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# Dynamical modeling of non-proportionally damped multibody systems using a modal Udwadia-Kalaba formulation based on complex modes of the dissipative subsystems

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## Abstract

This paper presents a modal Udwadia-Kalaba formulation based on complex modal analysis. This alternative formulation is developed in the general case of vibro-acoustic systems with an internal cavity. In a similar manner to the original formulation, the global response of a constrained multibody system is expressed as a sum of its unconstrained response and a corrective term allowing the enforcement of constraints. The use of complex modes of the dissipative substructures has the advantage of leading to a set of ordinary differential equations, regarding the unconstrained response, even in the case of non-classically damped substructures. Moreover, in the frame of experimental substructuring, the estimation of complex modal parameters of a state-space representation is more straightforward than for real ones of the equivalent second order model, hence this alternative formulation is of prac-

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tical interest in this context. An academic example with non-proportional damping allows to validate the proposed modal formulation. Then two experimentally identified models of a guitar and a harp from Central Africa are coupled to a stiff string whose modal parameters are known from the analytic theory. The guitar model contains an acoustic degree of freedom and allows to illustrate the suitability of the Udwadia-Kalaba formulation to model coupled vibro-acoustic substructures. The harp model includes highly complex modes which allows to highlight the advantage of expressing the coupled response of the global system in terms of its complex modal coordinates.

*Keywords:* Udwadia-Kalaba, Substructuring, Internal vibro-acoustics, Non-proportional damping, Complex modes

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## 1. Introduction

When performing the dynamical analysis of a structure, it may necessary to split it into substructures whose behavior is easier to characterize. Such situations might arise for example, in experimental configuration, when the complete structure is too large to be analyzed as a whole, or in a numerical case, when the number of degrees of freedom (dofs) leads to unreasonable computation time. This approach of splitting the problem into subparts is known as dynamic substructuring. When the global analysis presents no particular difficulty, dynamic substructuring may still be advantageous to conduct a parametric study componentwise or to build hybrid assemblies combining (discrete or continuous/analytical) model(s) and experimentally identified parts.

Constrained multibody systems have been studied for many years and

14 multiple methods were developed to deal with rigid bodies until recent decades.  
15 In the last 60 years, efforts have been put to take into account the elastic  
16 behavior of substructures thus further increasing the list of available formu-  
17 lations. de Klerk et al. [1] proposed a classification of these formulations in  
18 a general framework in which the structural dynamics are analyzed in three  
19 distinct domains: the physical, modal and frequency domains.

20 The physical domain includes the well known finite element method and  
21 others originally developed for rigid substructures, whose review was pro-  
22 posed by Lalausa and Bachau [2], such as Maggi's method [3, 4] and the  
23 Udwadia-Kalaba [5–9] (U-K) formulation. These methods require the knowl-  
24 edge of physical mass, stiffness and damping matrices for each substructure.  
25 In the frame of experimental substructuring, identification techniques [10]  
26 allow to estimate physical matrices but the damping matrix remains delicate  
27 to obtain, due to its high sensitivity to noise and inconsistencies in the data,  
28 making physical domain formulations unpractical.

29 On the other hand, frequency based substructuring (FBS) does not rely  
30 on the separate knowledge of mass, stiffness and damping properties. Instead,  
31 the dynamic behavior of the substructures is described by transfer functions  
32 which presents no particular difficulty to measure. Typical FBS methods  
33 are the admittance coupling [11, 12], the impedance coupling [13] and the  
34 Lagrange multipliers FBS (LM-FBS) coupling[14, 15]. Obviously, in order  
35 to transform the equations of motion from the time domain to the frequency  
36 domain, each substructure needs to be linear time invariant (LTI) and in  
37 steady-state which limits the case of applications.

38 Another option lies in the class of modal domain formulations, known

39 as component-mode synthesis (CMS). Although the use of CMS is not as  
40 straight forward as FBS for experimental substructuring, since it requires  
41 to perform modal analysis (and induces truncation effects due to the order  
42 reduction), it remains a useful solution for analyzing non-LTI multibody sys-  
43 tems. Methods based on all kinds of structural modes have been proposed  
44 during the last 60 years to which an overview was given by Craig [16]. The  
45 most widespread CMS formulations are the Craig-Bampton [17] (combina-  
46 tion of constraint modes and fixed-interface normal modes), MacNeal [18]  
47 and Rubin [19] (combination of attachment modes and free-interface normal  
48 modes) methods. In the last decade, with a view to modeling musical instru-  
49 ments as constrained multibody systems, Antunes and Debut [20] adapted  
50 the U-K formalism to continuous flexible systems using free-interface normal  
51 modes. The (modal) U-K formulation offers a compact and general solution  
52 (allowing for redundant and non-ideal constraints). The relevance of this  
53 new CMS method has been illustrated in several research works involving  
54 geometrical string non-linearities [21–23] and intermittent contacts [24].

55 Other promising approaches, developed for substructure-based system  
56 identification, constitute the class labeled state-space substructuring (SSS)  
57 which consists in assembling state-space models of substructures. Proposed  
58 in recent years, they can be linked to the physical or modal domain de-  
59 pending on the choice of state variables. Examples of such methods are the  
60 classical state-space substructuring (classical-SSS) method [12] and recently  
61 the Lagrange multipliers state-space substructuring (LM-SSS) method [25].

