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THE VOLUME INTRINSIC TO A COMMUTATIVE GRADED ALGEBRA

KARIM ALEXANDER ADIPRASITO, STAVROS ARGYRIOS PAPADAKIS,
AND VASILIKI PETROTOU

ABSTRACT. Recent works of the authors have demonstrated the usefulness of considering moduli spaces of Artinian reductions of a given ring when studying standard graded rings and their Lefschetz properties. This paper illuminates a key aspect of these works, the behaviour of the canonical module under deformations in this moduli space. We demonstrate that even when there is no natural geometry around, we can give a viewpoint that behaves like it, effectively constructing geometry out of nothing, giving interpretation to intersection numbers without cycles. Moreover, we explore some properties of this normalization.

1. INTRODUCTION

The fundamental class of a variety or manifold is an important invariant, not least of all because it allows us to formulate Poincaré Duality. And it has a natural geometric interpretation as well: in a d -dimensional smooth manifold, de Rham cohomology associates it with the integration of a d -form over the manifold. And at this point, we can study what happens to the fundamental class under small deformations of the manifold. And if it makes sense to fix a d -form ω under this deformation, how does the evaluation under the fundamental class change?

We are in particular inspired by another case: That of toric varieties. If we consider a complete, rationally smooth toric variety of dimension d , we can consider it as part of an entire moduli space: we consider the family of all toric varieties with the same underlying combinatorics, or in other words, the same equivariant cohomology, parametrized by the torus action. Say it makes sense to consider an element ω of the top cohomology in such a way that we can study it under a deformation of the torus action. How does the evaluation change when we vary in the moduli space?

This question is invaluable, in particular in recent works of the authors, and studying it led to advances in combinatorial Lefschetz theory. It is, moreover, a first step towards understanding these new theories.

Let us once more examine the case of toric varieties, and explain what we mean.

Consider a unimodular complete fan Σ in \mathbb{R}^d , and the toric variety X associated to it. There are natural ways of thinking about the cohomology ring of X_Σ .

Here are two:

First, we can consider the ring of conewise polynomial functions $\mathcal{P}(\Sigma)$; it is isomorphic to the equivariant cohomology ring of X_Σ .

The cohomology ring of X_Σ is equally easy to compute:

$$\mathcal{A}(\Sigma) := \mathcal{P}(\Sigma) / \langle \mathcal{G} \rangle$$

where \mathcal{G} is the ideal generated by global polynomials. It is naturally a Poincaré duality algebra, so that

$$\mathcal{A}(\Sigma)_d \cong \mathbb{R}.$$

But what is the map? What is

$$\text{vol} : \mathcal{A}(\Sigma)_d \rightarrow \mathbb{R}.$$

It is nothing more natural than to think about this map, also known as the volume map, in terms of de Rham cohomology. We then obtain, assuming the fan Σ is unimodular, or equivalently X_Σ is smooth, a natural normalization. Consider for instance the natural basis for $\mathcal{A}(\Sigma)_1$: the characteristic function of a ray χ_ρ is conewise linear and vanishes on each

ray of Σ but ρ , where it is positive. Let v_ρ denote the point on ρ where it attains the value 1.

Then, if $(\rho_i)_{i \in [d]}$ is a collection of d rays of Σ , then

$$\text{vol} \left(\prod_{i \in [d]} \chi_{\rho_i} \right) = \left| \frac{1}{\Delta v_{\rho_i}} \right|,$$

where Δv_{ρ_i} denotes the determinant of the $d \times d$ matrix with i -th row equal to v_{ρ_i} .

That is, in a circumvent way, what happens in toric varieties, where the above recovers the natural degree map coming from algebraic geometry, computing intersection numbers faithfully. As most of what we consider takes place in the realm of commutative algebra, and degree has two possible meanings, we shall forgo calling it degree throughout, and instead refer to this as the volume map throughout.

Second, to explain this another way, let us switch models, and discuss what happens in Stanley-Reisner rings, or rather, face rings.

Consider Σ a triangulated sphere of dimension $d-1$, and \mathbb{k} an infinite field. The face ring of Σ is $\mathbb{k}[\Sigma]$, the quotient of the polynomial ring $\mathbb{k}[\mathbf{x}]$ (with indeterminates indexed by the vertices) by the ideal I_Σ generated by nonfaces.

To obtain a Poincaré duality algebra, we consider the quotient of $\mathbb{k}[\Sigma]$ by $\Theta \mathbb{k}[\Sigma]$, where $\Theta = (\theta_{i,j}) \mathbf{x}$ is a collection of d linear forms, encoded by a matrix $(\theta_{i,j})$ of entries in \mathbb{k} whose columns are indexed by the vertices of Σ and the indeterminates of the face ring. If we think of the fundamental class of the simplicial homology as a cycle $\mu = \mu_\Sigma$, then for F a facet of Σ we have

$$\text{vol} \left(\prod_{i \in F} x_i \right) = \frac{\text{sgn}_\mu F}{\Delta((\theta_{i,j}))|_F}$$

where $\text{sgn}_\mu F$ is the oriented sign of F and $\Delta((\theta_{i,j}))|_F$ is the determinant of the minor of $(\theta_{i,j})$ cut out by F .

So, we have an, in several ways natural, normalization of the volume map. And we have established several times now how powerful understanding this function can be. Let us recall why, and then aim at understanding the volume map in general.

1.1. Motivation: Anisotropy and generic Artinian reductions. One of the most important motivations towards understanding the geometry of volume maps arises in Lefschetz theory: In [1], it was observed that the hard Lefschetz property for a Gorenstein standard graded ring R is true if for a sufficiently large set of ideals I in $\mathcal{A}(R)$, an Artinian reduction of R , the Poincaré pairing is non-degenerate when restricted to I . In [48], and subsequently [2, 3], one studies the most extreme form of this principle: non-degeneracy at principal ideals.

In other words, we consider u in $\mathcal{A}(R)$ of degree at most k , k less or equal to $d/2$, d being the socle degree, and investigate nonvanishing of u^2 . We may assume that $k = d/2$. Now, there is an issue here: How do we choose the appropriate Artinian reduction, indexed by the linear system of parameters $\Theta = (\theta_{i,j}) \mathbf{x}$? The trick is to pass from R , which is a quotient of a polynomial ring over a field \mathbb{k} , to a quotient of a polynomial ring over the purely transcendental field extension $\mathbb{k}(\theta_{i,j})$.

Now, the idea is simple: We want to understand $\text{vol}(u^2)$ as a rational function in the variables $(\theta_{i,j})$.

In [48] and [2], this is then understood by considering differential identities for $\text{vol}(u^2)$ to understand the case of face rings.

In [3], we explore instead semigroup algebras associated to lattice polytopes, and understand $\text{vol}(u^2)$ by exploring Parseval-Rayleigh identities for $\text{vol}(\cdot)$.

But that leaves out the main question: How does $\text{vol}(\cdot)$ behave at all, how does it depend on $\theta_{i,j}$? Our ultimate goal is to understand this problem towards the following conjecture:

Conjecture 1.1. Consider R a reduced standard graded Gorenstein ring. Then a generic Artinian reduction of R has the strong Lefschetz property.

This is motivated by a slightly less general conjecture of Stanley [20]: he asks whether the same is true for integral domains R (which is obviously more restrictive than reduced). Alas, to start with the methods of generic Lefschetz theory and attack either conjecture, the first question one must ask, and which we tackle here, is the question: What is $\text{vol}(\cdot)$? To learn to speak of it concretely is a main goal of this paper.

1.2. Plan for the paper. Philosophically speaking, our normalization is the minimal polynomial identity the volume map satisfies. We make this precise in the next section by introducing the notion of a generic polynomial reduction. We then introduce the Kustin-Miller, or KM normalization, using this notion, and immediately note two alternative ways of doing so. We then illustrate our theory on some important examples. To make the theory more flexible, we finally introduce relative normalization, and show that it explains several phenomena of the normalization more succinctly, and allows us to explain the normalization for semigroup algebras of lattice polytopes introduced and studied in [3].

2. GENERIC POLYNOMIAL REDUCTION

We introduce and study the notion of generic polynomial reduction of a graded algebra, which will play an essential role in the definition of volume normalization in Section 3.

Assume $m \geq 1$ and \mathbb{k} is a field. We consider the polynomial ring $\mathbb{k}[x_1, \dots, x_m]$, where the degree of the variable x_i is equal to 1, for all $1 \leq i \leq m$. Assume $I \subset \mathbb{k}[x_1, \dots, x_m]$ is a homogeneous ideal. We denote by d the Krull dimension of the quotient ring $\mathbb{k}[x_1, \dots, x_m]/I$ and assume $d \geq 1$.

We assume $\theta_{i,j}$ are new variables, and we set R_{up} to be the graded polynomial ring

$$R_{up} = \mathbb{k}[x_1, \dots, x_m, \theta_{i,j} : 1 \leq i \leq d, 1 \leq j \leq m].$$

We equip R_{up} with a bidegree as follows: We say that an element $u \in R_{up}$ is bihomogeneous of degree (b, c) if when considered as a polynomial in x_i is homogeneous of degree b and when considered as a polynomial in $\theta_{i,j}$ is homogeneous of degree c . We set, for $1 \leq i \leq d$,

$$f_i = \sum_{j=1}^m \theta_{i,j} x_j.$$

We denote by I_{up} the ideal of R_{up} generated by the subset $I \cup \{f_1, \dots, f_d\}$, and by N_{up} the ideal

$$N_{up} = I_{up} : (x_1, \dots, x_m).$$

Definition 2.1. We define the *generic polynomial reduction* of $\mathbb{k}[x_1, \dots, x_m]/I$ to be the R_{up} -algebra

$$R_{up}/I_{up}.$$

In Subsection 2.2 we will prove the following important proposition.

Proposition 2.2. *We have*

$$\dim(R_{up}/I_{up}) = dm.$$

In other words, the Krull dimension of R_{up}/I_{up} is equal to dm .

It is clear that the ideals I_{up} and N_{up} are bihomogeneous, hence it follows that N_{up}/I_{up} is a bihomogeneous ideal of the bigraded ring R_{up}/I_{up} . For $b \geq 0$ we set

$$(N_{up}/I_{up})_{(b,-)} = \bigoplus_{c \geq 0} (N_{up}/I_{up})_{(b,c)}.$$

Since $x_i(N_{up}/I_{up}) = 0$ for all $1 \leq i \leq m$, it follows that $(N_{up}/I_{up})_{(b,-)}$ is an R_{up} -submodule of N_{up}/I_{up} .

In the following, h_1, \dots, h_s denote s bihomogeneous elements of R_{up} with the property that $h_1 + I_{up}, \dots, h_s + I_{up}$ is a minimal generating set for the R_{up} -module N_{up}/I_{up} . Moreover, $\deg_x h_t$ denotes the degree of h_t with respect to the variables x_j .

We denote by $\mathbb{k}[\theta_{i,j}]$ the polynomial ring

$$\mathbb{k}[\theta_{i,j} : 1 \leq i \leq d, 1 \leq j \leq m],$$

by $E = \mathbb{k}(\theta_{i,j})$ the field of fractions of $\mathbb{k}[\theta_{i,j}]$, and we set

$$R_{down} = E[x_1, \dots, x_m],$$

with $\deg(x_i) = 1$ for all i .

We denote by I_{down} the ideal of R_{down} generated by the subset $I \cup \{f_1, \dots, f_d\}$, and by N_{down} the ideal

$$N_{down} = I_{down} : (x_1, \dots, x_m).$$

Remember now that the *generic Artinian reduction* \mathcal{A} of $\mathbb{k}[x_1, \dots, x_m]/I$ is the Artinian E -algebra

$$\mathcal{A} = R_{down}/I_{down}.$$

We denote by e the top degree of \mathcal{A} , this means that the e -th graded component \mathcal{A}_e of \mathcal{A} is nonzero and $\mathcal{A}_s = 0$ when $s \geq e + 1$. Obviously $\mathcal{A}_e \subset \text{Socle}(\mathcal{A})$, where

$$\text{Socle}(\mathcal{A}) = \{u \in \mathcal{A} : ux_i = 0 \text{ for all } 1 \leq i \leq m\}$$

denotes the socle of \mathcal{A} . Moreover, from the definition of N_{down} it follows that

$$N_{down}/I_{down} = \text{Socle}(\mathcal{A}).$$

We set

$$S = \mathbb{k}[\theta_{i,j} : 1 \leq i \leq d, 1 \leq j \leq m] \setminus \{0\}.$$

Clearly S is a multiplicatively closed subset of R_{up} . Every element of R_{down} can be written in the form p/u with $p \in R_{up}$ and $u \in S$. It follows that R_{down} and $S^{-1}R_{up}$ are naturally isomorphic. In particular, there exists a natural injective localization map $\Phi_1 : R_{up} \rightarrow R_{down}$ that sends p to p for all $p \in R_{up}$.

It is clear that $\Phi_1(I_{up}) \subset I_{down}$ and $\Phi_1(N_{up}) \subset N_{down}$. Consequently, there is an induced homomorphism

$$\Phi : N_{up}/I_{up} \rightarrow N_{down}/I_{down} = \text{Socle}(\mathcal{A})$$

such that $\Phi(u + I_{up}) = u + I_{down}$ for all $u \in N_{up}$. The map Φ respects degrees, in the sense that

$$\Phi((N_{up}/I_{up})_{(b,-)}) \subset (N_{down}/I_{down})_b$$

for all $b \geq 0$.

