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Abderrahmane Bendali

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PRIMAL HYBRID FINITE ELEMENTS ON EXACT CURVED MESHES

A. BENDALI

ABSTRACT. An adaptation of the techniques introduced in [C. Bernardi, SIAM J. Numer. Anal., 26 (1989) pp. 1212-1240] is used to define Raviart-Thomas finite elements on exact meshes of domains with curved boundaries. These finite elements are part of those involved in the approximation of the primal hybrid formulations of second-order elliptic boundary-value problems in two and three dimensions. Related interpolation operators are built and corresponding error estimates established.

INTRODUCTION

This study is devoted to an adaptation of the construction of finite element (FE) spaces on curved simplices of exact meshes of domains with curved boundaries by Bernardi [3] for the approximation of primal hybrid formulations of second-order elliptic boundary-value problems. Exact meshes are used in computations aimed at obtaining optimal-order approximations. They can be used also for theoretical purposes. For example, in the case of a FE approximation of a boundary value problem posed in terms of the Helmholtz equation, the regularity of the domain boundary and its exact approximation make it possible to track the dependence of stability and error estimates not only on the mesh size and but also on the wave number (cf., e.g., [11] for a standard FE approximation, and [2] for the primal hybrid one).

To be specific, consider the following boundary-value problem posed in a bounded Lipschitz domain Ω of \mathbb{R}^d ($d = 2, 3$)

$$(1) \quad \begin{cases} u \in H^1(\Omega) \\ -\Delta u + u = f \text{ in } \Omega, \\ \partial_{\mathbf{n}} u + u = g \text{ on } \partial\Omega. \end{cases}$$

In (1), we have avoided any difficulties that are not specific to the primal hybrid formulation, for example by considering an interior equation $-\Delta u + u = f$ leading to a direct estimate of coercivity in H^1 without having to resort to difficult Poincaré inequalities [5], which are clearly not at the heart of the FE approximation of the primal hybrid formulation. However, we added the term $+u$ in the boundary condition because the way it is treated is an important feature of this type of formulation. We have denoted by f and g the right-hand side data of the problem, respectively in $L^2(\Omega)$ and $L^2(\partial\Omega)$. Throughout the document, the symbol \mathbf{n} , with or without a subscript, indicating the domain concerned, here Ω , represents the unit normal to the domain boundary, here $\partial\Omega$, directed outwards from the domain. Consider a mesh \mathcal{T}^h of Ω in elements, generically denoted K , which will be more explicitly described below. Taking v in the broken Sobolev space

$$\mathbb{X} = \{v \in L^2(\Omega); v_K \in H^1(K), \forall K \in \mathcal{T}^h\},$$

and using a Green formula in each $K \in \mathcal{T}^h$, one readily comes to the primal hybrid formulation of the problem (1)

$$(2) \quad \begin{cases} u \in \mathbb{X}, \lambda \in \mathbf{M}, \\ a(u, v) + b(\lambda, v) = (f, v)_\Omega + (g, v)_{\partial\Omega}, \forall v \in \mathbb{X}, \\ b(\mu, u) = 0, \forall \mu \in \mathbf{M}. \end{cases}$$

The scalar product in $L^2(D)$ is denoted by $(\cdot, \cdot)_D$, where D is a domain of \mathbb{R}^d or a part of its boundary. In this last case, this scalar product will be the duality brackets $H^{-1/2}(D)$ - $H^{1/2}(D)$, when the first argument is in $H^{-1/2}(D)$ and the second one is in $H^{1/2}(D)$, D then being the boundary of a domain of \mathbb{R}^d . For simplicity, all the functions are supposed to be real-valued. The complex case can be handled with simple adaptations. Other notations are

- **Space for multipliers or interface unknowns**

$$\begin{aligned} \mathbb{M} &= \prod_K H^{-1/2}(\partial K), \\ H_0(\operatorname{div}, \Omega) &= \{ \mathbf{q} \in L^2(\Omega, \mathbb{R}^d); \nabla \cdot \mathbf{q} \in L^2(\Omega), \mathbf{q} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \}, \\ \mathbf{M} &= \{ \mu \in \mathbb{M}; \exists \mathbf{q} \in H_0(\operatorname{div}, \Omega), \mathbf{q}_K \cdot \mathbf{n}_K = \mu_K \text{ on } \partial K, \forall K \in \mathcal{T}^h \}. \end{aligned}$$

- **Bilinear forms**

$$\begin{aligned} a(u, v) &= \sum_{K \in \mathcal{T}^h} (\nabla u_K, \nabla v_K)_K + (u_K, v_K)_K + \sum_{e \in \mathcal{E}_{K, \partial\Omega}} (u_K, v_K)_e, \\ b(\lambda, v) &= -\sum_{K \in \mathcal{T}^h} (\lambda_K, v_K)_{\partial K}. \end{aligned}$$

The meaning of the set $\mathcal{E}_{K, \partial\Omega}$ of edges/faces e of K included in $\partial\Omega$ requires some developments and is postponed. By convention, the sum on the elements of $\mathcal{E}_{K, \partial\Omega}$ is zero when this set is empty.

Remark 1. *Two comments are in order.*

- (1) *The condition $\mathbf{q} \in H_0(\operatorname{div}, \Omega)$ implies that*

$$(3) \quad \mu_K|_e = 0, \forall e \in \mathcal{E}_{K, \partial\Omega}.$$

(All the sets related to the mesh as the elements K , the edges/faces e , etc., are supposed open, i.e. without their boundary. The restriction is thus taken in the sense of distributions.)

- (2) *For $\mu \in \mathbb{M}$ such that*

$$(4) \quad \mu_K \in L^1(\partial K), \forall K \in \mathcal{T}^h,$$

the property $\mathbf{q} \in H(\operatorname{div}, \Omega)$ with $\mu_K = \mathbf{q}_K \cdot \mathbf{n}_K$, is equivalent to

$$(5) \quad \mu_K|_e + \mu_L|_e = 0, \forall e \in \mathcal{E}_{K, \Omega} \cap \mathcal{E}_{L, \Omega}$$

where $\mathcal{E}_{K, \Omega}$ is the set of edges/faces e of K that are contained in Ω . Since the elements of FE spaces, for approximating the interface variable μ , satisfy (4), their characterization as elements of \mathbf{M} can be equivalently stated by (3) and (5). This will be the case for any FE subspace of \mathbf{M} .

Functions in \mathbb{X} can be approximated by a direct application of Bernardi's results [3]. Therefore, we mainly focus in this document on the FE approximation of the distributions in \mathbb{M} . As in the case of straight simplices [14, 6], this approximation is undertaken by considering them as normal traces of vector fields in $H(\operatorname{div}, K)$ and approximating these fields by Raviart-Thomas FEs. The construction is therefore reduced to that of Raviart-Thomas FE on curved simplices. Although obtained from techniques developed by Bernardi [3, p. 1217], it is not a direct application of them. Indeed, this author refers to F. Dubois's definition [9] for this construction. Unfortunately, F. Dubois' construction is limited to the lowest-order Raviart-Thomas FE and requires domains with globally smooth boundaries, contrary to the framework considered in [3] and retained here.

The outline is the following. The first section is devoted to introducing, essentially on the basis of Bernardi's results [3], the type of curved-boundary domains that are considered, the way in which these are exactly meshed by straight and curved simplices, the basic isomorphisms involved in parametric definitions of related FE spaces, and corresponding fundamental estimates. This will allow us to define Raviart-Thomas FE spaces on curved exact meshes with interpolation operators and to establish the related error estimates. In a next section, we extend the stability analysis and the error bounds for the FE approximation of the primal hybrid formulation (2) for standart meshes [15, 14, 6] to exact meshes of domains with curved boundaries.

1. PARAMETRIC FINITE ELEMENT SPACES

1.1. Curved meshes and related norms and semi-norms. In all what follows, we assume that Ω is a curved polygonal/polyhedral domain such that

$$(6) \quad \partial\Omega \text{ is piecewise of class } \mathcal{C}^{k+1,1}$$

where k is an integer ≥ 0 . The second superscript 1 means that the derivatives of order $k+1$ are Lipschitz functions. The domain Ω can hence be endowed with an exact triangulation \mathcal{T}^h in curved d -simplices K . Implicitly throughout the sequel, the curved d -simplices K of \mathcal{T}^h are assumed to be of class \mathcal{C}^{k+1} in Bernardi's meaning [3, Def. 2.2]. Parameter h is the mesh size, defined by

$$h = \max_{K \in \mathcal{T}^h} h_K,$$

where, following [3], h_K is the maximum distance between two vertices of $K \in \mathcal{T}^h$. In the same way, we denote by ϱ_K the diameter of the inscribed ball in the straight d -simplex \tilde{K} whose vertices are those of K . The definition of the vertices of K will be given later.

