Li and Chunhua Ma. Asymptotic properties of estimators in a stable Cox-Ingersoll-Ross model. *Stochastic Process. Appl.*, 125(8):3196–3233, 2015.
- [16] Fabian Mies. Rate-optimal estimation of the Blumenthal-Gettoor index of a Lévy process. *Electron. J. Stat.*, 14(2):4165–4206, 2020.
- [17] Fabian Mies and Mark Podolskij. Estimation of mixed fractional stable processes using high-frequency data. *Ann. Statist.*, 51(5):1946–1964, 2023.
- [18] Ken-iti Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2013. Translated from the 1990 Japanese original, Revised edition of the 1999 English translation.
- [19] Xu Yang. Maximum likelihood type estimation for discretely observed CIR model with small α -stable noises. *Statist. Probab. Lett.*, 120:18–27, 2017.