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On the cut-elimination of the modal μ -calculus: Linear Logic to the rescue

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Abstract

This paper presents a proof-theoretic analysis of the modal μ -calculus, well-known in verification theory and relevant to the model-checking problem. More precisely, we prove a syntactic cut-elimination for the non-wellfounded modal μ -calculus, using methods from linear logic and its exponential modalities. To achieve this, we introduce a new system, $\mu\text{LL}_{\square}^{\infty}$, which is a linear version of the modal μ -calculus, intertwining the modalities from the modal μ -calculus with the exponential modalities from linear logic. Our strategy for proving cut-elimination involves (i) proving cut-elimination for $\mu\text{LL}_{\square}^{\infty}$ and (ii) translating proofs of the modal μ -calculus into this new system via a "linear translation," allowing us to extract the cut-elimination result.

2012 ACM Subject Classification Theory of computation \rightarrow Proof theory; Theory of computation \rightarrow Modal and temporal logics; Theory of computation \rightarrow Linear logic; Theory of computation \rightarrow Logic and verification; Theory of computation \rightarrow Type theory

Keywords and phrases proof theory, verification theory, model checking theory, circular proofs, cut elimination, exponential modalities, fixed-points, linear logic, modal μ calculus, non-wellfounded proofs, sequent calculus

1 Introduction

Eliminability of cut and the modal μ -calculus. Since Kozen's seminal work on the *modal μ -calculus* [12], this logic based on modal logic extended with least and greatest fixed-points has been extremely fruitful for the study of computational systems, especially reactive systems, both for their specification and their verification. In addition to its wide expressive power, its deep roots in logic also allow for a number of fruitful approaches, be they model-theoretic, proof-theoretic, automata-theoretic, as well as complexity-theoretic.

Still, *cut-elimination* – a cornerstone of modern proof-theory – only received partial solutions [17, 5, 15, 16, 1], either as cut-admissibility statements (usually obtained as corollary of a completeness theorem with respect to a cut-free proof system – and therefore noneffective) or as syntactic cut-elimination results capturing only a fragment of the calculus. Some of them are admissibility results [17, 1], possibly using non-wellfounded or circular systems (allowing proof-trees with infinitely long branches). Systems with ω -rule (allowing infinitely branching proof-trees) also enjoy cut-admissibility (see [11] for instance), however a problem that arises when trying to describe a *syntactic cut-elimination* is the fact that a choice on the number of times a μ -rule must be made sometimes before knowing how many times it should be to fit each hypotheses of a ν -rule. In [5], the authors discuss a specific example where syntactic cut-elimination fails. Syntactic results of cut-elimination can still be found in ω -rule systems [16, 5, 15], however these systems are strict fragments of the modal μ -calculus. In fact, there is no syntactic cut-elimination theorem for the modal μ -calculus.

The present work establishes syntactic cut-elimination theorems for the modal μ -calculus by providing a proof-theoretic analysis grounded on linear logic stemming from the striking remark that logic presents at the same time a deep unity and a wide diversity.

On unity and diversity in computational logic. While the unity of logic is embodied by

44 its fundamental objects and questions¹, the vast diversity of logics and the broad nature of
 45 reasoning are a source of multiplicity of formalisms which often turn out to be incompatible.
 46 This leads, for instance, Miller [14], to argue that “*it is far more accurate to say that*
 47 *its universal character has been badly fractured in the past few decades*”, due to the wide
 48 range of its applications, the various families of logics that have emerged and the different
 49 computational tools that are in use often with little relationship. Miller’s analysis of this
 50 *fracture* goes to the point of proposing the following questions as the first of a list of
 51 “challenges”:

52 **Challenge 1:** *Unify a wide range of logical features into a single framework. How best can we*
 53 *explain the many enhancements that have been designed for logic: for example, classical /*
 54 *intuitionistic / linear, fixed points, first-order / higher-order quantification, modalities,*
 55 *and temporal operators? (...)*

56 In the present paper, we partially address Miller’s first challenge, providing a common
 57 framework for two of the main logics that emerged in the 1980s, Kozen’s modal μ -calculus [12]
 58 and Girard’s linear logic [9]. Working in the setting of circular and non-wellfounded proof
 59 systems for the above logics, we propose a so-called *linear decomposition* of the modal μ -
 60 calculus in linear logic with fixed-points. This proof-theoretic analysis of the modal μ -calculus
 61 allows us a finer-grained treatment of syntactic cut-elimination.

62 **Cut-admissibility vs cut-elimination.** The treatment of the cut-inference in sequent-
 63 based proof-systems follows two main traditions: (i) one can consider cut-free proofs as
 64 the primitive proof-objects, establishing that the cut-inference is admissible (according to
 65 that tradition, the cut-inference essentially lives at the meta level, ensuring compositionality
 66 of the logic) or (ii) one can consider that the cut inference lives at the object-level and
 67 is a fundamental piece of proofs, establishing that it is *eliminable* thus ensuring the sub-
 68 formula property (and its numerous important consequences, ranging from consistency to
 69 interpolation properties). This second tradition often comes with the investigation of a
 70 syntactic, or effective, approach to cut-elimination, consisting in a cut-reduction relation on
 71 proofs, shown to be (at least) weakly normalizing, the normal forms being cut-free proofs. In
 72 several settings (most notably LJ and LL [9]), such cut-reductions may have a computational
 73 interpretation that is the starting point of Curry-Howard correspondence built upon sequent
 74 calculus [6].

75 **Linear Logic.** Linear logic (LL) is often described as a resource-sensitive logic. It is more
 76 accurate, though, to view it as a logic designed for analyzing cut-elimination itself. Indeed, LL
 77 comes from an analysis of structural rules, aiming at controlling them rather than weakening
 78 them as in substructural logics. This solves some fundamental drawbacks of cut-elimination
 79 in classical logic, such as its non-termination or non-confluence. For instance, LL permits
 80 the decomposition of both intuitionistic and classical logic, in a structured and fine-grained
 81 manner allowing the refinement of the cut-elimination of those logics as well as their notion of
 82 model (allowing the building of a non-trivial denotational model of proofs for classical logic);
 83 the prototypical example of such a linear decomposition consists in decomposing the usual
 84 intuitionistic arrow (that is the function type of the λ -calculus), $A \Rightarrow B$, into a replication
 85 operator and a linear implication, $!A \multimap B$ [9, 7]. Further analyzes on these exponential
 86 modalities led to the finding of alternative presentations offering the possibility to tame their

¹ E.g. how to design logical languages and use logical consequence from a model- or a proof-theoretic perspective, what are the invariants emerging from models or proofs and how to provide algorithms and software to mechanize these studies and with what intrinsic expressiveness and complexity?

87 complexity in a flexible way, introducing *light logics*, but considering alternative – generally
 88 weaker – exponential modalities. The proof theory of LL was extended to LL extended with
 89 fixed-points in the finitary and non-wellfounded setting [2, 8, 20] and μLL^∞ allowed for the
 90 same kind of linear decomposition for (the non-wellfounded version of) μLJ and μLK . A
 91 natural question is therefore whether LL and its extensions with fixed-points can help us
 92 achieving syntactic cut-elimination for the modal μ -calculus as well.

93 **Contributions.** The discussion of the above paragraph suggests a first question: what
 94 would be a linear decomposition of the modal μ -calculus? The first contribution of this paper
 95 is to provide such a linear decomposition of the modal μ -calculus which is compatible with
 96 circular and non-wellfounded proof theory, $\mu\text{LL}_\square^\infty$. This linear-logical modal μ -calculus will
 97 allow us to complete the analysis of cut-elimination for the modal μ -calculus.

98 We therefore adopt the following roadmap in the body of the paper: in Section 2, we recall
 99 the necessary technical background about μLL^∞ and $\mu\text{LK}_\square^\infty$ (with list and sequence-based
 100 sequents). In Section Section 3, we motivate and introduce $\mu\text{LL}_\square^\infty$, the linear calculus in
 101 which one can decompose the modal μ -calculus. We then prove cut-elimination for $\mu\text{LL}_\square^\infty$ in
 102 Section 4 before defining the linear decomposition of $\mu\text{LK}_\square^\infty$ and concluding its cut-elimination
 103 theorem in Section 5 in the form of an infinitary weak-normalizing cut-reduction system.

104 2 Sequent calculi for (non-)wellfounded & circular proof systems

105 2.1 The Modal μ -calculus

Formulas. First we define the pre-formulas of the modal μ -calculus, $\mu\text{LK}_\square^\infty$:

$$F, G ::= a \in \mathcal{A} \mid X \in \mathcal{V} \mid \mu X.F \mid \nu X.F \mid \Box F \mid \Diamond F \mid F^\perp \mid F \rightarrow G \mid F \vee G \mid F \wedge G \mid \mathbf{F} \mid \mathbf{T}.$$

106 Knaster-Tarski's theorem guarantees the existence of extremal fixed-points for monotonic
 107 functions on complete lattices; monotonicity is reflected syntactically as a positivity condition
 108 on fixed-point variables. We therefore consider only pre-formulas satisfying this condition:

109 **► Definition 1** (Positive and negative occurrence of a fixed-point variable). *Let $X \in \mathcal{V}$ be a*
 110 *fixed-point variable, one defines the fact, for X , to occur positively (resp. negatively) in a*
 111 *pre-formula by induction on the structure of pre-formulas:*

- 112 **■** X occurs positively in X .
- 113 **■** X occurs positively (resp. negatively) in $c(F_1, \dots, F_n)$, if there is some $1 \leq i \leq n$ such
 114 that X occurs positively (resp. negatively) in F_i for $c \in \{\Box, \Diamond, \vee, \wedge\}$.
- 115 **■** X occurs positively (resp. negatively) in F^\perp if it occurs negatively (resp. positively) in F .
- 116 **■** X occurs positively (resp. negatively) in $F \rightarrow G$ if X occurs either positively (resp.
 117 negatively) in G or negatively (resp. positively) in F .
- 118 **■** X occurs positively (resp. negatively) in $\delta Y.G$ (with $Y \neq X$) if it occurs positively (resp.
 119 negatively) in G (for $\delta \in \{\mu, \nu\}$).

120 **► Definition 2** (Formulas). *A $\mu\text{LK}_\square^\infty$ formula F is a closed pre-formula such that for any*
 121 *sub-pre-formula of F of the form $\delta X.G$ (with $\delta \in \{\mu, \nu\}$), X does not occur negatively in G .*

122 By considering the μ, ν, X -free formulas of this system, we get a fixed-point-free version
 123 of the modal μ -calculus: LK_\square . By considering the \Box, \Diamond -free formulas of $\mu\text{LK}_\square^\infty$, we get the
 124 μ -calculus. Finally, the intersection of these two systems, is the propositional classical logic.

125 **Sequent calculus.** We now define the sequents, inference rules and proofs for $\mu\text{LK}_\square^\infty$.

■ **Figure 1** Rules of LK_{\Box}

■ **Figure 2** Rules for the fixed-point fragment

126 ► **Definition 3** (Sequent). A sequent is a pair of two lists of formulas Γ, Δ , that we usually
 127 write $\Gamma \vdash \Delta$. We call Γ the antecedent of the sequent and Δ the succedent of it. We also
 128 refer to the formulas of Γ (resp. Δ) as the hypotheses (resp. conclusions) of the sequent.

129 ► **Remark 1** (Derivation rules & ancestor relation). In the structural proof theory literature,
 130 inference rules are usually given together with an ancestor relation (or sub-occurrence relation)
 131 between formulas of the conclusion and formulas of the hypotheses. While this relation is
 132 often overlooked we provide some details here. Sequent being lists, we define the ancestor
 133 relation, to be a relation from the positions of the formula in the conclusion, to the positions
 134 of the formula in the hypotheses.

135 Those ancestor relations will be dealt graphically, by drawing the ancestor relation on
 136 sequents when needed and leaving it implicit when unambiguous.

137 We define inference rules for LK_{\Box} in figure 1. Rules for LK will be the \Box, \Diamond -free rules of
 138 LK_{\Box} . We add rules of figure 2 to LK , (resp. LK_{\Box}) to get the fixed-point version μLK^{∞} (resp.
 139 μLK_{\Box}^{∞}) of this system. The exchange rule (ex) from figures 1 and 3 allows one to derive
 140 the rule $\frac{\vdash \sigma(\Gamma)}{\vdash \Gamma} \text{ex}(\sigma)$ for any permutation σ of $\llbracket 1, \#(\Gamma) \rrbracket$, where $\sigma(\Gamma)$ designates the
 141 action of σ on the list Γ , with the induced ancestor relation. In the rest of the article, we
 142 will intentionally treat the exchange rule implicitly: the reader can consider that each of our
 143 rules are preceded and followed by a finite number of rule (ex).

144 Proofs of non fixed-point systems, LK, LK_{\Box} are the trees inductively generated by the
 145 corresponding set of rules of each of these systems. We can define a first notion of infinite
 146 derivations, pre-proofs, that will soon be refined:

147 ► **Definition 4** (Pre-proofs). Given a set of derivation rules, we define pre-proofs to be the
 148 trees co-inductively generated by rules of each of these systems.

149 ► **Example 1** (Regular pre-proof). Regular pre-proofs are those pre-proofs having a finite
 150 number of sub-proofs. We represent them with back-edges. Taking $F := \nu X. \Diamond X$, we give an

151 *example of regular proof:*

$$\frac{\frac{F \vdash F}{\diamond F \vdash \diamond F} \square_p \quad \frac{F \vdash F}{\diamond F \vdash \diamond F} \square_p}{F \vdash F} \nu_l, \nu_r \quad \text{cut}$$

▶ **Remark 2.** *The pre-proofs define an inconsistent system. In fact, any sequent is provable:*

$$\frac{\frac{\Gamma \vdash \nu X.X}{\Gamma \vdash \nu X.X} \nu_r \quad \frac{\nu X.X \vdash \Delta}{\nu X.X \vdash \Delta} \nu_l}{\Gamma \vdash \Delta} \text{cut}$$

152

153 To define a sound non-wellfounded proof system, we need a last definition:

154 ▶ **Definition 5** (Active & Principal occurrence of a rule). *We define active occurrences (resp. principal formula) of the rules of figures 1, 2, 3 and 4 to be the first occurrence (resp. formula) of each conclusion sequent of that rule except for:*

- 157 ■ (ex) which does not contain any active occurrences nor principal formulas;
- 158 ■ (cut) which does not contain any active occurrences but has F as principal formula;
- 159 ■ the modal rule (\square) where all the occurrences are active and $\square F$ is the principal formula.

160 From that, we define the proofs as a subset of the pre-proofs:

161 ▶ **Definition 6** (Validity and proofs). *Let $b = (s_i)_{i \in \omega}$ be a sequence of sequents defining an infinite branch in a pre-proof π . A thread of b is a sequence $(F_i \in s_i)_{i > n}$ of occurrences such that for each j , F_j and F_{j+1} are satisfying the ancestor relation. We say that a thread of b is valid if the minimal recurring formula of this sequence, for sub-formula ordering, exists and is (i) either a ν -formula and appearing on the succedent of its sequent or is a μ -formula and appearing in the antecedent of its sequent and (ii) the thread is infinitely often active (there are an infinite number of active formulas in it). A branch b is valid if there is a valid thread of b . A pre-proof is valid and is a proof if each of its infinite branches are valid.*

169 ▶ **Remark 3.** *The least (μ) and greatest (ν) fixed-point constructors have the same derivation rules: they will be distinguished thanks to the validity condition which is a parity condition akin to parity games for the μ -calculus.*

172 ▶ **Example 1.** *The pre-proofs of Example 1 and Remark 2 are respectively valid and invalid.*

▶ **Example 2** (Valid and not valid pre-proofs). *Here, we give some examples of infinite proof, we use a notation $\text{Nat} := \mu X.1 \vee X$, representing the type of natural numbers. We can represent any natural number n by a finite proof π_n :*

$$\pi_0 := \frac{\frac{\overline{1}}{\vdash 1} \mu_r^1}{\vdash \text{Nat}} \nu_r^1 \quad \pi_{n+1} := \frac{\pi_n}{\vdash \text{Nat}} \mu_r, \nu_r^2$$

173 *There also exists an infinite pre-proof on $\vdash \text{Nat}$ which is not valid:* $\frac{\vdash \text{Nat}}{\vdash \text{Nat}} \mu_r, \nu_r^2$. *The infinite branch here is supported by one only thread which is not valid as the minimal formula is a μ -formula appearing on the right of the proof. Which is coherent with the interpretation that μ is a least fixed-point, we want to reject an infinite natural number. Note that the same kind of proof with $\text{Nat} := \nu X.1 \vee X$ would have given a valid proof. The following pre-proof*

178 *(representing the double function) is valid:* $\frac{\frac{\frac{\overline{1 \vdash 1}}{\vdash 1 \vee \text{Nat}} \mu_r^1 \quad \frac{\text{Nat} \vdash \text{Nat}}{\text{Nat} \vdash \text{Nat}} \mu_r, \nu_r^2}{\vdash \text{Nat}} \nu_r^2}{\vdash \text{Nat}} \mu_l$ *The*

$$\begin{array}{c}
\frac{}{F \vdash F} \text{ax} \quad \frac{\Gamma_1 \vdash F, \Delta_1 \quad \Gamma_2 \vdash F^\perp, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{cut} \quad \frac{\Gamma \vdash \Delta_1, G, F, \Delta_2}{\Gamma \vdash \Delta_1, F, G, \Delta_2} \text{ex}_r \quad \frac{\Gamma_1, G, F, \Gamma_2 \vdash \Delta}{\Gamma_1, F, G, \Gamma_2 \vdash \Delta} \text{ex}_l \\
\frac{\Gamma \vdash F, G, \Delta}{\Gamma \vdash F \wp G, \Delta} \wp_r \quad \frac{\Gamma_1, F \vdash \Delta_1 \quad \Gamma_2, G \vdash \Delta_2}{\Gamma_1, \Gamma_2, F \wp G \vdash \Delta_1, \Delta_2} \wp_l \quad \frac{\Gamma, F, G \vdash \Delta}{\Gamma, F \otimes G \vdash \Delta} \otimes_l \quad \frac{\Gamma_1 \vdash F, \Delta_1 \quad \Gamma_2 \vdash G, \Delta_2}{\Gamma_1, \Gamma_2 \vdash F \otimes G, \Delta_1, \Delta_2} \otimes_r \\
\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta} \multimap_r \quad \frac{\Gamma_1, B \vdash \Delta_1 \quad \Gamma_2 \vdash A, \Delta_2}{\Gamma_1, \Gamma_2, A \multimap B \vdash \Delta_1, \Delta_2} \multimap_l \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^\perp, \Delta} (-)_r^\perp \quad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^\perp \vdash \Delta} (-)_l^\perp \\
\frac{\Gamma \vdash F_1, \Delta}{\Gamma \vdash F_1 \oplus F_2, \Delta} \oplus_r^1 \quad \frac{\Gamma \vdash F_2, \Delta}{\Gamma \vdash F_1 \oplus F_2, \Delta} \oplus_r^2 \quad \frac{\Gamma, F_1 \vdash \Delta \quad \Gamma, F_2 \vdash \Delta}{\Gamma, F_1 \oplus F_2 \vdash \Delta} \oplus_l \\
\frac{\Gamma, F_1 \vdash \Delta}{\Gamma, F_1 \& F_2 \vdash \Delta} \&_l^1 \quad \frac{\Gamma, F_2 \vdash \Delta}{\Gamma, F_1 \& F_2 \vdash \Delta} \&_l^2 \quad \frac{\Gamma \vdash F_1, \Delta \quad \Gamma \vdash F_2, \Delta}{\Gamma \vdash F_1 \& F_2, \Delta} \&_r \\
\frac{}{\vdash 1} 1_r \quad \frac{\Gamma \vdash \Delta}{\Gamma, 1 \vdash \Delta} 1_l \quad \frac{}{\perp \vdash} \perp_l \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \perp, \Delta} \perp_r \quad \frac{}{\Gamma \vdash \top, \Delta} \top \quad \frac{}{\Gamma, 0 \vdash \Delta} 0
\end{array}$$

■ **Figure 3** Rules of multiplicative and additive linear logic

179 *only infinite branch is the branch going infinitely on the right at the application of (\vee_l) -rule.*
180 *This branch is supported by the infinite thread in the antecedent of each sequents which has a*
181 *μ -formula as its minimal formula.*

182 2.2 Linear Logic

183 The main difference between LK (or LJ) and LL lies in the fact that formulas are not always
184 erasable nor duplicable. Hence, the sequent $A, B \vdash A$ is not always provable, neither is
185 $A \vdash A \otimes A$ (a sequent similar to $A \vdash A \wedge A$ in LK). This restriction allows LL to interpret
186 programs with finer resource control than LK (or LJ). Here we recall the usual definitions of
187 both the wellfounded and non-wellfounded systems of LL, following the definitions of the
188 previous section. (We are less colloquial: complete definitions can be found in appendices.)

