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Martin Mion-Mouton

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# RIGIDITY OF SINGULAR DE-SITTER TORI WITH RESPECT TO THEIR LIGHTLIKE BI-FOLIATION

MARTIN MION-MOUTON

ABSTRACT. In this paper, we introduce a natural notion of constant curvature Lorentzian surfaces with conical singularities, and provide a large class of examples of such structures. We moreover initiate the study of their global rigidity, by proving that de-Sitter tori with a single singularity are essentially determined by the topological equivalence class of their lightlike bi-foliation. While this result is reminiscent of Troyanov's work on Riemannian surfaces with conical singularities, the rigidity will come from topological dynamics in the Lorentzian case.

## 1. INTRODUCTION

A Lorentzian metric induces on a surface a pair of *lightlike foliations*, and the Poincaré-Hopf theorem implies therefore that the torus is the only closed and orientable Lorentzian surface. An analog of the Gauß-Bonnet formula shows moreover that the only constant curvature Lorentzian metrics on the torus are actually *flat* (see [Ave63, Che63]). It is then natural to try to widen this class of geometries, in order to obtain structures locally modelled on the *de-Sitter space*  $\mathbf{dS}^2$  – the Lorentzian homogeneous space of non-zero curvature, introduced in paragraph 2.1.3 below. This will obviously not be possible on a closed surface without removing some points, and a natural way to do this is to proceed as in the Riemannian case, by concentrating all the curvature in finitely many points where the metric has conical *singularities*.

The first goal of this paper is to introduce this new natural class of *singular constant curvature Lorentzian surfaces*, to provide examples of such structures, and to initiate their study by proving some of their fundamental properties. The second and main goal is to investigate in the de-Sitter case the relations of these geometrical objects with associated dynamical ones: their pair of lightlike foliations.

**1.1. Singular de-Sitter surfaces.** The Lorentzian singularities are defined analogously to the Riemannian ones, and their local definition already appeared in [BBS11]. The connected component of the identity in the isometry group of  $\mathbf{dS}^2$  is isomorphic to  $\mathrm{PSL}_2(\mathbb{R})$ , acts transitively on  $\mathbf{dS}^2$ , and the stabilizer of a point  $\mathfrak{o} \in \mathbf{dS}^2$  in  $\mathrm{PSL}_2(\mathbb{R})$  is a one-parameter hyperbolic group  $A = \{a^\theta\}_{\theta \in \mathbb{R}} \subset \mathrm{PSL}_2(\mathbb{R})$ . As in the Riemannian case, a natural way to describe a conical singularity in the de-Sitter space is to chose a non-trivial isometry  $a^\theta \in A$  and a geodesic ray  $\gamma$  emanating from  $\mathfrak{o}$ , to consider the sector from  $\gamma$  to  $a^\theta(\gamma)$  in  $\mathbf{dS}^2$ , and to glue its two boundary components by  $a^\theta$ . For simplicity we chose a *lightlike* half-geodesic  $\mathcal{F}_\alpha^+(\mathfrak{o})$ , and a phenomenon specific to the Lorentzian situation happens then:  $\mathcal{F}_\alpha^+(\mathfrak{o})$  is fixed by  $a^\theta$ . In other words, the sector described by  $a^\theta(\mathcal{F}_\alpha^+(\mathfrak{o})) = \mathcal{F}_\alpha^+(\mathfrak{o})$  is simply the surface  $\mathbf{dS}_*^2$  obtained by cutting  $\mathbf{dS}^2$  open along  $\mathcal{F}_\alpha^+(\mathfrak{o})$ . It contains two up and down copies  $\iota_\pm(\mathcal{F}_\alpha^+(\mathfrak{o}))$  of the initial geodesic ray as boundary components, which can be identified by  $\iota_+(x) \sim \iota_-(a^\theta(x))$  to obtain a topological disk  $\mathbf{dS}_\theta^2 = \mathbf{dS}_*^2 / \sim$  in the quotient. This identification space has a marked point  $\mathfrak{o}_\theta$  which is the projection of  $\mathfrak{o}$ , and the metric of  $\mathbf{dS}^2$  induces moreover a natural locally  $\mathbf{dS}^2$  Lorentzian metric on  $\mathbf{dS}_\theta^2 \setminus \mathfrak{o}_\theta$ , since the gluing was made by isometries. The point  $\mathfrak{o}_\theta$  is then defined as the local model of a *standard singularity of angle  $\theta$*  of the locally  $\mathbf{dS}^2$  surface  $\mathbf{dS}_\theta^2 \setminus \mathfrak{o}_\theta$ , and a *singular  $\mathbf{dS}^2$ -surface* is an orientable surface bearing a locally  $\mathbf{dS}^2$  Lorentzian metric outside of a discrete set of points which are standard singularities (see Definition 2.22 below). We refer to paragraph 2.2.1 for more

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details on this construction, analogously introduced in the case of zero curvature (namely for the Minkowski space), and illustrated in Figure 2.1 below.

One of the purposes of this work is to set the ground for the future investigation of singular constant curvature Lorentzian surfaces, and to be usable as a possible reference for basic results in future works on the subject. To this end, we carefully prove in paragraphs 2.2 and 2.3 below many structural properties of these structures, and furnish in Proposition 3.4 a general method to construct a large class of examples. We also give an interpretation of singular constant curvature Lorentzian surfaces analogous to the one of Riemannian metrics with conical singularities as *metric length spaces*. A natural Lorentzian counterpart of the latter notion was indeed introduced in [KS18] under the name of *Lorentzian length spaces*, which constitute as in the Riemannian case a synthetic geometric version of smooth Lorentzian manifolds focusing on their main geometrical byproducts (namely the causal structure and the Lorentzian length of curves). Lorentzian length spaces encompass the low regularity Lorentzian metrics in an unified way, and were partly designed to that purpose. To the best of our knowledge, singular constant curvature Lorentzian surfaces did however not appear so far as an object of independent interest, and we will see in the Appendix D that they furnish a large and natural class of examples of Lorentzian length spaces, apparently new in the literature.

**1.2. Dynamics of the lightlike foliations and geometric rigidity.** As we will see in paragraph 2.2.5, we could have used any geodesic to define a standard singularity in paragraph 1.1. The benefit of using a lightlike ray is however to observe naturally from the construction, that the lightlike foliations  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$  of  $\mathbf{dS}^2$  extend to two transverse (one-dimensional) *topological foliations* of  $\mathbf{dS}_\theta^2$  at the standard singularity  $\mathfrak{o}_\theta$  (a result properly proved in Proposition 2.13). Any singular  $\mathbf{dS}^2$ -structure on a surface induces thus a *lightlike bi-foliation*  $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ , and the torus remains therefore the only closed and orientable surface bearing a Lorentzian metric with constant curvature and standard singularities. The study of constant curvature Lorentzian metrics on higher genus surfaces requests the introduction of other types of singularities, which produce *singular* foliations. They will be the object of a future work, and we refer to Remark 3.6 for a discussion of such examples.

The seminal work of Troyanov in [Tro86, Tro91] described the main global rigidity properties of Riemannian surfaces with conical singularities. Troyanov proves therein that for any fixed set of singularities and angles on a closed orientable surface, any conformal class contains a unique metric of a given curvature having the prescribed singularities (with necessary conditions relating the angles, the constant curvature and the Euler characteristic of the surface, given by the Gauß-Bonnet formula). On the other hand, it is easily checked that two Lorentzian metrics  $\mu_1$  and  $\mu_2$  on a surface are conformal if, and only if they have identical lightlike bi-foliations. In the direction of Troyanov's results, is then natural to investigate the relation of singular constant curvature Lorentzian surfaces to their lightlike bi-foliations. The following theorem is the main result of this paper, and provides a partial answer to this question.

**Theorem A.** *Let  $S_1, S_2$  be two closed singular  $\mathbf{dS}^2$ -surfaces having a unique singularity of the same angle. Assume that the lightlike bi-foliations of  $S_1$  and  $S_2$  are minimal and topologically equivalent. Then  $S_1$  and  $S_2$  are isometric.*

We say that a lightlike bi-foliation  $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$  is *minimal* if both foliations are such, *i.e.* have all of their leaves dense. The lightlike bi-foliations of  $S_1$  and  $S_2$  are moreover said to be *topologically equivalent* if there exists a homeomorphism  $f: S_1 \rightarrow S_2$  which is a simultaneous equivalence of the  $\alpha$  and the  $\beta$ -foliations, *i.e.* such that  $f(\mathcal{F}_\alpha^{S_1}(x)) = \mathcal{F}_\alpha^{S_2}(f(x))$  and  $f(\mathcal{F}_\beta^{S_1}(x)) = \mathcal{F}_\beta^{S_2}(f(x))$  for any  $x \in S_1$ .

A crucial difference between Theorem A and Troyanov's work on the Riemannian case should be emphasized at this point: the isometry between the singular  $\mathbf{dS}^2$ -surfaces is obtained in the current work from an equivalence which is only *topological* between their lightlike bi-foliations. In particular, we deduce from a topological equivalence between the bi-foliations the existence of a smooth one, which may be seen as a *geometric rigidity* result for this class of bi-foliations (we refer the reader to the very pleasant presentation of the general problem of geometric rigidity for

dynamical systems given in [Gha21, p.468]). The former rigidity result would be of little interest without its companion existence result, given by the following Theorem.

**Theorem B.** *Let  $A_\alpha \neq A_\beta \in \mathbf{P}^+(\mathbf{H}_1(\mathbf{T}^2, \mathbb{R}))$  be two distinct irrational rays and  $\theta \in \mathbb{R}_+^*$ . Then there exists on  $\mathbf{T}^2$  a singular  $\mathbf{dS}^2$ -structure with a unique singularity of angle  $\theta$ , and whose lightlike foliations are suspensions of projective asymptotic cycles  $A^+(\mathcal{F}_\alpha) = A_\alpha$  and  $A^+(\mathcal{F}_\beta) = A_\beta$ . In particular,  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$  are both minimal.*

The main results of this paper may be seen as a global description of the *deformation space* of singular  $\mathbf{dS}^2$ -structures of the two-torus having a unique singularity of angle  $\theta$  at  $0 \in \mathbf{T}^2$ , denoted by  $\text{Def}_\theta(\mathbf{T}^2, 0)$  and introduced in Definition 3.29. The description is done in terms of the *projective asymptotic cycles* of the lightlike foliations, which is the topological invariant of oriented topological foliations of the torus. It can be seen as a global counterpart of the rotation number of the first-return map on a section, and will be introduced in paragraph 3.6 below. The projective asymptotic cycles of the lightlike foliations are well-defined for an equivalence class  $[\mu]$  of structures in  $\text{Def}_\theta(\mathbf{T}^2, 0)$  (see Remark 3.30), and the general question investigated in this paper may then be roughly summarized as follows: *to which extent is the map*

$$(1.1) \quad [\mu] \in \text{Def}_\theta(\mathbf{T}^2, 0) \mapsto (A^+(\mathcal{F}_\alpha^{[\mu]}), A^+(\mathcal{F}_\beta^{[\mu]})) \in \mathbf{P}^+(\mathbf{H}_1(\mathbf{T}^2, \mathbb{R}))^2$$

*bijective ?* This is in a sense a counterpart of Troyanov's description in [Tro86, Tro91], where the deformation space of Riemannian metrics with prescribed conical singularities is shown to identify with the one of conformal structures (namely with the Teichmüller space). Contrarily to Troyanov's work, the description is however done in the current paper in terms of a *topological dynamical invariant*: the projective asymptotic cycle.

While the map defined in (1.1) is not globally injective, we will prove in the two following results its surjectivity, as well as its injectivity on large parts of  $\text{Def}_\theta(\mathbf{T}^2, 0)$ .

**Theorem C.** *Let  $\theta \in \mathbb{R}_+^*$  and  $c_\alpha \neq c_\beta \in \pi_1(\mathbf{T}^2)$  be two distinct primitive elements. Then there exists in  $\text{Def}_\theta(\mathbf{T}^2, 0)$  a unique point  $[\mu]$  for which  $\mathcal{F}_\alpha(0)$  and  $\mathcal{F}_\beta(0)$  are closed and  $([\mathcal{F}_\alpha(0)], [\mathcal{F}_\beta(0)]) = (c_\alpha, c_\beta)$ . Moreover,  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$  are suspensions and  $(\mathbf{T}^2, [\mu])$  is isometric to a  $\mathbf{dS}^2$ -torus  $\mathcal{T}_{-\theta, x}$ .*

The  $\mathbf{dS}^2$ -tori  $\mathcal{T}_{-\theta, x}$  will be introduced below in Proposition 3.12.

**Theorem D.** *Let  $\theta \in \mathbb{R}_+^*$ ,  $c_\alpha \in \pi_1(\mathbf{T}^2)$  be a primitive element and  $A_\beta \in \mathbf{P}^+(\mathbf{H}_1(\mathbf{T}^2, \mathbb{R}))$  an irrational ray. Then there exists in  $\text{Def}_\theta(\mathbf{T}^2, 0)$  a unique point  $[\mu]$  such that:*

- (1)  $\mathcal{F}_\alpha(0)$  is closed and  $[\mathcal{F}_\alpha(0)] = c_\alpha$ ,
- (2)  $A^+(\mathcal{F}_\beta) = A_\beta$ .

Moreover,  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$  are suspensions,  $\mathcal{F}_\beta$  is minimal and  $(\mathbf{T}^2, [\mu])$  is isometric to a  $\mathbf{dS}^2$ -torus  $\mathcal{T}_{-\theta, x}$ . The obvious analogous statement holds by exchanging the roles of the  $\alpha$  and  $\beta$ -foliations.

Theorems A, C and D advertise the general idea that closed singular constant curvature Lorentzian surfaces are in a sense much more rigid than their Riemannian counterparts. This rigidity will be a leitmotiv in this text, and finds its origin in the existence of the two lightlike foliations – such a preferred pair of transverse foliations do not exist for singular Riemannian surfaces.

**1.3. Methods, and link with the smoothness of conjugacies for circle diffeomorphisms with breaks.** In [Tro86, Tro91], Troyanov translates the existence, in a given conformal class, of a unique constant curvature Riemannian metric with suitable singularities, into the existence of a unique solution for a differential equation involving the Laplacian. Using the well-behaved properties of the latter, he proves his results by relying mainly on analytical methods. Contrarily to the Riemannian one, the Lorentzian Laplacian is not widely studied, and is more importantly a *hyperbolic* differential operator and not anymore an elliptic one, which makes his use way less suited to our purpose. Moreover, the phenomena that we wish to highlight in this work are by nature dynamical, the geometric rigidity expressed by Theorem A coming from the topological dynamics of the lightlike foliations.

For this reason, we will use in this text a constant interaction of geometrical and dynamical methods. The former will seem more familiar to the readers used to more classical types of geometric structures on surfaces (for instance translation or dilation surfaces), while the latter will come from one-dimensional dynamics (namely piecewise Möbius interval exchange maps and their associated circle homeomorphisms) and will be used in link with the lightlike foliations through their first-return maps.

As we will see in Lemma 2.30, the first-return maps of lightlike foliations in a singular  $\mathbf{dS}^2$ -surface are not only continuous but actually *circle diffeomorphisms with breaks*, and while it may appear as a technical detail, this regularity actually gives a crucial dynamical information on the first-return map  $T$ . Indeed, the seminal work of Denjoy [Den32] implies then that  $T$  does not have an exceptional minimal set, and is thus topologically conjugated to a rigid rotation of the circle if it has moreover an irrational *rotation number*. Since  $T$  is piecewise smooth, it is natural to wonder at this point if  $T$  is actually *smoothly* conjugated to a rotation. But as naive as it may seem, this question is an old and deep one which remains still open in its full generality. If  $T$  is  $C^\infty$  and its rotation number *Diophantine*, Herman showed in [Her79] that it is  $C^\infty$ -conjugated to a rigid rotation, following the initial work of Arnol'd [Arn64] on this question. Since these founding works, the research on this subject never stopped to be intensively active and we do not pretend to cover its vast literature. The problem remains in any case unsolved for circle diffeomorphisms with breaks, about which the optimal result up to date appears in [KKM17] to the best of our knowledge, and answers the question in the case of a single singularity.

The main rigidity result proved in this paper happens to be similar in its philosophy to the problem of smoothness of the conjugacy to a rigid rotation for a circle diffeomorphism. Indeed, a topological equivalence between two pairs of foliations forces in Theorem A the existence of a smooth one – hence of a smooth conjugacy between the first-return maps. This link between singular  $\mathbf{dS}^2$ -structures of the torus and circle diffeomorphisms with breaks is one of our motivations for this subject, and we wish to investigate it more precisely in a future work.

**1.4. Strategies of the main proofs and generalization to multiple singularities.** The first step to prove Theorem D is to reduce it to a one-parameter family of singular  $\mathbf{dS}^2$ -tori introduced in paragraph 3.2, which are identification spaces of lightlike rectangles of  $\mathbf{dS}^2$ , illustrated in Figure 3.1 below. The uniqueness claim is translated in this way in Proposition 3.23 into a statement about a one-parameter family of circle maps – the first-returns of the  $\beta$ -lightlike foliation on the closed  $\alpha$ -leaf. In the end, the latter statement eventually follows from a well-known fact of one-dimensional dynamics: the rotation number of a monotonic one-parameter family of circle homeomorphisms increases *strictly* at irrational points (see Lemma B.1). This scheme of proof may serve as a paradigm for the geometrico-dynamical arguments used in the present paper and for the efficiency of their interactions – geometrical statements becoming easy consequences of dynamical ones once suitably translated.

The general strategy to prove Theorem A is then to show that two structures  $\mu_1$  and  $\mu_2$  with topologically equivalent lightlike foliations admit arbitrarily close *deformations*  $\mu_1^n$  and  $\mu_2^n$ , having a closed  $\alpha$ -leaf at the singularity and identical irrational asymptotic cycles of their  $\beta$ -foliations. We can then rely on Theorem D to say that  $[\mu_1^n] = [\mu_2^n]$  in the deformation space, and conclude that  $[\mu_1] = [\mu_2]$  since the latter sequence converges, by construction, both to  $[\mu_1]$  and to  $[\mu_2]$ .

This strategy of proof of Theorem A will essentially persist for any number of singularities. Indeed, the two main tools developed in this paper to apply the previously summarized strategy are the existence of simple closed timelike geodesics in Appendix A, and the construction of suitable *surgeries* in paragraph 4.3 – and both of them are proved in full generality. The existence of closed geodesics is known for regular Lorentzian manifolds (see for instance [Tip79, Gal86, Suh13]), and we prove in Appendix A that the usual tools and arguments concerning the causal structure and the Lorentzian length remain available in the setting of singular constant curvature Lorentzian surfaces, allowing us to follow the standard proof and to obtain the expected result.

It is actually the proof of Theorem D, and more precisely the one of the dynamical Lemma B.1 which fails for  $n \geq 2$  singularities, and explains that the present paper focuses on the case of a single singularity. Indeed, the rough description that we gave previously hid a fundamental aspect of the proof of Theorem D: after the geometrical reduction to identification spaces of

polygons, the number of parameters of the resulting family of circle maps is equal to the number of singularities of the initial structure. And while the strict monotonicity of the rotation number at irrational points is easily shown for a *one*-parameter family, essentially everything can happen for generic *two*-parameter families of circle maps – which is mainly due to the naive but fundamental observation that the rotation number is itself a *one*-dimensional invariant. The investigation of the rigidity of  $\mathbf{dS}^2$ -tori with multiple singularities requests therefore a new method to handle this dynamical difficulty, which is the content of a work in progress of the author in collaboration with Selim Ghazouani.

**1.5. Perspectives on singular flat tori and an open question.** We will prove in Proposition 2.32 a version of the Gauß-Bonnet formula, showing in particular that a constant curvature Lorentzian metric on the torus with exactly one singularity necessarily has non-zero curvature. The (temporary) restriction to the case of a single singularity in this paper explains thus why we focused here on singular  $\mathbf{dS}^2$ -structures, and not on *flat* ones. Singular flat tori will be independently investigated in a future work, with a more analytical point of view closer to the one of [Tro86, Tro91] concerning singular Riemannian surfaces. Note that in the case of the Minkowski space, the Laplacian is simply the d’Alembertian  $\partial_x^2 - \partial_y^2$ .

To conclude this introduction, we emphasize that in all the examples of singular  $\mathbf{dS}^2$ -tori constructed in this text, both lightlike foliations are suspensions of circle homeomorphisms. The author does not know if there exists a singular  $\mathbf{dS}^2$ -structure on  $\mathbf{T}^2$ , one of whose lightlike foliations has a Reeb component.

**1.6. Organization of the paper.** Basic definitions and properties of singular constant curvature Lorentzian surfaces are introduced and proved in section 2. Section 3 is then concerned with the construction of such structures, and we give in Proposition 3.4 a general existence result of surfaces obtained as identification spaces of polygons with lightlike geodesic edges. In the remainder of section 3 we study thoroughly the properties of a one-parameter and a two-parameter family of  $\mathbf{dS}^2$ -tori with one singularity, which allows us to conclude in paragraph 3.8 the proof of the existence parts of Theorems B, C and D (we prove actually a more refined statement given in Theorem 3.1). The proof of the uniqueness parts of Theorems A, C and D is concluded in section 4. Along the way, we introduce in paragraph 4.3 a notion of surgery, and prove in Appendix A the existence of closed definite geodesics, for general singular constant curvature Lorentzian surfaces. We also prove in Appendix B the main technical result used on the rotation number (which is classical), and in Appendix C that holonomies of lightlike foliations are piecewise Möbius. Lastly, we explain in Appendix D how singular constant curvature Lorentzian surfaces can be seen as Lorentzian length spaces.

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**Some usual notations and a standing assumption.** If  $X$  is a space endowed with an equivalence relation  $\sim$ , then we generally denote by  $\pi: X \rightarrow X/\sim$  the canonical projection when we need it, and also use the notation  $[x] = \pi(x) \in X/\sim$  for  $x \in X$ . For any subset  $P$  of a topological space  $X$ , we denote by  $\text{Int}(P)$  the interior of  $P$ , by  $\text{Cl}(P)$  its closure and by  $\partial P$  its boundary.

*All the surfaces (and any other manifolds) considered in this text are assumed to be connected, orientable and boundaryless, unless explicitly stated otherwise.*

## 2. SINGULAR CONSTANT CURVATURE LORENTZIAN SURFACES

This section is devoted to define and prove the fundamental notions and properties concerning singular constant curvature Lorentzian surfaces.

**2.1. Constant curvature Lorentzian surfaces.** As a preparation to consider singular structures, we first focus in this subsection on regular ones. We define the main Lorentzian notions that will be used throughout the text, and introduce the two 2-dimensional Lorentzian homogeneous spaces as well as the surfaces modelled on them.

**2.1.1. Lorentzian surfaces, time and space-orientation, and lightlike foliations.** A quadratic form is said *Lorentzian* if it is non-degenerate and of signature  $(1, n) = (-, +, \dots, +)$ . A *Lorentzian metric* of class  $\mathcal{C}^k$  on a manifold  $M$  is a  $\mathcal{C}^k$  field  $\mu$  of Lorentzian quadratic forms on the tangent bundle of  $M$ . Usually, we will denote by  $g = g_\mu$  the bilinear form associated to  $\mu$ , so that  $\mu(u) = g(u, u)$ . Observe that if  $\mu$  is a Lorentzian metric on a surface  $S$ , then  $-\mu$  is also a Lorentzian metric on  $S$ .

Any Lorentzian vector space  $(V, q)$  (or tangent space of a Lorentzian manifold) is decomposed according to the sign of  $q, u \in V$  being called:

- (1) *spacelike* if  $q(u) > 0$ ,
- (2) *timelike* if  $q(u) < 0$ ,
- (3) *lightlike* if  $q(u) = 0$ ,
- (4) *causal* is  $q(u) \leq 0$ ,
- (5) and *definite* if it is timelike or spacelike.

These denominations of *signatures* of vectors in Lorentzian tangent spaces will be used in the natural compatible way for line fields and curves.

A *time-orientation* on a Lorentzian surface  $(S, \mu)$  is a continuous choice among one of the two connected components of the cone  $\mu_x^{-1}(\mathbb{R}_-) \setminus \{0\}$  of non-zero timelike vectors, which is called the *future cone*. We will also talk without distinction of the associated *future causal cone*, closure of the future timelike one, and use the obvious similar notion of *space-orientation* in a Lorentzian surface (namely a continuous choice among one of the two connected components of  $\mu_x^{-1}(\mathbb{R}_+) \setminus \{0\}$ ). Not any Lorentzian surface bears a time-orientation, and it is said *time-orientable* if it does. An orientable Lorentzian surface is time-orientable if, and only if it is space-orientable.

Any Lorentzian surface  $S$  bears locally two (unique) *lightlike line fields*, which are globally well-defined if, and only if  $S$  is oriented. In the latter case, they give rise to two *lightlike foliations* on the surface of which we always choose an ordering  $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$  (defined in paragraph 2.1.5 for the surfaces studied in this text), and this ordered pair of foliations will be called the *lightlike bi-foliation* of the surface. If  $S$  is furthermore time-oriented, then these lightlike foliations are themselves orientable. We will always use the convention for which the orientation of the lightlike bi-foliation  $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$  is both compatible with the orientation of  $S$  and with its time-orientation, as illustrated in Figure 2.1 below. In other words with these conventions, a time-orientation and an ordering  $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$  of the lightlike foliations of an oriented Lorentzian surface, induce a space-orientation and an orientation of  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$ .

We will call *quadrant at  $x \in S$*  the four connected components of  $T_x S \setminus \{\mu^{-1}(0)\}$ , or of  $D \setminus (\mathcal{F}_\alpha(x) \cup \mathcal{F}_\beta(x))$  for  $D$  a disk around  $x$  small enough for  $(x, D, I_\alpha, I_\beta)$  to be topologically equivalent to  $(0, ]0; 1[{}^2, ]0; 1[ \times \{0\}, \{0\} \times ]0; 1[)$ , with  $I_{\alpha/\beta}$  the respective connected components of  $D \cap \mathcal{F}_{\alpha/\beta}(x)$  containing  $x$ .

**2.1.2. The Minkowski space.** The flat model space of Lorentzian metrics is the Minkowski space  $\mathbb{R}^{1,n}$ , *i.e.* the vector space  $\mathbb{R}^{n+1}$  endowed with a Lorentzian quadratic form  $q_{1,n}$ . In this text we will be interested in Lorentzian surfaces, and we thus focus now on the Minkowski plane  $\mathbb{R}^{1,1}$  that we endow with the quadratic form  $q_{1,1}(x, y) = 2xy$  and the induced left-invariant Lorentzian metric  $\mu_{\mathbb{R}^{1,1}}$ . We fix on  $\mathbb{R}^{1,1}$  the standard orientation of  $\mathbb{R}^2$ , and the time-orientation (respectively space-orientation) for which the set of future timelike (resp. spacelike) vectors is the top left quadrant  $\{(u, v) \mid u < 0, v > 0\}$  (resp. top right quadrant  $\{(u, v) \mid u > 0, v > 0\}$ ).

The connected component of the identity in the orthogonal group of  $q_{1,1}$  is the subgroup

$$(2.1) \quad \text{SO}^0(1, 1) := \left\{ a^t \mid t \in \mathbb{R} \right\} \subset \text{SL}_2(\mathbb{R}) \text{ with } a^t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Since  $q_{1,1}$  is by construction preserved by translations, the subgroup  $\mathbb{R}^{1,1} \rtimes \text{SO}^0(1, 2)$  of affine transformations preserves  $q_{1,1}$  and its time-orientation, and equals in fact the group  $\text{Isom}^0(\mathbb{R}^{1,1})$

of orientation and time-orientation preserving isometries of  $\mathbb{R}^{1,1}$ . In particular,  $\text{Isom}^0(\mathbb{R}^{1,1})$  acts transitively on  $\mathbb{R}^{1,1}$  with stabilizer  $\text{SO}^0(1,1)$  at  $\mathbf{0} = (0,0)$ , which induces a  $\mathbb{R}^{1,1} \times \text{SO}^0(1,2)$ -equivariant identification of  $\mathbb{R}^{1,1}$  with the homogeneous space  $\mathbb{R}^{1,1} \times \text{SO}^0(1,2)/\text{SO}^0(1,1)$ .

**2.1.3. The de-Sitter space.** We now introduce the Lorentzian homogeneous space of non-zero constant curvature. We will denote by  $[S]$  the projection of  $S \subset \mathbb{R}^{n+1} \setminus \{0\}$  in the projective space  $\mathbb{RP}^n$ , by  $(e_i)$  the standard basis of  $\mathbb{R}^n$ , and use the identification

$$(2.2) \quad \varphi_0: \begin{cases} t \in \mathbb{R} & \mapsto \hat{t} := [t : 1] \in \mathbb{RP}^1 \setminus [e_1] \\ \infty & \mapsto \hat{\infty} := [e_1] \end{cases}$$

between  $\mathbb{R} \cup \{\infty\}$  and  $\mathbb{RP}^1$ . Since any pair of distinct points of  $\mathbb{RP}^1$  is contained in the image  $U$  of the map  $\varphi := g \circ \varphi_0|_{\mathbb{R}}: \mathbb{R} \rightarrow U$  for some  $g \in \text{PSL}_2(\mathbb{R})$ , the set

$$\mathbf{dS}^2 := (\mathbb{RP}^1 \times \mathbb{RP}^1) \setminus \Delta \text{ with } \Delta := \{(p, p) \mid p \in \mathbb{RP}^1\}$$

is covered by the domains of maps of the form

$$(2.3) \quad \phi: (p, q) \in \mathcal{U} := (U \times U) \setminus \Delta \mapsto (\varphi^{-1}(p), \varphi^{-1}(q)) \in \mathbb{R}^2 \setminus \{\text{diagonal}\}$$

which we will call *affine charts of  $\mathbf{dS}^2$* . The transition map between any two such affine charts is by construction of the form  $(x, y) \in I^2 \setminus \{\text{diagonal}\} \mapsto (g(x), g(y)) \in \mathbb{R}^2$ , with  $I \subset \mathbb{R}$  some interval, and  $g$  abusively denoting the homography

$$(2.4) \quad g(t) := \frac{at + b}{ct + d} \text{ associated to } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{R}),$$

characterized by the relation  $g(\hat{t}) = \widehat{g(t)}$ . A direct computation shows that the Lorentzian metric

$$\mu_{\mathbf{dS}^2}^0 := \frac{1}{|x - y|^2} dx dy$$

on  $\mathbb{R}^2 \setminus \{\text{diagonal}\}$  is preserved by the transition maps  $g \times g$  (2.4) between affine charts of  $\mathbf{dS}^2$ , which allows the following.

**Definition 2.1.**  $\mu$  is defined as the Lorentzian metric of  $\mathbf{dS}^2$  equaling  $\phi^* \mu_{\mathbf{dS}^2}^0$  on the domain of any affine chart  $\phi$  of the form (2.3). The Lorentzian surface  $(\mathbf{dS}^2, \mu)$  will be called the *de-Sitter space*.

We endow  $\mathbb{RP}^1$  with the  $\text{PSL}_2(\mathbb{R})$ -invariant orientation induced by the standard one of  $\mathbb{R}$  through the identification (2.2), and  $\mathbf{dS}^2 \subset \mathbb{RP}^1 \times \mathbb{RP}^1$  with the orientation induced by the one of  $\mathbb{RP}^1$ . We also endow  $\mathbf{dS}^2$  with the time-orientation (respectively space-orientation) for which the set of future timelike (resp. spacelike) vectors is the top left quadrant  $\{(u, v) \mid u < 0, v > 0\}$  (resp. top right quadrant  $\{(u, v) \mid u > 0, v > 0\}$ ), in a tangent space endowed with the coordinates coming from an affine chart (2.3).

By construction,  $\mu$  is invariant by the diagonal action  $g(x, y) := (g(x), g(y))$  of  $\text{PSL}_2(\mathbb{R})$  on  $\mathbf{dS}^2$ . This action is moreover transitive and the stabilizer of  $\mathfrak{o} := ([e_1], [e_2]) \in \mathbf{dS}^2$  is the diagonal group

$$A := \{a^t \mid t \in \mathbb{R}\},$$

hence  $\mathbf{dS}^2$  is identified with  $\text{PSL}_2(\mathbb{R})/A$  in a  $\text{PSL}_2(\mathbb{R})$ -equivariant way. Note that the projection  $\text{SL}_2(\mathbb{R}) \rightarrow \text{PSL}_2(\mathbb{R})$  induces an isomorphism from  $\text{SO}^0(1,1)$  defined in (2.1) with  $A$ .

We now give another (more usual) description of the de-Sitter space. The quadratic form  $q_{1,2}$  of the Minkowski space  $\mathbb{R}^{1,2}$  equips (by restriction to its tangent bundle) the quadric

$$\mathbf{dS}^2 := \{x \in \mathbb{R}^3 \mid q_{1,2}(x) = 1\}$$

with a Lorentzian metric  $\mu_{\mathbf{dS}^2}$  of sectional curvature constant equal to 1 (see for instance [O'N83, Proposition 4.29]), and the Lorentzian surface  $(\mathbf{dS}^2, \mu_{\mathbf{dS}^2})$  is the two-dimensional *hyperboloid model of the de-Sitter space*. Observe that endowing  $\mathbf{dS}^2$  with the restriction of the quadratic form  $q_{2,1} := -q_{1,2}$  defines a Lorentzian metric of constant curvature equal to  $-1$ . In other words,



the de-Sitter and anti-de-Sitter spaces are anti-isometric in dimension 2 and have thus the same geometry.

**Lemma 2.2.** (1)  $\mathrm{PSL}_2(\mathbb{R})$  is the subgroup of isometries of  $(\mathbf{dS}^2, \boldsymbol{\mu})$  preserving both its orientation and time-orientation.  
 (2)  $(\mathbf{dS}^2, \boldsymbol{\mu})$  is isometric to  $(\mathrm{dS}^2, \mu_{\mathrm{dS}^2})$  up to a multiplicative constant. For the sake of clarity, we normalize henceforth  $(\mathbf{dS}^2, \boldsymbol{\mu})$  to have constant curvature 1.

*Proof.* (1) This claim follows from the facts that  $\mathrm{PSL}_2(\mathbb{R})$  acts transitively on  $\mathbf{dS}^2$ , that the stabilizer of points in  $\mathrm{PSL}_2(\mathbb{R})$  realize all linear isometries (*i.e.* that  $a \in A \mapsto D_{\mathfrak{o}}a \in \mathrm{O}(\mathrm{T}_{\mathfrak{o}}\mathbf{dS}^2, \boldsymbol{\mu}_{\mathfrak{o}})$  is surjective), and that the one-jet determines pseudo-Riemannian isometries (a local isometry defined on a connected open subset, fixing a point  $x$  and of trivial differential at  $x$ , is the identity). (2) One checks that the stabilizer in  $\mathrm{SO}^0(1, 2)$  of a point of  $\mathrm{dS}^2$  is a one-parameter hyperbolic subgroup, which gives an identification between  $\mathrm{dS}^2$  and  $\mathrm{PSL}_2(\mathbb{R})/A$ , equivariant with respect to some isomorphism between  $\mathrm{SO}^0(1, 2)$  and  $\mathrm{PSL}_2(\mathbb{R})$ . This yields two  $\mathrm{PSL}_2(\mathbb{R})$ -invariant Lorentzian metrics on  $\mathrm{PSL}_2(\mathbb{R})/A$ , respectively coming from the identifications with  $(\mathrm{dS}^2, \mu_{\mathrm{dS}^2})$  and  $(\mathbf{dS}^2, \boldsymbol{\mu})$ . But up to multiplication by a constant,  $\mathfrak{sl}_2/\mathfrak{a}$  admits a unique Lorentzian quadratic form which is invariant by the adjoint action of  $A$ , and  $\mathrm{PSL}_2(\mathbb{R})/A$  admits therefore a unique  $\mathrm{PSL}_2(\mathbb{R})$ -invariant Lorentzian metric up to multiplication by a constant.  $\square$

*Remark 2.3.* We emphasize that  $\mathcal{C} := \mathbf{P}^+(q_{1,2}^{-1}(0)) = \{l \subset \mathbb{R}^{1,2} \mid \text{null half-line}\}$  can be naturally interpreted as the *conformal boundary* of  $\mathrm{dS}^2$ , and that this interpretation yields a natural identification of  $\mathrm{dS}^2$  with  $\mathbf{dS}^2$  where each  $\mathbb{RP}^1$  appears as a connected component of  $\mathcal{C}$ . We refer to the proof of Proposition C.2 for more details on this construction.

2.1.4. *Lorentzian  $\mathbf{X}$ -surfaces.* We will be interested in this paper in the Lorentzian surfaces locally modelled on one of the two formerly introduced homogeneous spaces. Denoting henceforth by  $(\mathbf{G}, \mathbf{X})$  one of the pairs  $(\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1, 2), \mathbb{R}^{1,1})$  or  $(\mathrm{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$ , we will use in this text the convenient language of  $(\mathbf{G}, \mathbf{X})$ -structures that we now introduce.

**Definition 2.4.** A  $(\mathbf{G}, \mathbf{X})$ -atlas on an oriented topological surface  $S$  is an atlas of orientation-preserving  $\mathcal{C}^0$ -charts from connected open subsets of  $S$  to  $\mathbf{X}$ , whose transition maps are restrictions of elements of  $\mathbf{G}$  (any two open domains of the atlas being always assumed to have a connected intersection). A  $(\mathbf{G}, \mathbf{X})$ -structure is a maximal  $(\mathbf{G}, \mathbf{X})$ -atlas, and a  $(\mathbf{G}, \mathbf{X})$ -surface an oriented surface endowed with a  $(\mathbf{G}, \mathbf{X})$ -structure. A  $(\mathbf{G}, \mathbf{X})$ -morphism between two  $(\mathbf{G}, \mathbf{X})$ -surfaces is a map which reads in any connected  $(\mathbf{G}, \mathbf{X})$ -chart as the restriction of an element of  $\mathbf{G}$ .

**Convention 2.5.** All along this paper,  $\mathbf{X}$  will be considered solely with the action of the group  $\mathbf{G}$ . In order to make the text lighter, we thus drop henceforth  $\mathbf{G}$  from our notations, and talk simply of  $\mathbf{X}$ -chart,  $\mathbf{X}$ -structure,  $\mathbf{X}$ -surface and  $\mathbf{X}$ -morphism.

For any  $\mathbf{X}$ -structure on a surface  $S$ , each covering  $\pi: S' \rightarrow S$  of  $S$  is induced with the unique  $\mathbf{X}$ -structure for which  $\pi$  is a  $\mathbf{X}$ -morphism. In particular,  $\pi_1(S)$  acts on the universal cover  $\tilde{S}$  by  $\mathbf{X}$ -morphisms of its  $\mathbf{X}$ -structure. Moreover for any  $\mathbf{X}$ -morphism  $f$  from a connected open subset  $U \subset \tilde{S}$  to  $\mathbf{X}$ , there exists a unique extension

$$\delta: \tilde{S} \rightarrow \mathbf{X}$$

of  $f$  to a  $\mathbf{X}$ -morphism defined on  $\tilde{S}$ , and such a map is called a *developing map* of  $S$ . For any developing map  $\delta$ , there exists furthermore a unique group morphism

$$\rho: \pi_1(S) \rightarrow \mathbf{G}$$

with respect to which  $\delta$  is equivariant, called the *holonomy morphism* associated to  $\delta$ . Such a pair  $(\delta, \rho)$  associated to the  $\mathbf{X}$ -structure of  $S$  is unique up to the action  $g \cdot (\delta, \rho) := (g \circ \delta, g\rho g^{-1})$  of  $\mathbf{G}$ , and reciprocally any such pair defines a unique compatible  $\mathbf{X}$ -structure on  $S$ . We refer the reader to [Thu97, CEG87] for more details on  $(G, X)$ -structures.

The core idea of  $\mathbf{X}$ -surfaces is that any  $\mathbf{G}$ -invariant geometric object on  $\mathbf{X}$  gives rise to a corresponding object on any  $\mathbf{X}$ -surface. In particular, any  $\mathbf{X}$ -surface bears a unique Lorentzian metric for which the  $\mathbf{X}$ -charts are local isometries to  $\mathbf{X}$ , as well as a time and space-orientation

transported to the ones of  $\mathbf{X}$  by the  $\mathbf{X}$ -charts. Let  $\varepsilon_{\mathbf{X}}$  denote the constant sectional curvature of  $\mathbf{X}$ .

**Proposition-Definition 2.6.** *On any orientable surface  $S$ ,  $\mathbf{X}$ -structures are in equivalence with time-oriented Lorentzian metrics of constant curvature  $\varepsilon_{\mathbf{X}}$  in the following way.*

- (1) *For any  $\mathbf{X}$ -structure on  $S$ , there exists a unique Lorentzian metric for which  $(\mathbf{G}, \mathbf{X})$ -charts are local isometries. The latter metric is time-oriented and has constant curvature  $\varepsilon_{\mathbf{X}}$ .*
- (2) *Conversely, any time-oriented Lorentzian metric of constant curvature  $\varepsilon_{\mathbf{X}}$  on  $S$  is induced by a unique  $\mathbf{X}$ -structure.*
- (3) *Moreover under this correspondence, the  $\mathbf{X}$ -morphisms between  $\mathbf{X}$ -surfaces are exactly their orientation-preserving isometries between connected open subsets.*

*Proof.* (1) Since  $\mathbf{G}$  preserves the time-orientation of  $\mathbf{X}$ , the Lorentzian metric induced by a  $\mathbf{X}$ -structure is time-oriented, and of constant curvature  $\varepsilon_{\mathbf{X}}$ .

(2) Let  $\mu$  be a time-oriented Lorentzian metric on  $S$  of constant sectional curvature  $\varepsilon_{\mathbf{X}}$ . Then it is locally isometric to  $\mathbf{X}$  according to [O'N83, Corollary 8.15], and there exists thus an atlas of local isometric charts of  $S$  to  $\mathbf{X}$  preserving both orientation and time-orientation. We claim that the transition maps of such an atlas and between two such atlases are restrictions of elements of  $\mathbf{G}$ , which will prove the claim. This is essentially due to the counterpart of the *Liouville theorem* for  $(\mathbf{G}, \mathbf{X})$ , claiming that any orientation and time-orientation preserving local isometry between two connected open subsets of  $\mathbf{X}$ , is the restriction of an element of  $\mathbf{G}$ . This last claim is easily obtained from the proof of Lemma 2.2.(2).

(3) Liouville theorem proves in particular the last claim. □

2.1.5. *Lightlike  $\alpha$  and  $\beta$ -foliations of  $\mathbf{X}$ -surfaces.* We now describe the lightlike foliations of our models.

**Definition 2.7.** We will call  $\alpha$  and  $\beta$ -foliation and denote by  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$  the foliations of  $\mathbf{dS}^2$  (respectively  $\mathbb{R}^{1,1}$ ) whose leaves are the respective fibers of the second and first projections of  $\mathbf{dS}^2 \subset \mathbb{RP}^1 \times \mathbb{RP}^1$  to  $\mathbb{RP}^1$  (resp. the horizontal and vertical affine lines of  $\mathbb{R}^{1,1}$ ). We call and denote in the same way the lightlike foliations induced by the latter on any  $\mathbf{dS}^2$ -surface (resp.  $\mathbb{R}^{1,1}$ -surface).

In other words, the  $\alpha$ -leaves (resp.  $\beta$ -leaves) of  $\mathbf{dS}^2$  read as horizontal (resp. vertical) lines in any affine chart (2.3) (hence the denomination to match the one for  $\mathbb{R}^{1,1}$ ). Observe that the action of  $\mathrm{PSL}_2(\mathbb{R})$  on  $\mathbf{dS}^2$  (respectively of  $\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1, 2)$  on  $\mathbb{R}^{1,1}$ ) preserve both the  $\alpha$  and the  $\beta$ -foliation, which induce thus indeed foliations on any  $\mathbf{dS}^2$ -surface (resp.  $\mathbb{R}^{1,1}$ -surface).

We endow the lightlike leaves of  $\mathbf{dS}^2$  with the  $\mathrm{PSL}_2(\mathbb{R})$ -invariant orientation induced by the one of  $\mathbb{RP}^1$ , and the lightlike leaves  $\mathbb{R} \times \{b\}$  and  $\{a\} \times \mathbb{R}$  of  $\mathbb{R}^{1,1}$  with the  $\mathbb{R}^{1,1} \rtimes \mathrm{SO}^0(1, 2)$ -one induced by  $\mathbb{R}$ . This further induces an orientation on the lightlike foliations of any  $\mathbf{X}$ -surface, compatible with its orientation, time-orientation and space-orientation as illustrated by Figure 2.1 below.

The lightlike leaves of  $\mathbf{dS}^2$  and  $\mathbb{R}^{1,1}$  are embeddings of  $\mathbb{R}$ , and we denote by  $\mathcal{F}_\alpha^{+*}(p)$  and  $\mathcal{F}_\alpha^{-*}(p)$  the *half  $\alpha$ -leaves*, i.e. the two connected components of  $\mathcal{F}_\alpha(p) \setminus \{p\}$  emanating respectively in the positive and negative directions, by  $\mathcal{F}_\alpha^+(p)$  and  $\mathcal{F}_\alpha^-(p)$  their closures, and accordingly for  $\mathcal{F}_\beta^\pm(p)$ . Note that the lightlike leaves are the lightlike geodesics of the metric (for  $\mathbf{dS}^2$ , this is most easily seen in the hyperboloid model, where the geodesics are the intersection of planes with the hyperboloid).

2.1.6. *Cyclic order, intervals of a circle and rectangles of  $\mathbf{dS}^2$ .* The circles  $\mathbb{RP}^1$  and  $\mathbf{S}^1$  inherit from their orientation a *cyclic ordering*, i.e. a partition of triplets  $(x_1, x_2, x_3) \in (\mathbb{RP}^1)^3$  (respectively  $(\mathbf{S}^1)^3$ ) between *positive* and *negative* ones which is invariant by cyclic permutations, exchanged by transpositions and defined in the following way. Any  $n$ -tuple ( $n \geq 3$ ) of two-by-two distinct points of  $\mathbb{RP}^1$  has an ordering  $(x_1, \dots, x_n)$ , unique up to the  $n$  cyclic permutations  $(1, \dots, n)^k$  for  $1 \leq k \leq n$ , such that for any  $1 \leq i \leq n-1$ , the positively oriented injective path of  $\mathbb{RP}^1$  from  $x_i$  to  $x_{i+1}$  does not meet any of the  $x_j$  for  $j \notin \{i, i+1\}$ . In this case  $(x_1, \dots, x_n)$  is said to be *positively cyclically ordered*, and two  $n$ -tuples  $(x_1, \dots, x_n)$  and  $(y_1, \dots, y_n)$  are said

to have *the same cyclic order* if there exists a permutation  $\sigma$  such that  $(x_{\sigma(1)}, \dots, x_{\sigma(n)})$  and  $(y_{\sigma(1)}, \dots, y_{\sigma(n)})$  are both positive. For any  $x, y \in \mathbb{RP}^1$ , we denote

$$[x; y] := \{x, y\} \cup \left\{ z \in \mathbb{RP}^1 \mid (x, z, y) \text{ is positively cyclically ordered} \right\} \subset \mathbb{RP}^1$$

with  $[x; y] = \{x\}$  if  $x = y$ , and adopt the same notation for any oriented topological circle. For any  $p = (x_p, y_p), q = (x_q, y_q) \in \mathbf{dS}^2$  such that  $q \in \mathcal{F}_\alpha^+(p)$  – respectively  $q \in \mathcal{F}_\beta^+(p)$  – we denote

$$[p; q]_\alpha := \{(x, y_p) \mid x \in [x_p; x_q]\} - \text{resp. } [p; q]_\beta := \{(x_p, y) \mid y \in [y_p; y_q]\},$$

with obvious corresponding notations in  $\mathbb{R}^{1,1}$ , and for open or half-open intervals of extremities  $p$  and  $q$ . More generally in any  $\mathbf{X}$ -surface,  $[p; q]_{\alpha/\beta}$  denotes the portion of  $\mathcal{F}_{\alpha/\beta}(p)$  from  $p$  to  $q$  for the natural orientation of  $\mathcal{F}_{\alpha/\beta}(p)$ .

**Definition 2.8.** For any four distinct points  $(A, B, C, D) \in \mathbf{dS}^2$  such that  $(x_A, y_A) = A = \mathcal{F}_\alpha^-(B) \cap \mathcal{F}_\beta^-(D)$  and  $(x_C, y_C) = C = \mathcal{F}_\beta^+(B) \cap \mathcal{F}_\alpha^+(D)$ ,

$$\mathcal{R}_{ABCD} = \mathcal{R}_{(x_A, x_C, y_A, y_C)} := [x_A; x_C] \times [y_A; y_C]$$

will be called a *rectangle of  $\mathbf{dS}^2$  with lightlike boundary*.

Note that by convention, the rectangles that we consider are non-degenerated (*i.e.* have distinct edges), and that we name the vertices of a rectangle  $\mathcal{R}_{ABCD}$  of  $\mathbf{dS}^2$  in the positive cyclic order by starting with its “bottom-left” vertex  $A$ . The area of an orientable surface  $S$  for the area form induced by a Lorentzian metric  $\mu$  (which, by definition, gives volume 1 to an orthogonal basis of norms  $(1, -1)$  for  $\mu$ ), will be denoted by  $A_\mu(S)$ .