62 The present paper deals with an alternative modal U-K formulation al-  
63 lowing to simplify and optimize the dynamic substructuring of constrained

64 multibody systems with non-proportional damping, especially in the case of  
65 experimental substructuring. The use of complex modes of unconstrained  
66 dissipative substructures allows to always obtain a system of uncoupled Or-  
67 dinary Differential Equations (ODEs) with respect to complex modal coordi-  
68 nates, contrary to existing CMS methods which end up with a fully populated  
69 modal damping matrix in case of general viscous damping. Another devel-  
70 opment is found in the extension of the original U-K formulation to cover  
71 the case of vibro-acoustic substructures with an internal cavity. Indeed this  
72 type of problem is frequently encountered for example in musical acoustics  
73 or practical engineering [26], and the U-K formalism provides an elegant so-  
74 lution expressed in terms of eigenmodes of the unconstrained substructures.

75 The first section includes a brief recall of the original U-K formulation  
76 [5, 9], followed by an extension taking into account vibro-acoustic substruc-  
77 tures. Then, the modal domain adaptation developed by Antunes and Debut  
78 [20] is touched on as an intermediary step to present the novel modal U-K  
79 formulation based on complex modal analysis. In section 3, the validity  
80 of this new formulation is assessed through an application to an academic  
81 test case with non-proportional damping (also called non-classical damping).  
82 Then a practical example, from musical acoustics, based on the combination  
83 of analytical and experimental data allows to demonstrate the relevance of  
84 this formulation.

## 85 **2. Udwadia-Kalaba formalism**

86 In order to develop the newly proposed formulation, the original U-K  
87 formulation [5–9] is briefly recalled. This allows to introduce an extension

88 of the latter to subsystems involving internal vibro-acoustic problems. Then  
 89 the recently developed modal formulation [20] based on modal bases of asso-  
 90 ciated conservative subsystems is briefly covered for completeness. Finally,  
 91 an alternative modal formulation based on modal bases of under-damped dis-  
 92 sipative subsystems is presented and two solutions are proposed depending  
 93 on the presence/absence of acoustic dofs in the subsystems.

## 94 2.1. Physical space

### 95 2.1.1. Original formulation

96 Let  $\mathbf{y}_s(t)$  represent the response, and  $\mathbf{x}_s(t)$  the degrees of freedom, of a  
 97 discrete structural mechanical system of mass, damping and stiffness matrices  
 98  $\mathbf{M}_s$ ,  $\mathbf{C}_s$  and  $\mathbf{K}_s$ , which consists of  $J$  constrained subsystems via constraining  
 99 forces  $\mathbf{f}_c$ , subjected to external constraint-independent forces  $\mathbf{f}_{nc}$ . The U-K  
 100 formulation derives from the second order model

$$\begin{aligned} \mathbf{M}_s \ddot{\mathbf{x}}_s(t) + \mathbf{C}_s \dot{\mathbf{x}}_s(t) + \mathbf{K}_s \mathbf{x}_s(t) &= \mathbf{D}_s^T (\mathbf{f}_{nc}(\mathbf{x}_s, \dot{\mathbf{x}}_s, t) + \mathbf{f}_c(\mathbf{x}_s, \dot{\mathbf{x}}_s, t)) \\ \mathbf{y}_s(t) &= \mathbf{D}_s \mathbf{x}_s(t) \end{aligned} \quad (1)$$

101 with  $\mathbf{D}_s$  being the output (or input) shape matrix. While the original for-  
 102 mulation was obtained with  $\mathbf{D}_s$  being the identity matrix, thus assuming the  
 103 number of responses equals the number of dofs, it is relevant to develop it for  
 104 any arbitrary matrix (hence any choice of dofs) thus differentiating system  
 105 responses and dofs.

106 Constraining forces can be expressed through Lagrange multipliers [27]  $\lambda$  as

$$\mathbf{f}_c = -\mathbf{A}^T \lambda \quad (2)$$

107 where  $\mathbf{A}(\dot{\mathbf{x}}_s, \mathbf{x}_s, t)$  is the constraint matrix, associated with  $\mathbf{b}(\dot{\mathbf{x}}_s, \mathbf{x}_s, t)$  a vec-  
 108 tor function of the motion, corresponding to the following system of P holo-

109 nomic and non-holonomic constraints in terms of accelerations

$$\mathbf{A}\ddot{\mathbf{y}}_s = \mathbf{b} . \quad (3)$$

110 By combining Eqs. (1–3), the following augmented differential-algebraic  
111 equation (DAE) may be built

$$\begin{bmatrix} \mathbf{M}_s & (\mathbf{A}\mathbf{D}_s)^T \\ \mathbf{A}\mathbf{D}_s & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{x}}_s \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \mathbf{D}_s^T \mathbf{f}_{\text{nc}} - \mathbf{C}_s \dot{\mathbf{x}}_s - \mathbf{K}_s \mathbf{x}_s \\ \mathbf{b} \end{Bmatrix} \quad (4)$$

$$\mathbf{y}_s = \mathbf{D}_s \mathbf{x}_s.$$

112 Solving Eq. (4), in the least square sense [28] since  $\mathbf{A}$  may be rank deficient,  
113 gives the explicit expression of Lagrange multipliers

$$\lambda = - \left( \tilde{\mathbf{A}} \mathbf{M}_s^{-1} \tilde{\mathbf{A}}^T \right)^\dagger \left( \mathbf{b} - \tilde{\mathbf{A}} \mathbf{M}_s^{-1} \left( \mathbf{D}_s^T \mathbf{f}_{\text{nc}} - \mathbf{C}_s \dot{\mathbf{x}}_s - \mathbf{K}_s \mathbf{x}_s \right) \right) \text{ where } \tilde{\mathbf{A}} = \mathbf{A}\mathbf{D}_s. \quad (5)$$