The various norms and semi-norms of v , defined on K , are denoted by

$$|v|_{0,K} = \left(\int_K |v|^2 dx \right)^{1/2}, \quad |v|_{j,K} = \left(\sum_{|\alpha|=j} |\partial_x v|_{0,K}^2 \right)^{1/2}, \quad j = 1, 2, \dots$$

For a vector field $\mathbf{q} \in L^2(K, \mathbb{C}^d)$, we use similar notation substituting the euclidian norm $|\mathbf{q}|$ of \mathbf{q} for the absolute value of v_K involved in the integral defining the L^2 -norm of v .

The numerical analysis of the primal hybrid formulation makes use of the two following norms on $H^1(K)$ [15, 14]: the usual norm

$$(7) \quad \|v\|_{1,K} = \left(|v|_{1,K}^2 + |v|_{0,K}^2 \right)^{1/2}$$

and the h_K -dependent norm

$$(8) \quad \|v\|_{1,h_K^{-1},K} = \left(|v|_{1,K}^2 + h_K^{-2} |v|_{0,K}^2 \right)^{1/2}.$$

This induces two norms on \mathbb{X}

$$(9) \quad |v|_{\mathbb{X}} = \left(\sum_{K \in \mathcal{T}^h} \|v_K\|_{1,K}^2 \right)^{1/12}, \quad \text{and} \quad \|v\|_{\mathbb{X}} = \left(\sum_{K \in \mathcal{T}^h} \|v_K\|_{1,h_K^{-1},K}^2 \right)^{1/12}$$

satisfying

$$(10) \quad |v|_{\mathbb{X}} \lesssim \|v\|_{\mathbb{X}}.$$

We use the symbol \lesssim (resp. \gtrsim) for right-hand (resp. left-hand) estimates independent of the mesh size h and the estimated functions or distributions.

The norm in $H^{-1/2}(\partial K)$ [15, 14] is similarly defined by

$$\|\mu\|_{-1/2,\partial K} = \inf_{\mathbf{q} \cdot \mathbf{n}_K = \mu} \|\mathbf{q}\|_{H(\operatorname{div},K)}, \quad \|\mathbf{q}\|_{H(\operatorname{div},K)} = \left(|\mathbf{q}|_{0,K}^2 + h_K^2 |\nabla \cdot \mathbf{q}|_{0,K}^2 \right)^{1/2}.$$

These norms induce the following norm on \mathbb{M}

$$(11) \quad \|\mu\|_{\mathbb{M}} = \left(\sum_{K \in \mathcal{T}^h} \|\mu_K\|_{-1/2,K}^2 \right)^{1/2}.$$

1.2. Basic isomorphisms. Now, we explain what is meant by a curved mesh in Bernardi's sense [3, Def. 2.2]. Any d -simplex K of the mesh \mathcal{T}^h is parametrized as follows. Let \widehat{K} be the unit straight d -simplex consisting of vectors $\widehat{x} \in \mathbb{R}^d$, whose components \widehat{x}_i are positive and such that $\widehat{x}_1 + \dots + \widehat{x}_d < 1$. A d -simplex K of \mathbb{R}^d is the image $F_K(\widehat{K})$ of \widehat{K} by

$$F_K : \widehat{K} \xrightarrow{C^{k+1}\text{-diffeomorphism}} K,$$

where

$$F_K \widehat{x} = B_K \widehat{x} + \Phi_K(\widehat{x}),$$

with B_K an invertible linear-affine map from \mathbb{R}^d to \mathbb{R}^d . The integer k is that involved in the description of the piecewise regularity of $\partial\Omega$ in (6). This parametrization of K by \widehat{K} , by means of F_K , is furthermore subject to the following condition

$$c_K = \sup_{\widehat{x} \in \widehat{K}} \left| (B'_K)^{-1} F'_K(\widehat{x}) - I_d \right| < 1.$$

The linear-affine part B_K of F_K transforms \widehat{K} in the straight d -simplex \widetilde{K} , associated with K above. Recall the elementary notation: $F'_K(\widehat{x})$ is the differential of F_K at the point \widehat{x} and B'_K is the linear part of the linear-affine map B_K .

The d -simplex is straight if $\Phi_K = 0$ and curved otherwise.

The vertices of \widetilde{K} and the edges/faces e of K , already mentioned above, are simply the images by B_K and F_K of respectively the vertices and edges/faces of \widehat{K} . It is worth mentioning that only simplices with at least two vertices on $\partial\Omega$ can be curved.

The following isomorphisms, as they appear in [15, Chap II Sect 5], [6, Section III.1.3], for straight d -simplices K , and in [9] for the curved ones, are used to define spaces on $K \in \mathcal{T}^h$ or its boundary ∂K in terms of those defined on the reference d -simplex \widehat{K} or its boundary $\partial\widehat{K}$

$$\begin{cases} \mathcal{F}_K : L^1(\widehat{K}) \rightarrow L^1(K), & v = \mathcal{F}_K \widehat{v}, & v(F_K(\widehat{x})) = \widehat{v}(\widehat{x}), \\ \mathcal{P}_K : L^1(\widehat{K}, \mathbb{C}^d) \rightarrow L^1(K, \mathbb{C}^d), & \mathbf{q} = \mathcal{P}_K \widehat{\mathbf{q}}, & \mathbf{q}(F_K(\widehat{x})) = (J_K^{-1} F'_K \widehat{\mathbf{q}})(\widehat{x}), \\ \mathcal{G}_K : L^1(\partial\widehat{K}) \rightarrow L^1(\partial K), & \mu = \mathcal{G}_K \widehat{\mu}, & \mu(F_K(\widehat{x})) = (J_{\partial K}^{-1} \widehat{\mu})(\widehat{x}). \end{cases}$$

Vector field \mathbf{q} is obtained from $\widehat{\mathbf{q}}$ by the Piola (contravariant) transform \mathcal{P}_K . Above, $J_K = \det F'_K$ and $J_{\partial K} = |\mathcal{P}_K^{-1} \mathbf{n}_K \cdot \mathbf{n}_{\widehat{K}}|$ express the link of the elementary area/volume of K and \widehat{K} ($dx = J_K d\widehat{x}$) and the elementary length/area of ∂K and $\partial\widehat{K}$ ($ds = J_{\partial K} d\widehat{s}$) respectively. We make the assumption that the numbering of the vertices of K are compatible with the orientation of the space so that $J_K > 0$. Useful links between gradients, divergences, and normal traces related to these isomorphisms are given by (cf., e.g., [6, Lemma III.1.5])

$$(12) \quad \nabla_x v = \mathcal{F}_K \left((F'_K)^{-T} \nabla_{\widehat{x}} \widehat{v} \right), \quad \nabla \cdot \mathbf{q} = \mathcal{F}_K \left(J_K^{-1} \nabla_{\widehat{x}} \cdot \widehat{\mathbf{q}} \right), \quad \mathbf{q} \cdot \mathbf{n}_K = \mathcal{G}_K \left(\widehat{\mathbf{q}} \cdot \mathbf{n}_{\widehat{K}} \right),$$

where $(F'_K)^{-T}$ is the transpose of the inverse of F'_K , \mathcal{F}_K being extended to vector-valued functions component by component.

Recall that the normal trace $\mathbf{q} \cdot \mathbf{n}_K$ of $\mathbf{q} \in H(\text{div}, K)$ is defined in $H^{-1/2}(\partial K)$ for $\mathbf{q} \in H(\text{div}, K)$ and that the duality product $(\mu, v)_{\partial K}$ reduces to the L^2 scalar product when $\mu \in L^2(\partial K)$. Similar definitions and notations are associated with \widehat{K} .

The above isomorphisms induce important bounds involved in the approximation properties of the usual and the Raviart-Thomas FE spaces.

