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# Linear Canonical Transformations in Phase Space: the Gabor Frames Approach

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## Abstract

Linear Canonical Transformations are phase space deformations that can be used to describe linear optical systems. In geometric optics they correspond to the transformations of the light rays during propagation in arbitrary media, while in wave optics they describe the evolution of the space-frequency components. Although their meaning is non-ambiguous in the former case, in the latter it depends on the specific space-frequency decomposition of the wavefield. In this paper we gather several results from theoretical time-frequency analysis with Gabor Frames and provide a preliminary analysis on the implementation of phase space deformation for linear optical systems.

**Keywords :** Linear Optics, Phase Space, Symplectic Geometry

## 1 Introduction

Canonical Transformations have a long history of describing optical phenomena in phase space: before being used in particle physics and extended to quantum mechanics, they were originally designed for describing the ray propagation in geometric optics [9]. In this particular case, the phase space is the domain where the light rays are represented by the position and angle at which they intersect a reference plane, and canonical transformation describe the physically admissible transformations when the light travels from a plane to another. A very similar setting can be constructed for wave optics, in which rays are replaced by light beams; although the propagation is still represented by canonical transformations, they are now associated to integral operators acting on the wavefield section over reference planes. The Linear subset of canonical transformations corresponds to the special case of paraxial optics; although it is commonly recognized that it is largely underexploited [14], it has been the subject of intensive research in the past decades, either in terms of purely optics considerations or computational efficiency [10].

Linear Canonical Transformations (LCTs) elegantly relate ray-transfer matrices, acting in phase space, to integral operators acting in space or frequency domain. Compositions of optical transformations are then expressed as matrix multiplications in the former, and operator composition in the latter. This makes LCTs a powerful tool for computing light propagation through complex optical systems. However, although the integral operator is defined uniquely (up to an amplitude factor), the action of the ray-transfer matrix in phase space has to be interpreted according of the type of phase space representation: this is due to the fact that there is no standard space / frequency representation, mainly because of the Heisenberg inequalities that emerge in the wave optics context. Yet, in spite of their powerful correspondence with integral operators, LCTs are fundamentally phase space transformations, and as such, they allow a much better comprehension of the effect of the propagation through an optical system than the wave propagation operators; indeed, spatial representations of wavefields only depict chaotic diffraction patterns that show no direct correlation with the objects or scenes they originate from, whereas the representation of light in terms of beams is very similar to a decomposition into a set of rays, although beams are subject to a trade-off between localization in space and frequency.

In this article we argue that the new usages related to digital holography will heavily rely on phase space representations and we report on-going study aiming to provide a guideline for an efficient application of LCTs on a particular space / frequency representation, namely the *Gabor Frames*.

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All along the article, we will stress the fact that although the canonical transformations and their operators counterparts have to be handled by tough mathematical theories in the general case [19, 17], the linear case allows much simpler treatments. Indeed, the purpose of this communication is to gather various studies on the relations between canonical transforms and Gabor representations in the domains of optics, applied mathematics and signal processing, in order to provide a unified vision of the concepts involved in designing a real-time computation of light propagation in phase space. For this, we will first depict in Section 2 an informative application involving the phase space representation of a hologram and illustrate the importance of acting on its Gabor expansion rather than processing it in space domain. After reviewing the related topics and works involved in such a processing, we provide an elementary reminder on LCTs and their interpretation in optics in Section 3. Having stressed the importance of a sparse representation, we introduce in Section 4 the Gabor Transform and relate it to optical transformations. Finally, in Section 5 we discuss the different choices that emerge from covariance formulas describing the evolution of the Gabor coefficients.

Throughout the various topics considered in the article, we will continuously highlight the advantages of handling *linear* transformations compared to the generic case. However, we hope that the designed path could be a starting point to treat the non-linear case as well, i.e. come up with tractable algorithms for generic phase space propagation.

## 2 Motivation and related works

In this section we explain the motivation for handling phase space representations and deforming them with canonical transformations, and mention the major related works upon which our study is based.

Planar sections of monochromatic wavefields constitute the modern definition of *holograms*, extending the traditional diffraction pattern to a more generic and capture-agnostic representation. However, this representation is highly chaotic and it is difficult to relate portions of a hologram to meaningful parts of the 3D data that generated the wavefield. Instead, localized data in 3D space turn out to be spread over the whole surface of the hologram. On the contrary, in phase space, i.e. in the space-frequency domain, local information correspond to familiar primitives such as spherical waves, planar waves or oriented beams. Such semantic representation can be exploited for information compression or advanced editing, among others.

A typical example of a scenario involving at the same time compression and phase space processing is illustrated in Fig. 1. It represents an application delivering animated holograms from a server where the visual data have been encoded and stored in a compressed and semantically ordered manner. For this, holograms have been decomposed into Gabor Frames in order to produce coefficients that can be directly related to the position and orientation of the light beam. Doing so, it is easy to discard coefficients that would not be relevant for a given viewer, providing some sort of directional scalability [4].