Formulas. The pre-formulas of the non-wellfounded linear logic, μLL^∞ are:

$$\begin{aligned}
F, G ::= & a \in \mathcal{A} \mid X \in \mathcal{V} \mid \mu X.F \mid \nu X.F \mid F^\perp \mid F \multimap G \mid \\
& F \wp G \mid F \otimes G \mid \perp \mid 1 \mid F \oplus G \mid F \& G \mid 0 \mid \top \mid ?F \mid !F.
\end{aligned}$$

189 Positivity of those pre-formulas are defined the same way as for $\mu\text{LK}_\square^\infty$ formulas, identifying
190 \multimap to \rightarrow (see details in appendix A.1). Formulas are defined similarly to $\mu\text{LK}_\square^\infty$.

191 The $!, ?$ -free formulas of μLL^∞ are the formulas of μMALL^∞ , the multiplicative and
192 additive fragment of non-wellfounded linear logic. The μ, ν, X -free fragment of formulas of
193 μLL^∞ are the formulas of linear logic LL. The $!, ?$ -free formulas of μLL^∞ are the formulas
194 of μMALL^∞ , the multiplicative and additive fragment of non-wellfounded linear logic. The
195 μ, ν, X -free fragment of formulas of μLL^∞ are the formulas of linear logic LL. The intersection
196 of these two fragment is MALL, the multiplicative and additive fragment of linear logic.

197 **Sequent calculus.** The definition of sequent is the same as for $\mu\text{LK}_\square^\infty$. The rules of MALL
198 are given by figure 3, the rules of LL are the rules of MALL together with the rules of figure 4.
199 We add rules of figure 2 to MALL (resp. LL) obtain μMALL^∞ (resp. μLL^∞). Pre-proofs,
200 active $\&$ principal occurrence as well as validity are defined as in the previous section.

201 In linear logic, the property of being duplicable or erasable can be obtained via $?$ (read
202 *why not*) and $!$ (read *of course*) modalities. If a hypothesis is preceded by an $!$ modality, one

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash ?F, \Delta} ?_w \quad \frac{\Gamma \vdash \Delta}{\Gamma, !F \vdash \Delta} !_w \quad \frac{\Gamma \vdash ?F, ?F, \Delta}{\Gamma \vdash ?F, \Delta} ?_c \quad \frac{\Gamma, !F, !F \vdash \Delta}{\Gamma, ?F \vdash \Delta} !_c \quad \frac{\Gamma \vdash F, \Delta}{\Gamma \vdash ?F, \Delta} ?_d \quad \frac{\Gamma, F \vdash \Delta}{\Gamma, !F \vdash \Delta} !_d \quad \frac{!F \vdash F, ?\Delta}{!F \vdash !F, ?\Delta} !_p \quad \frac{!F, F \vdash ?\Delta}{!F, ?F \vdash ?\Delta} ?_p$$

■ **Figure 4** Exponential fragment of LL

203 can duplicate or erase it via the $(!_c)$ (*contraction*) and the $(!_w)$ (*weakening*) rules, respectively.
 204 Additionally, the $(!_d)$ (*dereliction*) rule allows to use the hypothesis inside the $!$. Finally, the
 205 $(!_p)$ (*promotion*) rule is the only right-rule on the $!$ modality and allows one to use a conclusion
 206 formula preceded by an $!$ if and only if each hypothesis in the sequent is preceded by an $!$.
 207 Therefore, a formula $!A$ can be produced if each hypotheses producing it are duplicable
 208 and erasable. Therefore, in linear logic, contractions and weakenings are possible but in a
 209 more controlled way, especially when it comes to cut-reduction sequences: having modalities
 210 for contractions and weakening sequentializes certain reductions. This gives the property
 211 of strong normalization to LL [19], which can not be obtained from LK (see section 5.4.4
 212 in [18]). However, the good normalization properties of LL can be recovered by using a linear
 213 translation from LK to LL, similar to the double negation translations from LK to LJ. Indeed,
 214 every formula, every sequent and every proof of LK can be translated into a proof in LL by
 215 adding $?$ and $!$ modalities, for instance:

► **Example 3** (Linear translation example). *We translate each connectives $c(A_1, \dots, A_n)$ by $!(c(?A_1, \dots, ?A_n))$ and add a $?$ on the succedents (the additional rules are shown in red):*

$$\frac{\frac{\frac{A \vdash A}{\vdash A^\perp, A} ax \quad (-)_r^\perp \quad \frac{A \vdash A}{A \vdash A} ax}{A^\perp \rightarrow A \vdash A, A} c}{A^\perp \rightarrow A \vdash A} \rightarrow_l \rightsquigarrow \frac{\frac{\frac{A \vdash A}{?A \vdash ?A} ax \quad ?_p, ?_d \quad \frac{\vdash (?A)^\perp, ?A}{\vdash ?!(?A)^\perp, ?A} (-)_r^\perp}{\vdash ?!(?A)^\perp, ?A} ?_d, !_p \quad \frac{A \vdash A}{?A \vdash ?A} ax \quad ?_p, ?_d}{?!(?A)^\perp \multimap ?A \vdash ?A, ?A} \multimap_l}{\frac{?!(?A)^\perp \multimap ?A \vdash ?A, ?A}{!(?!(?A)^\perp \multimap ?A) \vdash ?A, ?A} !_d}{!(?!(?A)^\perp \multimap ?A) \vdash ?A} ?_c} \rightarrow_l$$

216 By taking any maximal sequence of cut reduction on such proofs in LL and using the strong
 217 normalization property, we find a cut-free proof of the same sequent in LL. This projects to
 218 an LK proof of the original sequent, by simply *forgetting* superfluous modalities. In Section 5,
 219 we will use this technique to prove the cut elimination of the modal μ -calculus.

3 A linear-logical modal mu-calculus

221 In the present section, we introduce an extension of μLL^∞ with modalities akin to the modal
 222 mu-calculus. The term *logical* emphasizes the fact that the logic is linear in the use of
 223 resources, not in the structures of its models as in LTL or linear-time μ -calculus [21].

224 To motivate a linear-logical modal μ -calculus, we need to understand what problem will
 225 be encountered by the translation of $\mu\text{LK}_\square^\infty$ into a linear logic system extended with \square and \diamond
 226 modalities (the aim of this linear-logical modal μ -calculus is ultimately to refine $\mu\text{LK}_\square^\infty$). Let
 227 us add them in the system as well as the modal rule, and let us extend the $(-)^{\bullet}$ translation
 228 on modalities the same way as in Example 3: $(\diamond A)^{\bullet} := !\diamond ?A^{\bullet}$ $(\square A)^{\bullet} := !\square ?A^{\bullet}$.

229 Let us consider the modal rule \square_p , $\frac{\Gamma \vdash A, \Delta}{\square \Gamma \vdash \square A, \diamond \Delta} \square_p$, with an instance of the rule

230 where $\Gamma = []$ and $\Delta = [B]$: $\frac{\vdash A, B}{\vdash \square A, \diamond B} \square_p$. Following the usual LK to LL sequent translation,

$$\begin{array}{c}
\frac{\Gamma \vdash A, \Delta}{\Box \Gamma \vdash \Box A, \Diamond \Delta} \Box_p, \quad \frac{\Gamma \vdash \Diamond A, \Diamond A, \Delta}{\Gamma \vdash \Diamond A, \Delta} \Diamond_c, \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Diamond A, \Delta} \Diamond_w, \quad \frac{\Box \Gamma', !\Gamma \vdash A, ?\Delta, \Diamond \Delta'}{\Box \Gamma', !\Gamma \vdash !A, ?\Delta, \Diamond \Delta'} !_p^\Diamond \\
\frac{\Gamma \vdash A, \Delta}{\Box \Gamma, \Diamond A \vdash \Diamond \Delta} \Diamond_p, \quad \frac{\Gamma, \Box A, \Box A \vdash \Delta}{\Gamma, \Box A \vdash \Delta} \Box_c, \quad \frac{\Gamma \vdash \Delta}{\Gamma, \Box A \vdash \Delta} \Diamond_w, \quad \frac{\Box \Gamma', !\Gamma, A \vdash ?\Delta, \Diamond \Delta'}{\Box \Gamma', !\Gamma, ?A \vdash ?\Delta, \Diamond \Delta'} ?_p^\Box
\end{array}$$

■ **Figure 5** Rules involving modalities for μLL_\Box^∞

231 we should start with the sequent (from bottom to top) $\vdash ?!\Box?A^\bullet, ?!\Diamond?B^\bullet$ and end up with
232 $\vdash ?A^\bullet, ?B^\bullet$.

233 To get to $\vdash ?A^\bullet, ?B^\bullet$, we will have to remove both the \Diamond and the \Box with a modal rule.
234 However, whatever sequence of rule that we use, we always get to a sequent containing an !

235 and a modality \Box or \Diamond . Here is an example of such a derivation: $\frac{\frac{\frac{\vdash ?!\Box?A^\bullet, \Diamond?B^\bullet}{\vdash ?!\Box?A^\bullet, \Diamond?B^\bullet} ?_d}{\vdash ?!\Box?A^\bullet, !\Diamond?B^\bullet} !_p}{\vdash ?!\Box?A^\bullet, ?!\Diamond?B^\bullet} ?_d$. We

236 are stuck at this point: the top sequent is conclusion of no rule of the system (except for the
237 cut and exchange of course).

238 In our attempt to translate this rule we are left with an unprovable sequent where a
239 !-formula is in a context with a \Diamond -formula, not ?-formulas. It would therefore be convenient
240 to have promotion with right contexts possibly prefixed with \Diamond , dually with left contexts

241 prefixed with \Box : $\frac{\Box \Gamma', !\Gamma \vdash A, ?\Delta, \Diamond \Delta'}{\Box \Gamma', !\Gamma \vdash !A, ?\Delta, \Diamond \Delta'} !_p^\Diamond$. Considering such a promotion rule allows us to

242 finish the derivation of our (\Box_p) instance: $\frac{\frac{\frac{\vdash ?A^\bullet, ?B^\bullet}{\vdash \Box?A^\bullet, \Diamond?B^\bullet} \Box_p}{\vdash !\Box?A^\bullet, \Diamond?B^\bullet} !_p}{\vdash ?!\Box?A^\bullet, \Diamond?B^\bullet} ?_d$. Note that this translation
 $\frac{\frac{\frac{\vdash ?!\Box?A^\bullet, !\Diamond?B^\bullet}{\vdash ?!\Box?A^\bullet, ?!\Diamond?B^\bullet} !_p}{\vdash ?!\Box?A^\bullet, ?!\Diamond?B^\bullet} ?_d$

243 does not work when there are two or more \Box -formulas at the left of the sequent. We will
244 give the correct translation for a non-empty antecedent later.

245 Allowing \Diamond -formulas in the succedent and \Box -formulas in the antecedent of the sequent of
246 a promotion has implications for the system's robustness to cut-elimination.

For instance, taking the $(?_p/?_w)$ principal case and adding modal-formulas to the context,
naturally requires to be able to weaken \Diamond/\Box -formulas (this corresponds to the reason for the
design of the promotion rule in LL):

$$\frac{\frac{\Gamma_1 \vdash \Delta_1}{\Gamma_1 \vdash ?C, \Delta_1} ?_w \quad \frac{!\Gamma_2, \Box \Gamma_3, C \vdash ?\Delta_2, \Diamond \Delta_3}{!\Gamma_2, \Box \Gamma_3, ?C \vdash ?\Delta_2, \Diamond \Delta_3} ?_p}{\Gamma_1, !\Gamma_2, \Box \Gamma_3 \vdash \Delta_2, ?\Delta_2, \Diamond \Delta_3} \text{cut} \rightsquigarrow \frac{\Gamma_1 \vdash \Delta_1}{\Gamma_1, !\Gamma_2, \Box \Gamma_3 \vdash \Delta_2, ?\Delta_2, \Diamond \Delta_3} ?_w, !_w, \Box_w, \Diamond_w$$

247 Similarly, the $(?_c/?_p)$ key-case naturally asks to be able to contract \Diamond -formulas. More details
248 can be found in appendix B.1.

249 We now give a formal definition of the linear-logical modal μ -calculus: μLL_\Box^∞ . Pre-formulas
250 of μLL_\Box^∞ are defined as: $F, G ::= a \in \mathcal{A} \mid X \in \mathcal{V} \mid \mu X.F \mid \nu X.F \mid F^\perp \mid F \multimap G \mid F \wp G$
251 $\mid F \otimes G \mid \perp \mid 1 \mid F \oplus G \mid F \& G \mid 0 \mid \top \mid ?F \mid !F \mid \Diamond F \mid \Box F$.

252 We get positivity of an occurrence and the definition of formulas of μLL_\Box^∞ in the same
253 way as for μLL^∞ and μLK_\Box^∞ . Rules for μLL_\Box^∞ are the rules for μLL^∞ together with rules
254 depicted in figure 5. Note that $(!_p)$ (resp. $(?_p)$) is a special case of $(!_p^\Diamond)$ (resp. $(?_p^\Box)$). From
255 these rules, we define the sequents, pre-proofs and proofs similarly to systems of Section 2.

4 Cut-elimination for $\mu\text{LL}_{\square}^{\infty}$

In order to eliminate cuts for $\mu\text{LL}_{\square}^{\infty}$ we shall use a generalization of the cut inference, multicuts, as done in previous works on similar non-wellfounded proof systems [8, 4, 3]. The multicut is extensively defined in appendix C.1.

► **Definition 7** (Multicut rule). *The multicut rule is a rule with an arbitrary number of hypotheses:*

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \dots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta} \text{mcut}(\iota, \perp\!\!\!\perp)$$

The ancestor relation ι sends one formula of the conclusion to exactly one formula of the hypotheses; whereas the $\perp\!\!\!\perp$ -relation links cut-formulas together.

► **Remark 4.** *The idea of the multicut is to abstract a finite tree of binary cuts quotiented by cut-commutation rule. We give an example of a multicut rule and represent graphically ι in red and $\perp\!\!\!\perp$ in blue.*

$$\frac{\vdash A, B \quad B \vdash C \quad C \vdash D}{\vdash A, D} \text{mcut}(\iota, \perp\!\!\!\perp)$$

We can understand the multicut rule as a tree of binary cuts through the (cut/mcut)-principal case:

$$\frac{\mathcal{C} \quad \frac{\Gamma_1 \vdash F, \Delta_1 \quad \Gamma_2, F \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{cut}}{\Gamma \vdash \Delta} \text{mcut}(\iota, \perp\!\!\!\perp) \quad \rightsquigarrow \quad \frac{\mathcal{C} \quad \Gamma_1 \vdash F, \Delta_1 \quad \Gamma_2, F \vdash \Delta_2}{\Gamma \vdash \Delta} \text{mcut}(\iota', \perp\!\!\!\perp')$$

Here, ι' sends on \mathcal{C} formulas that were sent on \mathcal{C} by ι , either it uses the ancestor relation of the cut-rule that has been merged. The relation $\perp\!\!\!\perp'$ is obtained from $\perp\!\!\!\perp$ by adding $F \perp\!\!\!\perp' F^{\perp}$.

4.1 The (mcut) reduction steps

We will use a multicut reduction strategy. We first describe the steps of reduction. To describe these mcut-steps of reduction, we will use a notation and a definition:

► **Notation 1** ((!)-contexts). $\mathcal{C}^{!/\square}$ denotes a list of $\mu\text{LL}_{\square}^{\infty}$ -proofs, all concluded by a (\diamond_p^{\diamond}) , a $(?_p^{\square})$, a (\square_p) or a (\diamond_p) -rule.

\mathcal{C}^{\square} denotes a list of $\mu\text{LL}_{\square}^{\infty}$ -proofs which are all concluded either by a (\square_p) -rule or a (\diamond_p) -rule.

In the latter case, \mathcal{C} denotes the list of $\mu\text{LL}_{\square}^{\infty}$ -proofs formed by gathering the immediate subproofs of the last (\square_p) or (\diamond_p) -rule.

► **Definition 8** (Restriction of a multicut context). Let $\frac{\mathcal{C}}{s} \text{mcut}(\iota, \perp\!\!\!\perp)$ be a multicut occurrence such that $\mathcal{C} = s_1 \dots s_n$ and let $s_i := \vdash F_1, \dots, F_{k_i}$, we define \mathcal{C}_{F_j} to be the sequents linked to the formula F_j with the $\perp\!\!\!\perp$ -relation. We extend this definition to contexts of formulas. (More details are provided in Appendix C.2.)