Proposition 2.3. *We have*

$$(\Phi(N_{up}/I_{up})) = \text{Socle}(\mathcal{A}).$$

In other words, the ideal of \mathcal{A} generated by the image of Φ is equal to $\text{Socle}(\mathcal{A})$.

Proof. Since $I_{down} = S^{-1}I_{up}$, the result follows from [6, p. 42, Remark 1]. \square

Corollary 2.4. *Assume $i \geq 0$. Then*

$$\{ \Phi(h_t + I_{up}) : \deg_x h_t = i \}$$

is a generating set for the E -vector space $(\text{Socle}(\mathcal{A}))_i$.

Proof. Since for all $1 \leq j \leq m$ we have $x_j(h_t + I_{up}) = 0 + I_{up}$, we have that the result follows from Proposition 2.3. \square

Remark 2.5. An immediate consequence of Corollary 2.4 is that if $i \geq 0$ has the property that $(\text{Socle}(\mathcal{A}))_i$ is nonzero, then there exists t such that $\deg_x h_t = i$. In particular, since $(\text{Socle}(\mathcal{A}))_e = \mathcal{A}_e \neq 0$, there exists h_t with $\deg_x h_t = e$.

Remark 2.6. See Examples 2.14 and 2.15 for two examples where Φ is not injective and (for suitable i) the set

$$\{ \Phi(h_t + I_{up}) : \deg_x h_t = i \}$$

is not a basis of \mathcal{A}_i as E -vector space.

Corollary 2.7. *The ideal (x_1, \dots, x_m) of R_{up} is a minimal associated prime ideal of R_{up}/I_{up} .*

Proof. We have $I_{up} \subset (x_1, \dots, x_m)$. Since $\dim R_{up}/(x_1, \dots, x_m) = dm$ and by, Proposition 2.2, $\dim R_{up}/I_{up} = dm$, we get that the ideal (x_1, \dots, x_m) is a minimal prime of I_{up} . Hence, by [27, Theorem 3.1 a.] it is also an associated prime. \square

Corollary 2.8. *Assume $\mathbb{k}[x_1, \dots, x_m]/I$ is Cohen-Macaulay. Then R_{up}/I_{up} is Cohen-Macaulay.*

Proof. Using the assumption, we get by Proposition 2.2 that R_{up}/I_{up} is Cohen-Macaulay, since it is the quotient of $R_{up}/(I)$ by a homogeneous regular sequence. \square

Proposition 2.9. *Assume R_{up}/I_{up} has no embedded associated prime ideals. Then we have that the map $\Phi : N_{up}/I_{up} \rightarrow \text{Socle}(\mathcal{A})$ is injective.*

Proof. Suppose $u \in N_{up} \setminus I_{up}$ has the property that $\Phi(u + I_{up}) = 0$. Since, by [6, p. 39, Corollary 3.4 iii)], Φ is a localization map there exists, by [6, p. 37], nonzero $p \in \mathbb{k}[\theta_{i,j}]$ such that $pu \in I_{up}$. Hence

$$(x_1, \dots, x_m, p) \subset I_{up} : (u).$$

By [27, Proposition 3.4] there exists an associated prime q of R_{up}/I_{up} such that

$$(x_1, \dots, x_m, p) \subset q.$$

This is a contradiction, since by Corollary 2.7 (x_1, \dots, x_m) is an associated prime of R_{up}/I_{up} and, by assumption, R_{up}/I_{up} has no embedded associated prime ideals. \square

Remark 2.10. See Examples 2.14 and 2.15 for two examples where Φ is not injective.

Corollary 2.11. *Assume $\mathbb{k}[x_1, \dots, x_m]/I$ is Cohen-Macaulay. Then we have that the map $\Phi : N_{up}/I_{up} \rightarrow \text{Socle}(\mathcal{A})$ is injective.*

Proof. By Corollary 2.8 R_{up}/I_{up} is Cohen-Macaulay, hence by [7, p. 58, Theorem 2.1.2] it has no embedded associated prime ideals. The result follows from Proposition 2.9. \square

Since $\mathbb{k}[\theta_{i,j}]$ is a subring of R_{up} , it is natural to consider, for $b \geq 0$, $(N_{up}/I_{up})_{(b,-)}$ also as a $\mathbb{k}[\theta_{i,j}]$ -module.

Proposition 2.12. *Assume that the R_{up} -module $(N_{up}/I_{up})_{(e,-)}$ is cyclic and denote by u a generator. Then the multiplication map*

$$s_u : \mathbb{k}[\theta_{i,j}] \rightarrow (N_{up}/I_{up})_{(e,-)}$$

with

$$s_u(p(\theta_{i,j})) = p(\theta_{i,j})u$$

for all $p(\theta_{i,j}) \in \mathbb{k}[\theta_{i,j}]$, is an isomorphism of $\mathbb{k}[\theta_{i,j}]$ -modules.

Proof. It is clear that s_u is a homomorphism of $\mathbb{k}[\theta_{i,j}]$ -modules. Since $x_j(N_{up}/I_{up}) = 0$ for all $1 \leq j \leq m$, the map s_u is surjective by the assumption that u generates $(N_{up}/I_{up})_{(e,-)}$ as an R_{up} -module.

Assume $p(\theta_{i,j}) \in \mathbb{k}[\theta_{i,j}]$ is an element in the kernel s_u . By Corollary 2.4, we have that \mathcal{A}_e is a 1-dimensional vector space over E and $\Phi(u)$ is an E -basis of \mathcal{A}_e . Hence, $s_u(p(\theta_{i,j})) = 0$ implies $p(\theta_{i,j})\Phi(u) = 0$ which implies $p(\theta_{i,j}) = 0$. Therefore, s_u is also injective. \square

2.1. Examples and Questions. In this subsection we discuss a number of examples and questions about the generic polynomial reduction. Related Macaulay2 [28] code is contained in the file <http://users.uoi.gr/spapadakis/volumem2codev1.txt>.

Example 2.13. Assume \mathbb{k} is any field and $\mathbb{k}[x_1, \dots, x_n]/I$ is Gorenstein. Recall that e denotes the top degree of the generic Artinian reduction \mathcal{A} of $\mathbb{k}[x_1, \dots, x_n]/I$. We will prove in Section 5 that the R_{up} -module N_{up}/I_{up} is cyclic and generated by an element $u \in (N_{up}/I_{up})_{(e,-)}$. For an explicit computation of u in the case where $\mathbb{k}[x_1, \dots, x_n]/I$ is the face ring of a simplicial sphere, see Section 6.

Example 2.14. Assume \mathbb{k} is any field and $I = (x_1^5, x_1x_2) \subset \mathbb{k}[x_1, x_2]$. Then, Macaulay2 computations suggest the following: The generic Artinian reduction \mathcal{A} of $\mathbb{k}[x_1, x_2]/I$ has Hilbert function $(1, 1, 0, 0, \dots)$, hence the top degree of \mathcal{A} is 1. The R_{up} -module N_{up}/I_{up} is minimally generated by its subset

$$\{ \theta_{1,2}x_2 + I_{up}, \quad x_1^4 + I_{up} \}.$$

This implies that $(N_{up}/I_{up})_{(1,-)}$ is a cyclic R_{up} -module with generator $\theta_{1,2}x_2 + I_{up}$ and that Φ is not injective, since $x_1^4 + I_{up}$ is a nonzero element in the kernel.

Example 2.15. Assume \mathbb{k} is any field and $I = (x_1^5, x_1x_2, x_1x_3) \subset \mathbb{k}[x_1, x_2, x_3]$. Macaulay2 computations suggest the following: The generic Artinian reduction \mathcal{A} of $\mathbb{k}[x_1, x_2, x_3]/I$ has Hilbert function $(1, 1, 0, 0, \dots)$, hence the top degree of \mathcal{A} is 1. The R_{up} -module N_{up}/I_{up} is minimally generated by its subset $\{h_1 + I_{up}, h_2 + I_{up}, h_3 + I_{up}\}$, where

$$h_1 = \theta_{1,2}x_2 + \theta_{1,3}x_3, \quad h_2 = \theta_{2,2}x_2 + \theta_{2,3}x_3, \quad h_3 = x_1^4.$$

It holds $\theta_{2,1}(h_1 + I_{up}) - \theta_{1,1}(h_2 + I_{up}) = 0$ and Φ restricted to $(N_{up}/I_{up})_{(1,-)}$ is injective. Therefore, $(N_{up}/I_{up})_{(1,-)}$ is not a cyclic R_{up} -module and Φ is not injective, since $h_3 + I_{up}$ is a nonzero element in the kernel.

Example 2.16. Assume \mathbb{k} is any field and D is the simplicial complex triangulating the torus $S^1 \times S^1$ described in [53, p. 70, Exerc. 5.1.8]. We denote by $I \subset \mathbb{k}[x_1, \dots, x_{10}]$ the Stanley-Reisner ideal of D . We remark that by Reisner's criterion ([7, p. 235, Corollary 5.3.9]) $\mathbb{k}[x_1, \dots, x_{10}]/I$ is not Cohen-Macaulay. Macaulay2 computations suggest the following: The generic Artinian reduction \mathcal{A} of $\mathbb{k}[x_1, \dots, x_{10}]/I$ has Hilbert function $(1, 7, 13, 1, 0, 0, \dots)$, while the socle of \mathcal{A} has Hilbert function $(0, 0, 6, 1, 0, 0, \dots)$. The R_{up} -module N_{up}/I_{up} is minimally generated by 7 homogeneous elements u_1, \dots, u_7 such that $u_i \in (N_{up}/I_{up})_{(2,-)}$ for $1 \leq i \leq 6$ and

$$u_7 = [8, 9, 10]x_8x_9x_{10} + I_{up} \in (N_{up}/I_{up})_{(3,-)}.$$

Here $[8, 9, 10]$ denotes the determinant of the 3×3 submatrix of the 3×10 matrix $[\theta_{i,j}]$ obtained by keeping columns 8, 9 and 10. Moreover, we have that

$$\Phi(u_1), \dots, \Phi(u_7)$$

is an E -basis of $\text{Socle}(\mathcal{A})$. We remark that (up to sign) in the paper [2] the volume normalization on \mathcal{A} , which was denoted there by deg , was defined by the condition that $\Phi(u_7)$ maps to 1.

Example 2.17. Assume that \mathbb{k} is any field, $R = \mathbb{k}[x_1, x_2, g_{1,1}, g_{1,2}, g_{2,2}]$ and

$$I = (g) \subset R,$$

where

$$g = g_{1,1}x_1^2 + g_{1,2}x_1x_2 + g_{2,2}x_2^2$$

is the generic degree 2 hypersurface in 2 variables x_i . We write $g = p_1x_1 + p_2x_2$, for some (non-unique) $p_1, p_2 \in R$ and set

$$u = (-p_1\theta_{1,2} + p_2\theta_{1,1}) \in R_{up}.$$

Then, by Subsection 5.1 below we have that $(I_{up} : (x_1, x_2))/I_{up}$ is a cyclic R_{up} -module generated by $u + I_{up}$. We denote by $\text{vol} : \mathcal{A}_1 \rightarrow E$ the volume normalization linear isomorphism

uniquely specified by the condition that $u + I_{down}$ maps to 1. In the special case where the field \mathbb{k} has characteristic 2, one can show the *Parseval Equations*

$$(1) \quad \text{vol}(x_1) = g_{1,1}\theta_{1,2}(\text{vol}(x_1))^2 + g_{1,2}\theta_{1,1}(\text{vol}(x_1))^2 + g_{2,2}\theta_{1,2}(\text{vol}(x_2))^2$$

and

$$(2) \quad \text{vol}(x_2) = g_{1,1}\theta_{1,1}(\text{vol}(x_1))^2 + g_{1,2}\theta_{1,2}(\text{vol}(x_2))^2 + g_{2,2}\theta_{1,1}(\text{vol}(x_2))^2,$$

where, for $1 \leq i \leq 2$, by abuse of notation we denote $\text{vol}(x_i + I_{down})$ by $\text{vol}(x_i)$. It is interesting to compare Equations (1) and (2) with the Parseval Equations obtained in the reference [3, Section 5] and the Differential Identities conjectured in [48, Section 14] and proven in [36, Section 4].

Question 2.18. The strongest one can hope is that

$$\Phi(h_1 + I_{up}), \dots, \Phi(h_s + I_{up})$$

is an E -basis of $\text{Socle}(\mathcal{A})$. This happens, by the results in Section 5, if $\mathbb{k}[x_1, \dots, x_m]/I$ is Gorenstein. It also happens in the torus triangulation Example 2.16. However, it fails in Examples 2.14 and 2.15. We believe it will be interesting to investigate under which conditions on I it holds.

Question 2.19. Another interesting question is under which conditions on I the map $\Phi : N_{up}/I_{up} \rightarrow \text{Socle}(\mathcal{A})$ is injective. By Corollary 2.11 this is the case if $\mathbb{k}[x_1, \dots, x_m]/I$ is Cohen-Macaulay. It also happens in the non-Cohen-Macaulay torus triangulation Example 2.16, but it fails in Examples 2.14 and 2.15.

Question 2.20. Assume that the dimension of the top degree \mathcal{A}_e as E -vector space is 1. We believe it will be worth to investigate under which conditions the R_{up} -module $(N_{up}/I_{up})_{(e,-)}$ is cyclic. This is the case, by the results in Section 5, if $\mathbb{k}[x_1, \dots, x_m]/I$ is Gorenstein, in the torus triangulation Example 2.16 and in Example 2.14. However, it fails in an interesting way in Example 2.15.