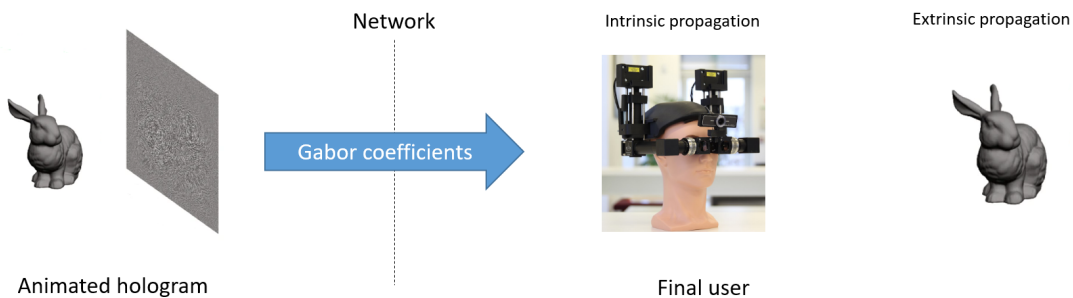


Figure 1: Typical hologram video transmission and recomputation on the fly from the phase space representation. Gabor coefficients are streamed through a network according to the visualization of the final user, equipped with an AR headset. The Gabor representation allows a viewpoint scalability. At the user end, a recomputed hologram is needed in order to take into account the traversal of the headset’s optical elements and the movements of the observer.

At the user end, the hologram is visualized on an augmented or virtual reality headset. Such a headset is likely to gain popularity in the future considering the recent progresses in terms of hologram displays and the current lacks of traditional stereoscopic headsets when it comes to vergence-accommodation conflict [5]. At arrival, the perspective on the virtual 3D scene has to be changed according to the user position and orientation. Hence a new hologram has to be recomputed and displayed on the local screen of the headset, taking into account:

- the internal optics of the headset, forming what we will call the *intrinsic propagation*
- the rigid 3D transform defined by the user position and orientation at a given time, that will give rise to *extrinsic propagation*.

The intrinsic propagation does not depend on time; it can be either computed as an operator in space domain or a canonical transformation in phase space domain. On the contrary, the extrinsic parameters will vary at each displacement of the user and each transition from a video frame to another.

Canonical transformations and their integral operators counterparts allow to model globally the successive steps of the hologram propagation from the original one to the displayed one when traversing the various optical elements within the headset and free space outside of it. In our example, they will operate on the coefficients so that the final hologram is obtained by a simple synthesis with the appropriate basis functions. We see that in this context, the phase space representations facilitates the compression, the adaptive transmission and the recomputation of the hologram; canonical transformations are the tools that enable processing the representation according for modelling the propagation in the optical system constituted by the headset, including free space propagation before entering the display device. The canonical transformation will also be helpful for pruning the set of coefficients sent at a given time: since the application targets a single user whose viewpoint can be located in a known viewing window, the mentioned LCT will provide a viewing window in phase space related to the *original hologram*, leading to a sub-hologram specific to the user requirements. We see that the stake here is to be able to reconstruct this sub-hologram in real-time. Since there exist fast algorithms to compute LCTs [18], the computation in phase space should be compared to the complexity of the mere propagation in space domain of the sub-hologram.

We collected results from various disciplines involving phase space representations in order to highlight the specificities of the linear setting and the simplifications over the general case, so as to come up with algorithms that can be run in real-time for applications such as the one presented above. Namely, the reference works our study is based upon concern symplectic geometry [9], metaplectic group [10], Gabor Frames and their relations to operators [8, 1, 11, 2] and Gabor synthesis on non-separable lattices [15, 16]. Note that since Linear Canonical Transformations constitute the subset of Canonical Transformations that model the paraxial optics, the illustrative virtue of this application example has its limitations; in practice it is likely that the some optical elements within the headsets will not suit to paraxial optics, because of the form factor that constraints the spacing between the elements. Also, the rotation of the headset is not a linear transformation, and its paraxial approximation is only relevant when the rotation angle is small. A theoretical study of the corresponding operator in phase space in the general case can be found in [6]. Hence our study aims to provide a basis for tackling the non-linear case as well, which is essential for other upcoming applications such as hologram video coding [13] or all kinds of hologram editing [6]. In order to simplify the formulas and illustrations, we will assume that the optical elements obey to Gaussian optics, i.e. that they have a rotation symmetry along their optical axis. This allows to reduce the phase space from a 4D to a 2D space wich is easier to depict.

### 3 Linear canonical transformations

Canonical Transformations, also called *symplectomorphisms* in mathematics, describe the physically acceptable evolution of a system in phase space, i.e. a space that has twice the dimensions of the space domain itself and that includes a momentum associated to every space dimension. In the case of geometric optics, the system is represented by a collection of light rays, and the (position  $\times$  momentum) representation is given by the location and orientation of the rays when they intersect a given surface. Generally, these surfaces are parallel planes, and the canonical transformation associated to a given optical system explicitly provides these generalized coordinates in the target plane in function of those in the source plane. This paradigm appears in many other domains of physics such as particle mechanics, quantum physics and wave optics. They share the same mathematical foundations constituted by the *symplectic geometry*. In a nutshell, saying that these transformations are symplectic means that they preserve a *symplectic form* that appears in the *Hamilton Equations* that rule the system evolution. Such a mapping is illustrated in Fig. 2.