We give the reduction steps of the exponential fragment of $\mu\text{LL}_{\square}^{\infty}$ in figures 6 and 7. For commutative steps, we only describe steps where the principal formula of the rule that will be commuted is in the succedent of the sequent. For principal steps, we only describe steps where the cut-formula on which the step is applied is a $?$ or \diamond -formula. The rest of the cases can be retrieved by duality or can be found in Appendix C.3 as well as rules for the non-exponential fragment.

$$\begin{array}{c}
\frac{\pi}{\frac{! \Gamma_1, \Box \Gamma_2 \vdash A, ? \Delta_1, \Diamond \Delta_2}{! \Gamma_1, \Box \Gamma_2 \vdash ! A, ? \Delta_1, \Diamond \Delta_2} \uparrow_{\text{P}}^{\Diamond} \quad \frac{\pi}{! \Gamma', \Box \Gamma'' \vdash ! A, ? \Delta, \Diamond \Delta'} \uparrow_{\text{P}}^{\Diamond} \quad \mathcal{C}^{!/\Box}}{\text{mcut}(\iota, \perp\!\!\!\perp)} \quad \rightsquigarrow \quad \frac{\pi}{! \Gamma_1, \Box \Gamma_2 \vdash A, ? \Delta_1, \Diamond \Delta_2} \mathcal{C}^{!/\Box} \quad \text{mcut}(\iota, \perp\!\!\!\perp)}{! \Gamma', \Box \Gamma'' \vdash ! A, ? \Delta, \Diamond \Delta'} \uparrow_{\text{P}}^{\Diamond}
\end{array}$$

$$\begin{array}{c}
\frac{\pi}{\frac{\Gamma \vdash A, \Delta}{\Box \Gamma \vdash \Box A, \Diamond \Delta} \Box_{\text{P}} \quad \mathcal{C}^{\Box}}{\text{mcut}(\iota, \perp\!\!\!\perp)} \quad \rightsquigarrow \quad \frac{\pi}{\frac{\Gamma \vdash A, \Delta}{\Gamma' \vdash A, \Delta'} \mathcal{C} \quad \text{mcut}(\iota, \perp\!\!\!\perp)}{\Box \Gamma' \vdash \Box A, \Diamond \Delta'} \Box_{\text{P}}
\end{array}$$

$$\begin{array}{c}
\frac{\pi}{\frac{\Gamma \vdash \delta A, \Delta}{\Gamma \vdash \delta A, \Delta} \delta_{\text{c}} \quad \mathcal{C}}{\text{mcut}(\iota, \perp\!\!\!\perp)} \quad \rightsquigarrow \quad \frac{\pi}{\frac{\Gamma \vdash \delta A, \delta A, \Delta}{\Gamma' \vdash \delta A, \delta A, \Delta'} \mathcal{C} \quad \text{mcut}(\iota', \perp\!\!\!\perp')}}{\Gamma \vdash \delta A, \Delta} \delta_{\text{c}}
\end{array}$$

$$\begin{array}{c}
\frac{\pi}{\frac{\Gamma \vdash \delta A, \Delta}{\Gamma' \vdash \delta A, \Delta'} \delta_{\text{w}} \quad \mathcal{C}}{\text{mcut}(\iota, \perp\!\!\!\perp)} \quad \rightsquigarrow \quad \frac{\pi}{\frac{\Gamma \vdash \Delta}{\Gamma' \vdash \Delta'} \mathcal{C} \quad \text{mcut}(\iota', \perp\!\!\!\perp')}}{\Gamma \vdash \delta A, \Delta} \delta_{\text{w}}
\end{array}$$

$$\begin{array}{c}
\frac{\pi}{\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash ? A, \Delta} ?_{\text{d}} \quad \mathcal{C}}{\text{mcut}(\iota, \perp\!\!\!\perp)} \quad \rightsquigarrow \quad \frac{\pi}{\frac{\Gamma \vdash A, \Delta}{\Gamma' \vdash ? A, \Delta'} ?_{\text{d}} \quad \mathcal{C} \quad \text{mcut}(\iota', \perp\!\!\!\perp')}}{\Gamma \vdash ? A, \Delta} ?_{\text{d}}
\end{array}$$

■ **Figure 6** $\mu\text{LL}_{\Box}^{\infty}$ commutative cut-elimination steps (commutation with right rules) – $\delta \in \{?, \Diamond\}$

$$\begin{array}{c}
\frac{\mathcal{C}_{\Gamma, \Delta} \quad \frac{\pi}{\frac{\Gamma \vdash \delta A, \delta A, \Delta}{\Gamma \vdash \delta A, \Delta} \delta_{\text{c}}} \quad \mathcal{C}^{!/\Box}}{! \Gamma_1, \Box \Gamma_2, \Gamma_3 \vdash \Delta_1, ? \Delta_2, \Diamond \Delta_3} \text{mcut}(\iota, \perp\!\!\!\perp)} \rightsquigarrow \frac{\mathcal{C}_{\Gamma, \Delta} \quad \frac{\pi}{\frac{\Gamma \vdash \delta A, \delta A, \Delta}{\Gamma \vdash \delta A, \Delta} \delta_{\text{c}}} \quad \mathcal{C}^{!/\Box} \quad \mathcal{C}^{!/\Box}}{! \Gamma_1, \Box \Gamma_2, ! \Gamma_1, \Box \Gamma_2, \Gamma_3 \vdash \Delta_3, ? \Delta_1, \Diamond \Delta_2, ? \Delta_1, \Diamond \Delta_2} \text{mcut}(\iota', \perp\!\!\!\perp')}}{! \Gamma_1, \Box \Gamma_2, \Box \Gamma_2, \Gamma_3 \vdash \Delta_3, ? \Gamma_1, \Diamond \Gamma_2, \Diamond \Gamma_2} ?_{\text{c}, !_{\text{c}}} \quad \Diamond_{\text{c}}, \Box_{\text{c}}
\end{array}$$

$$\begin{array}{c}
\frac{\mathcal{C}_{\Gamma, \Delta} \quad \frac{\pi}{\frac{\Gamma \vdash \Delta}{\Gamma \vdash \delta A, \Delta} \delta_{\text{w}}} \quad \mathcal{C}^{!/\Box}}{! \Gamma_1, \Box \Gamma_2, \Gamma_3 \vdash \Delta_1, ? \Delta_2, \Diamond \Delta_3} \text{mcut}(\iota, \perp\!\!\!\perp)} \rightsquigarrow \frac{\mathcal{C}_{\Gamma, \Delta} \quad \frac{\pi}{\frac{\Gamma \vdash \Delta}{\Gamma_3 \vdash \Delta_3} \text{mcut}(\iota', \perp\!\!\!\perp')}}{! \Gamma_1, \Gamma_3 \vdash \Delta_1, ? \Delta_2} ?_{\text{w}, !_{\text{w}}} \quad \Diamond_{\text{w}}, \Box_{\text{w}}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\Gamma_1 \vdash A, \Delta_1}{\Gamma_1 \vdash ? A, \Delta_1} ?_{\text{d}} \quad \frac{! \Gamma_2, \Box \Gamma_3, A \vdash ? \Delta_2, \Diamond \Delta_3}{! \Gamma_2, \Box \Gamma_3, ? A \vdash ? \Delta_2, \Diamond \Delta_3} ?_{\text{P}}^{\Box} \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(\iota, \perp\!\!\!\perp)} \rightsquigarrow \frac{\Gamma_1 \vdash A, \Delta_1 \quad ! \Gamma_2, \Box \Gamma_3, A \vdash ? \Delta_2, \Diamond \Delta_3 \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(\iota', \perp\!\!\!\perp')}
\end{array}$$

■ **Figure 7** One side of the $\mu\text{LL}_{\Box}^{\infty}$ principal cut-elimination steps – in all these proofs, $\delta \in \{?, \Diamond\}$

284 ► **Definition 9** (Reduction sequence). A reduction sequence $(\pi_i)_{i \in 1+\lambda}$ ($\lambda \in \omega + 1$) is a \rightsquigarrow
285 sequence such that π_0 does not contain more than one (*mcut*) rule per branches.

286 We want to prove that each reduction sequence converges to a cut-free proof. However,
287 the theorem is certainly not true as such, even for infinite reduction sequences: one can
288 well apply infinitely many reductions only on some part of the proof, without reducing
289 some cuts in another part of the proof. Therefore, we need to be finer, motivating the
290 following definition, directly borrowed from [4, 3] (the notion of residual is the usual one
291 from rewriting):

292 ► **Definition 10** (Fair reduction sequences [4, 3]). A reduction sequence $(\pi_i)_{i \in \omega}$ is fair, if for
293 each π_i such that there is a reduction \mathcal{R} to a proof π' , there exist a $j > i$ such that π_j does
294 not contain any residual of \mathcal{R} .

295 We can now state our cut-elimination theorem:

296 ► **Theorem 1** (Cut-elimination for $\mu\text{LL}_{\Box}^{\infty}$). Every fair (*mcut*)-reduction sequence of $\mu\text{LL}_{\Box}^{\infty}$
297 valid proofs converges to a cut-free valid proof.

298 To prove it, we will translate formulas, proofs and (mcut)-steps of $\mu\text{LL}_{\square}^{\infty}$ into μLL^{∞} and use
299 the following cut-elimination result from [20]:

300 ► **Theorem 2** (Cut-elimination for μLL^{∞}). *Every fair (mcut)-reduction sequence of μLL^{∞}
301 valid proofs converges to a cut-free valid proof.*

302 In [20], exponential formulas, proofs and cut-steps are encoded into μMALL^{∞} . We could have
303 made the choice to encode the modalities of $\mu\text{LL}_{\square}^{\infty}$ directly into μMALL^{∞} , replaying the
304 proof of [20] to get cut-elimination. However, using the μLL^{∞} cut-elimination theorem as
305 such, makes our approach more modular and more easy to adapt to future extensions of
306 μLL^{∞} validity condition or variants of its cut-elimination proof.

307 4.2 Translation of $\mu\text{LL}_{\square}^{\infty}$ into μLL^{∞}

308 We give a translation of $\mu\text{LL}_{\square}^{\infty}$ into μLL^{∞} (more details can be found in appendix C.4):

309 ► **Definition 11** (Translation of $\mu\text{LL}_{\square}^{\infty}$ into μLL^{∞}). *Translation of formula is defined
310 inductively on the formula:*

- 311 ■ *Translations of \diamond and \square -formulas:* $(\diamond A)^{\circ} := ?A^{\circ}$ and $(\square A)^{\circ} := !A^{\circ}$.
- 312 ■ *Translations of atomic and unit formulas and variables f :* $f^{\circ} := f$.
- 313 ■ *Translations of other non-fixed-point connectives:* $c(A_1, \dots, A_n)^{\circ} := c(A_1^{\circ}, \dots, A_n^{\circ})$.
- 314 ■ *Translations of fixed-point connectives are given by:* $(\delta X.F)^{\circ} := \delta X.F^{\circ}$ (with $\delta \in \{\mu, \nu\}$).

*Translation of structural rules for modalities, (\diamond_c) , (\diamond_w) , (\square_c) and (\square_w) are respectively
(?_c), (?_w), (!_c) and (!_w). Translation for the promotions $(!_p^{\diamond})$ and $(?_p^{\square})$ are respectively $(!_p)$
and $(?_p)$. Translation of the modal rules are given by:*

$$\frac{\Gamma \vdash A, \Delta}{\square \Gamma \vdash \square A, \diamond \Delta} \square_p \rightsquigarrow \frac{\frac{\Gamma^{\circ} \vdash A^{\circ}, \Delta^{\circ}}{! \Gamma^{\circ} \vdash A^{\circ}, ? \Delta^{\circ}} !_d, ?_d}{! \Gamma^{\circ} \vdash ! A^{\circ}, ? \Delta^{\circ}} !_p \quad \frac{! \Gamma_1, \square \Gamma_2, A \vdash ? \Delta_1, \diamond \Delta_2}{! \Gamma_1, \square \Gamma_2, ? A \vdash ? \Delta_1, \diamond \Delta_2} !_p^{\diamond} \rightsquigarrow \frac{! \Gamma_1^{\circ}, ! \Gamma_2^{\circ}, A^{\circ} \vdash ? \Delta_1^{\circ}, ? \Delta_2^{\circ}}{! \Gamma_1^{\circ}, ! \Gamma_2^{\circ}, ! A^{\circ} \vdash ? \Delta_1^{\circ}, ? \Delta_2^{\circ}} !_p$$

315 *Translation of other inference rules (r) are (r) themselves.*

316 *Translation of pre-proofs are defined co-inductively using translations of rules.*

317 The translation preserves validity both ways:

318 ► **Lemma 1** (Validity robustness to $(-)^{\circ}$ translation). *Let π be a $\mu\text{LL}_{\square}^{\infty}$ pre-proof, then π is
319 valid if and only if π° is.*

320 **Proof.** Let B a branch of π , we have that B is validated by a thread (A_i) if and only if B°
321 is validated by (A_i°) as the minimal recurring fixed point formula is a ν on the right (resp. μ
322 on the left) in (A_i) if it is in (A_i°) . ◀

323 Finally, we have to make sure (mcut)-reduction sequences are robust under this translation.
324 In our proof of the final theorem, we also need one-step reduction-rules to be simulated by a
325 finite number of reduction steps in the translation.

326 ► **Lemma 2.** *Consider a $\mu\text{LL}_{\square}^{\infty}$ reduction step $\pi_0 \rightsquigarrow \pi_1$, there exist a finite number of μLL^{∞}
327 proofs $\theta_0, \dots, \theta_n$ such that:* $\pi_0^{\circ} = \theta_0 \rightarrow \theta_1 \rightarrow \dots \rightarrow \theta_{n-1} \rightarrow \theta_n = \pi_1^{\circ}$.

Proof. We only give a sketch of the proof, full proof can be found in appendix C.5. Reductions
from the non-exponential part of $\mu\text{LL}_{\square}^{\infty}$ translates easily to one step of reduction in μLL^{∞} .
The same is true for the exponential part except for the commutation of the modal rule.

The translation of the left proof of this step is of the form (we only do the case of \diamond_p , \square_p is similar):

$$\frac{\frac{\frac{\pi_1^\circ}{\Gamma_1^\circ \vdash A_1^\circ, \Delta_1^\circ} \text{!}_d, ?_d}{\Gamma_1^\circ \vdash A_1^\circ, ?\Delta_1^\circ} \text{!}_p \quad \dots \quad \frac{\frac{\pi_n^\circ}{\Gamma_n^\circ \vdash A_n^\circ, \Delta_n^\circ} \text{!}_d, ?_d}{\Gamma_n^\circ \vdash A_n^\circ, ?\Delta_n^\circ} \text{!}_p \quad \frac{\frac{\pi_{n+1}^\circ}{\Gamma_{n+1}^\circ, A_{n+1}^\circ \vdash \Delta_{n+1}^\circ} \text{!}_d, ?_d}{\Gamma_{n+1}^\circ, A_{n+1}^\circ \vdash ?\Delta_{n+1}^\circ} ?_p \quad \dots \quad \frac{\frac{\pi_{n+m}^\circ}{\Gamma_{n+m}^\circ, A_{n+m}^\circ \vdash \Delta_{n+m}^\circ} \text{!}_d, ?_d}{\Gamma_{n+m}^\circ, A_{n+m}^\circ \vdash ?\Delta_{n+m}^\circ} ?_p}{\Gamma \vdash !A^\circ, ?\Gamma^\circ} \text{mcut}(t, \perp\!\!\!\perp)$$

328 Here, we notice that for each dereliction on a cut-formula there exists a corresponding
 329 promotion that will be erased by a dereliction/promotion key-case. The first promotion will
 330 therefore commute under the cut and then each dereliction on formula of the conclusion will
 331 commute as well and each dereliction and each promotion on cut-formulas will be erased,
 332 finally we commute the translation of the modal rule under the multicut. \blacktriangleleft

333 Now that we know a step of (mcut)-reduction in $\mu\text{LL}_{\square}^\infty$ translates to one or more μLL^∞
 334 (mcut)-reduction steps, it is easy to translate each reduction sequence of $\mu\text{LL}_{\square}^\infty$ into reduction
 335 sequence of μLL^∞ . However, to use the cut-elimination theorem of μLL^∞ , we need the
 336 reduction sequence to be fair. The purpose of the following lemma is to control the fairness
 337 of the translated reduction sequence:

338 **► Lemma 3** (Completeness of the (mcut)-reduction system). *Let π and π' be two $\mu\text{LL}_{\square}^\infty$ proofs.*
 339 *If there is a μLL^∞ -redex \mathcal{R} sending π° to π'° then there is also a $\mu\text{LL}_{\square}^\infty$ -redex \mathcal{R}' sending π*
 340 *to a proof π'' , such that in the translation of \mathcal{R}' , \mathcal{R} is reduced.*

341 **Proof.** We only prove the exponential cases, the non-exponential cases being immediate. We
 342 have several cases:

- 343 ■ If the case is the commutative step of a weakening (resp. a contraction, resp. a dereliction)
 344 (r), as it is on top of a (mcut), it necessarily means that (r) comes from a rule (r') being
 345 the translation of a contraction (resp. a weakening, resp. a dereliction) which is also on
 346 top of an (mcut) in π , we can take \mathcal{R}' as the step commuting (r') under the cut.
- 347 ■ If it is a principal case again on a contraction or a weakening (r) on a formula $?A$ (resp.
 348 $!A$), it means that each proofs cut-connected to $?A$ (resp. $!A$) ends with a promotion. As
 349 π° is the translation of a $\mu\text{LL}_{\square}^\infty$ -proof, it means that (r) is the translation of a weakening
 350 or contraction rule (r') on a formula $?A'$ (resp. $!A'$) or $\diamond A'$ (resp. $\square A'$) on top of a
 351 (mcut). It also means that all the proofs cut-connected to these formulas are promotions
 352 or modal rules (no other rules than a modal rule nor a promotion in $\mu\text{LL}_{\square}^\infty$ translates to
 353 a derivation ending with a promotion). Therefore, the principal case on (r') is possible,
 354 we define \mathcal{R}' to be this principal case.
- 355 ■ If it's a principal case on a dereliction, we have that it comes from a dereliction and a
 356 promotion in the original proof, and we can take \mathcal{R}' to be this redex.
- 357 ■ If it is the commutative step of a promotion (r), it means that all the proofs of the
 358 contexts of the (mcut) are promotions or modal rules. Meaning that (r) is contained in
 359 the translation of a promotion (r') on top of the (mcut). We also have that the context
 360 of this (mcut) are only proofs ending with promotions. We therefore need to make sure
 361 that each (mcut) with a context full of promotions or modal rules are covered by the
 362 \rightsquigarrow -relation. Looking back at figure 6 together with conditions given by each corresponding
 363 lemmas, we have that:
 - 364 ■ The commutation of ($!_p^\diamond$) (or ($?_p^\square$)) is covered by the first commutative case in 6.
 - 365 ■ If it is a modal rule that is ready to be commuted, then other rules are necessarily
 366 modal rules and therefore is covered by the second commutative case in 6.

367

We use the two previous lemma, to prove the following:

368

► **Corollary 1.** *For every fair $\mu\text{LL}_{\square}^{\infty}$ reduction sequence $(\pi_i)_{i \in 1+\lambda}$ ($\lambda \in \omega + 1$), we have:*

369

■ *a fair μLL^{∞} reduction sequence $(\theta_i)_{i \in 1+\lambda'}$ ($\lambda' \in \omega + 1$);*

370

■ *a sequence of strictly increasing $(\varphi(i))_{i \in 1+\lambda}$ of elements of $1 + \lambda'$;*

371

■ *for each i , $\theta_{\varphi(i)} \in \pi_i^{\circ}$.*

372

Proof. We construct the sequence by induction on the steps of reductions of $(\pi_i)_{i \in 1+\lambda}$.