114 Thus, the dynamic response of the constrained system verifies the following  
115 differential equation

$$\mathbf{M}_s \ddot{\mathbf{x}}_s = \left( \mathbf{D}_s^T \mathbf{f}_{\text{nc}} - \mathbf{C}_s \dot{\mathbf{x}}_s - \mathbf{K}_s \mathbf{x}_s \right) + \mathbf{\Delta} \left( \mathbf{b} - \tilde{\mathbf{A}} \mathbf{M}_s^{-1} \left( \mathbf{D}_s^T \mathbf{f}_{\text{nc}} - \mathbf{C}_s \dot{\mathbf{x}}_s - \mathbf{K}_s \mathbf{x}_s \right) \right)$$

$$\mathbf{y}_s = \mathbf{D}_s \mathbf{x}_s. \quad (6)$$

116 with  $\mathbf{\Delta} = \tilde{\mathbf{A}}^T (\tilde{\mathbf{A}} \mathbf{M}_s^{-1} \tilde{\mathbf{A}}^T)^\dagger$ . The main result from Udwadia and Kalaba [5, 7]  
117 lies in the restatement of Eq. (6), in terms of the unconstrained acceleration

$$118 \quad \ddot{\mathbf{x}}_u = \mathbf{M}_s^{-1} \left( \mathbf{D}_s^T \mathbf{f}_{\text{nc}} - \mathbf{C}_s \dot{\mathbf{x}}_s - \mathbf{K}_s \mathbf{x}_s \right),$$

$$\ddot{\mathbf{x}}_s = \ddot{\mathbf{x}}_u + \mathbf{M}_s^{-1/2} \mathbf{B}^\dagger (\mathbf{b} - \tilde{\mathbf{A}} \ddot{\mathbf{x}}_u)$$

$$\mathbf{y}_s = \mathbf{D}_s \mathbf{x}_s. \quad (7)$$

119 with  $\mathbf{B}^\dagger$  denoting the Moore-Penrose pseudo-inverse of  $\mathbf{B} = \tilde{\mathbf{A}} \mathbf{M}_s^{-1/2}$ .

120



121 *2.1.2. Extension to vibro-acoustic substructures*

122 In the case of subsystems involving internal vibro-acoustic problems, as-  
 123 suming lossless fluid-structure interactions, a discrete formulation may be  
 124 provided by the following matrix system [29]

$$\begin{aligned} \begin{bmatrix} \mathbf{M}_s & \mathbf{0} \\ \mathbf{L}^T & \mathbf{M}_a \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{x}}_s \\ \ddot{\mathbf{x}}_a \end{Bmatrix} + \begin{bmatrix} \mathbf{C}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_a \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{x}}_s \\ \dot{\mathbf{x}}_a \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_s & -\mathbf{L} \\ \mathbf{0} & \mathbf{K}_a \end{bmatrix} \begin{Bmatrix} \mathbf{x}_s \\ \mathbf{x}_a \end{Bmatrix} = \begin{bmatrix} \mathbf{D}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_a \end{bmatrix}^T \left( \begin{Bmatrix} \mathbf{f}_s \\ \dot{\mathbf{g}}_a \end{Bmatrix} + \mathbf{f}_c \right) \\ \begin{Bmatrix} \mathbf{y}_s \\ \mathbf{y}_a \end{Bmatrix} = \begin{bmatrix} \mathbf{D}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_a \end{bmatrix} \begin{Bmatrix} \mathbf{x}_s \\ \mathbf{x}_a \end{Bmatrix} \end{aligned} \quad (8)$$

125 where  $\mathbf{y}_s$  and  $\mathbf{y}_a$  are vectors denoting the structural and acoustic responses  
 126 (at the displacement and pressure levels),  $\mathbf{x}_s$  and  $\mathbf{x}_a$  represent the structural  
 127 dofs and the acoustic dofs. The output shape matrix  $\mathbf{D}$  is composed of sub-  
 128 matrices  $\mathbf{D}_s$  and  $\mathbf{D}_a$ .  $\mathbf{M}_s$ ,  $\mathbf{M}_a$ ,  $\mathbf{K}_s$ ,  $\mathbf{K}_a$ ,  $\mathbf{C}_s$  and  $\mathbf{C}_a$  are, respectively, the  
 129 mass, stiffness and damping matrices of the structure and the fluid domain.  
 130 Note that  $\mathbf{M}_a$ ,  $\mathbf{K}_a$  and  $\mathbf{C}_a$  are not homogeneous to a mass, a stiffness and a  
 131 damping but these notations are used by analogy with the solid domain.  $\mathbf{L}$  is  
 132 the vibro-acoustic coupling matrix.  $\mathbf{f}_s$  is the vector of external forces applied  
 133 on the structural components and  $\dot{\mathbf{g}}_a$  is associated to acoustic sources in the  
 134 cavity.

135 The assembled vectors and matrices of Eq. (8) are defined

$$\begin{aligned} \mathbf{x}_\star \equiv \begin{Bmatrix} \mathbf{x}_\star^1 \\ \vdots \\ \mathbf{x}_\star^J \end{Bmatrix}, \mathbf{f}_s \equiv \begin{Bmatrix} \mathbf{f}_s^1 \\ \vdots \\ \mathbf{f}_s^J \end{Bmatrix}, \mathbf{g}_a \equiv \begin{Bmatrix} \mathbf{g}_a^1 \\ \vdots \\ \mathbf{g}_a^J \end{Bmatrix}, \mathbf{M}_\star \equiv \text{diag}(\mathbf{M}_\star^1, \dots, \mathbf{M}_\star^J) = \begin{bmatrix} \mathbf{M}_\star^1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{M}_\star^J \end{bmatrix}, \\ \mathbf{C}_\star \equiv \text{diag}(\mathbf{C}_\star^1, \dots, \mathbf{C}_\star^J), \mathbf{K}_\star \equiv \text{diag}(\mathbf{K}_\star^1, \dots, \mathbf{K}_\star^J), \mathbf{L} \equiv \text{diag}(\mathbf{L}^1, \dots, \mathbf{L}^J) \end{aligned} \quad (9)$$

136 with  $\star$  denoting either  $s$  or  $a$ .