**Lemma 2.9.** *Two rectangles of  $\mathbf{dS}^2$  with lightlike boundaries are in the same orbit under  $\text{PSL}_2(\mathbb{R})$  if, and only if they have the same area.*

*Proof.* For any rectangle  $\mathcal{R}_{(x_A, x_C, y_A, y_C)}$ ,  $(y_A, y_C, x_A)$  is a positively cyclically ordered triplet of  $\mathbb{RP}^1$ , and we can thus assume without loss of generality that  $\mathcal{R}_{(x_A, x_C, y_A, y_C)} = \mathcal{R}_{(\hat{1}, \hat{t}, \hat{\infty}, \hat{0})}$ . Since  $t \in ]1; +\infty[ \mapsto A_\mu(\mathcal{R}_{(\hat{1}, \hat{t}, \hat{\infty}, \hat{0})}) \in \mathbb{R}_+^*$  is bijective, two rectangles have the same area if, and only if the 4-tuples defining them have the same cross-ratio, which happens if and only if they are in the same orbit under  $\text{PSL}_2(\mathbb{R})$ .  $\square$

**2.2. The local model of standard singularities.** We define in this subsection the local singularities that will be considered in this text (which appeared in [BBS11, §3.3]), and prove some of their fundamental properties.

$(\mathbf{G}, \mathbf{X})$  denotes one of the pairs  $(\mathbb{R}^{1,1} \rtimes \text{SO}^0(1, 2), \mathbb{R}^{1,1})$  or  $(\text{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$ ,  $\mu$  the Lorentzian metric of  $\mathbf{X}$ , and  $g_\mu$  its associated bilinear form. We also fix a base-point  $\mathfrak{o} \in \mathbf{X}$ , respectively equal to  $(0, 0)$  or  $([e_1], [e_2])$ , and denote by  $A = \{a^t\}_{t \in \mathbb{R}}$  its stabilizer in  $\mathbf{G}$ .

**Convention 2.10.** Henceforth, we will use the unique parametrization of  $A = \{a^t\}_{t \in \mathbb{R}}$  satisfying the following for any non-zero future spacelike vector  $u \in \text{T}_\mathfrak{o}\mathbf{X}$ .

- (1) With  $u_t$  the unique point of  $\mathbb{R}^+ \text{D}_\mathfrak{o} a^t(u)$  belonging to the unit circle  $C$  of a fixed Euclidean quadratic form on  $\text{T}_\mathfrak{o}\mathbf{X}$ ,  $t \mapsto u_t$  is a positively oriented curve on  $C$  (endowed with the orientation induced from the one of  $\mathbf{X}$ ).
- (2) Moreover denoting by  $\cosh$  the hyperbolic cosine function, for any  $t \in \mathbb{R}$  we have:

$$\frac{g_\mu(u, a^t(u))}{\mu(u)} = \cosh(t).$$

This convention will be crucial for the correspondence (2.5) between angles and areas given below by Gauß-Bonnet formula. Apart from this formula, the convention does not matter. We emphasize that for  $\mathbf{X} = \mathbb{R}^{1,1}$ , the parametrization is simply the usual one given by (2.1).

2.2.1. *Standard singularities as identification spaces.* We denote by  $\mathbf{X}_*$  the surface with boundary and one conical point obtained from  $\mathbf{X}$  by cutting it along  $\mathcal{F}_\alpha^{+*}(\mathfrak{o})$ . The interior of  $\mathbf{X}_*$  is identified with  $\mathbf{X} \setminus \mathcal{F}_\alpha^+(\mathfrak{o})$ , its conical point  $\mathfrak{o}'$  with  $\mathfrak{o}$ , and its two boundary components are “upper” and “lower” embeddings  $\iota_\pm: \mathcal{F}_\alpha^+(\mathfrak{o}) \rightarrow \mathbf{X}_*$  of  $\mathcal{F}_\alpha^+(\mathfrak{o})$  with  $\iota_\pm(\mathfrak{o}) = \mathfrak{o}'$ . Furthermore  $\mathbf{X}_*$  is endowed with an action of the diagonal subgroup  $A$  for which the embeddings  $\iota_\pm$  are equivariant.

For  $\theta \in \mathbb{R}$ , we introduce the equivalence relation generated by the relations  $\iota_+(x) \sim_\theta \iota_-(a^\theta(x))$  for any  $x \in \mathcal{F}_\alpha^{+*}(\mathfrak{o})$ , and we denote by

$$\pi_\theta: \mathbf{X}_* \rightarrow \mathbf{X}_\theta = \mathbf{X}_*/\sim_\theta$$

the canonical projection onto the topological quotient of  $\mathbf{X}_*$  by  $\sim_\theta$ . This identification space is illustrated in Figure 2.1.

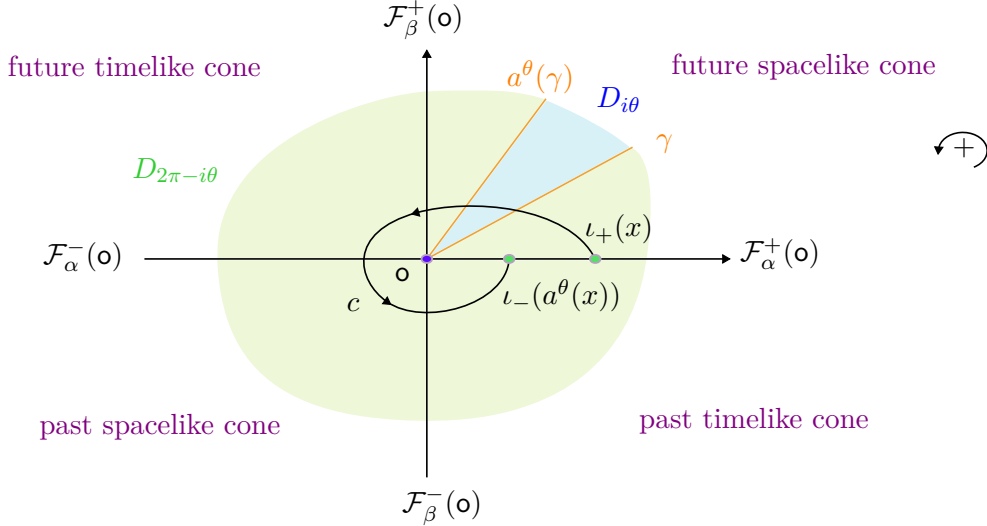


FIGURE 2.1. Standard singularity, quadrants and orientations.

We define  $\mathfrak{o}_\theta := \pi_\theta(\mathfrak{o}')$  and endow  $\mathbf{X}_\theta \setminus \{\mathfrak{o}_\theta\}$  with its *standard  $\mathbf{X}$ -structure* defined by the following atlas.

- (1) For any open set  $U \subset \mathbf{X} \setminus \mathcal{F}_\alpha^+(\mathfrak{o})$ , we consider the chart  $\varphi_{\pi_\theta(U)}: \pi_\theta(U) \rightarrow U$  satisfying  $\varphi_{\pi_\theta(U)} \circ \pi_\theta|_U = \text{id}|_U$ .
- (2) Let  $U \subset \mathbf{X} \setminus \{\mathfrak{o}\}$  be an open set such that  $U \setminus \mathcal{F}_\alpha^+(\mathfrak{o})$  has two respectively up and down connected components  $U_+$  and  $U_-$ , and  $a^\theta(U) \cap U = \emptyset$ . Then we consider the open set  $V = \pi_\theta(U_+ \cup \iota_+(U \cap \mathcal{F}_\alpha^+(\mathfrak{o})) \cup a^\theta(U_-))$  of  $\mathbf{X}_\theta$ , and the chart  $\varphi_V: V \rightarrow U$  satisfying:
  - $\varphi_V \circ \pi_\theta = \text{id}$  in restriction to  $U_+ \cup \iota_+(U \cap \mathcal{F}_\alpha^+(\mathfrak{o}))$ ,
  - and  $\varphi_V \circ \pi_\theta = a^{-\theta}$  in restriction to  $a^\theta(U_-)$ .

**Definition 2.11.** The *standard  $\mathbf{X}$ -cone of angle  $\theta$*  is the oriented topological surface  $\mathbf{X}_\theta$  endowed with its marked point  $\mathfrak{o}_\theta$ , its standard  $\mathbf{X}$ -structure on  $\mathbf{X}_\theta \setminus \{\mathfrak{o}_\theta\}$  and its associated Lorentzian metric denoted by  $\mu_\theta$ .

Note that our definition makes sense for  $\theta = 0$ , and that in this case  $\mathbf{X}_0 = \mathbf{X}$ .

*Remark 2.12.* The standard cones that we have introduced do not exhaust the natural geometric singularities, and we refer to Remark 3.6 for a discussion of other kind of examples. However these singularities are the *dynamically natural* ones: they are essentially the only ones at which the lightlike foliations extend to two continuous foliations, in a sense made more precise in Lemma 2.14. The existence of these continuous foliations is our main motivation for considering this specific type of singularities, and is the subject of the next paragraph.

2.2.2. *Lightlike foliations at a standard singularity.* To investigate the behaviour of the lightlike foliations at the singularity, we consider a continuous chart of  $\mathbf{X}_\theta$  at  $\mathfrak{o}_\theta$  defined as follows. Let  $\text{exp}_\mathfrak{o}: T_\mathfrak{o}\mathbf{X} \rightarrow \mathbf{X}$  denote the exponential chart of  $\mathbf{X}$  at  $\mathfrak{o}$ , and  $d_\nu \subset T_\mathfrak{o}\mathbf{X}$  the half-open line making a positive euclidean angle  $\nu \in [0; 2\pi[$  with  $d_0$ , where  $\text{exp}_\mathfrak{o}(d_0) \subset \mathcal{F}_\alpha^+(\mathfrak{o})$ . Note that

$a^\theta \circ \exp_{\mathfrak{o}} = \exp_{\mathfrak{o}} \circ D_{\mathfrak{o}} a^\theta$ , hence with  $\theta' \in \mathbb{R}$  characterized by  $D_{\mathfrak{o}} a^\theta(X) = e^{-2\theta'} X$  for  $X \in T_{\mathfrak{o}} \mathcal{F}_\alpha(\mathfrak{o})$ , we have  $\iota_+(\exp_{\mathfrak{o}}(X)) \sim_\theta \iota_-(\exp_{\mathfrak{o}}(e^{-2\theta'} X))$ . With  $D$  an open disk centered at  $0$  in  $T_{\mathfrak{o}} \mathbf{X}$ , we consider the open neighbourhood

$$U := \iota_+ \circ \exp_{\mathfrak{o}}(d_0 \cap D) \cup \bigcup_{\nu \in ]0; 2\pi[} \exp_{\mathfrak{o}}(e^{-\frac{\nu}{\pi} \theta'}(d_\nu \cap D))$$

of  $\mathfrak{o}'$  in  $\mathbf{X}_*$ , so that  $V = \pi_\theta(U)$  is an open neighbourhood of  $\mathfrak{o}_\theta$  in  $\mathbf{X}_\theta$ . We define then a map  $\psi_\theta: V \rightarrow D$ , for any  $\nu \in ]0; 2\pi[$  and  $X \in e^{-\frac{\nu}{\pi} \theta'}(d_\nu \cap D)$ , by

$$\psi_\theta \circ \pi_\theta(\exp_{\mathfrak{o}}(X)) = e^{\frac{\nu}{\pi} \theta'} X.$$

In the above equation for  $p \in \mathcal{F}_\alpha^+(\mathfrak{o})$ , we denoted  $\iota_+(p)$  simply by  $p$ . It is easily checked that  $\psi$  is a homeomorphism from  $V$  to  $D$ .

**Proposition 2.13.** *The lightlike foliations of  $\mathbf{X}_\theta \setminus \{\mathfrak{o}_\theta\}$  extend uniquely to two topological one-dimensional foliations on  $\mathbf{X}_\theta$ , that we call the lightlike foliations of  $\mathbf{X}_\theta$  and continue to denote by  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$ . Moreover for any small enough open neighbourhoods  $I$  and  $J$  of  $\mathfrak{o}_\theta$  in  $\mathcal{F}_\alpha(\mathfrak{o}_\theta)$  and  $\mathcal{F}_\beta(\mathfrak{o}_\theta)$ ,*

$$\Phi: (x, y) \in I \times J \mapsto \mathcal{F}_\beta(x) \cap \mathcal{F}_\alpha(y)$$

*is a homeomorphism onto its image, restricting outside of  $\mathfrak{o}_\theta$  to a  $\mathcal{C}^\infty$ -diffeomorphism onto its image. The continuous  $\alpha$  and  $\beta$ -foliations are thus transverse in the sense that  $\Phi$  defines a simultaneous  $\mathcal{C}^0$  foliated chart.*

*Proof.* Since  $\psi_\theta(\pi_\theta(\iota_+(\mathcal{F}_\alpha^{+*}(\mathfrak{o})) \cup \mathcal{F}_\alpha^{-*}(\mathfrak{o}))) = \mathbb{R} \cdot d_0 \setminus \{0\}$  and  $\psi_\theta(\pi_\theta(\mathcal{F}_\beta^{+*}(\mathfrak{o}) \cup \mathcal{F}_\beta^{-*}(\mathfrak{o}))) = \mathbb{R} \cdot d_\beta \setminus \{0\}$  where  $\exp_{\mathfrak{o}}(\mathbb{R} \cdot d_\beta) = \mathcal{F}_\beta(\mathfrak{o})$ , the only possible definition of the  $\alpha$  and  $\beta$ -leaves of  $\mathfrak{o}_\theta$  for it to define a foliation with continuous leaves, is:  $\mathcal{F}_\alpha(\mathfrak{o}_\theta) = \pi_\theta \circ \iota_+(\mathcal{F}_\alpha^+(\mathfrak{o})) \cup \pi_\theta(\mathcal{F}_\alpha^{-*}(\mathfrak{o}))$  and  $\mathcal{F}_\beta(\mathfrak{o}_\theta) = \{\mathfrak{o}_\theta\} \cup \pi_\theta(\mathcal{F}_\beta^{+*}(\mathfrak{o}) \cup \mathcal{F}_\beta^{-*}(\mathfrak{o}))$ . This makes  $\mathcal{F}_\alpha(\mathfrak{o}_\theta)$  and  $\mathcal{F}_\beta(\mathfrak{o}_\theta)$  two topological 1-manifolds. Now for any small enough open neighbourhoods  $I$  and  $J$  of  $\mathfrak{o}_\theta$  in  $\mathcal{F}_\alpha(\mathfrak{o}_\theta)$  and  $\mathcal{F}_\beta(\mathfrak{o}_\theta)$ , and any  $(x, y) \in I \times J$ :  $\mathcal{F}_\beta(x) \cap \mathcal{F}_\alpha(y)$  is a single point which we denote by  $[x, y]$ . Moreover for  $x, x' \in \mathcal{F}_\alpha(\mathfrak{o}_\theta)$ ,  $x \neq x'$  implies  $\mathcal{F}_\beta(x) \cap \mathcal{F}_\beta(x') = \emptyset$ , and similarly for  $y \neq y' \in \mathcal{F}_\beta(\mathfrak{o}_\theta)$ . Therefore  $\Phi: (x, y) \in I \times J \mapsto [x, y]$  is an injective map from  $I \times J$  to the topological surface  $\mathbf{X}_\theta$ , which is clearly continuous, and  $\Phi(\mathfrak{o}_\theta, \mathfrak{o}_\theta) = \mathfrak{o}_\theta$ . By Brouwer's invariance of domain theorem,  $\Phi$  is thus a homeomorphism onto its image  $U$ , which is an open neighbourhood of  $\mathfrak{o}_\theta$ . Observe moreover that  $\Phi$  is a  $\mathcal{C}^\infty$ -diffeomorphism onto its image on restriction to any small enough open subset of  $\mathbf{X}_\theta \setminus \{\mathfrak{o}_\theta\}$ , since it is so in  $\mathbf{X}$ . Furthermore  $\Phi(\{x\} \times J)$  contains an open neighbourhood of  $x$  in  $\mathcal{F}_\beta(x)$ , and  $\Phi(I \times \{y\})$  an open neighbourhood of  $y$  in  $\mathcal{F}_\alpha(y)$ . The restriction of  $\Phi$  to suitable subsets defines thus a simultaneous continuous foliated chart for the  $\alpha$  and  $\beta$ -foliations, which concludes the proof.  $\square$

**2.2.3. Characterization of standard singularities by the developing map.** We now characterize the singularity  $\mathfrak{o}_\theta$  of  $\mathbf{X}_\theta$  among the  $\mathbf{X}$ -structures of a punctured disk. Let us call *slit neighbourhood of  $\mathbf{X}$*  an open set of the form  $U' = U \setminus \mathcal{F}_\alpha^+(p)$  for  $U$  an open neighbourhood of a point  $p \in \mathbf{X}$ .

**Lemma 2.14.** *Let  $D$  be an oriented topological disk,  $x \in D$ , and  $D^* := D \setminus \{x\}$  be endowed with a  $\mathbf{X}$ -structure. Let  $R$  denote the positive generator of  $\pi_1(D^*)$ , i.e. the homotopy class of a positively oriented closed loop around  $x$  generating  $\pi_1(D^*)$ . Then the following properties (1) and (2) are equivalent.*

- (1) *There exists  $\theta \in \mathbb{R}$ , and a homeomorphism  $\varphi$  from an open neighbourhood  $U$  of  $x$  to an open neighbourhood of  $\mathfrak{o}_\theta$  in  $\mathbf{X}_\theta$ , such that:  $\varphi(x) = \mathfrak{o}_\theta$ , and  $\varphi$  is a  $\mathbf{X}$ -morphism in restriction to  $U^* = U \setminus \{x\}$ .*
- (2) (a) *The lightlike foliations of  $D^*$  extend uniquely to two continuous 1-dimensional foliations of  $D$ ;*  
 (b) *and there exists an open disk  $U \subset D$  containing  $x$ , and a  $\mathbf{X}$ -isomorphism  $\psi$  from  $U' = U \setminus \mathcal{F}_\alpha^+(x)$  to a slit neighbourhood of  $\mathfrak{o}$ .*

*Furthermore property (1) for  $\theta \in \mathbb{R}$  is equivalent to (2).(a) and (2).(b) together with:*

- (2).(c)  *$\rho(R) = a^\theta$ , with  $\rho$  the holonomy morphism associated to the developing map extending the lift of a  $\mathbf{X}$ -morphism  $\psi$  to a slit neighbourhood of  $\mathfrak{o}$  like in (2).(b).*

In particular, there exists at most one  $\theta \in \mathbb{R}$  for which the equivalent properties (1) and (2) can be satisfied for  $\theta$ .

**Definition 2.15.** Let  $D^* := D \setminus x$  be an oriented topological punctured disk endowed with a  $\mathbf{X}$ -structure. We will say that  $x$  is a *standard singularity of angle  $\theta$*  of  $D$  if the equivalent properties (1) and (2).(a)-(c) of Lemma 2.14 are satisfied at  $x$  for  $\theta \in \mathbb{R}$ . A developing map of  $D^*$  extending a lift of  $\varphi$  like in (1) (equivalently of  $\psi$  like in (2).(b)) and its holonomy morphism are said *compatible at  $x$* .

*Remark 2.16.* The holonomy of a positively oriented loop around a singularity is well defined only *up to conjugacy*, and for  $\theta \in \mathbb{R}$  and  $g \in \mathrm{PSL}_2(\mathbb{R})$ :  $a^\theta = ga^{-\theta}g^{-1}$  if, and only if  $g$  is an anti-diagonal matrix. Hence if the angle of singularities were to be simply defined as the latter holonomy conjugacy class, then it would be well-defined only *up to sign*. For this reason one has to consider specific developing maps around a standard singularity  $x$  to define the sign of its angle: the *compatible* ones as introduced in Definition 2.15. Let  $\pi: E \rightarrow D^* = D \setminus \{x\}$  be the universal covering of a singular  $\mathbf{X}$ -disk with a single singularity at  $x$ , and  $F \subset E$  be a closed fundamental domain of  $\pi$ , such that  $\pi|_{\mathrm{Int} F}$  is injective,  $\pi(F) = D^*$  and  $\partial F$  is a copy of two lifts  $I^d$  and  $I^u = R(I^d)$  of  $\mathcal{F}_\alpha^{+*}(x)$ . Then a developing map  $\delta: E \rightarrow \mathbf{X}$  is compatible at  $x$  if, and only if  $\delta(\mathrm{Int} F)$  is a slit neighbourhood of  $\mathfrak{o}$ . We will see in Lemma 2.20 and Remark 2.21 another intrinsic characterization of the angle of a singularity.

Lemma 2.14 implies directly the following results.

**Corollary 2.17.** Let  $D^* := D \setminus x$  be an oriented punctured disk endowed with a  $\mathbf{X}$ -structure. If  $x$  is a standard singularity of angle 0, equivalently a standard singularity of trivial holonomy, then the  $\mathbf{X}$ -structure of  $D^*$  uniquely extends to  $D$ .

**Corollary 2.18.** Let  $x$  be a standard singularity of a  $\mathbf{X}$ -structure on an oriented punctured disk  $D^* := D \setminus x$ ,  $\rho: \pi_1(D^*) \rightarrow \mathbf{G}$  be a compatible holonomy map at  $x$ , and  $c$  a positively oriented loop of  $D^*$  whose homotopy class  $[c]$  generates  $\pi_1(D^*)$ . Then  $x$  is of angle  $\theta \in \mathbb{R}$  if, and only if  $\rho([c]) = a^\theta$ .

The interpretation of the angle  $\theta$  of a standard singularity  $x$  as the holonomy of a positive closed loop  $c$  around it is illustrated in Figure 2.1.

*Proof of Lemma 2.14. (1) for  $\theta \Rightarrow$  (2).(a),(b)&(c).* The unique continuous extension of the lightlike foliations follows from Proposition 2.13. The restriction of the map  $\varphi$  of (1) to a slit neighbourhood  $U'$  of  $x$  is a  $\mathbf{X}$ -isomorphism to a slit neighbourhood of  $\mathfrak{o}_\theta$  which is canonically identified with a slit neighbourhood of  $\mathfrak{o}$  by the projection map  $\pi_\theta$ , giving us the desired map  $\psi$ . Now let  $O$  be an open subset of the universal cover of  $D^*$  projecting homeomorphically to  $U'$ , and  $\delta$  be the developing map extending the lift  $\tilde{\psi}: O \rightarrow \mathbf{X}$  of  $\psi$  to  $O$ . Then  $\delta$  satisfies  $\delta \circ R = a^\theta \circ \delta$  (on the non-empty open subset where this equality is well-defined) by the very definition of  $\mathbf{X}_\theta$ , which shows that  $\rho(R) = a^\theta$  and concludes the proof of this implication.

*(2).(a)&(b)  $\Rightarrow$  (1) for some  $\theta$ .* Let  $\pi: E \rightarrow U^* = U \setminus \{x\}$  be the universal covering map of  $U^*$ , and  $O \subset E$  be an open set such that  $\pi|_O$  is a diffeomorphism onto  $U' = U \setminus \mathcal{F}_\alpha^+(x)$ . The existence of  $\psi$  shows that the restriction of the developing map  $\delta: E \rightarrow \mathbf{X}$  to  $O$  is an isometry onto  $V' = V \setminus \mathcal{F}_\alpha^+(\mathfrak{o})$ , with  $V$  an open neighbourhood of  $\mathfrak{o}$ . The lightlike leaf spaces of  $V'$  have the following description:

- the leaf space  $\mathcal{L}_\beta$  of the  $\beta$ -foliation of  $V'$  is homeomorphic to the non-Hausdorff topological 1-manifold  $(L^+ \cup L^-) / \sim$ , with  $L^\pm$  two copies of  $\mathbb{R}$  and  $p^- \sim p^+$  for  $p \in \mathbb{R}_{<0}$ , the special points  $0^\pm$  corresponding to the special leaves  $J_\beta^\pm := \mathcal{F}_\beta^\pm(\mathfrak{o}) \cap V'$ ;
- the leaf space of the  $\alpha$ -foliation of  $V'$  has one specific point  $J_\alpha^- := \mathcal{F}_\alpha^-(\mathfrak{o}) \cap V'$ , which is the only  $\alpha$ -leaf intersecting none of the leaves  $p^\pm \in \mathcal{L}_\beta$  for  $p \geq 0$ .

Since the lightlike foliations of  $D^*$  extend by assumption to continuous foliations of  $D$ , we can choose  $U$  to be a small enough neighbourhood of  $x$  for it to be a trivialization domain of both lightlike foliations of  $D$ . The same description holds then for the lightlike leaf spaces of  $U'$  than for the ones of  $V'$ . Let us denote by  $I_\beta^\pm$ , respectively  $I_\alpha^-$  the lifts of  $\mathcal{F}_\beta^\pm(x) \cap U$ , resp.  $\mathcal{F}_\alpha^-(x) \cap U$

in  $O$ , and by  $I_\alpha^{d/u}$  the “down and up” lifts of  $\mathcal{F}_\alpha^+(x) \cap U$ , so that  $\partial O = I_\alpha^d \cup I_\alpha^u$  and  $R(I_\alpha^d) = I_\alpha^u$ . Then since  $\delta$  is a simultaneous equivalence between the lightlike foliations, the descriptions of the leaf spaces impose  $\delta(I_\beta^\pm) = J_\beta^\pm$ ,  $\delta(I_\alpha^-) = J_\alpha^-$  and  $\delta(I_\alpha^{d/u}) = ]\mathfrak{o}; p^{d/u}[_\alpha$  with  $p^{d/u} \in \mathcal{F}_\alpha^{+*}(\mathfrak{o})$ . With  $\rho$  the holonomy morphism associated to  $\delta$  we have thus  $\rho(R)(] \mathfrak{o}; p^d[_\alpha) = ] \mathfrak{o}; p^u[_\alpha$ , which shows that  $\rho(R)$  fixes  $\mathfrak{o}$ , *i.e.*  $\rho(R) = a^\theta$  for some  $\theta$ , and thus  $\delta \circ R = a^\theta \circ \delta$ .

We now define a map  $\varphi: U \rightarrow \mathbf{X}_\theta$  by:

- $\varphi(x) = \mathfrak{o}_\theta$ ;
- $\varphi \circ \pi = \pi_\theta \circ \delta$  on  $O$ ;
- $\varphi \circ \pi = \pi_\theta \circ \iota_+ \circ \delta$  on  $I_\alpha^d$ ;

and show that  $\varphi$  satisfy the properties of (1). Let  $W$  be an open neighbourhood of  $p \in I_\alpha^d$  so that  $\pi|_W$  is a diffeomorphism onto  $\pi(W)$ , and  $W \setminus I_\alpha^d$  has two connected components  $W^\pm$ , with  $W^+ \subset O$  and  $R(W^-) \subset O$ . Since  $\delta \circ R = a^\theta \circ \delta$ , we have  $\varphi \circ \pi = \pi_\theta \circ a^\theta \circ \delta$  on  $W^-$ ,  $\varphi \circ \pi = \pi_\theta \circ \iota_+ \circ \delta$  on  $I_\alpha^d \cap W$  and  $\varphi \circ \pi = \pi_\theta \circ \delta$  on  $W^+$ , which shows that  $\varphi$  is a  $\mathbf{X}$ -morphism into  $\mathbf{X}_\theta$  on the neighbourhood of  $\pi(p)$ .

It thus only remains to show that  $\varphi$  is continuous at  $x$ . Our former description shows that  $\varphi(\mathcal{F}_{\alpha/\beta}(x) \cap U) = \mathcal{F}_{\alpha/\beta}(\mathfrak{o}_\theta)$ , and thus that  $\varphi$  induces two maps  $\phi_{\alpha/\beta}$  between the respective leaf spaces of the  $\alpha$ , resp.  $\beta$ -foliations of  $U$  and  $\varphi(U) \subset \mathbf{X}_\theta$ . These foliations being continuous and transverse, it moreover suffices to show that the maps  $\phi_{\alpha/\beta}$  induced by  $\varphi$  between the leaf spaces are continuous at  $\mathcal{F}_{\alpha/\beta}(x) \cap U$ , to conclude that  $\varphi$  is continuous at  $x$ . But our former description of the leaf spaces of  $U'$  and  $V'$  showed that  $\delta(I_\alpha^-) = J_\alpha^-$ , and thus for any sequence  $L_n$  of  $\alpha$ -leaves contained in  $U'$  and converging to  $\mathcal{F}^\alpha(x) \cap U$ ,  $\varphi(L_n)$  converges to  $\mathcal{F}_\alpha^-(\mathfrak{o}_\theta)$ , which shows the continuity of  $\phi_\alpha$  at  $\mathcal{F}^\alpha(x) \cap U$ . In the same way, the fact that  $\delta(I_\beta^\pm) = J_\beta^\pm$  shows that  $\phi_\beta$  is continuous at  $\mathcal{F}^\beta(x) \cap U$ , which concludes the proof of the second implication.

**Unicity of  $\theta$ .** If  $\theta_1$  and  $\theta_2$  both satisfy the equivalent properties (1) and (2), then the holonomy morphism of a developing map extending the lift of a  $\mathbf{X}$ -isomorphism like in (b) should satisfy  $a^{\theta_1} = \rho(R) = a^{\theta_2}$  according to (c) (note that (b) is indeed independent of  $\theta$ ). Hence  $\theta_1 = \theta_2$ , which concludes the proof of the Lemma.  $\square$

**2.2.4. Standard singularities as quotients.** Let  $D$  be an open disk around  $\mathfrak{o}$  in  $\mathbf{X}$ , and  $E$  be the universal cover of  $D^* := D \setminus \{\mathfrak{o}\}$ . Since  $a^\theta$  fixes  $\mathfrak{o}$ , it induces an isometry of  $D^*$  which lifts to a unique isometry  $\widetilde{a}^\theta$  of  $E$  fixing each lift of the punctured lightlike leaves of  $\mathfrak{o}$ . On the other hand,  $E$  admits also a preferred isometry  $R$  which is the positive generator of its covering automorphism group.

**Lemma 2.19.**  *$\widetilde{a}^\theta \circ R$  acts properly discontinuously on  $E$ , and  $E/\langle \widetilde{a}^\theta \circ R \rangle$  is  $\mathbf{X}$ -isomorphic to  $\mathbf{X}_\theta \setminus \{\mathfrak{o}_\theta\}$ . More precisely, there is a natural embedding of  $E/\langle \widetilde{a}^\theta \circ R \rangle$  as the complement of a point  $\mathfrak{o}_\theta$  in a topological disk  $\bar{E}$ , for which  $\mathfrak{o}_\theta$  is a standard singularity of angle  $\theta$  of  $\bar{E}$ .*

*Proof.* Any lift  $\tilde{\mathcal{F}}_\alpha$  of  $\mathcal{F}_\alpha^{+*}(\mathfrak{o})$  is an embedding of  $\mathbb{R}$  separating  $E \simeq \mathbb{R}^2$  in two connected components, and since  $\langle R \rangle \simeq \mathbb{Z}$  acts properly discontinuously on  $E$ , the images of  $\tilde{\mathcal{F}}_\alpha$  by  $\langle R \rangle$  are pairwise disjoint and form a discrete set. The complement of  $\langle R \rangle \cdot \tilde{\mathcal{F}}_\alpha$  in  $E$  is a disjoint union of topological disks, the boundary of each of them being the disjoint union of an upper and a lower translate of  $\tilde{\mathcal{F}}_\alpha$ , and the closure of any of these connected components is a fundamental domain for the action of  $\langle R \rangle$  on  $E$ . Now the important observation is that by definition,  $\widetilde{a}^\theta$  preserves the interior and the boundary of any of these fundamental domains, showing that  $\widetilde{a}^\theta \circ R$  acts indeed properly discontinuously on  $E$ .

We add to  $E/\langle \widetilde{a}^\theta \circ R \rangle$  a point  $\mathfrak{o}_\theta$ , with a neighbourhood basis composed of images of sets of the form  $U \cup \{\mathfrak{o}_\theta\}$ , for  $\widetilde{a}^\theta \circ R$ -invariant open sets  $U \subset E$  projecting to punctured neighbourhoods of  $\mathfrak{o}$  in  $D$ . This defines a topological disk  $\bar{E}$ , in which the lightlike foliations of  $E/\langle \widetilde{a}^\theta \circ R \rangle = \bar{E} \setminus \{\mathfrak{o}_\theta\}$  extend to two continuous transverse foliations. The complement of  $\tilde{\mathcal{F}}_\alpha = \mathcal{F}_\alpha^{+*}(\mathfrak{o}_\theta)$  in  $\bar{E}$  is  $\mathbf{X}$ -isomorphic to the interior of one of the previously described fundamental domains, themselves isomorphic to the slit neighbourhood  $D \setminus \mathcal{F}_\alpha^{+*}(\mathfrak{o})$  in  $\mathbf{X}$ . The result now follows from Lemma 2.14.  $\square$

2.2.5. *Standard singularities as angle defaults.* Let now  $D$  be a small disk around  $\mathfrak{o}$  in  $\mathbf{X}$ ,  $\gamma$  be a half-open future-oriented spacelike geodesic starting from  $\mathfrak{o}$ ,  $\theta > 0$  and  $\gamma_\theta := a^\theta(\gamma)$ . Then  $D \setminus (\gamma \cup \gamma_\theta)$  has two connected components. One of them is contained in the future spacelike quadrant of  $\mathfrak{o}$  and its closure is denoted by  $D_{i\theta}$ . The other one contains the three other quadrants and its closure is denoted by  $D_{2\pi-i\theta}$ . We denote by  $\bar{D}_{2\pi-i\theta}$  the quotient of  $D_{2\pi-i\theta}$  by the relation  $\gamma \ni x \sim a^\theta(x) \in \gamma_\theta$  on its boundary (in particular  $\mathfrak{o} \sim \mathfrak{o}$ ). As we did in paragraph 2.2.1, we consider the surface  $D_*$  obtained from  $D$  by cutting it open along  $\gamma \setminus \{\mathfrak{o}\}$ , with two upper and lower boundary components  $\iota_\pm: \gamma \rightarrow D_*$ . We can now form the quotient  $\bar{D}_{2\pi+i\theta}$  of  $D_* \cup D_{i\theta}$  by the relation:  $\iota_-(x) \sim x \in \gamma$  and  $\iota_+(x) \sim a^\theta(x) \in \gamma_\theta$  for  $x \in \gamma$ . Both topological disks  $\bar{D}_{2\pi\pm i\theta}$  have a marked point  $o_\theta$ , image of  $\mathfrak{o}$ , and bear a natural  $\mathbf{X}$ -structure on  $\bar{D}_{2\pi\pm i\theta} \setminus \{o_\theta\}$  which is defined as in paragraph 2.2.1. These constructions are illustrated in Figure 2.1.

**Lemma 2.20.**  *$o_\theta$  is a standard singularity of angle  $\theta$  (respectively  $-\theta$ ) of  $\bar{D}_{2\pi-i\theta} \setminus \{o_\theta\}$  (resp. of  $\bar{D}_{2\pi+i\theta} \setminus \{o_\theta\}$ ). The obvious analogous statement can be given for any two half-geodesics of the same signature and orientation. In particular, any lightlike half-leaf can be used to define a standard singularity.*

*Proof.* The first important observation is that both  $D_{2\pi-i\theta}$  and  $D_*$  contain three quadrants of  $D$  at  $\mathfrak{o}$ , and thus that the lightlike foliations of  $\bar{D}_{2\pi\pm i\theta} \setminus \{o_\theta\}$  extend to two transverse continuous foliations of  $\bar{D}_{2\pi\pm i\theta}$ . With  $E$  the universal cover of  $D \setminus \{\mathfrak{o}\}$ ,  $\tilde{a}^\theta$  the lift of  $a^\theta$  fixing each lift of the punctured lightlike leaves of  $\mathfrak{o}$ , and  $R$  the positive generator of the automorphism group of  $E$ ,  $\bar{D}_{2\pi-i\theta} \setminus \{o_\theta\}$  is the quotient of  $E$  by  $\langle \tilde{a}^\theta \circ R \rangle$ , and  $\bar{D}_{2\pi+i\theta} \setminus \{o_\theta\}$  the quotient of  $E$  by  $\langle \tilde{a}^{-\theta} \circ R \rangle$ . The claim is now a consequence of Lemma 2.19.  $\square$

*Remark 2.21.* Lemma 2.20 provides us with the Lorentzian counterpart of the usual interpretation of Riemannian singularities as *angles defaults*. Indeed, we will see in the proof of Proposition 2.32 that for a natural notion of *Lorentzian angle* (for which angles are *complex* numbers),  $D_{i\theta}$  is a sector of angle  $i\theta$  (oriented from  $\gamma$  to  $a^\theta(\gamma)$ ), and  $D_{2\pi-i\theta}$  a sector of angle  $2\pi - i\theta$  (oriented from  $a^\theta(\gamma)$  to  $\gamma$ ). Hence a standard singularity  $x$  has angle  $\nu \in \mathbb{R}$  if, and only if the total angle around  $x$  is  $2\pi - i\nu$ . This gives in particular a new intrinsic characterization of the angle of a standard singularity (and especially of its sign).

Our main interest being in this text for the extension of the lightlike foliations at the singularities as topological foliations, it seems to us that the use of lightlike geodesics is clearer at first sight. However the point of view of definite geodesics will be useful for some aspects. We emphasize that contrarily to the Riemannian case, the *same* (lightlike) geodesic can be used in the Lorentzian setting to define a singularity of non-zero cone angle.

**2.3. Singular  $\mathbf{X}$ -surfaces.** We use in this subsection the local model of singularities described in paragraph 2.2, to define singular  $\mathbf{X}$ -surfaces and to prove some of their fundamental properties.

**Definition 2.22.** A *singular  $\mathbf{X}$ -structure* on an oriented topological surface  $S$ , is the data of:

- (1) a set  $\Sigma \subset S$  of *singular points* in  $S$ ;
- (2) and a  $\mathbf{X}$ -structure on  $S \setminus \Sigma$  for which any  $x \in \Sigma$  is a *standard singularity*, *i.e.* for which there exists  $\theta_x \in \mathbb{R}$  (the *angle* at  $x$ ) and a homeomorphism  $\varphi$  from an open neighbourhood  $U \subset S$  of  $x$  to an open neighbourhood  $V$  of  $\mathfrak{o}_\theta$  in  $\mathbf{X}_{\theta_x}$ , such that:
  - (a)  $U \cap \Sigma = \{x\}$ ,
  - (b)  $\varphi(x) = \mathfrak{o}_{\theta_x}$ ,
  - (c) and  $\varphi$  is a  $\mathbf{X}$ -morphism in restriction to  $U \setminus \{x\}$ .

Such a map  $\varphi$  is called a *singular  $\mathbf{X}$ -chart* at  $x$ .

A *singular  $\mathbf{X}$ -surface*  $(S, \Sigma)$  is an oriented topological surface  $S$  endowed with a singular  $\mathbf{X}$ -structure of singular set  $\Sigma$ . The  $\mathbf{X}$ -structure of  $S^* := S \setminus \Sigma$  and its underlying Lorentzian metric will be equally denoted by  $\mu$  (or unspecified if no confusion is possible in the context).  $S^*$  will always be endowed with the  $\mathcal{C}^\infty$  structure defined by its  $\mathbf{X}$ -structure, and  $S$  with a  $\mathcal{C}^\infty$  structure extending the one of  $S^*$  (see for instance [Hat]). The points of  $S$  which are not singular are called *regular*, and  $S$  itself is said *regular* if it does not have any singular point (*i.e.* if it is a



$\mathbf{X}$ -surface). If we want to specify them, we will denote by  $\Theta$  the (ordered) set of angles of the (ordered) singularities  $\Sigma$ .

A *singular  $\mathbf{X}$ -atlas*  $(\varphi_i, U_i)$  on  $S$  is an atlas of  $\mathcal{C}^0$ -charts  $\varphi_i: U_i \rightarrow V_i$  from connected open subsets  $U_i$  of  $S$  to either  $\mathbf{X}$  (*regular charts*) or some  $\mathbf{X}_\theta$  (*singular charts*), such that:

- (1) any two distinct singular chart domains are disjoint,
- (2) regular charts cover  $S \setminus \Sigma$ , with  $\Sigma = \{\varphi^{-1}(\mathfrak{o}_\theta) \mid \varphi \text{ singular chart to } \mathbf{X}_\theta\}$  the set of *singularities* of the atlas,
- (3) the transition map between any two charts is a  $\mathbf{X}$ -morphism (which makes sense since  $U_i \cap U_j \cap \Sigma = \emptyset$  for any two distinct chart domains  $U_i, U_j$ ).

An *isometry* between two singular  $\mathbf{X}$ -surfaces  $(S_i, \Sigma_i, \mu_i)_{i=1,2}$  is a homeomorphism  $f: S_1 \rightarrow S_2$  such that:

- (1)  $f(\Sigma_1) = \Sigma_2$ ;
- (2)  $f$  is a  $\mathbf{X}$ -morphism in restriction to  $S_1 \setminus \Sigma_1$ .

The *area* of a singular  $\mathbf{X}$ -surface  $(S, \Sigma)$  is the area of  $S \setminus \Sigma$  for  $\mu$ .

*Remark 2.23.* Let us say that a time-oriented Lorentzian metric  $\mu$  of constant sectional curvature  $\varepsilon_{\mathbf{X}}$  defined on the complement of a discrete subset  $\Sigma$  of an orientable surface  $S$  is *singular*, if it is induced by a singular  $\mathbf{X}$ -structure. Then according to Proposition 2.6, time-oriented singular Lorentzian metric of constant sectional curvature  $\varepsilon_{\mathbf{X}}$  are equivalent to singular  $\mathbf{X}$ -structures.

Note moreover that an isometry  $f: S \rightarrow S$  between two singular  $\mathbf{X}$ -structures  $\mu_1$  and  $\mu_2$  on a surface  $S$  sharing the same singular set  $\Sigma$ , is simply an isometry between their underlying Lorentzian metrics of  $S \setminus \Sigma$ , which is  $\mathcal{C}^\infty$  for the (compatible)  $\mathcal{C}^\infty$ -structures defined by  $\mu_1$  and  $\mu_2$  and extends to a homeomorphism of  $S$  fixing  $\Sigma$ .

2.3.1. *First properties of singular  $\mathbf{X}$ -surfaces.* We prove now some elementary but fundamental properties of singular  $\mathbf{X}$ -surfaces.

**Lemma 2.24.** *Let  $(S, \Sigma)$  be a singular  $\mathbf{X}$ -surface.*

- (1)  $\Sigma$  is discrete, hence finite if  $S$  is closed.
- (2) For any singularity  $x \in \Sigma$  of angle  $\theta_x$ ,  $\rho: \pi_1(S \setminus \Sigma) \rightarrow \mathbf{G}$  a holonomy representation of  $S^*$  compatible at  $x$  (see Definition 2.15), and  $[\gamma] \in \pi_1(S \setminus \Sigma)$  the homotopy class of a positively oriented loop around  $x$  homotopic to  $x$  in  $S$ :  $\rho([\gamma]) = a^{\theta_x}$ . In particular,  $\rho([\gamma])$  is conjugated to  $a^{\theta_x}$ .
- (3) If  $S$  is closed, then the area of  $(S, \Sigma)$  is finite.

*Proof.* (1) Any singular  $\mathbf{X}$ -chart contains indeed a unique singularity.

(2) Since  $x$  is a standard singularity of angle  $\theta_x$ , this is a direct consequence of Lemma 2.14.

(3) For any compact measurable subset  $K \subset S \setminus \Sigma$ ,  $\mathcal{A}_{\mu_S}(K)$  is finite, and the claim follows thus from the fact that for any compact neighbourhood  $K$  of  $\mathfrak{o}_\theta$  in  $\mathbf{X}_\theta$ , the area of  $K \setminus \{\mathfrak{o}_\theta\}$  equals the one of  $K$  and is thus finite.  $\square$

We emphasize that the second claim of Lemma 2.24 shows that the singularities and their angles are characterized by  $\mu_S$ , and are geometrical invariants in the following sense.

**Corollary 2.25.** *Let  $f: S_1 \rightarrow S_2$  be an isometry between two singular  $\mathbf{X}$ -surfaces. Then for any singular point  $x$  of  $S_1$ ,  $x \in \Sigma_1$  and  $f(x) \in \Sigma_2$  have the same angle:  $\theta_x = \theta_{f(x)}$ .*

*Proof.* Let  $[\gamma] \in \pi_1(S_1 \setminus \Sigma_1)$  be the homotopy class of a positively oriented loop homotopic to  $x$ , and  $\rho: \pi_1(S_1 \setminus \Sigma_1) \rightarrow \mathbf{G}$  be a compatible holonomy representation of  $S_1$  at  $x$ . Then  $[f(\gamma)] \in \pi_1(S_2 \setminus \Sigma_2)$  and the morphism  $\rho \circ f_*^{-1}: \pi_1(S_2 \setminus \Sigma_2) \rightarrow \mathbf{G}$  induced by  $f$  has the same properties with respect to  $f(x)$ , hence  $a^{\theta_x} = \rho([\gamma]) = \rho \circ f_*^{-1}([f \circ \gamma]) = a^{\theta_{f(x)}}$ , i.e.  $\theta_x = \theta_{f(x)}$ .  $\square$

Observe that for any  $u \in \mathbb{R}$ ,  $a^u$  preserves the equivalence relation  $\sim_\theta$  used to define  $\mathbf{X}_\theta$ . It induces thus a map on  $\mathbf{X}_\theta$  preserving  $\mathfrak{o}_\theta$  that we denote by  $\bar{a}^u$ , characterized by  $\bar{a}^u \circ \pi_\theta = \pi_\theta \circ a^u$ .

**Proposition 2.26.** *Let  $\varphi$  be a singular  $\mathbf{X}$ -chart of  $\mathbf{X}_\theta$  at  $\mathfrak{o}_\theta$ , or equivalently a local isometry of  $\mathbf{X}_\theta$  defined on a connected neighbourhood of  $\mathfrak{o}_\theta$  and fixing  $\mathfrak{o}_\theta$ . Then  $\varphi$  is the restriction of some  $\bar{a}^u$ .*

*Proof.* First according to Corollary 2.25, a singular  $\mathbf{X}$ -chart of  $\mathbf{X}_\theta$  at  $\mathfrak{o}_\theta$  is indeed a local isometry of  $\mathbf{X}_\theta$  fixing  $\mathfrak{o}_\theta$ . Denoting  $U^* := U \setminus \{\mathfrak{o}_\theta\}$  we can assume without loss of generality that  $\mathcal{F}_\beta(\mathfrak{o}_\theta) \cap U^*$  is the union of two down and up connected components  $I_- = ]x; \mathfrak{o}_\theta[_\beta$  and  $I_+ = ]\mathfrak{o}_\theta; y[_\beta$ . The first natural but important observation is that  $\varphi$  preserves both ends of  $\mathcal{F}_\beta^*(\mathfrak{o}_\theta)$  in the sense that  $\varphi(I_-) = ]x'; \mathfrak{o}_\theta[_\beta$  and  $\varphi(I_+) = ]\mathfrak{o}_\theta; y'[_\beta$  for some  $x'$  and  $y'$ . Likewise both ends of  $\mathcal{F}_\alpha^*(\mathfrak{o}_\theta)$  are preserved, the proof being identical. Indeed  $\varphi(I_-)$  and  $\varphi(I_+)$  are intervals of  $\beta$ -leaves since  $\varphi|_{U^*}$  is a  $\mathbf{X}$ -morphism, containing furthermore  $\mathfrak{o}_\theta$  in their closure since  $\varphi(\mathfrak{o}_\theta) = \mathfrak{o}_\theta$ . Hence the only alternative to the above claim is that  $\varphi(I_-) = ]\mathfrak{o}_\theta; x'[_\beta$  and  $\varphi(I_+) = ]y'; \mathfrak{o}_\theta[_\beta$  for some  $x'$  and  $y'$ . But then since  $\varphi(\mathfrak{o}_\theta) = \mathfrak{o}_\theta$ ,  $\varphi$  would reverse the canonical orientation defined on  $\beta$ -leaves by the  $\mathbf{X}$ -structure of  $U^*$  (see paragraph 2.1.5), which contradicts the fact that  $\varphi|_{U^*}$  is a  $\mathbf{X}$ -morphism. Let  $\mathcal{U}$  be an open neighbourhood of  $\mathfrak{o}$  in  $\mathbf{X}$ , so that with  $U' := \mathcal{U} \setminus \mathcal{F}_\alpha^+(\mathfrak{o})$  and  $\mathcal{U}_* = U' \cup \iota_-(U \cap \mathcal{F}_\alpha^+(\mathfrak{o})) \cup \iota_+(U \cap \mathcal{F}_\alpha^+(\mathfrak{o})) \subset \mathbf{X}_*$ ,  $U = \pi_\theta(U')$ . Then the restriction to  $U'$  (resp.  $V'$ ) of  $\pi_\theta$  are  $\mathbf{X}$ -morphisms, and  $\pi_\theta|_{V'}^{-1} \circ \varphi \circ \pi_\theta$  is thus the restriction of an element  $g \in \mathbf{G}$ . But our previous claim shows then that  $g$  is simultaneously in the stabilizer of  $\mathcal{F}_\alpha(\mathfrak{o})$  and  $\mathcal{F}_\beta(\mathfrak{o})$  whose intersection is  $\text{Stab}(\mathfrak{o}) = A$ . In other words there exists  $u \in \mathbb{R}$  so that  $\varphi = a^u$  on  $U^*$  and thus on  $U$ , which concludes the proof.  $\square$

For any  $\mathbf{X}$ -surface  $(S, \Sigma)$ , the union of a  $\mathbf{X}$ -atlas of  $S \setminus \Sigma$  with a (small enough) singular  $\mathbf{X}$ -chart at each singularity defines a singular  $\mathbf{X}$ -atlas of  $S$ . Conversely, any singular  $\mathbf{X}$ -atlas of  $S$  defines of course on  $S$  a singular  $\mathbf{X}$ -structure with the same singularities. The following result follows directly from Proposition 2.26.

**Corollary 2.27.** *Let  $S$  be an oriented topological surface. Then the transition maps between any two singular  $\mathbf{X}$ -atlases defining the same singular  $\mathbf{X}$ -structure on  $S$  are:*

- either restrictions of some  $a^u$  between two singular charts at the same singularity,
- or  $\mathbf{X}$ -morphisms outside of singularities.