Proposition 2. *The following formulas and inequalities hold true*

$$(\mathbf{q}, \nabla_x v)_K = (\widehat{\mathbf{q}}, \nabla_{\widehat{x}} \widehat{v})_{\widehat{K}}, \quad (\nabla_x \cdot \mathbf{q}, v)_K = (\nabla_{\widehat{x}} \cdot \widehat{\mathbf{q}}, \widehat{v})_{\widehat{K}}, \quad (\mu, v)_{\partial K} = (\widehat{\mu}, \widehat{v})_{\partial \widehat{K}}$$

$$\begin{aligned} |v|_{0,K} &\leq C_K h_K^{d/2} |\widehat{v}|_{0,\widehat{K}}, & |\widehat{v}|_{0,\widehat{K}} &\leq C_K \varrho_K^{-d/2} |v|_{0,K}, \\ |v|_{1,K} &\leq C_K h_K^{d/2} \varrho_K^{-1} |\widehat{v}|_{1,\widehat{K}}, & |\widehat{v}|_{1,\widehat{K}} &\leq C_K h_K \varrho_K^{-d/2} |v|_{1,K}, \\ |\mathbf{q}|_{0,K} &\leq C_K h_K \varrho_K^{-d/2} |\widehat{\mathbf{q}}|_{0,\widehat{K}}, & |\widehat{\mathbf{q}}|_{0,\widehat{K}} &\leq C_K h_K^{d/2} \varrho_K^{-1} |\mathbf{q}|_{0,K}, \\ |\nabla_x \cdot \mathbf{q}|_{0,K} &\leq C_K \varrho_K^{-d/2} |\nabla_{\widehat{x}} \cdot \widehat{\mathbf{q}}|_{0,\widehat{K}}, & |\nabla_{\widehat{x}} \cdot \widehat{\mathbf{q}}|_{0,\widehat{K}} &\leq C_K h_K^{d/2} |\nabla_x \cdot \mathbf{q}|_{0,K}, \\ \|v\|_{1/2, h_K^{-1}, \partial K} &\leq C_K h_K^{d/2} \varrho_K^{-1} \|\widehat{v}\|_{1/2, \partial \widehat{K}}, & \|\widehat{v}\|_{1/2, \partial \widehat{K}} &\leq C_K h_K \varrho_K^{-d/2} \|v\|_{1/2, h_K^{-1}, \partial K}, \\ \|\mu\|_{-1/2, h_K^{-1}, \partial K} &\leq & \|\widehat{\mu}\|_{-1/2, \partial \widehat{K}} &\leq \\ &C_K h_K \varrho_K^{-d/2} \|\widehat{\mu}\|_{-1/2, \partial \widehat{K}}, & &C_K h_K^{d/2} \varrho_K^{-1} \|\mu\|_{-1/2, \partial K}, \end{aligned}$$

with

$$v = \mathcal{F}_K \widehat{v}, \quad \mathbf{q} = \mathcal{P}_K \widehat{\mathbf{q}}, \quad \mu = \mathcal{G}_K \widehat{\mu}_K,$$

and where C_K stands for a constant not the same in all instances depending only on $1 + c_K$ or $1/(1 - c_K)$

Proof. Directly follows from the above definitions, relations and the properties of d -simplices either curved or straight. \square

We need the following extension to semi-norms of order m , $2 \leq m \leq k$, which can be directly deduced from Bernardi's results [3, Est. (2.13)]

$$(13) \quad |\widehat{v}|_{m,\widehat{K}} \leq C_K \varrho_K^{-d/2} h_K^m \sum_{0 \leq r \leq m} |v|_{0,K}.$$

They are based on the fundamental bounds [3, Est. (2.7) and (2.8)]

$$(14) \quad \sup_{\widehat{x} \in \widehat{K}} \left| \partial_{\widehat{x}_{\ell_1} \dots \widehat{x}_{\ell_p}}^p F_K(\widehat{x}) \right| \leq c_p(K) h_K^p, \quad 2 \leq p \leq k.$$

Estimates for the vector field \mathbf{q} are more involved. They are set out in the following proposition.

Proposition 3. *Under the above general notations, we have*

$$(15) \quad |\widehat{\mathbf{q}}|_{m,\widehat{K}} \leq C_{K,m} (h_K / \varrho_K)^{m+d/2+1} h_K^{m+d/2-1} (|\mathbf{q}|_{0,K} + \dots + |\mathbf{q}|_{m,K})$$

where $C_{K,m}$ refers to a constant depending only on $c_K, c_2(K), \dots, c_m(K)$.

Proof. From the expression of the Piola transform of $\widehat{\mathbf{q}}$, we have

$$\widehat{\mathbf{q}} = J_K(\widehat{x}) (F'_K(\widehat{x}))^{-1} \mathbf{q}(F_K(\widehat{x})),$$

which can also written as follows

$$\widehat{\mathbf{q}} = J_K A^{-1} \mathbf{q} \circ F_K$$

with $A = F'_K$ to lighten the notation. Now, the elementary formula of the derivative of a product gives that the following derivative can be decomposed in 3^m terms

$$\partial_{\widehat{x}_{\ell_1} \dots \widehat{x}_{\ell_m}}^m (J_K A^{-1} \mathbf{q} \circ F_K) = \sum_{r+s+t=m} \partial_{\widehat{x}_{\lambda_1} \dots \widehat{x}_{\lambda_r}}^r J_K \partial_{\widehat{x}_{\mu_1} \dots \widehat{x}_{\mu_s}}^s A^{-1} \partial_{\widehat{x}_{\nu_1} \dots \widehat{x}_{\nu_t}}^t \mathbf{q} \circ F_K$$

where $\{\{\lambda_1, \dots, \lambda_r\}, \{\mu_1, \dots, \mu_s\}, \{\nu_1, \dots, \nu_t\}\}$ is a partition in three sets of the set $\{\ell_1, \dots, \ell_m\}$, one or two of which can be empty, indicating then a derivative or order zero. Based on the following formulas

$$\begin{aligned} \partial_{\hat{x}_\ell} J_K &= J_K \operatorname{tr}(A^{-1} \partial_{\hat{x}_\ell} A), \\ \partial_{\hat{x}_{\mu_1}} (A^{-1} \partial_{\hat{x}_{\mu_2} \dots \hat{x}_{\mu_n}}^n A) &= -A^{-1} \partial_{\hat{x}_{\mu_1}} A A^{-1} \partial_{\hat{x}_{\mu_2} \dots \hat{x}_{\mu_n}}^n A + A^{-1} \partial_{\hat{x}_{\mu_1} \dots \hat{x}_{\mu_n}}^n A, \end{aligned}$$

inequalities (14), (13), and $h_K/\varrho_K \geq 1$, an induction argument allows us to obtain (15) by cumbersome but simple calculations. \square

Remark 4. *Under the condition $h_K/\varrho_K \lesssim 1$, estimates (15) are compatible with the usual case when K is a straight d -simplex [15, Est. III.4.30a and 4.25a].*

1.3. Right and left bounds on the sesquilinear forms. As an illustration of the necessity to consider the two norms (7) and (8), we prove the following result relative to a right-bound on the sesquilinear form $(u, v)_{\partial\Omega}$ for u and v in the broken Sobolev space \mathbb{X} . This bound is fundamental to the analysis below. For this purpose, we need the following usual assumption, which is assumed throughout the rest of this document:

\mathcal{T}^h is a regular exact mesh of Ω in the meaning of Def. 3.1 in [3].

This in particular means that

$$(16) \quad h_K/\varrho_K \lesssim 1$$

and that the above constants $c_K, c_2(K), \dots, c_k(K)$ satisfy the uniform bounds

$$(17) \quad c_K \leq c < 1, \quad c_2(K), \dots, c_k(K) \lesssim 1$$

where k is the integer fixing the piecewise regularity of $\partial\Omega$ given in (6).

Proposition 5. *Under the above assumptions, the following bound holds true*

$$(18) \quad |(u, v)_e| \lesssim \|u\|_{1, h_K^{-1}, K} \|v\|_{1, K}, \quad \forall e \in \mathcal{E}_{\partial K}$$

for all u and $v \in H^1(K)$; $\mathcal{E}_{\partial K}$ is the set of edges/faces, straight or curved, of K .