373

■ For $i = 0$: $\theta_0 = \pi_0^{\circ}$, $\varphi(0) = 0$ and $k_0 = 0$:

374

■ For $i + 1$, suppose we constructed everything up to rank i . We use lemma 2 on the step

375

$\pi_i \rightsquigarrow \pi_{i+1}$ to get a finite sequence of reduction $\pi_i = \theta'_0 \rightsquigarrow \dots \rightsquigarrow \theta'_n = \pi_{i+1}$. We then

376

construct both sequences by setting $\varphi(i + 1) := \varphi(i) + n$, $\theta_{\varphi(i)+j} := \theta'_j$ (for $j \in \llbracket 0, n \rrbracket$).

377

We get fairness of $(\theta_i)_{i \in 1+\lambda'}$ from lemma 3 and from the fact that after the translation of an

378

(mcut)-step, $\pi^{\circ} \rightsquigarrow \pi'^{\circ}$, each residual of a redex \mathcal{R} of π° , is contained in the translations of

379

residuals of the associated redex \mathcal{R}' of lemma 3. ◀

380

4.3 Cut-elimination for $\mu\text{LL}_{\square}^{\infty}$

381

Finally, we can prove the main theorem of the section:

382

Proof of Theorem 1. Let $(\pi_i)_{i \in 1+\lambda}$ be a $\mu\text{LL}_{\square}^{\infty}$ reduction sequence. We use corollary 1 and

383

get a fair μLL^{∞} reduction sequence $(\theta_i)_{i \in 1+\lambda'}$ and a sequence $(\varphi(i))_{i \in 1+\lambda}$ of natural numbers.

384

By theorem 2, we know that $(\theta_i)_{i \in \omega}$ converges to a cut-free proof θ of μLL^{∞} . Now suppose

385

for the sake of contradiction that (π_i) does not converge to a (mcut)-free pre-proof, meaning

386

that there is a j and a path p such that for each proof $\pi_{j'}$, with $j' \geq j$, there is an (mcut)-rule

387

at the end of path p . This means that the translation of p leads to an (mcut) for each proof

388

$\theta_{j'}$ with $j' \geq \varphi(j)$, contradicting the convergence of (θ_j) to a cut-free proof. We also have

389

that (π_i) converges to a pre-proof π such that $\pi^{\circ} = \theta$, as $\theta_{\varphi(j)}$ is equal to π_j° under the

390

multicuts. Moreover by lemma 1, π is valid. It is also cut-free, by cut-freeness of θ . ◀

391

5 Cut-elimination of $\mu\text{LK}_{\square}^{\infty}$

392

We extend the translation from [20] of μLK^{∞} to $\mu\text{LK}_{\square}^{\infty}$ to obtain a translation into $\mu\text{LL}_{\square}^{\infty}$. We

already gave a translation at the beginning of Section 3. However, this translation only worked

when there was an empty antecedent. Taking back our example and adding the left side to it:

$$\frac{\Gamma \vdash F, \Delta}{\square\Gamma \vdash \square F, \diamond\Delta} \square_p \text{ would end up on something like this: } \frac{\frac{\frac{\Gamma^{\bullet} \vdash ?A^{\bullet}, ?\Delta^{\bullet}}{\square? \Gamma^{\bullet} \vdash \square?A^{\bullet}, \diamond? \Delta^{\bullet}} \square_p}{\square? \Gamma^{\bullet} \vdash ?!\square?A^{\bullet}, \diamond? \Delta^{\bullet}} ?_d, !_p^{\diamond}}{\square! \Gamma^{\bullet} \vdash ?!\square?A^{\bullet}, ?!\diamond? \Delta^{\bullet}} ?_d, !_p^{\diamond}, !_d$$

Now, if Γ contains more than two formulas, we are not able to apply $(?_p)$ on it. By adding

an !-connective in the translation of \square -formulas: $\square A^{\bullet} := !\square!A^{\bullet}$, we can conclude:

$$\frac{\frac{\frac{\frac{\Gamma^{\bullet} \vdash ?A^{\bullet}, ?\Delta^{\bullet}}{!\square? \Gamma^{\bullet} \vdash ?! \square?A^{\bullet}, ?\Delta^{\bullet}} !_d, ?_p}{!\square? \Gamma^{\bullet} \vdash ?! \square?A^{\bullet}, ?\Delta^{\bullet}} !_p^{\diamond}}{\square! \square? \Gamma^{\bullet} \vdash \square! \square?A^{\bullet}, \diamond? \Delta^{\bullet}} \square_p}{\square! \square? \Gamma^{\bullet} \vdash ?!\square! \square?A^{\bullet}, \diamond? \Delta^{\bullet}} ?_d, !_p^{\diamond}}{\square! \square! \square? \Gamma^{\bullet} \vdash ?!\square! \square?A^{\bullet}, ?!\diamond? \Delta^{\bullet}} ?_d, !_p^{\diamond}, !_d$$

393

Based on this, we define a translation from $\mu\text{LK}_{\square}^{\infty}$ into $\mu\text{LL}_{\square}^{\infty}$:

$$\begin{array}{c}
 \frac{\Delta \vdash F, \Gamma}{\Box \Delta \vdash \Box F, \Diamond \Gamma} \Box_P \quad \rightsquigarrow \quad \frac{\frac{\frac{\frac{\Delta^\bullet \vdash ?F^\bullet, ?\Gamma^\bullet}{!?\Delta^\bullet \vdash ?F^\bullet, ?\Gamma^\bullet} !_d, ?_p}{!?\Delta^\bullet \vdash !?F^\bullet, ?\Gamma^\bullet} !_p}{\Box !?\Delta^\bullet \vdash \Box !?F^\bullet, \Diamond ?\Gamma^\bullet} \Box_P}{\frac{\Box !?\Delta^\bullet \vdash \Box !?F^\bullet, \Diamond ?\Gamma^\bullet}{! \Box !?\Delta^\bullet \vdash \Box !?F^\bullet, \Diamond ?\Gamma^\bullet} !_d}{! \Box !?\Delta^\bullet \vdash ?! \Box !?F^\bullet, ?! \Diamond ?\Gamma^\bullet} ?_d, !_\Diamond^P \\
 \\
 \frac{\Delta, F \vdash \Gamma}{\Box \Delta, \Diamond F \vdash \Diamond \Gamma} \Diamond_P \quad \rightsquigarrow \quad \frac{\frac{\frac{\frac{\Delta^\bullet, F^\bullet \vdash ?\Gamma^\bullet}{!?\Delta^\bullet, F^\bullet \vdash ?\Gamma^\bullet} !_d, ?_p}{!?\Delta^\bullet, ?F^\bullet \vdash ?\Gamma^\bullet} ?_p}{\Box !?\Delta^\bullet, \Diamond ?F^\bullet \vdash \Diamond ?\Gamma^\bullet} \Diamond_P}{\frac{\Box !?\Delta^\bullet, \Diamond ?F^\bullet \vdash \Diamond ?\Gamma^\bullet}{! \Box !?\Delta^\bullet, ! \Diamond ?F^\bullet \vdash \Diamond ?\Gamma^\bullet} !_d}{! \Box !?\Delta^\bullet, ! \Diamond ?F^\bullet \vdash ?! \Diamond ?\Gamma^\bullet} ?_d, !_\Diamond^P
 \end{array}$$

■ **Figure 8** Linear translation of the modal rules

394 ▶ **Definition 12** (Linear translation of $\mu\text{LK}_{\Box}^\infty$). We define the translation $(-)^{\bullet}$ from formulas
 395 of $\mu\text{LK}_{\Box}^\infty$ to formulas of $\mu\text{LL}_{\Box}^\infty$ by induction on these formulas in the following way:

$$\begin{array}{lll}
 396 & (A_1 \rightarrow A_2)^{\bullet} := !(?A_1^{\bullet} \multimap ?A_2^{\bullet}) & X^{\bullet} := !X & (\mu X.A)^{\bullet} := !\mu X.?A^{\bullet} \\
 397 & (A_1 \wedge A_2)^{\bullet} := !(?A_1^{\bullet} \& ?A_2^{\bullet}) & \top^{\bullet} := !\top & (\nu X.A)^{\bullet} := !\nu X.?A^{\bullet} \\
 398 & (A_1 \vee A_2)^{\bullet} := !(?A_1^{\bullet} \oplus ?A_2^{\bullet}) & \text{F}^{\bullet} := !0 & (\Diamond A)^{\bullet} := !\Diamond ?A^{\bullet} \\
 399 & (A^\perp)^{\bullet} := !(?A^{\bullet})^\perp & a^{\bullet} := !a & (\Box A)^{\bullet} := !\Box ?A^{\bullet}
 \end{array}$$

400 We also have a translation for sequents: $(\Gamma \vdash \Delta)^{\bullet} := \Gamma^{\bullet} \vdash ?\Delta^{\bullet}$.

401 We have the following property that must be kept in mind when defining rule translations:

402 ▶ **Proposition 1.** Let A be a $\mu\text{LK}_{\Box}^\infty$ formula, then A^{\bullet} is an $!$ -formula.

403 We give the translation of modal rules in figure 8 and the rest in the appendix D.1. We then
 404 define translations of proofs coinductively on the proofs using the translation of each rule.
 405 As the smallest formula (for inclusion ordering) of a totally ordered set of translations is the
 406 translation of the smallest formula, and that a branch of π^{\bullet} contains all the translations of
 407 threads from π and vice-versa, we have the following:

408 ▶ **Lemma 4** (Robustness of $(-)^{\bullet}$ to validity). If π is a valid pre-proof iff π^{\bullet} is valid.

409 We define a translation $\text{SK}(-)$ going from $\mu\text{LL}_{\Box}^\infty$ formulas and pre-proofs to $\mu\text{LK}_{\Box}^\infty$
 410 formulas and pre-proofs, by forgetting linear information from formulas and pre-proofs
 411 (ie erasing exponential modalities, as well as dereliction and promotion, and projecting
 412 other connectives or inferences to the corresponding $\mu\text{LK}_{\Box}^\infty$ connectives and inferences):
 413 $\text{SK}(!A) := \text{SK}(A)$ $\text{SK}(?A) := \text{SK}(A)$. As our goal is to come back to $\mu\text{LK}_{\Box}^\infty$ from a
 414 translation into $\mu\text{LL}_{\Box}^\infty$, we can restrict our translation to the $\otimes, \wp, \perp, 1$ -free fragment of
 415 $\mu\text{LL}_{\Box}^\infty$. The full definition of $\text{SK}((-))$ translation and proofs of the two following lemmas
 416 can be found in appendix D.2.

417 ▶ **Lemma 5** (Robustness of the skeleton to validity). If π is a $\mu\text{LL}_{\Box}^\infty$ valid pre-proof then
 418 $\text{SK}(\pi)$ is a $\mu\text{LK}_{\Box}^\infty$ valid pre-proof, and vice-versa.

419 ▶ **Lemma 6** (Composition of $\text{SK}(-)$ and of $(-)^{\bullet}$). Let π be a $\mu\text{LK}_{\Box}^\infty$ pre-proof. We have that
 420 $\text{SK}(\pi^{\bullet})$ is equal to π .

421 We define our rewriting system using the $\text{SK}()$ translation:

422 ▶ **Definition 13** ((mcut)-rewriting system of $\mu\text{LK}_{\Box}^\infty$). We define (mcut)-rewriting system of
 423 $\mu\text{LK}_{\Box}^\infty$ to be the (mcut)-system obtained from $\mu\text{LL}_{\Box}^\infty$ (mcut)-system by forgetting the linear
 424 information of proofs of this system.

425 Finally, we have the following theorem:

426 ► **Theorem 3.** *The (mcut)-reduction system of $\mu\text{LK}_{\square}^{\infty}$ is an infinitary weakly-normalizing.*

427 **Proof.** Consider a $\mu\text{LK}_{\square}^{\infty}$ proof π and a fair reduction sequence σ_{L} from π^{\bullet} . By theorem 1,
 428 σ_{L} converges to a cut-free $\mu\text{LL}_{\square}^{\infty}$ proof. By applying $\text{SK}(-)$ to each proof in the sequence,
 429 we obtain a sequence of $\mu\text{LK}_{\square}^{\infty}$ valid proofs which are all valid and such that either $\text{SK}(\pi_i) =$
 430 $\text{SK}(\pi_{i+1})$ or $\text{SK}(\pi_i)$ reduces to $\text{SK}(\pi_{i+1})$ with one step of $\mu\text{LK}_{\square}^{\infty}$ mcut-reduction. By
 431 dropping the equality cases, we obtain a $\mu\text{LK}_{\square}^{\infty}$ cut-reduction sequence σ_{K} that is infinite
 432 and converges to a valid, cut-free $\mu\text{LK}_{\square}^{\infty}$ proof. ◀

433 6 Conclusion

434 We have introduced $\mu\text{LL}_{\square}^{\infty}$ a linear version of the modal μ -calculus as well as its circular
 435 and non-wellfounded system, and we proved a cut-elimination theorem with respect to
 436 the call of fair cut-elimination reduction sequences, generalizing previous results on the
 437 non-wellfounded proof theory of linear logic. By means of a linear translation of the circular
 438 and non-wellfounded proof systems for the modal μ -calculus, $\mu\text{LK}_{\square}^{\infty}$, to $\mu\text{LL}_{\square}^{\infty}$, we obtained
 439 a cut-elimination theorem the non-wellfounded sequent calculus for the modal μ -calculus.

440 In our opinion, this work presents a new and interesting application of linear logic to
 441 modal μ -calculus, developing proof theories in both domains and highlighting the potential
 442 for cross-fertilization for the two communities. Indeed, this constitutes the first full syntactic
 443 cut-elimination theorem for a proof system modelling the full modal μ -calculus.

444 Moreover, due to the fine-grained cut-elimination inherited from linear logic, one can
 445 hope to have a non-trivial cut-elimination equivalence on $\mu\text{LK}_{\square}^{\infty}$ proofs and therefore to be
 446 able to design a denotational semantics for proofs of modal μ ; such a question was, till now,
 447 beyond reach not only due to the lack of a syntactic cut-elimination theorem, but also due
 448 to the lack of structure in proofs of the modal μ -calculus.

449 From the linear logic-theoretic point of view, our system $\mu\text{LL}_{\square}^{\infty}$ can be viewed as a
 450 linear logic with two sets of exponential modalities satisfying different structural rules
 451 and exponential. This is akin to so-called *light logics* [10, 13], that are variants of linear
 452 logics developed by taming the power of exponential modalities in order to control the
 453 complexity of cut-elimination (for instance constraining the $?$ -context of a promotion to
 454 be immediately derelicted after a promotion ensures that typable programs have at most
 455 elementary complexity [10]). Still, our calculus is new in this respect as, to our knowledge, it
 456 has never been considered in the realm of light logics. We are pursuing a detailed investigation
 457 of those light logics in presence of fixed-points in order to generalize the above cut-elimination
 458 results in a uniform way.

459 Another important direction for future work is whether our linear-logical modal μ -
 460 calculus can be adapted to the wellfounded proof-systems of linear logic with fixed-points in
 461 a μLL_{\square} sequent calculus and whether one can adapt our methodology to obtain, via a linear
 462 translation from μLK_{\square} to μLL_{\square} a cut-elimination theorem for the finitary sequent calculus
 463 for modal μ . This question is highly challenging due to the complex structure of the rules
 464 for fixed-points in finitary μLL . While the skeletons obtaining by forgetting linear information
 465 from μLL_{\square} proofs should readily provide μLK_{\square} proofs and simulates the cut-elimination, is
 466 a far less obvious whether one can design a linear translation from μLK_{\square} to μLL_{\square} which
 467 commutes with cut-elimination as in the present paper.

References

- 468 —
- 469 1 Bahareh Afshari and Graham E. Leigh. Cut-free completeness for modal mu-calculus. In *2017*
470 *32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–12, 2017.
471 doi:10.1109/LICS.2017.8005088.
- 472 2 David Baelde. Least and greatest fixed points in linear logic. *ACM Trans. Comput. Log.*,
473 13(1):2:1–2:44, 2012. doi:10.1145/2071368.2071370.
- 474 3 David Baelde, Amina Doumane, Denis Kuperberg, and Alexis Saurin. Bouncing threads
475 for circular and non-wellfounded proofs: Towards compositionality with circular proofs. In
476 *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS*
477 '22, New York, NY, USA, 2022. Association for Computing Machinery. doi:10.1145/3531130.
478 3533375.
- 479 4 David Baelde, Amina Doumane, and Alexis Saurin. Infinitary Proof Theory: the Multiplicative
480 Additive Case. In *CSL 2016*, volume 62 of *LIPICs*, pages 42:1–42:17, 2016. URL: [http://](http://drops.dagstuhl.de/opus/volltexte/2016/6582)
481 drops.dagstuhl.de/opus/volltexte/2016/6582, doi:10.4230/LIPICs.CSL.2016.42.
- 482 5 Kai Br unnler and Thomas Studer. Syntactic cut-elimination for a fragment of the modal
483 mu-calculus. *Annals of Pure and Applied Logic*, 163(12):1838–1853, 2012. URL: [https://](https://www.sciencedirect.com/science/article/pii/S0168007212000760)
484 www.sciencedirect.com/science/article/pii/S0168007212000760, doi:10.1016/j.apal.
485 2012.04.006.
- 486 6 Pierre-Louis Curien and Hugo Herbelin. The duality of computation. In *ICFP 2000*, pages
487 233–243. ACM, 2000. doi:10.1145/351240.351262.
- 488 7 Vincent Danos, Jean-Baptiste Joinet, and Harold Schellinx. A new deconstructive logic: Linear
489 logic. *J. Symb. Log.*, 62(3):755–807, 1997. doi:10.2307/2275572.
- 490 8 Amina Doumane. *On the infinitary proof theory of logics with fixed points*. Phd thesis, Paris
491 Diderot University, 2017.
- 492 9 Jean-Yves Girard. Linear logic. *Theor. Comput. Sci.*, 50:1–102, 1987. doi:10.1016/
493 0304-3975(87)90045-4.
- 494 10 Jean-Yves Girard. Light linear logic. 143(2):175–204, June 1998. doi:10.1006/inco.1998.
495 2700.
- 496 11 Gerhard J ager, Mathis Kretz, and Thomas Studer. Canonical completeness of infinitary μ .
497 *The Journal of Logic and Algebraic Programming*, 76(2):270–292, 2008. Logic and Information:
498 From Logic to Constructive Reasoning. URL: [https://www.sciencedirect.com/science/](https://www.sciencedirect.com/science/article/pii/S1567832608000209)
499 [article/pii/S1567832608000209](https://www.sciencedirect.com/science/article/pii/S1567832608000209), doi:10.1016/j.jlap.2008.02.005.
- 500 12 Dexter Kozen. Results on the propositional mu-calculus. *Theor. Comput. Sci.*, 27:333–354,
501 1983. doi:10.1016/0304-3975(82)90125-6.
- 502 13 Yves Lafont. Soft linear logic and polynomial time. *Theoretical Computer Science*, 318(1–
503 2):163–180, June 2004. doi:10.1016/j.tcs.2003.10.018.
- 504 14 Dale Miller. Finding unity in computational logic. In *Proceedings of the 2010 ACM-BCS*
505 *Visions of Computer Science Conference*, ACM-BCS '10, Swindon, GBR, 2010. BCS Learning
506 & Development Ltd.
- 507 15 Grigori Mints. Effective cut-elimination for a fragment of modal mu-calculus. *Studia Logica:*
508 *An International Journal for Symbolic Logic*, 100(1/2):279–287, 2012. URL: [http://www.](http://www.jstor.org/stable/41475226)
509 [jstor.org/stable/41475226](http://www.jstor.org/stable/41475226).
- 510 16 Grigori Mints and Thomas Studer. Cut-elimination for the mu-calculus with one variable.
511 *Electronic Proceedings in Theoretical Computer Science*, 77:47–54, February 2012. URL:
512 <http://dx.doi.org/10.4204/EPTCS.77.7>, doi:10.4204/eptcs.77.7.
- 513 17 Damian Niwiński and Igor Walukiewicz. Games for the μ -calculus. *Theoretical Computer*
514 *Science*, 163(1):99–116, 1996. URL: [https://www.sciencedirect.com/science/article/](https://www.sciencedirect.com/science/article/pii/0304397595001360)
515 [pii/0304397595001360](https://www.sciencedirect.com/science/article/pii/0304397595001360), doi:10.1016/0304-3975(95)00136-0.
- 516 18 Karim Nour, René David, Christophe Raffalli, and Pierre-Louis Curien. *Introduction à la*
517 *logique : Théorie de la démonstration - Cours et exercices corrigés*. Dunod, 2e edition, 2004.
- 518 19 Michele Pagani and Lorenzo Tortora de Falco. Strong normalization property for second order
519 linear logic. *Theoretical Computer Science*, 411(2):410–444, 2010.