*Two singular  $\mathbf{X}$ -atlases whose transition maps are of this form are said equivalent, and singular  $\mathbf{X}$ -structures are in correspondence with equivalence classes of singular  $\mathbf{X}$ -atlases.*

Consequently, any  $\mathbf{G}$ -invariant object or notion on  $\mathbf{X}$  which projects well to  $\mathbf{X}_\theta$  through  $\pi_\theta$  will make sense on any singular  $\mathbf{X}$ -structure. The main application of this vague remark will be the Definition A.5 given below of geodesics in singular  $\mathbf{X}$ -surfaces.

**2.3.2. First-return maps, suspensions and regularity of the lightlike foliations.** If  $T$  is a homeomorphism of the circle  $\mathbf{S}^1$ , the vertical foliation of  $\mathbf{S}^1 \times [0; 1]$  of leaves  $\{p\} \times [0; 1]$  induces on the quotient  $M_T := \mathbf{S}^1 \times [0; 1] / \{(1, p) \sim (0, T(p))\}$ , homeomorphic to a torus, a foliation  $\mathcal{F}_T$  called the *suspension of  $T$* . We will be interested in this text with lightlike foliations of singular  $\mathbf{X}$ -structures which are suspensions of circle homeomorphisms, and it happens that the dynamics of a circle homeomorphism  $T$ , hence of its suspension, is highly dependent of the regularity of  $T$ . Indeed, circle homeomorphisms can in general have pathological behaviours by admitting *exceptional minimal sets* (see [HH86, Chapter I §5]), but the seminal work of Herman [Her79] showed that regular enough circle homeomorphisms behave nicely. In this paragraph we give the main technical properties of the lightlike foliations of a singular  $\mathbf{X}$ -surface, and show in particular that if they are suspensions of a circle homeomorphism  $T$ , then  $T$  is a  $\mathcal{C}^2$  diffeomorphism with breaks.

**Definition 2.28.** A homeomorphism  $f: I = [a; b] \rightarrow J$  between two intervals of  $\mathbb{R}$  is an *orientation-preserving  $\mathcal{C}^k$ -diffeomorphism with breaks* ( $1 \leq k \leq \infty$ ) if there exists a finite number of points  $a = x_0 < \dots < x_N = b$  in  $I$  such that for any  $1 \leq i \leq N$ :

- (1)  $f|_{]x_{i-1}; x_i[}$  is an orientation-preserving  $\mathcal{C}^k$ -diffeomorphism onto its image,
- (2) for any  $1 \leq l \leq k$ , the  $l^{\text{th}}$  derivative of  $f$  has finite limites from above at  $x_{i-1}$  and from below at  $x_i$ ,
- (3)  $f'_+(x_{i-1}) := \lim_{t \rightarrow x_{i-1}^+} f'(t)$  and  $f'_-(x_i) := \lim_{t \rightarrow x_i^-} f'(t)$  are  $> 0$ .

If  $f'_+(x_{i-1}) \neq f'_-(x_i)$ , then  $x_i$  is a *break point* of  $f$ . A homeomorphism of  $\mathbf{S}^1$  is a  *$\mathcal{C}^k$ -diffeomorphism with breaks* if it is so in restriction to any small enough interval of  $\mathbf{S}^1$ .

The following naive observation will be useful to us.

**Lemma 2.29.** *Let two consecutive intervals  $[a; b]$  and  $[b; c]$  of  $\mathbb{R}$  be endowed with  $C^\infty$ -structures  $C^0$ -compatible with the topology of  $\mathbb{R}$ , and  $\varphi: [a; c] \rightarrow I \subset \mathbb{R}$  be a homeomorphism. Then for any  $1 \leq k \leq \infty$ , the following are equivalent.*

- (1)  $\varphi$  restricts to  $C^k$ -diffeomorphisms with breaks of  $[a; b]$  and  $[b; c]$ , and  $\lim_{t \rightarrow b^\pm} \varphi'(t) > 0$ .
- (2) In a  $C^\infty$ -structure of  $[a; c]$  which is  $C^\infty$ -compatible with the structures of both of its subintervals,  $\varphi$  is a  $C^k$ -diffeomorphism with breaks.

Let  $\mathcal{F}$  be an oriented  $C^0$  one-dimensional foliation on a surface  $S$ ,  $I$  and  $J$  be two transversals of  $\mathcal{F}$ , i.e.  $C^0$  one-dimensional submanifolds transverse to  $\mathcal{F}$  in a foliation chart, and  $x \in I$  such that  $\mathcal{F}(x) \cap J \neq \emptyset$ . Then by transversality,  $\mathcal{F}(x)$  has a *first intersection point* denoted by  $H(x)$  with  $J$  (with respect to the orientation of  $\mathcal{F}$ ), and there exists an open neighbourhood  $I'$  of  $x$  in  $I$  such that  $H(y) \in J$  is well-defined for any  $y \in I'$ . The map  $H: I' \rightarrow J$  obtained in this way is a homeomorphism onto its image (an open neighbourhood of  $H(x)$ ) which is called the *holonomy of  $\mathcal{F}$  from  $I$  to  $J$* . We refer to [CLN85, §IV.1] for more details. A *section* of  $\mathcal{F}$  is a simple closed curve  $\gamma$  of  $S$  transverse to  $\mathcal{F}$  and intersecting any of its leaves. In this case, if the holonomy of  $\mathcal{F}$  from  $\gamma$  to itself is well-defined, it will be called the *first-return map* of  $\mathcal{F}$  on  $\gamma$  and be denoted by  $P_\mathcal{F}^\gamma$  (in reference to Poincaré). We recall that a homeomorphism (respectively a foliation) of a manifold  $M$  is said *minimal* if all its orbits (resp. leaves) are dense in  $M$ .

**Lemma 2.30.** *Let  $(S, \Sigma)$  be a singular  $\mathbf{X}$ -surface.*

- (1) *The lightlike foliations of  $S \setminus \Sigma$  extend uniquely to two one-dimensional continuous foliations on  $S$ , still denoted by  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$ .*
- (2) *There exists at any point of  $S$  a simultaneous  $C^0$  foliation chart for  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$  (in the sense of Proposition 2.13).*

Let  $\mathcal{F}$  be one of the lightlike foliations of  $S$ .

- (3) *Let  $T_1, T_2 \subset S$  be two small  $C^\infty$  transversals of  $\mathcal{F}$  such that  $T_1 \cap \Sigma = \{x\}$  and  $T_2 \subset S \setminus \Sigma$  intersects  $\mathcal{F}(x)$ , and  $H: T_1 \rightarrow T_2$  be the holonomy of  $\mathcal{F}$  from  $T_1$  to  $T_2$ . Then  $H$  is a  $C^\infty$ -diffeomorphism with breaks.*
- (4) *If  $S$  is homeomorphic to  $\mathbf{T}^2$  and  $\mathcal{F}$   $C^0$ -conjugated to the suspension of an orientation-preserving homeomorphism  $H$  of  $\mathbf{S}^1$ , then  $H$  is  $C^0$ -conjugated to a  $C^\infty$ -diffeomorphism with breaks of  $\mathbf{S}^1$ , and has no exceptional minimal set. If  $H$  has moreover irrational rotation number  $\rho \in \mathbf{S}^1$ , then  $H$  is  $C^0$ -conjugated to the rotation  $R_\rho: x \in \mathbf{S}^1 \mapsto x + \rho \in \mathbf{S}^1$  and is thus minimal. In particular  $\mathcal{F}$  is then  $C^0$ -equivalent to the corresponding linear foliation of  $\mathbf{T}^2$ , hence minimal.*

The notion of rotation number is recalled in Proposition-Definition 3.18. We will prove below in Proposition C.2 a “geometric version” of claims (3) and (4) of the following Lemma, in the case where the transverse curves are geodesics of the surface.

*Proof of Lemma 2.30.* (1) follows directly from Proposition 2.13, using singular  $\mathbf{X}$ -charts at the singularities.

(2) follows from Proposition 2.13 at the singularities and from the  $\mathbf{X}$ -charts at regular points. Indeed the affine charts (2.3) are simultaneous foliated charts of the lightlike foliations of  $\mathbf{X}$ .

(3) Without loss of generality, we can assume that  $S = \mathbf{X}_\theta$ ,  $x = \mathfrak{o}_\theta$ ,  $\mathcal{F} = \mathcal{F}_\beta$ , and that  $T_1 = \mathcal{F}_\beta(\mathfrak{o}_\theta)$  and  $T_2 = \mathcal{F}_\beta(p)$  with  $p \in \mathcal{F}_\alpha^+(\mathfrak{o}_\theta)$ . These reductions being done, and since the  $C^\infty$  structure of  $S$  is by definition compatible with the  $\mathbf{X}$ -structure of  $S \setminus \Sigma$ , it only remains to check according to Lemma 2.29 that the restriction of  $H$  to the closure of each component of  $\mathcal{F}_\beta(\mathfrak{o}_\theta) \setminus \{\mathfrak{o}_\theta\}$  is a  $C^\infty$ -diffeomorphism with breaks, with a positive limit of the derivative at  $\mathfrak{o}_\theta$  from below and above. We do it for  $\mathcal{F}_\beta^+(\mathfrak{o}_\theta)$ , the case of the other component being analogous. According to Proposition 2.13, for  $I$  and  $J$  small open neighbourhoods of  $\mathfrak{o}_\theta$  in  $\mathcal{F}_\alpha(\mathfrak{o}_\theta)$  and  $\mathcal{F}_\beta(\mathfrak{o}_\theta)$ , the map  $(x, y) \in I \times J \mapsto \mathcal{F}_\beta(x) \cap \mathcal{F}_\alpha(y)$  defines outside of  $\mathfrak{o}_\theta$  a smooth diffeomorphism onto a punctured open neighbourhood of  $\mathfrak{o}_\theta$  in  $\mathbf{X}_\theta$ . But the holonomy  $H$  reads in this chart as the identity of the vertical factor, and extends thus clearly on the closure  $I^+$  of the upper component to a  $C^\infty$ -diffeomorphism whose derivative has a positive limit at  $\mathfrak{o}_\theta$ .

(4) Since  $\Sigma \cap S$  is finite and  $\mathcal{F}$  is by assumption a suspension, there exists a  $C^\infty$  section  $T \subset S \setminus \Sigma$  of  $\mathcal{F}$ . The first-return map  $H: T \rightarrow T$  of  $\mathcal{F}$  on  $T$  is then well-defined, and is according to (3) a  $C^2$ -diffeomorphism with breaks as a composition of such homeomorphisms. The two last claims follow then from Denjoy Theorem [Den32] (see also [Her79, Théorème VI.5.5 p.76]): if an orientation-preserving homeomorphism  $T$  of  $\mathbf{S}^1$  is a  $C^2$ -diffeomorphism with breaks, then it has no exceptional minimal set. If  $T$  has moreover irrational rotation number  $\rho$ , then it is  $C^0$ -conjugated to the rotation  $R_\rho$ .<sup>1</sup>  $\square$

**Corollary 2.31.** *Any closed connected orientable surface which bears a singular  $\mathbf{X}$ -structure, is homeomorphic to a torus.*

*Proof.* According to [HH86, Theorem 2.4.6], any closed connected orientable surface bearing a topological foliation is indeed homeomorphic to a torus.  $\square$

This corollary shows the necessity of introducing *branched covers* of the standard singularities to obtain singular  $\mathbf{X}$ -structures on higher-genus surfaces.

**2.3.3. Gauß-Bonnet formula.** The standard Riemannian Gauß-Bonnet formula has a natural counterpart for singular constant curvature Lorentzian surfaces, which imposes a relation between the singularities and the area of a singular  $\mathbf{X}$ -torus. We recall that  $\varepsilon_{\mathbf{X}}$  denotes the constant sectional curvature of  $\mathbf{X}$ :  $\varepsilon_{\mathbb{R}^{1,1}} = 0$  and  $\varepsilon_{\mathbf{dS}^2} = 1$ .

**Proposition 2.32** (Gauß-Bonnet formula). *Let a closed connected orientable surface  $S$  be endowed with a singular  $\mathbf{X}$ -structure of area  $\mathcal{A}(S) \in \mathbb{R}_+^*$ , having  $n \in \mathbb{N}^*$  singularities of angles  $(\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ . Then:*

$$(2.5) \quad \varepsilon_{\mathbf{X}} \cdot \mathcal{A}(S) = - \sum_{i=1}^n \theta_i.$$

*In particular, we have the following.*

- (1) *If  $S$  is a closed singular  $\mathbb{R}^{1,1}$ -surface, then:*
  - (a) *either  $S$  is regular, i.e. is a flat Lorentzian torus;*
  - (b) *or  $S$  has exactly two singularities of opposite signs;*
  - (c) *or else  $S$  has at least three singularities.*
- (2) *The area of a closed singular  $\mathbf{dS}^2$ -surface is entirely determined by the angles at its singularities.*
- (3) *If a closed singular  $\mathbf{dS}^2$ -surface  $S$  has a single singularity  $x$ , then  $x$  has negative angle  $-\mathcal{A}(S) \in \mathbb{R}_-^*$ .*

*Proof.* Let us denote by  $\Sigma$  the singular set of  $S$ , and by  $S^* = S \setminus \Sigma$  the  $\mathbf{X}$ -surface associated to  $S$ . A general topological fact ensures that  $S$  admits a finite triangulation *subordinate* to any given covering, i.e. each of which triangle is contained in an open set of the chosen covering. Let us choose a singular  $\mathbf{X}$ -atlas of  $S$ , each of which chart domain is a normal convex neighbourhood of any of its points. Around a singular point of  $S$ , we use a natural generalization in the singular setting of the usual notion of normal convex neighbourhood, introduced in Proposition A.7 below. This allows us to consider a triangulation  $\mathcal{T}$  of  $S$ , whose set of vertices, edges and faces (namely triangles) are respectively denoted by  $\mathcal{V}$ ,  $\mathcal{E}$  and  $\mathcal{F}$ , such that:

- (1)  $\Sigma$  is contained in the vertex set  $\mathcal{V}$ ;
- (2) the interior of any edge  $e \in \mathcal{E}$  is a geodesic interval of timelike or spacelike signature.

Formula (2.5) will follow from a Lorentzian counterpart of the Gauß-Bonnet formula, proved in [Dza84, p.225] for compact subsets of regular Lorentzian surfaces whose boundary are piecewise smooth timelike or spacelike curves, taking into account the angles between consecutive smooth segments at the breaking points (see also [Ave63, Che63] for analogous formula in any signatures and dimensions with intrinsic proofs, but in the boundaryless setting). This formula needs thus the definition of *angles* between tangent vectors of Lorentzian manifolds, which is done in [Dza84,

<sup>1</sup>Note that this theorem of Denjoy holds more generally for the so-called *class P homeomorphisms*, of which  $C^2$ -diffeomorphisms with breaks are specific examples.

§3 p.217]. For any non-zero future-oriented spacelike vector  $u \in T_{\mathfrak{o}}\mathbf{X}$ , the angle between  $u$  and  $D_{\mathfrak{o}}a^t(u)$  in Dzan's convention is simply given by

$$(2.6) \quad \varnothing(u, D_{\mathfrak{o}}a^t(u)) = i.t.$$

This relation follows from our Convention 2.10 on the parametrization of the stabilizer  $\{a^t\}_t$  of  $\mathfrak{o}$ . We draw the attention of the reader on the fact (surprising at first sight) that Lorentzian angles have *complex* values (for instance pure imaginary in (2.6)). One can then define the angle axiomatically by stating that it is additive in the usual sense (see [Dza84, Definition 7 p.220]), and that  $\varnothing(u, v) = \frac{\pi}{2}$  if  $g_{\mu}(u, v) = 0$  for  $u$  and  $v$  non-zero. Let  $T^i$  be a vertex of a triangle  $T \in \mathcal{F}$ , and  $(e_{-}^i, e_{+}^i)$  be the two edges of  $T$  incident to  $T^i$ , each of them being oriented from  $T^i$  to its other extremity. Then with  $u_{\pm}^i$  a vector at  $T^i$  tangent to  $e_{\pm}^i$  and compatible with its orientation, the *interior* and *exterior angles* at  $T^i$  are naturally defined by

$$(2.7) \quad \alpha(T^i) = \varnothing(u_{-}^i, u_{+}^i) \text{ and } \lambda(T^i) := \pi - \alpha(T^i).$$

If  $T^i$  is a singular point then the tangent vectors  $u_{\pm}^i$  are well-defined in any singular chart at  $T^i$ , and the angle  $\varnothing(u_{-}^i, u_{+}^i)$  being invariant by isometry, it will not depend on the chosen singular chart according to Proposition 2.26. Therefore, this definition still makes sense at a singular vertex. Denoting by  $(T^1, T^2, T^3)$  the vertices of  $T \in \mathcal{F}$ , the Gauß-Bonnet formula proved in [Dza84, p.225] becomes then:

$$(2.8) \quad i\varepsilon_{\mathbf{X}}\mathcal{A}_{\mu}(T) + \sum_{i=1}^3 \lambda(T^i) = 2\pi$$

with  $\mathcal{A}_{\mu}(T)$  the area of  $T$ . To translate Dzan's formula into the equation (2.8) for our geodesic triangle  $T$ , the following remarks are in order about the successive terms of the left-hand-side of [Dza84, Gauß-Bonnet formula p.225]:

- (1) the area element  $dS$  appearing in the formula is purely imaginary, equal to  $idS_0$  with  $dS_0$  the standard area element of  $S$  (see [Dza84, (55) p.224]);
- (2) the edges of  $T$  being geodesic, the integral of the geodesic curvature  $k_g$  vanishes;
- (3) the "directed sectorial measure of the exterior angle  $\lambda_i$ " at  $T^i$ , equals our exterior angle  $\lambda(T^i)$  defined in (2.7).

For any  $v \in \mathcal{V}$ , we denote by  $\mathcal{F}_v$  the set of triangles containing  $v$  as a vertex, and for  $T \in \mathcal{F}_v$ , by  $T^{i_v}$  the (unique) vertex of  $T$  equal to  $v$ . The remark preceding [Dza84, Definition 3 p.218] and the additivity of the Lorentzian angle imply then that the total angle at any regular vertex  $v \in \mathcal{V}$  is  $2\pi$ , *i.e.* that:

$$\sum_{T \in \mathcal{F}_v} \alpha(T^{i_v}) = 2\pi.$$

Thanks to the interpretation of standard singularities as angle defaults in Remark 2.21, this relation becomes:

$$(2.9) \quad \sum_{T \in \mathcal{F}_v} \alpha(T^{i_v}) = 2\pi - i\theta_v.$$

at a singular point  $v \in \mathcal{V}$  of angle  $\theta_v$ .

We are finally ready to sum the formula (2.8) on the faces of our triangulation. To this end, we denote by  $V$ ,  $E$ ,  $F$  and  $F_v$  the respective cardinals of the sets  $\mathcal{V}$ ,  $\mathcal{E}$ ,  $\mathcal{F}$  and  $\mathcal{F}_v$  for any  $v \in \mathcal{V}$ . We first translate (2.9) into:

$$\sum_{T \in \mathcal{F}_v} \lambda(T^{i_v}) = \pi(F_v - 2) + i\theta_v,$$

which gives

$$\begin{aligned}
 \sum_{T \in \mathcal{F}} \sum_{i=1}^3 \lambda(T^i) &= \sum_{v \in \mathcal{V}} \sum_{T \in \mathcal{F}_v} \lambda(T^{iv}) \\
 &= \sum_{v \in \mathcal{V}} (\pi(F_v - 2) + i\theta_v) \\
 (2.10) \qquad \qquad \qquad &= \pi(3F - 2V) + i \sum_{v \in \mathcal{V}} \theta_v
 \end{aligned}$$

by summing on the vertices. In the last equality, we used the obvious relation  $\sum_{v \in \mathcal{V}} F_v = \sum_{T \in \mathcal{F}} 3 = 3F$ . Using (2.10), we obtain from (2.8):

$$\begin{aligned}
 \sum_{T \in \mathcal{F}} i\varepsilon_{\mathbf{X}} \mathcal{A}_\mu(T) + \sum_{T \in \mathcal{F}} \sum_{i=1}^3 \lambda(T^i) &= \sum_{T \in \mathcal{F}} 2\pi \\
 \Leftrightarrow i\varepsilon_{\mathbf{X}} \mathcal{A}_\mu(S) + \pi(3F - 2V) + i \sum_{v \in \mathcal{V}} \theta_v &= 2\pi F \\
 (2.11) \qquad \qquad \qquad \Leftrightarrow i\varepsilon_{\mathbf{X}} \mathcal{A}_\mu(S) + \pi(F - 2V) + i \sum_{v \in \mathcal{V}} \theta_v &= 0.
 \end{aligned}$$

Since  $\mathcal{T}$  is a triangulation, each of its edge belongs to exactly two of its faces, which translates as  $\sum_{e \in \mathcal{E}} 2 = \sum_{T \in \mathcal{F}} 3$  and thus  $E = \frac{3F}{2}$ . Hence  $\pi(F - 2V) = 2\pi(-F + E - V) = -2\pi\chi(S)$  with  $\chi(S)$  the Euler characteristic of  $S$ , and (2.11) becomes thus:

$$(2.12) \qquad \qquad \qquad i \left( \varepsilon_{\mathbf{X}} \mathcal{A}_\mu(S) + \sum_{v \in \mathcal{V}} \theta_v \right) = 2\pi\chi(S).$$

But  $S$  is homeomorphic to a torus according to Corollary 2.31, hence  $\chi(S) = 0$ , and (2.12) yields the expected formula (2.5) which concludes the proof of the Proposition.  $\square$

### 3. CONSTRUCTIONS OF SINGULAR $\mathbf{dS}^2$ -TORI

In this section, we present the constructions of  $\mathbf{dS}^2$ -tori with one singularity yielding the existence results from Theorem B, C and D. More precisely, we will prove the following.

**Theorem 3.1.** *Let  $\theta \in \mathbb{R}_+^*$ ,  $c_\alpha \neq c_\beta \in \pi_1(\mathbf{T}^2)$  be two distinct primitive elements, and  $A_\alpha \neq A_\beta \in \mathbf{P}^+(\mathbf{H}_1(\mathbf{T}^2, \mathbb{R}))$  be two distinct irrational rays. Then there exists on  $\mathbf{T}^2$  a singular  $\mathbf{dS}^2$ -structure having a unique singularity of angle  $\theta$  at  $\mathbf{0} = [0, 0]$ , whose lightlike foliations are suspensions of circle homeomorphisms, and moreover satisfy any of the following properties.*

- (1)  $\mathcal{F}_\alpha(0)$  and  $\mathcal{F}_\beta(0)$  are closed leaves of  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$ , and  $([\mathcal{F}_\alpha(0)], [\mathcal{F}_\beta(0)]) = (c_\alpha, c_\beta)$ . We can moreover assume that either  $\mathcal{F}_\alpha(0)$  or  $\mathcal{F}_\beta(0)$  is the unique closed leaf of its foliation, and that both of them are such if  $(c_\alpha, c_\beta)$  is a basis of  $\pi_1(\mathbf{T}^2)$ .
- (2)  $([\mathcal{F}_\alpha(0)], A^+(\mathcal{F}_\beta)) = (c_\alpha, A_\beta)$  (in particular,  $\mathcal{F}_\beta$  is minimal), and  $\mathcal{F}_\alpha(0)$  is the unique closed leaf of  $\mathcal{F}_\alpha$ .
- (3)  $(A^+(\mathcal{F}_\alpha), A^+(\mathcal{F}_\beta)) = (A_\alpha, A_\beta)$  (in particular,  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$  are both minimal).

We recall that according to Proposition 2.32, the negative angles are the only one which can be realized by a single singularity of a  $\mathbf{dS}^2$ -torus, hence the necessary condition  $\theta \in \mathbb{R}_+^*$  which is not a restriction. The proof of Theorem 3.1 will be concluded in paragraph 3.8.

$A^+(\mathcal{F}) \in \mathbf{P}^+(\mathbf{H}_1(\mathbf{T}^2, \mathbb{R}))$  denotes the *oriented projective asymptotic cycle* of the oriented foliation  $\mathcal{F}$ , which will be introduced in paragraph 3.6. An element  $\alpha \in \pi_1(\mathbf{T}^2)$  is *primitive* if it cannot be written as  $\alpha = \beta^k$  with  $\beta \in \pi_1(\mathbf{T}^2)$  and  $k \geq 2$  – equivalently if  $\alpha$  is represented by simple closed curves of  $\mathbf{T}^2$ . We denote by  $[\gamma]$  the homotopy class of a curve  $\gamma$  in  $\pi_1(\mathbf{T}^2)$ . A line  $l \in \mathbf{P}(\mathbf{H}_1(\mathbf{T}^2, \mathbb{R}))$  is *rational* if  $l = \mathbb{R}\alpha$  with  $\alpha \in \pi_1(\mathbf{T}^2) \cong \mathbf{H}_1(\mathbf{T}^2, \mathbb{Z}) \subset \mathbf{H}_1(\mathbf{T}^2, \mathbb{R})$ , and *irrational* otherwise.

We fix for the whole subsection an area  $\mathcal{A} \in \mathbb{R}_+^*$  and the corresponding angle  $\theta = -\mathcal{A} \in \mathbb{R}_+^*$  given by Gauß-Bonnet formula (2.5) in Lemma 2.32.(3), so that if a singular  $\mathbf{dS}^2$ -torus of area  $\mathcal{A}$  has a single singularity, then it is of angle  $\theta$ . We also identify in the whole section  $\mathbb{R}\mathbf{P}^1$  with

$\mathbb{R} \cup \{\infty\}$  and elements of  $\mathrm{PSL}_2(\mathbb{R})$  with their associated homography of  $\mathbb{R} \cup \{\infty\}$ , as defined in (2.2) and (2.4).

**3.1. Gluings of polygons in  $\mathbf{dS}^2$ .** Let us denote by  $y_{\mathcal{A}} := 1 - e^{-\mathcal{A}} \in ]0; 1[$  the unique number such that  $\mathcal{A}_{\mu}(\mathcal{R}_{(1,\infty,0,y_{\mathcal{A}})}) = \mathcal{A}$ . According to Lemma 2.9,  $\mathcal{R}_{\mathcal{A}} := \mathcal{R}_{(1,\infty,0,y_{\mathcal{A}})}$  is, up to action of  $\mathrm{PSL}_2(\mathbb{R})$ , the unique rectangle with lightlike edges and area  $\mathcal{A}$  in  $\mathbf{dS}^2$ . Our goal is to define a quotient of it with only one singularity, which will *a posteriori* necessarily have angle  $\theta_{\mathcal{A}}$  by Gauß-Bonnet formula (2.5). A first easy way to do this is to consider the unique elements  $g = g_{\mathcal{A}}$  and  $h_{\mathcal{A}}$  of  $\mathrm{PSL}_2(\mathbb{R})$  such that

$$(3.1) \quad g(1, 0, y_{\mathcal{A}}) = (\infty, 0, y_{\mathcal{A}}) \text{ and } h_{\mathcal{A}}(1, \infty, 0) = (1, \infty, y_{\mathcal{A}}),$$

and to form the quotient of  $\mathcal{R}_{\mathcal{A}}$  by gluing its edges through  $g$  and  $h_{\mathcal{A}}$  (see Figure 3.1). The gluing being made by isometries, the  $\mathbf{dS}^2$ -torus obtained in this way will have, as sought, a unique singularity at the class of the vertices. However by such a construction, both lightlike leaves of the singularity will always be closed. To obtain a structure with a minimal lightlike foliation, it is thus necessary to consider another type of gluing.

**3.1.1. Suspension of homographic interval exchange transformations.** Inspired from the constructions of translation surfaces as “suspensions” of (classical) interval exchange transformations, a natural idea to obtain minimal lightlike foliations is to keep gluing the  $\beta$ -edges of  $\mathcal{R}_{\mathcal{A}}$  through  $g$ , but to glue its two  $\alpha$ -edges through a *homographic interval exchange transformation (HIET)* with two components. Such a map is a bijection of an interval  $I$  of  $\mathbb{RP}^1$  exchanging the components of two partitions of  $I$  traditionally called *top* and *bottom* partitions, and which is homographic, *i.e.* equals the restriction of an element of  $\mathrm{PSL}_2(\mathbb{R})$ , on each component of the start partition. The notion of HIET is both a natural generalization of the ones of (classical) IET and affine IET, and a restriction of the notion of *generalized interval exchange transformation (GIET)*. We refer the reader to the excellent [Yoca, Yocb] for more informations on these notions (which will however not be needed in this text).

For any  $x, x' \in ]1; \infty[$ , we introduce the following subintervals of  $I = [1; \infty[$ :

$$(3.2) \quad I_1^t = [1; x'], I_2^t = [x'; \infty[, I_1^b = [1; x], I_2^b = [x; \infty[,$$

delimiting a *top* partition  $I = I_1^t \sqcup I_2^t$  and a *bottom* partition  $I = I_1^b \sqcup I_2^b$  of  $I$ . By three-transitivity of  $\mathrm{PSL}_2(\mathbb{R})$  on  $\mathbb{RP}^1$ , there exists a unique pair  $h_1, h_2$  of elements of  $\mathrm{PSL}_2(\mathbb{R})$  such that  $h_k(0) = y_{\mathcal{A}}$  for  $k = 1, 2$ ,  $h_1(I_1^t) = I_2^b$  and  $h_2(I_2^t) = I_1^b$ , and we define a HIET  $E: I \rightarrow I$  by:

$$E|_{I_1^t} = h_1|_{I_1^t}, E|_{I_2^t} = h_2|_{I_2^t}.$$

We now “suspend” this HIET  $E$ , obtaining the quotient  $\mathcal{T}_{\mathcal{A},E}$  of the rectangle  $\mathcal{R}_{\mathcal{A}}$  by the following edges identifications:

$$\begin{cases} [1; \infty[ \times \{0\} \ni (p, 0) \sim (E(p), y_{\mathcal{A}}) \in [1; \infty[ \times \{y_{\mathcal{A}}\}, \\ \{1\} \times [0; y_{\mathcal{A}}] \ni (1, p) \sim (\infty, g(p)) \in \{\infty\} \times [0; y_{\mathcal{A}}]. \end{cases}$$

These gluings, illustrated in Figure 3.1, give us a first family of examples of singular  $\mathbf{dS}^2$ -tori. Vertices of  $\mathcal{R}_{\mathcal{A}}$  of the same color indicate points identified in the quotient  $\mathcal{T}_{\mathcal{A},E}$ .

**Proposition 3.2.** *For any  $\mathcal{A} \in \mathbb{R}_+^*$  and  $x, x' \in ]1; \infty[$ ,  $\mathcal{T}_{\mathcal{A},E}$  is homeomorphic to  $\mathbf{T}^2$  and the  $\mathbf{dS}^2$ -structure of the interior of  $\mathcal{R}_{\mathcal{A}}$  extends to a unique singular  $\mathbf{dS}^2$ -structure on  $\mathcal{T}_{\mathcal{A},E}$ . The latter has area  $\mathcal{A}$ , the  $\alpha$ -leaf of  $[\infty, 0]$  is closed, its unique (potentially) singular points are  $[\infty, 0]$  and  $[x', 0]$ , and the holonomies of small positively oriented loops around them are:*

- (1) *holonomy around  $(\infty, 0) = h_2^{-1}h_1g^{-1}$ ,*
- (2) *holonomy around  $(x', 0) = h_1^{-1}gh_2$ .*

*Proof.* Let us denote by  $\pi: \mathcal{R}_{\mathcal{A}} \rightarrow \mathcal{T}_{\mathcal{A},E}$  the canonical projection, and  $[a, b] = \pi(a, b)$ . We first observe that the gluing of the edges are well-defined for the quotient to be topologically a torus, as a Euler characteristic computation directly shows. The edges being moreover identified by elements of  $\mathrm{PSL}_2(\mathbb{R})$ , the  $\mathbf{dS}^2$ -structure of  $\pi(\mathrm{Int}(\mathcal{R}_{\mathcal{A}}))$  for which  $\pi|_{\mathrm{Int}(\mathcal{R}_{\mathcal{A}})}$  is a  $\mathbf{dS}^2$ -morphism extends on the complement of the vertices, *i.e.* on  $\mathcal{T}_{\mathcal{A},E} \setminus \{[\infty, 0], [x', 0]\}$  to a structure of area

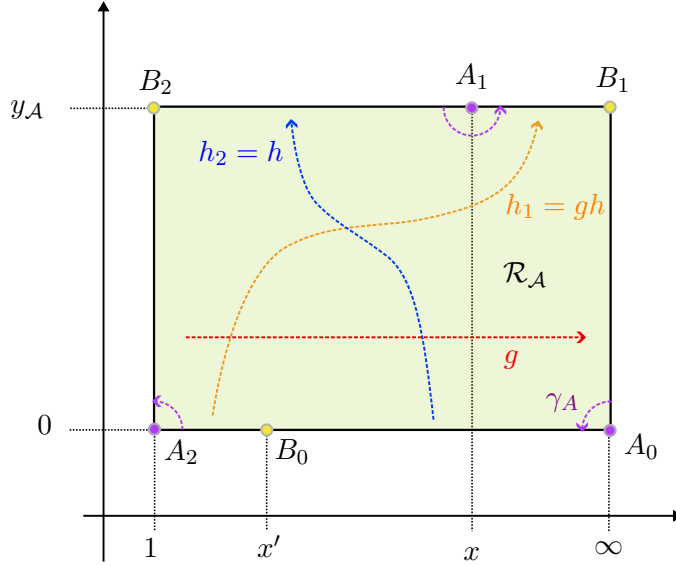


FIGURE 3.1.  $\mathbf{dS}^2$ -torus with one singularity and a closed  $\alpha$ -lightlike leaf.

$\mathcal{A}$ . Lastly, observe that the lightlike foliations of  $\pi(\text{Int}(\mathcal{R}_{\mathcal{A}}))$  clearly extend to two transverse continuous foliations of  $\mathcal{T}_{\mathcal{A},E}$ .

The top and bottom partitions (3.2) of  $[1; \infty[$  define associated partitions of the  $\alpha$  and  $\beta$  boundary parts of  $\mathcal{R}_{\mathcal{A}}$ , that we will call *edges*, and their extremities will be called *vertices*. Let us detail in the specific case of  $A = [\infty, 0] \in \mathcal{T}_{\mathcal{A},E}$  a general “recipe” to compute the holonomy around any vertex  $P$  of  $\mathcal{T}_{\mathcal{A},E}$ , illustrated in Figure 3.1. First of all, note that each vertex  $P$  is associated with a positively cyclically ordered periodic orbit  $(P_0, P_1, \dots, P_d)$ , which has length 2 for  $A$ . A small positively oriented closed loop  $\gamma_P$  around  $P$  defines indeed a cyclic ordering on the (finite) equivalence class of  $P$  for  $\sim$ , describing in which order the points are met in  $\mathcal{R}_{\mathcal{A}}$  when following  $\gamma_P$ . For instance in the case of  $A$  if we start with  $A_0 = (\infty, 0)$ , then we successively meet  $A_1 = (x, y_A)$ ,  $A_2 = (1, 0)$  and finally come back to  $A_0$ . Moreover at each step  $P_i$ ,  $i \geq 1$  of this periodic orbit,  $\gamma_P$  meets in  $\mathcal{T}_{\mathcal{A},E}$  an interval of a lightlike leaf of  $P$  corresponding to two top and bottom edges  $e_{P_i}^t$  and  $e_{P_i}^b$  of  $\mathcal{R}_{\mathcal{A}}$  having respectively  $P_i$  and  $P_{i-1}$  as one of their extremities: these are for instance  $e_{A_1}^t = [1; x] \times \{y_A\}$  ( $A_1$  as right extremity) and  $e_{A_1}^b = [x'; \infty] \times \{0\}$  ( $A_0$  as right extremity) for  $A_1$ . These edges are then identified by some  $f_{P_i} \in \text{PSL}_2(\mathbb{R})$ , characterized by  $f_{P_i}(e_{P_i}^t) = e_{P_i}^b$ :  $f_{A_1} = h_2^{-1}$  for instance. Lastly, each point  $P_i$  of the periodic orbit  $(P_0, P_1, \dots, P_d)$  contributes for a certain sequence  $Q_{P_i}$  of *quadrants* around  $P$ , ordered as they are met by  $\gamma_P$ . For instance for  $A$ ,  $Q_{A_0} = \text{future timelike}$ ,  $Q_{A_1} = (\text{past spacelike}, \text{past timelike})$  and  $Q_{A_2} = \text{future spacelike}$ . We will say *the identification of the quadrants around  $P$  is standard*, if the sequence  $(Q_{P_0}, \dots, Q_{P_d})$  equals the *standard sequence*: (future timelike, past spacelike, past timelike, future spacelike), up to cyclic permutations.

**Fact 3.3.** *Let assume that the identification of the quadrants around a vertex  $P$  is standard. Then  $P$  is a standard singularity of  $\mathcal{T}_{\mathcal{A},E}$ . Moreover with  $\rho$  the holonomy associated to the developing map extending the section  $s: \pi(\text{Int}(\mathcal{R}_{\mathcal{A}})) \rightarrow \text{Int}(\mathcal{R}_{\mathcal{A}})$  of  $\pi$ , we have:*

$$(3.3) \quad \rho(\gamma_P) = f_{P_1} f_{P_2} \dots f_{P_d} f_{P_0} \in \text{Stab}_{\text{PSL}_2(\mathbb{R})}(P_0).$$

*Proof.* For the sake of clarity, we write the proof in the specific case of  $A$ , but it is formally identical in any situation. We define  $\varphi_0 = s$  as a  $\mathbf{dS}^2$ -chart on  $\pi(U_0)$ , with  $U_0$  a small neighbourhood of  $A_0$  in  $\mathcal{R}_{\mathcal{A}}$ . Now let  $U_1$  be a small neighbourhood of  $A_1$  in  $\mathcal{R}_{\mathcal{A}}$ , and  $\varphi_1$  be a  $\mathbf{dS}^2$ -chart defined on a neighbourhood of  $\pi(U_1)$  in  $\mathcal{T}_{\mathcal{A},E} \setminus \{[\infty, 0], [x', 0]\}$ , and agreeing with  $\varphi_0$  on a neighbourhood of  $[\infty, 0]$  in  $\pi([1; \infty] \times \{0\})$ . Then  $\varphi_1 = g_{A_1} \circ s$  on  $\pi(U_1)$  with  $g_{A_1} \in \text{PSL}_2(\mathbb{R})$  agreeing with  $f_{A_1} = h_2^{-1}$  on a neighbourhood of  $A_1$  in  $[1; x] \times \{y_A\}$ . The naive but important observation is now that if  $g, g' \in \text{PSL}_2(\mathbb{R})$  have the same action on a non-empty open lightlike interval, then  $g = g'$ . Indeed, it is sufficient to check this for  $g, g' \in \text{Stab}(\mathfrak{o})$ , for which it simply follows from



the fact that a non-trivial element of  $\text{Stab}(\mathfrak{o})$  has a non-trivial action on any non-empty open lightlike interval of extremity  $\mathfrak{o}$ . This shows that  $g_{A_1} = f_{A_1}$ , *i.e.* that  $\varphi_1 = f_{A_1} \circ s$  on  $\pi(U_1)$ .

Continuing in the same way, we conclude that if  $U_2$  is a neighbourhood of  $A_2$  in  $\mathcal{R}_{\mathcal{A}}$ , and  $\varphi_2$  a  $\mathbf{dS}^2$ -chart defined on a neighbourhood of  $\overline{\pi(U_2)}$  and agreeing with  $\varphi_1$  on the suited  $\alpha$ -interval, then  $\varphi_2 = f_{A_1} \circ f_{A_2} \circ s$  on  $\pi(U_2)$ . To understand this relatively counter-intuitive order in the compositions, observe first that  $f_{A_2} \circ s|_{\pi(U_2)}$  and  $s|_{\pi(U_1)}$  glue together to define a  $\mathbf{dS}^2$ -chart on a punctured neighbourhood of  $[1, 0]$  in  $\pi([1; x'] \times \{0\})$ , hence that  $f_{A_1} \circ f_{A_2} \circ s$  and  $f_{A_1} \circ s = \varphi_1$  agree on the intersection of their domains.

In the end  $\varphi_3 = f_{A_1} \circ f_{A_2} \circ f_{A_0} \circ \varphi_0$ , and the maps  $\varphi_i$  for  $i = 0, \dots, 3$  agree on the intersection of their domains. They glue thus together to give a  $\mathbf{dS}^2$ -isomorphism  $\psi$  from a slit neighbourhood  $U' = U \setminus \mathcal{F}_{\alpha}([\infty, 0])$  of  $[\infty, 0]$  to a slit neighbourhood of  $(\infty, 0) = \mathfrak{o}$  in  $\mathbf{dS}^2$ . This map satisfies the hypotheses of Lemma 2.14.(2), and we conclude thus that  $[\infty, 0] = A$  is a standard singularity of the  $\mathbf{dS}^2$ -structure of  $\mathcal{T}_{\mathcal{A}, E} \setminus \{[1, 0], [x', 0]\}$ , and that  $\rho(\gamma_A) = f_{A_1} \circ f_{A_2} \circ f_{A_0} \in \text{Stab}(\mathfrak{o})$ .  $\square$

Fact 3.3 shows our claim for the vertices  $[\infty, 0]$  and  $[x', 0]$ , and concludes thus the proof of the proposition.  $\square$

3.1.2. *Further remarks on the gluings.* To clarify our exposition, avoid unnecessary notations and rather emphasize the main ideas, we chose to focus on the constructions of singular  $\mathbf{dS}^2$ -tori that will be developed in the sequel of the text in the case of one singularity. However, the same formal proof than the one of Fact 3.3 offers a general way of constructing singular  $\mathbf{X}$ -tori, and proves the following result. We refer to the proof of Proposition 3.2 for the definition of a *standard identification of quadrants around a vertex*, and of the related notions appearing in the statement. We will call *polygon* a compact connected subset of  $\mathbf{X}$ , homeomorphic to a closed disk, and whose boundary is a finite union of geodesic segments. We also denote by  $(\mathbf{G}, \mathbf{X})$  the pair  $(\text{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$  or  $(\mathbb{R}^{1,1} \rtimes \text{SO}^0(1, 1), \mathbb{R}^{1,1})$ .

**Proposition 3.4.** *Let  $\mathcal{P}$  be a polygon of  $\mathbf{X}$ , whose boundary is lightlike and endowed with:*

- (1) *a decomposition into an even number of edges which are segments of lightlike leaves,*
- (2) *and pairwise identifications between these edges by elements of  $\mathbf{G}$ .*

*Assume that the identification of the quadrants around each vertex is standard. Then the quotient of  $\mathcal{P}$  by the edges identifications is a torus endowed with a unique singular  $\mathbf{X}$ -structure compatible with the one of  $\mathcal{P}$ . This singular  $\mathbf{X}$ -torus has the same area than  $\mathcal{P}$ , and the holonomies at the vertices are given by the formula (3.3).*

*Remark 3.5.* Proposition 3.4 proves in particular the existence of singular  $\mathbb{R}^{1,1}$ -tori or *singular flat tori*, and offers a way to construct a large family of them. The investigation of singular flat tori will be considered in a future work with a more analytical point of view, closer to the one of [Tro86, Tro91] for singular Riemannian surfaces.

*Remark 3.6.* Proposition 3.4 could be stated more generally: the quotient of any connected polygon of  $\mathbf{X}$  whose boundary is lightlike and endowed with an even partition into edges, by any pairwise identifications of the edges by elements of  $\mathbf{G}$ , is endowed with a natural  $\mathbf{X}$ -structure on the complement of the vertices. But these vertices are not standard singularities as studied in this text if the total number of quadrants around them is not 4. For instance, non-standard singularities do not see four lightlike half-leaves emanating from them, and in particular the lightlike foliations do not extend to  $\mathcal{C}^0$  foliations there. This should however not exclude the attention for such examples, particularly interesting ones arising for instance when the lightlike foliations have themselves standard singularities at the singularities of the metric (*i.e.* when they are the stable and unstable foliations of a pseudo-Anosov map). The study of this very interesting class of examples will be the content of a future work.

Lastly, Lemma 2.20 shows that standard singularities do not need to be constructed from lightlike geodesics, and that definite geodesics work just as well. Therefore, we observe that the natural counterpart of Proposition 3.4 can be stated and proved in the same way for any polygon of  $\mathbf{X}$  with a geodesic boundary, a partition into an even number of edges and pairwise identifications by elements of  $\mathbf{G}$ .

All the graphs are assumed to be finite.

**Definition 3.7.** A graph  $C$  embedded in a singular  $\mathbf{X}$ -surface  $S$  is said *lightlike*, if any vertex of  $C$  has degree at least 2, and any edge of  $C$  is a connected subset of a lightlike geodesic of  $S$ . A surface  $S$  is *rectangular* if there exists a lightlike graph  $C$  embedded in  $S$  such that:

**Proposition 3.8.** *Any rectangular singular  $\mathbf{X}$ -torus is isometric to one given by Proposition 3.4.*

*Proof.* Let  $\mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$  be endowed with a rectangular singular  $\mathbf{X}$ -structure, and  $\bar{C} \subset \mathbf{T}^2$  be a graph as in Definition 3.7. We endow  $\mathbb{R}^2$  with the  $\mathbb{Z}^2$ -invariant singular  $\mathbf{X}$ -structure for which the universal covering  $\pi: \mathbb{R}^2 \rightarrow \mathbf{T}^2$  is a local isometry, and denote by  $\tilde{C} = \pi^{-1}(\bar{C})$  the lift of  $\bar{C}$ . This is an embedded graph in  $\mathbb{R}^2$  satisfying properties (2) and (3) of Definition 3.7 for  $S = \mathbb{R}^2$ , and such that each connected component of  $\mathbb{R}^2 \setminus \tilde{C}$  is a topological disk. We denote by  $E$  the closure of one of these connected components, and by  $C$  the subgraph of  $\tilde{C}$  which is the boundary of  $E$ . Then  $E$  is a fundamental domain for the action of  $\mathbb{Z}^2$  on  $\mathbb{R}^2$ , and  $\mathbf{T}^2$  is thus isometric to the quotient of  $E$  by the suitable identifications of the edges of  $C$  by the action of elements of  $\mathbb{Z}^2$ . Note that any edge of  $\tilde{C}$  has two lifts in  $C$ , hence  $C$  has an even number of edges.

(a) **Injectivity of the developing map on a fundamental domain.** Since the singularities  $\bar{\Sigma}$  of  $\mathbf{T}^2$  are by assumption contained in  $\bar{C}$ , the singularities  $\tilde{\Sigma} = \pi^{-1}(\bar{\Sigma})$  of  $\mathbb{R}^2$  are contained in  $\tilde{C}$ , and with  $\Sigma = \tilde{\Sigma} \cap C$ , we have  $\pi(\Sigma) = \bar{\Sigma}$ . In particular  $E^* := E \setminus \Sigma$  is contained in  $\mathbb{R}^2 \setminus \tilde{\Sigma}$ , and with  $U$  a simply connected open neighbourhood of  $E^* := E \setminus \Sigma$  contained in  $\mathbb{R}^2 \setminus \tilde{\Sigma}$ , there exists a  $\mathbf{X}$ -morphism

$$\delta: U \rightarrow \mathbf{X},$$

which is the developing map of the  $\mathbf{X}$ -structure of  $U$ . Note that  $U$  is a topological disk, as is any connected and simply connected open subset of the plane.

**Fact 3.9.**  $\delta$  extends to a continuous map  $D$  from a neighbourhood  $\mathcal{U}$  of  $E$  to  $\mathbf{X}$ . There exists moreover a lightlike rectangle  $E_0$  of  $\mathbf{X}$ , a decomposition of its boundary into a graph  $C_0$  whose edges are segments of lightlike leaves, and a subset  $\Sigma_0$  of the vertices of  $C_0$ , such that:

- (1)  $D(E) \subset E_0$ ,
- (2)  $D(\Sigma) = \Sigma_0$  and  $D$  is a graph morphism from  $C$  to  $C_0$ ,
- (3)  $D$  is injective in restriction to  $C$ .

*Proof.* By assumption, any vertex of  $\tilde{C}$  has degree at least 2, and since any edge is a segment of lightlike leaf, the vertices also have degree at most 4 inside  $\tilde{C}$  (in the maximal case, segments of the four lightlike half-leaves emanate from the vertex). But  $C$  being the boundary of  $E$  hence a topological circle, any vertex of  $C$  has of course degree exactly 2 inside  $C$ . Now we endow the circle  $C = \partial E$  with the orientation induced by the one of  $E$ , fix  $v \in \Sigma$  a singular vertex of  $C$ , and denote by  $e_-, e_+$  the two (closed) edges of  $C$  of extremity  $v$ ,  $e_+$  being met after  $e_-$  in the positive orientation of  $C$  ( $e_- \neq e_+$  since  $v$  has degree 2). Up to a cyclic permutation of the quadrants, the three following situations are the only one that can arise.

- (1)  $e_-$  is a segment of the  $\alpha$ -leaf of  $v$  denoted by  $[x_-; v]_\alpha$ , going from  $x_-$  to  $v$  for the positive orientation of  $C$ . Similarly,  $e_+$  is a segment of the  $\beta$ -leaf of  $v$  of the form  $[v; x_+]_\beta$ . Moreover,  $v$  admits an open neighbourhood  $Q_v$  in  $E$  which is a small future timelike quadrant, and such that  $Q_v \cap \Sigma = \{v\}$ .
- (2)  $e_-$  is an  $\alpha$ -segment  $[x_-; v]_\alpha$ ,  $e_+$  an  $\alpha$ -segment  $[v; x_+]_\alpha$ , and  $v$  admits an open neighbourhood  $Q_v \subset E^* \cup \{v\}$  in  $E$  which is the union of a small future timelike quadrant and of a small future spacelike quadrant.
- (3)  $e_-$  is an  $\alpha$ -segment  $[x_-; v]_\alpha$ ,  $e_+$  a  $\beta$ -segment  $[v; x_+]_\beta$ , and  $v$  admits an open neighbourhood  $Q_v \subset E^* \cup \{v\}$  in  $E$  which is the union of a small future timelike quadrant, a small future spacelike quadrant and a small past timelike quadrant.