Proof. The proof is based on the so-called multiplicative trace inequality (see, e.g., [10, Ineq. (2.11)])

$$|\hat{u}|_{0, \partial\hat{K}}^2 \lesssim |\hat{u}|_{0, \hat{K}} (|\hat{u}|_{1, \hat{K}} + |\hat{u}|_{0, \hat{K}})^{1/2}$$

and the bounds given in Proposition 2:

$$\begin{aligned} |u|_{0, \partial K}^2 &\lesssim \frac{h_K^d}{\varrho_K} |\hat{u}|_{0, \partial\hat{K}}^2 \lesssim \frac{h_K^d}{\varrho_K} |\hat{u}|_{0, \hat{K}} (|\hat{u}|_{1, \hat{K}}^2 + |\hat{u}|_{0, \hat{K}}^2)^{1/2} \\ &\lesssim \frac{h_K^d}{\varrho_K} \left(\frac{1}{\varrho_K^{d/2}} |u|_{0, K} \right) \left(\frac{h_K^2}{\varrho_K^d} |u|_{1, K}^2 + \frac{1}{\varrho_K^d} |u|_{0, K}^2 \right)^{1/2} \\ &\lesssim \frac{h_K^d}{\varrho_K^{d+1}} |u|_{0, K} \left(h_K^2 |u|_{1, K}^2 + |u|_{0, K}^2 \right)^{1/2}. \end{aligned}$$

The Cauchy-Schwarz inequality then gives

$$\begin{aligned}
|(u, v)_e|^2 &\leq |u|_{0,e}^2 |v|_{0,e}^2 \leq |u|_{0,\partial K}^2 |v|_{0,\partial K}^2 \\
&\lesssim \frac{h_K^{2d}}{\varrho_K^{2d+2}} |u|_{0,K} \left(h_K^2 |u|_{1,K}^2 + |u|_{0,K}^2 \right)^{1/2} |v|_{0,K} \left(h_K^2 |v|_{1,K}^2 + |v|_{0,K}^2 \right)^{1/2} \\
&\lesssim \frac{h_K^{2d+2}}{\varrho_K^{2d+2}} \frac{1}{h_K} |u|_{0,K} \|u\|_{1,h_K^{-1},K} |v|_{0,K} \left(h_K^2 |v|_{1,K}^2 + |v|_{0,K}^2 \right)^{1/2} \\
&\lesssim \frac{h_K^{2d+2}}{\varrho_K^{2d+2}} \|u\|_{1,h_K^{-1},K}^2 \|v\|_{1,K} \left(h_K^2 |v|_{1,K}^2 + |v|_{0,K}^2 \right)^{1/2}.
\end{aligned}$$

The proof is completed by noting that $h_K/\varrho_K \lesssim 1$ and $h_K \lesssim 1$. \square

The following proposition groups together the main right-bounds and left-bounds involved in the stability analysis of the numerical approximation of the primal hybrid formulation (2).

Proposition 6. *The following bounds hold true*

$$(19) \quad |a(u, v)| \lesssim \|u\|_{\mathbb{X}} |v|_{\mathbb{X}}, \quad |b(\mu, u)| \lesssim \|\mu\|_{\mathbb{M}} \|u\|_{\mathbb{X}},$$

$$(20) \quad a(u, u) \gtrsim |u|_{\mathbb{X}}^2, \quad \sup_{\|v\|_{\mathbb{X}} \leq 1} |b(\lambda, v)| \gtrsim \|\lambda\|_{\mathbb{M}},$$

where the norms $|v|_{\mathbb{X}}$ and $\|v\|_{\mathbb{X}}$ are defined in (9), $\|\mu\|_{\mathbb{M}}$ in (11).

Proof. All the estimates are direct consequence to the definition of the norms and estimate (18). \square

1.4. General Finite element spaces.

1.4.1. *Finite element approximation of the primal variable.* The general line of construction is in [2]. It is characterized by two features: an integer $m \geq 1$ related to the order of the approximation, which, in a sense, rests on polynomials of degree $\leq m$, and a finite-dimensional subspace $Y_{\widehat{K}}^m$ of $H^1(\widehat{K})$ ensuring the stability of the approximation of the saddle-point problem (2), as usual for saddle-point problems, in the form of a global uniform inf-sup condition. More precisely, let be given a finite-dimensional subspace $X_{\widehat{K}}^{m,h}$ of $H^1(\widehat{K})$ admitting the following algebraic decomposition

$$(21) \quad X_{\widehat{K}}^m = \mathbb{P}_{\widehat{K}}^m \oplus Y_{\widehat{K}}^m$$

where $\mathbb{P}_{\widehat{K}}^m$ is as usual the space of polynomial functions on \widehat{K} of degree $\leq m$ with complex coefficients. For compatibility with the fact that the d -simplices of \mathcal{T}^h are of class $k+1$, with k defined in (6), we implicitly assume below that

$$(22) \quad m \leq k.$$

The space $Y_{\widehat{K}}^m$ can be a space spanned by a system of homogeneous polynomials of degree $> m$. It can also be a subspace of a composite FE space $X_{\widehat{K}}^m$ built on a submesh \mathcal{T}_K of K . However, whether in the latter case [13], or in the former [1, 7], the decomposition (21), which plays an important role in both construction and analysis, has not been highlighted. Explicit choices for $X_{\widehat{K}}^m$ are given in [2]. The space $X_K^{m,h}$ is then defined by the above isomorphism $X_K^{m,h} = \mathcal{F}_K X_{\widehat{K}}^m$ yielding the finite-dimensional approximation of \mathbb{X}

$$\mathbb{X}^{m,h} = \left\{ v^h \in \mathbb{X}; v_K^h \in X_K^{m,h}, \forall K \in \mathcal{T}^h \right\}.$$

The approximation properties of these FE spaces are stated in the following proposition, and are a direct consequence of Bernardi's general results [3, Cor. 5.2].

Proposition 7. *Let \mathcal{L}_m^h be the Lagrange interpolation operator of Clément-type built on \mathcal{T}^h , which maps any $v \in L^1(\Omega)$ to an element $\mathcal{L}_m^h v$ in*

$$V^{m,h}(\Omega) = \left\{ v^h \in H^1(\Omega); v^h|_K \in \mathcal{F}_K \widehat{v}_K^h, \widehat{v}_K^h \in \mathbb{P}_K^m \right\}.$$

Then, for any integers $0 \leq \nu \leq j \leq m+1$, it verifies

$$\left(\sum_{K \in \mathcal{T}^h} |v - \mathcal{L}_m^h v|_{\nu,K}^2 \right)^{1/2} \lesssim h^{j-\nu} \|v\|_{H^j(\Omega)}, \quad \forall v \in H^j(\Omega),$$

with

$$\|v\|_{H^j(\Omega)} = \left(\sum_{0 \leq \nu \leq j} |v|_{\nu,\Omega}^2 \right)^{1/2},$$

and therefore

$$(23) \quad \|\mathcal{L}_m^h v - v\|_{\mathbb{X}} \lesssim h^j \|v\|_{H^{j+1}(\Omega)}, \quad \forall v \in H^{j+1}(\Omega), \quad 1 \leq j \leq m.$$

Proof. Based on (21), the construction of the Lagrange interpolation operator \mathcal{L}_m^h uses only functions belonging to $\mathcal{F}_K \mathbb{P}_K^m$. Proposition 5.1 and Corollary 6.1 in [3] then give the proof of the proposition. \square

1.4.2. *Finite element approximation of the dual variable.* We now turn to the finite element approximation of the space \mathbb{M} . In the case of a mesh in straight d -simplices, the approximating functions are taken as the normal traces of vector functions in the Raviart-Thomas FE spaces (cf., e.g., [14, Th. 6.5]). This construction is carried out here by extending the Raviart-Thomas FE method to curved elements.

Let n be a non-negative integer. The FE approximation $\mathbf{M}^{n,h}$ of \mathbf{M} is built from the Raviart-Thomas FE space of column-vectors of length d , defined by $\text{RT}_{\widehat{K}}^n = (\mathbb{P}_{\widehat{K}}^n)^d + \widehat{\mathbf{r}} \mathbb{P}_{\widehat{K}}^n$, where $\widehat{x} \rightarrow \widehat{\mathbf{r}}(\widehat{x})$ is the column-vector function on \widehat{K} whose component i is the i -th Cartesian coordinate \widehat{x}_i of \widehat{x} . It is important to note that there is a difference of 1 in the definition of the order of the Raviart-Thomas FE given in [14, Est. (6.19) and (6.20)] and that adopted here. Keeping the general setting in [3], and extending Dubois' definition [9] of lowest order curved Raviart-Thomas FE, or by adapting the isoparametric definition of higher-order Raviart-Thomas FE spaces for polynomially interpolated boundaries [4], we give the definition of these spaces for any straight or curved d -simplex K

$$(24) \quad \text{RT}_K^n = \mathcal{P}_K \text{RT}_{\widehat{K}}^n.$$

Note that, in two space dimensions, such a definition is also suggested by switching to the reference element [15, Chap. III - Sect. 2]. Bernardi in [3] refers for the properties of these spaces to [12], where only the case of meshes in straight d -simplices is treated. This is why a more detailed study of the properties of these spaces is necessary.