In the case where the considered transformation is linear, this simply implies that its matrix  $M$  in a given coordinate system preserves a *symplectic matrix*  $J$  in the sense that

$$MJM^t = J, \quad (1)$$

with

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (2)$$

A general non-linear transformation is symplectic when it locally operates as in Equation (1), i.e. the latter is satisfied by its Jacobian matrix. The set of matrices satisfying (1) form a group called the *symplectic group*.

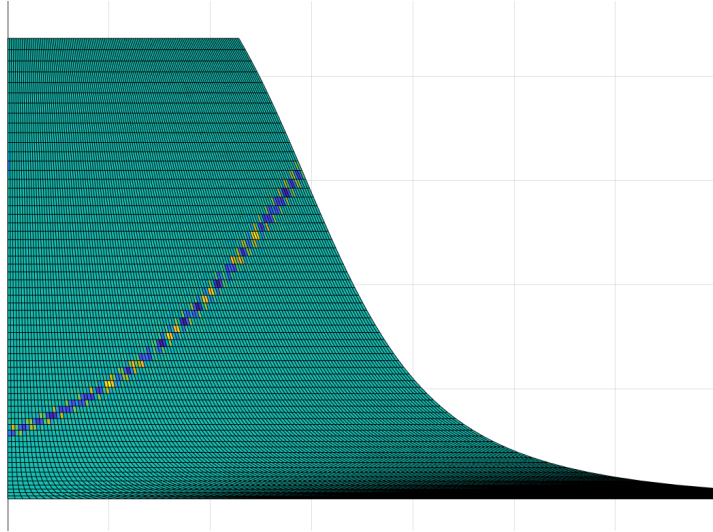


Figure 2: Deformation of a 2D phase space grid corresponding to a rotation of 25 degrees. The blue / yellow curve represents the phase space distribution of a point source, that is displaced accordingly. Although the mapping is not linear, it locally preserves the area, and more generally the symplectic form.

In particular the product of two symplectic matrices is again a symplectic matrix. It describes all paraxial optical systems and is much easier to understand and handle than the group of all canonical transformations. This simplicity especially emerges when trying to extend these transformations from geometric optics to monochromatic wave optics; it turns out that to every canonical transformation  $M$  one can associate an operator  $T_M$  that is either a quadratic phase modulation or an integral operator defined by

$$T_M\psi(x) = \int \psi(y)e^{\frac{i}{\lambda}\varphi(x,y)} dy, \quad (3)$$

where  $\psi$  is the section of the wavefield over the source plane,  $\lambda$  denotes the wavelength of the monochromatic wave and  $\varphi$  is a smooth function that depends only on  $M$ . The result of  $T_M$  represents the value of the wavefield at the target plane. The set of operators as in Equation (3) for which  $M$  is a linear transformation also form a group, called the *metaplectic group*, which is closely related to the symplectic group; in particular, matrix multiplications directly correspond to operator composition. An important fact is that this group is characterized by  $\varphi$  being a quadratic polynomial in  $x$  and  $y$ .

Although the mathematical theory justifying this correspondence is quite vast, the structure of the operator of Equation (3) can be understood intuitively by considering that  $\varphi$  represents the optical path of the equivalent ray in geometric optics. The integral can then be interpreted as the summation of the field amplitude and phase at the arrival of the ray. Similarly, the fact that the underlying canonical transformation displaces the space-frequency components is related to the fact that the momentum  $\sin \theta$  of a ray tilted by an angle  $\theta$  in geometric optics is equal to  $\lambda f$  by the well-known *grating equation*,  $f$  being the local frequency of the signal. This can be illustrated using the example described in the previous section: a typical optical system composing the headset is shown in Fig.3. According to the terminology defined previously, this will correspond to the *intrinsic* transformation.

The ray-transfer matrices, that correspond to the way the rays are transformed when traversing the optical elements, are precisely the elements of the symplectic group mentioned above. Catalogues of such correspondences are well documented in the literature [10]. In the example of Fig.3, the optical elements and matrices involved are:

- **Thin lens:** a thin lens of focal lens  $f$  is represented by the matrix  $\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$ .
- **Free space propagation:** propagation at distance  $d$  in Gauss conditions is represented by  $\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$ .
- **Beam-splitter:** the portion of light reflected by the mirror can be described by matrix  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ , and the portion passing through the splitter is simply described by the identity. The output waves are weighted by the respective transmission ratios.