Note that the two orientations of the segment  $[x_-; v]_\alpha$  respectively induced by the one of the lightlike foliation and by  $C$  coincide in these three cases, while for  $[v; x_+]_\alpha$  and  $[v; x_+]_\beta$  they coincide in cases (1) and (2) but are opposite in case (3).

Since  $v$  is a standard singularity, denoting by  $Q_o$  the union of quadrants at  $o$  corresponding to  $Q_v$ ,  $Q_v^* := Q_v \setminus \{v\}$  is isometric to  $Q_o^* := Q_o \setminus \{o\}$ . Namely, there exists an isometry  $\varphi$  from a

neighbourhood  $V \subset U \setminus \tilde{\Sigma}$  of  $Q_v^*$  in  $\mathbb{R}^2$  to a neighbourhood  $V_0$  of  $Q_0^*$  in  $\mathbf{X}$ , such that  $\varphi(Q_v^*) = Q_0^*$  (see Lemma 2.14). Since  $\delta|_V$  is another  $\mathbf{X}$ -morphism from  $V$ , there exists moreover  $g \in \mathbf{G}$  such that  $\delta|_V = g \circ \varphi$ . Hence  $\delta(Q_v^*) = g(Q_0^*) = Q_{v_0}^*$ , with  $Q_{v_0}$  the union of quadrants at  $v_0 := g(\mathfrak{o})$  corresponding to  $Q_v$ . In particular, this shows that  $\delta|_V$  extends to an injective continuous maps  $D_v$  from a neighbourhood  $W$  of  $Q$  to a neighbourhood  $W_0$  of  $Q_{v_0}$ , sending  $v$  to  $v_0$ .

We can now glue together these maps  $D_v$ , to define a map  $D$  from a neighbourhood  $\mathcal{U}$  of  $E$  to  $\mathbf{X}$ . Since  $\delta$  is a local diffeomorphism it is injective in restriction to any open edge of  $C$ , and  $D$  is thus injective to any closed edge since the lightlike leaves of  $\mathbf{X}$  are embeddings of  $\mathbb{R}$ . By construction,  $C_0 := D(C)$  is a lightlike rectangular closed loop in  $\mathbf{X}$ , and we define a decomposition of it by stating that  $D$  is a graph morphism (which makes sense since  $D$  is injective in restriction to any edge). A naive but important remark is now that any lightlike rectangular closed loop in  $\mathbf{X}$  is simple, *i.e.* without any self-intersection. Since  $E$  is moreover always on the same side of  $C$  by definition of its orientation,  $D(E)$  is always on the same side of  $C_0$ , hence  $D(E)$  is contained in the (unique) lightlike rectangle  $E_0$  of  $\mathbf{X}$  bounded by  $C_0$ .

We know at this stage that  $D|_C$  is a continuous map from the topological circle  $C = \partial E$  to the topological circle  $C_0$ , which is locally injective hence a local homeomorphism. But since the oriented graph  $C$  contains only one positively travelled  $\alpha$ -segment,  $D|_C$  also has degree one. In the end  $D|_C$  is injective, which concludes the proof of the fact.  $\square$

Now since the continuous map  $D|_E: E \rightarrow E_0$  is locally injective and injective in restriction to  $\partial D$ ,  $D|_E$  is injective according to [MO63, Theorem 1 p.75] (see also Definition 3 p.74 therein). And since  $\delta$  is a local diffeomorphism,  $D$  is actually injective in restriction to a small enough neighbourhood  $\mathcal{U}$  of  $E$ , and is thus a homeomorphism from  $\mathcal{U}$  to a neighbourhood  $\mathcal{U}_0$  of  $E_0$  in  $\mathbf{X}$  according to Brouwer's invariance of domain theorem. In particular,  $D(E)$  is a compact subset of  $E_0$  of boundary  $\partial E_0$ , *i.e.*  $D(E) = E_0$ .

**(b) Edges identifications.** Recall that  $C = \partial E$  has an even number of edges denoted by  $\{(e_i^t, e_i^b)\}_i$ , and that  $\mathbf{T}^2$  is isometric to the quotient  $\mathcal{E}$  of  $E$  by the identification of each  $e_i^t$  with the corresponding  $e_i^b$  through a translation  $T_{u_i}$  (where  $u_i \in \mathbb{Z}^2$  and  $T_{u_i}(e_i^t) = e_i^b$ ). Since integral translations are isometries of  $\mathbb{R}^2$ , there exists moreover unique elements  $g_i \in \mathbf{G}$  such that

$$\delta \circ T_{u_i} = g_i \circ \delta$$

in restriction to a connected neighbourhood of  $e_i^t$ . Since  $D$  is a graph morphism according to Fact 3.9, we can define a decomposition of  $C_0$  associated to the one of  $C$  by  $f_i^t = D(e_i^t)$  and  $f_i^b = D(e_i^b)$ . We have then  $g_i(f_i^t) = f_i^b$ , and we can thus form the quotient  $\mathcal{E}_0$  of  $\mathcal{E}$  by these edges identifications, given by Proposition 3.4. By construction,  $D$  induces then an isometry from  $\mathcal{E} \simeq \mathbf{T}^2$  to  $\mathcal{E}_0$ , which concludes the proof.  $\square$

**Lemma 3.10.** *Let  $S$  be a closed singular  $\mathbf{X}$ -torus with one singularity  $x$ , such that one of the lightlike leaves of  $x$  is closed and for the other lightlike foliation  $\mathcal{F}$ : either  $\mathcal{F}$  is minimal or  $\mathcal{F}(x)$  is closed. Then  $S$  is rectangular.*

*Proof.* We assume to fix the ideas that  $\mathcal{F}_\alpha(x)$  is closed. We then begin the construction of the graph  $C$  of Definition 3.7 with  $\mathcal{F}_\alpha(x)$ , which gives one vertex and one edge. Let us denote by  $y$  the first intersection point of  $\mathcal{F}_\beta(x)$  with  $\mathcal{F}_\alpha(x)$  (for the positive orientation of  $\mathcal{F}_\beta(x)$ ), and by  $e$  the positive  $\beta$ -segment from  $x$  to  $y$ . Note that  $y$  exists since either  $\mathcal{F}_\beta(x)$  is closed, or  $\mathcal{F}_\beta$  is minimal. We define then  $C = \mathcal{F}_\alpha(x) \cup e$ , with set of vertices  $V$  equal to  $\{x, y\}$  (if  $y = x$ , then we add a vertex anywhere on  $C \setminus \{x\}$  to have degree 2 at any vertex), and edges given by the connected components of  $C \setminus V$ . Since  $S \setminus \mathcal{F}_\alpha(x)$  is a cylinder,  $S \setminus C$  is indeed a topological disk, while the other properties of Definition 3.7 are clearly satisfied.  $\square$

Henceforth, we come back to the homogeneous model space  $(\mathbf{G}, \mathbf{X}) = (\mathrm{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$ , and investigate thoroughly two families of singular  $\mathbf{dS}^2$ -tori.

**3.2. A one-parameter family of  $\mathbf{dS}^2$ -tori with one singularity having a closed leaf.** We now apply Proposition 3.2 to obtain a first one-parameter family of  $\mathbf{dS}^2$ -tori.

3.2.1. *Definition of the one-parameter family.* For any  $x \in ]1; \infty]$  and  $x' \in [1; \infty[$ , let  $h = h_{(x,x')}$  be the unique element of  $\mathrm{PSL}_2(\mathbb{R})$  such that

$$(3.4) \quad h(x', \infty, 0) = (1, x, y_{\mathcal{A}}),$$

i.e.  $h = h_2$  in the notations of Proposition 3.2. Proposition 3.2 and Corollary 2.17 indicate us that  $[x', 0] \in \mathcal{T}_{\mathcal{A},E}$  is regular if, and only if  $h_1 = gh_2 = gh$ , or equivalently if:

$$(3.5) \quad gh(1, x', 0) = (x, \infty, y_{\mathcal{A}}).$$

Since  $gh(x', 0) = (\infty, y_{\mathcal{A}})$  is automatically satisfied according to the equations (3.4) and (3.1), the condition  $h_1 = gh_2$  eventually amounts to  $gh(1) = x$ .

**Lemma 3.11.**  *$gh(1) = x$  if, and only if  $x' = \frac{x}{x-1}$ . Moreover,  $g$  and  $h$  are hyperbolic.*

*Proof.* The last claim follows from a direct observation of the dynamics of  $g$  and  $h$  on  $\mathbb{RP}^1$ . With  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , the definition of  $g$  reads:  $c+d=0$ ,  $b=0$ ,  $ay_{\mathcal{A}}+b=y_{\mathcal{A}}(cy_{\mathcal{A}}+d)$ , i.e.  $y_{\mathcal{A}}(cy_{\mathcal{A}}-c-a)=0$  and thus  $a=c(y_{\mathcal{A}}-1)$ . Hence  $g = (1-y_{\mathcal{A}})^{-1/2} \begin{pmatrix} -(1-y_{\mathcal{A}}) & 0 \\ 1 & -1 \end{pmatrix}$  and  $g(t) = (y_{\mathcal{A}}-1)\frac{t}{t-1}$ . Now if  $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , the definition of  $h$  reads:  $ax'+b=cx'+d$ ,  $a=cx$ ,  $b=dy_{\mathcal{A}}$ , hence  $d = \frac{cx'(x-1)}{(1-y_{\mathcal{A}})}$  and thus

$$h(t) = \frac{x(1-y_{\mathcal{A}})t + x'(x-1)y_{\mathcal{A}}}{(1-y_{\mathcal{A}})t + x'(x-1)}.$$

A computation shows  $x-gh(1) = ((1+e^{\mathcal{A}}(-1+x))(x(-1+x')-x'))/(e^{\mathcal{A}}(-1+x)(-1+x'))$  and since  $x > 1 > 1-e^{-\mathcal{A}}$ , this vanishes if and only if  $x(-1+x')-x'=0$  i.e.  $x' = x/(x-1)$ .  $\square$

We now fix  $x \in [1; \infty]$  and denote:

- (1)  $x' = x'_x := \frac{x}{x-1} \in [1; \infty]$  ( $x'_\infty = 1$ ,  $x'_1 = \infty$ ),
- (2) and  $h = h_x := h_{(x,x'_x)}$  if  $x > 1$ , extended by  $h_1 := g^{-1}h_\infty$  for  $x = 1$ .

The equations (3.4) and (3.5) show that  $\lim_{x \rightarrow 1} gh_x = h_\infty$ , hence that  $\lim_{x \rightarrow 1} h_x = \lim_{x \rightarrow 1} g^{-1}(gh_x) = h_1$ , so that the maps

$$x \in [1; \infty] \mapsto h_x \in \mathrm{PSL}_2(\mathbb{R}), x \in [1; \infty] \mapsto gh_x \in \mathrm{PSL}_2(\mathbb{R})$$

are continuous. Using the top and bottom partitions (3.2) of  $I = [1; \infty[$ , we consider the HIET  $E = E_x: I \rightarrow I$  defined by

$$(3.6) \quad E_x|_{I_1^t} = gh_x|_{I_1^t}: I_1^t \rightarrow I_2^b \text{ and } E_x|_{I_2^t} = h_x|_{I_2^t}: I_2^t \rightarrow I_1^b,$$

and denote by  $\mathcal{T}_{\mathcal{A},x} := \mathcal{T}_{\mathcal{A},E_x}$  the suspension of  $E_x$  defined in Proposition 3.2, illustrated in Figure 3.1. Note that  $E_1 = E_\infty$  is simply the restriction of  $h_\infty$  to  $I$ , and thus that  $\mathcal{T}_{\mathcal{A},1} = \mathcal{T}_{\mathcal{A},\infty}$ . The following result summarizes the construction and follows from Proposition 3.2.

**Proposition 3.12.** *For any  $\mathcal{A} \in \mathbb{R}_+^*$  and  $x \in [1; \infty]$ ,  $\mathcal{T}_{\mathcal{A},x}$  is homeomorphic to  $\mathbf{T}^2$ , and the  $\mathbf{dS}^2$ -structure of the interior of  $\mathcal{R}_{\mathcal{A}}$  extends to a unique singular  $\mathbf{dS}^2$ -structure on  $\mathcal{T}_{\mathcal{A},x}$ . The latter has area  $\mathcal{A}$ , and its unique singular point  $[1, 0] = [\infty, 0]$  has a closed  $\alpha$ -leaf and angle  $\theta$ .*

**Corollary 3.13.** *Let  $S$  be a closed singular  $\mathbf{dS}^2$ -surface with a single singularity  $x$ , such that one of the lightlike leaves of  $x$  is closed and for the other lightlike foliation  $\mathcal{F}$ : either  $\mathcal{F}$  is minimal or  $\mathcal{F}(x)$  is closed. Then  $S$  is isometric to a torus  $\mathcal{T}_{\mathcal{A},x}$  given by Proposition 3.12.*

*Proof.* This is a direct consequence of Lemma 3.10 and Proposition 3.8.  $\square$

*Remark 3.14.* Of course, one can realize the symmetric construction to obtain a quotient of  $\mathcal{R}_{\mathcal{A}}$  with this time the  $\beta$ -leaf of  $[\infty, 0]$  being closed. This is done by gluing the  $\alpha$ -edges of  $\mathcal{R}_{\mathcal{A}}$  by the restriction of  $h_{\mathcal{A}}$  defined in (3.1), and its  $\beta$ -edges by a HIET with two components of  $J = \{1\} \times [0; y_{\mathcal{A}}]$  with top and bottom partitions

$$J_1^t = [0; y'[, J_2^t = [y'; y_{\mathcal{A}}[, J_1^b = [0; y[, J_2^b = [y; y_{\mathcal{A}}].$$

These  $\mathbf{dS}^2$ -tori of area  $\mathcal{A}$ , with one singularity at  $[\infty, 0]$  whose  $\beta$ -leaf is closed, will be denoted by  $\mathcal{T}_{\mathcal{A},y,*}$ .

**3.2.2. Investigation of the holonomy.** Let  $a$  denote the homotopy class of the positively oriented closed  $\alpha$ -lightlike leaf  $[1; \infty] \times \{0\}$  in  $\mathcal{T}_{\mathcal{A},x}^* := \mathcal{T}_{\mathcal{A},x} \setminus \{[\infty, 0]\}$ . Let  $b_1$  be the  $\beta$ -lightlike positively oriented geodesic segment  $\{1\} \times [0; y_{\mathcal{A}}]$  going from  $[1, 0]$  to  $[x', 0]$ , and  $b_2$  be the  $\alpha$ -lightlike negatively oriented geodesic segment  $[1; x'] \times \{0\}$  going from  $[x', 0]$  to  $[1, 0]$ . We denote by  $b$  the homotopy class of the piecewise geodesic closed loop  $b_1 b_2$  in  $\mathcal{T}_{\mathcal{A},x}^*$  ( $b_1$  followed by  $b_2$ ). Then  $a$  and  $b$  freely generate the rank-two free group  $\pi_1(\mathcal{T}_{\mathcal{A},x}^*) = \langle a, b \rangle$ , and  $K := aba^{-1}b^{-1}$  is the homotopy class of a small positively oriented closed loop around  $[\infty, 0]$  in  $\mathcal{T}_{\mathcal{A},x}^*$ .

With  $\rho: \pi_1(\mathcal{T}_{\mathcal{A},x}^*) \rightarrow \mathrm{PSL}_2(\mathbb{R})$  be the holonomy representation of  $\mathcal{T}_{\mathcal{A},x}^*$ , we have  $\rho(a) = g$ ,  $\rho(b) = h$  and thus

$$\rho(K) = ghg^{-1}h^{-1},$$

which is coherent with Proposition 3.2. A direct computation using the description of  $g$  and  $h$  in Lemma 3.11 shows moreover that

$$\mathrm{tr}(gh) = \frac{-\sqrt{x}(2 - y_{\mathcal{A}})}{\sqrt{(x - y_{\mathcal{A}})}}, \quad \mathrm{tr}(ghg^{-1}h^{-1}) = \frac{y_{\mathcal{A}}^2 - 2y_{\mathcal{A}} + 2}{1 - y_{\mathcal{A}}},$$

and in particular that for any  $\mathcal{A} \in \mathbb{R}_+^*$  (equivalently any  $\theta \in \mathbb{R}$ ):

- (1)  $\mathrm{tr}(\rho(K)) > 2$ ;
- (2)  $\mathrm{tr}(gh) < 0$ , and  $x \in ]1; \infty[ \mapsto \mathrm{tr}(gh_x) + 2$  take any sign, *i.e.*  $gh_x$  can be hyperbolic, elliptic or parabolic depending on the value of  $x$ .

We emphasize that, while the traces of  $g$  and  $h$  are not well-defined, any lifts of  $g$  and  $h$  to  $\mathrm{SL}_2(\mathbb{R})$  give the same  $\mathrm{tr}(ghg^{-1}h^{-1})$  (the signs vanishing in the commutator). This trace is thus a well-defined quantity associated to  $\mathcal{T}_{\mathcal{A},x}$ .

A particularly important class of  $\mathbf{dS}^2$ -surfaces are the *Kleinian* (or uniformizable) ones, of the form  $S = \Gamma \backslash \Omega$  with  $\Omega$  an open subset of  $\mathbf{dS}^2$  and  $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$  a discrete subgroup acting properly discontinuously on  $\Omega$ . In this case, the holonomy morphism  $\rho_S$  of  $S$  has image  $\Gamma$ , and is thus in particular discrete (though non necessarily faithful if  $\Omega$  is not simply connected). It is relatively easy to check that  $g$  and  $h_\infty$  satisfy a ping-pong configuration on  $\mathbf{dS}^2$ , and that  $\mathcal{T}_{\mathcal{A},\infty}^*$  is therefore a Kleinian punctured torus. However, the following claim shows that the behaviour of  $\rho$  and of the family of  $\mathbf{dS}^2$ -structures  $\mathcal{T}_{\mathcal{A},x}^*$  is very diverse.

**Lemma 3.15.** *Let  $x \in ]1; \infty[$  be such that  $-2 < \mathrm{tr}(gh_x)$ .*

- (1) *Then  $gh_x$  is elliptic, and  $\rho$  is not both discrete and faithful.*
- (2) *There exists  $x \in ]1; \infty[$  such that  $\mathrm{tr}(gh) \notin 2\cos(2\pi\mathbb{Q})$ , and then  $\mathcal{T}_{\mathcal{A},x}^*$  is not Kleinian.*

*Proof.* 1. Indeed if it was, then the subgroup  $\langle gh \rangle$  generated by  $gh$  would be both contained in the compact one-parameter elliptic subgroup containing  $gh$  and in the discrete subgroup  $\rho(\pi_1(\mathcal{T}_{\mathcal{A},x}^*))$ , and would thus be finite. In particular  $gh$  would have finite order, contradicting the fact that  $\rho$  is injective.

2. If  $\mathcal{T}_{\mathcal{A},x}^*$  is Kleinian, then  $\rho(\pi_1(\mathcal{T}_{\mathcal{A},x}^*))$  hence  $\langle gh \rangle$  is discrete. Since  $gh_x$  is elliptic, this forces it to have finite order, therefore  $\mathrm{tr}(gh_x) = 2\cos(\nu)$  for some angle  $\nu$  such that  $k\nu = 2n\pi$ , *i.e.*  $\mathrm{tr}(gh) \in 2\cos(2\pi\mathbb{Q})$ . By continuity of  $x \mapsto \mathrm{tr}(gh_x)$ , there exists  $x$  such that  $\mathrm{tr}(gh) \notin 2\cos(2\pi\mathbb{Q})$ , which concludes the proof of the claim.  $\square$

Since  $\pi_1(\mathcal{T}_{\mathcal{A},x}^*)$  is free,  $\rho: \pi_1(\mathcal{T}_{\mathcal{A},x}^*) \rightarrow \mathrm{PSL}_2(\mathbb{R})$  lifts to a representation into  $\mathrm{SL}_2(\mathbb{R})$ , and singular  $\mathbf{dS}^2$ -tori give thus a new geometric interpretation to the representations  $\rho$  of a rank-two free group  $\langle a, b \rangle$  into  $\mathrm{SL}_2(\mathbb{R})$ , for which  $\rho(a)$  and  $\rho(b)$  are hyperbolic and  $\mathrm{tr}(\rho(aba^{-1}b^{-1})) > 2$ . We refer to the seminal work [Gol03] where such representations were thoroughly studied.

**3.3. A two-parameter family of  $\mathbf{dS}^2$ -tori with one singularity.** Our goal being to construct singular  $\mathbf{dS}^2$ -tori with one singularity both of whose lightlike foliations are minimal, we should first make sure that both leaves of the singularity are non-closed. To this end we fix  $0 < y \leq y_{\mathcal{A}}$  and  $1 < x \leq \infty$ , and wish to apply Proposition 3.4 to the “L-shaped polygon”

$$\mathcal{L}_{\mathcal{A},x,y} := \mathcal{R}_{(1,\infty,0,y_+)} \setminus ]x; \infty] \times ]y; y_+] \subset \mathbf{dS}^2$$

of area  $\mathcal{A}$  illustrated in Figure 3.2. The point

$$y_+ = y_{+(x,y)} := \frac{-x + e^{\mathcal{A}}(x-y)}{-1 + e^{\mathcal{A}}(x-y)} \in [y_{\mathcal{A}}; 1[$$

is fixed by  $(x, y)$ , and is the unique one so that  $\mathcal{A}_{\mu}(\mathcal{L}_{\mathcal{A},x,y}) = \mathcal{A}$ . We emphasize that, conversely to  $\mathcal{R}_{\mathcal{A}}$ , the orbit space of L-shaped polygons of area  $\mathcal{A}$  under the action of  $\mathrm{PSL}_2(\mathbb{R})$  is non-trivial, and parametrized by  $(x, y)$ .

3.3.1. *A pair of HIETs.* As we previously did for the rectangle  $\mathcal{R}_{\mathcal{A}}$ , we want to glue the edges of  $\mathcal{L}_{\mathcal{A},x,y}$  through HIETs of the intervals  $I = [1; \infty[$  and  $J = [0; y_+[$ , exchanging the two components of the top and bottom partitions defined by

$$\begin{cases} I_1^t = [1; x'[ , I_2^t = [x'; \infty[ , I_1^b = [1; x[ , I_2^b = [x; \infty[ , \\ J_1^t = [0; y'[ , J_2^t = [y'; y_+[ , J_1^b = [0; y[ , J_2^b = [y; y_+[ , \end{cases}$$

for  $x' \in [1; \infty[$  and  $y' \in [0; y_+[$ . We denote by  $h_1 = h_{1(x,x',y)}$  and  $h_2 = h_{2(x,x',y)}$  the unique elements of  $\mathrm{PSL}_2(\mathbb{R})$  realizing the gluing of the  $\alpha$ -edges of  $\mathcal{L}_{\mathcal{A},x,y}$  according to these partitions, characterized by

$$h_1(I_1^t \times \{0\}) = I_2^b \times \{y\} \text{ and } h_2(I_2^t \times \{0\}) = I_1^b \times \{y_+\}$$

or equivalently by

$$(3.7) \quad h_1(1, x', 0) = (x, \infty, y) \text{ and } h_2(x', \infty, 0) = (1, x, y_+).$$

We denote in the same way by  $(g_1, g_2)$  the elements of  $\mathrm{PSL}_2(\mathbb{R})$  realizing the gluing of  $\beta$ -edges, indicated in Figure 3.2.

We can then form the quotient of  $\mathcal{L}_{\mathcal{A},x,y}$  by these gluings as described in Proposition 3.4, and compute the holonomy around the vertices of  $\mathcal{L}_{\mathcal{A},x,y}$ . Formula (3.3) indicate us that  $B = [x', 0]$  and  $C = [1, y']$  are regular points in the quotient if, and only if

$$g_1 = h_2 h_1 h_2^{-1} \text{ and } g_2 = h_1 h_2^{-1}.$$

These two relations impose two equations on  $(x, y, x', y')$ , given by the following Lemma which follows from computations similar to the ones detailed in Lemma 3.11.

**Lemma 3.16.** (1)  $h_1 h_2^{-1}$  and  $h_2$  are hyperbolic.

(2)  $h_2 h_1 h_2^{-1}(0) = y$  if, and only if  $x' = \frac{x}{e^{\mathcal{A}}(y-1)+x}$  ( $= 1$  if  $x = \infty$ ).

(3)  $\frac{x}{e^{\mathcal{A}}(y-1)+x} > 1$  if, and only if  $y > 1 - e^{-\mathcal{A}}x$ .

(4) If  $x' = \frac{x}{e^{\mathcal{A}}(y-1)+x}$  and  $y > 1 - e^{-\mathcal{A}}x$ , then  $h_2 h_1^{-1}(0) = \frac{x+e^{\mathcal{A}}x(y-1)}{1+e^{\mathcal{A}}x(y-1)+y(x-1)} \in [0; y_+[$ .

We thus fix henceforth  $x \in ]1; \infty[$  and  $y \in ]1 - e^{-\mathcal{A}}x; y_{\mathcal{A}}[$ , and define

$$(3.8) \quad \begin{cases} x' = x'_{(x,y)} := \frac{x}{e^{\mathcal{A}}(y-1)+x}, \\ h_1 = h_{1(x,y)} := h_1(x, x'_{(x,y)}, y), h_2 = h_{2(x,y)} := h_2(x, x'_{(x,y)}, y), \\ y' := h_2 h_1^{-1}(0) \\ g_1 := h_2 h_1 h_2^{-1}, g_2 := h_1 h_2^{-1}. \end{cases}$$

Then according to Lemma 3.16.(3) and (4):  $x' \in [1; \infty[$  and  $y' \in [0; y_+[$ . Moreover according to Lemma 3.16.(2) and the definition of  $h_1$  and  $h_2$  in (3.7) we have

$$g_1(1, 0, y') = (x, y, y_+) \text{ and } g_2(1, y', y_+) = (\infty, 0, y).$$

This allows us to define a pair  $E = E_{x,y}: I \rightarrow I$  and  $F = F_{x,y}: J \rightarrow J$  of HIET with two components by

$$(3.9) \quad \begin{cases} E_{x,y}|_{I_1^t} = h_{1(x,y)}|_{I_1^t}: I_1^t \rightarrow I_2^b \text{ and } E_{x,y}|_{I_2^t} = h_{2(x,y)}|_{I_2^t}: I_2^t \rightarrow I_1^b, \\ F_{x,y}|_{J_1^t} = g_{1(x,y)}|_{J_1^t}: J_1^t \rightarrow J_2^b \text{ and } F_{x,y}|_{J_2^t} = g_{2(x,y)}|_{J_2^t}: J_2^t \rightarrow J_1^b. \end{cases}$$



**3.4. Rotation numbers.** Our goal is to prescribe the dynamics of the lightlike foliations of the  $\mathbf{dS}^2$ -tori that we constructed. They will be essentially characterized by those of the HIET's that we suspended to construct our examples, and in the end by the dynamics of circle homeomorphisms induced by these HIET's. Therefore, we introduce now the basic invariant of circle homeomorphisms, namely the rotation number.

**3.4.1. From HIET to circle homeomorphisms and rotation numbers.** We see the circle as the additive group  $\mathbf{S}^1 = \mathbb{R}/\mathbb{Z}$ , denote by  $\pi: \mathbb{R} \rightarrow \mathbf{S}^1$  the canonical projection when we need it, and also use the notation  $[x] := \pi(x) \in \mathbf{S}^1$  for  $x \in \mathbb{R}$ . We endow  $\mathbf{S}^1$  with the orientation induced by the one of  $\mathbb{R}$ , for which a continuous map  $f: I \rightarrow \mathbf{S}^1$ ,  $I$  being an interval of  $\mathbb{R}$ , is *non-decreasing* if for any lift  $F: I \rightarrow \mathbb{R}$  of  $f$ ,  $F$  is non-decreasing. In the same way a continuous map  $f: \mathbf{S}^1 \rightarrow \mathbf{S}^1$  is non-decreasing if any lift  $F: \mathbb{R} \rightarrow \mathbb{R}$  of  $f$  is so. We adopt the natural analogous definitions for non-increasing, and strictly increasing or decreasing maps.

Any HIET  $E$  of an interval  $I = [a; b[ \subset \mathbb{R}\mathbf{P}^1$  with one or two components naturally induces a bijection  $\mathbf{E}$  of the quotient  $\mathbf{S}_I^1 := [a; b]/\{a \sim b\}$ , defined by

$$\forall p \in I, \mathbf{E}([p]) = [E(p)].$$

$\mathbf{S}_I^1$  is homeomorphic to the circle  $\mathbf{S}^1$  and bears a natural orientation induced from the one of  $I$ , and it is moreover easy to check that  $\mathbf{E}$  is an *orientation-preserving homeomorphism* of  $\mathbf{S}_I^1$  (since the HIET  $E$  exchanges at most two components).

If  $f \in \text{Homeo}^+(\mathbf{S}^1)$  is an orientation-preserving homeomorphism of the circle, then any lift  $F: \mathbb{R} \rightarrow \mathbb{R}$  of  $f$  is a strictly increasing homeomorphism of  $\mathbb{R}$  commuting with every integer translation  $T_n: x \in \mathbb{R} \mapsto x + n \in \mathbb{R}$  ( $n \in \mathbb{Z}$ ). Following [Her79] and the literature, we denote by  $\mathbf{D}(\mathbf{S}^1)$  the subgroup of all such homeomorphisms of  $\mathbb{R}$ , *i.e.* of all the lifts of elements of  $\text{Homeo}^+(\mathbf{S}^1)$  to  $\mathbb{R}$  ( $\mathbf{D}(\mathbf{S}^1)$  is precisely the centralizer in  $\text{Homeo}^+(\mathbb{R})$  of the translation  $T_1$ ). Denoting by  $\pi(F) \in \text{Homeo}^+(\mathbf{S}^1)$  the map  $\pi(F)([x]) = [F(x)]$ , we have a short exact sequence

$$0 \rightarrow \{T_n \mid n \in \mathbb{Z}\} \rightarrow \mathbf{D}(\mathbf{S}^1) \xrightarrow{\pi} \text{Homeo}^+(\mathbf{S}^1) \rightarrow 0.$$

The *translation number* of  $F \in \mathbf{D}(\mathbf{S}^1)$  is the asymptotic average amount by which  $F$  translates the points of  $\mathbb{R}$ . We refer to [Her79, II.2 p.20] and [dFG22, §2.1] for a proof of the following classical results.

**Proposition-Definition 3.18.** *Let  $f, g \in \text{Homeo}^+(\mathbf{S}^1)$  and  $F \in \mathbf{D}(\mathbf{S}^1)$  be any lift of  $f$ .*

(1) *The limit*

$$(3.12) \quad \tau(F) = \lim_{n \rightarrow \pm\infty} \frac{F^n(x) - x}{n}$$

*exists for any  $x \in \mathbb{R}$ , is independent of  $x$ , and uniform on  $\mathbb{R}$ . It is called the translation number of  $F$ .*

(2) *If  $G = F + d$  is another lift of  $f$  ( $d \in \mathbb{Z}$ ), then  $\tau(G) = \tau(F) + d$ , and*

$$\rho(f) = [\tau(F)] \in \mathbf{S}^1$$

*is called the rotation number of  $f$ .*

(3) *The maps  $F \in \mathbf{D}(\mathbf{S}^1) \rightarrow \tau(F) \in \mathbb{R}$  and  $f \in \text{Homeo}^+(\mathbf{S}^1) \rightarrow \rho(f) \in \mathbf{S}^1$  are continuous for the compact-open topology.*

(4) *Moreover  $\rho$  is a conjugacy invariant:  $\rho(g \circ f \circ g^{-1}) = \rho(f)$ .*

The following simple observation will be useful to us all along this text.

**Lemma 3.19.** *Let  $C$  be an oriented topological circle and  $f \in \text{Homeo}^+(C)$ . Then for any orientation-preserving homeomorphisms  $\varphi_1, \varphi_2: C \rightarrow \mathbf{S}^1: \rho(\varphi_1 \circ f \circ \varphi_1^{-1}) = \rho(\varphi_2 \circ f \circ \varphi_2^{-1})$ . This common number will still be called the rotation number of  $f$  and be denoted by  $\rho(f) \in \mathbf{S}^1$ .*

*Proof.* Since  $\varphi_2 \circ f \circ \varphi_2^{-1} = \varphi \circ (\varphi_1 \circ f \circ \varphi_1^{-1}) \circ \varphi^{-1}$  with  $\varphi = \varphi_2 \circ \varphi_1^{-1} \in \text{Homeo}^+(\mathbf{S}^1)$ , the claim follows from Proposition 3.18.(4).  $\square$

We will say that  $f \in \text{Homeo}^+(C)$  is *conjugated* to  $f_0 \in \text{Homeo}^+(\mathbf{S}^1)$  at the base-point  $o \in C$ , if  $f_0 = \varphi \circ f \circ \varphi^{-1}$  with  $\varphi: C \rightarrow \mathbf{S}^1$  an orientation-preserving homeomorphism such that  $\varphi(o) = [0]$ .



**Lemma 3.20.** *In the examples  $\mathcal{T}_{\mathcal{A},x}$  introduced in Proposition 3.12,  $\mathbf{E}_x^{-1} \in \text{Homeo}^+(\mathbf{S}_{[1;\infty]}^1)$  is the holonomy map of the  $\beta$ -foliation on the closed  $\alpha$ -leaf  $[1;\infty] \times \{y_{\mathcal{A}}\}$ . Moreover if  $\mathbf{E}$  has irrational rotation number  $\rho \in \mathbf{S}^1$  at the base point  $[1]$ , then it is  $\mathcal{C}^0$ -conjugated at the base point  $[1]$  to the rotation  $R_\rho: x \in \mathbf{S}^1 \mapsto x + \rho \in \mathbf{S}^1$ .*

*Proof.* The first claim follows directly from the construction of  $\mathcal{T}_{\mathcal{A},x}$ . Since  $\mathcal{F}_\beta$  is the suspension of  $\mathbf{E}$ , the second claim is a direct consequence of Lemma 2.30.(4).  $\square$

**3.4.2. Rotation numbers as cyclic ordering of the orbits.** For  $\theta \in \mathbb{R}$ , we say that a sequence  $(p_n)_{n \in \mathbb{Z}}$  in  $\mathbf{S}^1$  is of *cyclic order*  $[\theta] \in \mathbf{S}^1$  if it is cyclically ordered as an orbit of  $R_{[\theta]}$ , namely if for any  $(n_1, n_2, n_3) \in \mathbb{Z}^3$ : the three points  $(p_{n_1}, p_{n_2}, p_{n_3}) \in (\mathbf{S}^1)^3$  are two-by-two distinct and positively cyclically ordered if, and only if  $(R_{[\theta]}^{n_1}([0]), R_{[\theta]}^{n_2}([0]), R_{[\theta]}^{n_3}([0])) = ([n_1\theta], [n_2\theta], [n_3\theta])$  are so. We will henceforth assume every rational  $\frac{p}{q} \in \mathbb{Q}$  to be *written in reduced form*, i.e. such that:

- either  $\frac{p}{q} = 0$  and then  $(p, q) = (0, 1)$ ;
- or  $p \in \mathbb{Z}^*$ ,  $q \in \mathbb{N}^*$  and  $p, q$  are coprimes.

We refer to [dFG22, §1.1] and [dMvS93, I.1] for a proof of the following classical results.

**Proposition 3.21.** *Let  $T \in \text{Homeo}^+(\mathbf{S}^1)$ .*

- (1)  $\rho(T) = [\frac{p}{q}] \in [\mathbb{Q}]$  if, and only if there exists a periodic orbit of  $T$  of minimal period  $q$  and cyclic order  $[\frac{p}{q}]$ . Moreover if this is the case, then any periodic orbit of  $T$  is of this form. In particular,  $\rho(T) = [0]$  if, and only if  $T$  has a fixed point.
- (2)  $\rho(T) = \theta \in [\mathbb{R} \setminus \mathbb{Q}]$  if, and only if any orbit of  $T$  is of cyclic order  $\theta$ .

**3.5. Realization of rotation numbers.** We now come back to the HIETs that we suspended in paragraphs 3.2 and 3.3, and show existence results for their rotation numbers.

**3.5.1. Rotation number for a single HIET.** We now use the notations of the paragraph 3.2.1. For any  $x \in [1; \infty]$ , we consider the orientation-preserving homeomorphism  $\mathbf{E}_x$  of  $\mathbf{S}_I^1 := [1; \infty]/\{1 \sim \infty\}$  induced by the HIET  $E_x$  of  $I = [1; \infty[$  defined in (3.6).

Note that when  $x$  converges to 1,  $x'_x$  converges to  $\infty$  and  $gh_x$  to  $h_\infty = gh_1$ , since

$$gh_x(1, x'_x, 0) = (x, \infty, y_{\mathcal{A}}).$$

Hence  $\mathbf{E}_x$  converges to  $\mathbf{E}_1 = \mathbf{E}_\infty$  for the compact-open topology of  $\text{Homeo}^+(\mathbf{S}_I^1)$  when  $x \rightarrow 1$ , and the map

$$\mathbf{E}: [x] \in \mathbf{S}_I^1 \mapsto \mathbf{E}_x \in \text{Homeo}^+(\mathbf{S}_I^1)$$

is thus easily seen to be continuous.

Let  $\{g^t\}_{t \in \mathbb{R}} \subset \text{PSL}_2(\mathbb{R})$  denote the one-parameter hyperbolic subgroup containing  $g$  (defined by (3.1)), parametrized so that  $g = g^1$ .

**Lemma 3.22.** *Let  $x_1 \leq x_2 \in [1; \infty]$ .*

- (1)  $h_{x_1}^{-1}gh_{x_1}g^{-1} = h_{x_2}^{-1}gh_{x_2}g^{-1}$ .
- (2) There exists a unique  $\tau \in [0; 1]$  such that  $x_2 = g^\tau(x_1)$ , and  $h_{x_2} = g^\tau h_{x_1}$ .
- (3) Moreover  $E_{x_2} = S_\tau \circ E_{x_1}$ , with  $S_\tau$  the HIET defined by

$$\begin{cases} \forall p \in [1; E_{x_1}(x'_2)[, S_\tau(p) = g^\tau(p) \in [g^\tau(1); \infty[, \\ \forall p \in [E_{x_1}(x'_2); \infty[, S_\tau(p) = g^{\tau-1}(p) \in [1; g^\tau(1)[. \end{cases}$$

*Proof.* (1) According to Proposition 3.2, the holonomy around  $[\infty, 0]$  in  $\mathcal{T}_{\mathcal{A},x_i}$  is equal to  $h_{x_i}^{-1}gh_{x_i}g^{-1}$  (for a developing map compatible at  $[\infty, 0]$ , see Lemma 2.14), hence  $h_{x_1}^{-1}gh_{x_1}g^{-1} = a^\theta = h_{x_2}^{-1}gh_{x_2}g^{-1}$ . Note that this extends to the case  $x_1 = 1$  since by definition of  $h_1$  we have  $h_1^{-1}gh_1g^{-1} = (h_\infty^{-1}g)g(g^{-1}h_\infty)g^{-1} = h_\infty^{-1}gh_\infty g^{-1}$ .

(2) According to (1),  $hgh^{-1} = g$  with  $h = h_{x_2}h_{x_1}^{-1}$ . Hence  $h$  is in the centralizer of  $g$  in  $\text{PSL}_2(\mathbb{R})$ , which is equal to  $\{g^t\}_{t \in \mathbb{R}}$ . Now if  $h_2 = g^\tau h_1$  we obtain directly from (3.1) and (3.4) that  $x_2 = g^\tau(x_1)$ .

(3) Indeed for any  $p \in [1; x_1[, E_{x_1}^{-1}(p) = H_1^{-1}(p) \in [x'_1; \infty[$ , and  $x'_2 < x'_1$  so  $E_{x_2} \circ E_{x_1}^{-1}(p) = H_2 H_1^{-1}(p) = g^\tau(p) \in [g^\tau(1); x_2[$ . Note that  $gH_1(x'_2) \in ]x_1; \infty[$ , so for  $p \in [x_1; gH_1(x'_2)[$ ,  $E_{x_1}^{-1}(p) = H_1^{-1}g^{-1}(p) \in [1; x'_2[$  and  $E_{x_2} \circ E_{x_1}^{-1}(p) = gH_2 H_1^{-1}g^{-1}(p) = g^\tau(p) \in [x_2; \infty[$ . Lastly for

$p \in [gH_1(x'_2); \infty[$ ,  $E_{x_1}^{-1}(p) = H_1^{-1}g^{-1}(p) \in [x'_2; x'_1[$ , and thus  $E_{x_2} \circ E_{x_1}^{-1}(p) = g^\tau H_1 H_1^{-1} g^{-1}(p) = g^{\tau-1}(p) \in [x_2; \infty[$ .  $\square$

**Proposition 3.23.** *The map  $[x] \in \mathbf{S}_I^1 \mapsto \rho(\mathbf{E}_x) \in \mathbf{S}^1$  is continuous, non-decreasing, and has degree one (in particular, it is surjective). Moreover it is strictly increasing at any  $x$  for which  $\rho(\mathbf{E}_x) \in [\mathbb{R} \setminus \mathbb{Q}]$ . In particular for any  $u \in [\mathbb{R} \setminus \mathbb{Q}]$ , there exists a unique  $[x] \in \mathbf{S}_I^1$  such that  $\rho(\mathbf{E}_x) = u$ . Lastly, for any  $r \in [\mathbb{Q}]$  there exists  $x \in [1; \infty]$  such that the orbit of  $[1, 0]$  under  $\mathbf{E}_x$  is periodic and of cyclic order  $r$ .*

*Proof.* The continuity of  $x \in [1; \infty] \mapsto \rho(\mathbf{E}_x) \in \mathbf{S}^1$  follows from the continuity of  $\mathbf{E}$  and of the rotation number itself (see for instance [Her79, Proposition 2.7]), for the compact-open topology of  $\text{Homeo}^+(\mathbf{S}_I^1)$ . Note that both  $\mathbf{E}_1$  and  $\mathbf{E}_\infty$  have  $[1] \in \mathbf{S}_I^1$  as a fixed point, and thus that  $\rho(\mathbf{E}_1) = \rho(\mathbf{E}_\infty) = [0] \in \mathbf{S}^1$ . On the other hand it is easily checked that for any  $x \in ]1; \infty[$ ,  $\mathbf{E}_x$  does not have any fixed point and thus that  $\rho(\mathbf{E}_x) \neq [0]$ . In particular,  $x \mapsto \rho(\mathbf{E}_x)$  is not constant.

According to Lemma 3.22.(4), we have moreover  $\mathbf{E}_{g^\tau(1)} = S_\tau \circ \mathbf{E}_1$  with  $\tau \in [0; 1] \mapsto S_\tau \in \text{Homeo}^+(\mathbf{S}_I^1)$  a continuous map such that  $\tau \in [0; 1] \mapsto S_\tau(p) \in \mathbf{S}_I^1$  is strictly increasing for any  $p \in \mathbf{S}_I^1$ . According to Lemma B.1,  $x \in [1; \infty] \mapsto \rho(\mathbf{E}_x) \in \mathbf{S}^1$  is thus continuous and non-decreasing. But since it is also not constant and attains the same value  $[0]$  at 1 and  $\infty$ , it is surjective according to the Intermediate value theorem. The value  $[0]$  being attained only at the point  $[1] = [\infty]$  of  $\mathbf{S}_I^1$ , the map  $[x] \in \mathbf{S}_I^1 \mapsto \rho(\mathbf{E}_x) \in \mathbf{S}^1$  moreover has degree one. It is also strictly increasing at any  $x$  for which  $\rho(\mathbf{E}_x) \in [\mathbb{R} \setminus \mathbb{Q}]$  according to Lemma B.1, which forbids any element of  $[\mathbb{R} \setminus \mathbb{Q}]$  to have more than one pre-image in  $\mathbf{S}_I^1$  since the map has degree one, and concludes the proof of the three first claims.

Lastly, the existence of periodic orbits of  $[1, 0]$  of any rational cyclic order under the maps  $\mathbf{E}_x$  follows from Lemma B.1.(4), which concludes the proof.  $\square$

**3.5.2. Rotation numbers for a pair of HIET.** We now want to realize rotation numbers for the pair  $(E, F)$  of HIETs introduced in paragraph 3.3.1. For any  $(x, y) \in \mathcal{D}$  (defined in (3.10)), we consider the orientation-preserving homeomorphisms  $\mathbf{E}_{x,y}$  of  $\mathbf{S}_I^1 := [1; \infty]/\{1 \sim \infty\}$  and  $\mathbf{F}_{x,y}$  of  $\mathbf{S}_J^1 := [0; y_+]/\{0 \sim y_+\}$  induced by the HIETs  $E_{x,y}$  and  $F_{x,y}$  defined in (3.9). Note that with the definitions introduced in paragraph 3.3.3 for  $(x, y) \in \text{Cl}(\mathcal{D})$ ,  $\mathbf{F}_{x,y}$  is a well-defined orientation-preserving homeomorphism of  $\mathbf{S}_J^1$ . On the other hand for  $x \in ]1; e^A]$  and  $y = 1 - e^{-A}x$ ,  $\mathbf{E}_{x,y}^{-1}$  is a well-defined orientation-preserving endomorphism of  $\mathbf{S}_I^1$ , i.e. by definition a continuous, degree-one and orientation-preserving self-map of  $\mathbf{S}_I^1$ . Equivalently,  $f$  is an orientation-preserving endomorphism of  $\mathbf{S}^1$  if it admits a lift  $F$  to  $\mathbb{R}$  which is a continuous, non-decreasing self-map of  $\mathbb{R}$  commuting with integer translations. According to [PJM82, Appendix Lemma 3] and [NPT83, Chapter III Proposition 3.3], the Proposition-Definition 3.18 extend to endomorphisms of  $\mathbf{S}^1$ , and the rotation number  $\rho(\mathbf{E}_{x,y}^{-1})$  is thus well-defined. Lastly, the maps

$$\mathbf{E}^{-1}: (x, y) \in \text{Cl}(\mathcal{D}) \mapsto \mathbf{E}_{x,y}^{-1} \in \text{End}^+(\mathbf{S}_I^1) \text{ and } \mathbf{F}: (x, y) \in \text{Cl}(\mathcal{D}) \mapsto \mathbf{F}_{x,y} \in \text{Homeo}^+(\mathbf{S}_J^1)$$

are continuous. The author want to thank Florestan Martin-Baillon, who helped him to obtain a more elegant proof for this result than in a first version.

**Proposition 3.24.** *The map  $(x, y) \in \mathcal{D} \mapsto (\rho(\mathbf{E}_{x,y}), \rho(\mathbf{F}_{x,y})) \in (\mathbf{S}^1)^2$  is continuous and surjective.*

*Proof.* Since the maps  $\mathbf{E}^{-1}$  and  $\mathbf{F}$  are continuous, and such is the rotation number as well according to [NPT83, Chapter III Proposition 3.3], the map

$$R: (x, y) \in \text{Cl}(\mathcal{D}) \mapsto (\rho(\mathbf{E}_{x,y}^{-1}), \rho(\mathbf{F}_{x,y})) \in (\mathbf{S}^1)^2$$

is continuous. We recall that  $\rho(T^{-1}) = \rho(T)^{-1}$  for any  $T \in \text{Homeo}^+(\mathbf{S}^1)$  (see for instance [dFG22, §2.1]). We begin by investigating what happens for the rotation numbers on the boundary of  $\mathcal{D}$ , as we did in paragraph 3.3.3.

**If  $x = \infty$  and  $y \in [0; y_A]$ :** Then  $\rho(\mathbf{E}_{\infty,y}^{-1}) = [0]$  since  $[1]$  is a fixed point of  $\mathbf{E}_{\infty,y}^{-1}$ , and

$$y \in \mathbf{S}_J^1 \mapsto \rho(\mathbf{F}_{\infty,y}) \in \mathbf{S}^1$$

is a continuous degree-one map as we proved in Lemma 3.23.

**If  $x \in [1; \infty]$  and  $y = y_A$ :** Then  $\rho(F_{\infty, y}) = [0]$  since  $[0]$  is a fixed point of  $F_{x, y_A}$ , and

$$x \in \mathbf{S}_I^1 \mapsto \rho(E_{x, y_A}^{-1}) \in \mathbf{S}^1$$

is a continuous degree-one map as we proved in Lemma 3.23.

**If  $x \in [e^A; \infty[$  and  $y = 0$ :** Then  $\rho(F_{\infty, y}) = [0]$  since  $[0]$  is a fixed point of  $F_{x, y_A}$ . On the other hand  $x \in [e^A; \infty[ \mapsto x'_{x,0} = \frac{x}{x-e^A} \in [1; \infty]$  is surjective (see (3.8)),  $\rho(E_{e^A, 0}^{-1}) = \rho(E_{\infty, 0}^{-1}) = [0]$  since  $[\infty]$  is a fixed point of both, and  $\rho(E_{x,0}^{-1}) \neq [0]$  for any  $x \in ]e^A; \infty[$  since  $E_{x,0}^{-1}$  has no fixed points. Therefore, the same argument than in Lemma 3.23 shows that

$$[x] \in [e^A; \infty[ / \{e^A \sim \infty\} \mapsto \rho(E_{x,0}^{-1}) \in \mathbf{S}^1$$

is a continuous monotonous map with value  $[0]$  only at  $x = e^A$  and  $x = \infty$ , hence a degree-one map.

**If  $x \in ]1; e^A]$  and  $y = 1 - e^{-A}x$ :** Then  $x' = \infty$ , hence  $[1]$  is a fixed point of  $E_{\infty, y}^{-1}$  and thus  $\rho(E_{x, 1-e^{-A}x}^{-1}) = [0]$ . On the other hand  $x \in [1; e^A] \mapsto y(x) := 1 - e^{-A}x \in [0; y_A]$  is surjective, with  $y = y_+ = y_A$  for  $x = 1$ , and  $(y = 0, y' = y_+)$  for  $x = e^A$ . The same argument than in Lemma 3.23 shows thus that

$$[x] \in [1; e^A] / \{1 \sim e^A\} \mapsto \rho(F_{\infty, y}) \in \mathbf{S}^1$$

is a continuous monotonous map with value  $[0]$  only at  $x = 1$  and  $x = e^A$ , hence a degree-one map.