First, the following lemma shows that the definition (24) is invariant by a linear-affine change of variable transforming \widehat{K} into itself. This is called “*independent of the reference element*” in [3, Rem. 2.3].

Lemma 8. *Definition (24) is invariant by a linear-affine variable change keeping \widehat{K} unchanged.*

Proof. We have to check that RT_K^n remains invariant under a linear-affine variable change $\widehat{x} = A x^\# + a$ keeping \widehat{K} unchanged. Let $x^\# \rightarrow \mathbf{q}^\#(x^\#)$ be an element of $\text{RT}_{\widehat{K}}^n$. Define a

parametrization of K by \widehat{K} from the relation $F_K^\#(x^\#) = F_K(Ax^\# + a)$. Since $(F_K^\#)' = F_K' A$ and $\det(F_K^\#)' = \det F_K' \det A$, we get

$$\mathcal{P}_K^\# \mathbf{q}^\# = \frac{1}{\det F_K' \det A} F_K' A \mathbf{q}^\# = \mathcal{P}_K \frac{1}{\det A} A \mathbf{q}^\#.$$

The proof is completed by noting that the transform $\mathbf{q}^\# \rightarrow \frac{1}{\det A} A \mathbf{q}^\#$ is automorphic in the usual $\text{RT}_{\widehat{K}}^n$ space. \square

The following remark will be important later to establish that the above curved Raviart-Thomas finite elements are $H(\text{div}, \Omega)$ -conforming.

Remark 9. *In the two-dimensional case, i.e. for $d = 2$, every edge shared by two triangles, either straight or curved, is a straight edge, and Φ_K is zero on this edge. For $d = 3$, a face shared by two tetrahedrons can be curved. However, the parametrization of this face from its two sides are identical when written in terms of the barycentric coordinates ξ_1, \dots, ξ_d of its vertices (cf. [3, Rem. 6.7]).*

To extend the definition of the usual Raviart-Thomas finite elements to the curved case, we now proceed to define the set Σ_K^n of degrees of freedom attached to them. This set splits into two classes $\Sigma_K^n = \Sigma_{\text{int}, K}^n \cup (\cup_{e \in \mathcal{E}_{\partial K}} \Sigma_{e, K}^n)$. For $\mathbf{q} \in \mathcal{C}^0(\overline{K}; \mathbb{C}^d)$, these classes are respectively given by

$$\begin{aligned} \Sigma_{\text{int}, K}^n &= \left\{ \mu_{\alpha, K} \mathbf{q} = \int_K ((F_K^{-1})' \mathbf{q}) \mathcal{F}_K \widehat{x}^\alpha dx, |\alpha| \leq n-1 \right\}, \\ \Sigma_{e, K}^n &= \left\{ \mu_{\alpha, e, K} \mathbf{q} = \int_e \mathbf{q} \cdot \mathbf{n}_K \mathcal{F}_K \widehat{x}^\alpha ds, |\alpha| \leq n, \right\}. \end{aligned}$$

Note that $\Sigma_{\text{int}, K}^0$ is empty. The degrees of freedom $\cup_{e \in \mathcal{E}_{\partial K}} \Sigma_{e, K}^n$ exactly coincide with the usual ones when K is a straight d -simplex (cf., e.g., [6, Formulas III.3.43]), while the set of degrees of freedom $\Sigma_{\text{int}, K}^n$, defined here, only spans the same vectorial space.

The following proposition collects the most important properties of the triplet $\{K, \text{RT}_K^n, \Sigma_K^n\}$.

Proposition 10. *The following properties hold true.*

- (1) *The triplet $\{K, \text{RT}_K^n, \Sigma_K^n\}$ defines a FE on K .*
- (2) *For each $e \in \mathcal{E}_{\partial K}$, $\{e, \text{RT}_K^n \cdot \mathbf{n}_K|_e, \Sigma_{e, K}^n\}$ is a finite element.*
- (3) *If e is shared by K_1, K_2 in \mathcal{T}^h , then $\text{RT}_{K_1}^n \cdot \mathbf{n}_{K_1}|_e = \text{RT}_{K_2}^n \cdot \mathbf{n}_{K_2}|_e$ and, Σ_{e, K_1}^n and Σ_{e, K_2}^n span the same vectorial space.*

Proof. The first two properties result from the fact that $\{\widehat{K}, \text{RT}_{\widehat{K}}^n, \Sigma_{\widehat{K}}^n\}$ is the usual Raviart-Thomas finite element of order n and the formulas

$$\int_e \mathcal{P}_K \widehat{\mathbf{q}} \cdot \mathbf{n}_K \mathcal{F}_K \widehat{x}^\alpha ds = \int_{\widehat{e}} \widehat{\mathbf{q}} \cdot \mathbf{n}_{\widehat{K}} \widehat{x}^\alpha d\widehat{s}, \quad \int_K (F_K^{-1})' \mathcal{P}_K \widehat{\mathbf{q}} \mathcal{F}_K \widehat{x}^\alpha dx = \int_{\widehat{K}} \widehat{\mathbf{q}} \widehat{x}^\alpha d\widehat{x}.$$

To establish the last property, we first use the invariance established in Lemma (8). We are therefore led to assume that the parametrization of K_1 and K_2 are respectively given by

$$x = F_{K_j}(\widehat{x}_1, \dots, \widehat{x}_d), \quad j = 1, 2$$

where $\widehat{x}_1, \dots, \widehat{x}_d, \widehat{x}_{d+1} = 1 - \widehat{x}_1 - \dots - \widehat{x}_d$ are the barycentric coordinates of the vertices $a_{1, K_j}, \dots, a_{d+1, K_j}$ of \widehat{K}_j , and $a_{1, \widehat{K}_1} = a_{1, \widehat{K}_2} \stackrel{\text{def}}{=} a_1, \dots, a_{d, \widehat{K}_1} = a_{d, \widehat{K}_2} \stackrel{\text{def}}{=} a_d$, being the vertices of the straight face \widehat{e} shared by \widehat{K}_1 and \widehat{K}_2 . In view of the remark 9, we have either from the parametrization of K_1 or that of K_2

$$(25) \quad x = a_d + \widehat{x}_1(a_1 - a_d) + \dots + \widehat{x}_{d-1}(a_{d-1} - a_d) + \Phi(\widehat{x}_1, \dots, \widehat{x}_{d-1}), \quad x \in e.$$

This shows that

$$J_{\partial K_1} \circ F_{K_1}^{-1}|_e = J_{\partial K_2} \circ F_{K_2}^{-1}|_e \stackrel{\text{def}}{=} J_e.$$

Now, let $\mathbf{q}_{K_j} \in \text{RT}_{K_j}^n$, $j = 1, 2$. Hence, $\mathbf{q}_{K_j} = \mathcal{P}_{K_j} \widehat{\mathbf{q}}_j$, with $\widehat{\mathbf{q}}_j \in \text{RT}_{\widehat{K}}^n$, for $j = 1, 2$. Using the last formula in (12), we hence get that $\mathbf{q}_{K_j} \cdot \mathbf{n}_{K_j}|_e \in J_e^{-1} \text{RT}_{\widehat{K}}^n|_{\widehat{x}_{d+1}=0}$, $j = 1, 2$. This proves that $\mathbf{q}_{K_1} \cdot \mathbf{n}_{K_1}|_e$ and $\mathbf{q}_{K_2} \cdot \mathbf{n}_{K_2}|_e$ belong to the same vectorial space. For the second part of the last property, observe that, if the parametrization satisfy (25), we have directly got that $\mu_{\alpha,e,K_1} \mathbf{q} + \mu_{\alpha,e,K_2} \mathbf{q} = 0$. The fact that Σ_{e,K_1}^n and Σ_{e,K_2}^n span the same vectorial space then results from the property that the space $\mathbb{P}_K^n = \mathcal{F}_K \mathbb{P}^n$ is independent of a linear-affine variable change keeping \widehat{K} unchanged, as this can be established directly from the property of the usual polynomial spaces of being invariant under such a variable change. \square

A direct consequence of the previous proposition is the construction of interpolation operators on spaces of vectorial fields a bit smoother than those in $H(\text{div}, \Omega)$.