- 520 20 Alexis Saurin. A linear perspective on cut-elimination for non-wellfounded sequent calculi
521 with least and greatest fixed points. TABLEAUX '23. Springer, 2023. URL: [https://hal.
522 science/hal-04169137](https://hal.science/hal-04169137).
- 523 21 Colin Stirling. Modal and Temporal Logics. In *Handbook of Logic in Computer Science*.
524 Oxford University Press, 12 1992. [arXiv:https://academic.oup.com/book/0/chapter/
525 421961577/chapter-pdf/52352328/isbn-9780198537618-book-part-5.pdf](https://academic.oup.com/book/0/chapter/421961577/chapter-pdf/52352328/isbn-9780198537618-book-part-5.pdf), doi:10.1093/
526 oso/9780198537618.003.0005.

527 **A** Appendix on the section 2528 **A.1** Details of definitions of Linear Logic

529 ► **Definition 14** (Positive and negative occurrence of a fixed-point variable). *Let $X \in \mathcal{V}$ be a*
 530 *fixed-point variable, one defines the fact, for X , to occur positively (resp. negatively) in a*
 531 *pre-formula by induction on the structure of pre-formulas:*

- 532 ■ *The variable X occurs positively in X .*
 533 ■ *The variable X occurs positively (resp. negatively) in $c(F_1, \dots, F_n)$, if there is some*
 534 *$1 \leq i \leq n$ such that X occurs positively (resp. negatively) in F_i for $c \in \{\otimes, \wp, \&, \oplus, !, ?\}$.*
 535 ■ *The variable X occurs positively (resp. negatively) in F^\perp if X occurs negatively (resp.*
 536 *positively) in F .*
 537 ■ *The variable X occurs positively (resp. negatively) in $F \multimap G$ if X occurs either positively*
 538 *(resp. negatively) in G or negatively (resp. positively) in F .*
 539 ■ *The variable X occurs positively (resp. negatively) in $\delta Y.G$ (with $Y \neq X$) if it occurs*
 540 *positively (resp. negatively) in G (for $\delta \in \{\mu, \nu\}$).*

541 **B** Appendix on the section 3542 **B.1** Details on the discussion about robustness of $\mu\text{LL}_\square^\infty$

As said in the core of the paper, taking the $(?_p/?_w)$ principal case:

$$\frac{\frac{\Gamma_1 \vdash \Delta_1}{\Gamma_1 \vdash ?C, \Delta_1} ?_w \quad \frac{! \Gamma_2, C \vdash ? \Delta_2}{! \Gamma_2 ? C \vdash ? \Delta_2} ?_p}{\Gamma_1, ! \Gamma_2 \vdash \Delta_1, ? \Delta_2} \text{cut} \rightsquigarrow \frac{\Gamma_1 \vdash \Delta_1}{\Gamma_1, ! \Gamma_2 \vdash \Delta_2, ? \Delta_2} ?_w, !_w$$

and adding modal-formulas to the context, naturally requires to be able to weaken \diamond/\square -formulas (this corresponds to the reason for the design of the promotion rule in LL):

$$\frac{\frac{\Gamma_1 \vdash \Delta_1}{\Gamma_1 \vdash ?C, \Delta_1} ?_w \quad \frac{! \Gamma_2, \square \Gamma_3, C \vdash ? \Delta_2, \diamond \Delta_3}{! \Gamma_2, \square \Gamma_3, ?C \vdash ? \Delta_2, \diamond \Delta_3} ?_p}{\Gamma_1, ! \Gamma_2, \square \Gamma_3 \vdash \Delta_2, ? \Delta_2, \diamond \Delta_3} \text{cut} \rightsquigarrow \frac{\Gamma_1 \vdash \Delta_1}{\Gamma_1, ! \Gamma_2, \square \Gamma_3 \vdash \Delta_2, ? \Delta_2, \diamond \Delta_3} ?_w, !_w, \square_w, \diamond_w$$

Moreover, the weakening on \diamond (and dually on \square) is necessary to preserve the cut-elimination property, as the sequent $\vdash \diamond \perp, 1$ is provable with (cut) and without (\diamond_w):

$$\frac{\frac{\vdash 1}{\vdash 1, ? \diamond \perp} 1 \quad \frac{\diamond \perp \vdash \diamond \perp}{? \diamond \perp \vdash \diamond \perp} \text{ax}}{\vdash \diamond \perp, 1} ?_w \quad \frac{\vdash 1 \quad \diamond \perp \vdash \diamond \perp}{? \diamond \perp \vdash \diamond \perp} ?_p}{\vdash \diamond \perp, 1} \text{cut}$$

but is unprovable without (cut) and without (\diamond_w) as we cannot apply any rules on such a sequent. Similarly, the $(?_c/?_p)$ key-case naturally asks to be able to contract \diamond -formulas.

$$\frac{\frac{\Gamma_1 \vdash ?C, ?C, \Delta_1}{\Gamma_1 \vdash ?C, \Delta_1} ?_c \quad \frac{! \Gamma_2, \square \Gamma_3, C \vdash ? \Delta_2, \diamond \Delta_3}{! \Gamma_2, \square \Gamma_3, ?C \vdash ? \Delta_2, \diamond \Delta_3} ?_p}{\Gamma_1, ! \Gamma_2, \square \Gamma_3 \vdash \Delta_1, ? \Delta_2, \diamond \Delta_3} \text{cut} \rightsquigarrow$$

$$\frac{\frac{\Gamma_1 \vdash ?C, ?C, \Delta_1}{\Gamma_1, ! \Gamma_2, \square \Gamma_3 \vdash ?C, \Delta_1, ? \Delta_2, \diamond \Delta_3} ?_p \quad \frac{! \Gamma_2, \square \Gamma_3, C \vdash ? \Delta_2, \diamond \Delta_3}{! \Gamma_2, \square \Gamma_3, ?C \vdash ? \Delta_2, \diamond \Delta_3} ?_p}{\Gamma_1, ! \Gamma_2, ! \Gamma_2, \square \Gamma_3, \square \Gamma_3 \vdash \Delta_1, ? \Delta_2, ? \Delta_2, \diamond \Delta_3, \diamond \Delta_3} \text{cut} \quad \frac{! \Gamma_2, \square \Gamma_3, C \vdash ? \Delta_2, \diamond \Delta_3}{! \Gamma_2, \square \Gamma_3, ?C \vdash ? \Delta_2, \diamond \Delta_3} ?_p}{\Gamma_1, ! \Gamma_2, ! \Gamma_2, \square \Gamma_3, \square \Gamma_3 \vdash \Delta_1, ? \Delta_2, ? \Delta_2, \diamond \Delta_3, \diamond \Delta_3} \diamond_c$$

$$\frac{\Gamma_1, ! \Gamma_2, ! \Gamma_2, \square \Gamma_3 \vdash \Delta_1, ? \Delta_2, ? \Delta_2, \diamond \Delta_3}{\Gamma_1, ! \Gamma_2, \square \Gamma_3 \vdash \Delta_1, ? \Delta_2, \diamond \Delta_3} ?_c, !_c$$

565 **C.3 Full (mcut)-reduction steps**

566 ► **Definition 16** (μMALL^∞ & μLL^∞ (mcut)-reduction steps). *Reduction steps of μMALL^∞*
 567 *are given in figures 9, 10 and 11.*

568 *Reduction steps of μLL^∞ are the reduction steps of μMALL^∞ together with the steps of*
 569 *figures 12, 13, 14, 15.*

570 **C.4 Details on $(-)^{\circ}$ -translation**

571 ► **Definition 17** (Translation of $\mu\text{LL}_{\square}^\infty$ into μLL^∞). *Translation of formula is defined*
 572 *inductively on the formula:*

- 573 ■ *Translations of \diamond and \square -formulas:* $(\diamond A)^{\circ} := ?A^{\circ}$ and $(\square A)^{\circ} := !A^{\circ}$.
 574 ■ *Translations of atomic and unit formulas and variables f :* $f^{\circ} := f$.
 575 ■ *Translations of other non-fixed-point connectives:* $c(A_1, \dots, A_n)^{\circ} := c(A_1^{\circ}, \dots, A_n^{\circ})$.
 576 ■ *Translations of fixed-point connectives are given by:* $(\delta X.F)^{\circ} := \delta X.F^{\circ}$ (with $\delta \in \{\mu, \nu\}$).

Translation of structural rules for modalities, (\diamond_c) , (\diamond_w) , (\square_c) and (\square_w) are respectively
 $(?_c)$, $(?_w)$, $(!_c)$ and $(!_w)$:

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \diamond A, \Delta} \diamond_w \rightsquigarrow^{\circ} \frac{\Gamma^{\circ} \vdash \Delta^{\circ}}{\Gamma^{\circ} \vdash ?A^{\circ}, \Delta^{\circ}} ?_w \quad \frac{\Gamma \vdash \diamond A, \diamond A, \Delta}{\Gamma \vdash \diamond A, \Delta} \diamond_c \rightsquigarrow^{\circ} \frac{\Gamma^{\circ} \vdash ?A^{\circ}, ?A^{\circ}, \Delta^{\circ}}{\Gamma^{\circ} \vdash ?A^{\circ}, \Delta^{\circ}} ?_c$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \square A \vdash \Delta} \square_w \rightsquigarrow^{\circ} \frac{\Gamma^{\circ} \vdash \Delta^{\circ}}{\Gamma^{\circ}, !A^{\circ} \vdash \Delta^{\circ}} !_w \quad \frac{\Gamma, \square A, \square A \vdash \Delta}{\Gamma, \square A \vdash \Delta} \square_c \rightsquigarrow^{\circ} \frac{\Gamma^{\circ}, !A^{\circ}, !A^{\circ} \vdash \Delta^{\circ}}{\Gamma^{\circ}, !A^{\circ} \vdash \Delta^{\circ}} !_c$$

Translation of the modal rules and promotion rules are given by:

$$\frac{\Gamma \vdash A, \Delta}{\square \Gamma \vdash \square A, \diamond \Delta} \square_p \rightsquigarrow \frac{\frac{\Gamma^{\circ} \vdash A^{\circ}, \Delta^{\circ}}{! \Gamma^{\circ} \vdash A^{\circ}, ? \Delta^{\circ}} !_d, ?_d}{! \Gamma^{\circ} \vdash !A^{\circ}, ? \Delta^{\circ}} !_p \quad \frac{! \Gamma_1, \square \Gamma_2 \vdash A, ? \Delta_1, \diamond \Delta_2}{! \Gamma_1, \square \Gamma_2 \vdash !A, ? \Delta_1, \diamond \Delta_2} !_{\diamond} \rightsquigarrow \frac{! \Gamma_1^{\circ}, ! \Gamma_2^{\circ} \vdash A^{\circ}, ? \Delta_1^{\circ}, ? \Delta_2^{\circ}}{! \Gamma_1^{\circ}, ! \Gamma_2^{\circ} \vdash !A^{\circ}, ? \Delta_1^{\circ}, ? \Delta_2^{\circ}} !_p$$

$$\frac{\Gamma, A \vdash \Delta}{\square \Gamma, \diamond A \vdash \diamond \Delta} \diamond_p \rightsquigarrow \frac{\frac{\Gamma^{\circ}, A^{\circ} \vdash \Delta^{\circ}}{! \Gamma^{\circ}, A^{\circ} \vdash ? \Delta^{\circ}} !_d, ?_d}{! \Gamma^{\circ}, ?A^{\circ} \vdash ? \Delta^{\circ}} !_p \quad \frac{! \Gamma_1, \square \Gamma_2, A \vdash ? \Delta_1, \diamond \Delta_2}{! \Gamma_1, \square \Gamma_2, ?A \vdash ? \Delta_1, \diamond \Delta_2} !_{\diamond} \rightsquigarrow \frac{! \Gamma_1^{\circ}, ! \Gamma_2^{\circ}, A^{\circ} \vdash ? \Delta_1^{\circ}, ? \Delta_2^{\circ}}{! \Gamma_1^{\circ}, ! \Gamma_2^{\circ}, !A^{\circ} \vdash ? \Delta_1^{\circ}, ? \Delta_2^{\circ}} !_p$$

577 *Translation of other inference rules (r) are (r) themselves.*

578 *Translation of pre-proofs are defined co-inductively using translations of rules.*

579 **C.5 Proof of lemma 2**

► **Lemma 7.** *Consider a $\mu\text{LL}_{\square}^\infty$ reduction step $\pi_0 \rightsquigarrow \pi_1$, there exist a finite number of μLL^∞*
proofs $\theta_0, \dots, \theta_n$ such that:

$$\pi_0^{\circ} = \theta_0 \rightarrow \theta_1 \rightarrow \dots \rightarrow \theta_{n-1} \rightarrow \theta_n = \pi_1^{\circ}.$$

Proof. Reductions from the non-exponential part of $\mu\text{LL}_{\square}^\infty$ ($\mathcal{E}, \leq_g, \leq_f, \leq_u$) translates easily to one step of reduction in μLL^∞ . The same is true for the exponential part except for the commutation of the modal rule. The translation of the left proof of it is of the form (we only do the case of \diamond_p , \square_p is similar):

$$\frac{\frac{\frac{\pi_1^{\circ}}{! \Gamma_1^{\circ} \vdash A_1^{\circ}, \Delta_1^{\circ}} !_d, ?_d}{! \Gamma_1^{\circ} \vdash !A_1^{\circ}, ? \Delta_1^{\circ}} !_p}{\Gamma_1^{\circ} \vdash !A_1^{\circ}, ? \Delta_1^{\circ}} \dots \frac{\frac{\frac{\pi_n^{\circ}}{\Gamma_n^{\circ} \vdash A_n^{\circ}, \Delta_n^{\circ}} !_d, ?_d}{! \Gamma_n^{\circ} \vdash !A_n^{\circ}, ? \Delta_n^{\circ}} !_p}{\Gamma_n^{\circ} \vdash !A_n^{\circ}, ? \Delta_n^{\circ}} \dots \frac{\frac{\frac{\pi_{n+1}^{\circ}}{! \Gamma_{n+1}^{\circ}, A_{n+1}^{\circ} \vdash \Delta_{n+1}^{\circ}} !_d, ?_d}{! \Gamma_{n+1}^{\circ}, A_{n+1}^{\circ} \vdash ? \Delta_{n+1}^{\circ}} ?_p}{! \Gamma_{n+1}^{\circ}, ?A_{n+1}^{\circ} \vdash ? \Delta_{n+1}^{\circ}} \dots \frac{\frac{\frac{\pi_{n+m}^{\circ}}{! \Gamma_{n+m}^{\circ}, A_{n+m}^{\circ} \vdash \Delta_{n+m}^{\circ}} !_d, ?_d}{! \Gamma_{n+m}^{\circ}, A_{n+m}^{\circ} \vdash ? \Delta_{n+m}^{\circ}} ?_p}{! \Gamma_{n+m}^{\circ}, ?A_{n+m}^{\circ} \vdash ? \Delta_{n+m}^{\circ}} \text{mcut}(\iota, \perp\!\!\!\perp)$$