137 While Eq. (8) directly expresses the coupling between structural displace-  
 138 ment and acoustic pressure, an alternative formulation [30] with a positive  
 139 definite mass matrix is preferred in the following of this paper

$$\begin{aligned}
 & \underbrace{\begin{bmatrix} \mathbf{M}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_a \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{Bmatrix} \ddot{\mathbf{x}}_s \\ \ddot{\mathbf{z}}_a \end{Bmatrix}}_{\ddot{\mathbf{x}}} + \underbrace{\begin{bmatrix} \mathbf{C}_s & -\mathbf{L} \\ \mathbf{L}^T & \mathbf{C}_a \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{Bmatrix} \dot{\mathbf{x}}_s \\ \dot{\mathbf{z}}_a \end{Bmatrix}}_{\dot{\mathbf{x}}} + \underbrace{\begin{bmatrix} \mathbf{K}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_a \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{Bmatrix} \mathbf{x}_s \\ \mathbf{z}_a \end{Bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} \mathbf{D}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_a \end{bmatrix}}_{\mathbf{D}}^T \left( \underbrace{\begin{Bmatrix} \mathbf{f}_s \\ \mathbf{f}_a \end{Bmatrix}}_{\mathbf{f}_{\text{nc}}} + \tilde{\mathbf{f}}_{\text{c}} \right) \\
 & \underbrace{\begin{Bmatrix} \mathbf{y}_s \\ \tilde{\mathbf{y}}_a \end{Bmatrix}}_{\mathbf{y}} = \begin{bmatrix} \mathbf{D}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_a \end{bmatrix} \begin{Bmatrix} \mathbf{x}_s \\ \mathbf{z}_a \end{Bmatrix}
 \end{aligned} \tag{10}$$

140 where  $\tilde{\mathbf{y}}_a$  is (up to a multiplicative constant) the acoustic velocity poten-  
 141 tial such that  $\dot{\mathbf{z}}_a = \mathbf{x}_a$  and  $\mathbf{f}_a(t) = \mathbf{g}_a(t) - \mathbf{g}_a(0)$ .

142 Assuming that the constraints between vibro-acoustic subsystems may  
 143 be expressed in the form of Eq. (3) with respect to the response vector  $\mathbf{y}$ ,  
 144 constraining forces  $\tilde{\mathbf{f}}_{\text{c}}$  are expressed as in Eq. (2). Building and solving the  
 145 augmented DAE of Eq. (4) leads to the differential equation (7) and the  
 146 result from Udwadia and Kalaba is thus recovered.

147 While the formulation of Eq. (7) is perfectly valid from a mathematical  
 148 point of view, in practice it has the disadvantage of requesting the knowledge  
 149 of physical matrices  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$  and while efficient experimental methods  
 150 have been developed to estimate mass and stiffness matrices, the experimen-  
 151 tal estimation of the damping matrix  $\mathbf{C}$  remains a difficult task. Moreover,  
 152 from a computational point of view, the unconstrained acceleration  $\ddot{\mathbf{x}}_u$  in

153 Eq. (7) involves a system of coupled Ordinary Differential Equations (ODE)  
 154 whose resolution, for subsystems containing a large amount of dofs, might be  
 155 time consuming. This motivates the use of modal expansion techniques, cov-  
 156 ered in section 2.2, to transform this system of coupled ODE to an uncoupled  
 157 one.

## 158 2.2. Modal space

### 159 2.2.1. Normal modes

160 In case of subsystems with a classical damping, Antunes & Debut [20]  
 161 adapted the U-K formulation to continuous flexible system by means of a  
 162 modal expansion on the modal bases of associated conservative subsystems  
 163 (principal coordinates). In a similar manner to Eq. (7), they obtained in  
 164 principal coordinates the following expression for the modal response  $\underline{\mathbf{q}}^{\text{p}}$  of  
 165 the constrained system

$$\begin{aligned} \underline{\ddot{\mathbf{q}}}^{\text{p}} &= \underline{\ddot{\mathbf{q}}}_u^{\text{p}} + (\underline{\mathbf{M}}^{\text{p}})^{-1/2} \underline{\mathbf{B}}^\dagger (\mathbf{b} - \underline{\mathbf{A}} \underline{\ddot{\mathbf{q}}}_u^{\text{p}}) \\ \mathbf{y} &= \underline{\mathbf{D}} \underline{\Phi} \underline{\mathbf{q}}^{\text{p}} \end{aligned} \quad (11)$$