Corollary 11. *Let us denote by*

$$H_{0,\partial\Omega}(\text{div}, \Omega) = \{\mathbf{q} \in H(\text{div}, \Omega); \mathbf{q} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

and by

$$\mathcal{H}_{0,\partial\Omega}(\text{div}, \Omega) = \{\mathbf{q} \in H_{0,\partial\Omega}(\text{div}, \Omega); \mathbf{q}_K \cdot \mathbf{n}_K \in L^1(\partial K), \forall K \in \mathcal{T}^h\}.$$

An interpolation operator $\Pi^{n,h} : \mathcal{H}_{0,\partial\Omega}(\text{div}, \Omega) \rightarrow \text{RT}_{0,\partial\Omega}^{n,h}(\Omega)$, with

$$\text{RT}_{0,\partial\Omega}^{n,h}(\Omega) = \text{RT}^{n,h}(\Omega) \cap H_{0,\partial\Omega}(\text{div}, \Omega).$$

The space

$$\text{RT}^{n,h}(\Omega) = \left\{ \mathbf{q}^h \in H(\text{div}, \Omega); \mathbf{q}^h|_K \in \text{RT}_K^{n,h}, \forall K \in \mathcal{T}^h \right\},$$

is defined as follows

$$\mathbf{q}_K^h = \Pi_K^n(\mathbf{q}|_K), \forall K \in \mathcal{T}^h.$$

It has the same dimension than the usual similar space constructed on the polygonal/polyhedral domain Ω_h , which is meshed by the straight d -simplices $\widehat{F}_K \widehat{K}$, $K \in \mathcal{T}^h$. Moreover, under the above general assumptions, in particular that \mathcal{T}^h is regular, we have the following error bound

$$(26) \quad \sum_{K \in \mathcal{T}^h} \left(|\mathbf{q} - \Pi_K^n \mathbf{q}|_{0,K} + h_K |\nabla \cdot \mathbf{q} - \nabla \cdot \Pi_K^n \mathbf{q}|_{0,K} \right) \lesssim h^{n+1} \|\mathbf{q}\|_{H^{n+1}(\Omega)}$$

for all $\mathbf{q} \in H^{n+1}(\Omega; \mathbb{C}^d) \cap H_{0,\partial\Omega}(\text{div}, \Omega)$.

Proof. All properties, except the error bound, immediately follow from the third property in Lemma 10, which ensures that the curved Raviart-Thomas finite elements $\{K, \text{RT}_K^n, \Sigma_K^n\}$ are $H(\text{div}, \Omega)$ -conforming [3, Def. 5.2 and Cor. 5.1]. Using the proposition 2, we get

$$\begin{aligned} \left| \mathbf{q} - \Pi_K^{n,h} \mathbf{q} \right|_{0,K} + h_K \left| \nabla \cdot \left(\mathbf{q} - \Pi_K^{n,h} \mathbf{q} \right) \right|_{0,K} &\lesssim \\ &h_K \varrho_K^{-d/2} \left(\left| \widehat{\mathbf{q}} - \Pi_{\widehat{K}}^n \widehat{\mathbf{q}} \right|_{0,\widehat{K}} + \left| \nabla \cdot \left(\widehat{\mathbf{q}} - \Pi_{\widehat{K}}^n \widehat{\mathbf{q}} \right) \right|_{0,\widehat{K}} \right). \end{aligned}$$

Note that the factor h_K in front of the term relative to the divergence compensates for the missed h_K of the bound $|\nabla_x \cdot \mathbf{q}|_{0,K} \leq C_K \varrho_K^{-d/2} |\nabla_{\widehat{x}} \cdot \widehat{\mathbf{q}}|_{0,\widehat{K}}$. It is well-known [15, p. III.28], and easy to establish with Green's formula, that

$$(27) \quad \int_{\widehat{K}} \widehat{x}^\alpha \nabla_{\widehat{x}} \cdot \left(\widehat{\mathbf{q}} - \Pi_{\widehat{K}}^n \widehat{\mathbf{q}} \right) d\widehat{x} = 0, \forall \alpha, |\alpha| \leq n.$$

Bramble-Hilbert lemma then gives that

$$(28) \quad \left| \nabla_{\hat{x}} \cdot \left(\hat{\mathbf{q}} - \Pi_{\hat{K}}^{n,h} \hat{\mathbf{q}} \right) \right|_{0,\hat{K}} \lesssim |\nabla_{\hat{x}} \cdot \hat{\mathbf{q}}|_{n,\hat{K}}.$$

Since $\hat{\mathbf{q}} - \Pi_{\hat{K}}^{n,h} \hat{\mathbf{q}} = 0$ if each component of $\hat{\mathbf{q}}$ is a polynomial of degree $\leq n$, Bramble-Hilbert lemma yields

$$\left| \hat{\mathbf{q}} - \Pi_{\hat{K}}^{n,h} \hat{\mathbf{q}} \right|_{0,\hat{K}} + \left| \nabla_{\hat{x}} \cdot \left(\hat{\mathbf{q}} - \Pi_{\hat{K}}^{n,h} \hat{\mathbf{q}} \right) \right|_{0,\hat{K}} \lesssim |\hat{\mathbf{q}}|_{n+1,\hat{K}} + |\nabla \cdot \hat{\mathbf{q}}|_{n,\hat{K}} \lesssim |\hat{\mathbf{q}}|_{n+1,\hat{K}}.$$

Bound (15) then gives

$$\left| \mathbf{q} - \Pi_K^{n,h} \mathbf{q} \right|_{0,K} + h_K \left| \nabla \cdot \left(\mathbf{q} - \Pi_K^{n,h} \mathbf{q} \right) \right|_{0,K} \lesssim h_K \varrho_K^{-d/2} (h_K / \varrho_K)^{n+2+d/2} h_K^{n+d/2} \|\mathbf{q}\|_{H^{n+1}(K)}.$$

Condition (16) then directly leads to (26). \square

Remark 12. In the above proof, we have not used the full information contained in (27), which would give the term $|\nabla_{\hat{x}} \cdot \hat{\mathbf{q}}|_{n+1,\hat{K}}$ instead of $|\nabla_{\hat{x}} \cdot \hat{\mathbf{q}}|_{n,\hat{K}}$ in the right-hand side of (28). However, for the application aimed here, the regularity of $\nabla_{\hat{x}} \cdot \hat{\mathbf{q}}$ cannot be greater than that induced by $\hat{\mathbf{q}}$. Such a loss has already been encountered in [15, III.(4.40)] in the case of straight simplices.

We can now define the following FE subspace of \mathbf{M}

$$\begin{aligned} \mathbf{M}^{n,h} = \{ & \mu^h = (\mu_K^h)_{K \in \mathcal{T}^h}; \mu_K^h = \mathbf{q}^h|_K \cdot \mathbf{n}_K \text{ on } \partial K, \\ & \mathbf{q}^h|_K \in \mathbf{RT}_K^{n,h}, \forall K \in \mathcal{T}^h, \mathbf{q}^h \in H(\operatorname{div}, \Omega), \mathbf{q}^h \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \}. \end{aligned}$$

The following proposition shows how to construct a related interpolation operator with error bounds.