580 We use a more general lemma:

$$\begin{array}{c}
\frac{\frac{\Gamma_1 \vdash A_1, \Delta_1}{\Gamma_1, \Gamma_2 \vdash A_1 \otimes A_2, \Delta_1, \Delta_2} \otimes_r \quad \frac{\Gamma_2 \vdash A_2, \Delta_2}{\Gamma_3, A_1 \otimes A_2 \vdash \Delta_3} \otimes_l \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t, \perp\!\!\!\perp) \rightsquigarrow \\
\frac{\Gamma_1 \vdash A_1, \Delta_1 \quad \Gamma_2 \vdash A_2, \Delta_2 \quad \Gamma_3, A_1, A_2 \vdash \Delta_3 \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t', \perp\!\!\!\perp) \\
\frac{\frac{\Gamma_1, A_1 \vdash \Delta_1}{\Gamma_1, \Gamma_2, A_1 \wp A_2 \vdash \Delta_1, \Delta_2} \wp_l \quad \frac{\Gamma_2, A_2 \vdash \Delta_2}{\Gamma_3 \wp A_1 \wp A_2, \Delta_3} \wp_r \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t, \perp\!\!\!\perp) \rightsquigarrow \\
\frac{\Gamma_1, A_1 \vdash \Delta_1 \quad \Gamma_2, A_2 \vdash \Delta_2 \quad \Gamma_3 \wp A_1, A_2, \Delta_3 \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t', \perp\!\!\!\perp) \\
\frac{\frac{\Gamma_1 \vdash A_1, \Delta_1}{\Gamma_1, \Gamma_2, A_1 \multimap A_2 \vdash \Delta_1, \Delta_2} \multimap_l \quad \frac{\Gamma_2, A_2 \vdash \Delta_2}{\Gamma_3 \vdash A_1 \multimap A_2, \Delta_3} \multimap_r \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t, \perp\!\!\!\perp) \rightsquigarrow \\
\frac{\Gamma_1 \vdash A_1, \Delta_1 \quad \Gamma_2, A_2 \vdash \Delta_2 \quad \Gamma_3, A_1 \vdash A_2, \Delta_3 \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t', \perp\!\!\!\perp) \\
\frac{\frac{\Gamma_1 \vdash \Delta_1}{\Gamma_1, 1 \vdash \Delta_1} 1_l \quad \frac{}{\vdash 1} 1_r \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t, \perp\!\!\!\perp) \rightsquigarrow \frac{\Gamma_1 \vdash \Delta_1 \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t', \perp\!\!\!\perp) \\
\frac{\frac{\Gamma_1 \vdash \Delta_1}{\Gamma_1 \vdash \perp, \Delta_1} \perp_r \quad \frac{}{\perp \vdash} \perp_l \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t, \perp\!\!\!\perp) \rightsquigarrow \frac{\Gamma_1 \vdash \Delta_1 \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t', \perp\!\!\!\perp) \\
\frac{\frac{\Gamma_1 \vdash F_i, \Delta_1}{\Gamma_1 \vdash F_1 \oplus F_2, \Delta_1} \oplus_r^i \quad \frac{\Gamma_2, F_1 \vdash \Delta_2 \quad \Gamma_2, F_2 \vdash \Delta_2}{\Gamma_2, F_1 \oplus F_2 \vdash \Delta_2} \oplus_l \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t, \perp\!\!\!\perp) \rightsquigarrow \\
\frac{\Gamma_1 \vdash F_i, \Delta_1 \quad \Gamma_2, F_i \vdash \Delta_2 \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t', \perp\!\!\!\perp) \\
\frac{\frac{\Gamma_1, F_i \vdash \Delta_1}{\Gamma_1, F_1 \& F_2 \vdash \Delta_1} \&_l^i \quad \frac{\Gamma_2 \vdash F_1, \Delta_2 \quad \Gamma_2 \vdash F_2, \Delta_2}{\Gamma_2 \vdash F_1 \& F_2, \Delta_2} \&_r \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t, \perp\!\!\!\perp) \rightsquigarrow \\
\frac{\Gamma_1, F_i \vdash \Delta_1 \quad \Gamma_2 \vdash F_i, \Delta_2 \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t', \perp\!\!\!\perp) \\
\frac{\frac{\Gamma_1, A \vdash \Delta_1}{\Gamma_1 \vdash A^\perp, \Delta_1} (-)_r^\perp \quad \frac{\Gamma_2 \vdash A, \Delta_2}{\Gamma_2, A^\perp \vdash \Delta_2} (-)_l^\perp \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t, \perp\!\!\!\perp) \rightsquigarrow \\
\frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2 \vdash A, \Delta_2 \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t', \perp\!\!\!\perp) \\
\frac{}{\Gamma \vdash A} \text{ax} \quad \mathcal{C} \text{mcut}(t, \perp\!\!\!\perp) \rightsquigarrow \frac{\mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t', \perp\!\!\!\perp) \\
\frac{\frac{\Gamma_1 \vdash F, \Delta_1 \quad \Gamma_2, F \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{cut} \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t, \perp\!\!\!\perp) \rightsquigarrow \\
\frac{\Gamma_1 \vdash F, \Delta_1 \quad \Gamma_2, F \vdash \Delta_2 \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t', \perp\!\!\!\perp) \\
\frac{\frac{\Gamma_1, F[X := \delta X.F] \vdash \Delta_1}{\Gamma_1, \delta X.F \vdash \Delta_1} \delta_l \quad \frac{\Gamma_2 \vdash F[X := \delta X.F], \Delta_2}{\Gamma_1 \vdash \delta X.F, \Delta_1} \delta_r \quad \mathcal{C}}{\Gamma \vdash \Delta} \text{mcut}(t, \perp\!\!\!\perp) \rightsquigarrow \\
\frac{\Gamma_1, F[X := \delta X.F] \vdash \Delta_1 \quad \Gamma_2 \vdash F[X := \delta X.F], \Delta_2 \quad \mathcal{C}}{\Gamma_1, \delta X.F \vdash \Delta_1 \quad \Gamma_1 \vdash \delta X.F, \Delta_1} \delta_l \quad \delta_r \text{mcut}(t', \perp\!\!\!\perp) \\
\delta \in \{\mu, \nu\}
\end{array}$$

■ **Figure 9** Principal (mcut)-step of μMALL^∞

$$\begin{array}{c}
\frac{}{A \vdash A} \text{ax} \quad \text{mcut}(\iota, \perp\!\!\perp) \rightsquigarrow \frac{}{A \vdash A} \text{ax} \\
\frac{}{\vdash 1} 1_r \quad \text{mcut}(\iota, \perp\!\!\perp) \rightsquigarrow \frac{}{\vdash 1} 1_r \\
\frac{}{\perp \vdash} \perp_l \quad \text{mcut}(\iota, \perp\!\!\perp) \rightsquigarrow \frac{}{\perp \vdash} \perp_l \\
\frac{\frac{\Gamma \vdash A_1, A_2, \Delta}{\Gamma \vdash A_1 \wp A_2, \Delta} \wp_r \quad \mathcal{C}}{\Gamma' \vdash A_1 \wp A_2, \Delta'} \text{mcut}(\iota, \perp\!\!\perp) \rightsquigarrow \frac{\frac{\Gamma \vdash A_1, A_2, \Delta}{\Gamma' \vdash A_1, A_2, \Delta'} \mathcal{C}}{\Gamma' \vdash A_1 \wp A_2, \Delta'} \text{mcut}(\iota', \perp\!\!\perp') \\
\frac{\frac{\Gamma, A_1, A_2 \vdash \Delta}{\Gamma, A_1 \otimes A_2 \vdash \Delta} \otimes_l \quad \mathcal{C}}{\Gamma', A_1 \otimes A_2 \vdash \Delta'} \text{mcut}(\iota, \perp\!\!\perp) \rightsquigarrow \frac{\frac{\Gamma, A_1, A_2 \vdash \Delta}{\Gamma', A_1 \otimes A_2 \vdash \Delta'} \mathcal{C}}{\Gamma', A_1 \otimes A_2 \vdash \Delta'} \otimes_l \text{mcut}(\iota', \perp\!\!\perp') \\
\frac{\frac{\Gamma, A_1 \vdash A_2, \Delta}{\Gamma \vdash A_1 \multimap A_2, \Delta} \multimap_r \quad \mathcal{C}}{\Gamma' \vdash A_1 \multimap A_2, \Delta'} \text{mcut}(\iota, \perp\!\!\perp) \rightsquigarrow \frac{\frac{\Gamma, A_1 \vdash A_2, \Delta}{\Gamma', A_1 \vdash A_2, \Delta'} \mathcal{C}}{\Gamma' \vdash A_1 \multimap A_2, \Delta'} \multimap_r \text{mcut}(\iota', \perp\!\!\perp') \\
\frac{\frac{\Gamma', A \vdash \Delta'}{\Gamma' \vdash A^\perp, \Delta'} (-)_r^\perp \quad \mathcal{C}}{\Gamma \vdash A^\perp, \Delta} \text{mcut}(\iota, \perp\!\!\perp) \rightsquigarrow \frac{\frac{\Gamma', A \vdash \Delta'}{\Gamma, A \vdash \Delta} \mathcal{C}}{\Gamma \vdash A^\perp, \Delta} (-)_r^\perp \text{mcut}(\iota', \perp\!\!\perp') \\
\frac{\frac{\Gamma' \vdash A, \Delta'}{\Gamma', A^\perp \vdash \Delta'} (-)_l^\perp \quad \mathcal{C}}{\Gamma, A^\perp \vdash \Delta} \text{mcut}(\iota, \perp\!\!\perp) \rightsquigarrow \frac{\frac{\Gamma' \vdash A, \Delta'}{\Gamma \vdash A, \Delta} \mathcal{C}}{\Gamma, A^\perp \vdash \Delta} (-)_l^\perp \text{mcut}(\iota', \perp\!\!\perp') \\
\frac{\frac{\frac{\Gamma'_1 \vdash A_1, \Delta'_1 \quad \Gamma'_2 \vdash A_2, \Delta'_2}{\Gamma'_1, \Gamma'_2 \vdash A_1 \otimes A_2, \Delta'_1, \Delta'_2} \otimes_r \quad \mathcal{C}_{\Gamma'_1, \Delta'_1} \quad \mathcal{C}_{\Gamma'_2, \Delta'_2}}{\Gamma_1, \Gamma_2 \vdash A_1 \otimes A_2, \Delta_1, \Delta_2} \text{mcut}(\iota, \perp\!\!\perp) \rightsquigarrow \frac{\frac{\frac{\Gamma'_1 \vdash A_1, \Delta'_1}{\Gamma_1 \vdash A_1, \Delta_1} \mathcal{C}_{\Gamma'_1, \Delta'_1} \text{mcut}(\iota', \perp\!\!\perp') \quad \frac{\Gamma'_2 \vdash A_2, \Delta'_2}{\Gamma_2 \vdash A_2, \Delta_2} \mathcal{C}_{\Gamma'_2, \Delta'_2} \text{mcut}(\iota'', \perp\!\!\perp'')}{\Gamma_1, \Gamma_2 \vdash A_1 \otimes A_2, \Delta_1, \Delta_2} \otimes_r \\
\frac{\frac{\frac{\Gamma'_1, A_1 \vdash \Delta'_1 \quad \Gamma'_2, A_2 \vdash \Delta'_2}{\Gamma'_1, \Gamma'_2, A_1 \wp A_2 \vdash \Delta'_1, \Delta'_2} \wp_l \quad \mathcal{C}_{\Gamma'_1, \Delta'_1} \quad \mathcal{C}_{\Gamma'_2, \Delta'_2}}{\Gamma_1, \Gamma_2, A_1 \wp A_2 \vdash \Delta_1, \Delta_2} \text{mcut}(\iota, \perp\!\!\perp) \rightsquigarrow \frac{\frac{\frac{\Gamma'_1, A_1 \vdash \Delta'_1}{\Gamma_1, A_1 \vdash \Delta_1} \mathcal{C}_{\Gamma'_1, \Delta'_1} \text{mcut}(\iota', \perp\!\!\perp') \quad \frac{\Gamma'_2, A_2 \vdash \Delta'_2}{\Gamma_2, A_2 \vdash \Delta_2} \mathcal{C}_{\Gamma'_2, \Delta'_2} \text{mcut}(\iota'', \perp\!\!\perp'')}{\Gamma_1, \Gamma_2, A_1 \wp A_2 \vdash \Delta_1, \Delta_2} \wp_l \\
\frac{\frac{\frac{\Gamma'_1 \vdash A_1, \Delta'_1 \quad \Gamma'_2, A_2 \vdash \Delta'_2}{\Gamma'_1, \Gamma'_2, A_1 \multimap A_2 \vdash \Delta'_1, \Delta'_2} \multimap_l \quad \mathcal{C}_{\Gamma'_1, \Delta'_1} \quad \mathcal{C}_{\Gamma'_2, \Delta'_2}}{\Gamma_1, \Gamma_2, A_1 \multimap A_2 \vdash \Delta_1, \Delta_2} \text{mcut}(\iota, \perp\!\!\perp) \rightsquigarrow \frac{\frac{\frac{\Gamma'_1 \vdash A_1, \Delta'_1}{\Gamma_1 \vdash A_1, \Delta_1} \mathcal{C}_{\Gamma'_1, \Delta'_1} \text{mcut}(\iota', \perp\!\!\perp') \quad \frac{\Gamma'_2, A_2 \vdash \Delta'_2}{\Gamma_2, A_2 \vdash \Delta_2} \mathcal{C}_{\Gamma'_2, \Delta'_2} \text{mcut}(\iota'', \perp\!\!\perp'')}{\Gamma_1, \Gamma_2, A_1 \multimap A_2 \vdash \Delta_1, \Delta_2} \multimap_l
\end{array}$$

■ **Figure 10** Commutative (mcut)-step of the multiplicative fragment of MALL

$$\begin{array}{c}
\frac{\overline{\Gamma' \vdash \top, \Delta'} \top_r}{\Gamma \vdash \top, \Delta} \mathcal{C} \text{ mcut}(\iota, \perp\!\!\!\perp) \rightsquigarrow \overline{\Gamma \vdash \top, \Delta} \top_r \\
\frac{\overline{\Gamma', 0 \vdash \Delta'} 0_l}{\Gamma, 0 \vdash \Delta} \mathcal{C} \text{ mcut}(\iota, \perp\!\!\!\perp) \rightsquigarrow \overline{\Gamma, 0 \vdash \Delta} 0_l \\
\frac{\frac{\Gamma \vdash F', \Delta}{\Gamma \vdash F, \Delta} r}{\Gamma' \vdash F, \Delta'} \mathcal{C} \text{ mcut}(\iota, \perp\!\!\!\perp) \rightsquigarrow \frac{\frac{\Gamma \vdash F', \Delta}{\Gamma' \vdash F', \Delta'} \mathcal{C}}{\Gamma' \vdash F, \Delta'} r \text{ mcut}(\iota, \perp\!\!\!\perp) \quad (r \in \{\oplus_r^i, \mu_r, \nu_r\}) \\
\frac{\frac{\Gamma, F' \vdash \Delta}{\Gamma, F \vdash \Delta} r}{\Gamma', F \vdash \Delta'} \mathcal{C} \text{ mcut}(\iota, \perp\!\!\!\perp) \rightsquigarrow \frac{\frac{\Gamma, F' \vdash \Delta}{\Gamma', F' \vdash \Delta'} \mathcal{C}}{\Gamma', F \vdash \Delta'} r \text{ mcut}(\iota, \perp\!\!\!\perp) \quad (r \in \{\oplus_l^i, \mu_l, \nu_l\}) \\
\frac{\frac{\Gamma' \vdash A_1, \Delta' \quad \Gamma' \vdash A_2, \Delta'}{\Gamma' \vdash A_1 \& A_2, \Delta'} \&_r}{\Gamma \vdash A_1 \& A_2, \Delta} \mathcal{C} \text{ mcut}(\iota, \perp\!\!\!\perp) \rightsquigarrow \frac{\frac{\Gamma' \vdash A_1, \Delta'}{\Gamma \vdash A_1, \Delta} \mathcal{C} \text{ mcut}(\iota', \perp\!\!\!\perp') \quad \frac{\Gamma' \vdash A_2, \Delta'}{\Gamma \vdash A_2, \Delta} \mathcal{C} \text{ mcut}(\iota'', \perp\!\!\!\perp'')}{\Gamma \vdash A_1 \& A_2, \Delta} \&_r \\
\frac{\frac{\Gamma', A_1 \vdash \Delta' \quad \Gamma', A_2 \vdash \Delta'}{\Gamma', A_1 \oplus A_2 \vdash \Delta'} \oplus_l}{\Gamma, A_1 \oplus A_2 \vdash \Delta} \mathcal{C} \text{ mcut}(\iota, \perp\!\!\!\perp) \rightsquigarrow \frac{\frac{\Gamma', A_1 \vdash \Delta'}{\Gamma, A_1 \vdash \Delta} \mathcal{C} \text{ mcut}(\iota', \perp\!\!\!\perp') \quad \frac{\Gamma', A_2 \vdash \Delta'}{\Gamma, A_2 \vdash \Delta} \mathcal{C} \text{ mcut}(\iota'', \perp\!\!\!\perp'')}{\Gamma, A_1 \oplus A_2 \vdash \Delta} \oplus_l
\end{array}$$

■ **Figure 11** Commutative (mcut)-step of the additive fragment of μMALL^∞

$$\begin{array}{c}
\frac{\frac{\frac{\pi}{!\Gamma_1, \Box\Gamma_2 \vdash A, ?\Delta_1, \Diamond\Delta_2} \text{!}_p^{\Diamond} \quad \mathcal{C}'/\Box}{!\Gamma', \Box\Gamma'' \vdash !A, ?\Delta, \Diamond\Delta'} \text{mcut}(\iota, \perp\!\!\!\perp)}{\frac{\frac{\pi}{\Gamma \vdash A, \Delta} \Box_p \quad \mathcal{C}^\Box}{\Box\Gamma' \vdash \Box A, \Diamond\Delta'} \text{mcut}(\iota, \perp\!\!\!\perp)} \text{mcut}(\iota, \perp\!\!\!\perp)} \rightsquigarrow \frac{\frac{\frac{\pi}{!\Gamma_1, \Box\Gamma_2 \vdash A, ?\Delta_1, \Diamond\Delta_2} \mathcal{C}'/\Box \quad \text{mcut}(\iota, \perp\!\!\!\perp)}{!\Gamma', \Box\Gamma'' \vdash A, ?\Delta, \Diamond\Delta'} \text{!}_p^{\Diamond}}{!\Gamma', \Box\Gamma'' \vdash !A, ?\Delta, \Diamond\Delta'} \text{mcut}(\iota, \perp\!\!\!\perp)} \\
\rightsquigarrow \frac{\frac{\frac{\pi}{\Gamma \vdash A, \Delta} \mathcal{C} \quad \text{mcut}(\iota, \perp\!\!\!\perp)}{\Gamma' \vdash A, \Delta'} \Box_p}{\Box\Gamma' \vdash \Box A, \Diamond\Delta'} \text{mcut}(\iota, \perp\!\!\!\perp)} \\
\rightsquigarrow \frac{\frac{\frac{\pi}{\Gamma \vdash \Delta} \mathcal{C} \quad \text{mcut}(\iota', \perp\!\!\!\perp')}{\Gamma' \vdash \Delta'} \delta_w}{\Gamma' \vdash \delta A, \Delta'} \text{mcut}(\iota, \perp\!\!\!\perp)} \\
\rightsquigarrow \frac{\frac{\frac{\pi}{\Gamma \vdash \delta A, \delta A, \Delta} \mathcal{C} \quad \text{mcut}(\iota, \perp\!\!\!\perp)}{\Gamma \vdash \delta A, \Delta} \delta_c}{\Gamma' \vdash \delta A, \Delta'} \text{mcut}(\iota, \perp\!\!\!\perp)} \\
\rightsquigarrow \frac{\frac{\frac{\pi}{\Gamma \vdash A, \Delta} \text{?}_d \quad \mathcal{C} \quad \text{mcut}(\iota, \perp\!\!\!\perp)}{\Gamma \vdash ?A, \Delta} \text{?}_d}{\Gamma' \vdash ?A, \Delta'} \text{mcut}(\iota, \perp\!\!\!\perp)} \\
\delta \in \{?, \Diamond\}
\end{array}$$