We conclude from this description that there exists continuous monotonous and surjective maps  $f_h: [e^A; \infty] \times \{0\} \rightarrow [1; \infty] \times \{y_A\}$  and  $f_v: \{(x, 1 - e^{-A}x) \mid x \in [1; e^A]\} \rightarrow \{\infty\} \times [0; y_A]$  between the horizontal and vertical edges of  $\partial\mathcal{D}$ , such that  $R \circ f_h = R$  on  $[e^A; \infty] \times \{0\}$  and  $R \circ f_v = R$  on  $\{(x, 1 - e^{-A}x) \mid x \in [1; e^A]\}$ . In other words,  $R$  induces a continuous map

$$\bar{R}: \mathcal{T} \rightarrow (\mathbf{S}^1)^2$$

such that  $\bar{R} \circ \pi = R$ , with  $\pi: \text{Cl}(\mathcal{D}) \rightarrow \mathcal{T}$  the quotient of  $\text{Cl}(\mathcal{D})$  by the identifications  $p \sim f_h(p)$  and  $p \sim f_v(p)$  of its edges under  $f_h$  and  $f_v$ , which is homeomorphic to a torus.

Assume now by contradiction that the restriction of  $R$  to  $\mathcal{D}$  misses a point in the torus  $(\mathbf{S}^1)^2$ . Since our previous description of  $R|_{\partial\mathcal{D}}$  shows that  $\mathbf{S}^1 \times [0] \cup [0] \times \mathbf{S}^1 \subset R((\{\infty\} \times [0; y_A]) \cup ([1; \infty] \times \{y_A\})) \subset R(\mathcal{D})$ , we have thus  $\bar{R}(\mathcal{T}) \subset (\mathbf{S}^1)^2 \setminus \{p\}$  for some  $p \in (\mathbf{S}^1)^2 \setminus (\mathbf{S}^1 \times [0] \cup [0] \times \mathbf{S}^1)$ . Now  $(\mathbf{S}^1)^2 \setminus \{p\}$  retracts to a bouquet of two circles, hence its fundamental group is a free group  $F_2$  in two generators represented by the loops  $\mathbf{S}^1 \times [0]$  and  $[0] \times \mathbf{S}^1$ , and  $\bar{R}$  induces moreover in homotopy a morphism  $\bar{R}_*$  from  $\pi_1(\mathcal{T}) \simeq \mathbb{Z}^2$  to  $\pi_1((\mathbf{S}^1)^2 \setminus \{p\}) = F_2$ . The image of this morphism is then an abelian subgroup of  $F_2$ , and since every subgroup of  $F_2$  is free according to the Nielsen-Schreier theorem [MKS04, Corollary 2.9, p. 95], it is actually a free abelian group and is thus contained in a copy of  $\mathbb{Z}$  in  $F_2$ . But  $\bar{R}$  sends the horizontal and vertical generators of  $\pi_1(\mathcal{T})$ , given by the projections of the horizontal and vertical edges of  $\partial\mathcal{D}$ , to the respective generators  $\mathbf{S}^1 \times [0]$  and  $[0] \times \mathbf{S}^1$  of the free group  $F_2 = \pi_1((\mathbf{S}^1)^2 \setminus \{p\})$ . Obviously, the latter elements do not belong to a common infinite cyclic subgroup of  $F_2 = \pi_1((\mathbf{S}^1)^2 \setminus \{p\})$ , and this contradiction concludes the proof of the Proposition.  $\square$

**3.6. Projective asymptotic cycles and class A structures.** Our goal is to prove the existence of singular  $d\mathbf{S}^2$ -tori whose lightlike foliations are prescribed in terms of an invariant which is in a sense a global version of the rotation number of the first-return map: the *projective asymptotic cycle*. The notion of *asymptotic cycle* was introduced by Schwartzman in [Sch57]. It associates to any suitable orbit  $O$  of a topological flow on a closed manifold  $M$ , an element of the first homology group of  $M$  which is in a sense the “best approximation of  $O$  by a closed loop in homology”. This notion has a natural projective counterpart for the leaves of an oriented topological one-dimensional foliation  $\mathcal{F}$ , that we now quickly describe, referring to [Sch57, Yan85] for more details.

We consider an auxiliary smooth Riemannian metric  $\mu$  on  $M$ , the induced metric and its induced distance  $d_{\mathcal{F}}$  on the leaves of  $\mathcal{F}$ . For  $x \in M$  and  $T \in \mathbb{R}$  we denote by  $\gamma_{T,x}$  the closed

curve on  $M$  obtained by: first following  $\mathcal{F}(x)$  from  $x$  in the positive direction until the unique  $y \in \mathcal{F}(x)$  such that  $d_{\mathcal{F}}(x, y) = T$ , and then closing the curve by following the minimal geodesic of  $\mu$  from  $y$  to  $x$ . Following [Yan85], we then define the *oriented projective asymptotic cycle* of  $\mathcal{F}$  at  $x$  as the half-line

$$(3.13) \quad A_{\mathcal{F}}^{\pm}(x) := \mathbb{R}^+ \left\{ \lim_{T \rightarrow +\infty} \frac{1}{T} [\gamma_{T,p}] \right\} \in \mathbf{P}^+(H_1(M, \mathbb{R}))$$

in the first homology group of  $M$ , if this limit exists and is non-zero. Note that the orientation of  $A_{\mathcal{F}}^{\pm}(x)$  obviously depends of the orientation of the foliation  $\mathcal{F}$ , and is reversed when the orientation of  $\mathcal{F}$  is. We also denote by  $A_{\mathcal{F}}(x) = \mathbb{R}A_{\mathcal{F}}^+(x)$  the unoriented *projective asymptotic cycle*. This line (if it exists) is by definition constant on leaves, does not depend on the auxiliary Riemannian metric, and is moreover natural with respect to any homeomorphism  $f$ :

$$(3.14) \quad A_{f_*\mathcal{F}}^+(f(x)) = f_*(A_{\mathcal{F}}^+(x)).$$

In particular, any homeomorphism isotopic to the identity acts trivially on projective asymptotic cycles. For these properties of asymptotic cycles, we refer to [Sch57, Theorem p.275] proving the equivalence between the geometric interpretation (3.13) and the equivariant definition.

In the case of foliations on the torus, the following result is a reformulation of [Yan85, Theorem 6.1 and Theorem 6.2]. We identify henceforth  $H_1(\mathbf{T}^2, \mathbb{R})$  with  $\mathbb{R}^2$  through the isomorphism induced by the covering map  $\mathbb{R}^2 \rightarrow \mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ , and we say that a line in  $H_1(\mathbf{T}^2, \mathbb{R})$  is *rational* if it passes through a point of the lattice  $H_1(\mathbf{T}^2, \mathbb{Z}) = \mathbb{Z}^2$ .

**Proposition 3.25** ([Yan85]). *Let  $\mathcal{F}$  be an oriented topological one-dimensional foliation of  $\mathbf{T}^2$ .*

- (1)  $A_{\mathcal{F}}^+(x)$  exists at any  $x \in \mathbf{T}^2$ , it is moreover constant on  $\mathbf{T}^2$  and will be denoted by  $A^+(\mathcal{F})$  (respectively  $A(\mathcal{F}) = \mathbb{R}A^+(\mathcal{F})$  for the unoriented asymptotic cycle).
- (2) If  $\mathcal{F}$  has a closed leaf  $F$ ,  $A^+(\mathcal{F})$  equals its homology class  $[F]$  and is thus rational.
- (3) If  $\mathcal{F}$  is the linear oriented foliation induced by a half-line  $l \subset \mathbb{R}^2$ , then  $A^+(\mathcal{F}) = l$ .

We will later apply the notion of projective asymptotic cycle to the lightlike foliations of  $\mathbf{dS}^2$ -structures which are suspensions of circle homeomorphisms. According to Lemma 2.30, the foliations are in this case topologically equivalent to suspensions of  $\mathcal{C}^2$ -diffeomorphisms with breaks and have thus no exceptional minimal set. It will be useful to have in mind a rough classification of such suspensions, that we summarize in the following statement. Those results are well-known, and are for instance proved in [HH86, §4]. We recall that a foliation (respectively a homeomorphism) is said *minimal* if all its leaves (resp. orbits) are dense.

**Proposition 3.26.** *Let  $\mathcal{F}$  be a topological foliation of  $\mathbf{T}^2$ . Then:*

- (1) either  $\mathcal{F}$  has at least one Reeb component, and in this case it is not minimal;
- (2) or  $\mathcal{F}$  is a suspension.

Assume now that  $\mathcal{F}$  is the suspension of a  $\mathcal{C}^2$  circle diffeomorphism  $T$  with breaks. Then one of the two following exclusive situations arise.

- (1) Either  $T$  has rational rotation number, and then  $\mathcal{F}$  has closed leaves, all of which are freely homotopic, and every non-closed leaf is past- and future-asymptotic to one of these closed leaves.
- (2) Or  $T$  has irrational rotation number  $\rho$ , and then  $\mathcal{F}$  is minimal and topologically equivalent to a linear foliation of slope  $\rho$ .

The following result is classical, and we recall its statement for the convenience of the reader.

**Lemma 3.27.** *Let  $\mathcal{F}_1, \mathcal{F}_2$  be two oriented topological foliations of  $\mathbf{T}^2$  having the same oriented projective asymptotic cycles, and  $\gamma_1, \gamma_2$  be freely homotopic oriented sections of  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . Then the first-return maps on  $\gamma_1$  and  $\gamma_2$  have the same rotation number:*

$$\rho(P_{\mathcal{F}_1}^{\gamma_1}) = \rho(P_{\mathcal{F}_2}^{\gamma_2}).$$

The next result state that conversely, the rotation number of the first-return map is locally equivalent to the oriented asymptotic cycle. While well-known by experts of the area, we give a short sketch of proof of this fact for the convenience of the reader.

**Lemma 3.28.** *Let  $\mathcal{F}_1, \mathcal{F}_2$  be two oriented topological foliations of  $\mathbf{T}^2$  having the same oriented projective asymptotic cycle, and  $\gamma_1, \gamma_2$  be two freely homotopic oriented sections of  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . Then for any oriented foliations  $\mathcal{F}'_i$  sufficiently close to  $\mathcal{F}_i$ :*

$$\rho(P_{\mathcal{F}'_1}^{\gamma_1}) = \rho(P_{\mathcal{F}'_2}^{\gamma_2}) \Rightarrow A^+(\mathcal{F}'_1) = A^+(\mathcal{F}'_2).$$

*Proof.* The first step is the well-known fact that for suspensions in  $\mathbf{T}^2$ , equality of the oriented projective asymptotic cycles is equivalent to be topologically conjugated by a homeomorphism of  $\mathbf{T}^2$  isotopic to the identity. We can thus assume without loss of generality that  $\mathcal{F}_1 = \mathcal{F}_2$  is the oriented linear foliation  $\mathcal{F}_l$  of  $\mathbf{T}^2$  induced by some half-line  $l \in \mathbf{P}^+(\mathbb{R}^2)$ , and since the rotation number of the first-return map is invariant by free homotopy of the section, we can also assume that  $\gamma_1 = \gamma_2$  are a common simple closed curve  $b \in \pi_1(\mathbf{T}^2) \cong \mathbb{Z}^2$ . Let  $a \in \pi_1(\mathbf{T}^2)$  be any simple closed curve completing  $b$  into a basis  $(a, b)$  of  $H_1(\mathbf{T}^2, \mathbb{R}) \cong \mathbb{R}^2$ . Then lifting a suspension  $\mathcal{G}$  of  $\mathbf{T}^2$  to  $\mathbb{R}^2$ , it is easily checked that  $\rho(P_{\mathcal{G}}^b) = [u]$  if, and only if there exists  $n \in \mathbb{Z}$  such that  $A^+(\mathcal{G}) = \mathbb{R}^+[\pm(a + (u + n)b)]$ . But on the other hand there exists a neighbourhood of  $l$  in  $\mathbf{P}^+(H_1(\mathbf{T}^2, \mathbb{R}))$ , containing at most one of the half-lines  $\{\mathbb{R}^+[\pm(a + (u + n)b)] \mid n \in \mathbb{Z}\}$  for any  $[u] \in \mathbf{S}^1$ . Since the oriented asymptotic cycle vary continuously with the foliation, this shows that for any oriented foliations  $\mathcal{F}'_1, \mathcal{F}'_2$  in a neighbourhood of  $\mathcal{F}$ :  $\rho(P_{\mathcal{F}'_1}^b) = \rho(P_{\mathcal{F}'_2}^b)$  implies  $A^+(\mathcal{F}'_1) = A^+(\mathcal{F}'_2)$ .  $\square$

We will say, following [Suh13], that a singular  $\mathbf{X}$ -surface  $S$  is *class A* if the projective asymptotic cycles of its  $\alpha$  and  $\beta$  lightlike foliations are distinct:  $A(\mathcal{F}_\alpha) \neq A(\mathcal{F}_\beta)$ ; and that it is *class B* otherwise. All of the structures studied in this text are class A (see Lemma A.15 for more details) and both of their lightlike foliations are moreover suspensions.

**3.7. Deformation space and marking of the structures.** We now want to define a *marking* of the families of singular  $\mathbf{dS}^2$ -tori that we have constructed in Propositions 3.12 and 3.17. To this end, we first have to introduce a deformation space to work in.

**3.7.1. Deformation space of singular  $\mathbf{dS}^2$ -structures.** For any oriented surface  $S$  and any set  $\Theta = \{\theta_i\}_i$  of angles  $\theta_i \in \mathbb{R}$ , we denote by  $\mathcal{S}(S, \Theta)$  the set of singular  $\mathbf{dS}^2$ -structures on  $S$  whose singular points angles are given by  $\Theta$ . We will endow  $\mathcal{S}(S, \Theta)$  with the usual topology on  $(G, X)$ -structures, defined as follows (see [CEG87, §1.5] for more details).

Let  $(S, \Sigma, \mu)$  be a singular  $\mathbf{dS}^2$ -surface of singular  $\mathbf{dS}^2$ -atlas  $(\varphi_i: U_i \rightarrow X_i)_i$ , where  $X_i = \mathbf{dS}^2$  if  $\varphi_i$  is a regular chart, and  $X_i = \mathbf{dS}^2_{\theta_i}$  at a singular point  $x_i$  of angle  $\theta_i$ . Let  $(U'_i)_i$  be a shrinking of  $(U_i)_i$ , *i.e.* a covering such that  $\overline{U'_i} \subset U_i$  for each  $i$ , and assume that  $U'_i$  contains the singular point  $x_i$  at any singular chart (note that the  $\overline{U'_i}$  for singular charts are pairwise disjoint, since the associated  $U_i$  are such and  $\overline{U'_i} \subset U_i$ ). Lastly, let  $\mathcal{V}_i$  be for any  $i$  an open neighbourhood of  $\varphi_i|_{U'_i}$  in the compact-open topology of  $C(U'_i, X_i)$ , small enough so that  $\mathbf{o}_{\theta_i} \in \psi(U'_i)$  at any singular point of angle  $\theta_i$  and for any  $\psi \in \mathcal{V}_i$ .

**Definition 3.29.** The set  $\mathcal{S}(S, \Theta)$  of singular  $\mathbf{dS}^2$ -structures of angles  $\Theta$  on an oriented surface  $S$ , is endowed with the topology for which the sets of the form

$$\left\{ \mu' \text{ defined by a singular } \mathbf{dS}^2 \text{ atlas } \psi_i: U'_i \rightarrow X_i \mid \psi_i \in \mathcal{V}_i \right\}$$

for any initial singular  $\mathbf{dS}^2$ -structure  $(\Sigma, \mu)$  on  $S$  and any choice of shrinking  $U'_i$  and compact-open neighbourhoods  $\mathcal{V}_i$  as above, form a sub-basis of the topology. We denote by  $\mathcal{S}(S, \Sigma, \Theta)$  the subspace of singular  $\mathbf{dS}^2$ -structures on  $S$  of (ordered) singular set  $\Sigma$  with (ordered) angles  $\Theta$ .

Let  $(S, \Sigma, \mu)$  be a singular  $\mathbf{dS}^2$ -surface of singular  $\mathbf{dS}^2$ -atlas  $(\varphi_i, U_i)$ . If  $f$  is a homeomorphism of  $S$  acting as the identity on  $\Sigma$ , then the singular  $\mathbf{dS}^2$ -structure  $f^*\mu$  of  $S$  is defined by the singular  $\mathbf{dS}^2$ -atlas  $(\varphi_i \circ f, f^{-1}(U_i))$ . It has the same singular set  $\Sigma$  and the same angle at each singular point. This defines a right action of the subgroup  $\text{Homeo}(S, \Sigma)$  of homeomorphisms of  $S$  acting as the identity on  $\Sigma$ , on each  $\mathcal{S}(S, \Sigma, \Theta)$ .

The *deformation space* of singular  $\mathbf{dS}^2$ -structures on  $S$  with singular set  $\Sigma$  of angles  $\Theta$ , denoted by  $\text{Def}_\Theta(S, \Sigma)$ , is defined as the quotient of  $\mathcal{S}(S, \Sigma, \Theta)$  by the subgroup  $\text{Homeo}^0(S, \Sigma)$  of homeomorphisms of  $\Sigma$  isotopic to the identity relative to  $\Sigma$ .

With  $\text{Homeo}^+(S, \Sigma)$  the group of orientation-preserving homeomorphisms of  $S$  acting as the identity on  $\Sigma$ , the quotient  $\text{PMod}(S, \Sigma)$  of  $\text{Homeo}^+(S, \Sigma)$  by  $\text{Homeo}^0(S, \Sigma)$  is called the *pure mapping class group* of  $(S, \Sigma)$ , and acts naturally on  $\text{Def}_\Theta(S, \Sigma)$ . The quotient of this action is the *moduli space* of  $\mathbf{dS}^2$ -structures on  $S$  with singular set  $\Sigma$  of angles  $\Theta$ .

*Remark 3.30.* The projective asymptotic cycle of the lightlike foliations of a point  $[\mu] \in \text{Def}_\Theta(\mathbf{T}^2, \Sigma)$  in the deformation space is well-defined, since homeomorphisms isotopic to the identity act trivially on projective asymptotic cycles according to (3.14). In particular, the notion of class A and B structures is invariant by the action of  $\text{Homeo}^0(S, \Sigma)$ , and makes thus sense in  $\text{Def}_\Theta(\mathbf{T}^2, \Sigma)$ .

**Lemma 3.31.** *The subset  $\text{Def}_\Theta(\mathbf{T}^2, \Sigma)^A$  of class A (respectively class B) singular  $\mathbf{dS}^2$ -structures on  $\mathbf{T}^2$  is a union of connected components of  $\text{Def}_\Theta(\mathbf{T}^2, \Sigma)$ .*

*Proof.* The condition  $A(\mathcal{F}_\alpha) \neq A(\mathcal{F}_\beta)$  of class A structures being clearly open, the set of class A structures is open. In the other hand according to Lemma A.15, if a structure  $\mu$  is class B then both of its lightlike foliations have closed leaves  $F_\alpha$  and  $F_\beta$ , such that  $F_\alpha$  is freely homotopic to  $\pm[F_\beta]$ . An important remark at this point, is that this situation cannot happen if one of the lightlike foliations has only closed leaves. Indeed assume  $\mathcal{F}_\alpha$  has only closed leaves. Then all of them are freely homotopic to  $F_\alpha$ , which prevents  $F_\beta$  to be transverse to  $\mathcal{F}_\alpha$ . Since both  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$  have non-closed leaves, they are stable in the sense that any small deformation of any of them still contains a closed leaf freely homotopic to the original closed curve  $F_{\alpha/\beta}$ . Therefore any small deformation of  $\mu$  remains class B, which shows that the subset of class B structures is open. Since class A and B structures form a partition of all singular  $\mathbf{dS}^2$ -structures in  $\mathcal{S}_\Theta(\mathbf{T}^2, \Sigma)$ , this shows in the end that the set of class A (respectively class B) structures is both open and closed, *i.e.* is a union of connected components of  $\mathcal{S}_\Theta(\mathbf{T}^2, \Sigma)$ .  $\square$

3.7.2. *Parameter families in the deformation space.* Taking the homotopy classes in  $\pi_1(\mathcal{T}_{\mathcal{A},x}) \simeq \mathbb{Z}^2$  and not anymore in  $\pi_1(\mathcal{T}_{\mathcal{A},x}^*)$ , the pair

$$m_x := (a, b)$$

defined in paragraph 3.2.2 is a basis of  $\pi_1(\mathcal{T}_{\mathcal{A},x})$ . It is in fact easy to check that  $b$  is freely homotopic to a closed path  $b'$  intersecting  $a$  only at  $[1, 0]$ .

In the same way with  $a_1$  the positively oriented  $\alpha$ -lightlike segment of  $\mathcal{T}_{\mathcal{A},x,y}$  from  $[1, 0]$  to  $[\infty, 0]$  and  $a_2$  the negatively oriented  $\beta$ -lightlike segment from  $[\infty, 0] = [1, y']$  to  $[1, 0]$ , we denote by  $a$  the homotopy class of  $a_1 a_2$ . Lastly with  $b_1$  the positively oriented  $\beta$ -lightlike segment from  $[1, 0]$  to  $[1, y_+]$  and  $b_2$  the negatively oriented  $\alpha$ -lightlike segment from  $[1, y_+] = [x', 0]$  to  $[1, 0]$ , we denote by  $b$  the homotopy class of  $b_1 b_2$ . The marking of  $\mathcal{T}_{\mathcal{A},x,y}$  is then

$$(3.15) \quad m_{x,y} := (a, b).$$

To see that  $(a, b)$  indeed generate  $\pi_1(\mathcal{T}_{\mathcal{A},x,y})$ , one easily check that  $a$  and  $b$  are freely homotopic to two transverse closed curves  $a'$  and  $b'$  indicated in Figure 3.2 which have algebraic intersection number 1 (the signs of their three intersection points being indicated in brown).

We also fix a marking  $m^0 = ([1, 0], [0, 1])$  of  $\mathbf{T}^2$ , and denote by  $0 = [0, 0]$  its origin. We recall that a homeomorphism  $f$  of a surface  $S$  acting as the identity on a discrete set  $\Sigma$  is said *isotopic to the identity relative to  $\Sigma$* , if there exists a continuous family  $t \in [0; 1] \mapsto f_t \in \text{Homeo}(\Sigma)$ , such that  $f_0 = f$ ,  $f_1 = \text{id}_S$  and for any  $t$ ,  $f_t|_\Sigma = \text{id}_\Sigma$ .

**Lemma 3.32.** *Up to pre-composition by homeomorphisms of  $\mathbf{T}^2$  isotopic to the identity relative to  $0$ , there exists:*

- (1) for any fixed  $x \in [1; \infty]$ , a unique homeomorphism  $M_x: \mathbf{T}^2 \rightarrow \mathcal{T}_{\mathcal{A},x}$  such that  $M_x(0) = [1, 0]$  and whose action in homotopy sends  $m^0$  to  $m_x$ ;
- (2) for any fixed  $(x, y) \in \mathcal{D}$ , a unique homeomorphism  $M_{x,y}: \mathbf{T}^2 \rightarrow \mathcal{T}_{\mathcal{A},x,y}$  such that  $M_{x,y}(0) = [1, 0]$  and whose action in homotopy sends  $m^0$  to  $m_{x,y}$ .

*Proof.* The existence being clear, we only have to prove that a homeomorphism of  $\mathbf{T}^2$  fixing  $0$  and acting trivially in homotopy, is isotopic to the identity relative to  $0$ . This fact is well-known but we outline here the proof for sake of completeness. First, for a homeomorphism  $f$  of  $\mathbf{T}^2$  fixing

0 and with  $h$  the restriction of  $f$  to  $\mathbf{T}^2 \setminus \{0\}$ ,  $f$  is isotopic to  $\text{id}_{\mathbf{T}^2}$  relative to 0 if and only if  $h$  is isotopic to  $\text{id}_{\mathbf{T}^2 \setminus \{0\}}$  (see for instance [BCLR20, Proposition 1.6]). Then,  $h$  is isotopic to  $\text{id}_{\mathbf{T}^2 \setminus \{0\}}$  if, and only if it is homotopic to  $\text{id}_{\mathbf{T}^2 \setminus \{0\}}$ , due to a result of Epstein in [Eps66] (see also [BCLR20, Theorem 2]). Lastly,  $h$  is homotopic to  $\text{id}_{\mathbf{T}^2 \setminus \{0\}}$  if, and only if it acts trivially on  $\pi_1(\mathbf{T}^2 \setminus \{0\})$  (see [BCLR20, Theorem 2 and §2.2]). But if  $f$  acts trivially on  $\pi_1(\mathbf{T}^2)$ , then  $h$  acts trivially on  $\pi_1(\mathbf{T}^2 \setminus \{0\})$ , which concludes the proof.  $\square$

For any fixed  $x \in [1; \infty]$  (respectively  $(x, y) \in \mathcal{D}$ ), the homeomorphisms  $M_x$  (resp.  $M_{x,y}$ ) given by Lemma 3.32 define thus the same point  $[M_x^* \mathcal{T}_{\mathcal{A},x}]$  (resp.  $[M_{x,y}^* \mathcal{T}_{\mathcal{A},x,y}]$ ) in  $\text{Def}_\theta(\mathbf{T}^2, 0)$ , which will be denoted by

$$\mu_{\theta,x} \text{ (resp. } \mu_{\theta,x,y} \text{)}.$$

We summarize the constructions of this paragraph in the following result.

**Proposition 3.33.** *The maps*

$$x \in [1; \infty] \mapsto \mu_{\theta,x} \in \text{Def}_\theta(\mathbf{T}^2, 0) \text{ and } (x, y) \in \mathcal{D} \mapsto \mu_{\theta,x,y} \in \text{Def}_\theta(\mathbf{T}^2, 0)$$

are continuous.

*Proof.* This follows from the continuity of the HIETs proved in paragraphs 3.5.1 and 3.5.2.  $\square$

*Remark 3.34.* We emphasize that  $\mu_{\theta,1} \neq \mu_{\theta,\infty}$ . Indeed  $\mathcal{T}_{\mathcal{A},1} = \mathcal{T}_{\mathcal{A},\infty}$  and  $m_1 = (a_1, b_1) = (a_\infty, -a_\infty + b_\infty)$  with the notations of paragraph 3.2.2. Hence with  $\Phi$  the element of the pure mapping class group of  $(\mathbf{T}^2, 0)$  defined by the matrix

$$\phi = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),$$

$\mu_{\theta,1} = \Phi^*(\mu_{\theta,\infty})$ . In other words,  $\mu_{\theta,x}$  does not define a closed loop, but a path in  $\text{Def}_\theta(\mathbf{T}^2, 0)$ .

**3.7.3. The deformation space is a topological surface.** The only topological property of the deformation space that we will use in this text is that  $\text{Def}_\theta(\mathbf{T}^2, 0)$  is *Hausdorff*. This standard fact is obtained for singular  $\mathbf{dS}^2$ -surfaces in the same way than in the classical case of Riemannian constant curvature surfaces with conical singularities. More precisely, one usually deduces the Hausdorff nature of  $\text{Def}_\theta(\mathbf{T}^2, \Sigma)$  from the fact that it is a topological manifold, a property that we will never use in this text. Since this remains the most natural way to reach the fact that  $\text{Def}_\theta(\mathbf{T}^2, 0)$  is Hausdorff, we prove however this property now in the case that we are interested with, namely the one of singular  $\mathbf{dS}^2$ -structures on  $\mathbf{T}^2$  with a single singularity, both of which lightlike foliations are minimal.

A natural geometrical strategy to prove that a given deformation space is a topological manifold and to compute its dimension, is to express this deformation space as a space of polygons in the model space, with prescribed identifications of their edges. We refer the reader to [FM11, §10.4.2] for a very nice presentation of this type of arguments in the classical case of hyperbolic structures on surfaces. In the case that we are interested with we can actually do better, and rely on the specific two-parameter family of structures that we previously constructed.

**Lemma 3.35.** *Any singular  $\mathbf{dS}^2$ -structure on  $\mathbf{T}^2$  with a single singularity, both of which lightlike foliations are minimal, is isometric to a structure  $\mathcal{T}_{\mathcal{A},x,y}$  given by Proposition 3.17.*

*Proof.* The first step is to adapt Lemma 3.10, by proving that such a structure  $\mu$  admits an embedded lightlike graph  $C$  containing the unique singularity  $x$ , such that  $\mathbf{T}^2 \setminus C$  is a topological disk, and such that the oriented boundary of the surface  $\mathbf{T}^2 \setminus C$  obtained by cutting  $\mathbf{T}^2$  along  $C$  is a lightlike ‘‘L-shaped’’ polygon. Let  $\varphi_{\alpha/\beta}^u$  be topological flows parametrizing  $\mathcal{F}_{\alpha/\beta}$ . It is easy to deduce from the minimality of  $\mathcal{F}_{\alpha/\beta}$  the existence of  $s, t > 0$  and  $s' \in [0; t]$ ,  $t' \in [0; s]$  such that:

- (1)  $\varphi_\alpha^s(x) = \varphi_\beta^{t'}(x)$ ,
- (2)  $\varphi_\beta^t(x) = \varphi_\alpha^{s'}(x)$ ,
- (3) and  $\varphi_\alpha^{[0;s]}(x) \cap \varphi_\beta^{[0;t]}(x) = \{x, \varphi_\alpha^s(x), \varphi_\beta^t(x)\}$ ,

with possibly  $x = \varphi_\alpha^s(x)$  and/or  $x = \varphi_\beta^t(x)$ . Let  $a$  denote the loop at  $x$  obtained by following  $\varphi_\alpha^{[0;s]}(x)$  in the positive direction and then  $\varphi_\beta^{[0;t']}(x)$  in the negative direction, and  $b$  denote the loop at  $x$  obtained by following  $\varphi_\beta^{[0;t]}(x)$  in the positive direction and then  $\varphi_\alpha^{[0;s']}(x)$  in the negative direction. Then  $a$  and  $b$  can be slightly deformed to two loops  $a'$  and  $b'$  intersecting transversally and only at  $x$ . Indeed,  $a$  can be slightly deformed to a loop  $a'$  at  $x$  obtained by following  $\varphi_\alpha^{[0;s-\varepsilon]}(x)$  in the positive direction for a small  $\varepsilon > 0$ , and then a path transverse to  $\mathcal{F}_\beta$  and close to  $\varphi_\beta^{[0;t']}(x)$ . In the same way, we can deform  $b$  to a path  $b' = \varphi_\beta^{[0;t-\varepsilon]}(x) \cdot \gamma$  with  $\gamma$  a path transverse to  $\mathcal{F}_\alpha$  and close to  $\varphi_\alpha^{[0;s']}(x)$ . Such deformations make disappear the common sub-paths  $\varphi_\alpha^{[0;s']}(x)$  and  $\varphi_\beta^{[0;t']}(x)$  of  $a$  and  $b$ , and create no new intersections if they are small enough (this can be done by compactness of the loops). Hence  $a'$  and  $b'$  have algebraic intersection number 1 (with suited orientations conventions) and therefore  $\mathbf{T}^2 \setminus C$  is a topological disk with  $C = a \cup b$ . The boundary of  $\mathbf{T}^2 \setminus C$  is moreover by construction L-shaped, which completes the first step.

The second step is then to show that  $\mathbf{T}^2 \setminus C$  is isometric to a L-shaped polygon  $\mathcal{L}$  in  $\mathbf{dS}^2$ , and to deduce from there that  $(\mathbf{T}^2, \mu)$  is isometric to a quotient of  $\mathcal{L}$  by an identification of its edges. This part is a straightforward adaptation of the arguments carefully described in Proposition 3.8 in the case of rectangular surfaces. We recall from paragraph 3.3.1 that the identification of the edges of  $\mathcal{L}$  is entirely determined by  $\mathcal{L}$ .  $\square$

**Proposition 3.36.** *In the neighbourhood of any structure whose lightlike foliations are both minimal,  $\text{Def}_\theta(\mathbf{T}^2, 0)$  is a topological surface.*

*Proof.* Let  $\mu_0 \in \text{Def}_\theta(\mathbf{T}^2, 0)$  have both of its lightlike foliations minimal. Then  $(\mathbf{T}^2, \mu_0)$  is isometric to a structure  $\mathcal{T}_{\mathcal{A},x,y}$  according to Lemma 3.35, and so is any close enough structure since this property is clearly open. In other words, the continuous map  $(x, y) \mapsto \mu_{\theta,x,y}$  from Proposition 3.33 is surjective from an open disk  $D \subset \mathcal{D}$  to an open neighbourhood  $U \subset \text{Def}_\theta(\mathbf{T}^2, 0)$  of  $\mu_0$ , and it only remains to prove that this map is injective if  $D$  is small enough. But if  $\mu_1$  and  $\mu_2$  define the same point of  $\text{Def}_\theta(\mathbf{T}^2, 0)$ , then the  $\mathbf{dS}^2$ -structures  $\mu_1^*$  and  $\mu_2^*$  of  $\mathbf{T}^2 \setminus \{0\}$  are in particular isotopic, by an isotopy as close to the identity as one wants on the neighbourhood of 0. According to the Ehresman-Thurston deformation principle, the space of  $\mathbf{dS}^2$ -structures of  $\mathbf{T}^2 \setminus \{0\}$  modulo isotopy is moreover locally homeomorphic to the space of holonomy morphisms modulo conjugation (see [Gol88, Deformation Theorem p.178] or [BG04]). In particular, we deduce from this that if two close enough singular  $\mathbf{dS}^2$ -structures  $\mu_1$  and  $\mu_2$  define the same point in  $\text{Def}_\theta(\mathbf{T}^2, 0)$ , then they have conjugated holonomy morphisms. But it follows from a direct computation that for any two close enough  $(x_1, y_1)$  and  $(x_2, y_2)$ ,  $\text{tr}(h_2(x_1, y_1)) = \text{tr}(h_2(x_2, y_2))$  and  $\text{tr}(g_2(x_1, y_1)) = \text{tr}(g_2(x_2, y_2))$  implies  $(x_1, y_1) = (x_2, y_2)$ , hence  $[\mu_{\theta,x_1,y_1}] = [\mu_{\theta,x_2,y_2}]$  implies  $(x_1, y_1) = (x_2, y_2)$ . This shows that  $(x, y) \in D \mapsto [\mu_{\theta,x,y}] \in U$  is continuous, surjective and locally injective. Since it is also open by definition of the topology of  $\text{Def}_\theta(\mathbf{T}^2, 0)$  it is thus a local homeomorphism, which concludes the proof.  $\square$

*Remark 3.37.* Another way to obtain this result is to deduce it directly from the topology of relative character varieties. Indeed, Ehresman-Thurston Deformation principle shows that on a neighbourhood of the singular  $\mathbf{dS}^2$ -structures that we are studying, the deformation space of singular  $\mathbf{dS}^2$ -structures with one singularity of fixed angle  $\theta$  at 0 is locally homeomorphic to a relative character variety. Namely the quotient space (modulo conjugations) of the space of morphisms from  $\pi_1(\mathbf{T}^2 \setminus 0)$  to  $\text{PSL}_2(\mathbb{R})$  having fixed trace  $e^\theta + e^{-\theta}$  for the peripheral homotopy class (see for instance [Gol88, Deformation Theorem p.178] or [BG04]). Note that this local homeomorphism does not hold everywhere, but holds on the open subset of stable conjugacy orbits, to which belong the holonomies of the singular  $\mathbf{dS}^2$ -structures studied in this paper (see for instance the discussion at the top of [Gol88, p.179]). It is then possible to show that this character variety is indeed a two-dimensional topological manifold, on the neighbourhood of the holonomy maps appearing here (see for instance the seminal works [Gol84, Gol03]).

**3.8. Conclusion of the proof of Theorem 3.1.** We can now use the structures constructed in Propositions 3.12 and 3.17 to conclude the proof of the existence Theorem 3.1.



Let us consider two closed curves  $(a', b')$  of  $\mathcal{T}_{\mathcal{A},x,y}$ , belonging to the marking  $m = (a, b)$  defined in (3.15) and respectively transverse to the  $\beta$  and the  $\alpha$ -foliation. To fix the ideas, we define

- (1) for  $t \in [1; x]$ ,  $a'_t$  as the closed loop obtained by following positively  $[t; \infty] \times \{0\}$  and then the segment of  $\mathcal{L}_{\mathcal{A},x,y}$  from  $(1, y')$  to  $(t, 0)$ ;
- (2) for  $t \in [0; y]$ ,  $b'_t$  as the closed loop obtained by following positively  $\{1\} \times [t; y_+]$  and then the segment of  $\mathcal{L}_{\mathcal{A},x,y}$  from  $(x', 0)$  to  $(1, t)$ .

Then  $t \mapsto a'_t$  and  $t \mapsto b'_t$  are homotopies, respectively beginning at  $a'_1 = a$  and  $b'_0 = b$ . See Figure 3.2 where  $a'$  and  $b'$  are indicated. We moreover fix an identification from  $a$  to  $a'$ , given by the identity on common points of the loops, and by following  $\beta$ -leaves positively until the first meeting point elsewhere. Accordingly, we fix an identification from  $b$  to  $b'$  given by following  $\alpha$ -leaves positively. Through these identifications, the homeomorphisms  $\mathbf{E}$  and  $\mathbf{F}$  of  $\mathbf{S}_I^1 = [1; \infty]/\{1 \sim \infty\}$  and  $\mathbf{S}_J^1 = [0; y_+]/\{0 \sim y_+\}$  introduced in paragraph 3.5.2 induce homeomorphisms  $\mathbf{E}'$  and  $\mathbf{F}'$  of  $a'$  and  $b'$ . Let

$$P_\beta^{a'} : a' \rightarrow a' \text{ and } P_\alpha^{b'} : b' \rightarrow b'$$

denote the respective holonomies of  $\mathcal{F}_\beta$  on  $a'$  and of  $\mathcal{F}_\alpha$  on  $b'$ .

**Lemma 3.38.**  $P_\beta^{a'} = \mathbf{E}'^{-1}$  and  $P_\alpha^{b'} = \mathbf{F}'^{-1}$ .

*Proof.* The fact that  $\mathbf{E}'^{-1}$  and  $\mathbf{F}'^{-1}$  are the respective holonomies of the  $\beta$  and  $\alpha$  foliations directly follows from an observation of the Figure 3.2.  $\square$

With obvious corresponding notations, we will use in any torus  $\mathcal{T}_{\mathcal{A},x}$  a closed loop  $b'$  homotopic to  $b$  and transverse to  $\mathcal{F}_\alpha$ , denote by  $\bar{g}'$  the homeomorphism of  $b'$  induced by  $g$ , and by  $P_\alpha^{b'}$ ,  $P_\beta^a$  the holonomies of  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$  on  $b'$  and  $a$ . The following result is then proved in the same way than Lemma 3.38.

**Lemma 3.39.**  $P_\beta^a = \mathbf{E}^{-1}$  and  $P_\alpha^{b'} = \bar{g}'^{-1}$ .

*Conclusion of the proof of Theorem 3.1.* We will repeatedly use the Propositions 3.25 and 3.26 to translate the dynamics of a torus foliation into its projective asymptotic cycle. We will realize the non-oriented projective asymptotic cycles, which yields all the expected oriented projective asymptotic cycles by composing with orientation-reversing maps.

(1) It is clear from the dynamics of  $g$  and  $h_1$  that  $\mathcal{F}_\alpha([1, 0])$  (respectively  $\mathcal{F}_\beta([1, 0])$ ) is the unique closed  $\alpha$ -leaf (resp.  $\beta$ -leaf) of the torus  $\mathcal{T}_{\mathcal{A},1}$ , and by acting with the pure mapping class group of  $(\mathcal{T}_{\mathcal{A},1}, [1, 0])$ , one obtains any basis of  $\pi_1(\mathcal{T}_{\mathcal{A},1})$ . On the other hand, Proposition 3.23 and Lemma 3.39 show that any periodic cyclic order for the orbit of  $[1, 0]$  under the first-return map  $\mathbf{E}_x$  of  $\mathcal{F}_\beta$  on  $a_x$  is reached. Since Dehn twists around  $a_x$  belong to the pure mapping class group of  $(\mathcal{T}_{\mathcal{A},x}, [1, 0])$  and fix  $a_x$ , we can act by such Dehn twists to obtain points  $[\mu] \in \text{Def}_\theta(\mathbf{T}^2, 0)$  so that  $[\mathcal{F}_\alpha^\mu(0)] = [1, 0]$  and  $[\mathcal{F}_\beta^\mu(0)]$  is any primitive element of  $\pi_1(\mathbf{T}^2)$  distinct from  $(1, 0)$ . We lastly observe that  $\mathcal{F}_\alpha([1, 0])$  remains the unique closed  $\alpha$ -leaf of  $\mathcal{T}_{\mathcal{A},x}$ , and that the same can be achieved for  $\mathcal{F}_\beta$  according to Remark 3.14. This concludes the first claim.

(2) The first-return map of the  $\beta$ -foliation is given by the map  $\mathbf{E}$  according to Lemma 3.38. Proposition 3.23 shows thus in particular that the map  $x \in [0; \infty] \mapsto A(\mathcal{F}_\beta^{\mu_\theta, x}) \in \mathbf{P}(\mathbf{H}_1(\mathbf{T}^2, \mathbb{R})) \cong \mathbb{R}\mathbf{P}^1$  is continuous, monotonous and non-constant. Therefore  $A(\mathcal{F}_\beta^{\mu_\theta, x})$  reaches an interval  $I \subset \mathbf{P}(\mathbf{H}_1(\mathbf{T}^2, \mathbb{R}))$  of non-empty interior, hence containing irrational lines. On the other hand, Remark 3.34 shows that  $\mu_{\theta, \infty} = \Phi_*(\mu_{\theta, 1})$  with  $\Phi$  a Dehn twist around  $(1, 0)$  fixing  $A(\mathcal{F}_\alpha^{\mu_\theta, x}) = [\mathcal{F}_\alpha^{\mu_\theta, x}](0) = [1, 0]$ . It is now easily checked that the translates of  $I$  by the iterates of  $\Phi$  cover  $\mathbf{P}(\mathbf{H}_1(\mathbf{T}^2, \mathbb{R})) \setminus \{[1, 0]\}$ , which shows the first claim of (2). The fact that  $\mathcal{F}_\alpha(0)$  is the unique closed leaf of  $\mathcal{F}_\alpha$  follows again from the fact that  $[0] = [y_{\mathcal{A}}]$  is the unique periodic point of  $g$ .

(3) As for (2), the first-return maps of the  $\alpha$  and  $\beta$  foliations are given by the maps  $\mathbf{F}$  and  $\mathbf{E}$  according to Lemma 3.38. Proposition 3.24 shows thus that  $(A(\mathcal{F}_\alpha^{\mu_\theta, x, y}), A(\mathcal{F}_\beta^{\mu_\theta, x, y}))$  reaches a subset  $K \subset \mathbf{P}(\mathbf{H}_1(\mathbf{T}^2, \mathbb{R}))^2 \setminus \{\text{diagonal}\}$  of non-empty interior, hence containing pairs of irrational lines. As for (2), the claim follows then from the fact that the translates of  $K$  by the action of the pure mapping group of  $(\mathbf{T}^2, 0)$  cover the pairs of distinct irrational lines in  $\mathbf{P}(\mathbf{H}_1(\mathbf{T}^2, \mathbb{R}))^2$ .  $\square$

4. RIGIDITY OF SINGULAR  $\mathbf{dS}^2$ -TORI

We conclude in this section the proofs of the rigidity Theorems A, C and D.

**4.1. Conclusion of the proof of Theorem C.** The existence part was proved in Theorem 3.1. Let  $\mu_1, \mu_2$  be two singular  $\mathbf{dS}^2$ -structures on  $\mathbf{T}^2$  with a unique singularity of angle  $\theta$  at  $0$ , whose lightlike leaves of  $0$  are closed and homotopic:

$$([\mathcal{F}_\alpha^{\mu_1}(0)], [\mathcal{F}_\beta^{\mu_1}(0)]) = ([\mathcal{F}_\alpha^{\mu_2}(0)], [\mathcal{F}_\beta^{\mu_2}(0)]).$$

According to Lemma 3.10 and the proof of Proposition 3.8, there exists homotopic isometries respectively sending  $\mu_1$  and  $\mu_2$  to structures  $\mu_{\theta, x_1}$  and  $\mu_{\theta, x_2}$  with  $x_1, x_2 \in [1; \infty[$  such that  $\mathcal{F}_\beta^{\mu_{\theta, x_1}}(0)$  and  $\mathcal{F}_\beta^{\mu_{\theta, x_2}}(0)$  are closed and homotopic, and it only remains to prove that  $x_1 = x_2$ . Indeed this will show that  $\mu_2 = \varphi^* \mu_1$  for some homeomorphism of  $\mathbf{T}^2$  fixing  $0$ , but (4.3) will then imply that  $\varphi$  acts trivially in homotopy *i.e.* is isotopic to identity relative to  $0$  (see the proof of Lemma 3.32 for more details), hence  $[\mu_1] = [\mu_2]$  in  $\text{Def}_\theta(\mathbf{T}^2, 0)$ .

From now on we implicitly identify  $(\mathbf{T}^2, \mu_{\theta, x_i})$  and  $\mathcal{T}_{-\theta, x_i}$  as explained in Lemma 3.32, to simplify notations. The first return map of  $\mathcal{F}_\beta^{\mu_{\theta, x_i}}$  on  $\mathcal{F}_\alpha^{\mu_{\theta, x_i}}(0)$  being  $E_{x_i}^{-1}$  according to Lemma 3.39, we can translate the fact that  $\mathcal{F}_\beta^{\mu_{\theta, x_1}}(0)$  and  $\mathcal{F}_\beta^{\mu_{\theta, x_2}}(0)$  are closed and homotopic, to the one that the orbits of  $[1] \in \overline{[1; \infty]} := [1; \infty]/\{1 \sim \infty\}$  under  $E_{x_1}$  and  $E_{x_2}$  are periodic, say of minimal period  $q \in \mathbb{N}^*$ , and of the same cyclic order on the circle  $\overline{[1; \infty]}$ . We can moreover assume that  $x_1, x_2 \in ]1; \infty[$  and that  $q \geq 2$ , since  $E_x$  has no fixed points unless  $x = 0$ . For  $p \in \overline{[1; \infty]}$ , let us denote:

- (1)  $l(p) = a$  if  $p \in [1; x'_i[$ , equivalently if  $E_{x_i}(p) = gh_{x_i}(p)$ ;
- (2) and  $l(p) = b$  if  $p \in [x'_i; \infty[$ , equivalently if  $E_{x_i}(p) = h_{x_i}(p)$ .

Then with  $l_1 = l([1])$  and  $l_{k+1} = l(l_k([1]))$ , the word  $w = l_q \dots l_1$  in the letters  $a$  and  $b$  is the *coding* of the periodic orbit of  $[1]$  under  $E_{x_i}$ , and is equivalent to its cyclic ordering. In other words the codings of  $[1]$  under  $E_{x_1}$  and  $E_{x_2}$  equal a common word  $w$ , characterized by

$$(4.1) \quad E_{x_i}^k([1]) = w_k(gh, h)([1])$$

where  $w_k = l_k \dots l_1$ , and  $v(A, B) \in \text{PSL}_2(\mathbb{R})$  is obtained for any  $A, B \in \text{PSL}_2(\mathbb{R})$  from a word  $v$  in the letters  $a$  and  $b$  by replacing  $a$  by  $A$  and  $b$  by  $B$ .

According to Lemma 3.22 there exists  $T \in [0; 1]$  such that  $x_2 = g^T(x_1)$  and  $h_{x_2} = g^T h_{x_1}$ , and we thus want to show that  $T = 0$ . From now on we denote  $h := h_{x_1}$  to simplify notations, and work in  $\mathbb{R} \cup \{\infty\}$  identified with  $\mathbb{RP}^1$  (in the same  $\text{PSL}_2(\mathbb{R})$ -equivariant way (2.2) than usually). The equalities (4.1) translate then as:

$$(4.2) \quad \begin{cases} w(gh, h)(1) = w(g^{T+1}h, g^T h)(1) = 1 \\ \forall k \in \{1, \dots, q-1\} : w_k(gh, h)(1) \text{ and } w_k(g^{T+1}h, g^T h)(1) \in ]1; \infty[. \end{cases}$$

**Fact 4.1.** *For any  $k \in \{1, \dots, q\}$ , the map  $s \in [0; T] \mapsto w_k(g^{s+1}h, g^s h)(1)$  is strictly increasing and has values in  $]1; \infty[$ .*

Fact 4.1 concludes the proof, since  $s \in [0; T] \mapsto w_q(g^{s+1}h, g^s h)(1) = w(g^{s+1}h, g^s h)(1)$  is in particular strictly increasing but has according to (4.2) the same value 1 at  $s = 0$  and  $s = T$ , which implies  $T = 0$ .

*Proof of Fact 4.1.* We prove the claim by recurrence on  $k$ .

**Case  $k = 1$ .** Then  $w_1 = l_1 = a$  and since  $gh(1) \in ]1; \infty[$ ,  $s \in \mathbb{R} \mapsto w_1(g^{s+1}h, g^s h)(1) = g^{s+1}h(1)$  is strictly increasing in  $\mathbb{R} \cup \{\infty\}$ . Since  $g^{T+1}h(1) \in ]1; \infty[$  as well according to (4.2), we have thus  $g^{s+1}h(1) \in ]1; \infty[$  for any  $s \in [0; T]$  by the intermediate values Theorem.