2.2. Proof of Proposition 2.2.

Proposition 2.21. *Assume R is a commutative ring of finite Krull dimension, I is a proper ideal of R , $s \geq 1$ and $f_{(i,1)}, f_{(i,2)} \in R$, for $1 \leq i \leq s$. We set, for $1 \leq i \leq s$,*

$$g_i = f_{(i,1)}f_{(i,2)}.$$

We assume that for any sequence j_1, \dots, j_s , with $j_t \in \{1, 2\}$ for all t , we have

$$\dim R/(I, f_{(1,j_1)}, f_{(2,j_2)}, \dots, f_{(s,j_s)}) = \dim R/I - s.$$

Then,

$$\dim R/(I, g_1, \dots, g_s) = \dim R/I - s.$$

Proof. By the definition of Krull dimension

$$\dim R/I = \sup \{ \dim R/p : p \in \text{Spec}(R), I \subset p \}.$$

Assume $p \in \text{Spec} R$ with $(I, g_1, \dots, g_s) \subset p$. Since p is prime and $g_i \in p$ for all i , there exists a sequence j_1, \dots, j_s with $j_t \in \{1, 2\}$ for all t , such that

$$(I, f_{(1,j_1)}, f_{(2,j_2)}, \dots, f_{(s,j_s)}) \subset p.$$

By the assumptions

$$\dim R/(I, f_{(1,j_1)}, f_{(2,j_2)}, \dots, f_{(s,j_s)}) = \dim R/I - s,$$

hence $\dim R/p \leq \dim R/I - s$, which implies that

$$(3) \quad \dim R/(I, g_1, \dots, g_s) \leq \dim R/I - s.$$

Conversely, there exists $p \in \text{Spec}(R)$ with

$$(I, f_{(1,1)}, f_{(2,1)}, \dots, f_{(s,1)}) \subset p$$

and $\dim R/p = \dim R/I - s$. Since

$$(I, g_1, \dots, g_s) \subset (I, f_{(1,1)}, f_{(2,1)}, \dots, f_{(s,1)}) \subset p$$

we get

$$(4) \quad \dim R/(I, g_1, \dots, g_s) \geq \dim R/I - s$$

Combining Inequalities (3) and (4) the result follows. \square

Remark 2.22. Assume \mathbb{k} is a field, $\mathbb{k} \subset F$ is a field extension and \mathcal{A} is a finitely generated \mathbb{k} -algebra. Then, by [41, Tag 00P3]

$$\dim(\mathcal{A} \otimes_{\mathbb{k}} F) = \dim \mathcal{A}.$$

We now give the proof of Proposition 2.2.

STEP 1. Using Remark 2.22, by passing to an infinite field extension of \mathbb{k} (for example the field $\mathbb{k}(t)$ of rational functions over \mathbb{k} in one variable) it is enough to prove the proposition assuming that \mathbb{k} is infinite.

STEP 2. Since $R_{up}/(I)$ is isomorphic to

$$(\mathbb{k}[x_1, \dots, x_m]/I)[\theta_{i,j} : 1 \leq i \leq d, 1 \leq j \leq m]$$

we have that

$$\dim R_{up}/(I) = d + dm.$$

Hence, by [7, p. 414, Proposition A.4] we have

$$(5) \quad \dim R_{up}/I_{up} \geq d + dm - d = dm.$$

STEP 3. Assume $c_{i,j} \in \mathbb{k}$ are Zariski general and consider the ideal Q of R_{up} , where

$$Q = I_{up} + (\theta_{i,j} - c_{i,j}\theta_{i,1} : 1 \leq i \leq d, 2 \leq j \leq m).$$

It is clear that R_{up}/Q is isomorphic to

$$\mathbb{k}[x_1, \dots, x_m, \theta_{i,1} : 1 \leq i \leq d]/(I, \theta_{1,1}q_1, \dots, \theta_{d,1}q_d),$$

where, for $1 \leq i \leq d$, we have $q_i = x_1 + \sum_{2 \leq j \leq m} c_{i,j}x_j$. Using that the $c_{i,j}$ are Zariski general and [7, p. 37, Theorem 1.5.17 (c)], we get from Proposition 2.21 that

$$\dim R_{up}/Q = d.$$

Hence, [7, p. 414, Proposition A.4] implies that

$$(6) \quad \dim R_{up}/I_{up} \leq d + (m-1)d = dm.$$

Proposition 2.2 follows by combining Inequalities (5) and (6).

3. KM NORMALIZATION

We continue using the notations of Section 2. Motivated by Proposition 2.12, Example 2.16, Example 2.17 and Remark 6.3, we give the following definition of volume normalization.

Definition 3.1 (KM normalization). Assume that the R_{up} -module $(N_{up}/I_{up})_{(e,-)}$ is cyclic, and denote by u a generator. By Corollary 2.4, we have that \mathcal{A}_e is a 1-dimensional vector space over E and $\Phi(u)$ is an E -basis of \mathcal{A}_e . We define the volume normalization linear isomorphism $\text{vol} = \text{vol}_u : \mathcal{A}_e \rightarrow E$ to be the unique linear map with the property that $\Phi(u)$ maps to 1.

Remark 3.2. Assume that the R_{up} -module $(N_{up}/I_{up})_{(e,-)}$ is cyclic, and denote by u_1, u_2 two generators. Since R_{up} is bigraded with minimum bidegree $(0,0)$ and $(R_{up})_{(0,0)} = \mathbb{k}$, it follows that there exist $c \in \mathbb{k} \setminus \{0\}$ such that $u_2 = cu_1$. Hence, $\text{vol}_{u_1} = c\text{vol}_{u_2}$.

Remark 3.3. The assumption that the R_{up} -module $(N_{up}/I_{up})_{(e,-)}$ is cyclic is satisfied in the important special case that $\mathbb{k}[x_1, \dots, x_m]/I$ is Gorenstein. We study this case in more detail in Section 5, where we relate it with the degenerate Kustin-Miller unprojection which is introduced in Section 4. Moreover, in the even more special case of complete intersections, we give explicit computations in Subsection 5.1 and relate it to the theory of multidimensional residues.

Remark 3.4. Assume $\mathbb{k}[x_1, \dots, x_m]/I$ is the face ring of a simplicial sphere D . In Section 6 we give explicit computations of the volume normalization linear isomorphism and we prove that it is exactly the isomorphism chosen usually in the case of toric varieties, and coincides with the one arising from de Rham cohomology.

3.1. The viewpoint of Kustin-Miller unprojection. Kustin-Miller unprojection is a technique originally invented to construct Gorenstein schemes from simpler ones, see [33] and [49]. In our setting, it has a new use. We now relate the volume normalization to Kustin-Miller unprojection, therefore justifying the name. We refer to p. 21 for a survey of Kustin-Miller unprojection, and concern ourselves only briefly with providing two alternate viewpoints.

We start with a Gorenstein ring $S = \mathbb{k}[\mathbf{x}]/I$, $\mathbf{x} = (x_1, \dots, x_m)$ over \mathbb{k} of Krull dimension d . As a next step, we introduce additional variables $\theta_{i,j}$, where $1 \leq j \leq m$ and $1 \leq i \leq d$. We then create new rings, by considering the polynomial ring $\mathbb{k}[\mathbf{x}, \theta_{i,j}, z]$ and the polynomials

$$\theta_i = \sum_j \theta_{i,j} x_j.$$

We set

$$R = \mathbb{k}[\mathbf{x}, \theta_{i,j}, z]/I + (\theta_i)$$

and

$$J = (z, x_1, \dots, x_m)$$

The main observation is that we just constructed new Gorenstein rings.

Proposition 3.5. *The rings R and R/J are Gorenstein, and J is of codimension one in R*

Using the functor $\text{Hom}_R(-, R)$ to the natural short exact sequence

$$0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$$

we obtain

$$0 \longrightarrow R \longrightarrow \text{Hom}_R(J, R) \longrightarrow R/J \longrightarrow 0$$

with the last map corresponding to the Poincaré residue map of complex geometry.

In other words, there is a homomorphism $\varphi : J \longrightarrow R$ that together with the inclusion $\iota : J \hookrightarrow R$ generates the R -module $\text{Hom}_R(J, R)$. Hence, we may find h in $\mathbb{k}[\mathbf{x}, \theta_{i,j}]$ with $\varphi(z) = h$.

Definition 3.6 (KM normalization, the second). We set $\text{vol}'(h) = 1$.

The following is immediate.

Proposition 3.7. *This normalization is well-defined up to an element in \mathbb{k}^* , and coincides with the previous definition up to a factor in \mathbb{k}^* .*

3.2. Once more. We give a useful alternative view towards this: Consider the polynomial ring $\mathbb{k}[\mathbf{x}, \theta_{i,j}]$ and the quotient

$$T = \mathbb{k}[\mathbf{x}, \theta_{i,j}]/I + (\theta_i)$$

as well as the quotient

$$U = \mathbb{k}[\mathbf{x}, \theta_{i,j}]/(x_i).$$

We have a natural surjection

$$T \longrightarrow U$$

and hence a map of minimal free resolutions

$$\mathcal{F}_\bullet(T) \longrightarrow \mathcal{F}_\bullet(U)$$

Consider the top nontrivial entry of the resolution; it is

$$\mathcal{F}_d(T) \cong \mathcal{F}_d(U) \cong \mathbb{k}[\mathbf{x}, \theta_{i,j}]$$

but the map of free resolutions induced

$$\varrho: \mathcal{F}_d(T) \longrightarrow \mathcal{F}_d(U)$$

is nontrivial: We consider the image $\varrho(1)$, a homogeneous polynomial in the latter, and set

$$\text{vol}''(\varrho(1)) = 1$$

. From the general theory of Kustin-Miller unprojection (compare the appendix on p. 21) we have:

Theorem 3.8. *The maps vol , vol' and vol'' coincide (up to a factor in \mathbb{k}^*).*

4. DEGENERATE KUSTIN-MILLER UNPROJECTION

We refer the reader to the appendix on p. 21 for a survey of Kustin-Miller unprojection. In the present section we introduce the notion of degenerate Kustin-Miller unprojection. Our motivation is that in Section 5 this notion will play a key role in the study of the volume normalization of commutative Gorenstein algebras. We remark that an example of degenerate Kustin-Miller unprojection appeared in [13].

Assume R_{small} is a local Noetherian ring and $I_{\text{small}} \subset J_{\text{small}}$ are two ideals of R_{small} with the property that both quotient $R_{\text{small}}/I_{\text{small}}$ and $R_{\text{small}}/J_{\text{small}}$ are Gorenstein of the same Krull dimension. We call the pair $I_{\text{small}} \subset J_{\text{small}}$ the *predata for a degenerate Kustin-Miller unprojection*.

We assume z is a new variable and we set $R = R_{\text{small}}[z]$. We denote by I the ideal of R generated by the set I_{small} , and by J the ideal of R generated by the set $J_{\text{small}} \cup \{z\}$. We fix $h_1, \dots, h_s \in R_{\text{small}}$ that generate the ideal I_{small} of R_{small} , clearly they also generate the ideal I of R .

Proposition 4.1. *We have that the pair $(J/I, R/I)$ satisfies the Kustin-Miller unprojection assumptions defined in [49] (see also Subsection 8.4). In other words, R/I is Gorenstein and J/I is a codimension 1 ideal of R/I with the quotient being Gorenstein.*

Proof. We denote by d the common Krull dimension of $R_{\text{small}}/I_{\text{small}}$ and $R_{\text{small}}/J_{\text{small}}$. Since I is generated as an ideal of R by the elements h_i of R_{small} it follows that R/I is isomorphic to the polynomial ring $(R_{\text{small}}/I_{\text{small}})[z]$. This implies that R/I is Gorenstein (since $R_{\text{small}}/I_{\text{small}}$ is) of Krull dimension $d + 1$. The quotient $(R/I)/(J/I)$ is isomorphic to $R_{\text{small}}/J_{\text{small}}$ which is Gorenstein of Krull dimension d . \square

We call the pair $(J/I, R/I)$ a *degenerate Kustin-Miller unprojection pair*. Combining Proposition 4.1 with [49, p. 563] (compare also Subsection 8.4) there exists a short exact sequence

$$(7) \quad 0 \rightarrow R/I \rightarrow \text{Hom}_{R/I}(J/I, R/I) \rightarrow R/J \rightarrow 0$$

where the first nonzero map sends $u \in R/I$ to the map $J/I \rightarrow R/I$ which is multiplication by u and the second nonzero map corresponds to the Poincaré residue map.

Recall that if R is a commutative ring with unit, the subset S_R of R consisting of all $a \in R$ such that the multiplication by a map $R \rightarrow R$ is injective, is a multiplicatively closed subset of R and the natural map R to the localization $(S_R)^{-1}R$, with $a \mapsto a/1$ is injective. The ring $(S_R)^{-1}R$ is called the total ring of fractions of R .

We denote by $Q(R/I)$ the total ring of fractions of R/I . Since the variable z does not appear in the generating set I_{small} of I , we have that the multiplication by z map $R/I \rightarrow R/I$ is injective. Consequently, z is an invertible element of $Q(R/I)$.