166 with  $\underline{\mathbf{A}} = \tilde{\underline{\mathbf{A}}} \underline{\Phi}$  being the modal constraint matrix and  $\underline{\mathbf{B}} = \underline{\mathbf{A}} (\underline{\mathbf{M}}^{\text{p}})^{-1/2}$ .  $\underline{\ddot{\mathbf{q}}}_u^{\text{p}}$  is  
 167 the modal acceleration of the unconstrained system governed by the equation

$$\underline{\mathbf{M}}^{\text{p}} \underline{\ddot{\mathbf{q}}}_u^{\text{p}} = \underline{\Phi}^T \underline{\mathbf{D}}^T \mathbf{f}_{\text{nc}} - \underline{\mathbf{C}}^{\text{p}} \underline{\dot{\mathbf{q}}}_u^{\text{p}} - \underline{\mathbf{K}}^{\text{p}} \underline{\mathbf{q}}_u^{\text{p}} \quad (12)$$

168 where  $\underline{\mathbf{M}}^{\text{p}}$ ,  $\underline{\mathbf{C}}^{\text{p}}$ ,  $\underline{\mathbf{K}}^{\text{p}}$  and  $\underline{\Phi}$  are block diagonal matrices containing, respec-  
 169 tively, modal mass, stiffness, damping and shape matrices of all subsystems  
 170 on their diagonal. As for Eq. 7, Antunes & Debut obtained Eq. 11 with  $\underline{\mathbf{D}}$   
 171 equal to the identity matrix, however including this matrix in the formula-  
 172 tion explicitly shows it is actually valid for any choice of output shape matrix

173 as well as if the number of system responses does not match the number of  
174 dofs.

### 175 *2.2.2. Complex modes*

176 If the  $J$  subsystems do not verify the assumption of classical damping, the  
177 system of equation (12) on unconstrained modal coordinates is non-diagonal  
178 which may lead to significant computation time for large subsystems. Apart  
179 from computation time considerations, in case of experimental substructur-  
180 ing, the physical damping matrix estimation presents many difficulties which  
181 make the use of equation (12) unpractical. The development of another  
182 modal formulation based on complex modal analysis (CMA) is thus proposed  
183 in this section in order to obtain a diagonal system of unconstrained modal  
184 coordinates even for non-proportionally damped substructures, allowing at  
185 the same time to shortcut the need for physical damping matrix estimation.  
186 At this point it should be emphasized that the use of CMA implies that the  
187  $J$  subsystems do not yield rigid body modes.

188 The usual modal expansion on the basis of modal coordinates of the  
189 dissipative system is written

$$\begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{Bmatrix} = \mathbf{\Upsilon}_r \underline{\mathbf{q}} \quad (13)$$

190 where, for  $j = 1, \dots, J$  constrained subsystems, the vectors that assemble  
191 the corresponding physical responses  $\mathbf{x}^j(t)$  and modal responses  $\underline{\mathbf{q}}^j(t)$ , as  
192 well as the matrices that assemble the eigenvalues  $\underline{\Lambda}^j$  and the left and right

193 eigenvectors  $\Upsilon_*^j$  (with  $*$  being either  $l$  or  $r$ ), are defined

$$\begin{aligned} \mathbf{x} &\equiv \begin{Bmatrix} \mathbf{x}^1 \\ \vdots \\ \mathbf{x}^J \end{Bmatrix}, \underline{\mathbf{q}} \equiv \begin{Bmatrix} \underline{\mathbf{q}}^1 \\ \vdots \\ \underline{\mathbf{q}}^J \end{Bmatrix}, \Psi_* \equiv \begin{bmatrix} \Psi_*^1 & \mathbf{0} \\ & \ddots \\ \mathbf{0} & \Psi_*^J \end{bmatrix}, \underline{\Lambda} \equiv \begin{bmatrix} \underline{\Lambda}^1 & \mathbf{0} \\ & \ddots \\ \mathbf{0} & \underline{\Lambda}^J \end{bmatrix}, \\ \underline{\mathbf{q}} &\equiv \begin{Bmatrix} \underline{\mathbf{q}} \\ \underline{\bar{\mathbf{q}}} \end{Bmatrix}, \Psi_* \equiv \begin{bmatrix} \Psi_* & \bar{\Psi}_* \end{bmatrix}, \underline{\Lambda} \equiv \begin{bmatrix} \underline{\Lambda} & 0 \\ 0 & \bar{\underline{\Lambda}} \end{bmatrix}, \Upsilon_* \equiv \begin{bmatrix} \Psi_* \\ \Psi_* \underline{\Lambda} \end{bmatrix} \end{aligned} \quad (14)$$

194 with, for each subsystem, the modal basis of  $2N^j$  unconstrained modes de-  
195 fined at  $R^j$  physical coordinates

$$\begin{aligned} \underline{\mathbf{q}}^j &\equiv \begin{Bmatrix} \underline{\mathbf{q}}_1^j \\ \vdots \\ \underline{\mathbf{q}}_{N^j}^j \end{Bmatrix}, \Psi_*^j \equiv \left[ \begin{Bmatrix} \psi_{*,1}^j(\bar{\mathbf{r}}_1^j) \\ \vdots \\ \psi_{*,1}^j(\bar{\mathbf{r}}_{R^j}^j) \end{Bmatrix} \cdots \begin{Bmatrix} \psi_{*,N^j}^j(\bar{\mathbf{r}}_1^j) \\ \vdots \\ \psi_{*,N^j}^j(\bar{\mathbf{r}}_{R^j}^j) \end{Bmatrix} \right], \underline{\Lambda}^j \equiv \begin{bmatrix} \lambda_1^j & \mathbf{0} \\ & \ddots \\ \mathbf{0} & \lambda_{N^j}^j \end{bmatrix}, \\ \underline{\mathbf{q}}^j &\equiv \begin{Bmatrix} \underline{\mathbf{q}}^j \\ \underline{\bar{\mathbf{q}}^j} \end{Bmatrix}, \Psi_*^j \equiv \begin{bmatrix} \Psi_*^j & \bar{\Psi}_*^j \end{bmatrix}, \underline{\Lambda}^j \equiv \begin{bmatrix} \underline{\Lambda}^j & 0 \\ 0 & \bar{\underline{\Lambda}}^j \end{bmatrix}, \Upsilon_*^j \equiv \begin{bmatrix} \Psi_*^j \\ \Psi_*^j \underline{\Lambda}^j \end{bmatrix}. \end{aligned} \quad (15)$$