**From  $k \in \{1, \dots, q-1\}$  to  $k+1$ .** Then  $w_{k+1}(g^{s+1}h, g^s h)(1) = l_{k+1}(g, \text{id})g^s h(\alpha(s))$  for  $s \in [0; T]$ , with  $\alpha(s) := w_k(g^{s+1}h, g^s h)(1)$  strictly increasing and with values in  $]1; \infty[$  by recurrence. Since  $h$  is orientation-preserving,  $s \in [0; T] \mapsto h \circ \alpha(s)$  is strictly increasing. Moreover the dynamics of  $h$  show that its attractive and repulsive fixed points respectively satisfy  $h_+ \in ]y_A; 1[$  and  $h_- \in ]\infty; 0[$ , and the attractive and repulsive fixed points of  $g$  are on the other hand  $0$  and  $y_A$ . We have thus  $h \circ \alpha([0; T]) \subset ]h_+; \infty[ \subset [y_A; 0]$ , and with  $G(s, p) = g^s(p)$  for any

$(s, p) \in \mathbb{R} \times ]y_A; 0[$  we have:  $\frac{\partial G}{\partial s}(s, p) > 0$  due to the dynamics of  $g$ , and  $\frac{\partial G}{\partial p}(s, p) > 0$  due to the fact that  $g^s$  is orientation-preserving. Therefore

$$\frac{d}{ds}g^s h(\alpha(s)) = \frac{d}{ds}G(s, h(\alpha(s))) = \frac{\partial G}{\partial s}(s, h(\alpha(s))) + \frac{d}{ds}h(\alpha(s)) \frac{\partial G}{\partial p}(s, h(\alpha(s)))$$

is strictly positive for any  $s \in [0; T]$  as a sum of strictly positive terms, hence  $s \in [0; T] \mapsto w_{k+1}(g^{s+1}h, g^s h)(1)$  is strictly increasing since  $g$  is orientation preserving. Since  $w_{k+1}(gh, h)(1)$  and  $w_{k+1}(g^{T+1}h, g^T h)(1)$  are moreover in  $[1; \infty[$  according to (4.2), we have thus  $g^{s+1}h(1) \in [1; \infty[$  for any  $s \in [0; T]$ , which concludes the proof.  $\square$

**4.2. Conclusion of the proof of Theorem D.** The existence part is given by Theorem 3.1. Let  $\mu_1, \mu_2$  be two singular  $\mathbf{dS}^2$ -structures on  $\mathbf{T}^2$  with a unique singularity of angle  $\theta$  at  $0$ , whose  $\alpha$  leaves of  $0$  are closed and such that:

$$(4.3) \quad ([\mathcal{F}_\alpha^{\mu_1}(0)], A^+(\mathcal{F}_\beta^{\mu_1})) = ([\mathcal{F}_\alpha^{\mu_2}(0)], A^+(\mathcal{F}_\beta^{\mu_2}))$$

with  $A(\mathcal{F}_\beta^{\mu_i})$  an irrational line. Then as in the beginning of paragraph 4.1, there exists homotopic isometries respectively sending  $\mu_1$  and  $\mu_2$  to structures  $\mu_{\theta, x_1}$  and  $\mu_{\theta, x_2}$  with  $x_1, x_2 \in [1; \infty[$  such that  $A^+(\mathcal{F}_\beta^{\mu_{\theta, x_1}}) = A^+(\mathcal{F}_\beta^{\mu_{\theta, x_2}})$ , and it only remains to prove that  $x_1 = x_2$ . Indeed this will show that  $\mu_2 = \varphi^* \mu_1$  for some homeomorphism of  $\mathbf{T}^2$  fixing  $0$ , and (4.3) will then imply that  $\varphi$  is isotopic to identity relative to  $0$ , hence that  $[\mu_1] = [\mu_2]$  in  $\text{Def}_\theta(\mathbf{T}^2, 0)$ .

Let us denote by  $\gamma_i: [0; 1] \rightarrow \mathbf{T}^2$  the unique future affine parametrization of  $\mathcal{F}_\alpha^{\mu_{\theta, x_i}}(0)$  such that  $\gamma_i(0) = 0$  and by  $P_i$  the first-return map of  $\mathcal{F}_\beta^{\mu_{\theta, x_i}}$  on  $\mathcal{F}_\alpha^{\mu_{\theta, x_i}}(0)$  (well-defined since  $\mathcal{F}_\beta^{\mu_{\theta, x_i}}$  is minimal). Then since  $[\mathcal{F}_\alpha^{\mu_{\theta, x_1}}(0)]$  and  $[\mathcal{F}_\alpha^{\mu_{\theta, x_2}}(0)]$  are homotopic, the equality of the oriented projective asymptotic cycles of the  $\beta$ -foliations implies that  $\rho(P_1) = \rho(P_2)$  according to Lemma 3.27. This implies in turn that  $x_1 = x_2$  according to Proposition 3.23, which concludes the proof.

**4.3. Surgeries of singular constant curvature Lorentzian surfaces.** In this paragraph we introduce a useful notion of surgery in the general setting of singular  $\mathbf{X}$ -surfaces,  $(\mathbf{G}, \mathbf{X})$  denoting as usual the pair  $(\text{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$  or  $(\mathbb{R}^{1,1} \rtimes \text{SO}^0(1, 1), \mathbb{R}^{1,1})$ . Let  $\gamma: [0; l] \rightarrow S$  denote for this whole subsection an affinely parametrized simple closed timelike geodesic in a singular  $\mathbf{X}$ -surface  $(S, \Sigma)$  (with  $\gamma(0) = \gamma(l)$  and  $\gamma|_{]0; l[}$  injective). Whenever convenient, we define  $\gamma(u + nl) := \gamma(u)$  for any  $u \in [0; l]$  and  $n \in \mathbb{Z}$ . Observe that all the results of the subsection remain true for a spacelike simple closed geodesic, by changing the metric of  $S$  to its opposite. For  $u \in [0; l]$  we denote by

$$R_u^\gamma: \gamma \rightarrow \gamma$$

the affine transformation of  $\gamma$  defined by  $R_u^\gamma(\gamma(t)) = \gamma(t + u)$  (namely the rotation of parameter  $u$ ). Moreover if it is well-defined, then we denote by

$$P_\varepsilon^\gamma: \gamma \rightarrow \gamma$$

the first-return map of the lightlike foliation  $\mathcal{F}_\varepsilon$  (with  $\varepsilon = \alpha$  or  $\beta$ ) on  $\gamma$ . It is characterized by the fact that for any  $x \in \gamma$ ,  $P_\varepsilon^\gamma(x)$  is the first intersection point of  $\mathcal{F}_\varepsilon(x)$  with  $\gamma$  starting from  $x$  (for the orientation of  $\mathcal{F}_\varepsilon$ ). Our goal is to prove the following result.

**Proposition 4.2.** *Let  $\gamma: [0; l] \rightarrow S$  be an affinely parametrized simple closed geodesic, in a singular  $\mathbf{X}$ -surface  $(S, \Sigma, \mu)$  of ordered angle set  $\Theta$ . Then for any  $t \in [0; l]$ , there exists a singular  $\mathbf{X}$ -structure  $\mu_t$  on  $S$  such that:*

- (1) *the ordered singular set of  $\mu_t$  is  $\Sigma$ , and its ordered angle set  $\Theta$ ;*
- (2)  *$t \in [0; l] \mapsto \mu_t \in \mathcal{S}(S, \Theta)$  is continuous, and  $\mu_0 = \mu$ ;*
- (3)  *$\gamma$  remains a (simple closed) affinely parametrized geodesic of  $\mu_t$  of the same signature;*
- (4) *if the first-return map  $P_{\alpha/\beta, \mu}^\gamma: \gamma \rightarrow \gamma$  of a lightlike foliation of  $\mu$  is well-defined on  $\gamma$ , then the one of  $\mu_t$  is also well-defined on  $\gamma$  and moreover equals  $P_{\alpha/\beta, \mu_t}^\gamma = R_t^\gamma \circ P_{\alpha/\beta, \mu}^\gamma$ .*

Moreover,  $\mu_t$  can be chosen to coincide with  $\mu$  outside of a tubular neighbourhood of  $\gamma$  as small as one wants.

We emphasize that this surgery construction is by no mean canonical, which does however not prevent it to be very useful.

4.3.1. *A one-parameter family of isometries.* If  $\gamma$  is definite, then a natural parametrization is of course the one for which  $|\boldsymbol{\mu}(\gamma'(t))| \equiv 1$ , but if  $\gamma$  is a *lightlike* geodesic then it does not have a preferred parametrization preserved by all the local isometries stabilizing it. We fix henceforth an affine parametrization  $\gamma: [0; l] \rightarrow S$  of  $\gamma$  in order to select a preferred one-parameter group in its stabilizer, even when  $\gamma$  is lightlike.

**Lemma 4.3.** *Let  $\gamma: \mathbb{R} \rightarrow \mathbf{X}$  be an affinely parametrized maximal geodesic of  $\mathbf{X}$ . Then there exists a unique parametrized one-dimensional subgroup  $t \in \mathbb{R} \mapsto g^t \in \mathbf{G}$ , for which there exists  $\varepsilon > 0$  and an open non-empty interval  $I \subset \mathbb{R}$ , such that for any  $t \in ]0; \varepsilon[$  and  $u \in I$ :  $g^t \circ \gamma(u) = \gamma(u+t)$ . Moreover  $g^t \circ \gamma(u) = \gamma(u+t)$  for any  $t, u \in \mathbb{R}$ .*

*Proof.* The claim is obvious if  $\gamma$  is definite, since  $\text{Stab}_{\mathbf{G}}(\gamma(I))$  is then a one-parameter subgroup preserving the parametrization up to translation and acting freely transitively on the maximal geodesic  $L = \gamma(I)$ . We assume thus that  $L$  is a lightlike leaf. If  $\mathbf{X} = \mathbb{R}^{1,1}$  then  $\gamma$  is an affinely parametrized horizontal or vertical line  $L$ . The set of  $g \in \text{Stab}_{\mathbf{G}}(\gamma(I))$  which preserve the parametrization of  $\gamma$  up to translation is simply the translation subgroup  $(T_u)_{u \in L} \subset \mathbb{R}^{1,1} \rtimes \text{SO}^0(1, 1)$ , and the claim obviously follows since  $(T_u)$  acts freely on  $L$ . If  $\mathbf{X} = \mathbf{dS}^2$ , we can assume that  $L = \mathbb{RP}^1 \setminus \{[e_1]\} \times [e_1]$  by transitivity of  $\text{PSL}_2(\mathbb{R})$ , in which case the affine structure of  $L$  is the one induced by  $\mathbb{RP}^1 \setminus \{[e_1]\}$  through the identification  $p \in \mathbb{RP}^1 \setminus \{[e_1]\} \mapsto (p, [e_1]) \in L$ . In other words we can assume without loss of generality that  $\gamma(t) = ([t : 1], [e_1])$ , and for

$$g = g_{\lambda, u} := \begin{bmatrix} \lambda & u \\ 0 & \lambda^{-1} \end{bmatrix} \in \text{Stab}_{\text{PSL}_2(\mathbb{R})}(L)$$

(with  $\lambda \in \mathbb{R}_*^+$  and  $u \in \mathbb{R}$ ) we have thus  $g \circ \gamma(t) = \gamma(\lambda^2 t + \lambda u)$ . Therefore  $g$  preserves the parametrization of  $\gamma$  up to translation if, and only if  $\lambda = 1$ , and since  $(g_{1, u})_{u \in \mathbb{R}}$  again acts freely transitively on  $L$ , this proves the claim.  $\square$

An important property to be emphasized at this point, is that if two  $\mathbf{X}$ -morphisms  $f_1: U_1 \rightarrow V_1$  and  $f_2: U_2 \rightarrow V_2$  coincide on a non-empty connected open subset  $U \subset U_1 \cap U_2$ , then  $f_1$  and  $f_2$  coincide on the connected component of  $U_1 \cap U_2$  containing  $U$ . This is well-known and due to the analyticity of the action of  $\mathbf{G}$  on  $\mathbf{X}$ .

**Lemma 4.4.** *Let  $A, B$  be two small open tubular neighbourhoods of  $\gamma$ , such that  $\text{Cl}(A) \subset B$  and  $B \cap \Sigma = \gamma \cap \Sigma$ . Then there exists  $\varepsilon > 0$  and a continuous family  $\Phi: t \in [0; \varepsilon] \mapsto \Phi^t \in C(A^+ \cup \gamma, B^+ \cup \gamma)$  of continuous maps  $\Phi^t$  defined on  $A^+ \cup \gamma$  and with values in  $B^+ \cup \gamma$ , which are homeomorphisms onto their images,  $\mathbf{X}$ -morphisms in restriction to  $A^+$ , and such that  $\Phi^t(\gamma(u)) = \gamma(u+t)$  for any  $u \in [0; l]$ .*

Observe that  $\gamma$  has indeed *two-sided* tubular neighbourhoods since  $S$  is orientable. We denoted in the above statement by  $A^+$  and  $A^-$  the up and down connected components of  $A \setminus \gamma$  (with respect to the orientation of  $\gamma$  and the one of  $S$ ) and likewise for  $B^\pm$ , and will adopt henceforth obvious similar notations.

*Proof of Lemma 4.4.* Let  $U$  be a topological disk of closure contained in  $B \setminus \Sigma$  and such that  $U \cap \gamma \neq \emptyset$  is connected. The Lemma 4.3 shows then the existence of  $\varepsilon > 0$  and of a unique continuous family  $t \in [0; \varepsilon] \mapsto \phi^t$  of  $\mathbf{X}$ -morphisms defined on  $U$  and with values in  $B$ , such that  $\phi^t \circ \gamma(u) = \gamma(t+u)$  for any  $\gamma(u) \in \gamma \cap U$  and  $t \in [0; \varepsilon]$ . Indeed if  $\phi_1^t$  and  $\phi_2^t$  are two such morphisms, then in any  $\mathbf{X}$ -chart  $\varphi: V \rightarrow \mathbf{X}$  from an open set  $V \subset U$ ,  $\varphi \circ \phi_1^t \circ \varphi^{-1}$  is the restriction of a  $g_1^t \in \mathbf{G}$  such that  $g_1^t \circ \varphi \circ \gamma(u) = \varphi \circ \gamma(u+t)$ , hence  $g_1^t = g_2^t$  according to Lemma 4.3, and thus  $\phi_1^t = \phi_2^t$ . The uniqueness on any small enough connected open subset of  $U$  gives the existence on  $U$  by gluing these  $\mathbf{X}$ -morphisms  $\phi_i^t$  together.

Since  $\Sigma$  is discrete and  $\gamma$  compact,  $\gamma \cap \Sigma$  is finite and  $\gamma \setminus \Sigma$  a finite union of intervals. We assume now that  $x \in \gamma \cap \Sigma$  is a singular point of angle  $\theta$ , let  $V \subset B$  be a normal convex neighbourhood of  $x$  and denote by  $I^+$  the positive half-leaf of  $\mathcal{F}_\alpha(x) \cap V$  starting from  $x$  for the orientation of  $\mathcal{F}_\alpha$ . Let  $U_1, U_2$  be two topological disks of closures contained in  $V \setminus I^+$ , such that  $U_1 \cap U_2 \neq \emptyset$  and  $U_i \cap \gamma \neq \emptyset$  are connected, and such that with  $\gamma_i$  the connected component of  $\gamma \setminus \Sigma$  containing  $U_i \cap \gamma$ :  $x$  is the future (respectively past) endpoint of  $\gamma_1$  (resp.  $\gamma_2$ ). Observe that possibly  $\gamma_1 = \gamma_2$  if  $\gamma \cap \Sigma$  is a point, in which case  $x$  is both a future and past endpoint.

We showed in the previous paragraph the existence for  $i = 1$  and  $2$  of unique continuous families  $t \in [0; \varepsilon] \mapsto \phi_i^t$  of  $\mathbf{X}$ -morphisms defined on  $U_i$  and with values in  $B$ , such that  $\phi_i^t \circ \gamma(u) = \gamma(u+t)$  for any  $\gamma(u) \in \gamma \cap U_i$  and  $t \in [0; \varepsilon]$ . Let now  $\varphi: V \setminus I^+ \rightarrow \mathbf{X}$  be a  $\mathbf{X}$ -chart of  $S$  such that  $\varphi(V \setminus I^+) = V_0 \setminus \mathcal{F}_\alpha^+(\mathfrak{o})$  with  $V_0$  a connected neighbourhood of  $\mathfrak{o}$ , and such that  $\pi_\theta \circ \varphi$  extends to a singular  $\mathbf{X}$ -chart at  $x$  with values in  $\mathbf{X}_\theta$ . This exists by definition of a standard singularity, see Lemma 2.14 for more details. Then  $\varphi \circ \phi_i^t \circ \varphi^{-1}$  is the restriction of a  $g_i^t \in \mathbf{G}$  such that

$$(4.4) \quad g_i^t \circ \varphi \circ \gamma(u) = \varphi \circ \gamma(u+t)$$

for any  $\gamma(u) \in \gamma \cap U_i$  and  $t \in [0; \varepsilon]$ . On the other hand, by Definition A.5 of a geodesic and since  $\pi_\theta \circ \varphi$  extends to a singular  $\mathbf{X}$ -chart at  $x$ ,  $\varphi(\gamma_1 \cap V)$  and  $\varphi(\gamma_2 \cap V)$  are two intervals of a common timelike geodesic  $L \subset \mathbf{X}$  through  $\mathfrak{o}$ , and of future (resp. past) endpoint  $\mathfrak{o}$ . The equalities (4.4) imply thus  $g_1^t = g_2^t$  according to Lemma 4.3, and therefore  $\phi_1^t = \phi_2^t$ . This argument was done for any two topological disks  $U_1, U_2 \subset V \setminus I^+$  with  $V$  a normal convex neighbourhood of  $x \in \gamma \cap \Sigma$ , and such topological disks  $U_i$  cover  $\text{Cl}(A^+) \setminus \Sigma$  if  $A$  is chosen small enough. The uniqueness yields thus the existence of  $\Phi^t$  on  $A^+$  by gluing these  $\phi_i^t$  together.

We eventually emphasize that  $\Phi^t(\gamma(u)) = \gamma(u+t)$  whenever  $\gamma([u; u+t]) \cap \Sigma = \emptyset$ , and that this equality extends thus on  $\gamma$  by continuity, which concludes the proof of the Lemma.  $\square$

4.3.2. *Proof of Proposition 4.2.* We first observe that it is sufficient to construct the surgery  $\mu_t$  for any  $t \in [0; \varepsilon]$  with some  $\varepsilon > 0$  depending only on  $\gamma$ , since one only has to apply later the same construction to  $\mu_\varepsilon$  and to compose with further surgeries. There exists a small open tubular neighbourhood  $A \subset S$  of  $\gamma$  whose two boundary components are transverse to whichever lightlike foliation  $\gamma$  is transverse to, *i.e.* to both lightlike foliations if  $\gamma$  is definite, and to  $\mathcal{F}_\beta$  (respectively  $\mathcal{F}_\alpha$ ) if  $\gamma$  is an  $\alpha$  (resp.  $\beta$ ) leaf. We moreover assume that  $\text{Cl}(A) \cap \Sigma = \gamma \cap \Sigma$ , that  $A \setminus \gamma$  has two connected components  $A^\pm$ ,  $A^+$  being the upper component with respect to the orientation of  $S$  and the future-orientation of  $\gamma$  (*i.e.* with  $N$  a normal to  $\gamma'(t)$  pointing to  $A^+$ ,  $(\gamma'(t), N)$  yields the positive orientation of  $S$ ). There exists also a closed curve  $\sigma \subset A^+$  freely homotopic to  $\gamma$  within  $A^+$  and that we orient compatibly with  $\gamma$ , that is transverse to whichever lightlike foliation  $\gamma$  is transverse to and such that  $A^+$  is itself a tubular neighbourhood of  $\sigma$ , *i.e.*  $A^+ \setminus \sigma$  is the union of two up and down boundary components  $A_1^+$  and  $A_2^+$ . If  $A$  is chosen small enough, there exists moreover an open tubular neighbourhood  $B$  of  $\text{Cl}(A)$  such that  $\text{Cl}(B) \cap \Sigma = \gamma \cap \Sigma$ , and  $\varepsilon > 0$  such that the continuous map  $\Phi: t \mapsto \Phi^t$  given by Lemma 4.4 is defined for any  $t \in [0; \varepsilon]$ , where each  $\Phi^t$  is defined on  $A^+ \cup \gamma$  with values in  $B^+ \cup \gamma$ , and is a  $\mathbf{X}$ -morphism on  $A^+$  (with the obvious similar notations for  $B^\pm$  than the one we defined for  $A^\pm$ ). We can moreover chose  $\sigma$  close enough to  $\gamma$ , so that for any  $t$  we have:

$$(4.5) \quad \max_{x \in \sigma} d_\sigma(\Phi^t(x), x) \leq 2 \max_{x \in \gamma} d_\gamma(\Phi^t(x), x),$$

where  $d_{\sigma/\gamma}$  are the distances induced on the curves by a fixed Riemannian metric  $h$  on  $S$ . Then there exists a continuous map  $F: t \in [0; \varepsilon] \mapsto F_t \in C(A, A \cup B^+)$ , such that  $F_0 = \text{id}_A$  and for any  $t \in [0; \varepsilon]$ :

- (1)  $F_t$  is an orientation-preserving homeomorphism onto its image,
- (2)  $F_t$  equals the identity on  $A^- \cup \gamma$ ,
- (3) and  $F_t$  equals  $\Phi^t$  on  $A_2^+ \cup \sigma$ .

We can moreover assume that

$$(4.6) \quad \max_{x \in A} d_S(F_t(x), x) \leq \max_{x \in \sigma} d_S(F_t(x), x),$$

with  $d_S$  the distance induced by  $h$  on  $S$ . Of course there exists many such maps  $F$ , but we fix one. We define then a singular  $\mathbf{X}$ -structure  $\mu_t^*$  on  $A$  by  $\mu_t^* = F_t^* \mu$ . Observe that:

- (1) since  $F_t|_\gamma = \text{id}|_\gamma$  and  $A \cap \Sigma = \gamma \cap \Sigma$ , the singular points of  $\mu_t^*$  and their angles coincide with the one of  $\mu|_A$ , and are thus contained in  $\gamma \cap \Sigma$ ;
- (2) since  $F_t|_{A^-} = \text{id}|_{A^-}$ ,  $\mu_t^*|_{A^-} = \mu|_{A^-}$ ;
- (3) since  $F_t|_{A_2^+} = \Phi^t|_{A_2^+}$  is a  $\mathbf{X}$ -morphism of  $\mu$ , the  $\mathbf{X}$ -atlas of  $\mu|_{A_2^+}$  is compatible with its pullback by  $F_t$ , in other words  $\mu_t^*|_{A_2^+} = \mu|_{A_2^+}$ .

Since  $(S \setminus \text{Cl}(A_1^+)) \cap A = A^- \sqcup A_2^+$ , the singular  $\mathbf{X}$ -structures  $\mu_t^*$  of  $A$  and  $\mu|_{S \setminus \text{Cl}(A_1^+)}$  of  $S \setminus \text{Cl}(A_1^+)$  are thus compatible, *i.e.* the union of their singular  $\mathbf{X}$ -atlases defines a singular  $\mathbf{X}$ -structure  $\mu_t$  on  $S = (S \setminus \text{Cl}(A_1^+)) \cup A$ . By construction, the singular points of  $\mu_t$  and their angles coincide with the ones of  $\mu$ , and  $t \mapsto \mu_t$  is moreover continuous since  $t \mapsto F_t$  is so. Furthermore for any small enough chart  $\varphi: U \rightarrow \mathbf{X}'$  of the singular  $\mathbf{X}$ -structure of  $S$  at a point  $x \in \gamma$  ( $\mathbf{X}' = \mathbf{X}$  if  $x$  is regular, and  $\mathbf{X}' = \mathbf{X}_\theta$  if  $x$  is singular of angle  $\theta$ ),  $U \subset F_t(A)$  and then  $F_t^{-1}(U)$  contains  $x = F_t(x)$ . Moreover  $\varphi \circ \gamma|_{\gamma^{-1}(U)}$  is an affinely parametrized timelike geodesic segment of  $\mathbf{X}'$  by Definition A.5 of a geodesic, and thus  $\varphi \circ F_t \circ \gamma|_{\gamma^{-1}(F_t^{-1}(U))}$  as well since  $F_t \circ \gamma(u) = \gamma(u+t)$ . This shows that  $\gamma$  remains an affinely parametrized timelike geodesic of  $\mu_t$  since  $\varphi \circ F_t$  is a chart of the singular  $\mathbf{X}$ -atlas of  $\mu_t$ .

We now prove the claim concerning the lightlike foliations, and it is sufficient to do so for  $\mathcal{F}_\alpha^\mu$  up to interverting the two foliations (note that if  $\gamma$  is lightlike, it is then a closed  $\beta$ -leaf). We assume thus that the first-return map  $P_\mu$  of  $\mathcal{F}_\alpha^\mu$  on  $\gamma$  is well-defined, and denote by  $P_\mu^1(x)$  the first intersection point of  $\mathcal{F}_\alpha^\mu(x)$  with  $\sigma$  for  $x \in \gamma$ . We recall that  $\gamma$  and  $\sigma$  are the lower and upper boundary components of  $A_1^+$ . Observe that, due to our orientation conventions,  $\mathcal{F}_\alpha(x)$  is transverse to  $\gamma$ , and leaves  $A^+$  when travelled positively. The direct consequence of this fact is the simple but crucial observation that the segments of  $\alpha$ -leaf from any  $x \in \gamma$  to  $P_\mu^1(x)$  coincide for  $\mu_t$  and  $\mu$ , since  $\mu_t|_{S \setminus \text{Cl}(A_1^+)} = \mu|_{S \setminus \text{Cl}(A_1^+)}$  by definition. We denote these segments by  $[x; P_\mu^1(x)]_\alpha^{\mu_t} = [x; P_\mu^1(x)]_\alpha^\mu$ , and we have thus showed that for any  $x \in \gamma$ :

$$(4.7) \quad P_{\mu_t}^1(x) = P_\mu^1(x).$$

By compactness of  $\gamma$ , we can assume  $\sigma$  close enough to  $\gamma$  so that for any  $y \in \sigma$ , there exists a  $\mathbf{X}$ -chart  $\varphi: U \rightarrow \mathbf{X}$  of  $\mu$  so that  $]y; P_\mu^2(y)]_\alpha^\mu \subset U$ , with  $]y; P_\mu^2(y)]_\alpha^\mu$  the connected component of  $\mathcal{F}_\alpha^\mu(y) \cap A_1^+$  containing  $y$  in its closure and  $P_\mu^2(y) \in \gamma$  the first intersection point of  $\mathcal{F}_\alpha(y)$  with  $\gamma$ . Adopting the obvious similar notations for  $\mu_t$  and with  $y' = (\Phi^t)^{-1}(y)$ , we have thus  $\varphi \circ F_t(]y'; P_{\mu_t}^2(y')]_\alpha^{\mu_t}) = \varphi(]y; P_\mu^2(y)]_\alpha^\mu)$ , by definition of  $\mu_t$  and since  $F_t(y') = y$ . Rearranging this equality, we obtain  $]y; P_{\mu_t}^2(y)]_\alpha^{\mu_t} = F_t^{-1}(]y; P_\mu^2(y)]_\alpha^\mu)$  for any  $y \in \sigma$ . In particular  $s \in [0; t] \mapsto ]y; P_{\mu_s}^2(y)]_\alpha^{\mu_s}$  is a homotopy from  $]y; P_\mu^2(y)]_\alpha^\mu$  to  $]y; P_{\mu_t}^2(y)]_\alpha^{\mu_t}$ , and in the end  $P_{\mu_t}^2(y) = F_t^{-1}(P_\mu^2(\Phi^t(y))) = P_\mu^2(\Phi^t(y))$ . The last equality is due to the fact that  $P_\mu^2(\Phi^t(y)) \in \gamma$  and  $F_t|_\gamma = \text{id}|_\gamma$ . But  $\Phi^t$  is a  $\mathbf{X}$ -morphism on  $A^+$  and sends therefore segments of  $\alpha$ -leaves to segments of  $\alpha$ -leaves, hence  $]y; P_{\mu_t}^2(y)]_\alpha^{\mu_t} = \Phi^t(]y; P_\mu^2(y)]_\alpha^\mu)$  and thus  $P_{\mu_t}^2(y) = \Phi^t(P_\mu^2(y))$ . In the end

$$(4.8) \quad P_{\mu_t}^2(y) = R_t^\gamma(P_\mu^2(y))$$

for any  $y \in \sigma$ , since  $\Phi^t(\gamma(u)) = \gamma(u+t) = R_t^\gamma(\gamma(u))$  (see lemma 4.4).

We can now collect our previous separated analyses of the foliation  $\mathcal{F}_\alpha^{\mu_t}$  within  $S \setminus \text{Cl}(A_1^+)$  and  $A_1^+$ . For  $x \in \gamma$ , the first-return map of  $\mathcal{F}_\alpha(x)$  on  $\gamma$  is defined by  $P_\mu(x) = P_\mu^2 \circ P_\mu^1(x)$  and likewise for  $\mu_t$ . The equalities (4.7) and (4.8) show then that  $P_{\alpha, \mu_t}(x) = P_{\mu_t}^2(P_\mu^1(x)) = R_t^\gamma \circ P_\mu^2 \circ P_\mu^1(x) = R_t^\gamma \circ P_\mu(x)$ , which concludes the proof of Proposition 4.2.

**4.3.3. Bounding the size of a surgery.** The space  $\mathcal{S}(\mathbf{T}^2, \Sigma, \Theta)$  of singular  $\mathbf{X}$ -structures on  $\mathbf{T}^2$  is induced with a distance  $d$  defined as follows. Let  $(\varphi_i: U_i \rightarrow X_i)_i$  be a finite singular  $\mathbf{dS}^2$ -atlas of  $\mu \in \mathcal{S}(\mathbf{T}^2, \Sigma, \Theta)$  (where  $X_i = \mathbf{dS}^2$  if  $\varphi_i$  is a regular chart and  $X_i = \mathbf{dS}_{\theta_i}^2$  at a singular point of angle  $\theta_i$ ) and  $(U'_i)_i$  be a shrinking of  $(U_i)_i$  as in Definition 3.29. Then with  $d_i$  a fixed distance on  $X_i$  and  $d_i^\infty(f, g) = \max_{x \in U_i} d_i(f(x), g(x))$  the associated uniform distance on continuous maps from  $U'_i$  to  $X_i$ , for any  $\mu' \in \mathcal{S}(\mathbf{T}^2, \Sigma, \Theta)$  defined by a singular  $\mathbf{dS}^2$ -atlas  $\psi_i: U'_i \rightarrow X_i$ , we define:

$$d(\mu', \mu) = \begin{cases} 0 & \text{if } \mu' = \mu, \\ \max_i d_i^\infty(\varphi_i|_{U'_i}, \psi_i) & \text{otherwise.} \end{cases}$$

We fix a Riemannian metric  $h$  on  $\mathbf{T}^2$ , endow  $\mathbf{T}^2$  and any piecewise smooth curve  $\gamma \subset \mathbf{T}^2$  with the induced distances  $d_{\mathbf{T}^2}$  and  $d_\gamma$ , and the space of continuous maps of  $\gamma$  with the associated uniform distance  $d_\gamma^\infty$ .

**Lemma 4.5.** *Let  $\mu \in \mathcal{S}(\mathbf{T}^2, \Sigma, \Theta)$ , and  $\gamma$  be a simple closed timelike geodesic of  $\mu$ . Then there exists a constant  $C > 0$ , such that for any surgery  $\nu$  of  $\mu$  around  $\gamma$  given by Proposition 4.2 and having a closed  $\alpha$ -leaf  $\mathcal{F}$ , and for any surgery  $\nu'$  of  $\nu$  around  $\mathcal{F}$  given by Proposition 4.2:*

$$d(\nu, \nu') \leq Cd_{\mathcal{F}}^{\infty}(P_{\beta, \nu}^{\mathcal{F}}, P_{\beta, \nu'}^{\mathcal{F}}).$$

*Proof.* By construction  $\nu'$  coincides with  $\nu$  outside of  $A_2^-$  (we refer henceforth to the notations introduced in the proof of Proposition 4.2 for the surgery  $\nu'$  of  $\nu$ ). With  $F$  the homeomorphism used to define  $\nu'|_{A_2^-} = F^*\nu|_{A_2^-}$  on  $A_2^-$ , we thus want to prove that  $d(F^*\nu|_{A_2^-}, \nu|_{A_2^-}) \leq Cd_{\mathcal{F}}^{\infty}(P_{\beta, \nu}^{\mathcal{F}}, P_{\beta, \nu'}^{\mathcal{F}})$  for some constant  $C > 0$ . It is sufficient to prove this claim for any small enough surgery  $\nu'$  of  $\nu$ , since the inequality follows then for further surgeries by triangular inequality. With  $(\varphi_i: U_i \rightarrow X_i)_i$  a finite singular  $\mathbf{dS}^2$ -atlas of  $\nu$  and  $(U'_i)_i$  a shrinking of  $(U_i)_i$  as above, we can thus assume that  $F(U'_i) \subset U_i$ . By finiteness of the atlas and continuity of the  $\varphi_i$ 's, there exists a constant  $C > 0$  such that  $d_i^{\infty}(\varphi_i \circ F|_{U'_i}, \varphi_i|_{U'_i}) \leq Cd_{\mathbf{T}^2}^{\infty}(F|_{U'_i}, \text{id}|_{U'_i})$  for any  $i$ , and thus

$$(4.9) \quad d(F^*\nu|_{A_2^-}, \nu|_{A_2^-}) \leq Cd_{\mathbf{T}^2}^{\infty}(F, \text{id}_{\mathbf{T}^2}).$$

Moreover  $F$  satisfies  $d_{\mathbf{T}^2}^{\infty}(F, \text{id}_{\mathbf{T}^2}) \leq d_{\sigma}^{\infty}(F|_{\sigma}, \text{id}_{\sigma})$  by assumption, see (4.6). Let  $\Phi$  be the isometry used to define  $F|_{\sigma \cup A_2^+} = \Phi|_{\sigma \cup A_2^+}$ , and such that the first-return maps of the  $\beta$  foliation on  $\mathcal{F}$  respectively satisfy  $P_{\beta, \nu'}^{\mathcal{F}} = \Phi \circ P_{\beta, \nu}^{\mathcal{F}}$  by definition of  $\nu$ . Then by continuity of  $\Phi$  and since the curve  $\sigma$  can be chosen as close as one wants from  $\mathcal{F}$ , we can assume that:

$$(4.10) \quad \begin{aligned} d_{\sigma}^{\infty}(F|_{\sigma}, \text{id}_{\sigma}) &= d_{\sigma}^{\infty}(\Phi|_{\sigma}, \text{id}_{\sigma}) \\ &\leq 2d_{\mathcal{F}}^{\infty}(\Phi|_{\mathcal{F}}, \text{id}_{\mathcal{F}}) \\ &= 2d_{\mathcal{F}}^{\infty}(P_{\beta, \nu'}^{\mathcal{F}}, P_{\beta, \nu}^{\mathcal{F}}) \end{aligned}$$

as we have seen previously in (4.5). In the end (4.9) and (4.10) imply together  $d(F^*\nu|_{A_2^-}, \nu|_{A_2^-}) \leq 2Cd_{\mathcal{F}}^{\infty}(P_{\beta, \nu'}^{\mathcal{F}}, P_{\beta, \nu}^{\mathcal{F}})$ , which concludes the proof.  $\square$

**4.4. Conclusion of the proof of Theorem A.** Let  $S_1$  and  $S_2$  be two closed singular  $\mathbf{dS}^2$ -surfaces having a unique singularity of angle  $\theta \in \mathbb{R}_+^*$ , and with minimal and topologically equivalent lightlike bifoliations. Without loss of generality, we can assume that  $S_1 = S_2 = \mathbf{T}^2$  and that the singular  $\mathbf{dS}^2$ -structures  $\mu_1$  and  $\mu_2$  have identical oriented lightlike bifoliations (this is possible since our definition of singular  $\mathbf{X}$ -structures authorizes  $\mathcal{C}^0$ -charts, hence singular  $\mathbf{X}$ -structures can be pulled back by homeomorphisms). Then  $\mu_1$  and  $\mu_2$  admit freely homotopic simple closed timelike geodesics  $\gamma_1$  and  $\gamma_2$  according to Theorem A.1 (since they are class A according to Lemma A.15). Up to translations we can assume that 0 is the unique singularity of both  $\mu_1$  and  $\mu_2$ , which does not change the existence of freely homotopic simple closed future timelike geodesics  $\gamma_1$  and  $\gamma_2$ , nor the equality

$$A^+(\mathcal{F}_{\alpha/\beta}^{\mu_1}) = A^+(\mathcal{F}_{\alpha/\beta}^{\mu_2})$$

of the oriented projective asymptotic cycles of the lightlike foliations. We denote by  $x_i$  the first intersection point of  $\mathcal{F}_{\alpha}^{\mu_i}(0)$  with  $\gamma_i$ , and fix for  $\gamma_i$  the unique simple future affine parametrization  $\gamma_i: [0; 1] \rightarrow \mathbf{T}^2$  such that  $\gamma_i(0) = x_i$  (i.e.  $\gamma_i(1) = \gamma_i(0)$  and  $\gamma_i|_{]0; 1[}$  is injective). Since  $\mathcal{F}_{\alpha}^{\mu_i}$  and  $\mathcal{F}_{\beta}^{\mu_i}$  are both assumed minimal, the first-return maps  $P_{\alpha/\beta, \mu_i}^{\gamma_i}: \gamma_i \rightarrow \gamma_i$  are well-defined, and moreover have the same rotation numbers

$$\rho(P_{\alpha/\beta, \mu_1}^{\gamma_1}) = \rho(P_{\alpha/\beta, \mu_2}^{\gamma_2})$$

according to Lemma 3.27, since  $\gamma_1$  and  $\gamma_2$  are freely homotopic.

According to Lemma B.1, there exists a sequence  $r_n \in \mathbf{S}^1$  of rationals converging to  $\rho(P_{\alpha, \mu_1}^{\gamma_1}) = \rho(P_{\alpha, \mu_2}^{\gamma_2}) \in [\mathbb{R} \setminus \mathbb{Q}]$  and sequences  $[u_i^n] \in \mathbf{S}^1$  converging to  $[0]$  such that for  $i = 1$  and  $2$ , the orbit of  $[0]$  for  $R_{[u_i^n]}^{\gamma_i} \circ P_{\alpha, \mu_i}^{\gamma_i}$  is periodic and of rational cyclic order  $r_n$ . For  $n$  large enough, Proposition 4.2 yields thus a surgery  $\mu_i^n = (\mu_i)_{u_i^n}$  of  $\mu_i$  around the geodesic  $\gamma_i$  such that:

- (1)  $\mu_i^n$  has a unique singularity of angle  $\theta$  at 0;
- (2)  $\mu_i^n$  converges to  $\mu_i$ ;
- (3)  $\gamma_i$  is still a timelike simple closed geodesic of  $\mu_i^n$ ;

(4) the first-return map of  $\mathcal{F}_{\alpha/\beta}^{\mu_i^n}$  on  $\gamma_i$  is well-defined and equals the circle homeomorphism

$$P_{\alpha/\beta, \mu_i^n}^{\gamma_i} = R_{[u_i^n]}^{\gamma_i} \circ P_{\alpha/\beta, \mu_i}^{\gamma_i}.$$

Since  $\mu_i^n$  converges to  $\mu_i$ ,  $\mathcal{F}_{\alpha/\beta}^{\mu_i^n}$  converges to  $\mathcal{F}_{\alpha/\beta}^{\mu_i}$ , and in particular  $A^+(\mathcal{F}_{\alpha/\beta}^{\mu_i^n})$  converges to  $A^+(\mathcal{F}_{\alpha/\beta}^{\mu_i})$ . Moreover according to property (4) and by definition of  $[u_i^n]$ , the orbits of  $[0]$  for  $P_{\alpha, \mu_1^n}^{\gamma_1}$  and  $P_{\alpha, \mu_2^n}^{\gamma_2}$  are periodic and of the same rational cyclic order  $r_n$ , hence  $\rho(P_{\alpha, \mu_1^n}^{\gamma_1}) = \rho(P_{\alpha, \mu_2^n}^{\gamma_2})$  according to Proposition 3.21, and the lightlike leaves  $\mathcal{F}_{\alpha^1}^{\mu_1^n}(0)$  and  $\mathcal{F}_{\alpha^2}^{\mu_2^n}(0)$  are closed. For any large enough  $n$ , Lemma 3.28 shows moreover that  $\rho(P_{\alpha, \mu_1^n}^{\gamma_1}) = \rho(P_{\alpha, \mu_2^n}^{\gamma_2})$  implies  $A^+(\mathcal{F}_{\alpha^1}^{\mu_1^n}) = A^+(\mathcal{F}_{\alpha^2}^{\mu_2^n})$  since  $\gamma_1$  and  $\gamma_2$  are freely homotopic, *i.e.* that the lightlike leaves  $\mathcal{F}_{\alpha^1}^{\mu_1^n}(0)$  and  $\mathcal{F}_{\alpha^2}^{\mu_2^n}(0)$  are freely homotopic since  $A^+(\mathcal{F}_{\alpha^1}^{\mu_1^n}) = [\mathcal{F}_{\alpha^1}^{\mu_1^n}(0)]$  according to Proposition 3.25.

**Fact 4.6.** *Possibly passing to a subsequence,  $\mathcal{F}_{\alpha^i}^{\mu_i^n}(0)$  is a section of  $\mathcal{F}_{\beta}^{\mu_i^n}$ .*

*Proof.* The surface with boundary  $A_n$  obtained from cutting  $\mathbf{T}^2$  along  $\mathcal{F}_{\alpha}^{\mu_i^n}(0)$  is an annulus whose boundary components are two copies of  $\mathcal{F}_{\alpha}^{\mu_i^n}(0)$ , and the foliation  $\mathcal{F}_{\beta}^{\mu_i^n}$  induces a foliation  $\mathcal{F}_n$  of  $A_n$  transverse to its boundary. It is known (see for instance [HH86, Remark 4.2.1 and Theorem 4.2.15]) that such a foliation is associated to a foliation  $\mathcal{F}'_n$  of  $A_n$  tangent to its boundary by “spiraling”  $\mathcal{F}_n$ , and that  $\mathcal{F}'_n$  is itself obtained by gluing together a finite number of Reeb components and suspensions. As a consequence, either  $\mathcal{F}_{\alpha}^{\mu_i^n}(0)$  intersects every leaf of  $\mathcal{F}_{\beta}^{\mu_i^n}$ , or else  $\mathcal{F}_n$  admits a closed leaf  $F_n^0$  in the interior of  $A_n$ , corresponding to a closed leaf  $F_n$  of  $\mathcal{F}_{\beta}^{\mu_i^n}$ . In the latter case  $F_n^0$  is freely homotopic to the boundary of  $A_n$  within  $A_n$ , and  $F_n$  is thus freely homotopic to  $\mathcal{F}_{\alpha}^{\mu_i^n}(0)$  in  $\mathbf{T}^2$ . In particular  $\mathcal{F}_{\alpha}^{\mu_i^n}$  and  $\mathcal{F}_{\beta}^{\mu_i^n}$  have then the same projective asymptotic cycle. But since  $A(\mathcal{F}_{\alpha/\beta}^{\mu_i^n})$  converges to  $A(\mathcal{F}_{\alpha/\beta}^{\mu_i})$ , and  $A(\mathcal{F}_{\alpha}^{\mu_i}) \neq A(\mathcal{F}_{\beta}^{\mu_i})$  as  $\mu_i$  is class A, there exists  $N \in \mathbb{N}$  such that for any  $n \geq N$ :  $A(\mathcal{F}_{\alpha}^{\mu_i^n}) \neq A(\mathcal{F}_{\beta}^{\mu_i^n})$ . This implies in turn that  $\mathcal{F}_{\alpha}^{\mu_i^n}(0)$  intersects every leaf of  $\mathcal{F}_{\beta}^{\mu_i^n}$  for any  $n \geq N$ , which concludes the proof of the Fact.  $\square$

There exists a continuous flow  $\varphi_i^t$  on  $\mathbf{T}^2$  such that  $t \in \mathbb{R} \mapsto \varphi_i^t(x)$  is a future affine parametrization of  $\mathcal{F}_{\alpha}^{\mu_i}(x)$  for any  $x \in \mathbf{T}^2$ . Denoting by  $U_i$  the fixed neighbourhood of  $\gamma_i$  such that  $\mu_i = \mu_i^n$  on  $\mathbf{T}^2 \setminus U_i$  (see Proposition 4.2), there exists then for any  $n$  a unique future affine parametrization  $\sigma_i^n : [0; l_i^n] \rightarrow \mathbf{T}^2$  of  $\mathcal{F}_{\alpha}^{\mu_i^n}(0)$  such that  $\sigma_i^n(0) = 0$ , and for which the maps  $F_{u_i^n}$  used on  $U_i$  to construct the surgeries  $\mu_i^n = (\mu_i)_{u_i^n}$  are affine maps (we refer to the construction of the surgery in paragraph 4.3.2 where these maps  $F_t$  are introduced). In particular,  $\mathcal{F}_{\alpha}^{\mu_i^n}(0) \setminus U_i$  is a union of segments of leaves of  $\mathcal{F}_{\alpha}^{\mu_i}$ , and  $\sigma_i^n$  coincide on  $\mathcal{F}_{\alpha}^{\mu_i^n}(0) \setminus U_i$  with the parametrization  $\varphi_i^t$  of  $\mathcal{F}_{\alpha}^{\mu_i}$  up to translation. The first-return map of  $\mathcal{F}_{\beta}^{\mu_i^n}$  on  $\sigma_i^n = \mathcal{F}_{\alpha}^{\mu_i^n}(0)$  is well-defined according to Fact 4.6, and for  $v \in ]-l_i^n; l_i^n[$  we denote by  $(\mu_i^n)_v$  the surgery of  $\mu_i^n$  around  $\sigma_i^n = \mathcal{F}_{\alpha}^{\mu_i^n}(0)$  given by Proposition 4.2, such that:

- (1)  $(\mu_i^n)_v$  has a unique singularity of angle  $\theta$  at 0;
- (2)  $\mathcal{F}_{\alpha}^{(\mu_i^n)_v}(0) = \mathcal{F}_{\alpha}^{\mu_i^n}(0) = \sigma_i^n$ ;
- (3) the first-return map of  $\mathcal{F}_{\beta}^{(\mu_i^n)_v}$  on  $\sigma_i^n$  is well-defined and equal to the circle homeomorphism

$$P_{\beta, (\mu_i^n)_v}^{\sigma_i^n} = R_{[v]}^{\sigma_i^n} \circ P_{\beta, \mu_i^n}^{\sigma_i^n} \text{ in the parametrization } \sigma_i^n.$$

The map  $v \in ]-l_i^n; l_i^n[ \mapsto A^+(\mathcal{F}_{\beta}^{(\mu_i^n)_v}) \in \mathbf{P}^+(\mathbf{H}_1(\mathbf{T}^2, \mathbb{R})) \equiv \mathbf{S}^1$  is continuous, non-decreasing and strictly increasing at any irrational line according to Lemma B.1. Since  $A^+(\mathcal{F}_{\beta}^{\mu_i^n})$  moreover converges to the irrational line  $A^+(\mathcal{F}_{\beta}^{\mu_i})$ , there exists for any large enough  $n$  some  $v \in ]-l_i^n; l_i^n[$  such that  $A^+(\mathcal{F}_{\beta}^{(\mu_i^n)_v}) = A^+(\mathcal{F}_{\beta}^{\mu_i})$ . There exists then a unique  $v_i^n \in ]-l_i^n; l_i^n[$  of smallest absolute value satisfying

$$(4.11) \quad A^+(\mathcal{F}_{\beta}^{\nu_i^n}) = A^+(\mathcal{F}_{\beta}^{\mu_i}) \text{ with } \nu_i^n := (\mu_i^n)_{v_i^n},$$



and in particular  $A^+(\mathcal{F}_\beta^{\nu_1^n}) = A^+(\mathcal{F}_\beta^{\nu_2^n})$ .

For  $x \in \gamma_i$ ,  $y \in \mathcal{F}_\beta^{\mu_i}(x)$  and  $t \in \mathbb{R}^*$ , let  $F_i(x, y, t) \in \mathbb{R}^*$  denote the unique time such that the holonomy

$$(4.12) \quad H_{\beta, \mu_i}: [x; R_{F_i(x, y, t)}^{\gamma_i}(x)]_{\gamma_i} \rightarrow [y; \varphi_i^t(y)]_\alpha$$

of  $\mathcal{F}_\beta^{\mu_i}$  is a homeomorphism between the segments  $[x; R_{F_i(x, y, t)}^{\gamma_i}(x)]_{\gamma_i}$  of  $\gamma_i$  and  $[y; \varphi_i^t(y)]_\alpha$  of  $\mathcal{F}_\alpha^{\mu_i}(y)$ . Since  $\sigma_i^n \setminus U_i$  is a union of segments of leaves of  $\mathcal{F}_\alpha^{\mu_i}$ , the first-return map of  $\mathcal{F}_\beta^{(\mu_i^n)_v}$  on the simple closed curve  $\gamma_i$  is well-defined (observe that  $\gamma_i$  is no longer a geodesic of  $(\mu_i^n)_v$ ), and moreover satisfies

$$(4.13) \quad P_{\beta, (\mu_i^n)_v}^{\gamma_i} = P_{\beta, \mu_i^n}^{\gamma_i} \circ F_{i, v}^n \text{ with } F_{i, v}^n: x \mapsto R_{F_i(x, H_i^n(x), v)}^{\gamma_i}(x)$$

where  $H_i^n$  is the holonomy of  $\mathcal{F}_\beta^{\mu_i^n}$  from  $\gamma_i$  to  $\sigma_i^n$ . Let  $d_{\sigma_i^n}^\infty$ ,  $d_{\gamma_i}^\infty$  denote the uniform distances on continuous maps of  $\sigma_i^n$  and  $\gamma_i$  induced by a fixed Riemannian metric on  $\mathbf{T}^2$ , and  $d_{\mathbf{S}^1}$  be a fixed distance on  $\mathbf{S}^1$ . Then there exists two constant  $C_1 > 0$  and  $C_2 > 0$  such that for any  $n$  and  $v \in ]-l_i^n; l_i^n[$ :

$$(4.14) \quad d_{\sigma_i^n}^\infty(R_{[v]}^{\sigma_i^n}, \text{id}_{\sigma_i^n}) \leq C_1 d_{\gamma_i}^\infty(F_{i, v}^n, \text{id}_{\gamma_i}),$$

$$(4.15) \quad d_{\gamma_i}^\infty(F_{i, v}^n, \text{id}_{\gamma_i}) \leq C_2 d_{\mathbf{S}^1}(\rho(P_{\beta, \mu_i^n}^{\gamma_i} \circ F_{i, v}^n), \rho(P_{\beta, \mu_i^n}^{\gamma_i})).$$

The existence of such constants for any fixed  $n$  is straightforward, and their uniformity in  $n$  follows from the definition (4.13) of  $F_{i, v}^n$ , and from the fact that the function  $F_i(x, y, t)$  defined in (4.12) satisfies

$$\sup \left\{ \left| \frac{t}{F_i(x, y, t)} \right| \mid x \in \gamma_i, y \in \mathcal{F}_\beta^{\mu_i}(x), t \in \mathbb{R}^* \right\} < +\infty$$

by compactness of  $\gamma_i$  and  $\mathbf{T}^2$ .