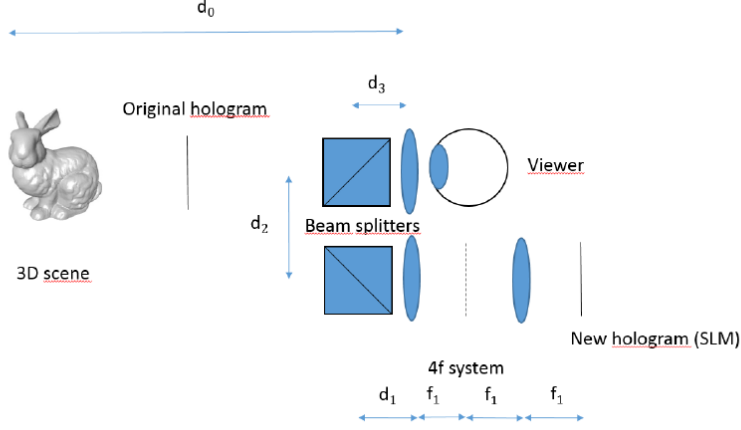


Figure 3: An example of internal optical system for a holographic AR headset. Each optical element is associated to a ray-transfer matrix, itself associated to an operator in space domain.

Because of the Gauss assumption, these matrices have size  $2 \times 2$  instead of  $4 \times 4$  in the general case. For a given wavelength  $\lambda$ , the resulting matrix  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is related to the metaplectic operator

$$T_M \psi(x) = (i\lambda b)^{-1} \int \psi(y) e^{\frac{i\pi}{b\lambda}(ay^2 + dx^2 - 2xy)} dy, \quad (4)$$

if  $b \neq 0$ , and

$$T_M \psi(x) = \sqrt{d} e^{\frac{i\pi}{\lambda} cdx^2} \psi(dx) \quad (5)$$

otherwise. Because the local frequency of a 2D wavefield section is not defined unambiguously, there is no standard way of transforming the space-frequency components in the same way that rays are transported by the underlying canonical transformation. Instead, the deformation of the space-frequency domain depends on the type of representation that is chosen for the signal in this domain. For example, a popular representation is the Wigner-Ville distribution function  $W$ , defined as:

$$W_\psi(x, \xi) = \int \overline{\psi(y - x/2)} \psi(y + x/2) e^{-2i\pi y \xi} dy. \quad (6)$$

We then have

$$W_{T\psi}(x, \xi) = W_\psi \circ M_\lambda^{-1}(x, \xi), \quad (7)$$

where

$$M_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix} M \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$$

is the normalized version of the ray-transfer matrix relating frequencies  $\xi$  to orientations  $\lambda\xi = \sin\theta$ . The fact that equality (7) is exact is again a consequence of the linearity of  $M$ . In the general case, this is only an approximation, that has to be corrected with terms that come from the wave nature of the light representation. However, the Wigner-Ville distribution is obtained by a continuous transform that is very greedy in terms of computation time. In the next section, we introduce Gabor Frames that will allow to remedy this issue.

## 4 Gabor frames and covariance

Gabor Frames constitute a discrete version of the *windowed Fourier Transform* (WFT), which is at the center of many space-frequency representations. The Wigner-Ville distribution mentioned in the previous section is a particular case of WFT. The WFT can be defined as

$$V_\psi^\phi(x, \xi) = \int \psi(y) \overline{\phi(y - x)} e^{-2i\pi y \xi} dy, \quad (8)$$



where  $\phi$  is a smooth function called the window of the transform. Expanding  $\psi$  in a Gabor Frame corresponds to discretizing  $V_\psi^\Phi$  and evaluating it on a lattice

$$\Lambda = \{(x, \xi) \in \mathbf{R}^4 : (x, \xi) = Au, u \in \mathbf{Z}^4\}, \quad (9)$$

where  $A$  is an invertible  $4 \times 4$  matrix or  $2 \times 2$  matrix in the Gaussian case. The initial wavefield  $\psi$  can be recovered by

$$\psi = \sum_{(x, \xi) \in \Lambda} V_\psi^{\tilde{\Phi}}(x, \xi) \pi(x, \xi) \tilde{\Phi}, \quad (10)$$

where  $\tilde{\Phi}$  is the *dual window* [7] of  $\Phi$  and  $\pi$  denotes the modulation / translation operator

$$\pi(x, \xi) f(y) = e^{2i\pi y \xi} f(y - x).$$

The theory [7] provides explicit conditions on  $\Lambda$  for the expansion to be exact. Intuitively, the lattice should be dense enough and satisfy criteria that are related to the Heisenberg inequalities : coarse sampling in the space domain should be compensated by a dense one in the frequency domain, and reciprocally. The special case when the inequality becomes equality is called the *critical sampling*. Just as in Equation (6), the linear case allows an exact expression of the space-frequency deformation; this starting point is a covariance formula [3] which relates the space-frequency shift  $\pi$ , the integral operator and the canonical transformation:

$$T\pi(x, \xi) = P(x, \xi) \pi(M_\lambda(x, \xi)^t) T, \quad (11)$$

where  $P$  is a phase factor that is a direct consequence of the non-commutativity of the modulation et translation part of  $\pi$ . For conciseness we have noted  $T$  instead of  $T_M$ . This formula can either be used at analysis or synthesis stage. At analysis we can write the Gabor coefficients in a given frame as

$$\begin{aligned} V_{T\psi}^{T\phi}(x, \xi) &= \langle T\psi, \pi(x, \xi) T\phi \rangle \\ &= \langle T\psi, T\pi(M_\lambda^{-1}(x, \xi)^t) \phi \rangle \end{aligned} \quad (12)$$