196 The total number of pairs of complex conjugate modes is thus  $N_{\text{tot}} = \sum_1^J N^j$ .

197 It can be shown, following the approach of [31], that left eigenvectors  $\Psi_l$  of  
198 Eq. (10) are linked to right eigenvectors  $\Psi_r$  in the following manner

$$\Psi_r = \begin{Bmatrix} \Psi_s \\ \Psi_a \end{Bmatrix} \text{ then } \Psi_l = \begin{Bmatrix} \Psi_s \\ -\Psi_a \end{Bmatrix} \quad (16)$$

199 with  $\Psi_s$  and  $\Psi_a$  corresponding, respectively, to structural and acoustic dofs.  
200 In the frame of experimental substructuring, the formulation with respect  
201 to displacements and pressures is usually preferred for experimental modal  
202 analysis, hence it should be emphasized that right eigenvectors  $\tilde{\Psi}_r$  of Eq. (8)

203 are linked to those of Eq. (10) through the relation

$$\tilde{\Psi}_r = \begin{bmatrix} \Psi_s \\ \Psi_a \Lambda \end{bmatrix}. \quad (17)$$

204 In order to develop the complex modal space version of Eq. (7), Eq. (6) is  
 205 augmented into a first order state-space model by grouping time derivatives  
 206 of  $\mathbf{x}$  on the left-hand side and associating the identity  $\mathbf{M}\dot{\mathbf{x}} = \mathbf{M}\dot{\mathbf{x}}$

$$\underbrace{\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix}}_{\mathbf{U}} \begin{Bmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{Bmatrix} - \left( \underbrace{\begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}}_{\mathbf{A}} + \begin{bmatrix} \mathbf{ZK} & \mathbf{ZC} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{Bmatrix} = \begin{Bmatrix} (\mathbb{I}_{N_{\text{dof}}} - \mathbf{Z}) \mathbf{D}^T \mathbf{f}_{\text{nc}} + \Delta \mathbf{b} \\ \mathbf{0} \end{Bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{Bmatrix} \quad (18)$$

207 with  $\mathbf{Z} = \Delta \tilde{\mathbf{A}} \mathbf{M}^{-1}$  and  $\mathbb{I}_{N_{\text{dof}}}$  the identity matrix of size  $N_{\text{dof}}$ , the total number  
 208 of dofs of the subsystems.

209 The projection of Eq. (18) on the bases of eigen modes of the dissipative  
 210 unconstrained subsystems leads to

$$\underline{\mathbf{\Pi}} \dot{\underline{\mathbf{q}}} - \left( \underline{\mathbf{\Pi}} \Lambda + \underbrace{\Upsilon_l^T \begin{bmatrix} \mathbf{ZK} & \mathbf{ZC} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \Upsilon_r}_{\mathbf{E}} \right) \underline{\mathbf{q}} = \underbrace{\Upsilon_l^T \begin{Bmatrix} (\mathbb{I}_{N_{\text{dof}}} - \mathbf{Z}) \mathbf{D}^T \mathbf{f}_{\text{nc}} + \Delta \mathbf{b} \\ \mathbf{0} \end{Bmatrix}}_{\mathbf{F}}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{D} & \mathbf{0} \end{bmatrix} \Upsilon_r \underline{\mathbf{q}} \quad (19)$$

211 using the orthogonality properties<sup>1</sup> [33]

$$\Upsilon_l^T \mathbf{U} \Upsilon_r = \Psi_l^T \mathbf{C} \Psi_r + \underline{\Lambda} \Psi_l^T \mathbf{M} \Psi_r + \Psi_l^T \mathbf{M} \Psi_r \underline{\Lambda} = \underline{\Pi}, \quad (20)$$

$$\Upsilon_l^T \underline{\Lambda} \Upsilon_r = -\Psi_l^T \mathbf{K} \Psi_r + \underline{\Lambda} \Psi_l^T \mathbf{M} \Psi_r \underline{\Lambda} = \underline{\Pi} \underline{\Lambda}.$$

212 where  $\underline{\Pi}$  is a diagonal matrix characterizing the normalization of complex  
 213 mode shapes. From Eq. (20), it can be shown that the inverse of the physical  
 214 mass matrix may be expressed

$$\mathbf{M}^{-1} = \Psi_r \underline{\Pi}^{-1} \underline{\Lambda} \Psi_l^T. \quad (21)$$

The terms  $\underline{\Xi}$  and  $\underline{\Gamma}$  of Eq. (19) may be simplified in the following manner  
 (see Appendix B for more details)

$$\underline{\Xi} = -\Psi_l^T \underline{\Delta} \tilde{\mathbf{A}} \Psi_r \underline{\Lambda}^2 \quad (22)$$

$$\text{and } \underline{\Gamma} = \Psi_l^T (\mathbb{I}_{N_{\text{dof}}} - \mathbf{Z}) \mathbf{D}^T \mathbf{f}_{\text{nc}} + \Psi_l^T \underline{\Delta} \mathbf{b}. \quad (23)$$

215 In the following, two cases are distinguished: first the subsystems are as-  
 216 sumed to contain acoustic dofs, a general formulation is obtained for vibro-  
 217 acoustic subsystems with internal fluid cavities, second the case of subsys-  
 218 tems with purely structural dofs is presented and leads to a more compact  
 219 formulation.