We know that  $\lim d_{\mathbf{S}^1}(\rho(P_{\beta, \mu_i}^{\gamma_i}), \rho(P_{\beta, \mu_i^n}^{\gamma_i})) = 0$  since  $\mu_i^n$  converges to  $\mu_i$ . On the other hand  $P_{\beta, \nu_i^n}^{\gamma_i} = P_{\beta, \mu_i^n}^{\gamma_i} \circ F_{i, \nu_i^n}^n$  according to (4.13), while the definition (4.11) of  $\nu_i^n$  and the Lemma 3.27 imply  $\rho(P_{\beta, \mu_i}^{\gamma_i}) = \rho(P_{\beta, \nu_i^n}^{\gamma_i}) = \rho(P_{\beta, \mu_i^n}^{\gamma_i} \circ F_{i, \nu_i^n}^n)$ . In the end  $\lim d_{\mathbf{S}^1}(\rho(P_{\beta, \mu_i^n}^{\gamma_i} \circ F_{i, \nu_i^n}^n), \rho(P_{\beta, \mu_i^n}^{\gamma_i})) = 0$ , which implies  $\lim d_{\sigma_i^n}^\infty(R_{[v_i^n]}^{\sigma_i^n}, \text{id}_{\sigma_i^n}) = 0$  according to (4.14) and (4.15), and thus:

$$\lim d_{\sigma_i^n}^\infty(P_{\beta, \nu_i^n}^{\sigma_i^n}, P_{\beta, \mu_i^n}^{\sigma_i^n}) = 0$$

with  $\sigma_i^n = \mathcal{F}_\alpha^{\mu_i^n}(0) = \mathcal{F}_\alpha^{\nu_i^n}(0)$ . Lemma 4.5 shows then that  $\lim d(\nu_i^n, \mu_i^n) = 0$  with  $d$  a distance on  $\mathcal{S}(\mathbf{T}^2, 0, \theta)$ , and  $\nu_i^n$  converges thus to  $\mu_i$ .

Since  $\sigma_1^n = \mathcal{F}_\alpha^{\nu_1^n}(0)$  and  $\sigma_2^n = \mathcal{F}_\alpha^{\nu_2^n}(0)$  are closed and freely homotopic and  $A^+(\mathcal{F}_\beta^{\nu_1^n}) = A^+(\mathcal{F}_\beta^{\nu_2^n})$ , Theorem D shows then that  $[\nu_1^n] = [\nu_2^n]$  in  $\text{Def}_\theta(\mathbf{T}^2, 0)$ . But  $\nu_i^n$  converges to  $\mu_i$  in  $\mathcal{S}(\mathbf{T}^2, 0, \theta)$ , and the sequence  $[\nu_1^n] = [\nu_2^n] \in \text{Def}_\theta(\mathbf{T}^2, 0)$  converges thus both to  $[\mu_1]$  and to  $[\mu_2]$ . Since  $\text{Def}_\theta(\mathbf{T}^2, 0)$  is Hausdorff in the neighbourhood of  $[\mu_1]$  and  $[\mu_2]$  (see Proposition 3.36), this shows that  $[\mu_1] = [\mu_2]$  and concludes the proof of Theorem A.

## APPENDIX A. SIMPLE CLOSED DEFINITE GEODESICS IN SINGULAR CONSTANT CURVATURE LORENTZIAN SURFACES

The main goal of this appendix is to prove the existence of simple closed definite geodesics in any closed constant curvature singular Lorentzian surface. This appendix is entirely independent from the rest of the paper, and the reader may thus choose to use Theorem A.1 below as a ‘‘black-box’’ in a first reading and come back to its proof later on. We will work in this section in the general setting of singular  $\mathbf{X}$ -surfaces,  $(\mathbf{G}, \mathbf{X})$  denoting as usual the pair  $(\text{PSL}_2(\mathbb{R}), \mathbf{dS}^2)$  or  $(\mathbb{R}^{1,1} \rtimes \text{SO}^0(1, 1), \mathbb{R}^{1,1})$ . *Geodesics* of singular  $\mathbf{X}$ -surfaces will be defined below in Definition A.5. The goal of the section is to prove the following existence result, which will be a direct consequence of the Proposition A.16 and the Theorem A.23 proved below.

**Theorem A.1.** *Let  $\mu_1$  and  $\mu_2$  be two class A singular  $\mathbf{X}$ -structures on a closed surface  $S$ , having identical oriented lightlike bi-foliations. Then  $\mu_1$  and  $\mu_2$  admit freely homotopic simple closed timelike geodesics, which are not null-homotopic. The obvious analogous statement holds for the spacelike signature.*

The existence of closed definite geodesics is known for regular Lorentzian surfaces, see for instance [Tip79, Gal86, Suh13]. However, it is *a priori* not clear that the usual tools and results of Lorentzian geometry can be used in our singular setting. Most notions and results of this section are classical in the setting of regular Lorentzian manifolds, and our goal is precisely to show that this toolbox persists in the setting of singular  $\mathbf{X}$ -surfaces – which is likely to have an independent interest in the future for their further study. Most proofs for classical results are adapted from [Min19] or [BEE96]. We then essentially follow the proof of [Tip79] to show Theorem A.1, with slight adaptations more suited to our setting. The main idea is to prove the existence of a simple closed timelike curve which *maximizes* the Lorentzian length, which is the extremal property of Lorentzian timelike geodesics in contrast with Riemannian ones.

**A.1. Geodesics of singular  $\mathbf{X}$ -surfaces.** We begin by defining the natural notion of geodesic in a singular  $\mathbf{X}$ -surface, and by recalling some easy facts on geodesics of  $\mathbf{X}$ . We call *affine structure* on an oriented topological one-dimensional manifold a maximal atlas of charts to  $\mathbb{R}$  whose transition maps are affine and orientation-preserving, in other words an  $(\text{Aff}^+(\mathbb{R}), \mathbb{R})$ -structure with  $\text{Aff}^+(\mathbb{R}) \simeq \mathbb{R} \times \mathbb{R}$  the group of affine transformations  $\lambda \text{id} + u: x \mapsto \lambda x + u$  of  $\mathbb{R}$  (with  $\lambda \in \mathbb{R}_+^*$ ,  $u \in \mathbb{R}$ ). We recall that geodesics of  $\mathbf{X}$  have a fixed signature, and that lightlike geodesics are simply the lightlike leaves. Geodesics moreover have a natural affine structure (as for any affine connection), given by the parametrizations satisfying the geodesic equation. For  $\mathbf{X} = \mathbb{R}^{1,1}$ , the affinely parametrized geodesics are simply the affinely parametrized affine segments. The following characterization for  $\mathbf{X} = \mathbf{dS}^2$  follows from straightforward computations.

**Lemma A.2.** *Let  $\gamma$  be a geodesic of  $\mathbf{X}$ .*

- (1) *The stabilizer of  $\gamma$  in  $\mathbf{G}$  acts transitively on  $\gamma$ . For  $\mathbf{X} = \mathbf{dS}^2$ , it is moreover:*
  - (a) *a hyperbolic one-parameter group if  $\gamma$  is timelike,*
  - (b) *an elliptic one-parameter group if  $\gamma$  is spacelike,*
  - (c) *a parabolic subgroup (i.e. conjugated to a triangular subgroup) if  $\gamma$  is lightlike.*
- (2) *There exists for any  $x \in \gamma$  a one-parameter subgroup ( $g^t$ ) stabilizing  $\gamma$  and acting freely at  $x$ , and  $t \in \mathbb{R} \mapsto g^t(x) \in \gamma$  is then an affine parametrization of an open subset of  $\gamma$ .*

*Proof.* (1) For  $\mathbf{X} = \mathbf{dS}^2$  we can work with the hyperboloid model  $\mathbf{dS}^2$  thanks to Lemma 2.2. The stabilizer of a plane  $P \subset \mathbb{R}^{1,2}$  is also the one of its orthogonal for  $q_{1,2}$ , which is respectively spacelike, timelike and lightlike in the three above cases. Straightforward computation show that these line stabilizers are of the announced form. The transitivity of the stabilizer on  $\gamma$  is checked by a direct computation (in the case of  $\mathbf{dS}^2$ , observe that  $\text{Stab}_{\text{PSL}_2(\mathbb{R})}(\gamma)$  preserves each connected component of  $P \cap \mathbf{dS}^2$ ).

(2) This fact follows easily from the identification of  $\mathbf{X}$  with  $\mathbf{G}/A$ . □

We observe now that any affinely parametrized geodesic interval  $\gamma: I \rightarrow \mathbf{X}$  passing through  $\mathfrak{o}$  avoids a quadrant, and without loss of generality we can thus assume that  $\gamma(I) \cap \mathcal{F}_\alpha^{+*}(\mathfrak{o}) = \emptyset$ . Using the projection  $\pi_\theta: \mathbf{X}_* \rightarrow \mathbf{X}_\theta$  introduced in paragraph 2.2.1 for the standard  $\mathbf{X}_\theta$ -cone,  $\pi_\theta \circ \gamma: I \rightarrow \mathbf{X}_\theta$  is in particular well-defined, and will be called an *affinely parametrized geodesic* of  $\mathbf{X}_\theta$ . Observe moreover that the orientations of time and space induce a natural notion of *future timelike* and *spacelike* geodesics in any  $\mathbf{X}$ -surface (the one whose derivative is future-pointing). This notion persists in  $\mathbf{X}_\theta$  by saying that a geodesic is *future timelike* (resp. *spacelike*) if it is the projection of such a geodesic of  $\mathbf{X}$ . For lightlike geodesics namely lightlike leaves, the *future* orientation is the positive orientation of the foliation.

**Lemma A.3.** (1) *Singular  $\mathbf{X}$ -charts of  $\mathbf{X}_\theta$  at  $\mathfrak{o}_\theta$  (equivalently homeomorphisms defined on a neighbourhood of  $\mathfrak{o}_\theta$  and fixing it, which are isometries on its complement) send future affinely parametrized geodesics of  $\mathbf{X}_\theta$  to future affinely parametrized geodesics of the same signature.*

- (2) Let  $\gamma$  be a parametrized curve of a singular  $\mathbf{X}$ -surface  $S$  passing through a singularity  $x$ . Then  $\gamma$  is mapped to a future affinely parametrized geodesic of  $\mathbf{X}_\theta$  by a singular  $\mathbf{X}$ -chart of  $S$  at  $x$ , if and only if it is mapped to a future affinely parametrized geodesic of the same signature in any singular  $\mathbf{X}$ -chart at  $x$ .

*Proof.* (1) According to Proposition 2.26, the singular  $\mathbf{X}$ -charts of  $\mathbf{X}_\theta$  at  $\mathfrak{o}_\theta$  are the maps  $\bar{a}$  induced by elements  $a \in \text{Stab}(\mathfrak{o})$  and characterized by  $\bar{a} \circ \pi_\theta = \pi_\theta \circ a$ . Since  $a$  preserve the affine structure of any geodesic of  $\mathbf{X}$  and its future orientation,  $\bar{a} \circ \pi_\theta \circ \gamma = \pi_\theta \circ a \circ \gamma$  is thus a future affinely parametrized parametrization for any future affinely parametrized geodesic  $\gamma$ .

(2) This is a direct consequence of (1).  $\square$

*Remark A.4.* The fundamental consequence of Lemma A.3 is that, contrarily to the Riemannian case, the notion of *straight* geodesic segment through a singular point *always* makes sense in a singular Lorentzian surface. In other words, every future timelike geodesic segment  $I^-$  converging to a singular point  $x$  has a preferred associated timelike geodesic segment  $I^+$  arising from  $x$ , such that  $I^- \cup I^+$  is a geodesic through  $x$ . This is a new manifestation of the higher rigidity of singular Lorentzian surfaces compared to their Riemannian counterparts.

**Definition A.5.** A (*maximal*) *affinely parametrized geodesic* of a singular  $\mathbf{X}$ -surface  $(S, \Sigma)$  is a (*maximal*) parametrized curve  $\gamma$ , which maps in any regular (respectively singular) chart of the singular  $\mathbf{X}$ -atlas of  $S$  to a future affinely parametrized geodesic interval of  $\mathbf{X}$  (resp. of  $\mathbf{X}_\theta$ ). The *signature* (timelike, spacelike or lightlike) of  $\gamma$  is the one of any of its images in such an atlas. The *affine structure* of  $\gamma$  is the one given by its parametrization. A *geodesic* is the image of a parametrized geodesic.

*Remark A.6.* There is also of course a natural notion of *piecewise geodesic* of a singular  $\mathbf{X}$ -surface  $(S, \Sigma)$ , namely a curve  $\gamma$  such that any connected component of  $\gamma \setminus \Sigma$  is a geodesic of  $S \setminus \Sigma$ . However this notion will not play any role in this text.

**Proposition A.7.** Let  $(S, \Sigma)$  be a singular  $\mathbf{X}$ -surface.

- (1) Geodesics of  $S$  are  $\mathcal{C}^0$  one-dimensional submanifolds, which are  $\mathcal{C}^\infty$  in  $S \setminus \Sigma$ .
- (2) Any geodesic of  $S$  is contained in a unique maximal geodesic.
- (3) Any point  $x \in S$  admits a connected open neighbourhood  $U$  homeomorphic to a disk, and such that:
  - (a)  $U$  is the domain of a chart of the singular  $\mathbf{X}$ -atlas centered at  $x$ ;
  - (b)  $U$  is the domain of a  $\mathcal{C}^0$  simultaneous foliated chart of the lightlike foliations;
  - (c)  $U \setminus (\mathcal{F}_\alpha(x) \cup \mathcal{F}_\beta(x))$  has four connected components, called the (open) quadrants of  $U$  at  $x$ ;
  - (d) for any two points  $y \neq z \in U$ , there exists a unique geodesic segment  $[y; z]_U \subset U$  of endpoints  $y$  and  $z$ , and  $[y; z]_U$  is moreover disjoint from (at least) one of the open quadrants at  $x$ .

Such an  $U$  will be called a *normal convex neighbourhood* of  $x$ . Moreover quadrants are themselves convex, i.e. if  $y, z$  are in a same open quadrant  $Q$  of  $U$  at  $x$ , then  $[y; z]_U \subset Q$ .

A quadrant of  $U$  will be said *future timelike* and denoted by  $U^+$  (respectively *past timelike*  $U^-$ ) according to the notations of Figure 2.1, namely if it is crossed by a future timelike geodesic segment starting at  $x$  of the same signature. Obvious similar denominations are used for spacelike and causal quadrants.

*Proof of Proposition A.7.* (1) This is immediate from the definition.

(2) This claim is of course true in  $\mathbf{X}$  and thus on  $S \setminus \Sigma$ , and we only have to prove it at a singular point  $x \in \Sigma$ . Namely for two geodesics  $\gamma_1, \gamma_2$  such that  $\gamma_1 \cap \gamma_2$  contains a geodesic interval  $I$  having  $x$  as one of its endpoints, we want to prove that  $\gamma_1 \cap \gamma_2$  contains a geodesic interval  $J$  containing  $x$  in its interior. With  $\varphi: U \rightarrow \mathbf{X}_\theta$  a singular  $\mathbf{X}$ -chart at  $x$ ,  $\varphi(I)$  is contained in the projection in  $\mathbf{X}_\theta$  of a maximal geodesic  $C$  of  $\mathbf{X}$ . In the same way,  $\varphi(\gamma_1)$  (resp.  $\varphi(\gamma_2)$ ) is contained in the projection of a maximal geodesic of  $\mathbf{X}$  which intersects  $C$  on the open interval  $\pi_\theta^{-1}(C)$ , but  $C$  is the only such geodesic of  $\mathbf{X}$ . In the end  $\varphi(\gamma_1 \cap \gamma_2)$  contains some neighbourhood of  $x$  in  $\pi_\theta(C)$ , which proves our claim.

(3) This claim is easily proved in  $\mathbf{X}$ , and thus on  $S \setminus \Sigma$  by using a standard normal convex neighbourhood. On the other hand (2) proves it on the neighbourhood of a singular point.  $\square$

Geodesics are natural in the following sense.

**Lemma A.8.** *Let  $f: S_1 \rightarrow S_2$  be an isometry of singular  $\mathbf{X}$ -surfaces. Then for any affine parametrization of a geodesic  $\gamma$  of  $S_1$ ,  $f \circ \gamma$  is an affine parametrization of a geodesic of  $S_2$  of the same type that  $\gamma$ .*

*Proof.* If  $(\varphi_i: U_i \rightarrow V_i)_i$  is a singular  $\mathbf{X}$ -atlas of  $S_1$ , then  $(\varphi_i \circ f^{-1}: f^{-1}(U_i) \rightarrow V_i)_i$  is a singular  $\mathbf{X}$ -atlas of  $S_2$ . Hence  $\gamma$  and  $f \circ \gamma$  read as the same path in these singular  $\mathbf{X}$ -atlases, which directly implies the claim.  $\square$

**A.2. Timelike curves and causality notions.** The following definition is identical to the classical one, to the exception of condition (1) handling the singular points.

**Definition A.9.** In a singular  $\mathbf{X}$ -surface  $(S, \Sigma)$ , a *timelike* (respectively *causal*, *spacelike*) curve is a continuous curve  $\sigma: [a; b] \rightarrow S$  such that:

- (1) for any  $t_0 \in [a; b]$  for which  $\gamma(t_0) \in \Sigma$ , there exists  $\varepsilon > 0$  and a normal convex neighbourhood  $U$  of  $\gamma(t_0)$  such that  $\gamma|_{]t_0-\varepsilon; t_0[} \subset U^-$  and  $\gamma|_{]t_0; t_0+\varepsilon[} \subset U^+$ , with  $U^-$  and  $U^+$  the past and future timelike (resp. spacelike, causal) quadrants in  $U$ ;
- (2)  $\sigma$  is locally Lipschitz;
- (3)  $\sigma'(t)$  is almost everywhere non-zero, future-directed and timelike (resp. causal, spacelike).

We emphasize that timelike, causal and spacelike curves are in particular always assumed to be relatively compact and future-oriented, unless explicitly stated otherwise. They are moreover not *trivial* (i.e. reduced to a point), and  $\sigma^{-1}(\Sigma)$  is discrete according to (1), hence finite.  $S$  will always be endowed with an auxiliary  $C^\infty$  Riemannian metric  $h$  and its induced distance  $d$ , with respect to which the Lipschitz conditions are considered. Note that  $\sigma$  is compact and locally Lipschitz, hence Lipschitz. A locally Lipschitz function being almost everywhere differentiable according to Rademacher's Theorem,  $\sigma'(t)$  is almost everywhere defined which gives sense to the condition (3). *Past* timelike, causal and spacelike curves are defined as future-oriented curves of the same signature traversed in the opposite direction.

**Definition A.10.** In a singular  $\mathbf{X}$ -surface  $S$ , we denote for  $x \in S$  by:

- (1)  $I^+(x)$  (respectively  $I^-(x)$ ) the set of points that can be reached from  $x$  by a timelike (resp. past timelike) curve;
- (2)  $J^+(x)$  (respectively  $J^-(x)$ ) the set of points that can be reached from  $x$  by a causal (resp. past causal) curve.

We will denote  $I_S^+(x)$  and likewise for the other notions, to specify that the curves are assumed to be contained in  $S$ . An open set  $U$  of a singular  $\mathbf{X}$ -surface  $S$  is *causally convex* if there exists no causal curve of  $S$  which intersects  $U$  in a disconnected set.  $S$  is said *strongly causal* if any point of  $S$  admits arbitrarily small causally convex open neighbourhoods. In particular  $S$  is then *causal*, i.e. admits no closed causal curves.  $S$  is *globally hyperbolic* if it is strongly causal, and if for any  $p, q \in S$ ,  $J^+(p) \cap J^-(q)$  is relatively compact.

Observe that for any convex normal neighbourhood  $U$  of  $x$  of future and past timelike quadrant  $U^+$  and  $U^-$ ,  $I_U^\pm(x) = U^\pm$ . This is classical in the regular Lorentzian setting (see for instance [Min19, Theorem 2.9 p.29]) and follows from our definition of timelike and causal curves at a singular point. Observe moreover that a  $\mathbf{X}$ -structure on  $\mathbb{R}^2$  has no closed lightlike leaves, as a consequence of the classical Poincaré-Hopf theorem for topological foliations proved for instance in [HH86, Theorem 2.4.6]. The two following results are well-known for regular Lorentzian metrics on  $\mathbb{R}^2$ , and the proofs respectively given in [BEE96, Proposition 3.42 and Corollary 3.44] persist in our singular setting. We repeat below a quick version of these proofs, and refer to the above reference for more details.

**Lemma A.11.** *In a singular  $\mathbf{X}$ -surface homeomorphic to  $\mathbb{R}^2$ , a timelike curve intersects a given lightlike leaf at most once.*

*Proof.* We henceforth endow  $\mathbb{R}^2$  with a singular  $\mathbf{X}$ -structure, and assume by contradiction the existence of a timelike curve intersecting a given lightlike leaf at least twice. It is well-known that timelike curves of regular Lorentzian manifolds are locally injective, and on the other hand it follows readily from our definition of a timelike curve  $\sigma$  in a singular  $\mathbf{X}$ -surface that it is also locally injective at a singularity (since  $\sigma^{-1}(\Sigma)$  is discrete). Hence timelike curves are locally injective (see also Fact A.20 below for an alternative proof), and it is thus easy to reduce the proof to the case of an injective timelike curve  $\sigma: [a; b] \rightarrow \mathbb{R}^2$  such that  $\sigma(a)$  and  $\sigma(b)$  belong to a leaf  $F$  of the, say  $\alpha$ -foliation, and such that  $\sigma|_{]a; b[}$  does not intersect  $F$ . Traversing  $\sigma$  from  $\sigma(a)$  to  $\sigma(b)$  followed by  $F$  from  $\sigma(b)$  to  $\sigma(a)$  (in the negative direction due to our orientation conventions) defines then a Jordan curve in  $\mathbb{R}^2$ , bounding a unique compact subset  $E \subset \mathbb{R}^2$ . Since  $\sigma$  is timelike for any  $t_1 \in ]a; b[$ ,  $\sigma(t)$  admits a punctured neighbourhood in  $\mathcal{F}_\alpha^+(\sigma(t))$  contained in  $\text{Int}(E)$ , and the first point of  $\mathcal{F}_\alpha^+(\sigma(t))$  contained in  $\partial E$  is then necessarily of the form  $\sigma(t'_1)$  for  $t'_1 \in ]t; b[$  (the existence of  $t'_1$  follows from the existence of foliated charts and from the compactness of  $E$ ). From there we construct recursively  $t_{n+1} = \frac{t_n + t'_n}{2}$ , and using the same notations we obtain sequences  $t_n, t'_n \in ]a; b[$  converging to a same point  $t_0 \in ]a; b[$  and such that  $t_n < t_{n+1} < t'_{n+1} < t'_n$ . Hence for  $n$  big enough,  $\sigma([t_n; t'_n])$  is contained in a normal convex neighbourhood  $U$  of  $\sigma(t'_n)$ , and  $\sigma(t'_n) \in \mathcal{F}_\alpha(\sigma(t_n)) \cap I_U^+(\sigma(t_n))$ . But we have seen that  $I_U^+(\sigma(t_n))$  is the future timelike quadrant  $U^+$ , which does not contain any point of  $\mathcal{F}_\alpha(\sigma(t_n))$ . This contradiction concludes the proof.  $\square$

Lemma A.11 implies in particular that for any lightlike leaf  $F$  of a  $\mathbf{X}$ -structure on  $\mathbb{R}^2$ ,  $F$  has for any  $x \in F$  a transversal  $T$  to the foliation of  $F$  (namely a timelike curve through  $x$ ) intersecting  $F$  only at  $x$ . This means by definition that the lightlike leaves of a  $\mathbf{X}$ -structure on  $\mathbb{R}^2$  are *proper*.

**Corollary A.12.** *In a singular  $\mathbf{X}$ -surface homeomorphic to  $\mathbb{R}^2$ , two distinct lightlike leaves intersect at most once.*

*Proof.* Assume by contradiction that two distinct lightlike leaves intersect at least two times. Then these are necessarily leaves  $F_\alpha$  and  $F_\beta$  of distinct lightlike foliations, and there exists  $x, y \in F_\alpha \cap F_\beta$  such that the open intervals  $]x; y[_{\alpha/\beta}$  of  $F_{\alpha/\beta}$  from  $x$  to  $y$  are disjoint. To fix the ideas we furthermore assume that these intervals are positively oriented, which can be achieved by inverting the orientations. The curve  $J$  formed by following  $]x; y[_\alpha$  from  $x$  to  $y$  and then  $]x; y]_\beta$  from  $y$  to  $x$  is then a Jordan curve of the  $\mathbf{X}$ -surface  $S \simeq \mathbb{R}^2$ , bounding a unique compact domain  $E$ . With  $\gamma$  a timelike curve starting from  $x$ ,  $\gamma$  enters  $E$  and cannot leave it, or it would intersect  $\partial E = F_\alpha \cup F_\beta$  and contradict Lemma A.11. We can obviously extend  $\gamma$  at its endpoint to a larger point, and we obtain thus timelike curves of arbitrarily large arclength with respect to a fixed Riemannian metric, and contained in  $E$ . But since  $E$  is compact, this contradicts the Fact A.20 which will be independently proved below, and conclude the proof.  $\square$

**Corollary A.13.** *Any singular  $\mathbf{X}$ -surface homeomorphic to  $\mathbb{R}^2$  is strongly causal.*

*Proof.* Assume by contradiction that a singular  $\mathbf{X}$ -structure on  $\mathbb{R}^2$  is not strongly causal. Then there exists a point  $x \in \mathbb{R}^2$ , a normal convex neighbourhood  $U$  of  $x$ , and a causal curve starting from  $x$ , leaving  $U$  and returning to it. It is easy to deform this curve to a timelike curve  $\sigma$  with the same properties. We can moreover choose the boundary of  $U$  to be the union of lightlike segments, and denote by  $I$  one of these segments which is first met by  $\sigma$  when it leaves  $U$ . We can then clearly extend  $\sigma$  if necessary, for it to be a timelike curve intersecting  $I$  twice. This contradicts Lemma A.11 and concludes the proof.  $\square$

**Corollary A.14.** *A singular  $\mathbf{X}$ -surface of universal cover homeomorphic to  $\mathbb{R}^2$  does not admit any null-homotopic closed causal curve.*

*Proof.* Indeed, such a null-homotopic closed causal curve would lift to a closed causal curve of a singular  $\mathbf{X}$ -structure on  $\mathbb{R}^2$ , contradicting Corollary A.13.  $\square$

We recall that for  $S$  a closed singular  $\mathbf{X}$ -surface, a line  $l$  in  $H_1(S, \mathbb{R}) \simeq \mathbb{R}^2$  is said *rational* if it passes through  $H_1(S, \mathbb{Z}^2) \simeq \mathbb{Z}^2$ , and *irrational* otherwise.

**Lemma A.15.** *A closed singular  $\mathbf{X}$ -surface  $S$  is class B if, and only if both of its lightlike foliations have closed leaves which are freely homotopic up to orientation, and is class A otherwise.*

In particular, if one of the lightlike foliations has irrational projective asymptotic cycle, then  $S$  is class A.

*Proof.* If the lightlike foliations have closed leaves which are not freely homotopic up to orientation, then since two primitive element  $c_\alpha \neq \pm c_\beta$  of  $\pi_1(\mathbf{T}^2)$  are not proportional in  $H_1(\mathbf{T}^2, \mathbb{R})$ , the projective asymptotic cycles are distinct according to Lemma 3.25. If only one of the lightlike foliations has a closed leaf, then it has a rational projective asymptotic cycle while the other lightlike foliation has an irrational cycle, hence  $A(\mathcal{F}_\alpha) \neq A(\mathcal{F}_\beta)$ .

If none of the lightlike foliations have closed leaves, then none of them has a Reeb component, hence both of them is a suspension of a homeomorphism with irrational rotation number according to Proposition 3.26. The latter is a  $\mathcal{C}^2$  diffeomorphism with breaks and is thus minimal according to Lemma 2.30.(4). By definition (3.13) of the asymptotic cycle, and because any line of  $H_1(\mathbf{T}^2, \mathbb{R})$  is the limit of a sequence of rational lines, there exists a smooth simple closed curve  $a$  representing  $A(\mathcal{F}_\alpha)$  and as close as one wants to a segment of a leave of  $\mathcal{F}_\alpha$ . In particular we can assume  $a$  to be transverse to  $\mathcal{F}_\beta$ . Moreover  $a$  meets all the leaves of  $\mathcal{F}_\beta$  since the latter is minimal, and  $\mathcal{F}_\beta$  is therefore the suspension of a homeomorphism of  $a$ . There exists thus a simple closed curve  $b$  representing  $A(\mathcal{F}_\beta)$ , whose class generates  $H_1(\mathbf{T}^2, \mathbb{R})$  together with  $[a]$ . In particular  $\mathbb{R}[a] \neq \mathbb{R}[b]$ , which shows that  $A(\mathcal{F}_\alpha) \neq A(\mathcal{F}_\beta)$  and concludes the proof of the Lemma.  $\square$

The oriented projective asymptotic cycles of the lightlike foliations of a class A singular  $\mathbf{X}$ -surface  $(S, \mu)$  delimit an open *timelike cone*

$$(A.1) \quad \mathcal{C}_\mu = \text{Int}(\text{conv}(A^+(\mathcal{F}_\beta) \cup (-A^+(\mathcal{F}_\alpha)))) \subset H_1(S, \mathbb{R})$$

in the homology, and likewise a *spacelike cone*  $\mathcal{C}_\mu^{\text{space}} = \text{Int}(\text{conv}(A^+(\mathcal{F}_\alpha) \cup A^+(\mathcal{F}_\beta)))$ .

**Proposition A.16.** *Let  $\mu_1$  and  $\mu_2$  be two class A singular  $\mathbf{X}$ -structures on a closed surface  $S$  with identical oriented lightlike bi-foliations. Then for any  $x \in S$  we have the following.*

- (1)  $\mu_1$  and  $\mu_2$  admit freely homotopic simple closed timelike curves passing through  $x$  which are not null-homotopic.
- (2) For any simple closed timelike curve  $a$  of  $\mu_1$  (respectively  $\mu_2$ ), there exists a simple closed spacelike curve  $b$  intersecting  $a$  in a single point.
- (3) There exists simple closed timelike and spacelike curves  $(a_1, b_1)$  of  $\mu_1$  (resp.  $(a_2, b_2)$  of  $\mu_2$ ) such that  $a_1$  is freely homotopic to  $a_2$ ,  $b_1$  freely homotopic to  $b_2$ , and  $([a_i], [b_i])$  is a basis of  $\pi_1(\mathbf{T}^2)$  for  $i = 1$  and  $2$ .

*Proof.* (1) Since  $S$  is homeomorphic to a torus we let  $S$  be equal to  $\mathbf{T}^2$  to fix the ideas, identify the action of  $\pi_1(\mathbf{T}^2)$  on the universal cover  $\mathbb{R}^2$  with the translation action of  $\mathbb{Z}^2$ , and endow  $\mathbb{R}^2$  with the induced singular  $\mathbf{X}$ -structures  $\tilde{\mu}_i$  and a  $\mathbb{Z}^2$ -invariant auxiliary Riemannian metric. With  $\mathcal{F}_\alpha$  and  $\mathcal{F}_\beta$  the common lightlike foliations of  $\tilde{\mu}_1$  and  $\tilde{\mu}_2$ , the half-leaves  $\mathcal{F}_\beta^+(p)$  and  $\mathcal{F}_\alpha^-(p)$  are for any  $p \in \mathbb{R}^2$  proper embeddings of  $\mathbb{R}^+$ . They intersect furthermore only at  $p$  according to Corollary A.12, and delimit thus a compact subset  $C_p \subset \mathbb{R}^2$  of boundary  $\mathcal{F}_\alpha^-(p) \cup \mathcal{F}_\beta^+(p)$ , containing all the timelike curves emanating from  $p$ . On the other hand there exists a constant  $K > 0$  such that for any  $p \in \mathbb{R}^2$ ,  $\mathcal{F}_\alpha(p)$  and  $\mathcal{F}_\beta(p)$  are respectively contained in the  $K$ -neighbourhoods of the affine lines  $p + A(\mathcal{F}_\alpha)$  and  $p + A(\mathcal{F}_\beta)$ . This property follows from the equivalence between asymptotic cycles and winding numbers [Sch57, p. 278], and is also very well explained in [Suh13, §3.1]. In particular, there exists thus an affine sub-cone  $\mathcal{C}'$  of non-empty interior of the timelike cone  $\mathcal{C}_{\mu_1} = \mathcal{C}_{\mu_2}$  in homology defined in (A.1), such that  $x + \mathcal{C}' \subset \text{Int}(C_x)$  for any  $x \in \mathbb{R}^2$ . We have then  $x + c \in \text{Int}(C_x)$ , and thus  $x + c \notin \mathcal{F}_\alpha(x) \cup \mathcal{F}_\beta(x)$ . Moreover the half-leaves  $\mathcal{F}_\beta^-(x + c)$  and  $\mathcal{F}_\alpha^-(x)$  intersect, at a unique point  $y$  according to Corollary A.12, and  $y \notin \{x, x + c\}$  since  $x + c \notin \mathcal{F}_\alpha(x) \cup \mathcal{F}_\beta(x)$ .

The piecewise lightlike curve  $\tilde{\nu}$  of  $(\mathbb{R}^2, \tilde{\mu}_i)$  ( $i = 1$  or  $2$ ) from  $x$  to  $x + c$  formed by following  $\mathcal{F}_\alpha^-(x)$  from  $x$  to  $y$  and then  $\mathcal{F}_\beta^+(y)$  from  $y$  to  $x + c$  is then contained in the closure of the cone  $C_x \subset \mathbb{R}^2$ , and is in particular *not* entirely contained in a lightlike leaf  $\mathcal{F}_\alpha(x)$  or  $\mathcal{F}_\beta(x + c)$  since  $y \notin \{x, x + c\}$ . Let  $\nu$  be its projection to  $\mathbf{T}^2$ , which a piecewise lightlike closed curve for  $\mu_1$  and  $\mu_2$ . Let  $t = \sup \left\{ s \in [0; 1] \mid \nu|_{[0; s[} \text{ is injective} \right\}$  (note that  $t > 0$  since the lightlike leaves are proper)

so that  $\nu(t)$  is the first self-intersection point of  $\nu$  with itself, and let  $u \in [0; t[$  be the unique time for which  $\nu(t) = \nu(u)$ . If  $u = 0$ , *i.e.*  $\nu(t) = \nu(u) = \nu(0)$ , then we define  $\sigma = \nu|_{[0; t]}$ . If not, then we define  $\nu_1$  as the curve constituted by  $\nu|_{[0; u]}$  followed by  $\nu|_{[t; 1]}$ , and repeat the process on  $\nu_1$ . Using for instance Fact A.20 to be proved below, there exists  $\varepsilon > 0$  such that for any  $s \in [0; 1]$ ,  $\nu|_{]s-\varepsilon; s+\varepsilon[}$  is injective. Hence this process finishes in a finite number of steps by compactness of  $\nu$ , and yields a simple subcurve  $\sigma$  of  $\nu$  passing through  $\bar{x} \in \mathbf{T}^2$ .

Since the piecewise lightlike simple closed curve  $\sigma$  of  $\mu_i$  is not entirely contained in one lightlike leaf, it can be slightly deformed to a freely homotopic timelike curve  $\eta_i$  of the singular  $\mathbf{X}$ -structure  $\mu_i$ , passing through  $\bar{x}$  and homotopic to  $\sigma$ . The freely homotopic simple closed timelike curves  $\eta_1$  and  $\eta_2$  of  $\mu_1$  and  $\mu_2$  through  $\bar{x}$  cannot be null-homotopic according to Corollary A.14, which concludes the proof.

(2) Let  $\mathcal{C}'$  be the sub-cone introduced previously of the future spacelike cone  $\mathcal{C}_S^{\text{space}}$  in homology, such that  $p + \mathcal{C}' \subset \text{Int}(C_p^{\text{space}})$  for any  $p \in \mathbb{R}^2$  with  $C_p^{\text{space}} \subset \mathbb{R}^2$  the compact subset of boundary  $\mathcal{F}_\alpha^+(p) \cup \mathcal{F}_\beta^+(p)$ . Then there exists a free homotopy class  $c \in \pi_1(S)$  contained in  $\mathcal{C}'$  and of algebraic intersection number  $\hat{i}(c, [a]) = 1$  with  $[a]$ . The proof of the first claim of the Proposition yields a closed piecewise lightlike curve  $\nu$  through  $x = a(0)$  in the free homotopy class  $c$ , not entirely contained in a single lightlike leaf. Since  $\nu$  and  $a$  intersect only transversally, and with a positive sign according to our orientations conventions (see Figure 2.1),  $\hat{i}([\nu], [a]) = 1$  implies moreover that  $\nu$  and  $a$  intersect actually only at  $x$ . With  $\sigma$  the simple closed subcurve of  $\nu$  through  $x$  constructed in the first part of the proof,  $a$  and  $\sigma$  intersect then only at  $x = a(0) = \sigma(0)$ , and have in particular algebraic intersection number  $\hat{i}([\sigma], [a]) = 1$ . Since  $\sigma$  is not entirely contained in a single lightlike leaf, we can now as before slightly deform  $\sigma$  to a freely homotopic simple closed spacelike curve  $b$ , intersecting  $a$  only at  $x$ .

(3) This last claim is a direct consequence of the two first ones.  $\square$

**A.3. Lorentzian length, time-separation and maximizing causal curves.** We define the *Lorentzian length* of a causal curve  $\gamma: [0; l] \rightarrow S$  in a singular  $\mathbf{X}$ -surface  $(S, \Sigma)$  by

$$L(\gamma) := \int_0^l \sqrt{-\mu_S(\gamma'(t))} dt \in [0; +\infty].$$

Causal curves being almost everywhere differentiable (see paragraph A.2 for more details), this quantity is well-defined and moreover independent of the (locally Lipschitz) parametrization of  $\gamma$  thanks to the change of variable formula. An important remark to keep in mind for this whole paragraph is that singular points do not play any role in the length of a causal curve  $\gamma$  in  $S$ . Indeed since  $\gamma^{-1}(\Sigma)$  is finite,  $\gamma$  is the concatenation of a finite number  $n$  of *regular pieces*, namely the connected components  $\gamma_i$  of  $\gamma \cap S^* = S \setminus \Sigma$ , and we have

$$(A.2) \quad L(\gamma) = \sum_{i=1}^n L(\gamma_i),$$

the lengths appearing in the right-hand finite sum being computed in the regular Lorentzian surface  $S^*$ . The Lorentzian length allows us to define on  $S \times S$  the *time-separation function* by

$$(A.3) \quad \tau_S(x, y) := \sup_{\sigma} L_S(\sigma) \in [0; +\infty],$$

the sup being taken on all future causal curves in  $S$  going from  $x$  to  $y$  if such a curve exists (*i.e.* if  $y \in J^+(x)$ ), and by  $\tau_S(x, y) = 0$  otherwise. To avoid any confusion, we emphasize that, on the contrary to  $\tau_S$ , the Lorentzian length  $L(\gamma)$  computed in any open subset  $U \subset S$  of course agrees with the one computed in  $S$ , which is why we do not bother to specify  $S$  in the notation  $L(\gamma)$ .

**Lemma A.17.** *Let  $y \in J^+(x)$  and  $z \in J^+(y)$ , then  $\tau_S(x, z) \geq \tau_S(x, y) + \tau_S(y, z)$ .*

*Proof.* The same exact proof than in the regular setting (see for instance [Min19, Theorem 2.32]) works in our case, and we repeat it here for the reader to get a grasp of the Lorentzian specificities. If  $\tau(x, y)$  or  $\tau(y, z)$  is infinite, then one construct easily by concatenation of segments of them causal curves of arbitrarily large length going from  $x$  to  $z$ , which prove the inequality (with equality sign). Assume now that  $\tau(x, y)$  and  $\tau(y, z)$  are both finite, let  $\varepsilon > 0$  and  $\gamma, \sigma$  be

causal curves respectively from  $x$  to  $y$  and from  $y$  to  $z$  such that  $L(\gamma) \geq \tau_S(x, y) - \varepsilon$  and  $L(\sigma) \geq \tau_S(y, z) - \varepsilon$ . Then the causal curve  $\nu$  equal to the concatenation of  $\gamma$  and  $\sigma$  goes from  $x$  to  $z$ , hence  $\tau_S(x, z) \geq L(\nu) = L(\gamma) + L(\sigma) \geq \tau_S(x, y) + \tau_S(y, z) - 2\varepsilon$  by the definition of  $\tau_S$ , which proves the claim by letting  $\varepsilon$  converge to 0.  $\square$

It is important to keep in mind that all the usual inequalities, suprema and infima encountered in Riemannian geometry when dealing with lengths of curves and geodesics are exchanged in Lorentzian geometry (for causal curves), as the reverse triangle inequality of Lemma A.17 already showed. The best way to understand this phenomenon (confusing at first sight), is for the reader to explicitly check in the case of the Minkowski plane  $\mathbb{R}^{1,1}$  that timelike geodesics realize the maximal length of a causal curve between two points. We now generalize this observation in the following classical result.

**Proposition A.18.** *In a singular  $\mathbf{X}$ -surface  $S$ , a future causal curve  $\gamma: I \rightarrow S$  is geodesic up to reparametrization if, and only if it is locally maximizing, namely if for any  $t \in I$  there exists a connected neighbourhood  $I_t = [a_t; b_t]$  of  $t$  in  $I$  and a connected open neighbourhood  $U_t$  of  $\gamma(t)$  in  $S$ , such that  $\gamma(I_t) \subset U_t$  and*

$$L(\gamma|_{I_t}) = \tau_{U_t}(\gamma(a_t), \gamma(b_t)).$$

If  $I = [a; b]$  and  $L(\gamma) = \tau_S(\gamma(a), \gamma(b))$  then we say that  $\gamma$  is maximizing. In this case  $\gamma$  is in particular locally maximizing, hence a geodesic (of timelike signature if moreover  $L(\gamma) > 0$ ).

*Proof.* We first prove that a maximizing causal curve  $\gamma: [a; b] \rightarrow S$  is locally maximizing. For any  $a < t < b$  we have:

$$(A.4) \quad L(\gamma|_{[a;t]}) + L(\gamma|_{[t;b]}) = L(\gamma) = \tau_S(\gamma(a), \gamma(b)) \geq \tau_S(\gamma(a), \gamma(t)) + \tau_S(\gamma(t), \gamma(b))$$

according to the reverse triangular inequality (Lemma A.17). Since on the other hand  $L(\gamma|_{[a;t]}) \leq \tau_S(\gamma(a), \gamma(t))$  and  $L(\gamma|_{[t;b]}) \leq \tau_S(\gamma(t), \gamma(b))$  by the definition of  $\tau_S$ , both of the latter inequalities have to be equalities to match (A.4). Applying twice this argument to  $a_t \in [a; b]$  and then  $b_t \in [a_t; b]$  we obtain  $L(\gamma|_{[a_t; b_t]}) = \tau_S(\gamma(a_t), \gamma(b_t)) \geq \tau_{U_t}(\gamma(a_t), \gamma(b_t))$ , the latter inequality following from the definition of  $\tau$  as a supremum. On the other hand  $L(\gamma|_{[a_t; b_t]}) \leq \tau_{U_t}(\gamma(a_t), \gamma(b_t))$  by definition of  $\tau_{U_t}$ , hence  $L(\gamma|_{[a_t; b_t]}) = \tau_{U_t}(\gamma(a_t), \gamma(b_t))$ , i.e.  $\gamma$  is locally maximizing.

The first claim of the Proposition is classical for causal curves of regular Lorentzian manifolds, and is for instance proved in [Min19, Theorem 2.9 and 2.20]. Since a causal curve  $\gamma$  of a singular  $\mathbf{X}$ -surface is clearly locally maximizing (respectively geodesic) if, and only if every of its regular pieces  $\gamma_i$  appearing in (A.2) are so, this concludes the proof of the Proposition.  $\square$

The following result is well-known in the classical setting of regular Lorentzian manifolds where it falls as a particular case of the *Limit curve theorems*. We give here the main arguments of its proof to make it clear that it persists in our singular setting, referring for instance to [Min19, §2.11 and Theorem 2.53] for more details.

**Lemma A.19.** *Let  $\gamma_n$  be a sequence of causal curves in a globally hyperbolic singular  $\mathbf{X}$ -surface  $S$  joining two points  $x$  and  $y$ . The  $(\gamma_n)$  have then uniformly bounded arclength with respect to a fixed Riemannian metric  $h$  on  $S$ . Let  $\sigma_n$  denote the reparametrization of  $\gamma_n$  by  $h$ -arclength. Then there exists a causal curve  $\sigma$  from  $x$  to  $y$  and a subsequence  $\sigma_{n_k}$  of  $\sigma_n$  converging to  $\sigma$  in the  $C^0$ -topology. Moreover  $\limsup L(\sigma_{n_k}) \leq L(\sigma) < +\infty$ .*

*Proof.* The first important and classical fact is:

**Fact A.20.** *For any relatively compact normal convex neighbourhood  $U$  of a  $\mathbf{X}$ -surface  $S$  (not necessarily globally hyperbolic), causal curves contained in  $U$  are equi-Lipschitz, of uniformly bounded Riemannian length, and leave  $U$  in a uniform bounded time. Namely there exists a constant  $K > 0$ , a time-function  $f$  and a Riemannian metric  $h$  on  $U$ , such that for any causal curve  $\gamma$  in  $U$ :*

- (1)  $\gamma$  may be reparametrized by  $f$  to be  $K$ -Lipschitz,
- (2) with this reparametrization,  $\gamma$  leaves  $U$  in a time bounded by  $K$ ,
- (3) the arclength of  $\gamma$  for  $h$  is bounded by  $K$ .



*Proof.* We explain the main idea leading to these properties for a causal curve  $\gamma$  contained in a relatively compact normal convex neighbourhood  $U$  of  $p \in S^*$ , and refer to [BEE96, p.75] and [Min19, Theorem 1.35, Remark 1.36 and Theorem 2.12] for more details. Denoting by  $g$  the Lorentzian metric of  $S^*$ , let  $x = (x_1, x_2)$  be coordinates on  $U$  such that  $g_p(\partial x_1, \partial x_1) = -1$ ,  $g_p(\partial x_2, \partial x_2) = 1$  and  $g_p(\partial x_1, \partial x_2) = 0$ . Then there exists  $\varepsilon > 0$  such that, possibly shrinking  $U$  further around  $p$ , the timelike cones of the Lorentzian metric  $-(1+\varepsilon)dx_1^2 + dx_2^2$  of  $U$  strictly contain the causal cones of  $g$  (this is indeed true at  $p$  by assumption, hence on a neighbourhood of it by continuity of  $g$ ). Introducing the Riemannian metric  $h = dx_1^2 + dx_2^2$  on  $U$  and  $K_0 := \sqrt{2+\varepsilon} > 0$ , this inclusion translates as  $\|u\|_h < K_0 dx_1(u)$  for any  $g$ -causal vector  $u$ , hence as

$$(A.5) \quad \int_0^t \|\gamma'(t)\|_h < K_0(x_1(\gamma(t)) - x_1(\gamma(0)))$$

by integration. This last inequality shows that that the arclength of causal curves contained in  $U$  for  $h$  is uniformly bounded, that  $x_1$  is strictly increasing over them, hence that they leave  $U$  in a uniformly bounded time when reparametrized by  $x_1$ , and that they are moreover equi-Lipschitz for this reparametrization. Note that for any function  $f$  sufficiently close to  $x_1$ , the causal curves in  $U$  retain these uniform properties when reparametrized by  $f$  (possibly changing the constants).