Proposition 13. Let us denote by $r : H_{0,\partial\Omega}(\operatorname{div}, \Omega) \rightarrow \mathbf{M}$ the surjective map defined by $\mu = r\mathbf{q}$, $\mu_K = \mathbf{q} \cdot \mathbf{n}_K$, $\forall K \in \mathcal{T}^h$, and by $\mathbf{M}^{(n+1)}$ the subspace of \mathbf{M} defined by

$$\mathbf{M}^{(n+1)} = \{ \mu \in \mathbb{M}; \mu = r\mathbf{q}, \mathbf{q} \in H^{n+1}(\Omega, \mathbb{C}^d) \cap H_{0,\partial\Omega}(\operatorname{div}, \Omega) \}.$$

There exists an interpolation operator $\Xi_n^h : \mathbf{M}^{(n+1)} \rightarrow \mathbf{M}^{n,h}$ such that

$$(29) \quad \|\mu - \Xi_n^h \mu\|_{\mathbb{M}} \lesssim h^{n+1} \inf_{r\mathbf{q}=\mu} \|\mathbf{q}\|_{H^{n+1}(\Omega)}.$$

Proof. The proof is immediate by defining $\Xi_n^h \mu$ by

$$(\Xi_n^h \mu)_K = \Pi_K^n \mathbf{q} \cdot \mathbf{n}_K, \forall K \in \mathcal{T}^h,$$

and making use of the bound (26) and the definition of $\|\mu\|_{\mathbb{M}}$ given in (11). \square

Remark 14. An inspection of the proofs, which are mainly based on the Bramble-Hilbert lemma, shows that, similarly to (23), a less sharp estimate is still preserved for a lower order of regularity for \mathbf{q}

$$\|\mu - \Xi_n^h \mu\|_{\mathbb{M}} \lesssim h^{n'+1} \inf_{r\mathbf{q}=\mu} \|\mathbf{q}\|_{H^{n'+1}(\Omega)},$$

when $\mathbf{q} \in \mathbf{M}^{(n'+1)}$ with $0 \leq n' \leq n$. Moreover, if $\mu_K = \partial_{\mathbf{n}_K} w$, $K \in \mathcal{T}^h$ with $w \in H^{n'+2}(\Omega)$, then, setting $m' = n' + 1$, we get the following estimate

$$(30) \quad \|\mu - \Xi_n^h \mu\|_{\mathbb{M}} \lesssim h^{m'} \|w\|_{H^{m'+1}(\Omega)}, \quad 1 \leq m' \leq n+1.$$

Remark 15. *The interpolation errors (29) and (30) can be obtained without appealing to Raviart-Thomas FE spaces for a triangle with one curved edge [7, Appendix B] by adapting a technique used for straight triangles in [13, Lem. 9], conceived in another context in [8]. However, using the Raviart-Thomas FE approach avoids the recourse to the construction of a smooth field approximating the unit normal \mathbf{n}_K on the boundary ∂K of K .*

1.4.3. *Inf-sup condition.* We now turn to the most important property of the above definition of the FE spaces $\mathbb{X}^{m,h}$ and $\mathbf{M}^{n,h}$. The following lemma shows that it is sufficient to establish an algebraic local inf-sup condition at the element level to get a global inf-sup condition. A more detailed discussion and effective conditions ensuring that this property holds true are given in [2].

Lemma 16. *If the local algebraic inf-sup condition*

$$(31) \quad \left\{ \widehat{\mu} \in L^2(\partial\widehat{K}); \widehat{\mu}|_e \in \mathbb{P}^n, \forall e \in \mathcal{E}_{\partial\widehat{K}}, (\widehat{\mu}, \widehat{v})_{\partial\widehat{K}} = 0, \forall \widehat{v} \in X_{\widehat{K}}^m \right\} = \{0\},$$

is satisfied, then the following global inf-sup condition holds true

$$(32) \quad \forall \mu^h \in \mathbf{M}^{n,h}, \quad \sup_{v^h \in \mathbb{X}^{h,m}} \frac{1}{\|v^h\|_{\mathbb{X}}} |b(\mu^h, v^h)| \gtrsim \|v^h\|_{\mathbb{M}}$$

Proof. Since both $\widehat{\mu}^h$ and \widehat{v}^h belong to a finite-dimensional space, the condition (31) implies the following inf-sup condition on the reference element

$$\sup_{\widehat{v}^h \in \mathbb{X}_{\widehat{K}}^m} \frac{1}{\|\widehat{v}^h\|_{1/2, \partial\widehat{K}}} |(\widehat{\mu}^h, \widehat{v}^h)_{\partial\widehat{K}}| \geq \beta \|\widehat{\mu}^h\|_{-1/2, \partial\widehat{K}}$$

where β is a positive constant depending only on \widehat{K} , n and m . In view of their definition, we have also $(\widehat{\mu}, \widehat{v})_{\partial\widehat{K}} = (\mu, v)_{\partial K}$ so that the bounds in the proposition 2 yield the following local inf-sup condition

$$\sup_{v^h \in X_{K^*}^{m,h}} \frac{1}{\|v^h\|_{1/2, h_K^{-1}, \partial K}} (\mu^h, v^h)_{\partial K} \gtrsim \beta \|\mu^h\|_{-1/2, \partial K}.$$

The rest of the proof can be obtained by means of an easy adaptation of the proof of the proposition 4.2 in [15]. \square

The following remark is fundamental in the analysis of the FE approximation of the saddle-point problems related to the above FE approximation of the spaces \mathbb{X} and \mathbf{M} .

Remark 17. *Let*

$$\mathbb{V}^{m,h} = \left\{ v^h \in \mathbb{X}^{m,h}; b(\mu^h, v^h) = 0, \forall \mu^h \in \mathbf{M}^{n,h} \right\}$$

be the so-called “kernel” of the sesquilinear form $b(\mu^h, v^h)$. Each $v^h \in \mathbb{V}^{m,h}$ is in $H^1(\Omega)$ and hence is also in $\mathbb{V}^{m,h}$. In this respect, the interpolate $\mathcal{L}_m^h v$ of any $v \in H^j(\Omega)$, for $j \geq 0$, is in $\mathbb{V}^{m,h}$.

2. FINITE ELEMENT SOLUTION OF THE PRIMAL HYBRID FORMULATION

The analysis of the FE approximation of the primal hybrid formulation can be performed in almost the same way as that corresponding to meshes with straight simplices. The main difference lies in the way an average value of the primal unknown involved in the error estimates is evaluated. Furthermore, The way this analysis is presented here put a clear insight into the involvement of the two norms: $|\cdot|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{X}}$ in stability estimates.

2.1. **The discrete problem.** To shorten the notation, we define

$$B(\{u, \lambda\}, \{v, \mu\}) = a(u, v) + b(\lambda, v) + b(\mu, u).$$

In all what follows, we assume that $\mathbb{X}^{m,h}$ and $\mathbf{M}^{n,h}$ satisfy (21) and (31).

Furthermore, given that the optimal order of the FE approximation in \mathbb{X} and \mathbb{M} are respectively in h^m and h^{n+1} , we also assume that the following compatibility condition

$$(33) \quad m = n + 1.$$

is satisfied.

The discrete problem related to the approximation of the formulation (2) can then be stated as follows

$$(34) \quad \begin{cases} \{u^h, \lambda^h\} \in \mathbb{X}^{m,h} \times \mathbf{M}^{n,h}, \forall \{v^h, \mu^h\} \in \mathbb{X}^{m,h} \times \mathbf{M}^{n,h}, \\ B(\{u^h, \lambda^h\}, \{v^h, \mu^h\}) = (f, v^h)_\Omega + (g, v^h)_{\partial\Omega}. \end{cases}$$

2.2. **Stability estimates.** The following lemma establishes the stability of the FE solution of the primal hybrid formulation (2).

Lemma 18. *Let ℓ be a sesquilinear form on \mathbb{X} satisfying the following bound*

$$|\ell v| \leq \ell_h |v|_{\mathbb{X}}, \quad \forall v \in \mathbb{X}.$$

The saddle-point problem

$$(35) \quad \begin{cases} \{w^h, \zeta^h\} \in \mathbb{X}^{m,h} \times \mathbf{M}^{n,h}, \\ B(\{w^h, \zeta^h\}, \{v^h, \mu^h\}) = \ell v^h, \quad \forall \{v^h, \mu^h\} \in \mathbb{X}^{m,h} \times \mathbf{M}^{n,h}, \end{cases}$$

has one and only solution satisfying

$$(36) \quad |w^h|_{\mathbb{X}} + \|\lambda^h\|_{\mathbb{M}} \lesssim \ell_h.$$

Proof. Variational equation (35) $a(w^h, w^h) = \ell_h w^h$ and the left-bounds (20) yield $|w^h|_{\mathbb{X}}^2 \leq \ell_h |w^h|_{\mathbb{X}}$, i.e., $|w^h|_{\mathbb{X}} \leq \ell_h$. Coming back to the problem (35), and using again (20), we get

$$|b(\zeta^h, v^h)| \leq |\ell v^h| + |a(w^h, v^h)| \lesssim \ell_h |v^h|_{\mathbb{X}} + |w^h|_{\mathbb{X}} \|v^h\|_{\mathbb{X}}.$$

The comparison of norms (10) then yields $|b(\zeta^h, v^h)| \lesssim \ell_h \|v^h\|_{\mathbb{X}}$. The discrete inf-sup condition (32) then gives $\|\zeta^h\|_{\mathbb{M}} \lesssim \ell_h$ completing the proof of the lemma. \square

Remark 19. *The stability estimate (36) ensures in particular that the discrete problem (34) has one and only one solution.*

2.3. **Error estimates.** First, we have the following corollary. In all the sequel, $\{u, \lambda\}$ is the solution of the primal hybrid formulation (2) and $\{u^h, \lambda^h\}$ that of its FE discretization (34).