220 *2.2.2.1. Presence of acoustic dofs.*

221

---

<sup>1</sup>For equivalent expressions in the case of continuous subsystems, the authors recom-  
 mend the paper by Krenk [32].

222 Developing  $\underline{\Delta}$  using Eq. (21) and introducing the modal constraint Ja-  
 223 cobian matrix  $\underline{\mathbf{A}}_* = \begin{bmatrix} \underline{\mathbf{A}}_* & \overline{\underline{\mathbf{A}}_*} \end{bmatrix} = \tilde{\mathbf{A}}\Psi_*$  (\* being either  $r$  or  $l$ ) gives

$$\Psi_l^T \underline{\Delta} = \underline{\mathbf{A}}_l^T \left( \underline{\mathbf{A}}_r \underline{\Pi}^{-1} \underline{\Lambda} \underline{\mathbf{A}}_l^T \right)^\dagger. \quad (24)$$

224 and defining now  $\underline{\mathbf{B}}_* = \underline{\mathbf{A}}_* \left( \underline{\Pi}^{-1} \underline{\Lambda} \right)^{1/2}$ ,

$$\Psi_l^T \underline{\Delta} = \left( \underline{\Pi}^{-1} \underline{\Lambda} \right)^{-1/2} \underline{\mathbf{B}}_l^T \left( \underline{\mathbf{B}}_r \underline{\mathbf{B}}_l^T \right)^\dagger. \quad (25)$$

225 By left multiplying Eq. (19) by  $\underline{\Pi}^{-1}$ , the main result finally emerges

$$\begin{aligned} \underline{\dot{\mathbf{q}}} &= \left( \mathbb{I}_{2N_{\text{tot}}} - \left( \underline{\Pi} \underline{\Lambda} \right)^{-1/2} \underline{\mathbf{B}}_l^T \left( \underline{\mathbf{B}}_r \underline{\mathbf{B}}_l^T \right)^\dagger \underline{\mathbf{A}}_r \underline{\Lambda} \right) \left( \underline{\Lambda} \underline{\mathbf{q}} + \underline{\Pi}^{-1} \Psi_l^T \mathbf{D}^T \mathbf{f}_{\text{nc}} \right) \\ &\quad + \left( \underline{\Pi} \underline{\Lambda} \right)^{-1/2} \underline{\mathbf{B}}_l^T \left( \underline{\mathbf{B}}_r \underline{\mathbf{B}}_l^T \right)^\dagger \mathbf{b} \quad (26) \\ \mathbf{y} &= \begin{bmatrix} \mathbf{D} & \mathbf{0} \end{bmatrix} \Upsilon_r \underline{\mathbf{q}} \end{aligned}$$

226 which can be restated, in terms of unconstrained modal coordinates governed  
 227 by  $\underline{\Pi} \underline{\dot{\mathbf{q}}}_u = \underline{\Pi} \underline{\Lambda} \underline{\mathbf{q}} + \Psi_l^T \mathbf{D}^T \mathbf{f}_{\text{nc}}$ , as

$$\begin{aligned} \underline{\dot{\mathbf{q}}} &= \underline{\dot{\mathbf{q}}}_u + \left( \underline{\Pi} \underline{\Lambda} \right)^{-1/2} \underline{\mathbf{B}}_l^T \left( \underline{\mathbf{B}}_r \underline{\mathbf{B}}_l^T \right)^\dagger \left( \mathbf{b} - \underline{\mathbf{A}}_r \underline{\Lambda} \underline{\dot{\mathbf{q}}}_u \right) \\ \mathbf{y} &= \begin{bmatrix} \mathbf{D} & \mathbf{0} \end{bmatrix} \Upsilon_r \underline{\mathbf{q}}. \quad (27) \end{aligned}$$

228 Eq. (27) thus allows to compute the constrained dynamic response of  
 229 mechanical systems involving internal vibro-acoustic problems. It should  
 230 be noted that the validity of this equation stays unchanged in absence of  
 231 acoustic dofs. However, a more compact form can be expressed in such a  
 232 case as presented below.

233 *2.2.2.2. Purely structural dofs.*

234



235 If the subsystems contain only structural dofs, the problem of Eq. (1) is  
 236 recovered, left and right eigenvectors are thus equal. Similarly to Eq. 21,  
 237 defining a hermitian modal mass matrix  $\underline{\mathbf{M}}$  as  $\underline{\mathbf{M}} = \underline{\Psi}^H \mathbf{M} \underline{\Psi}$ , one obtains

$$\mathbf{M}^{-1} = \underline{\Psi} \underline{\mathbf{M}}^{-1} \underline{\Psi}^H. \quad (28)$$

238 Note that in absence of acoustics dofs the matrix  $\underline{\mathbf{M}}$  is semi-definite positive.  
 239 When performing the complex modal analysis of a structure, the matrix  $\underline{\mathbf{\Pi}}$  is  
 240 set by prescribing the normalization of complex mode shapes. As developed  
 241 in Appendix A the matrix  $\underline{\mathbf{M}}$  can then be obtained from  $\underline{\mathbf{\Pi}}$ , assuming  $\underline{\Psi}$  is  
 242 full column rank (at least as many measured responses as identified pairs of  
 243 complex modes).