Proposition 4.2. *Assume $u \in (I_{small} : J_{small})$. Then, the multiplication by u/z map $Q(R/I) \rightarrow Q(R/I)$ sends the subset J/I inside $R/I \subset Q(R/I)$. Moreover, $(u/z)(w + I) = 0$ for all $w \in J_{small}$.*

Proof. Clearly $(u/z)(z + I) = u + I \in R/I$. Assume $w \in J_{small}$, then since $uw \in I_{small} \subset I$ we have $(u/z)(w + I) = (uw + I)/(z + I) = 0 \in Q(R/I)$. Since J is generated as an ideal of R by $J_{small} \cup \{z\}$ the proposition follows. \square

As a consequence, given $u \in (I_{small} : J_{small})$ there exists a well-defined element $T_u \in \text{Hom}_{R/I}(J/I, R/I)$ such that $T_u(z + I) = u$ and $T_u(w) = 0$ for all $w \in (J_{small} + I)/I$.

Proposition 4.3. *We have*

$$I \cap R_{small} = I_{small}.$$

Proof. By the definition of I we have $I_{small} \subset I$, hence $I_{small} \subset (I \cap R_{small})$. For the other inclusion, we assume $w \in (I \cap R_{small})$. Hence, there exist $m_i \in R$ such that

$$w = \sum_{1 \leq i \leq s} m_i h_i,$$

with equality in R . Substituting 0 for the variable z , and using that $w, h_i \in R_{small}$ we get that $w \in I_{small}$. \square

Proposition 4.4. *Assume $\phi \in \text{Hom}_{R/I}(J/I, R/I)$ has the properties $\phi(w) = 0$ for all $w \in (J_{small} + I)/I$ and that there exists $u \in R_{small}$ such that $\phi(z + I) = u + I$. Then $u \in (I_{small} : J_{small})$ and $\phi = T_u$.*

Proof. Assume $w \in J_{small}$. We have

$$0 = (z + I)\phi(w + I) = \phi((z + I)(w + I)) = (w + I)\phi(z + I) = wu + I.$$

Hence $wu \in I$. Since $wu \in R_{small}$, it follows that $wu \in (I \cap R_{small})$. Using Proposition 4.3 $wu \in I_{small}$, which implies that $u \in (I_{small} : J_{small})$. Since J is generated as an ideal of R by $J_{small} \cup \{z\}$ the equality $\phi = T_u$ follows. \square

We will now prove that the second assumption in the previous proposition implies the first.

Proposition 4.5. *Assume $\phi \in \text{Hom}_{R/I}(J/I, R/I)$ has the property that there exists $u \in R_{small}$ such that $\phi(z + I) = u + I$. Then $\phi(w) = 0$ for all $w \in (J_{small} + I)/I$. In addition, $u \in (I_{small} : J_{small})$ and $\phi = T_u$.*

Proof. Using Proposition 4.4 it is enough to prove that $\phi(w) = 0$ for all $w \in (J_{small} + I)/I$.

Assume $w \in J_{small}$. We fix $r \in R$ with $\phi(w + I) = r + I$. We have

$$zr + I = (z + I)\phi(w + I) = \phi((z + I)(w + I)) = (w + I)\phi(z + I) = wu + I.$$

Hence, $wu - zr \in I$, which implies that there exist $m_i \in R$ such that

$$wu - zr = \sum_{1 \leq i \leq s} m_i h_i$$

with equality in R . We write, for $1 \leq i \leq s$,

$$m_i = \sum_{1 \leq i \leq s} m_{1,i}z + m_{2,i},$$

with $m_{1,i} \in R$ and $m_{2,i} \in R_{small}$. Hence,

$$wu - zr = z \sum_{1 \leq i \leq s} m_{1,i} h_i + \sum_{1 \leq i \leq s} m_{2,i} h_i.$$

Since $w, u, h_i, m_{2,i} \in R_{small}$, looking at the coefficients of z it follows that

$$r = - \sum_{1 \leq i \leq s} m_{1,i} h_i,$$

with equality in R . Hence $r \in I$, which implies that $\phi(w + I) = 0$. \square

Proposition 4.6. *Assume $\phi \in \text{Hom}_{R/I}(J/I, R/I)$. Then there exist $c \in R$ and an element $u \in (I_{small} : J_{small})$ such that*

$$\phi = T_u + ci,$$

where $i : J/I \rightarrow R/I$ denotes the natural inclusion.

Proof. We fix $r \in R$ such that $\phi(z + I) = r + I$. We write

$$r = cz + u,$$

with $c \in R$ and $u \in R_{small}$ and set $\phi_1 = \phi - ci$. We have that $\phi_1 \in \text{Hom}_{R/I}(J/I, R/I)$ and

$$\phi_1(z + I) = \phi(z + I) - c(z + I) = u + I.$$

By Proposition 4.5, $u \in (I_{small} : J_{small})$ and $\phi_1 = T_u$. \square

Proposition 4.7. *Assume $u, v \in (I_{small} : J_{small})$. Then $T_u = T_v$ if and only if $u - v \in I_{small}$.*

Proof. Assume first that $u - v \in I_{small}$. Since $I_{small} \subset I$ we get $u + I = v + I$. Since J is the ideal of R generated by the subset $J_{small} \cup \{z\}$ and both T_u and T_v are zero on $(J_{small} + I)/I$, to prove $T_u = T_v$ it is enough to prove that if $T_u(z + I) = T_v(z + I)$. This holds, since $T_u(z + I) = u + I$ and $T_v(z + I) = v + I$.

Conversely, we assume that $T_u = T_v$. Evaluating at $z + I$ we get

$$u + I = T_u(z + I) = T_v(z + I) = v + I,$$

which implies that $u - v \in I$. By assumption $u - v \in R_{small}$. Proposition 4.3 now implies that $u - v \in I_{small}$. \square

We consider the R_{small} -module

$$L = (I_{small} : J_{small})/I_{small}.$$

Theorem 4.8. *The R_{small} -module L is cyclic. Assume $u \in (I_{small} : J_{small})$ has the property that $u + I_{small}$ generate L . Then T_u together with inclusion map $i : J/I \rightarrow R/I$ generate the R/I -module $\text{Hom}_{R/I}(J/I, R/I)$.*

Proof. Using the short exact sequence (7), there exists $\phi \in \text{Hom}_{R/I}(J/I, R/I)$ that together with the inclusion generate $\text{Hom}_{R/I}(J/I, R/I)$ as R/I -module. Using Proposition 4.6, there exist $c \in R$ and $v \in (I_{small} : J_{small})$ such that

$$\phi = T_v + ci.$$

As a consequence, T_v together with the inclusion generate $\text{Hom}_{R/I}(J/I, R/I)$ as R/I -module.

We claim that $v + I_{small}$ generate the R_{small} -module L . Indeed, assume $w \in (I_{small} : J_{small})$. Then $T_w \in \text{Hom}_{R/I}(J/I, R/I)$, hence there exist $c_1, c_2 \in R$ such that

$$T_w = c_1 T_v + c_2 i.$$

Consequently

$$w + I = T_w(z + I) = c_1 T_v(z + I) + c_2 i(z + I) = (c_1 v + c_2 z) + I.$$

Hence, there exist $m_i \in R$ such that

$$w - (c_1 v + c_2 z) = \sum_{1 \leq i \leq s} m_i h_i,$$

with equality in R . Substituting 0 for the variable z , using that $w, h_i, u \in R_{small}$ we get that $w - c_3 v \in I$, where $c_3 \in R_{small}$ is the result of the substitution of 0 for z in c_1 . Since $w, c_3, v \in R_{small}$, Proposition 4.3 implies that $w - c_3 v \in I_{small}$. Hence, $w + I_{small} = c_3 v + I_{small}$, therefore L is cyclic with generator $v + I_{small}$.

Conversely, assume that $u \in (I_{small} : J_{small})$ has the property that $u + I_{small}$ generate L . This implies that there exists $c_4 \in R_{small}$ such that

$$v + I_{small} = c_4(u + I_{small}),$$

hence

$$T_v = c_4 T_u.$$

Since T_v together with the inclusion generate $\text{Hom}_{R/I}(J/I, R/I)$ we get that also T_u together with the inclusion generate $\text{Hom}_{R/I}(J/I, R/I)$. \square

Remark 4.9. Assume $u \in (I_{small} : J_{small})$ has the property that T_u together with inclusion map $i : J/I \rightarrow R/I$ generate the R/I -module $\text{Hom}_{R/I}(J/I, R/I)$. The arguments in the proof of Theorem 4.8 show that $u + I_{small}$ generates L as R_{small} -module.

Remark 4.10. Assume $u \in (I_{small} : J_{small})$ has the property that $u + I_{small}$ generate L . This implies that

$$(I_{small} : J_{small}) = I_{small} + (u),$$

where (u) denotes the ideal of R_{small} generated by u . Then

$$(I_{small} : (u)) = (I_{small} : (I_{small} + (u))) = (I_{small} : (I_{small} : J_{small})) = J_{small},$$

where for the last equality we used [37, p. 115, Proposition 5.2.3 d)].

5. THE GORENSTEIN CASE OF VOLUME NORMALIZATION

In the present section we discuss the volume normalization theory of $\mathbb{k}[x_1, \dots, x_m]/I$ under the additional assumption that the quotient ring is Gorenstein. We base our treatment on the notion of degenerate Kustin-Miller unprojection introduced in Section 4.

Assume $m \geq 1$ and \mathbb{k} is a field. We consider the polynomial ring $\mathbb{k}[x_1, \dots, x_m]$, where the degree of the variable x_i is equal to 1, for all $1 \leq i \leq m$. Assume $I \subset \mathbb{k}[x_1, \dots, x_m]$ is a homogeneous ideal with $\mathbb{k}[x_1, \dots, x_m]/I$ Gorenstein. We denote by d the Krull dimension of the quotient ring $\mathbb{k}[x_1, \dots, x_m]/I$ and assume $d \geq 1$.

We will use the notations $R_{up}, I_{up}, R_{down}, I_{down}, \mathcal{A}, e, S, E, \Phi$, defined in Section 2. Using the assumption that the quotient ring $\mathbb{k}[x_1, \dots, x_m]/I$ is Gorenstein, we get by Proposition 2.2 that R_{up}/I_{up} is Gorenstein, since it is the quotient of $R_{up}/(I)$ by a homogeneous regular sequence. Consequently, the pair $I_{up} \subset (x_1, \dots, x_m)$ of ideals of R_{up} is the predata for a degenerate Kustin-Miller unprojection (see Section 4).

By Theorem 4.8, the R_{up} -module $(I_{up} : (x_1, \dots, x_m))/I_{up}$ is cyclic, and we fix an element $h \in (I_{up} : (x_1, \dots, x_m))$ such that $h + I_{up}$ generates $(I_{up} : (x_1, \dots, x_m))/I_{up}$. We get the following equality of ideals of R_{up}

$$(8) \quad (I_{up} : (x_1, \dots, x_m)) = I_{up} + (h).$$

Theorem 5.1. *The element $\Phi(h + I_{up})$ is a nonzero element of $\text{Socle}(\mathcal{A})$. Hence, it is a basis of $\text{Socle}(\mathcal{A})$ considered as E -vector space.*

Proof. Since \mathcal{A} is a graded Artinian Gorenstein E -algebra, $\text{Socle}(\mathcal{A})$ is a 1-dimensional E -vector space and $\text{Socle}(\mathcal{A}) = \mathcal{A}_e$. The result follows by Corollary 2.4. \square

Using the theorem, there exists a unique volume normalization linear isomorphism $\mathcal{A}_e \rightarrow E$ such that it sends $\Phi(h + I_{up})$ to 1.

5.1. The complete intersection case of volume normalization and relation with residues. In this subsection we discuss the volume normalization theory of the algebra $\mathbb{k}[x_1, \dots, x_m]/I$ under the additional assumption that I is a homogeneous complete intersection ideal. In addition, we relate it to the theory of multidimensional residues.

Assume $m \geq 1$ and \mathbb{k} is a field. We consider the polynomial ring $\mathbb{k}[x_1, \dots, x_m]$, where the degree of the variable x_i is equal to 1, for all $1 \leq i \leq m$. Assume $d \geq 1$ and

$$I = (H_1, \dots, H_{m-d}) \subset \mathbb{k}[x_1, \dots, x_m]$$

is a homogeneous ideal with the property that H_1, \dots, H_{m-d} is a homogeneous regular sequence of $\mathbb{k}[x_1, \dots, x_m]$. As a consequence, the Krull dimension of the quotient ring $\mathbb{k}[x_1, \dots, x_m]/I$ is d .

We will use the notations $R_{up}, I_{up}, \mathcal{A}, e, E, \Phi$, defined in Section 2.

Using the assumption that the H_1, \dots, H_{m-d} is a homogeneous regular sequence, we get that

$$H_1, \dots, H_{m-d}, f_1, \dots, f_d$$

is a homogeneous regular sequence of R_{up} . In particular, R_{up}/I_{up} is Gorenstein.

We fix an $m \times m$ matrix M such that

$$(H_1, \dots, H_{m-d}, f_1, \dots, f_d)^t = M(x_1, \dots, x_m)^t,$$

where N^t denotes the transpose of the matrix N . We denote by Δ the determinant of the matrix M . Combining the computations in [43, Section 4] (see also Subsection 8.5) with the discussion in Section 5 we get that

$$(I_{up} : (x_1, \dots, x_m)) = I_{up} + (\Delta).$$

As a consequence, we have the following proposition.

Proposition 5.2. *There exists a volume normalization linear isomorphism $\mathcal{A}_e \rightarrow E$ uniquely specified by the property that $\Phi(\Delta)$ maps to 1.*

Comparing Proposition 5.2 with [21, p. 42, Equation (1.50)] we get a connection, in the complete intersection setting, between the volume normalization theory and the theory of multidimensional residues.