To conclude the proof we only have to argue that these properties persist on the neighbourhood of a singular point  $p$ . We first consider normal convex neighbourhoods  $U^-$  and  $U^+$  in  $S^*$ , respectively avoiding the future and past timelike quadrants at  $p$  and such that  $U := U^- \cup U^+ \cup \{p\}$  is a neighbourhood of  $p$ . We next choose coordinates  $(x_1, x_2)$  on  $U$  so that  $x_1$  is sufficiently close to the respective functions  $x_1^\pm$  of the previous discussion on the neighbourhoods  $U^\pm$  for the uniform behaviours to happen. Property (1) of Definition A.9 implies then that  $x_1$  is strictly increasing on any causal curve  $\gamma$  in  $U$ , hence that  $\gamma$  leaves  $U$  in uniformly bounded time. When reparametrized by  $x_1$ , the causal curves of  $U$  are moreover clearly equi-Lipschitz and of uniform bounded length for a fixed Riemannian metric, since the inequality (A.5) does not take into account the singular point  $p$ .  $\square$

Since the causal curves  $\gamma_n$  join  $x$  to  $y$ , they are contained in  $J^+(x) \cap J^-(y)$  which is relatively compact by global hyperbolicity. Since  $S$  is in particular strongly causal by global hyperbolicity, we can moreover cover  $J^+(x) \cap J^-(y)$  by a finite number of normal convex neighbourhoods  $U_i$  which are causally convex. We reparametrize then each  $\gamma_n$  as given by Fact A.20 in any of the  $U_i$ , obtaining in this way an equi-Lipschitz family. Since each of the  $\gamma_n$  meets a given  $U_i$  at most once by causal convexity, Fact A.20 moreover indicates us that the arclength of the  $\gamma_n$  is uniformly bounded for a fixed Riemannian metric  $h$  on  $S$ . In particular, the sequence of causal curves  $\sigma_n: [0; a_n] \rightarrow S$  obtained by reparametrizing the  $\gamma_n$  by  $h$ -arclength remains equi-Lipschitz (since the changes of parametrizations are themselves equi-Lipschitz by boundedness of the arclengths). Since the arclengths  $a_n$  are bounded, we can moreover assume that  $(a_n)$  converges to  $a \in ]0; +\infty[$  by passing to a subsequence. We now extend the  $\sigma_n$  to *future inextendible* causal curves  $\nu_n: \mathbb{R}^+ \rightarrow S$ , *i.e.* such that  $\nu_n(t)$  has no limit when  $t \rightarrow +\infty$ . One easily proves using Fact A.20 that the  $h$ -arclength of the  $\nu_n$  is infinite, and we can thus reparametrize them by  $h$ -arclength on  $[a_n; \infty]$  to an equi-Lipschitz family  $\nu_n: \mathbb{R}^+ \rightarrow S$  of causal curves.

For any  $m \in \mathbb{N}$ , we can now apply Arzela-Ascoli theorem to the restriction of  $(\nu_n)$  to  $[0; m]$ . This shows that a subsequence of  $\nu_n|_{[0; m]}$  uniformly converges to a continuous curve  $\nu_\infty^m$  in  $S$ , which is still Lipschitz as a uniform limit of equi-Lipschitz curves. By a diagonal argument, we conclude to the existence of a subsequence  $(\nu_{n_k})$  of  $(\nu_n)$  and of a continuous curve  $\nu_\infty: \mathbb{R}^+ \rightarrow S$  obtained as the union of the  $\nu_\infty^m$ , such that  $\nu_{n_k}|_I$  uniformly converges to  $\nu_\infty|_I$  for any compact interval  $I \subset \mathbb{R}^+$ . It is easy to show that  $\nu_\infty$  is actually a causal curve as a uniform limit of such curves (see for instance [Min19, top of p.46]), and with  $\sigma$  the restriction of  $\nu_\infty$  to  $[0; a]$ , the subsequence  $(\sigma_{n_k})$  uniformly converges to  $\sigma$ .

Lastly the proof that  $\limsup L(\gamma_{n_k}) \leq L(\sigma)$  given in [Min19, Theorem 2.41] works without any variation in our singular setting, using the decomposition (A.2) of the length into the ones of its regular pieces.  $\square$

**A.4. Conclusion of the proof of Theorem A.1.** Let  $S$  be a closed singular  $\mathbf{X}$ -surface of class A, hence homeomorphic to  $\mathbf{T}^2$ ,  $\bar{b}$  be a simple closed spacelike curve in  $S$ , and  $\pi_C: C \rightarrow S$  be the  $\mathbb{Z}$ -covering of  $S$  for which  $\pi_{C*}(\pi_1(C))$  is generated by  $[\bar{b}]$ , endowed with its induced singular  $\mathbf{X}$ -structure.

**Lemma A.21.**  *$C$  is a globally hyperbolic singular  $\mathbf{X}$ -surface.*

*Proof.* With  $T$  the automorphism of the universal cover  $\Pi: \tilde{S} \rightarrow C$  of  $C$  induced by  $\bar{b}$ ,  $\Pi$  induces a homeomorphism from the quotient  $\tilde{S}/\langle T \rangle$  to  $C$ . Since  $\bar{b}$  lifts to spacelike curve in  $\tilde{S}$  and  $\tilde{S}$  is a class A surface, each point  $x \in C$  admits a connected open neighbourhood  $U$  such that  $\Pi^{-1}(U)$  is the disjoint union of open sets  $U_i \subset \tilde{S}$ , so that any two distinct  $U_i \neq U_j$  are not causally related, *i.e.* there exists no causal curve joining a point of  $U_i$  to a point of  $U_j$ . Since  $\tilde{S}$  is moreover strongly causal according to Corollary A.13, we can further choose  $U$  arbitrarily small and so that each  $U_i$  is causally convex in  $\tilde{S}$ . Now let  $\gamma: [0; 1] \rightarrow C$  be a causal curve joining  $p = \gamma(0) \in U$  to  $q = \gamma(1) \in U$ . Then  $\gamma$  lifts to a causal curve  $\tilde{\gamma}$  from  $p'$  in some  $U_i$  to  $q'$  in some  $U_j$ . By definition of the  $U_k$ 's this implies that  $U_i = U_j$ , hence that  $\tilde{\gamma} \subset U_i$  since  $U_i$  is causally convex. In the end  $\gamma \subset U$ , which shows the strong causality of  $C$  since  $x \in C$  is arbitrary and  $U$  can be chosen arbitrarily small.

Let  $R$  denote the generator of the automorphism group of  $\pi_C$ , positive in the sense that it is induced by the action on  $\tilde{S}$  of a homotopy class  $[\bar{a}]$  of closed curves of  $S$  of algebraic intersection number  $\hat{i}([\bar{b}], [\bar{a}]) = 1$  with  $\bar{b}$ . We denote in the same way the action of  $R$  on  $\tilde{S}$  and on  $C$ . Then for  $x, y \in C$ , there exists a lift  $b$  of  $\bar{b}$  in  $C$  and  $k \in \mathbb{N}^*$  such that  $x$  and  $y$  are contained in the interior of the unique connected compact annulus  $E \subset C$  bounded by  $b$  and  $R^k(b)$  (this is so by compactness of  $S$ ). For  $\gamma: [0; 1] \rightarrow C$  a causal curve from  $x$  to  $y$ , we show now that  $\gamma$  is contained in  $E$ . This will prove the relative compactness of  $J^+(x) \cap J^-(y)$  and conclude the proof of the Lemma. Since  $x, y \in \text{Int}(E)$  by assumption, there exists  $\varepsilon > 0$  such that  $\gamma([0; \varepsilon]) \subset \text{Int}(E)$  and  $\gamma([1 - \varepsilon; 1]) \subset \text{Int}(E)$ . Furthermore by connectedness,  $\gamma$  cannot leave  $\text{Int}(E)$  before meeting  $b$  or  $R^k(b)$ . Moreover since  $C \setminus E$  has two connected components having respectively  $b$  and  $R^k(b)$  as unique boundary components, if  $\gamma$  meets  $b$  (*resp.*  $R^k(b)$ ) then it does so in two opposite directions. But this contradicts the fact that  $\gamma$  is future-oriented, hence  $\gamma \subset E$  as claimed previously.  $\square$

Let  $\bar{a}$  be a closed timelike curve of  $S$  intersecting  $\bar{b}$  at a point  $\bar{x} = \bar{a}(0) = \bar{b}(0)$ , and of algebraic intersection number  $\hat{i}([\bar{b}], [\bar{a}]) = 1$  with it. In particular  $([\bar{a}], [\bar{b}])$  is a basis of  $\pi_1(S) \simeq \mathbb{Z}^2$ . We fix a lift  $x_1 \in \pi_C^{-1}(\bar{x})$  of  $\bar{x}$  in  $C$ , and denote by  $a: [0; 1] \rightarrow C$  and  $b_1: [0; 1] \rightarrow C$  the lifts of  $\bar{a}$  and  $\bar{b}$  starting from  $x_1 = a(0) = b_1(0)$ . By definition of  $C$  we have  $b_1(1) = x_1$ , *i.e.*  $b_1$  is a simple closed curve in  $C$ . On the other hand  $a$  is a simple segment but is not closed, and  $x_2 := a(1) = R(x_1)$  with  $R$  the positive generator of the covering automorphism group of  $\pi_C$  induced by  $[\bar{a}]$ . We denote by  $b_2: [0; 1] \rightarrow C$  the lift of  $\bar{b}$  starting from  $x_2$ , so that  $b_2 = R \circ b_1$ . For  $p \in b_1$  we denote by  $a_p$  the timelike curve which is the lift of  $\bar{a}$  starting from  $p$  (for instance  $a_{x_1} = a$ ), so that  $a_p(0) = p$  and  $a_p(1) = R(p)$ . We introduce then the set  $\mathcal{S}_p$  of causal curves of  $C$  from  $p$  to  $R(p)$  which are *causally homotopic* to  $a_p$ , *i.e.* homotopic through causal curves while fixing their extremities. The following result is a version of the classical Avez-Seifert theorem (see for instance [Min19, Theorem 4.123]), suitably adapted to our setting.

**Proposition A.22.** *The function*

$$(A.6) \quad F: p \in b_1 \mapsto \sup_{\sigma \in \mathcal{S}_p} L(\sigma) \in [0; \infty[$$

*has finite values, is continuous, and moreover for any  $p \in b_1$  there exists  $\sigma \in \mathcal{S}_p$  such that  $L(\sigma) = F(p)$ .*

*Proof.* Let  $p \in b_1$  and  $\sigma_n \in \mathcal{S}_p$  be a sequence of causal curves such that  $\lim L(\sigma_n) = F(p)$ . Since  $C$  is globally hyperbolic according to Lemma A.21, there exists according to Lemma A.19 a subsequence  $\sigma_{n_k}$  converging to a causal curve  $\sigma$  from  $p$  to  $R(p)$ . In any normal convex neighbourhood  $U$ , there exists  $\varepsilon_U > 0$  such that for any causal curve  $\gamma$  contained in  $U$  where  $\bar{V} \subset U$ , all the causal curves  $\varepsilon_U$ -close to  $\gamma$  (for a fixed auxiliary complete Riemannian metric and its induced distance on  $C$ ) are contained in  $U$  and causally homotopic to  $\gamma$ . Since  $J^-(p) \cap J^+(R(p))$  is compact by

global hyperbolicity and contains any curve of  $\mathcal{S}_p$ , we can cover  $J^-(p) \cap J^+(R(p))$  by a finite number of normal convex neighbourhoods as before and conclude to the existence of  $\varepsilon > 0$  such that for any  $\gamma \in \mathcal{S}_p$ , any causal curve  $\varepsilon$ -close to  $\gamma$  is causally homotopic to it. Hence for any large enough  $k$ ,  $\sigma$  is causally homotopic to  $\sigma_{n_k}$  and thus to  $a_p$ , *i.e.*  $\sigma \in \mathcal{S}_p$ . Hence  $L(\sigma) \leq F(p)$  by definition of  $F$ , and since  $F(p) = \lim L(\sigma_{n_k}) \leq L(\sigma)$  according to Lemma A.19 this shows that  $F(p) = L(\sigma) < +\infty$  and prove the first and third claims.

The proof that  $F$  is lower semi-continuous is a straightforward adaptation of [Min19, Theorem 2.32], to which we refer for more details. Let  $p \in b_1$ ,  $\varepsilon > 0$  be such that  $0 < 3\varepsilon < F(p)$  and  $\gamma \in \mathcal{S}_p$  so that  $L(\gamma) > F(p) - \varepsilon > 0$ . We slightly modify  $\gamma$  for it to be timelike and still satisfy this last inequality. We choose then  $p' \in \gamma$  close enough to  $p$  so that  $L(\gamma|_{[p;p']}) < \varepsilon$  and  $q' \in \gamma$  close enough to  $R(p)$  so that  $L(\gamma|_{[q';R(p)]}) < \varepsilon$ , hence  $L(\gamma|_{[p';q']}) > F(p) - 3\varepsilon > 0$ . If  $p'$  and  $q'$  are close enough to  $p$  and  $R(p)$ , then the respective past and future timelike quadrants  $U$  and  $V$  in normal convex neighbourhoods of  $p'$  and  $q'$  are neighbourhoods of  $p$  and  $R(p)$ ,  $I = U \cap b_1$  is a neighbourhood of  $p$  in  $b_1$ , and  $R(I)$  a neighbourhood of  $R(p)$  in  $b_2$ . For any  $x \in I$ , the causal curve  $\gamma_x$  from  $x$  to  $R(x)$  formed by first following the geodesic  $[x;p']_U$ , then  $\gamma|_{[p';q']}$  and finally  $[q';R(x)]_V$  is then causally homotopic to  $a_x$ , and  $F(x) \geq L(\gamma_x) \geq L(\gamma|_{[p';q']}) > F(p) - 3\varepsilon$  which proves the lower semi-continuity.

Assume now by contradiction that  $F$  is not upper semi-continuous, *i.e.* that there exists  $p_n \rightarrow p$  in  $b_1$  and  $\varepsilon > 0$  such that  $F(p_n) \geq F(p) + 2\varepsilon$  for any  $n$ . Then with  $\gamma_n \in \mathcal{S}_{p_n}$  such that  $L(\gamma_n) \geq F(p_n) - \varepsilon$ , since  $p_n$  converges to  $p$  and  $R(p_n)$  to  $R(p)$ , Lemma A.19 shows the existence of a causal curve  $\gamma$  from  $p$  to  $R(p)$  to which a subsequence  $\gamma_{n_k}$  converges. Indeed with  $p' \in I^-(p)$  and  $q' \in I^+(R(p))$  sufficiently close to  $p$  and  $R(p)$ , there exists for any large enough  $n$  timelike geodesics  $\gamma_n^-$  and  $\gamma_n^+$  respectively from  $p'$  to  $p_n$  and from  $R(p_n)$  to  $q'$ , contained in normal convex neighbourhoods of  $p'$  and  $q'$ . We can now directly apply Lemma A.19 to the sequence of causal curves formed by following  $\gamma_n^-$ ,  $\gamma_n$  and  $\gamma_n^+$ , and restrict the obtained limit curve to its segment from  $p$  to  $R(p)$ . We have thus according to Lemma A.19 and by assumption on  $L(\gamma_n)$  and  $F(p_n)$ :  $L(\gamma) \geq \limsup L(\gamma_{n_k}) \geq \limsup F(p_{n_k}) - \varepsilon \geq F(p) + \varepsilon$ . But the argument of the first paragraph of this proof shows that  $\gamma$  is causally homotopic to  $a_p$ , and this last inequality contradicts shows the definition of  $F(p)$ , which concludes the proof.  $\square$

We can finally conclude the proof of Theorem A.1 thanks to the following result.

**Theorem A.23.** *Let  $S$  be a closed singular  $\mathbf{X}$ -surface of class A. Then any simple closed timelike (resp. spacelike) curve in  $S$  admits a freely homotopic simple closed timelike (resp. spacelike) geodesic.*

*Proof.* We prove the claim for a simple closed timelike curve  $\bar{a}$ , and the proof follows then in the spacelike case by replacing the metric of  $S$  with its opposite. According to Proposition A.16, there exists a simple closed spacelike curve  $\bar{b}$  intersecting  $\bar{a}$  at a single point  $\bar{x} = \bar{a}(0) = \bar{b}(0)$ . We use the notations introduced before Proposition A.22 for the  $\mathbb{Z}$ -covering  $\pi_C: C \rightarrow S$  of  $S$  such that  $\pi_{C*}(\pi_1(C)) = \langle [\bar{b}] \rangle$ , for the lifts  $a$ ,  $b_i$  and  $x_i$  ( $i = 1, 2$ ) of  $\bar{a}$ ,  $\bar{b}$  and  $\bar{x}$ , and for the covering automorphism  $R$  induced by the action of  $[\bar{a}]$ . With this setup, we want to find a simple timelike geodesic segment  $\gamma: [0; l] \rightarrow C$ , such that  $\gamma(0) \in b_1$  and  $\gamma(l) = R(\gamma(0)) \in b_2$  and homotopic (while fixing its extremities) to  $a_{\gamma(0)}$ . We recall that  $a_p$  denotes for  $p \in b_1$  the unique lift of  $\bar{a}$  starting from  $p$ . According to Proposition A.22, the function  $F$  defined in (A.6) is continuous and finite on the compact set  $b_1$ , and reaches thus its maximum at a point  $p_0 \in b_1$ . There exists moreover according to the same Proposition a causal curve  $\gamma \in \mathcal{S}_{p_0}$  such that

$$(A.7) \quad L(\gamma) = F(p_0) = \sup_{p \in b_1} \sup_{\sigma \in \mathcal{S}_p} L(\sigma).$$

In particular, note that  $L(\gamma) \geq L(a) = L(\bar{a}) > 0$ .

We now prove that  $\gamma: [0; 1] \rightarrow C$  is locally maximizing, hence that it is a timelike geodesic up to reparametrization according to Proposition A.18. Indeed let  $t \in [0; 1]$ ,  $U$  be a normal convex neighbourhood of  $\gamma(t)$  and  $I = [a; b]$  be a connected neighbourhood of  $t$  in  $[0; 1]$  such that  $\gamma(I) \subset U$ . Then the unique geodesic segment  $[\gamma(a); \gamma(b)]_U$  of  $U$  from  $\gamma(a)$  to  $\gamma(b)$  is (future) timelike, and homotopic to  $\gamma|_I$  through causal curves while fixing the extremities. In other words the

curve  $\nu$  obtained by concatenating  $\gamma|_{[0;a]}$ ,  $[\gamma(a); \gamma(b)]_U$  and  $\gamma|_{[b;1]}$  is in  $\mathcal{S}_{p_0}$ , and thus  $L(\nu) \leq L(\gamma)$  according to (A.7). But on the other hand  $L([\gamma(a); \gamma(b)]_U) = \tau_U(\gamma(a), \gamma(b)) \geq L(\gamma|_{[a;b]})$  according to Proposition A.18, hence  $L(\nu) \geq L(\gamma)$ . The latter inequality has thus to be an equality, which can happen only if  $\tau_U(\gamma(a), \gamma(b)) \geq L(\gamma|_{[a;b]})$ . This proves that  $\gamma$  is locally maximizing, as we had claimed.

Let reparametrize  $\gamma: [0; l] \rightarrow C$  to be geodesic. Since  $C$  is strongly causal according to Lemma A.21, it contains in particular no closed timelike curve, and  $\gamma$  is thus injective. Furthermore,  $\gamma(]0; l[)$  is contained in the interior of the unique compact connected annulus  $E$  of  $C$  bounded by  $b_1$  and  $b_2$  as we have already seen in the second part of the proof of Lemma A.21, and in particular  $\gamma(]0; l[)$  is thus disjoint from  $b_1 \cup b_2$ . Since  $\pi_C: C \rightarrow S$  is injective in restriction to  $\text{Int}(E)$  and  $\pi_C(\gamma(0)) = \pi_C(\gamma(l))$ , this proves that  $\bar{\gamma} = \pi_C \circ \gamma: [0; l] \rightarrow S$  is a simple closed timelike curve of  $S$ , freely homotopic to  $a$  (since  $\gamma$  is freely homotopic to  $a_{p_0}$  while fixing their extremities), which is geodesic in restriction to  $]0; l[$ . In a small normal convex neighbourhood  $U$  of  $x := \bar{\gamma}(0)$ ,  $\bar{\gamma}$  is thus the union of two future timelike geodesic segments  $I_-$  and  $I_+$  of extremity  $x$ , and respectively contained in the past timelike and future timelike quadrants at  $x$ .

Assume by contradiction that  $\bar{\gamma}$  is not a geodesic, *i.e.* that  $I_{\pm}$  are not parts of the same geodesic segment of  $U$ . Then according to Proposition A.18, there exists two points  $x_{\pm} \in I_{\pm}$  distinct from  $x$  such that the unique geodesic segment  $[x_-; x_+]_U$  from  $x_-$  to  $x_+$  in  $U$  is future timelike and longer than the segment  $\bar{\gamma}|_{[x_-; x_]}$  of  $\bar{\gamma}$  going from  $x_-$  to  $x_+$ :

$$(A.8) \quad L([x_-; x_+]_U) > L(\bar{\gamma}|_{[x_-; x_]}).$$

With  $\bar{\gamma}_*$  the segment of  $\bar{\gamma}$  from  $x_+$  to  $x_-$ , the curve  $\bar{\nu}$  formed by following  $\bar{\gamma}_*$  and then  $[x_-; x_+]_U$  is thus future timelike, and satisfies  $L(\bar{\nu}) > L(\bar{\gamma})$  according to (A.8). We denote by  $\nu$  its lift in  $S$  starting from the lift  $y \in b_1$  of the (unique) intersection point of  $[x_-; x_+]_U$  with  $\bar{b}$ . Observe that, if  $x_{\pm}$  are chosen sufficiently close to  $x$  then  $\nu$  is freely homotopic to  $a_y$ , *i.e.*  $\nu \in \mathcal{S}_q$ . Since  $L(\nu) > L(\gamma)$  this contradicts the characterization of  $\gamma$  in (A.7) as the maximizer of  $L(\sigma)$  for  $p \in b_1$  and  $\sigma \in \mathcal{S}_p$ , which shows that  $\bar{\gamma}$  is a geodesic and concludes the proof.  $\square$

**Corollary A.24.** *Let  $c \in \text{Def}_{\Theta}(\mathbf{T}^2, \Sigma)$  be an equivalence class of class A singular  $\mathbf{X}$ -structures on a closed surface  $S$ . Then there exists a basis  $(A, B)$  of  $\pi_1(\mathbf{T}^2)$ , such that any  $\mu \in c$  admits simple closed timelike and spacelike geodesics  $a$  and  $b$  respectively freely homotopic to  $A$  and  $B$ . In particular  $a$  and  $b$  intersect at a single point.*

*Proof.* Theorem A.23 and Proposition A.16 yield a pair of simple closed timelike and spacelike geodesics defining a basis of  $\pi_1(\mathbf{T}^2)$ . In the other hand, the action of  $\text{Homeo}^0(S, \Sigma)$  sends such a pair on a freely homotopic one, which proves the claim.  $\square$

## APPENDIX B. SOME CLASSICAL RESULTS ON THE ROTATION NUMBER

We recall that for any orientation-preserving circle homeomorphism  $T \in \text{Homeo}^+(\mathbf{S}^1)$ , the minimal set of  $T$  (*i.e.* by definition, the minimal non-empty closed and  $T$ -invariant subset of  $\mathbf{S}^1$ ) is either a finite set if  $T$  has a periodic orbit,  $\mathbf{S}^1$  if  $T$  is minimal (*i.e.* has all its orbits dense), or is *exceptional* *i.e.* a Cantor set. The following result is classical, but we give here a proof for sake of completeness.

**Lemma B.1.** *Let  $f \in \text{Homeo}^+(\mathbf{S}^1)$ , and  $t \in [0; 1] \mapsto g_t \in \text{Homeo}^+(\mathbf{S}^1)$  be a continuous map such that:*

- $g_0 = \text{id}_{\mathbf{S}^1}$ ,
- $t \in [0; 1] \mapsto g_t(x) \in \mathbf{S}^1$  is increasing for any  $x \in \mathbf{S}^1$ ,
- and for any  $t \in [0; 1]$ ,  $f_t := g_t \circ f$  does not have an exceptional minimal set.

Then the map  $F: t \in [0; 1] \mapsto \rho(g_t \circ f) \in \mathbf{S}^1$  is:

- (1) continuous;
- (2) non-decreasing.

Assume moreover that  $t \in [0; 1] \mapsto g_t(x) \in \mathbf{S}^1$  is strictly increasing for any  $x \in \mathbf{S}^1$ . Then:

- (3)  $F$  is strictly increasing at any  $t_0 \in [0; 1]$  such that  $\rho(g_{t_0} \circ f) \in [\mathbb{R} \setminus \mathbb{Q}]$ , *i.e.* there exists  $\varepsilon > 0$  such that for any  $s \in [0; 1]$  with  $0 < |s - t_0| < \varepsilon$ :  $\rho(g_s \circ f) \neq \rho(g_{t_0} \circ f)$ .

Assume finally that  $\rho(f)$  is irrational. Then:

- (4) For any  $\varepsilon > 0$ , there exists  $\eta > 0$  such that for any rational  $r \in [\rho(f); \rho(f) + \eta] \subset \mathbf{S}^1$  and any  $x \in \mathbf{S}^1$ , there exists  $t \in [0; \varepsilon]$  such that the orbit of  $x$  under  $g_t \circ f$  is periodic and of cyclic order  $r$ . In particular  $\rho(g_t \circ f) = r$ .

The obvious analogous statements hold for decreasing maps, and for a family  $t \mapsto f \circ g_t$  of deformations.

We recall that in a singular closed  $\mathbf{X}$ -surface, the holonomy  $f$  of a lightlike foliation on any section  $S$  has according to Lemma 2.30 no exceptional minimal set, and satisfy thus the hypotheses of Lemma B.1 with the rotation  $g_t = R_{[t]}^S$  around  $S$  (for any continuous parametrization of  $S$ ).

*Proof of Lemma B.1.* The claim for  $t \mapsto f \circ g_t$  and increasing maps follows from the one concerning  $t \mapsto g_t \circ f$  and decreasing maps by taking the inverse, since  $\rho(f^{-1}) = -\rho(f)$  for any circle homeomorphism.

(1) The continuity follows readily from the ones of the rotation number (see Proposition 3.18) and of  $t \mapsto g_t$ .

(2) The assumptions on  $(g_t)$  ensure the existence of a family of lifts  $G_t \in D(\mathbf{S}^1)$  of  $g_t$  such that for any  $x \in \mathbb{R}$ :  $t \mapsto G_t(x)$  is non-decreasing. Let  $F$  be a lift of  $f$ , and  $s \leq t \in [0; 1]$ . Then  $G_s \circ F(0) \leq G_t \circ F(0)$  and if we assume that  $(G_s \circ F)^n(0) \leq (G_t \circ F)^n(0)$  for some  $n \in \mathbb{N}$ , then since  $F$  and the  $G_u$  are strictly increasing and  $x \mapsto G_u(x)$  is non-decreasing for any  $x \in \mathbb{R}$  we obtain:  $(G_s \circ F)^{n+1}(0) \leq G_t(F \circ (G_s \circ F)^n)(0) \leq (G_t \circ F)^{n+1}(0)$ . In the end  $(G_s \circ F)^n(0) \leq (G_t \circ F)^n(0)$  for any  $n \in \mathbb{N}$ , which shows that  $\tau(G_s \circ F) \leq \tau(G_t \circ F)$  according to (3.12). Hence  $u \in [0; 1] \mapsto \tau(G_u \circ F) \in \mathbb{R}$  is non-decreasing and is a lift of the map  $u \in [0; 1] \mapsto \rho(g_u \circ f) \in \mathbf{S}^1$ , which proves the claim.

(3) Since  $f_{t_0} = g_{t_0} \circ f$  has no exceptional minimal set and has irrational rotation number, it is minimal. There exists thus according to the pigeonhole principle a sequence  $n_k \in \mathbb{N}$  such that  $f_{t_0}^{n_k}([0])$  converges to  $[0]$ , and  $f_{t_0}^{n_{k+1}}([0]) \in ]f_{t_0}^{n_k}([0]); [0[$  for any  $k$ . Since  $t \in [0; 1] \mapsto f_t^{n_k}([0])$  is moreover strictly increasing and continuous for any  $k$ , for any  $\varepsilon > 0$  there exists  $K \in \mathbb{N}$  such that  $f_{t_0+\varepsilon}^{n_{K+1}}([0]) \in ]]0]; f_{t_0+\varepsilon}^{n_K}([0)[$ . This prevents in particular the orbits of  $[0]$  under  $f_{t_0+\varepsilon}$  and  $f_{t_0}$  to have the same cyclic orders, and therefore  $\rho(f_{t_0+\varepsilon}) \neq \rho(f_{t_0})$  according to Proposition 3.21. A symmetric argument can be done for  $t_0 - \varepsilon$  with a sequence converging to  $[0]$  from above.

(4) Since  $\rho(f)$  is irrational,  $F$  is not constant on a neighbourhood of 0 according to (3), and there exists thus by continuity of  $F$  some  $\eta > 0$  such that  $[\rho(f); \rho(f) + \eta] \subset [\rho(f); \rho(f_\varepsilon)]$ . Then for any rational  $r \in [\rho(f); \rho(f) + \eta]$ , there exists because of the continuity and the monotonicity of  $F$  some  $t_1 \leq t_2 \in ]0; \varepsilon]$  and some small  $\varepsilon' > 0$  such that:

- $F(t) \in [\rho(f); r[$  for any  $t \in [0; t_1[$ ,
- $F([t_1; t_2]) = \{r\}$ ,
- $F(t) \in ]r; \rho(f) + \eta]$  for any  $t \in ]t_2; t_2 + \varepsilon']$ .

Let  $x \in \mathbf{S}^1$ , and assume that  $x$  is not periodic for  $f_{t_1} = g_{t_1} \circ f$ . We first assume that  $r \neq [0]$ , which implies  $q \geq 2$  with  $r = [\frac{p}{q}]$  in reduced form. Denoting  $(x_1, \dots, x_n) \sim r$  if  $(x_1, \dots, x_n)$  has the same cyclic order than  $([0], r, 2r, \dots, (q-1)r)$ , we have for any  $\theta \in \mathbf{S}^1$ :

$$\left\{ ([0], \theta, \dots, (q-1)\theta) \sim r \text{ and } q\theta \in \text{Cl}(I_\theta^-) \right\} \Leftrightarrow \theta \in ]r - \frac{1}{q}; r],$$

$$\left\{ ([0], \theta, \dots, (q-1)\theta) \sim r \text{ and } q\theta \in \text{Cl}(I_\theta^+) \right\} \Leftrightarrow \theta \in [r; r + \frac{1}{q}[,$$

with  $I_\theta^-$  (respectively  $I_\theta^+$ ) the connected component of  $\mathbf{S}^1 \setminus \{0, \theta, \dots, (q-1)\theta\}$  having  $[0]$  as right extremity (respectively as left extremity). It is well-known that using the interpretation of the rotation number in terms of cyclic ordering of the orbits given by Proposition 3.21, the above equivalences adapt for any  $T \in \text{Homeo}^+(\mathbf{S}^1)$  to give the following:

$$(B.1a) \quad \left\{ (x, T(x), \dots, T^{q-1}(x)) \sim r \text{ and } T^q(x) \in \text{Cl}(I_T^-) \right\} \Leftrightarrow \rho(T) \in ]r - \frac{1}{q}; r],$$

$$(B.1b) \quad \left\{ (x, T(x), \dots, T^{q-1}(x)) \sim r \text{ and } T^q(x) \in \text{Cl}(I_T^+) \right\} \Leftrightarrow \rho(T) \in [r; r + \frac{1}{q}[,$$

with  $I_T^-$  (respectively  $I_T^+$ ) the connected component of  $\mathbf{S}^1 \setminus \{x, T(x), \dots, T^{q-1}(x)\}$  having  $x$  as right extremity (respectively as left extremity). Now  $f_{t_1}^q(x) \neq x$  since we assumed  $x$  to be non-periodic, and since moreover  $\rho(f_{t_1}) = r$ ,  $f_{t_1}^q(x)$  is actually either in  $I_{f_{t_1}}^-$  or in  $I_{f_{t_1}}^+$  according to (B.1a) and (B.1b). If  $f_{t_1}^q(x) \in I_{f_{t_1}}^+$ , then  $f_t^q(x) \in I_{f_t}^+$  for any  $t \in [0; t_1[$  sufficiently close to  $t_1$  by continuity of  $t \mapsto f_t^q(x)$ , which implies  $\rho(f_t) \in [r; r + \frac{1}{q}[$  for any such  $t$  according to (B.1b) and contradicts the definition of  $t_1$ . Therefore  $f_{t_1}^q(x) \in I_{f_{t_1}}^-$ , and since  $t \mapsto f_t^q(x)$  is continuous and increasing, with moreover  $\rho(f_t) = r$  for any  $t \in [t_1; t_2]$ : either  $f_t^q(x) = x$  for some  $t \in ]t_1; t_2]$ , or  $f_{t_2}^q(x)$  remains in  $I_{f_{t_2}}^-$ . In the latter case,  $f_t^q(x) \in I_{f_t}^-$  for any  $t \in [t_2; t_2 + \varepsilon']$  sufficiently close to  $t_2$ , which implies  $\rho(f_t) \in ]r - \frac{1}{q}; r]$  for such a  $t$  according to (B.1a) and contradicts the definition of  $t_2$ . In conclusion,  $f_t^q(x) = x$  for some  $t \in ]t_1; t_2]$ .

We assume now that  $\rho(f_{t_1}) = r = [0]$ . According to the interpretation of the rotation number in terms of cyclic ordering of the orbits given by Proposition 3.21 and equations (B.1a)- (B.1b), this is equivalent to say that the sequence  $(f_{t_1}^n(x))_{n \in \mathbb{N}}$  is monotonically cyclically ordered. More precisely, the cyclic monotonicity of  $(f_t^n(x))_{n \in \mathbb{N}}$  forces  $\rho(f_t)$  to be rational according to Proposition 3.21 and to be zero by equations (B.1a)- (B.1b), and reciprocally if  $(f_t^n(x))_{n \in \mathbb{N}}$  is not cyclically monotonous, then equations (B.1a)- (B.1b) implies that  $\rho(f_t) \neq [0]$ . Assume by contradiction that  $(f_{t_1}^n(x))_{n \in \mathbb{N}}$  is positively cyclically ordered, hence strictly since  $f_{t_1}(x) \neq x$  by assumption. Then since  $t \mapsto f_t^n(x)$  is increasing for any  $n$ , the sequence  $(f_t^n(x))_{n \in \mathbb{N}}$  is strictly positively cyclically ordered for any  $t \in [0; t_1[$  close enough to  $t_1$ . But this implies  $\rho(f_t) = [0]$  for such a  $t$  as we have seen previously, which contradicts the definition of  $t_1$ . Therefore  $(f_{t_1}^n(x))_{n \in \mathbb{N}}$  is negatively cyclically ordered, and thus using again that  $t \mapsto f_t^n(x)$  is increasing for any  $n$ : either  $f_t(x) = x$  for some  $t \in ]t_1; t_2]$ , or  $(f_{t_2}^n(x))_{n \in \mathbb{N}}$  remains strictly negatively cyclically ordered. But in the latter case  $(f_t^n(x))_{n \in \mathbb{N}}$  is strictly negatively cyclically ordered for any  $t \in [t_2; t_2 + \varepsilon']$  close enough to  $t_2$ , which implies  $\rho(f_t) = [0]$  for such a  $t$  and contradicts the definition of  $t_2$ . In conclusion  $f_t(x) = x$  for some  $t \in ]t_1; t_2]$ , which concludes the proof.  $\square$

## APPENDIX C. HOLONOMIES OF LIGHTLIKE FOLIATIONS ARE PIECEWISE MÖBIUS

We prove in this appendix that the holonomies of lightlike foliations in a singular  $\mathbf{X}$ -surface are piecewise Möbius maps.

A *projective structure* on a topological one-dimensional manifold is a  $(\mathrm{PSL}_2(\mathbb{R}), \mathbb{RP}^1)$ -structure consisting of orientation preserving charts, and we call *projective* the  $(\mathrm{PSL}_2(\mathbb{R}), \mathbb{RP}^1)$ -morphisms between two projective curves. We endow  $\mathbb{R}$  with its standard projective structure for which  $x \in \mathbb{R} \mapsto [x : 1] \in \mathbb{RP}^1$  is a global chart, so that projective morphisms between intervals of  $\mathbb{R}$  are precisely the (restrictions of) homographies. We recall that geodesics of singular  $\mathbf{dS}^2$ -surfaces have well-defined affine structures (see Definition A.5), and observe that an affine structure defines in particular a projective structure (through the embedding  $\mathbb{R} \hookrightarrow \mathbb{RP}^1$ , equivariant for the natural embedding  $\mathrm{Aff}^+(\mathbb{R}) \hookrightarrow \mathrm{PSL}_2(\mathbb{R})$ ).

**Definition C.1.** A homeomorphism  $F: I \rightarrow J$  between two projective 1-dimensional manifolds is *piecewise projective* if there exists a finite number of points  $x_1, \dots, x_N$  in  $I$ , called the *singular points* of  $F$ , such that  $F$  is projective in restriction to any connected component  $C$  of  $I \setminus \{x_1, \dots, x_N\}$ .

**Proposition C.2.** *Let  $H: I \rightarrow J$  be the holonomy of a lightlike foliation between two compact connected subsets  $I$  and  $J$  of geodesics in a singular  $\mathbf{X}$ -surface ( $I = J$  being allowed). Then  $H$  is piecewise projective in the affine parametrizations of  $I$  and  $J$ .*

*Proof. Case of  $\mathbb{R}^{1,1}$ .* In this case, the leaves of the  $\alpha$  and  $\beta$  foliations are the affine lines respectively parallel to the vector lines  $\mathbb{R}e_1$  and  $\mathbb{R}e_2$ . On the other hand the affinely parametrized geodesics are the affinely parametrized segments, and the holonomy between them is thus a dilation, *i.e.* an affine and in particular projective transformation.

**Case of  $\mathbf{dS}^2$ .** If the surface is  $\mathbf{dS}^2$ , the claim follows from a series of naive but fundamental observations. Thanks to Proposition 2.6 we can work with the hyperboloid model  $\mathbf{dS}^2$  of the de-Sitter space, that we will see here as the set  $\{l \in \mathbf{P}^+(\mathbb{R}^{1,2}) \mid \text{spacelike}\}$  of spacelike half-lines

of  $\mathbb{R}^{1,2}$ . We endow  $\mathbb{R}^3$  with its standard orientation,  $\mathbb{R}^{1,2}$  with its natural time-orientation for which  $(x_1, x_2, x_3) \in q_{1,2}^{-1}(\mathbb{R}_-) \setminus \{0\}$  is positive if  $x_1 > 0$ , any Lorentzian plane  $P \subset \mathbb{R}^{1,2}$  with the induced orientation for which  $(P_{\text{in}}^\perp, P)$  is positively oriented (with  $P_{\text{in}}^\perp$  the normal, for the standard euclidean metric, pointing inward the timelike cone), and lastly any plane which is either *null* (equivalently such that  $P \cap q_{1,2}^{-1}(0)$  is a line) or spacelike with the orientation such that  $(P_\perp^\perp, P)$  is positively oriented (with  $P_\perp^\perp$  the euclidean normal pointing in the positive causal cone). Note that all of these orientations are  $\text{SO}^0(1, 2)$ -invariant, and endow in particular  $\text{dS}^2$  and each of its geodesics with an orientation. The tangent spaces of  $\text{dS}^2$  are indeed identified with Lorentzian planes, and its geodesics with connected components of  $P \cap \text{dS}^2$  with  $P$  a plane of the same signature than the geodesic. Observe that we identified here  $P$  with the set of half-lines that it contains, a slight abuse of notations that we will frequently repeat below for any homothety-invariant subset of  $\mathbb{R}^{1,2}$ , in the hope to simplify the reading.

We now describe an affine parametrization of geodesics of  $\text{dS}^2$  by the  $(\text{SO}^0(1, 2)$ -invariant) positive copy  $\mathcal{C} := \{l \in \mathbf{P}^+(\mathbb{R}^{1,2}) \mid \text{lightlike and positive}\}$  of its conformal boundary. The latter is equipped with the *projective structure* for which  $t \in \mathbb{R} \mapsto g^t(l) \in \mathcal{C}$  is a projective parametrization for any one-parameter subgroup  $\{g^t\}_{t \in \mathbb{R}} \subset \text{SO}^0(1, 2)$  and  $l \in \mathcal{C}$ . To this end, we define as follows two  $\text{SO}^0(1, 2)$ -equivariant natural projections

$$\pi_{\alpha/\beta}: l \in \text{dS}^2 \mapsto l_{\alpha/\beta} \in \mathcal{C}$$

whose fibers are the  $\alpha$  and  $\beta$  lightlike foliations of  $\text{dS}^2$ . Any  $l \in \text{dS}^2$  is contained in exactly two null planes  $N_{\alpha/\beta}^l$  defining two lightlike geodesics  $n_{\alpha/\beta}^l$  containing  $l$  (the connected components of  $N_{\alpha/\beta}^l \cap \text{dS}^2$  containing  $l$ ), and we name them in such a way that with  $l_{\alpha/\beta} = N_{\alpha/\beta}^l \cap \mathcal{C}$ , the positive orientation of  $n_\alpha^l$  (respectively  $n_\beta^l$ ) goes from  $l$  to  $l_\alpha$  (resp.  $l_\beta$ ). We emphasize that  $\pi_\alpha(l) \neq \pi_\beta(l)$  and  $l = n_\alpha^l \cap n_\beta^l$  for any  $l \in \text{dS}^2$ , and that

$$l \in \text{dS}^2 \mapsto (\pi_\alpha(l), \pi_\beta(l)) \in \mathcal{C}^2 \setminus \{\text{diagonal}\} \equiv \mathbf{dS}^2$$

is a  $\text{SO}^0(1, 2)$ -bijection which naturally identifies  $\text{dS}^2$  with  $\mathbf{dS}^2$  once  $\mathcal{C}$  is projectively identified with  $\mathbb{RP}^1$  (compare with Remark 2.3).

For any plane  $S \subset \mathbb{R}^{1,2}$  and  $s \subset \text{dS}^2$  a geodesic defined by  $S$  (*i.e.* a connected component of  $S \cap \text{dS}^2$ ) which is not  $\alpha$ -lightlike, we now claim that the map  $\pi_\alpha|_s: s \rightarrow \mathcal{C}$  is projective for the affine parametrization of  $s$  and the projective structure of  $\mathcal{C}$ . The same proof shows of course that  $\pi_\beta|_s$  is projective if  $s$  is not  $\beta$ -lightlike. Indeed the stabilizer of  $S$  in  $\text{SO}^0(1, 2)$  contains a one-parameter subgroup  $(g^t)$  acting transitively on  $s$ , and  $t \in \mathbb{R} \mapsto g^t(x) \in s$  is an affine parametrization of  $s$  for any  $x \in s$ . The equivariance  $\pi_\alpha(g^t(x)) = g^t(\pi_\alpha(x))$  of  $\pi_\alpha$  concludes then the proof of the claim by definition of the projective structure of  $\mathcal{C}$ . Observe moreover that, unless  $s$  is  $\alpha$ -lightlike (in which case  $\pi_\alpha|_s$  is by definition constant),  $\pi_\alpha|_s$  is injective and defines thus a projective isomorphism onto its image (which equals  $\mathcal{C}$  if  $s$  is spacelike and an open proper subset in the remaining cases).

But for any two geodesics  $s_1, s_2$  of  $\text{dS}^2$  having the same signature, the holonomy  $H$  from  $s_1$  to  $s_2$  satisfies by definition the invariance  $\pi_\alpha|_{s_2} \circ H = \pi_\alpha|_{s_1}$  on the open subset where this equality is well-defined, showing that  $H$  is a projective isomorphism since the  $\pi_\alpha|_{s_i}$  are such.

**General case.** Let  $(S, \Sigma)$  be a singular  $\mathbf{X}$ -surface. Without lost of generality, we can assume that  $H$  is the holonomy of the  $\alpha$  foliation between compact connected subsets  $I$  and  $J$  of geodesics of  $S$ . Since  $\Sigma$  is discrete and  $\mathcal{F}_\alpha$  continuous, the set  $I_\Sigma$  of points  $p \in I$  such that  $[p; H(p)]_\alpha \cap \Sigma \neq \emptyset$  is discrete in  $I$ , hence finite (we denote by  $[p; H(p)]_\alpha$  the interval of the oriented leaf  $\mathcal{F}_\alpha(p)$  from  $p$  to  $H(p)$ ). Let  $C$  be a connected component of  $I \setminus I_\Sigma$ . Then for any  $x \in C$ , we can cover  $[x; H(x)]_\alpha$  by a finite chain of compatible regular  $\mathbf{X}$ -charts. This expresses  $H|_C$  as a finite composition of holonomies  $H_i$  between geodesics which are, for any  $i$ , contained in the domain of a given regular  $\mathbf{X}$ -chart. We proved previously that each  $H_i$  is projective, and  $H|_C$  is thus projective as a composition of such maps. This shows that  $H$  is piecewise projective and concludes the proof.  $\square$

## APPENDIX D. SINGULAR CONSTANT CURVATURE LORENTZIAN SURFACES AS LORENTZIAN LENGTH SPACES

We explain in this paragraph how globally hyperbolic singular  $\mathbf{X}$ -surfaces give examples of the *Lorentzian length spaces* introduced in [KS18].

The latter are natural Lorentzian counterparts of the usual metric length spaces (for which [BH99] is a classical reference), which give a synthetic approach to Lorentzian geometry by forgetting the metric itself and rather looking at its main geometric byproducts. Existing examples included for now (beyond smooth Lorentzian metrics) the Lorentzian metrics with low regularity, the cone structures [KS18, §5], the so-called “generalized cones” [AGKS21] and some gluing constructions [BR24]. To the best of our knowledge and understanding, the singular constant curvature Lorentzian surfaces as we introduce them here were not considered yet in the literature as examples of Lorentzian length spaces. It seems to us that they provide natural examples, as the constant curvature Riemannian metrics with conical singularities give important examples of metric length spaces.

We will quickly describe the relation with Lorentzian length spaces without entering into too much details, most of the technical work having been done in the Appendix A. Until the end of this section,  $S$  denotes a singular  $\mathbf{X}$ -surface endowed with the distance  $d_S$  induced by a fixed complete Riemannian metric.

The structure of a *causal space* on a set  $X$  is defined in [KS18, Definition 2.1] by a causal relation  $\leq$  (formally a reflexive and transitive relation) and a chronological relation  $\ll$  (formally a transitive relation contained in  $\leq$ ) on  $X$ . We endow of course our singular  $\mathbf{X}$ -surface  $S$  with the chronological and causal relations defined by the timelike and causal futures (see Definition A.10), namely by definition:

- (1)  $x \leq y$  if, and only if  $y \in J^+(x)$ ;
- (2)  $x \ll y$  if, and only if  $y \in I^+(x)$ .

On a metrizable causal space  $(X, d, \leq, \ll)$ , a *time-separation* function is then defined as a map  $\tau: X \times X \rightarrow [0; +\infty]$  such that  $x \not\leq y$  implies  $\tau(x, y) = 0$ ,  $\tau(x, y) > 0$  if and only if  $x \ll y$ ,  $\tau$  satisfies the reverse triangular inequality

$$(D.1) \quad \tau(x, z) \geq \tau(x, y) + \tau(y, z)$$

for any  $x \leq y \leq z$ , and  $\tau$  is lower semi-continuous. The two first conditions are by definition satisfied by the time-separation function  $\tau_S$  of  $S$  defined in (A.3), which also satisfies the reverse triangular inequality (D.1) according to Lemma A.17. Lastly, the lower semi-continuity of  $\tau_S$  is proved in the same way than the second part of the proof of Proposition A.22, which does not rely on global hyperbolicity (see also [Min19, Theorem 2.32]).  $(S, d_S, \leq, \ll, \tau_S)$  is then a *Lorentzian pre-length space* as defined in [KS18, Definition 2.8], and it is moreover automatically causally path connected as defined in [KS18, Definition 2.18, Definition 3.1].

We assume from now on that  $S$  is globally hyperbolic in the sense of Definition A.10. In this case the Lorentzian pre-length space  $(S, d_S, \leq, \ll, \tau_S)$  satisfies some additional nice properties. Lemma A.19 first shows that  $S$  is *causally closed* in the sense that if  $p_n \leq q_n$  respectively converge to  $p$  and  $q$ , then  $p \leq q$ . It is moreover easy to show that the restriction of  $\tau_S$  to a normal convex neighbourhood of  $S$  (see Proposition A.7) gives a localizing neighbourhood as defined in [KS18, Definition 3.16], hence that  $(S, d_S, \leq, \ll, \tau_S)$  is *strongly localizable*.

The last step to Lorentzian length spaces mimics the definition of usual metric length spaces. The  $\tau_S$ -length of a causal curve  $\gamma: [a; b] \rightarrow S$  is defined in [KS18, Definition 2.24] as

$$L_{\tau_S}(\gamma) = \inf \left\{ \sum_{i=0}^N \tau_S(\gamma(t_i), \gamma(t_{i+1})) \mid N \in \mathbb{N}, a = t_0 < t_1 < \dots < t_N = b \right\}.$$

Note that our usual notion of causal curve coincides with the one of [KS18, Definition 2.18] according to [KS18, Lemma 2.21]. Using [KS18, Proposition 2.32] and the decomposition (A.2) of the usual Lorentzian length  $L(\gamma)$  into the ones of its regular pieces, one easily shows that  $L(\gamma) = L_{\tau_S}(\gamma)$ . This last equality shows the following.



**Proposition D.1.** *Any globally hyperbolic singular  $\mathbf{X}$ -surface  $S$  has a natural structure of a regular Lorentzian length space  $(S, d_S, \leq, \ll, \tau_S)$  as defined in [KS18, Definition 3.22].*

We recall that according to Proposition A.16, class A closed singular  $\mathbf{X}$ -surfaces admit simple closed spacelike curves, and that  $\mathbb{Z}$ -coverings with respect to such curves give according to Lemma A.21 examples of globally hyperbolic singular  $\mathbf{X}$ -surfaces, hence of regular Lorentzian length spaces.

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MARTIN MION-MOUTON, MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES IN LEIPZIG

Email address: [martin.mion@mis.mpg.de](mailto:martin.mion@mis.mpg.de)

URL: <https://orcid.org/0000-0002-8814-0918>