Once again the linearity of the mappings makes things easier because metaplectic operators are unitary, which means that the operator's action on both sides within the brackets can be removed:

$$\begin{aligned} V_{T\psi}^{T\phi}(x, \xi) &= \langle \psi, \pi(M_\lambda^{-1}(x, \xi)^t) \phi \rangle \\ &= V_\psi^\phi(M_\lambda^{-1}(x, \xi)^t). \end{aligned} \quad (13)$$

By combining Equation (13) with the synthesis equation applied to  $T\psi$  given by

$$T\psi = \sum_{(x, \xi) \in \Lambda} V_{T\psi}^{T\phi}(x, \xi) \pi(x, \xi) \tilde{T}\phi, \quad (14)$$

we get

$$T\psi = \sum_{(x, \xi) \in \Lambda} V_\psi^\phi(M_\lambda^{-1}(x, \xi)^t) \pi(x, \xi) \tilde{T}\phi. \quad (15)$$

where we can replace  $\tilde{T}\phi$  by  $T\tilde{\phi}$  because of the unitarity of  $T$ .

Now starting from the synthesis equation applied to  $\psi$  and inserting the covariance formula leads to a slightly different deformation of the representation; we write

$$\begin{aligned} T\psi &= T \sum_{(x, \xi) \in \Lambda} V_\psi^\phi(x, \xi) \pi(x, \xi) \tilde{\phi} \\ &= \sum_{(x, \xi) \in \Lambda} V_\psi^\phi(x, \xi) T\pi(x, \xi) \tilde{\phi} \\ &= \sum_{(x, \xi) \in \Lambda} V_\psi^\phi(x, \xi) \pi(M_\lambda(x, \xi)^t) T\tilde{\phi}. \end{aligned} \quad (16)$$

Equations (15) and (16) are very close but differ in the type of deformation of the space-frequency representation: in Equation (15) what is deformed is the coefficient distribution  $V_\psi^\phi$  while in Equation (16) the canonical transformation acts on the lattice of the Gabor Frames. This leads to different constraints when implementing the synthesis. In particular, performing the synthesis on a square grid as in Eq. (15) can be done using tensor products of 1D transforms, i.e. with the same asymptotic computational complexity.

## 5 Interpolation vs basis displacement

In this section we discuss the benefits and drawbacks of using either Equations (15) or (16). Fig.4 shows a chirp signal corresponding to a point source in 2D space (in Gaussian optics this is a plane containing the optical axis) and its phase space representation under the form of coefficients in a Gabor Frame. The considered transformation is a mere propagation in free space. We will use this example to illustrate the various computation strategies associated to the covariance equations.

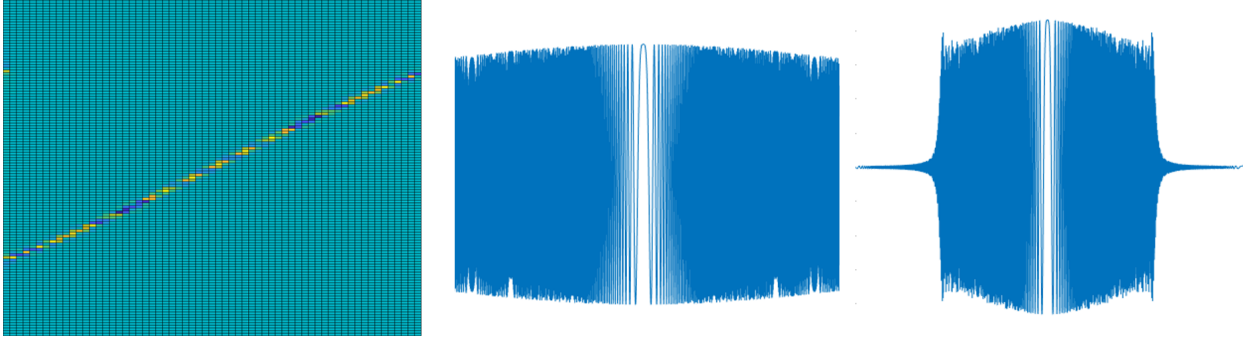


Figure 4: Example of a chirp signal corresponding to a point source (center). Left: phase space representation. The yellow / blue curve is the real part of the Gabor coefficients. The horizontal axis corresponds to space and the vertical one represents frequencies. Center: the function before propagation. Right: the function after propagation.