In particular, we can generalize the projective residue map $\text{res}_{\langle F_0, \dots, F_n \rangle}^{\mathbb{P}^n}$ defined in [21, p. 40] as follows.

Assume $m \geq 0$ and \mathbb{k} is a field. We consider the polynomial ring $R = \mathbb{k}[x_0, \dots, x_m]$, where the degree of the variable x_i is equal to 1, for all $0 \leq i \leq m$. Assume $I \subset R$ is a homogeneous ideal such that the quotient R/I is Artinian Gorenstein. We set $J = (x_0, \dots, x_m)$. Then, the pair $I \subset J$ is a predata for a degenerate Kustin-Miller unprojection (see Section 4). By Theorem 4.8, the R -module $N = (I : J)/I$ is cyclic, and we fix an element $h \in (I : J)$ such that $h + I$ generates N as an R -module.

Definition 5.3. Assume $Q \in R$ is a homogeneous element. If there exists $\lambda \in \mathbb{k}$ with $Q = \lambda h$ (equality in R/I), we set $\text{res}_{I,h}^{\mathbb{P}^n}(Q) = \lambda$, otherwise we set $\text{res}_{I,h}^{\mathbb{P}^n}(Q) = 0$.

Remark 5.4. Using Remark 3.2 we have the following. Assume $h_1, h_2 \in (I : J)$ have the property that, for all $1 \leq i \leq 2$, the element $h_i + I$ generates N as an R -module. Then there exists unique $c \in \mathbb{k} \setminus \{0\}$ such that

$$\text{res}_{I,h_2}^{\mathbb{P}^n} = c \text{res}_{I,h_1}^{\mathbb{P}^n}$$

In particular, if the field \mathbb{k} has two elements then Definition 5.3 does not depend on the choice of h .

6. VOLUME NORMALIZATION IN THE CASE OF SIMPLICIAL SPHERES

In the present section we compute the volume normalization of the face ring of a simplicial sphere.

Assume \mathbb{k} is a field and D is a simplicial sphere with vertex set $\{1, 2, \dots, m\}$ and dimension equal to $d - 1 \geq 1$. We denote by $I_D \subset \mathbb{k}[x_1, \dots, x_m]$ the Stanley-Reisner ideal of D and by $\mathbb{k}[D] = \mathbb{k}[x_1, \dots, x_m]/I_D$ the face ring of D over \mathbb{k} . We set

$$R_{up} = \mathbb{k}[x_1, \dots, x_m, \theta_{i,j} : 1 \leq i \leq d, 1 \leq j \leq m],$$

and, for $1 \leq i \leq d$, we set

$$f_i = \sum_{j=1}^m \theta_{i,j} x_j \in R_{up}.$$

We denote by I_{up} the ideal of R_{up} generated by $I_D \cup \{f_1, \dots, f_d\}$, and we set

$$L = (I_{up} : (x_1, \dots, x_m))/I_{up}.$$

Since D is a simplicial sphere, by [7, Section 5] $\mathbb{k}[D]$ is Gorenstein of Krull dimension d . As a consequence, the pair $I_{up} \subset (x_1, \dots, x_m)$ is a predata for a degenerate Kustin-Miller unprojection (see Section 4).

Proposition 6.1. *The R_{up} -module L is cyclic.*

Proof. It is an immediate application of Theorem 4.8. \square

We will now describe a generator of the cyclic module L . We fix an ordered facet $\sigma = (b_1, \dots, b_d)$ of D . This just means that the set $\{b_1, \dots, b_d\}$ is a facet of D . We set $x_\sigma = \prod_{1 \leq i \leq d} x_{b_i} \in R_{up}$. Moreover, we denote by $[\sigma]$ the determinant of the $d \times d$ matrix with (i, j) -entry equal to θ_{i,b_j} . Finally, we set

$$(9) \quad h = [\sigma]x_\sigma \in R_{up}.$$

The proof of the following theorem will be given in Subsection 6.1.

Theorem 6.2. *The element $h + I_{up}$ is a generator of the cyclic R_{up} -module L .*

Remark 6.3. We use the notations $E = \mathbb{k}(\theta_{i,j})$, $R_{down} = E[x_1, \dots, x_m]$, $I_{down} = (I_D) + (f_1, \dots, f_d) \subset R_{down}$, $\mathcal{A} = R_{down}/I_{down}$ defined in Section 2. We remark that (up to sign) the volume normalization linear isomorphism $\mathcal{A}_d \rightarrow E$ defined in [48, p. 6, Equation (1)] (and denoted there by the notation Ψ_e) is uniquely specified by the condition that the element $[\sigma]x_\sigma + I_{down}$ maps to 1.

6.1. Proof of Theorem 6.2. We assume z is a new variable and we set $R = R_{up}[z]$, hence

$$R = \mathbb{k}[z, x_1, \dots, x_m, \theta_{i,j} : 1 \leq i \leq d, 1 \leq j \leq m].$$

We consider the ideals $I = (I_D) + (f_1, \dots, f_d)$, $J = (x_1, \dots, x_m, z)$ and $P = (I_D)$ of R . Recall, that for any $w \in (I_{up} : (x_1, \dots, x_m))$ we defined in Section 4 an element $T_w \in \text{Hom}_{R/I}(J/I, R/I)$. We will prove in Proposition 6.12 that $h \in (I_{up} : (x_1, \dots, x_m))$. Using that and Remark 4.9, to show Theorem 6.2 it is enough to prove Proposition 6.13 below, which states that the map T_h together with the inclusion i generate $\text{Hom}_{R/I}(J/I, R/I)$ as R/I -module.

We set

$$c = \text{codim } I_D = \dim \mathbb{k}[x_1, \dots, x_m] - \dim(\mathbb{k}[x_1, \dots, x_m]/I_D) = m - d.$$

We use the following bidegree for R . For $a, b \in \mathbb{Z}$, we say that an element $w \in R$ has bidegree (a, b) if it is homogeneous of degree a with respect to the variables x_i and z and is homogeneous of degree b with respect to the variables $\theta_{i,j}$. We denote by $R_{a,b}$ the subset of R consisting of bihomogeneous elements of bidegree (a, b) , of course $R_{a,b} = 0$ if $a < 0$ or $b < 0$. For $a, b \in \mathbb{Z}$ we denote by $R(a, b)$ the bigraded R -module with $R(a, b)_{k,l} = R_{a+k, b+l}$ for all $k, l \in \mathbb{Z}$.

We denote by

C^1 : minimal graded free resolution of R/P as R -module.

C^2 : minimal graded free resolution of $R/(f_1, \dots, f_d)$ as R -module.

C^3 : minimal graded free resolution of R/I as R -module.

C^4 : minimal graded free resolution of R/J as R -module.

Proposition 6.4. C^1 is of the form

$$0 \rightarrow F_c = R(-m, 0) \rightarrow F_{c-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 = R$$

Proof. By [7, Lemma 5.6.4] we have that the socle degree of R/P is equal to $d = \dim D + 1$ and $d = m - c$. The result follows by the discussion in [37, p. 79]. \square

Proposition 6.5. C^2 is the Koszul complex on (f_1, \dots, f_d) . Hence, C^2 is

$$0 \rightarrow \tilde{F}_d = R(-d, -d) \rightarrow \tilde{F}_{d-1} \rightarrow \dots \rightarrow \tilde{F}_1 \rightarrow \tilde{F}_0 = R$$

Proposition 6.6. C^3 is the tensor product of C^1 and C^2 .

Proof. It follows by combining [26, p. 8] with [7, Exercise 1.1.12] and [7, Theorem 9.4.7]. \square

Hence, by Proposition 6.6 it follows that

Proposition 6.7. C^3 is

$$0 \rightarrow \hat{F}_m = R(-m - d, -d) \rightarrow \hat{F}_{m-1} \rightarrow \dots \rightarrow \hat{F}_1 \rightarrow \hat{F}_0 = R$$

Proposition 6.8. C^4 is the Koszul complex on (x_1, \dots, x_m, z) . Hence, C^4 is

$$0 \rightarrow \bar{F}_{m+1} = R(-m - 1, 0) \rightarrow \bar{F}_m = (R(-m, 0))^{m+1} \rightarrow \dots \rightarrow \bar{F}_1 \rightarrow \bar{F}_0 = R$$

Proposition 6.9. Since

$$I \subset J \subset R$$

there is a natural projection map

$$R/I \rightarrow R/J.$$

It induces a chain map $C^3 \rightarrow C^4$. Looking at $\hat{F}_m \simeq R \xrightarrow{h} \bar{F}_m \simeq R^{m+1}$ each entry of the matrix of h has bidegree (d, d) .

Proposition 6.10. If τ is an ordered facet of D , then there exists $\epsilon \in \{-1, 1\}$ such that

$$h - \epsilon[\tau]x_\tau$$

is an element of the ideal (f_1, \dots, f_d) of R .

Proof. See [48, Corollary 4.5]. \square

Proposition 6.11. The element $h + I$ of R/I is nonzero.

Proof. We use the notations $R_{\text{down}} = \mathbb{k}(\theta_{i,j})[x_1, \dots, x_m]$, $I_{\text{down}} = (I_D) + (f_1, \dots, f_d) \subset R_{\text{down}}$ defined in Section 2. By [48, p. 6], $h + I_{\text{down}}$ is a nonzero element of $R_{\text{down}}/I_{\text{down}}$. This implies that the element $h + I$ of R/I is nonzero. \square

Proposition 6.12. We have $h \in (I_{\text{up}} : (x_1, \dots, x_m))$.

Proof. Assume $1 \leq j \leq m$. We choose a facet τ of D such that j is not an element of τ . Using Proposition 6.10, there exists $\epsilon \in \{-1, 1\}$ such that

$$h - \epsilon[\tau]x_\tau \in I_{up}.$$

Moreover, since j is not an element of the facet τ , we have that $\tau \cup \{j\}$ is not a face of D , hence

$$x_\tau x_j \in I_D \subset I_{up}.$$

Hence, working on the quotient ring R/I_{up} we have

$$hx_j + I_{up} = \epsilon[\tau]x_\tau x_j + I_{up} = 0 + I_{up}.$$

This finishes the proof of the proposition. \square

Proposition 6.13. *The map T_h together with the inclusion i generate the R/I -module $\text{Hom}_{R/I}(J/I, R/I)$.*

Proof. We denote by $\langle i \rangle$ the R/I -submodule of $\text{Hom}_{R/I}(J/I, R/I)$ generated by the inclusion i . Proposition 6.9 together with the short exact sequence (7) and the general theory of Kustin-Miller unprojection (see [43, 49] and Subsection 8.8) implies that

$$\text{Hom}_{R/I}(J/I, R/I) / \langle i \rangle$$

is a cyclic module generated by the class of an element ϕ such that $\phi(z)$ has bidegree (d, d) . By Proposition 6.12 $h \in (I_{up} : (x_1, \dots, x_m))$, h clearly has bidegree (d, d) and by Proposition 6.11 h is nonzero in R/I . This finishes the proof of Proposition 6.13. \square

7. THE RELATIVE CASE AND LOCALITY

The volume normalization theory we have developed so far can be used under the assumption that we work with Gorenstein algebras. It applies, for example, for the class of face rings of simplicial spheres. The relative volume normalization setting, which we will now discuss, extends the theory. It is useful to handle situations where the socle is not one-dimensional and one may want to normalize with respect to a specific choice of fundamental class (such is the case when considering face rings of triangulated manifolds [1, 2]). Moreover, it allows us to understand the normalization in cases where it otherwise becomes rather challenging to access it directly, as in the case of lattice polytopes.

Let us try to establish that normalization is, in some way, local. We already saw a glimpse of this: In the case of simplicial spheres D , the volume map evaluated at \mathbf{x}_F , where F is a facet, only depended on the linear system of parameters at that facet; more precisely, on the indeterminates corresponding to that facet. But the why of this is a bit unclear from our calculation. We aim to explain that fact:

There are several ways to attempt to encode this: For instance, we could consider simplicial homeomorphisms $D' \rightarrow D$ that leave F invariant, and study the resulting map of rings. Alas, the class of objects we then understand is not of the desired generality. This section's main goal is to observe that sometimes, in order for practical calculations to succeed, it is useful to consider relative settings.

So we adopt a different perspective: consider a standard graded Gorenstein ring T as a module over a polynomial ring $R = \mathbb{k}[\mathbf{x}]$. We assume once more that T is of positive Krull dimension, and that the fundamental class lives in degree d . We finally consider a squarefree monomial \mathbf{x}_F of degree d in R , and set out to compute its volume map with respect to a normalization.

We call \mathbf{x}_F *isolated* in T if the ideal I_F generated by x_F in T is isomorphic to $\mathbb{k}[\mathbf{x}_{i \in F}]$.

Example 7.1. Consider a simplicial complex X , and let F denote a facet of X . Then \mathbf{x}_F is isolated in the face ring of X .

Naturally, the Krull dimension of T is d .

Moreover, T/I_F is Cohen-Macaulay, and I_F is Gorenstein.

Hence, the question arises:

What is the normalization, the value of the volume map, at \mathbf{x}_F ?

We can ask this question twice:

What is the value in I_F (which, naturally, is Gorenstein itself)?

And what is the value in T ?

Of course, we expect the answers to be the same. Let us do the math.

7.1. Relative KM Normalization in I_F . We adopt the viewpoint of Section 3.2. Consider $I_F = \langle \mathbf{x}_F \rangle$ in T .