■ **Figure 12** First side of commutative cut-elimination steps of $\mu\text{LL}_{\Box}^{\infty}$

$$\begin{array}{c}
\frac{\frac{\frac{\pi}{!\Gamma_1, \Box\Gamma_2, A \vdash ?\Delta_1, \Diamond\Delta_2} \text{?}_p^{\Box} \quad \mathcal{C}'/\Box}{!\Gamma', \Box\Gamma'', ?A \vdash ?\Delta, \Diamond\Delta'} \text{mcut}(\iota, \perp\!\!\!\perp)}{\frac{\frac{\frac{\pi}{\Gamma, a \vdash \Delta} \Diamond_p \quad \mathcal{C}^\Box}{\Box\Gamma, \Diamond A \vdash \Diamond\Delta} \text{mcut}(\iota, \perp\!\!\!\perp)}{\Box\Gamma', \Diamond A \vdash \Diamond\Delta'} \text{mcut}(\iota, \perp\!\!\!\perp)} \text{mcut}(\iota, \perp\!\!\!\perp)} \rightsquigarrow \frac{\frac{\frac{\pi}{!\Gamma_1, \Box\Gamma_2, A \vdash ?\Delta_1, \Diamond\Delta_2} \mathcal{C}'/\Box \quad \text{mcut}(\iota, \perp\!\!\!\perp)}{!\Gamma', \Box\Gamma'', A \vdash ?\Delta, \Diamond\Delta'} \text{!}_p^{\Box}}{!\Gamma', \Box\Gamma'', ?A \vdash ?\Delta, \Diamond\Delta'} \text{mcut}(\iota, \perp\!\!\!\perp)} \\
\rightsquigarrow \frac{\frac{\frac{\pi}{\Gamma, A \vdash \Delta} \mathcal{C} \quad \text{mcut}(\iota, \perp\!\!\!\perp)}{\Gamma', A \vdash \Delta'} \Diamond_p}{\Box\Gamma', \Diamond A \vdash \Diamond\Delta'} \text{mcut}(\iota, \perp\!\!\!\perp)} \\
\rightsquigarrow \frac{\frac{\frac{\pi}{\Gamma \vdash \Delta} \mathcal{C} \quad \text{mcut}(\iota', \perp\!\!\!\perp')}{\Gamma' \vdash \Delta'} \delta_w}{\Gamma', \delta A \vdash \Delta'} \text{mcut}(\iota, \perp\!\!\!\perp)} \\
\rightsquigarrow \frac{\frac{\frac{\pi}{\Gamma, \delta A, \delta A \vdash \Delta} \mathcal{C} \quad \text{mcut}(\iota, \perp\!\!\!\perp)}{\Gamma, \delta A \vdash \Delta} \delta_c}{\Gamma', \delta A \vdash \Delta'} \text{mcut}(\iota, \perp\!\!\!\perp)} \\
\rightsquigarrow \frac{\frac{\frac{\pi}{\Gamma \vdash A, \Delta} \text{!}_d \quad \mathcal{C} \quad \text{mcut}(\iota, \perp\!\!\!\perp)}{\Gamma, !A \vdash \Delta} \text{!}_d}{\Gamma', !A \vdash \Delta'} \text{mcut}(\iota, \perp\!\!\!\perp)} \\
\delta \in \{!, \Box\}
\end{array}$$

■ **Figure 13** Second side of commutative cut-elimination steps of $\mu\text{LL}_{\Box}^{\infty}$

581 ► **Lemma 8.** *Let $n \in \mathbb{N}$, let $d_1, \dots, d_n \in \mathbb{N}$ and let $p_1, \dots, p_n \in \{0, 1\}$. Let π be a μLL^∞ -proof
582 concluded by an (mcut)-rule, on top of which there is a list of n proofs π_1, \dots, π_n . We ask
583 for each π_i to be of one of the following forms depending on p_i :*

584 ■ *If $p_i = 1$, the $d_i + 1$ last rules of π_i are d_i derelictions ($(?_d)$ or $(!_d)$) and then a promotion
585 rule ($(!_p)$ or $(?_p)$). We ask for the principal formula of this promotion to be either a
586 formula of the conclusion, or to be cut with a formula being principal in a proof π_j on
587 one of the last $d_j + p_j$ rules.*

588 ■ *If $p_i = 0$, the d_i last rules of π_i are d_i derelictions.*

589 *In each of these two cases, we ask for π_i that each principal formulas of the d_i derelictions to
590 be either a formula of the conclusion of the multicut, either a cut-formula being cut with a
591 formula appearing in π_j such that $p_j = 1$. We prove that π reduces through a finite number
592 of mcut-reductions to a proof where each of the last $d_i + p_i$ rules either were eliminated by a
593 $(!_p/_!_d)$ -principal case, a $(?_p/wnde)$ -principal case or were commuted below the cut.*

594 **Proof.** We prove the property by induction on the sum of all the d_i and of all the p_i :

595 ■ (Initialization). As the sum of the d_i and p_i is 0, all d_i and p_i are equal to 0, meaning
596 that our statement is vacuously true.

597 ■ (Heredity). We have several cases:

598 ■ If the last rule of a proof π_i is a promotion or a dereliction for which the principal
599 formula is in the conclusion of the (mcut), we do a commutation step on this rule
600 obtaining π' . We apply our induction hypothesis on the proof ending with the (mcut);
601 and with parameters d'_1, \dots, d'_n as well as p'_1, \dots, p'_n and proofs π'_1, \dots, π'_n . To describe
602 these parameters we have two cases:

603 * If the rule is a promotion. We take for each $j \in \llbracket 1, n \rrbracket$, $d'_j = d_j$; $p'_j = p_j$ if $j \neq i$,
604 $p'_i = 0$; $\pi'_j = \pi_j$ if $j \neq i$.

605 * If the rule is a dereliction. We take for each $j \in \llbracket 1, n \rrbracket$, $d'_j = d_j$ if $j \neq i$, $d'_i = d_i - 1$;
606 $p'_j = p_j$.

607 The π'_j will be the hypotheses of the (mcut) of π'' . Note that $\sum d'_j + \sum p'_j =$
608 $\sum d_j + \sum p_j - 1$ meaning that we can apply our induction hypothesis. Combining our
609 reduction step with the reduction steps of the induction hypothesis, we obtain the
610 desired result.

611 ■ If there are no rules from the conclusion but that one π_i ends with $d_i > 0$ and
612 $p_i = 0$, meaning that the proof ends by a dereliction on a formula F . This means that
613 there is proof π_j such that $p_j = 1$ and such that F is cut with one of the formula
614 of π_j . As $p_j = 1$, F is the principal formula of the last rule applied on π_j . We
615 therefore can perform an promotion/dereliction principal case on the last rules from
616 π_i and π_j , leaving us with a proof π' with an (mcut) as conclusion. We apply the
617 induction hypothesis on this proof with parameters $d'_1 = d_1, \dots, d'_i = d_i - 1, \dots, d'_n = d_n$,
618 $p'_1 = p_1, \dots, p'_j = p_j - 1, \dots, p'_n = p_n$ and with the proofs being the hypotheses of
619 the multicut. Combining our steps with the steps from the induction hypotheses, we
620 obtain the desired result.

621 ■ We will show that the case where there are no rules from the conclusion and that no π_i
622 are such that $d_i > 0$ and $p_i = 0$, is impossible. Supposing, for the sake of contradiction,
623 that this case is possible. We will construct an infinite sequence of proofs $(\theta_i)_{i \in \mathbb{N}}$
624 all different and all being hypotheses of the multi-cut, which is impossible. We know
625 that there exist a proof $\theta_0 := \pi_j$ ending with a promotion on a formula F and that
626 this formula is not a formula from the conclusion. This proof is in relation by the
627 $\perp\!\!\!\perp$ -relation to another proof $\theta_1 := \pi_{j'}$. We know that this proof cannot be π_j because
628 the $\perp\!\!\!\perp$ -relation extended to sequents is acyclic. This proof also ends with a promotion

$$\begin{array}{c}
\frac{\Delta \vdash F, \Gamma}{\Box \Delta \vdash \Box F, \Diamond \Gamma} \Box_p \rightsquigarrow \frac{\frac{\frac{\frac{\Delta^\bullet \vdash ?F^\bullet, ?\Gamma^\bullet}{!?\Delta^\bullet \vdash ?F^\bullet, ?\Gamma^\bullet} !_d, ?_p}{!?\Delta^\bullet \vdash !?F^\bullet, ?\Gamma^\bullet} !_p}{\Box !?\Delta^\bullet \vdash \Box !?F^\bullet, \Diamond ?\Gamma^\bullet} \Box_p}{\frac{\frac{\frac{\Box !?\Delta^\bullet \vdash \Box !?F^\bullet, \Diamond ?\Gamma^\bullet}{! \Box !?\Delta^\bullet \vdash \Box !?F^\bullet, \Diamond ?\Gamma^\bullet} !_d}{! \Box !?\Delta^\bullet \vdash ?! \Box !?F^\bullet, ?! \Diamond ?\Gamma^\bullet} ?_d, !_p} \\
\frac{\Delta, F \vdash \Gamma}{\Box \Delta, \Diamond F \vdash \Diamond \Gamma} \Diamond_p \rightsquigarrow \frac{\frac{\frac{\frac{\Delta^\bullet, F^\bullet \vdash ?\Gamma^\bullet}{!?\Delta^\bullet, F^\bullet \vdash ?\Gamma^\bullet} !_d, ?_p}{!?\Delta^\bullet, ?F^\bullet \vdash ?\Gamma^\bullet} ?_p}{\Box !?\Delta^\bullet, \Diamond ?F^\bullet \vdash \Diamond ?\Gamma^\bullet} \Diamond_p}{\frac{\frac{\frac{\Box !?\Delta^\bullet, \Diamond ?F^\bullet \vdash \Diamond ?\Gamma^\bullet}{! \Box !?\Delta^\bullet, ! \Diamond ?F^\bullet \vdash \Diamond ?\Gamma^\bullet} !_d}{! \Box !?\Delta^\bullet, ! \Diamond ?F^\bullet \vdash ?! \Diamond ?\Gamma^\bullet} ?_d, !_p} \\
\frac{\frac{\pi}{\Gamma \vdash F[X := \delta X.F], \Delta} \delta_r}{\Gamma \vdash \delta X.F, \Delta} \delta_r \rightsquigarrow \frac{\frac{\frac{\pi^\bullet}{\Gamma^\bullet \vdash ?F^\bullet[X := \delta X.?F^\bullet], ?\Delta^\bullet} \delta_r}{\Gamma^\bullet \vdash \delta X.?F^\bullet, ?\Delta^\bullet} ?_d, !_p}{\Gamma^\bullet \vdash ?!(\delta X.?F^\bullet), ?\Delta^\bullet} \\
\frac{\frac{\pi}{\Gamma, F[X := \delta X.F] \vdash \Delta} \delta_l}{\Gamma, \delta X.F \vdash \Delta} \delta_l \rightsquigarrow \frac{\frac{\frac{\frac{\pi^\bullet}{\Gamma^\bullet, F^\bullet[X := \delta X.?F^\bullet] \vdash ?\Delta^\bullet} ?_p}{\Gamma^\bullet, ?F^\bullet[X := \delta X.?F^\bullet] \vdash ?\Delta^\bullet} \delta_l}{\frac{\frac{\Gamma^\bullet, \delta X.?F^\bullet \vdash ?\Delta^\bullet}{\Gamma^\bullet, !(\delta X.?F^\bullet) \vdash ?\Delta^\bullet} !_d}
\end{array}$$

with $\delta \in \{\mu, \nu\}$

■ **Figure 16** Translation of the fixed-point and modal fragment of rules of $\mu\text{LK}_{\Box}^\infty$ into $\mu\text{LL}_{\Box}^\infty$

629 on a principal formula which is not from the conclusion. By repeating this process, we
630 obtain the desired sequence $(\theta_i)_{i \in \mathbb{N}}$, giving us a contradiction.

631 The statement is therefore true by induction ◀

632 We apply this result on this proof with all the p_i being equal to 1 and with $d_i = \#(\Delta_i) + \#(\Gamma_i)$.
633 We can easily check that the conditions of the lemma are satisfied. Moreover, we notice that
634 there will be only one promotion rule commuting under the cut and that it commutes before
635 any dereliction, giving us one translation of the functorial promotion. ◀

636 **D** Appendix on the section 5

637 **D.1** Linear translation of rules

638 ▶ **Definition 18** ($(-)^{\bullet}$ -translation of rules). *The translation into $\mu\text{LL}_{\Box}^\infty$ of rules of $\mu\text{LK}_{\Box}^\infty$ are*
639 *depicted in figure 16, 17 and 18.*

640 **D.2** Details on SK(-)-translation

641 ▶ **Definition 19** ($\mu\text{LK}_{\Box}^\infty$ -Skeleton). *We define $\mu\text{LK}_{\Box}^\infty$ -skeleton on the $\otimes, \wp, \perp, 1$ -free fragment*
642 *of $\mu\text{LL}_{\Box}^\infty$ formula. We define this translation inductively as follows ($\delta \in \{\mu, \nu\}$):*

$$\begin{array}{c}
\frac{\pi}{\Gamma, F_1 \vdash F_2, \Delta} \rightarrow_r \rightsquigarrow \frac{\frac{\frac{\Gamma^\bullet, F_1^\bullet \vdash F_2^\bullet, ?\Delta^\bullet}{\Gamma^\bullet, ?F_1^\bullet \vdash ?F_2^\bullet, ?\Delta^\bullet} ?_p, ?_d}{\Gamma^\bullet \vdash ?F_1^\bullet \multimap ?F_2^\bullet, ?\Delta^\bullet} \multimap_r}{\Gamma^\bullet \vdash !(?F_1^\bullet \multimap ?F_2^\bullet), ?\Delta^\bullet} ?_d, !_p \\
\\
\frac{\frac{\pi_1}{\Gamma_1 \vdash F_1, \Delta_1} \quad \frac{\pi_2}{\Gamma_2, F_2 \vdash \Delta_2}}{\Gamma_1, \Gamma_2, F_1 \rightarrow F_2 \vdash \Delta_1, \Delta_2} \rightarrow_r \rightsquigarrow \frac{\frac{\frac{\frac{\Gamma_1^\bullet \vdash ?F_1^\bullet, ?\Delta_2^\bullet}{\Gamma_1^\bullet, \Gamma_2^\bullet, ?F_1^\bullet \multimap ?F_2^\bullet \vdash ?\Delta_1^\bullet, ?\Delta_2^\bullet} \multimap_r}{\Gamma_1^\bullet, \Gamma_2^\bullet, !(?F_1^\bullet \multimap ?F_2^\bullet) \vdash ?\Delta_1^\bullet, ?\Delta_2^\bullet} !_d}{\Gamma^\bullet \vdash ?F_1^\bullet, ?\Delta_2^\bullet} \frac{\frac{\frac{\Gamma_2^\bullet, F_2^\bullet \vdash ?\Delta_2^\bullet}{\Gamma_2^\bullet, ?F_2^\bullet \vdash ?\Delta_2^\bullet} ?_p}{\Gamma_2^\bullet, ?F_2^\bullet \vdash ?\Delta_2^\bullet} \multimap_r}{\Gamma^\bullet \vdash ?F_2^\bullet, ?\Delta_2^\bullet} ?_p}{\Gamma^\bullet \vdash !(?F_1^\bullet \multimap ?F_2^\bullet), ?\Delta_2^\bullet} !_d \\
\\
\frac{\frac{\pi_1}{\Gamma \vdash F_1, \Delta} \quad \frac{\pi_2}{\Gamma \vdash F_2, \Delta}}{\Gamma \vdash F_1 \wedge F_2, \Delta} \wedge_r \rightsquigarrow \frac{\frac{\frac{\Gamma^\bullet \vdash ?F_1^\bullet, ?\Delta^\bullet}{\Gamma^\bullet \vdash ?F_1^\bullet \& ?F_2^\bullet, ?\Delta^\bullet} \&_r}{\Gamma^\bullet \vdash !(?F_1^\bullet \& ?F_2^\bullet), ?\Delta^\bullet} ?_d, !_p}{\Gamma^\bullet \vdash ?F_1^\bullet, ?\Delta^\bullet} \frac{\frac{\frac{\Gamma^\bullet \vdash ?F_2^\bullet, ?\Delta^\bullet}{\Gamma^\bullet \vdash ?F_2^\bullet \& ?F_2^\bullet, ?\Delta^\bullet} \&_r}{\Gamma^\bullet \vdash ?F_2^\bullet, ?\Delta^\bullet} ?_d, !_p}{\Gamma^\bullet \vdash ?F_1^\bullet \& ?F_2^\bullet, ?\Delta^\bullet} \&_r \\
\\
\frac{\pi}{\Gamma, F_i \vdash \Delta} \wedge_l^i \rightsquigarrow \frac{\frac{\frac{\frac{\Gamma^\bullet, F_i^\bullet \vdash ?\Delta^\bullet}{\Gamma^\bullet, ?F_i^\bullet \vdash ?\Delta^\bullet} ?_p}{\Gamma^\bullet, ?F_1^\bullet \& ?F_2^\bullet \vdash ?\Delta^\bullet} \&_l^i}{\Gamma^\bullet, !(?F_1^\bullet \& ?F_2^\bullet) \vdash ?\Delta^\bullet} !_d}{\Gamma^\bullet, ?F_1^\bullet \& ?F_2^\bullet \vdash ?\Delta^\bullet} \&_l^i \\
\\
\frac{\frac{\pi_1}{\Gamma, F_1 \vdash \Delta} \quad \frac{\pi_2}{\Gamma, F_2 \vdash \Delta}}{\Gamma, F_1 \vee F_2 \vdash \Delta} \vee_l \rightsquigarrow \frac{\frac{\frac{\frac{\Gamma^\bullet, F_1^\bullet \vdash ?\Delta^\bullet}{\Gamma^\bullet, ?F_1^\bullet \vdash ?\Delta^\bullet} ?_p}{\Gamma^\bullet, ?F_1^\bullet \oplus ?F_2^\bullet \vdash ?\Delta^\bullet} \oplus_l}{\Gamma^\bullet, !(?F_1^\bullet \oplus ?F_2^\bullet) \vdash ?\Delta^\bullet} !_d}{\Gamma^\bullet, ?F_1^\bullet \oplus ?F_2^\bullet \vdash ?\Delta^\bullet} \frac{\frac{\frac{\Gamma^\bullet, F_2^\bullet \vdash ?\Delta^\bullet}{\Gamma^\bullet, ?F_2^\bullet \vdash ?\Delta^\bullet} ?_p}{\Gamma^\bullet, ?F_2^\bullet \vdash ?\Delta^\bullet} \oplus_l}{\Gamma^\bullet, ?F_1^\bullet \oplus ?F_2^\bullet \vdash ?\Delta^\bullet} \oplus_l}{\Gamma^\bullet, ?F_1^\bullet \oplus ?F_2^\bullet \vdash ?\Delta^\bullet} \oplus_l \\
\\
\frac{\pi}{\Gamma \vdash F_i, \Delta} \vee_r^i \rightsquigarrow \frac{\frac{\frac{\Gamma^\bullet \vdash ?F_i^\bullet, ?\Delta^\bullet}{\Gamma^\bullet \vdash ?F_1^\bullet \oplus ?F_2^\bullet, ?\Delta^\bullet} \oplus_r^i}{\Gamma^\bullet \vdash !(?F_1^\bullet \oplus ?F_2^\bullet), ?\Delta^\bullet} ?_d, !_p}{\Gamma^\bullet \vdash ?F_1^\bullet \oplus ?F_2^\bullet, ?\Delta^\bullet} \oplus_r^i
\end{array}$$