The spirit of Eq. (15) is to assign to each basis function  $\pi(x, \xi)T\tilde{\phi}$  the displaced coefficient  $V_{\psi}^{\phi}(M_{\lambda}^{-1}(x, \xi)^t)$ . A first way to recover  $V_{\psi}^{\phi}(M_{\lambda}^{-1}(x, \xi)^t)$  is to perform an analysis on  $M_{\lambda}^{-1}\Lambda$ . This is illustrated by Fig. 5 and we will call the corresponding process *coefficients recomputation*: the original signal is analyzed over the pre-deformed grid, and the synthesis is performed on the regular grid  $\Lambda$ . While this is the most accurate manner to recover  $V_{\psi}^{\phi}(M_{\lambda}^{-1}(x, \xi)^t)$ , this is at the cost of an analysis on a generally non-separable lattice. Indeed, separable lattices allow to perform the analysis in the space and frequency dimensions independently, which is not the case with non-separable lattices.

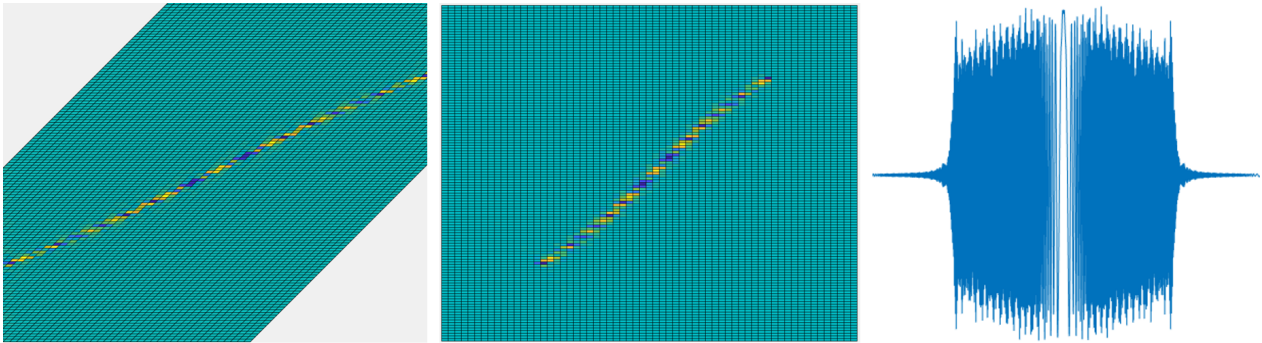


Figure 5: Recomputation of the Gabor coefficients. Left: coefficients distribution recomputed on deformed grid. Center: Recomputed coefficients on the original regular grid. Right: propagated signal obtained by synthesis over the regular lattice.

A faster algorithm to recover  $V$  on  $M_{\lambda}^{-1}\Lambda$  consists in interpolating the values of the initial phase space representation over a grid deformed by  $M_{\lambda}^{-1}$ , and perform a synthesis on the non-deformed grid just as in the recomputation case. For each element of  $M_{\lambda}^{-1}\Lambda$  it is enough to traverse and average its 4 neighbours (or 16 in the case of non-gaussian optics) on the regular grid  $\Lambda$  which are straightforward to identify. This process is illustrated in Fig.6. While in the linear case we couldn't exhibit significant interpolation errors, the study of the general case [6] shows that the rapidly varying phases can cause artifacts in the reconstructed signal. This is due to the difference of scale between the sampling performed by  $\Lambda$  and the rapid oscillations of  $V_{\psi}^{\phi}$ .



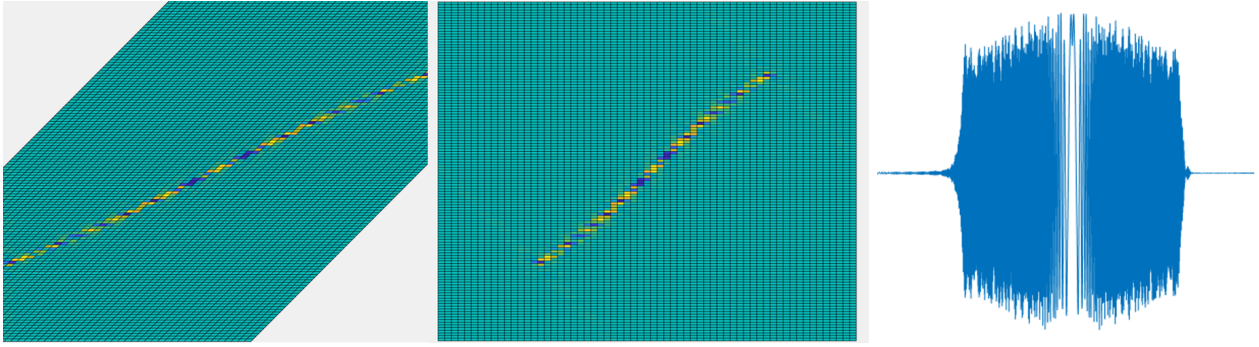


Figure 6: Interpolation of coefficients. Left: the original coefficients distribution is used to interpolate on the deformed grid. Center: the synthesis stage is the same as in the recomputation case. Right: reconstructed signal.