Let $R_{up} = R[\theta_{i,j}]$. Consider the minimal free resolution F_\bullet of the module $I_F/\Theta I_F$.

$$0 \longrightarrow F_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow I_F/\Theta I_F \longrightarrow 0$$

We also consider the minimal resolution G_\bullet of $R_{up}/\mathbf{x}R_{up}$. Since we have a natural map

$$I_F = F_0 \longrightarrow R_{up} = G_0$$

induced by the inclusion of I_F into S' , and in particular a map of the one-dimensional modules $\Phi : F_d \rightarrow G_d$, both of which are isomorphic to R_{up} .

Definition 7.2. The relative KM normalization sets $\text{vol}(\Phi(1)) = 1$.

We have that $\Phi(1) = \Delta(\theta_{i,j})_{F\mathbf{x}_F}$, and hence immediately obtain the KM normalization

$$\text{vol}(\Delta(\theta_{i,j})_{F\mathbf{x}_F}) = 1.$$

7.2. Normalization in T . We want to consider now T as an R_{up} -module, and compute the minimal free resolution H_\bullet of $T/\Theta T$. Note: we have a commutative triangle

$$\begin{array}{ccc} I_F/\Theta I_F & \hookrightarrow & T/\Theta T \\ & \searrow & \downarrow \\ & & R_{up}/\mathbf{x}R_{up} \end{array}$$

which of course induces a commutative triangle

$$\begin{array}{ccc} F_d & \longrightarrow & H_d \\ & \searrow & \downarrow \\ & & G_d \end{array}$$

of one dimensional R_{up} -modules. The top map is an isomorphism, hence the normalization coincides.

Example 7.3. If D is a simplicial sphere, and F is a facet, it follows that the KM normalization satisfies

$$\text{vol}(\Delta(\theta_{i,j})_{F\mathbf{x}_F}) = 1.$$

7.3. Locality. Let us reexamine what happened in the last section, and for this purpose, introduce the notion of a *dualizable Cohen-Macaulay module*: Such modules are characterized by being Cohen-Macaulay, and the socle of their Artinian reduction is of dimension one.

One such example is considering a Gorenstein ring T as above, and considering an ideal K in it such that T/K is Cohen-Macaulay. Such a module is dualizable Cohen-Macaulay, and, by defining the relative normalization as in Section 7.1, with I_F replacing K . Section 7.2 then shows that this normalization only depends on K , not on T . In particular, if two rings contain isomorphic dualizable Cohen-Macaulay ideals, then the KM normalization in this ideal only depends on K , and not on T or T' . Another way of saying it: If m is an element in the socle of \mathcal{K} , the Artinian reduction of K , then $\text{vol}(m)$ is independent of whether we are considering T or T' .

8. APPLICATIONS

Let us note the two main applications of this relative perspective.

8.1. Simplicial spheres to manifolds. One issue we had in the absolute perspective was to explain the KM normalization in the case of objects more general than manifolds, see 2.16; the relative normalization helps here. We can use it to explain the normalization for general (closed, compact orientable) manifolds, pseudomanifolds and more generally face rings of cycles [2].

Recall: Given a triangulated $(d - 1)$ -dimensional cycle μ over \mathbb{k} , we can consider the face ring T of its underlying simplicial complex. Consider any Artinian reduction \mathcal{A} of T . Note that μ defines a map

$$\mu^* : \mathcal{A}_d \longrightarrow \mathbb{k}$$

and we can consider the quotient \mathcal{B} of \mathcal{A} under the annihilator of this map. \mathcal{B} is Gorenstein of Krull dimension zero. This quotient is of course independent of the precise choice of μ^* , and the precise choice is nothing but the normalization of the volume map.

And we can consider any facet F of μ , and observe that if the coefficient of μ on F is μ_F , then this is reflected in the relative normalization with respect to I_F . And while it is a bit troublesome to explain the value of $\text{vol}(\mathbf{x}_F)$ in \mathcal{B} , we have a direct explanation within I_F , as I_F is simply isomorphic to the polynomial ring $\mathbb{k}[\mathbf{x}_{i,i \in F}]$. The value of $\text{vol}(\mathbf{x}_F)$ is therefore naturally subject to

$$\text{vol}(\Delta(\theta_{i,j})_F \mathbf{x}_F) = \mu_F.$$

In other words, relative KM normalization recovers the natural normalization for cycles.

8.2. Lattice polytopes. Consider next an IDP lattice polytope P of dimension $d - 1$, that is, a lattice polytope such that the semigroup algebra $\mathbb{k}[K_P]$ is standard graded (another word for generated in degree one). Here, K_P is the cone over $P \times \{1\}$ in $\Lambda \times \mathbb{Z}$, where Λ is the ambient lattice for P .

In [3], we studied the volume map of such semigroup algebras, and argued that a certain natural choice can be made. We argue here that it agrees with KM normalization.

Specifically, we are interested in the module $\mathbb{k}[K_P, \partial K_P]$ that is obtained as the kernel of the map

$$\mathbb{k}[K_P] \longrightarrow \mathbb{k}[\partial K_P].$$

The socle of the Artinian reduction \mathcal{A} of $\mathbb{k}[K_P, \partial K_P]$ is concentrated in degree d , and it is natural and imperative to understand the map

$$\mathcal{A}_d \longrightarrow \mathbb{k}$$

And while the normalization we gave in [3] fell a bit out of thin air, we reconstruct it here.

The key observation, **to start with**, is that if P is a unimodular lattice $(d - 1)$ -simplex, that is, a lattice simplex such that the vertices of $P \times \{1\}$ generate $K_P \cap (\Lambda \times \mathbb{Z})$, the semigroup algebra is but the face ring of a simplex, that is, it is the polynomial ring generated by d generators. And $\mathbb{k}[K_P, \partial K_P]$, in this case, is the ideal generated by \mathbf{x}_P , the product of these generators. We fully understand the normalization thanks to relative KM normalization.

Simple case. If P has a unimodular boundary facet τ in ∂P , then we obtain the desired normalization by matching the face ring picture and setting

$$(10) \quad 1 = \sum_{p \in (P \setminus \tau) \cap \Lambda} \text{vol}(\mathbf{x}_p \mathbf{x}_\tau) \Delta(\Theta_{|\tau, p})$$

where $\Theta = (\theta_{i,j})$ is the matrix of coefficients in the linear system of parameters.

General case. In general, consider a flag (τ_i) of faces of P such that $\tau_d = P$ and such that τ_i is a facet of τ_{i+1} . We say a set $\sigma = \{\sigma_0, \dots, \sigma_d\} \subset (P \cap \Lambda)^{d+1}$ without repetitions

is **coherent** with (τ_i) if it intersects τ_i in a set of cardinality $i + 1$. We then normalize by setting

$$(11) \quad 1 = \sum_{\sigma \text{ coherent with } (\tau_i)} \text{vol}(\mathbf{x}_\sigma) \Delta(\Theta|_\sigma)$$

We claim that this is consistent with KM normalization. Of course, at first, we do not have a Gorenstein ring. But we can create one: If we consider the complex of lattice polytopes $P \cup (v * \partial P)$, we have a lattice sphere Σ , and the resulting algebra is naturally Gorenstein, obtained by gluing together the semigroup algebras of the individual lattice polytopes, is Gorenstein [8].

If Σ has a unimodular facet, then we apply the relative normalization above to that facet. More precisely:

If P has a boundary consisting of unimodular simplices, then we consider the cone over P as a polytope; it follows that

$$(v * \partial P) \cup P$$

is a lattice complex (again, in the sense of [8]), and that it defines a Gorenstein ring, meaning that the volume on P extends uniquely to a volume on $(v * \partial P) \cup P$. For the simplices F of $v * \partial P$ the volume is uniquely determined and given by $\det^{-1}(\Theta|_F)$; the normalization in the simple case, that is, Identity (10) then follows by the following identity, see [48, Proposition 11.1].

Lemma 8.1. *Consider any Gorenstein ring T of socle degree d and Krull dimension d . Then in the generic Artinian reduction \mathcal{A} of T , and any homogeneous polynomial m of degree $d - 1$, we have*

$$\sum_{x_i, i \in I} \Delta(\theta|_{J,i}) \text{vol}(mx_i) = 0$$

where I indexes the indeterminates of the polynomial ring and J is any subset of I of size $d - 1$.

For an arbitrary IDP lattice polytope P , we argue in a similar fashion. Unfortunately, unless P has a unimodular facet,

$$(v * \partial P) \cup P$$

does not have a unimodular facet. However, we can use the fact that the faces of $v * \partial P$ are decidedly more simplicial than P : they are cones over polytopes of codimension one.

Let us spin this idea further: P is a d -dimensional lattice polytope, and we consider an additional set of vertices $V = \{v_1, \dots, v_d\}$.

Consider the polyhedral complex

$$\bigcup_{W \subset V} \left(\bigvee_{v \in W} v \right) * P^{(d - \#W)}$$

where $\bigvee_{v \in W} v$ denotes the free join over the vertices in W and $P^{(d - \#W)}$ consists of the faces of P of dimension $(d - \#W)$ or less.

This complex is grouped into polytopes of different kinds, depending on the size of W . For $W = V$, we obtain a unimodular simplex.

We observe that the algebra generated by this complex is Gorenstein, and that the KM normalization of the simplex $\bigvee_{v \in V} v$ is consistent with the observed rule, as it is just a unimodular simplex.

Assuming then that we have the normalization formula for W of cardinality at least s , Lemma 8.1 gives the normalization for cardinality $s - 1$. This recovers Identity (11), and the consistency of the volume normalization. Hence, the relative formalism has allowed us to recover the following

Corollary 8.2. *Consider Σ a sphere built out of IDP lattice polytopes. Then the algebra it generates is Gorenstein, and at any facet P of Σ satisfies Identity (11).*

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APPENDIX: A BRIEF SURVEY OF KUSTIN-MILLER UNPROJECTION

The present appendix is a brief survey of Kustin-Miller unprojection.

8.3. Introduction. The structure of Gorenstein rings of codimension ≤ 3 is well understood, (see e.g. [27, Corollary 21.20] and [7, Theorem 3.4.1]). More precisely, a codimension 1 Gorenstein ring is a hypersurface, a codimension 2 Gorenstein ring is a complete intersection and a codimension 3 Gorenstein ring is Pfaffian. In the 1980s, A. Kustin and M. Miller [30, 31, 32, 33, 34, 35] during their investigation of the structure of Gorenstein rings of codimension greater or equal than 4 introduced a method that constructs Gorenstein rings of more complicated form starting from simpler ones. This method is known as Kustin-Miller unprojection.

Some years later, M. Reid who was also interested to understand the structure of Gorenstein rings in low codimension, motivated by some problems coming from algebraic geometry, reinterpreted the theory of A. Kustin and M. Miller by giving his formulation. S. Papadakis and M. Reid [43, 49] developed further the theory of unprojection. Besides of Kustin-Miller unprojection, known also as type I unprojection, there are several other kinds of unprojection [44, 45, 46, 47, 52, 54]. However, in this survey we will only focus on the Kustin-Miller unprojection.

The assumptions of Kustin-Miller unprojection are that $I, J \subset R$ are two ideals of a positively graded Gorenstein ring R such that $R/I, R/J$ are Gorenstein, $I \subset J$ and $\dim R/I - \dim R/J = 1$. Then, there exists an R/I -module homomorphism $\phi: J \rightarrow R/I$ such that the R/I -module $\text{Hom}_{R/I}(J, R/I)$ is generated by ϕ and the inclusion $i: J \rightarrow R/I$. According to M. Reid, $\text{Hom}_{R/I}(J, R/I)$ contains all the important information for the construction of unprojection ring. Under these assumptions, M. Reid and S. Papadakis ([49, Theorem 1.5]), proved that the ring of unprojection is Gorenstein.

Kustin-Miller unprojection can be used several times, inductively, for the construction of Gorenstein rings of higher codimension. This process is called parallel Kustin-Miller unprojection, and was developed by J. Neves and S. Papadakis [39].

Unprojection theory can be considered as the algebraic language for the study of certain constructions in algebraic geometry. The range of its applications is wide. In explicit algebraic geometry, it is used in the study and the construction of some interesting geometric objects such as surfaces of general type, K3 surfaces, Fano 3-folds and Calabi-Yau 3-folds [4, 5, 18, 38, 39, 50, 51]. The graded ring database [16] contains lists of graded rings of such objects whose existence is conjectured and may possibly be proved via unprojection. Moreover, it is used in the further development of the Minimal Model Program by providing an effective way to study explicitly varieties and morphisms between them [17, 19, 24, 25]. Also, it appears in algebraic combinatorics, in the study of Stanley-Reisner ideal of cyclic polytopes and stellar subdivisions of Gorenstein complexes [11, 13, 14, 15].

Kustin-Miller unprojection is related to liaison theory (also known as linkage theory). For more details we refer to [34] and [29, Section 2.6].

The further development of unprojection theory is an active area of research. There are many recent contributions [22, 23, 50, 51, 54] on foundational questions, computational questions and applications of unprojection theory. In the current research work, we use unprojection theory and especially Kustin-Miller unprojection, to define the normalization of the volume map.