Corollary 20. *The following bound holds true*

$$(37) \quad |u - u^h|_{\mathbb{X}} + \|\lambda - \lambda^h\|_{\mathbb{M}} \lesssim \|u - z^h\|_{\mathbb{X}} + \|\lambda - \eta^h\|_{\mathbb{M}} + \sup_{|v|_{\mathbb{X}} \leq 1} |b(\lambda - \eta^h, v)|,$$

for all z^h in $\mathbb{V}^{m,h}$ and $\eta^h \in \mathbf{M}^{n,h}$. Moreover, this bound reduces to

$$(38) \quad |u - u^h|_{\mathbb{X}} + \|\lambda - \lambda^h\|_{\mathbb{M}} \lesssim \|u - z^h\|_{\mathbb{X}} + \|\lambda - \eta^h\|_{\mathbb{M}},$$

for all $z^h \in \mathbb{V}^{m,h}$ and all $\eta^h \in \mathbf{M}^{n,h}$ satisfying

$$(39) \quad (\eta_K^h, 1)_{\partial K} = (\lambda_K, 1)_{\partial K}, \quad \forall K \in \mathcal{T}^h.$$

In particular

$$(40) \quad |u - u^h|_{\mathbb{X}} + \|\lambda - \lambda^h\|_{\mathbb{M}} \lesssim \|u - \mathcal{L}_m^h u\|_{\mathbb{X}} + \|\lambda - \Xi_n^h \lambda\|_{\mathbb{M}}.$$

Proof. Let ℓ be defined by

$$\ell v^h = a(u - z^h, v^h) + b(\lambda - \eta^h, v^h), \quad \forall v^h \in \mathbb{X}^{m,h}.$$

Since $\{u^h, \lambda^h\}$ is the solution to (34) and $z^h \in \mathbb{V}^{m,h}$, we can write

$$B(\{u^h - z^h, \lambda^h - \eta^h\}, \{v^h, \mu^h\}) = \ell_h v^h, \quad \forall \{v^h, \mu^h\} \in \mathbb{X}^{m,h} \times \mathbf{M}^{n,h}.$$

Right-bounds (19) then give $|\ell v^h| \lesssim \ell_h |v^h|_{\mathbb{X}}$, $\forall v^h \in \mathbb{X}^{m,h}$, with

$$\ell_h = \|u - z^h\|_{\mathbb{X}} + \sup_{|v^h|_{\mathbb{X}} \leq 1} |b(\lambda - \eta^h, v^h)|.$$

Applying Corollary 18, we obtain

$$|u^h - z^h|_{\mathbb{X}} + \|\lambda^h - \eta^h\|_{\mathbb{M}} \lesssim \ell_h.$$

Estimate (37) is then obtained by the triangle inequality. The second estimate is handled similarly to the case when the mesh \mathcal{T}^h consists only of straight simplices [14, Est. (18.27)] with, as said above, the following adaptation of the average procedure. For $v \in \mathbb{X}$, define $\mathcal{M}_0 v \in \mathbb{X}$ such that

$$(\widehat{\mathcal{M}_0 v})_K = \frac{1}{|\widehat{K}|} \int_{\widehat{K}} \widehat{v}_K \, d\widehat{x}$$

with $v_K = \mathcal{F}_K \widehat{v}_K$ and $(\mathcal{M}_0 v)_K = \mathcal{F}_K (\widehat{\mathcal{M}_0 v})_K$. Note that when K is a straight d -simplex, due to the relation $J_K |\widehat{K}| = |K|$, $(\mathcal{M}_0 v)_K$ is then nothing other than the usual mean value of v_K on K . We have denoted by $|K|$ and $|\widehat{K}|$ the respective area/volume of K and \widehat{K} . Condition (39) then yields $b(\lambda - \eta^h, v) = b(\lambda - \eta^h, v - \mathcal{M}_0 v)$, and hence, using (19)

$$|b(\lambda - \eta^h, v)| \lesssim \|\lambda - \eta^h\|_{\mathbb{M}} \|v - \mathcal{M}_0 v\|_{\mathbb{X}}.$$

Now, in view of (9) and (8), we have

$$\|v_K - (\mathcal{M}_0 v)_K\|_{1, h_K^{-1}, K} \leq |v_K|_{1, K} + h_K^{-1} |v_K - (\mathcal{M}_0 v)_K|_{0, K}$$

and, using once more the proposition 2,

$$h_K^{-1} |v_K - (\mathcal{M}_0 v)_K|_{0, K} \lesssim h_K^{-1} h_K^{d/2} \left| \widehat{v}_K - (\widehat{\mathcal{M}_0 v})_K \right|_{0, \widehat{K}}.$$

Since $(\widehat{\mathcal{M}_0 v})_K$ is the mean-value of \widehat{v}_K on \widehat{K} , Bramble-Hilbert lemma gives

$$h_K^{-1} |v_K - (\mathcal{M}_0 v)_K|_{0, K} \lesssim h_K^{-1} h_K^{d/2} |\widehat{v}_K|_{1, \widehat{K}}.$$

Again using the proposition 2, we get this once

$$h_K^{-1} h_K^{d/2} |\widehat{v}_K|_{1, \widehat{K}} \lesssim (h_K / \varrho_K)^{d/2} |v_K|_{1, K} \leq (h_K / \varrho_K)^{d/2} |v_K|_{1, K} \leq (h_K / \varrho_K)^{d/2} \|v_K\|_{1, K}.$$

The estimate (38) follows from the condition (16) on the mesh \mathcal{T}^h . Finally, estimate (40) is obtained by taking $z^h = \mathcal{L}_m^h u$, since $\mathcal{L}_m^h u \in V^{m,h}$, hence in $\mathbb{V}^{m,h}$, and $\eta^h = \Xi_n^h \lambda$ since then η^h satisfies (39). \square

Remark 21. *The subspace $\mathbb{V}^{m,h}$ plays a key role in the above bound. Taking z^h in $\mathbb{X}^{m,h}$ only would result in an additional term in $b(\mu^h, z^h)$ and the linear form ℓ could not have been properly bounded. We see here how the algebraic decomposition (21) comes into play through the interpolate $\mathcal{L}_m^h u$ by first ensuring that this term is in $\mathbb{V}^{m,h}$ and next by exploiting the approximation properties of this interpolate.*

All the numerical analysis of the above approximation of the primal hybrid formulation is embodied in the following theorem.

Theorem 22. Assume that Ω is a bounded curved polygonal/polyhedral domain of \mathbb{R}^d , piecewise of class $\mathcal{C}^{k+1,1}$. Let \mathcal{T}^h be an exact mesh of Ω in curved d -simplices, furthermore assumed to be regular in the meaning of (17). Let $\mathbb{X}^{m,h} \times \mathbb{M}^{n,h}$ be a FE approximation of the space $\mathbb{X} \times \mathbb{M}$ in the form (21) and satisfying the compatibility condition (22) as well as the algebraic local inf-sup condition (31). We furthermore assume that the condition (33) of optimal polynomial approximation is satisfied. Then, the discrete problem (34) is well-posed and its solution (u^h, λ^h) satisfies the error bound (38). Moreover, if the solution u to the boundary-value problem (1) is in $H^{m+1}(\Omega)$, the following error bound holds true

$$|u - u^h|_{\mathbb{X}} + \|\lambda - \lambda^h\|_{\mathbb{M}} \lesssim h^m \|u\|_{H^{m+1}(\Omega)},$$

where (u, λ) is the solution the primal hybrid formulation (2), and the norms $|\cdot|_{\mathbb{X}}$, $\|\cdot\|_{\mathbb{M}}$, are defined in (9), (11).

Proof. Directly follows from (40) and interpolation error estimates (23) and (29). \square

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UNIV. TOULOUSE, INSA-TOULOUSE, IMT UMR CNRS 5219 135 AVENUE DE RANGUEIL, F-31400
TOULOUSE (FRANCE)
Email address: `abendali@insa-toulouse.fr`