Hence, although the interpolation is a fast process, considering Eq.(16) could be an alternative computation when the reconstruction errors are too important. Equation (16) has the advantage of re-using the coefficients computed on lattice  $\Lambda$  but in return it requires a synthesis over an irregular grid, potentially preventing to exploit the tensor product of 1D frames and the associated gain in computation time, as this was the case for the non-separable analysis of the recomputation strategy. Figure 7 shows the corresponding process: the initial distribution (on the left) is deformed according to  $M_\lambda$  (center) and the synthesis is performed on the irregular grid.

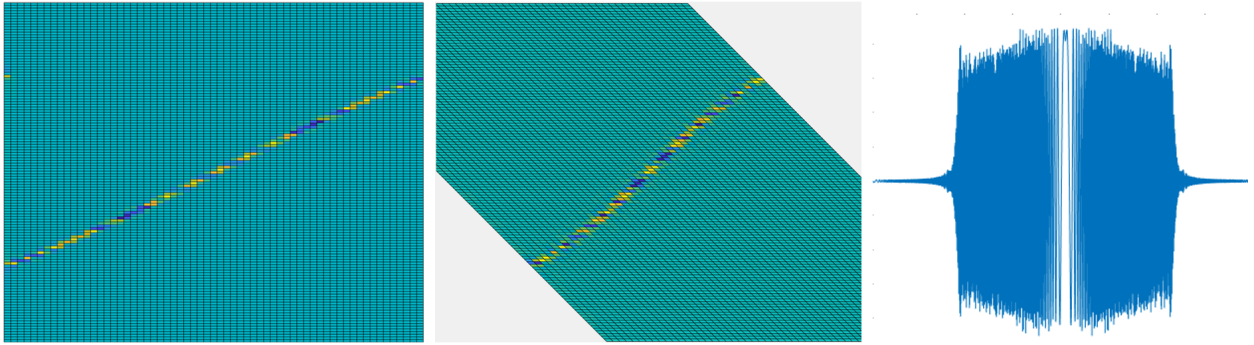


Figure 7: Basis displacement. Left: original coefficients on the regular grid. Center: displaced coefficients. Right: propagated signal after synthesis on the deformed grid.

What hinders the quick synthesis under the form of a tensor product is the fact that in the generic case the synthesis lattice is composed of irregularly spaced nodes. In the linear case,  $\Lambda$  is replaced by  $M_\lambda\Lambda$  which has regularly spaced nodes but since the dimensions are entangled the tensor product of Gabor synthesis cannot be straightforward applied. This issue is closely related to the design of Gabor Frames over non-separable lattices. Reducing the non-separable analysis and synthesis to separable ones have been the subject of several works that have approached the problem from various angles [15, 16], some of which have lead to efficient algorithms, among which those of the popular *lftat* matlab library:

- *Lattice partitioning*: when the the lattice can be reduced to an integer coordinates grid, it can be shown that it is the superposition of a finite number of separable lattices [12]. However, even if the general case could be approximated by an integer configuration, the number of sub-lattices can be unreasonably high since it depends on the profile of the resulting cells.
- *Phase Space deformation*: it can also be shown that, at least with a 2D phase space, a non-separable lattice can be deformed into a finite number of simple transformations such as sheers or transforms in the Special Linear Group. These mappings are contained in the symplectic group, hence at first sight this perfectly fits the requirements of our problem, but it actually tackles the opposite issue, i.e. reducing a non-separable analysis to a separable one.
- *Zak Transform*: the analysis, synthesis and window computation of Gabor Frames can be reformulated in a space / frequency domain and mapped to pointwise function products. While this interesting property

can be extended to the non-separable setting, it can only be exploited with appropriate sub-lattices just as in the lattice partitioning methods. This is due to the specific relations between the Zak transform and the space-frequency shifts, that is simplified only on regular lattices corresponding to the critical sampling of the Gabor Frames.

Indeed, other methods should be sought to reduce the computational complexity of the synthesis stage. A good candidate starting point could be the fact that the symplectic group is generated by shears, which benefit from efficient algorithms for Gabor synthesis, for example in the *lftat* library. Note that for either Eq. (15) or (16), we still have to apply the operator to  $\tilde{\phi}$ ; it is thus important that this step remains negligible in the whole computation, which can be realized by ensuring that  $\tilde{\phi}$  is well-localized in space. This can be enforced in two ways; the simplest one is to consider an oversampled Gabor Frame, for which the analysis and synthesis window will have good space localization. In the case of critical sampling, it is known that analysis and synthesis windows cannot be simultaneously well-localized in space. A solution then is to use badly localized window for the analysis since it is done offline. A third option is to impose a Gaussian function as the synthesis window. In this case the application of operator  $T$  can be derived exactly without the need to numerically compute the integral of formula (4).