The survey is organised as follows. In Subsection 8.4 we recall the main principles of unprojection as formulated by M. Reid and S. Papadakis. In Subsections 8.5, 8.6, we focus on

two kinds of Kustin-Miller unprojection which lead to the construction of Gorenstein rings of codimension 3 and 4 respectively. In Subsection 8.7, we refer briefly to the parallel Kustin-Miller unprojection of J. Neves and S. Papadakis and its applications. In Subsection 8.8, we present the original construction of A. Kustin and M. Miller known as the Kustin-Miller complex construction. In Subsection 8.9, we discuss the Macaulay2 package "KustinMiller", developed by J. Böhm and S. Papadakis, which implements the Kustin-Miller complex construction on the the computer algebra system Macaulay2 [28].

8.4. Kustin-Miller unprojection. In this subsection, we follow [49]. Assume R is a positively graded Gorenstein ring. Let I, J be homogeneous ideals of R such that R/I and R/J are Gorenstein, $I \subset J$ and $\dim R/I - \dim R/J = 1$. By duality theory, applying the functor $\text{Hom}_{R/I}(-, R/I)$ to the exact sequence

$$(12) \quad 0 \rightarrow J \rightarrow R/I \rightarrow R/J \rightarrow 0$$

we get the following exact sequence

$$(13) \quad 0 \rightarrow R/I \rightarrow \text{Hom}_{R/I}(J, R/I) \rightarrow R/J \rightarrow 0$$

where the second nonzero map corresponds to the Poincaré residue map.

Using that R/I and R/J are cyclic, we conclude that the R/I -module $\text{Hom}_{R/I}(J, R/I)$ is generated by two elements. Hence, there exists an R/I -module homomorphism $\phi : J \rightarrow R/I$ such that ϕ together with the inclusion map $i : J \rightarrow R/I$ generate $\text{Hom}_{R/I}(J, R/I)$ as R/I -module.

Definition 8.3. (*Reid*) Assume T is a new variable. We define as *Kustin-Miller unprojection ring* of the pair $J \subset R/I$ the quotient

$$\text{Unpr}(J, R/I) = \frac{R[T]}{I + (Tr - \phi(r) : r \in J)}$$

Theorem 8.4. (*Papadakis-Reid*) *The ring $\text{Unpr}(J, R/I)$ is Gorenstein.*

Remark 8.5. The Kustin-Miller unprojection ring $\text{Unpr}(J, R/I)$ does not depend (up to isomorphism) on the choice of the map ϕ .

The following example is the simplest example of Kustin-Miller unprojection. Although the algebra behind this example is quite simple, it has found many applications in birational geometry ([49, Applications 2.3.]).

Example 8.6. (*Reid's Ax-By argument*) Let $A, B \in R = \mathbb{k}[x, y, z, w]$. Assume that

$$X = V(Ax - By) \subset \mathbb{P}^3$$

is an irreducible cubic surface and

$$D = V(x, y)$$

is a codimension 1 subscheme contained in X .

Denote by I_X, I_D the ideals correspond to X and D respectively. We can easily check that the ideals I_X and I_D of R satisfy the conditions of Kustin-Miller unprojection.

Then, $\text{Hom}_{R/I_X}(I_D, R/I_X)$ is generated as R/I_X -module by the inclusion map i and ϕ . The R/I_X -module homomorphism $\phi : I_D \rightarrow R/I_X$ is the unique R/I_X -module homomorphism such that $\phi(x) = B + I_X$ and $\phi(y) = A + I_X$.

The Kustin-Miller unprojection ring of the pair $I_D \subset R/I_X$ is

$$\text{Unpr}(I_D, R/I_X) = \frac{R[T]}{I_X + (Tx - B, Ty - A)}.$$

Denote by $Y = V(I_X + (Tx - B, Ty - A)) \subset \mathbb{P}^4$. We note that Y is a codimension 2 complete intersection.

There is a natural projection

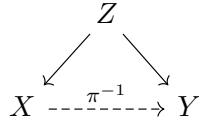
$$\pi : Y \dashrightarrow X \text{ with } [x, y, z, w, T] \mapsto [x, y, z, w]$$

with birational inverse the rational map (induced by the module homomorphism ϕ)

$$\pi^{-1} : X \dashrightarrow Y \text{ with } [x, y, z, w] \mapsto [x, y, z, w, T = \frac{A}{y} = \frac{B}{x}]$$

We have the following cases:

- (General Case) If X is smooth then π^{-1} is a regular map, the usual Castelnuovo blow-down of the (-1) -line D .
- (Special Case) Assume that $V(A, B) \cap V(x, y) \neq \emptyset$. For an explicit example, taken from [42, p. 21], assume $X = V(xz(w + x) - y(z + y)w)$. Then $V(A, B) \cap V(x, y) = \{P_1, P_2\}$, where P_1, P_2 are A_1 singularities of X . Denote by Z the graph of π . Then π^{-1} factorizes as follows:



The map $Z \rightarrow X$ is the blowup of X at the points P_1, P_2 and the map $Z \rightarrow Y$ is the blowdown of the strict transform of D .

Remark 8.7. In the above example, π^{-1} is the inverse of a projection, which explains the name unprojection.

8.5. Unprojection of a complete intersection inside a complete intersection. Assume R is a Gorenstein ring and I, J are complete intersection ideals of codimension r and $r + 1$ respectively with the property $I \subset J$. Following [43, Section 4], we describe the unprojection ring $\text{Unpr}(J, R/I)$.

The ideals I, J are generated by the regular sequences $\{u_1, \dots, u_r\}$ and $\{v_1, \dots, v_{r+1}\}$ respectively. That is,

$$I = (u_1, \dots, u_r), \quad J = (v_1, \dots, v_{r+1}).$$

Since $I \subset J$, there exists an $r \times (r + 1)$ matrix A with

$$\begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix} = A \begin{pmatrix} v_1 \\ \vdots \\ v_{r+1} \end{pmatrix}.$$

Denote by A_i the $r \times r$ submatrix of A obtained by removing the i -th column of A . We set for $1 \leq i \leq r + 1$, h_i to be the determinant of the matrix A_i .

Theorem 8.8. ([43, Theorem 4.3]) *Denote by*

$$\phi : J \rightarrow R/I$$

the map such that

$$\phi(v_i) = (-1)^{i+1} h_i, \text{ for } 1 \leq i \leq r + 1.$$

The R/I -module $\text{Hom}_{R/I}(J, R/I)$ is generated by ϕ and the canonical inclusion $i : J/I \rightarrow R/I$. Moreover the unprojection ring,

$$\text{Unpr}(J, R/I) = \frac{R[T]}{I + (Tv_i - (-1)^{i+1} h_i)}$$

is Gorenstein.

We finish this subsection by describing the unprojection ring in a specific example of a complete intersection inside a complete intersection.

Example 8.9. Let $R = \mathbb{k}[c_i, d_i, x_i]$, where $1 \leq i \leq 3$, be the standard graded polynomial ring in 9 variables over a field \mathbb{k} . Consider the ideals

$$I = (c_1x_1 + c_2x_2 + c_3x_3, d_1x_1 + d_2x_2 + d_3x_3), \quad J = (x_1, x_2, x_3)$$

of R . The ideals I, J of R satisfy the assumptions of Kustin-Miller unprojection.

We consider the 2×3 matrix

$$A = \begin{pmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix}$$

and, for $1 \leq i \leq 3$, we denote by A_i the 2×2 submatrix of A obtained by removing the i -th column of A . For $1 \leq i \leq 3$, denote by $|A_i|$ the determinant of A_i . We set

$$h_1 = |A_1| = c_2d_3 - c_3d_2,$$

$$h_2 = |A_2| = c_1d_3 - c_3d_1,$$

$$h_3 = |A_3| = c_1d_2 - c_2d_1,$$

Then, by [43, Theorem 4.3], the unprojection ring of the pair $J \subset R/I$ is

$$\text{Unpr}(J, R/I) = \frac{\mathbb{k}[c_1, c_2, c_3, d_1, d_2, d_3, x_1, x_2, x_3, T]}{I + (Tx_1 - h_1, Tx_2 - (-h_2), Tx_3 - h_3)}.$$

We remark that $\text{Unpr}(J, R/I)$ is not a complete intersection.

8.6. Tom and Jerry unprojections. Tom and Jerry families, defined and named by M. Reid, are two different families of unprojection which are used for the construction of a codimension 4 Gorenstein ideal with 9 equations and 16 first syzygies starting from a codimension 3 Gorenstein ideal. S. Papadakis [43] gave an explicit presentation of the unprojection ring of these families.

Before we introduce Tom and Jerry families of unprojection let us remind some preliminary notions.

Definition 8.10. A *skewsymmetric* matrix $M = [m_{ij}]$, $1 \leq i, j \leq n$ with entries in a commutative ring R is an $n \times n$ matrix such that $m_{i,i} = 0$ and $m_{ij} = -m_{ji}$.

For example, a 5×5 skewsymmetric matrix M is of the form

$$\begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} & m_{15} \\ -m_{12} & 0 & m_{23} & m_{24} & m_{25} \\ -m_{13} & -m_{23} & 0 & m_{34} & m_{35} \\ -m_{14} & -m_{24} & -m_{34} & 0 & m_{45} \\ -m_{15} & -m_{25} & -m_{35} & -m_{45} & 0 \end{pmatrix}$$

Definition 8.11. Let $M = [m_{ij}]$, $1 \leq i, j \leq n$ with entries in a commutative ring R be an $n \times n$ skewsymmetric matrix. Denote by M_i the skewsymmetric submatrix of M obtained by deleting the i -th row and i -th column of M and by $I_{n/2}$ the $n/2 \times n/2$ identity matrix.

- (1) If n is even, we call *Pfaffian* of the matrix M and denote it by $\text{Pf}(M)$, the unique polynomial with the properties

$$(\text{Pf}(M))^2 = \det M$$

and

$$\text{Pf}\left(\begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}\right) = 1$$

- (2) If n is odd, we call *Pfaffians* of the matrix M the set $\{\text{Pf}(M_1), \text{Pf}(M_2), \dots, \text{Pf}(M_n)\}$.

Example 8.12. If $n = 2$,

$$\text{Pf}(M) = m_{12}.$$

If $n = 5$,

$$\text{Pf}(M) = \{\text{Pf}(M_1), \text{Pf}(M_2), \dots, \text{Pf}(M_5)\}$$

where

$$\text{Pf}(M_1) = m_{23}m_{45} - m_{24}m_{35} + m_{25}m_{34},$$

$$\text{Pf}(M_2) = m_{13}m_{45} - m_{14}m_{35} + m_{15}m_{34},$$

$$\begin{aligned}\text{Pf}(M_3) &= m_{12}m_{45} - m_{14}m_{25} + m_{15}m_{24}, \\ \text{Pf}(M_4) &= m_{12}m_{35} - m_{13}m_{25} + m_{15}m_{23}, \\ \text{Pf}(M_5) &= m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23}.\end{aligned}$$

Let M be a 5×5 skewsymmetric matrix. Fix a codimension 4 complete intersection ideal J . We set the following question:

Question 8.13. What conditions should be satisfied by the entries of M such that the codimension 3 ideal generated by the Pfaffians of M is contained in J ?

Two different answers in this question are given by Tom and Jerry. According to our knowledge it is still an open question if Tom and Jerry are the only answers to the Question 8.13 ([52, Problem 8.2]).

Definition 8.14. (1) Assume $1 \leq i \leq 5$. The matrix M is called *Tom_i* in J if after we delete the i -th row and i -th column of M the remaining entries are elements of the codimension 4 ideal J .
(2) Assume $1 \leq i < j \leq 5$. The matrix M is called *Jerry_{ij}* in J if all the entries of M that belong to the i -th row or i -th column or j -th row or j -th column are elements of J , while there is no restriction for the remaining entries of M .

Example 8.15. We work over the polynomial ring $R = \mathbb{k}[y_k, z_k, m_{ij}^k]$ where $1 \leq k \leq 4$, $2 \leq i < j \leq 5$. Assume $J = (z_1, z_2, z_3, z_4)$ is a codimension 4 complete intersection ideal of R .

(1) The matrix

$$M = \begin{pmatrix} 0 & y_1 & y_2 & y_3 & y_4 \\ -y_1 & 0 & m_{23} & m_{24} & m_{25} \\ -y_2 & -m_{23} & 0 & m_{34} & m_{35} \\ -y_3 & -m_{24} & -m_{34} & 0 & m_{45} \\ -y_4 & -m_{25} & -m_{35} & -m_{45} & 0 \end{pmatrix},$$

where

$$m_{ij} = \sum_{k=1}^4 m_{ij}^k z_k.$$

is an example of a *Tom₁* matrix in J .

(2) The matrix

$$M = \begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} & m_{15} \\ -m_{12} & 0 & m_{23} & m_{24} & m_{25} \\ -m_{13} & -m_{23} & 0 & y_1 & y_2 \\ -m_{14} & -m_{24} & -y_1 & 0 & y_3 \\ -m_{15} & -m_{25} & -y_2 & -y_3 & 0 \end{pmatrix},$$

where

$$m_{ij} = \sum_{k=1}^4 m_{ij}^k z_k.$$

is an example of a *Jerry₁₂* matrix in J .

We finish this subsection by recalling the main ideas of Papadakis' calculation about Tom ([43, Section 3.3]).

Papadakis' Calculation about Tom We work over the polynomial ring $R = \mathbb{k}[y_k, z_k, m_{ij}^k]$ where $1 \leq k \leq 4$, $2 \leq i < j \leq 5$. Consider the *Tom₁* matrix M in the ideal $J = (z_1, z_2, z_3, z_4)$ of the first part of Example 8.15. Denote by I be the ideal generated by the Pfaffians $\text{Pf}(M_1), \text{Pf}(M_2), \dots, \text{Pf}(M_5)$ of M . It is easy to see that $I \subset J$.