■ **Figure 17** Translation of LK connective-rules of $\mu\text{LK}_\square^\infty$ into $\mu\text{LL}_\square^\infty$

$$\begin{array}{c}
\frac{}{F \vdash F} \text{ax} \rightsquigarrow \frac{}{F^\bullet \vdash F^\bullet} \text{ax} \\
\frac{}{F^\bullet \vdash ?F^\bullet} ?_d
\\
\frac{\frac{\pi_1}{\Gamma_1 \vdash F, \Delta_1} \quad \frac{\pi_2}{\Gamma_2, F \vdash \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{cut} \rightsquigarrow \frac{\frac{\pi_1^\bullet}{\Gamma_1^\bullet \vdash ?F^\bullet, ?\Delta_1^\bullet} \quad \frac{\pi_2^\bullet}{\Gamma_2^\bullet, ?F^\bullet \vdash ?\Delta_2^\bullet} ?_p}{\Gamma_1^\bullet, \Gamma_2^\bullet \vdash \Delta_1^\bullet, \Delta_2^\bullet} \text{cut}
\\
\frac{}{\Gamma \vdash \top, \Delta} \top_r \rightsquigarrow \frac{}{\Gamma^\bullet \vdash \top, ?\Delta^\bullet} \top_r \\
\frac{}{\Gamma^\bullet \vdash ?!\top, ?\Delta^\bullet} ?_d, !_p
\\
\frac{}{\Gamma, F \vdash \Delta} F_l \rightsquigarrow \frac{}{\Gamma^\bullet, 0 \vdash ?\Delta^\bullet} 0_l \\
\frac{}{\Gamma^\bullet, !0 \vdash ?\Delta^\bullet} !_d
\\
\frac{\frac{\pi}{\Gamma_1, G, F, \Gamma_2 \vdash \Delta} \text{ex}_l}{\Gamma_1, F, G, \Gamma_2 \vdash \Delta} \rightsquigarrow \frac{\frac{\pi^\bullet}{\Gamma_1^\bullet, G^\bullet, F^\bullet, \Gamma_2^\bullet \vdash ?\Delta^\bullet} \text{ex}_l}{\Gamma_1^\bullet, F^\bullet, G^\bullet, \Gamma_2^\bullet \vdash ?\Delta^\bullet}
\\
\frac{\frac{\pi}{\Gamma \vdash \Delta_1, G, F, \Delta_2} \text{ex}_r}{\Gamma \vdash \Delta_1, F, G, \Delta_2} \rightsquigarrow \frac{\frac{\pi^\bullet}{\Gamma^\bullet \vdash ?\Delta_1^\bullet, ?G^\bullet, ?F^\bullet, ?\Delta_2^\bullet} \text{ex}_r}{\Gamma^\bullet \vdash ?\Delta_1^\bullet, ?F^\bullet, ?G^\bullet, ?\Delta_2^\bullet}
\\
\frac{\frac{\pi}{\Gamma \vdash \Delta} w_l}{\Gamma, F \vdash \Delta} \rightsquigarrow \frac{\frac{\pi^\bullet}{\Gamma^\bullet \vdash ?\Delta^\bullet}}{\Gamma^\bullet, F^\bullet \vdash ?\Delta^\bullet} !_w
\\
\frac{\frac{\pi}{\Gamma \vdash \Delta} w_r}{\Gamma \vdash F, \Delta} \rightsquigarrow \frac{\frac{\pi^\bullet}{\Gamma^\bullet \vdash ?\Delta^\bullet}}{\Gamma^\bullet \vdash ?F^\bullet, ?\Delta^\bullet} ?_w
\\
\frac{\frac{\pi}{\Gamma, F, F \vdash \Delta} c_l}{\Gamma, F \vdash \Delta} \rightsquigarrow \frac{\frac{\pi^\bullet}{\Gamma^\bullet, F^\bullet, F^\bullet \vdash ?\Delta^\bullet}}{\Gamma^\bullet, F^\bullet \vdash ?\Delta^\bullet} !_c
\\
\frac{\frac{\pi}{\Gamma \vdash F, F, \Delta} c_r}{\Gamma \vdash F, \Delta} \rightsquigarrow \frac{\frac{\pi^\bullet}{\Gamma^\bullet \vdash ?F^\bullet, ?F^\bullet, ?\Delta^\bullet} ?_c}{\Gamma^\bullet \vdash ?F^\bullet, ?\Delta^\bullet}
\\
\frac{\frac{\pi}{\Gamma \vdash F, \Delta} \perp_l}{\Gamma, F^\perp \vdash \Delta} \rightsquigarrow \frac{\frac{\pi^\bullet}{\Gamma^\bullet \vdash ?F^\bullet, ?\Delta^\bullet} \perp_l}{\Gamma^\bullet, (?F^\bullet)^\perp \vdash ?\Delta^\bullet} \perp_l \\
\frac{}{\Gamma^\bullet, !((?F^\bullet)^\perp) \vdash ?\Delta^\bullet} !_d
\\
\frac{\frac{\pi}{\Gamma, F \vdash \Delta} \perp_r}{\Gamma \vdash F^\perp, \Delta} \rightsquigarrow \frac{\frac{\frac{\pi^\bullet}{\Gamma^\bullet, F^\bullet \vdash ?\Delta^\bullet} ?_p}{\Gamma^\bullet, ?F^\bullet \vdash ?\Delta^\bullet} \perp_r}{\Gamma^\bullet \vdash ?!(?F^\bullet)^\perp, ?\Delta^\bullet} ?_d, !_p
\end{array}$$

■ **Figure 18** Translation of structural & unit fragment of $\mu\text{LK}_\square^\infty$ into $\mu\text{LL}_\square^\infty$

$$\begin{array}{llll}
 SK(F \& G) = SK(F) \wedge SK(G) & SK(F \oplus G) = SK(F) \vee SK(G) & SK(F^\perp) = SK(F)^\perp & \\
 SK(F \multimap G) = SK(F) \rightarrow SK(G) & SK(\Box F) = \Box SK(F) & SK(\Diamond F) = \Diamond SK(F) & \\
 643 \quad SK(a) = a & SK(\top) = \top & SK(0) = \text{F} & \\
 SK(?F) = SK(F) & SK(!F) = SK(F) & & \\
 SK(\mu X.F) = \mu X.SK(F) & SK(\nu X.F) = \nu X.SK(F) & SK(X) = X & \\
 644 \quad \mu\text{LL}_{\Box}^\infty \text{ sequents are translated to sequent of skeletons of these formulas. Rules are translated} & & & \\
 645 \quad \text{straightforwardly by forgetting the linear information, translation are given in figures 19 and} & & & \\
 646 \quad \text{20..} & & & \\
 647 \quad \text{Translations of pre-proofs are obtained co-inductively by applying rule translations.} & & &
 \end{array}$$

648 D.3 Details on proofs of lemma 5 and 6

649 ► **Lemma 9** (Robustness of the skeleton to validity). *If π is a $\mu\text{LL}_{\Box}^\infty$ valid pre-proof then*
 650 *$SK(\pi)$ is a $\mu\text{LK}_{\Box}^\infty$ valid pre-proof, and vice-versa.*

651 **Proof.** This comes from the fact that (i) minimal formula of a set of translated formulas
 652 is the translation of the minimal formula of the set of initial formulas; (ii) translations of
 653 branches contains all the translations of formulas of the initial branch and vice-versa. ◀

654 ► **Lemma 10** (Composition of $SK(-)$ and of $(-)^{\bullet}$). *Let π be a $\mu\text{LK}_{\Box}^\infty$ pre-proof. We have*
 655 *that $SK(\pi^{\bullet})$ is equal to π .*

656 **Proof.** This comes from the fact that $(-)^{\bullet}$ -translation translates each rules (r) of $\mu\text{LK}_{\Box}^\infty$ to
 657 a derivation containing the pre-image of (r) by the translation $SK()$, adding only exponential
 658 rules. As exponential rules disappears from the proof by $SK()$, we get that $SK(r^{\bullet})$ is equal
 659 to (r). We coinductively apply this result on pre-proofs. ◀

$$\begin{array}{c}
\frac{}{F \vdash F} \text{ax} \rightsquigarrow \frac{}{\text{SK}(F) \vdash \text{SK}(F)} \text{ax} \\
\frac{\Gamma_1 \vdash F, \Delta_1 \quad \Gamma_2, F \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{cut} \rightsquigarrow \frac{\text{SK}(\Gamma_1) \vdash \text{SK}(F), \text{SK}(\Delta_1) \quad \text{SK}(\Gamma_2), \text{SK}(F) \vdash \text{SK}(\Delta_2)}{\text{SK}(\Gamma_1), \text{SK}(\Gamma_2) \vdash \text{SK}(\Delta_1), \text{SK}(\Delta_2)} \text{cut} \\
\frac{\Gamma \vdash F, \Delta \quad \Gamma \vdash G, \Delta}{\Gamma \vdash F \& G, \Delta} \&_r \rightsquigarrow \frac{\text{SK}(\Gamma) \vdash \text{SK}(F), \text{SK}(\Delta) \quad \text{SK}(\Gamma) \vdash \text{SK}(G), \text{SK}(\Delta)}{\text{SK}(\Gamma) \vdash \text{SK}(F) \wedge \text{SK}(G), \text{SK}(\Delta)} \wedge_r \\
\frac{\Gamma, F \vdash \Delta}{\Gamma, F \& G \vdash \Delta} \&_l^1 \rightsquigarrow \frac{\text{SK}(\Gamma), \text{SK}(F) \vdash \text{SK}(\Delta)}{\text{SK}(\Gamma), \text{SK}(F) \wedge \text{SK}(G) \vdash \text{SK}(\Delta)} \wedge_l^1 \\
\frac{\Gamma, G \vdash \Delta}{\Gamma, F \& G \vdash \Delta} \&_l^2 \rightsquigarrow \frac{\text{SK}(\Gamma), \text{SK}(G) \vdash \text{SK}(\Delta)}{\text{SK}(\Gamma), \text{SK}(F) \wedge \text{SK}(G) \vdash \text{SK}(\Delta)} \wedge_l^2 \\
\frac{}{\Gamma \vdash \top, \Delta} \top_r \rightsquigarrow \frac{}{\text{SK}(\Gamma) \vdash \top, \text{SK}(\Delta)} \top_r \\
\frac{\Gamma, F \vdash \Delta \quad \Gamma, G \vdash \Delta}{\Gamma, F \oplus G \vdash \Delta} \oplus_l \rightsquigarrow \frac{\text{SK}(\Gamma), \text{SK}(F) \vdash \text{SK}(\Delta) \quad \text{SK}(\Gamma), \text{SK}(G) \vdash \text{SK}(\Delta)}{\text{SK}(\Gamma), \text{SK}(F) \vee \text{SK}(G) \vdash \text{SK}(\Delta)} \vee_l \\
\frac{\Gamma \vdash F, \Delta}{\Gamma \vdash F \oplus G, \Delta} \oplus_r^1 \rightsquigarrow \frac{\text{SK}(\Gamma) \vdash \text{SK}(F), \text{SK}(\Delta)}{\text{SK}(\Gamma) \vdash \text{SK}(F) \vee \text{SK}(G), \text{SK}(\Delta)} \vee_r^1 \\
\frac{\Gamma \vdash G, \Delta}{\Gamma \vdash F \oplus G, \Delta} \oplus_r^2 \rightsquigarrow \frac{\text{SK}(\Gamma) \vdash \text{SK}(G), \text{SK}(\Delta)}{\text{SK}(\Gamma) \vdash \text{SK}(F) \vee \text{SK}(G), \text{SK}(\Delta)} \vee_r^2 \\
\frac{}{\Gamma, 0 \vdash \Delta} 0_l \rightsquigarrow \frac{}{\text{SK}(\Gamma), F \vdash \text{SK}(\Delta)} F_l \\
\frac{\Gamma, F \vdash G, \Delta}{\Gamma \vdash F \multimap G, \Delta} \multimap_r \rightsquigarrow \frac{\text{SK}(\Gamma), \text{SK}(F) \vdash \text{SK}(G), \text{SK}(\Delta)}{\text{SK}(\Gamma) \vdash \text{SK}(F) \multimap \text{SK}(G), \text{SK}(\Delta)} \multimap_r \\
\frac{\Gamma_1 \vdash F, \Delta_1 \quad \Gamma_2, G \vdash \Delta_2}{\Gamma_1, \Gamma_2, F \multimap G \vdash \Delta_1, \Delta_2} \multimap_l \rightsquigarrow \frac{\text{SK}(\Gamma_1) \vdash \text{SK}(F), \text{SK}(\Delta_1) \quad \text{SK}(\Gamma_2), \text{SK}(G) \vdash \text{SK}(\Delta_2)}{\text{SK}(\Gamma_1), \text{SK}(\Gamma_2), \text{SK}(F) \multimap \text{SK}(G) \vdash \text{SK}(\Delta_1), \text{SK}(\Delta_2)} \multimap_l \\
\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^\perp, \Delta} \perp_l \rightsquigarrow \frac{\text{SK}(\Gamma) \vdash \text{SK}(A), \text{SK}(\Delta)}{\text{SK}(\Gamma), \text{SK}(A)^\perp \vdash \text{SK}(\Delta)} \perp_l \\
\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^\perp, \Delta} \perp_r \rightsquigarrow \frac{\text{SK}(\Gamma), \text{SK}(A) \vdash \text{SK}(\Delta)}{\text{SK}(\Gamma) \vdash \text{SK}(A)^\perp, \text{SK}(\Delta)} \perp_r \\
\frac{\Gamma, F[\mu X.F/X] \vdash \Delta}{\Gamma, \mu X.F \vdash \Delta} \mu_l \rightsquigarrow \frac{\text{SK}(\Gamma), \text{SK}(F)[\mu X.\text{SK}(F)/X] \vdash \text{SK}(\Delta)}{\text{SK}(\Gamma), \mu X.\text{SK}(F) \vdash \text{SK}(\Delta)} \mu_l \\
\frac{\Gamma \vdash F[\mu X.F/X], \Delta}{\Gamma \vdash \mu X.F, \Delta} \mu_r \rightsquigarrow \frac{\text{SK}(\Gamma) \vdash \text{SK}(F)[\mu X.\text{SK}(F)/X], \text{SK}(\Delta)}{\text{SK}(\Gamma) \vdash \mu X.\text{SK}(F), \text{SK}(\Delta)} \mu_r \\
\frac{\Gamma, F[\nu X.F/X] \vdash \Delta}{\Gamma, \nu X.F \vdash \Delta} \nu_l \rightsquigarrow \frac{\text{SK}(\Gamma), \text{SK}(F)[\nu X.\text{SK}(F)/X] \vdash \text{SK}(\Delta)}{\text{SK}(\Gamma), \nu X.\text{SK}(F) \vdash \text{SK}(\Delta)} \nu_l \\
\frac{\Gamma \vdash F[\nu X.F/X], \Delta}{\Gamma \vdash \nu X.F, \Delta} \nu_r \rightsquigarrow \frac{\text{SK}(\Gamma) \vdash \text{SK}(F)[\nu X.\text{SK}(F)/X], \text{SK}(\Delta)}{\text{SK}(\Gamma) \vdash \nu X.\text{SK}(F), \text{SK}(\Delta)} \nu_r
\end{array}$$

■ **Figure 19** SK(-)-translation of non-modal rules

$$\begin{array}{l}
\frac{\Gamma \vdash \Delta}{\Gamma, !F \vdash \Delta} !_w \rightsquigarrow \frac{\text{SK}(\Gamma) \vdash \text{SK}(\Delta)}{\text{SK}(\Gamma), \text{SK}(F) \vdash \text{SK}(\Delta)} w_l \\
\frac{\Gamma, !F, !F \vdash \Delta}{\Gamma, !F \vdash \Delta} !_c \rightsquigarrow \frac{\text{SK}(\Gamma), \text{SK}(F), \text{SK}(F) \vdash \text{SK}(\Delta)}{\text{SK}(\Gamma), \text{SK}(F) \vdash \text{SK}(\Delta)} c_l \\
\frac{\Gamma, F \vdash \Delta}{\Gamma, !F \vdash \Delta} !_d \rightsquigarrow \text{SK}(\Gamma), \text{SK}(F) \vdash \text{SK}(\Delta) \\
\frac{! \Gamma \vdash F, ? \Delta}{! \Gamma \vdash !F, ? \Delta} !_p \rightsquigarrow \text{SK}(\Gamma) \vdash \text{SK}(F), \text{SK}(\Delta) \\
\frac{\Gamma \vdash \Delta}{\Gamma \vdash ?F, \Delta} ?_w \rightsquigarrow \frac{\text{SK}(\Gamma) \vdash \text{SK}(\Delta)}{\text{SK}(\Gamma), \text{SK}(F) \vdash \text{SK}(\Delta)} w_r \\
\frac{\Gamma \vdash ?F, ?F, \Delta}{\Gamma \vdash ?F, \Delta} ?_c \rightsquigarrow \frac{\text{SK}(\Gamma) \vdash \text{SK}(F), \text{SK}(F), \text{SK}(\Delta)}{\text{SK}(\Gamma) \vdash \text{SK}(F), \text{SK}(\Delta)} c_r \\
\frac{\Gamma \vdash F, \Delta}{\Gamma \vdash ?F, \Delta} ?_d \rightsquigarrow \text{SK}(\Gamma), \text{SK}(F) \vdash \text{SK}(\Delta) \\
\frac{! \Gamma, F \vdash ? \Delta}{! \Gamma, ?F \vdash ? \Delta} ?_p \rightsquigarrow \text{SK}(\Gamma), \text{SK}(F) \vdash \text{SK}(\Delta) \\
\frac{\Gamma \vdash A, \Delta}{\Box \Gamma \vdash \Box A, \Diamond \Delta} \Box_p \rightsquigarrow \frac{\text{SK}(\Gamma) \vdash \text{SK}(A), \text{SK}(\Delta)}{\Box \text{SK}(\Gamma) \vdash \Box \text{SK}(A), \Diamond \text{SK}(\Delta)} \Box_p \\
\frac{\Gamma, A \vdash \Delta}{\Box \Gamma, \Diamond A \vdash \Diamond \Delta} \Diamond_p \rightsquigarrow \frac{\text{SK}(\Gamma), \text{SK}(A) \vdash \text{SK}(\Delta)}{\Box \text{SK}(\Gamma), \Diamond \text{SK}(A) \vdash \Diamond \text{SK}(\Delta)} \Diamond_p
\end{array}$$

■ **Figure 20** SK(-)-translation of modal rules of μLL^∞

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