## 6 Conclusions and future work

We reported preliminary directions of a work aiming to gather results from applied mathematics, optics and signal processing in order to provide real-time algorithms for phase space computation of light propagation across optical elements in paraxial regime. We outlined a global methodology for the application of the canonical transformations on a set of coefficients obtained by Gabor transform, and identified the tracks to reducing the computational complexity of the involved processes.

Future work naturally include evaluating the various strategies with respect to reconstruction quality and time complexity, implementing the corresponding algorithms in an actual hologram streaming application, and investigating the extensions to the non-linear framework.

## References

- [1] Elena Cordero, Karlheinz Gröchenig, and Fabio Nicola. Approximation of Fourier Integral Operators by Gabor Multipliers. *Journal of Fourier Analysis and Applications*, 18(4):661–684, August 2012.
- [2] Maurice A. de Gosson. Symplectic and Hamiltonian Deformations of Gabor Frames. *arXiv:1305.1025 [math-ph, physics:quant-ph]*, May 2013. arXiv: 1305.1025.
- [3] Maurice A. de Gosson. Hamiltonian deformations of Gabor frames: First steps. *Applied and Computational Harmonic Analysis*, 38(2):196–221, March 2015.
- [4] Anas El Rhammad, Patrick Gioia, Antonin Gilles, Marco Cagnazzo, and Béatrice Pesquet-Popescu. View-dependent compression of digital hologram based on matching pursuit. In *Optics, Photonics, and Digital Technologies for Imaging Applications V*, volume 10679, page 106790L. International Society for Optics and Photonics, May 2018.
- [5] Antonin Gilles, Pierre Le Gargasson, Grégory Hocquet, and Patrick Gioia. Holographic near-eye display with real-time embedded rendering. In *SIGGRAPH Asia 2023 Conference Papers*, SA '23, New York, NY, USA, 2023. Association for Computing Machinery.
- [6] Patrick Gioia, Antonin Gilles, Anas El Rhammad, and San Vu Ngoc. Phase space formulation of light propagation on tilted planes. [https://preprints.opticaopen.org/articles/preprint/Phase\\_space\\_formulation\\_of\\_light\\_propagation\\_on\\_tilted\\_planes/25048025](https://preprints.opticaopen.org/articles/preprint/Phase_space_formulation_of_light_propagation_on_tilted_planes/25048025), 1 2024.
- [7] Karlheinz Gröchenig. *Foundations of Time-Frequency Analysis*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Boston, MA, 2001.
- [8] Karlheinz Gröchenig, Joaquim Ortega-Cerdà, and José Luis Romero. Deformation of Gabor systems. *Advances in Mathematics*, 277:388–425, June 2015. arXiv: 1311.3861.
- [9] V. Guillemin and S. Sternberg. *Symplectic Techniques in Physics*. Cambridge University Press, 1990.
- [10] John J. Healy, M. Alper Kutay, Haldun M. Ozaktas, and John T. Sheridan, editors. *Linear canonical transforms: theory and applications*. Number 198 in Springer series in optical sciences. Springer, New York, 2016. OCLC: ocn910411981.

- [11] Christopher Heil. Integral Operators, Pseudodifferential Operators, and Gabor Frames. In Hans G. Feichtinger and Thomas Strohmer, editors, *Advances in Gabor Analysis*, pages 153–169. Birkhäuser Boston, Boston, MA, 2003.
- [12] A.J. Leest, van. *Non-separable Gabor schemes : their design and implementation*. Phd thesis 1 (research tu/e / graduation tu/e), Electrical Engineering, 2001.
- [13] Raees Kizhakkumkara Muhamad, Raees Kizhakkumkara Muhamad, David Blinder, David Blinder, Athanasia Symeonidou, Athanasia Symeonidou, Tobias Birnbaum, Tobias Birnbaum, Osamu Watanabe, Osamu Watanabe, Colas Schretter, Colas Schretter, Peter Schelkens, and Peter Schelkens. Exact global motion compensation for holographic video compression. *Applied Optics*, 58(34):G204–G217, December 2019. Publisher: Optical Society of America.
- [14] Adrian Stern. Why is the Linear Canonical Transform so little known? *AIP Conference Proceedings*, 860(1):225–234, October 2006.
- [15] Arno J. van Leest and Martin J. Bastiaans. Implementations of non-separable gabor schemes. In *2004 12th European Signal Processing Conference*, pages 1565–1568, 2004.
- [16] Christoph Wiesmeyr, Nicki Holighaus, and Peter L. Sondergaard. Efficient Algorithms for Discrete Gabor Transforms on a Nonseparable Lattice. *IEEE Transactions on Signal Processing*, 61(20):5131–5142, October 2013.
- [17] Jared Wunsch. Microlocal analysis and evolution equations. *arXiv:0812.3181 [math]*, December 2008. arXiv: 0812.3181.
- [18] Feng Zhang, Ran Tao, and Yue Wang. Discrete linear canonical transform computation by adaptive method. *Optics Express*, 21(15):18138, July 2013.
- [19] M. Zworski. *Semiclassical Analysis*. Graduate studies in mathematics. American Mathematical Society, 2012.