Using that $\text{Pf}(M_2), \dots, \text{Pf}(M_5)$ are linear in z_1, z_2, z_3, z_4 , there exists a unique 4×4 matrix Q such that

$$\begin{pmatrix} \text{Pf}(M_2) \\ \text{Pf}(M_3) \\ \text{Pf}(M_4) \\ \text{Pf}(M_5) \end{pmatrix} = Q \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}.$$

For $i = 1, \dots, 4$, let H_i be the 1×4 matrix whose i -th entry is equal to $(-1)^{i+1}$ times the determinant of the submatrix of Q_i obtained by Q deleting the i -th column. We fix $1 \leq j \leq 4$. Using that for all i, j ,

$$y_i H_j = y_j H_i.$$

we can define four polynomials g_1, g_2, g_3, g_4 such that

$$(g_1, g_2, g_3, g_4) = H_j / y_j.$$

We note that this definition does not depend on the choice of j .

Denote by ϕ the map which is defined by

$$\begin{aligned} \phi: J &\rightarrow R/I \\ z_i &\mapsto g_i. \end{aligned}$$

Then, $\text{Hom}_{R/I}(J, R/I)$ is generated as R/I -module by the inclusion map i and ϕ . Denote by

$$I_{un} = I + (Tz_1 - g_1, Tz_2 - g_2, Tz_3 - g_3, Tz_4 - g_4).$$

The unprojection ring of the pair $J/I \subset R/I$

$$\text{Unpr}(J, R/I) = \frac{\mathbb{k}[y_k, z_k, m_{ij}^k, T]}{I_{un}}$$

is Gorenstein and the codimension of the ideal I_{un} is equal to 4.

Remark 8.16. According to our knowledge it is still an open question if every codimension 4 Gorenstein ring with a 9×16 resolution comes from Tom and Jerry unprojections ([52, Problem 8.3]).

8.7. Parallel Kustin-Miller unprojection. In Algebraic Geometry, especially for applications it is necessary to produce Gorenstein rings of higher codimension. For this aim, Kustin-Miller unprojection can be used more than one time in order to produce Gorenstein rings of arbitrary codimension whose properties are controlled by just a few equations as a number of new unprojection variables are adjoined.

In this direction, J. Neves and S. Papadakis [39] developed a theory which is called parallel Kustin-Miller unprojection. The initial data for parallel Kustin-Miller unprojection is a Gorenstein graded ring R and a finite set of codimension 1 ideals $\{J_1, \dots, J_n\}$ such that the quotients R/J_i are Gorenstein and satisfy some extra mild assumptions. The unprojection ring that is obtained by this process is defined as the quotient of $R[T_1, \dots, T_n]$, where T_1, \dots, T_n are new variables, by an ideal of simple form obtained from the initial data. Moreover, the new ring is Gorenstein ([39, Theorem 2.3]).

Parallel Kustin-Miller unprojection has found many applications in the construction of new interesting algebraic varieties [38, 39, 40, 50, 51]. For more details related to parallel Kustin-Miller unprojection we refer the reader to [39, 50].

8.8. Kustin-Miller complex construction. Kustin-Miller complex construction was introduced by A. Kustin and M. Miller [33] during their efforts to find a structure theorem of Gorenstein rings of codimension ≥ 4 . Given a pair of projective resolutions of Gorenstein rings that satisfy certain properties, this construction produces a new Gorenstein ring and its resolution.

Kustin-Miller complex construction has found many applications in algebraic geometry, for example in the construction of some interesting geometric objects such as Campedelli surfaces of general type and Calabi-Yau 3-folds of high codimension [38, 39]. In these cases, the numerical invariants of the varieties produced via Kustin-Miller unprojection

are controlled by the Kustin-Miller complex construction. Moreover, it has found many applications in algebraic combinatorics, for example in the study of face rings of cyclic polytopes and stellar subdivisions of Gorenstein complexes [11, 13, 14, 15].

J. Böhm and S. Papadakis [9, 10] developed the Macaulay2 [28] package "KustinMiller" which implements the Kustin-Miller complex construction.

In this subsection, we describe the Kustin-Miller complex construction, following [11, 10, 33].

We begin by recalling the assumptions of Kustin-Miller unprojection of Subsection 8.4.

Assume R is a positively graded polynomial ring over a field. Let I, J be homogeneous ideals of R such that R/I and R/J are Gorenstein, $I \subset J$ and $\dim R/I - \dim R/J = 1$.

Kustin-Miller complex construction is the construction of the graded free resolution of the Kustin-Miller unprojection ring $\text{Unpr}(J, R/I)$ (Definition 8.3) from graded free resolutions of R/I and R/J as R -modules.

Using that $R/I, R/J$ are Gorenstein rings, by [7, Proposition 3.6.11] there are integers k_1, k_2 such that $\omega_{R/I} = R/I(k_1)$ and $\omega_{R/J} = R/J(k_2)$. Assume that $k_1 > k_2$. The assumptions of Kustin-Miller unprojection are satisfied. Hence, we fix a graded homomorphism $\phi: J \rightarrow R/I$ of degree $k_1 - k_2$ such that $\text{Hom}_{R/I}(J, R/I)$ is generated as an R/I -module by ϕ and the canonical inclusion i . The Kustin-Miller unprojection ring of the pair $J \subset R/I$, $\text{Unpr}(J, R/I)$ defined by the ϕ (Definition 8.3), where T is a new variable of degree $k_1 - k_2$, is a positively graded algebra.

We now describe the construction given by A. Kustin and M. Miller [33] of the graded free resolution of the Kustin-Miller unprojection ring $\text{Unpr}(J, R/I)$ (Definition 8.3) from the graded free resolutions of R/I and R/J .

Denote by g the codimension of the ideal J of R . That is, $g = \dim R - \dim R/J$. Let

$$\begin{aligned} C_J : R/J \leftarrow \theta_0 = R \xleftarrow{\theta_1} \theta_1 \xleftarrow{\theta_2} \dots \xleftarrow{\theta_{g-1}} \theta_{g-1} \xleftarrow{\theta_g} \theta_g = R(-k_1 - \eta) \leftarrow 0 \\ C_I : R/I \leftarrow B_0 = R \xleftarrow{b_1} B_1 \xleftarrow{b_2} \dots \xleftarrow{b_{g-1}} B_{g-1} = R(-k_2 - \eta) \leftarrow 0 \end{aligned}$$

be minimal graded free resolutions of R/J and R/I respectively as R -modules, where η is the sum of the degrees of the variables of R . Due to Gorenstein property, C_I and C_J are self-dual ([27, Corollary 21.16]).

For an R -module M , we denote $M' := M \otimes_R R[T]$ which is an $R[T]$ -module. Consider the complex

$$C : \text{Unpr}(J, R/I) \leftarrow F_0 \xleftarrow{f_1} F_1 \xleftarrow{f_2} \dots \xleftarrow{f_{g-1}} F_{g-1} \xleftarrow{f_g} F_g \leftarrow 0$$

with the modules, when $g \geq 4$,

$$\begin{aligned} F_0 &= B'_0, & F_1 &= B'_1 \oplus \theta'_1(k_2 - k_1) \\ F_i &= B'_i \oplus \theta'_i(k_2 - k_1) \oplus B'_{i-1}(k_2 - k_1), & \text{for } 2 \leq i \leq g-2 \\ F_{g-1} &= \theta'_{g-1}(k_2 - k_1) \oplus B'_{g-2}(k_2 - k_1), & F_g &= B'_{g-1}(k_2 - k_1). \end{aligned}$$

If $g = 2$, we have

$$\begin{aligned} F_0 &= B'_0, & F_1 &= \theta'_1(k_2 - k_1) \\ F_2 &= B'_1(k_2 - k_1). \end{aligned}$$

If $g = 3$, we have

$$\begin{aligned} F_0 &= B'_0, & F_1 &= B'_1 \oplus \theta'_1(k_2 - k_1) \\ F_2 &= \theta'_2(k_2 - k_1) \oplus B'_1(k_2 - k_1), & F_3 &= B'_2(k_2 - k_1). \end{aligned}$$

We will define the differentials of the complex C by specifying chain maps $\alpha : C_I \rightarrow C_J$, $\beta : C_J \rightarrow C_I[-1]$ and a homotopy map $h : C_I \rightarrow C_I$ which is not required to be a chain map.

Let t_1 be the rank of the free R -module A_1 . The self-duality of C_J implies that t_1 is also the rank of the free R -module θ_{g-1} .

Fix R -module bases e_1, \dots, e_{t_1} and $\hat{e}_1, \dots, \hat{e}_{t_1}$ of θ_1 and θ_{g-1} respectively. For $1 \leq i \leq t_1$, we define $c_i, \hat{c}_i \in R$ such that

$$\theta_1(e_i) = c_i 1_R, \quad \theta_g(1_R) = \sum_{i=1}^{t_1} \hat{c}_i \hat{e}_i$$

where by Gorenstein property, $c_i, \hat{c}_i \in J$.

For $1 \leq i \leq t_1$, denote by $l_i, \hat{l}_i \in R$ lifts in R of $\phi(c_i)$ and $\phi(\hat{c}_i)$ respectively. For an R -module A , we set $A^* = \text{Hom}_R(A, R)$. For an R -basis f_1, \dots, f_t , denote by f_1^*, \dots, f_t^* the dual basis of A^* .

Let $\bar{a}_{g-1}^d: \theta_{g-1}^* \rightarrow R = B_{g-1}^*$ be the R -module homomorphism defined by $\bar{a}_{g-1}^d(\hat{e}_i^*) = \hat{l}_i 1_R$ for $1 \leq i \leq t_1$. Using that C_I, C_J are self-dual we get that \bar{a}_{g-1}^d extends to $\bar{a}^d: C_J^* \rightarrow C_I^*$. Denote by $\bar{a}: C_I \rightarrow C_J$ its dual.

By [43], the map $\bar{a}_0: B_0 = R \rightarrow R = \theta_0$ is multiplication by an invertible element of R , say u . We set $a = \bar{a}/u$.

Next step is to define a chain map $\beta: C_J \rightarrow C_I[-1]$ by extending $\beta_1: \theta_1 \rightarrow R = B_0$ defined by $\beta_1(e_i) = -l_i 1_R$.

The map $\beta_g: \theta_g = R \rightarrow R = B_{g-1}$ is multiplication by a nonzero constant. Then, by the proof of ([33, Theorem 1.4.]) there is a homotopy map $h: C_I \rightarrow C_I$ with $h_0: B_0 \rightarrow B_0$ and $h_{g-1}: B_{g-1} \rightarrow B_{g-1}$ being the zero maps and

$$\beta_i \alpha_i = h_{i-1} b_i + b_i h_i, \quad \text{for } 1 \leq i \leq g.$$

Denote by $r = \text{rank } B_t$. Let I_t be the identity $r \times r$ matrix. The differential maps $f_i: F_i \rightarrow F_{i-1}$ of the complex C are defined as:

$$\begin{aligned} f_1 &= \begin{pmatrix} b_1 & \beta_1 + T \cdot \theta_1 \end{pmatrix}, & f_2 &= \begin{pmatrix} b_2 & \beta_2 & h_1 + T \cdot I_1 \\ 0 & -\theta_2 & -\alpha_1 \end{pmatrix} \\ f_i &= \begin{pmatrix} b_i & \beta_i & h_{i-1} + (-1)^i T \cdot I_{i-1} \\ 0 & -\theta_i & -\alpha_{i-1} \\ 0 & 0 & b_{i-1} \end{pmatrix} & \text{for } 3 \leq i \leq g-2 \\ f_{g-1} &= \begin{pmatrix} \beta_{g-1} & h_{g-2} + (-1)^{g-1} T \cdot I_{g-2} \\ -\theta_{g-1} & -\alpha_{g-2} \\ 0 & b_{g-2} \end{pmatrix}, & f_g &= \begin{pmatrix} -\alpha_{g-1} + (-1)^g \frac{1}{\beta_g(1)} T \cdot \theta_g \\ b_{g-1} \end{pmatrix} \end{aligned}$$

Theorem 8.17. [33] *The chain complex C , known as the Kustin-Miller complex construction, is a graded free resolution of $\text{Unpr}(J, R/I)$ as $R[T]$ -module.*

Remark 8.18. In general, the resolution C is not minimal as indicated, for example, by [12, Subsection 5.1, Example 3]. However, it is minimal in some examples coming from algebraic geometry [38, 50, 51] and combinatorics [11, Theorem 6.1], [13]. For explicit examples of the Kustin-Miller complex construction, we refer the reader to [11, Subsection 6.2].

8.9. The Macaulay2 package KustinMiller. The Macaulay2 package "KustinMiller", developed by J. Böhm and S. Papadakis [10], is the implementation using Macaulay2 [28] of the Kustin-Miller complex construction. More precisely, the authors present an algorithm which has as input the resolutions C_I and C_J of I and J respectively and produces as output the Kustin-Miller complex C associated to I and J constructed in the previous paragraph.

To use the package, type in Macaulay2 the command line `needsPackage "KustinMiller"`.

The command `kustinMillerComplex` produces the Kustin-Miller complex C from the resolutions C_I, C_J . For an example, we refer to [10, Section 4]. Moreover, there are useful Macaulay2 commands such as `Hom(J, R^1/I)`, which determines the unprojection map ϕ (Subsection 8.4) and `extend` which extends homomorphisms to chain maps. For more details about the package we refer the reader to [9, 10].

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