244 Eq. 24 may be rewritten using Eq. (28) in the following manner

$$\underline{\Psi}^T \underline{\Delta} = \underline{\mathbf{A}}^T (\underline{\mathbf{A}} \underline{\mathbf{M}}^{-1} \underline{\mathbf{A}}^H)^\dagger. \quad (29)$$

245 Defining now  $\underline{\mathbf{E}} = \begin{bmatrix} \underline{\mathbf{E}} & \underline{\bar{\mathbf{E}}} \end{bmatrix} = \underline{\mathbf{A}} \begin{bmatrix} \underline{\mathbf{M}} & 0 \\ 0 & \underline{\bar{\mathbf{M}}} \end{bmatrix}^{-1/2}$  leads to

$$\underline{\Psi}^T \underline{\Delta} = \begin{bmatrix} \underline{\mathbf{M}}^{1/2} & 0 \\ 0 & \underline{\bar{\mathbf{M}}}^{1/2} \end{bmatrix}^T \underline{\mathbf{E}}^T (\underline{\mathbf{E}} \underline{\mathbf{E}}^H)^\dagger = \begin{bmatrix} \underline{\bar{\mathbf{M}}}^{1/2} \underline{\bar{\mathbf{E}}}^\dagger \\ \underline{\mathbf{M}}^{1/2} \underline{\mathbf{E}}^\dagger \end{bmatrix}. \quad (30)$$

246 By left multiplying Eq. (19) by  $\underline{\mathbf{\Pi}}^{-1}$ , the main result finally emerges

$$\begin{aligned} \underline{\dot{\mathbf{q}}} &= \left( \mathbb{I}_{2N_{\text{tot}}} - \underline{\mathbf{\Pi}}^{-1} \begin{bmatrix} \underline{\bar{\mathbf{M}}}^{1/2} \underline{\bar{\mathbf{E}}}^\dagger \\ \underline{\mathbf{M}}^{1/2} \underline{\mathbf{E}}^\dagger \end{bmatrix} \underline{\mathbf{A}} \underline{\mathbf{\Lambda}} \right) (\underline{\mathbf{\Lambda}} \underline{\mathbf{q}} + \underline{\mathbf{\Pi}}^{-1} \underline{\Psi}^T \mathbf{D}^T \mathbf{f}_{\text{nc}}) + \underline{\mathbf{\Pi}}^{-1} \begin{bmatrix} \underline{\bar{\mathbf{M}}}^{1/2} \underline{\bar{\mathbf{E}}}^\dagger \\ \underline{\mathbf{M}}^{1/2} \underline{\mathbf{E}}^\dagger \end{bmatrix} \mathbf{b} \\ & \mathbf{y} = \begin{bmatrix} \mathbf{D} & \mathbf{0} \end{bmatrix} \underline{\Upsilon} \underline{\mathbf{q}}. \end{aligned} \quad (31)$$

247 Thus the constrained response verifies

$$\begin{aligned} \underline{\dot{\mathbf{q}}} &= \underline{\dot{\mathbf{q}}}_u + \underline{\mathbf{\Pi}}^{-1} \begin{bmatrix} \underline{\mathbf{M}}^{1/2} \underline{\mathbf{E}}^\dagger \\ \underline{\mathbf{M}}^{1/2} \underline{\mathbf{E}}^\dagger \end{bmatrix} (\mathbf{b} - \underline{\mathbf{A}} \underline{\Lambda} \underline{\dot{\mathbf{q}}}_u) \\ \mathbf{y} &= \begin{bmatrix} \underline{\mathbf{D}} & \mathbf{0} \end{bmatrix} \underline{\Upsilon} \underline{\mathbf{q}} \end{aligned} \quad (32)$$

248 which is equivalent to Eq. (27) in absence of acoustic dofs but more com-  
 249 pact. When analyzing the dynamic behavior of multibody systems composed  
 250 of only structural dofs with non-proportional damping, if the modal mass  
 251 matrix  $\underline{\mathbf{M}}$  is known, the use of Eq. (32) is thus encouraged.

252 Eqs. (27) and (32) are quite similar to the previously proposed formula-  
 253 tions in Eqs. (7) and (11). Indeed, the constrained response is found to be  
 254 equal to the unconstrained response to which a corrective term, representing  
 255 the contribution of constraining forces, is added. However, an important dif-  
 256 ference resides in the order of the obtained system of ODE which was 2 for  
 257 previous formulations and is now 1. This is a direct consequence of the use  
 258 of the complex modal space which is made of eigenmodes of the state-space  
 259 problem of Eq. (18).

260 To summarize the results of this alternative formulation, and empha-  
 261 size on the added value with respect to the modal formulation based on  
 262 real modes, Fig. 1 presents block diagrams of the main steps of the U-K  
 263 substructuring, for both formulations, as well as those of the experimental  
 264 modal analysis. It is visible that both formulations involve the same steps:  
 265 first, input modal parameters of the unconstrained (free interface) substruc-  
 266 tures as well as constraint-independent forces have to be prescribed, then the  
 267 response of the subsystems to the constraint-independent forces is computed  
 268 and finally their constrained response verifying the constraint equations is

269 determined. In the authors opinion, the main advantage of the formulation  
 270 proposed in this paper, compared to the one based on real modes, lies in  
 271 the necessary input modal parameters. Indeed, the experimental estimation  
 272 of real modes (principal coordinates) of a structure results from the identi-  
 273 fication of its complex modal basis, mostly by assuming that the structure  
 274 verifies the hypothesis of proportional damping. Hence, using directly com-  
 275 plex modes of the substructures in the U-K formalism removes this step of the  
 276 experimental modal analysis and avoids resorting to this limiting assumption.

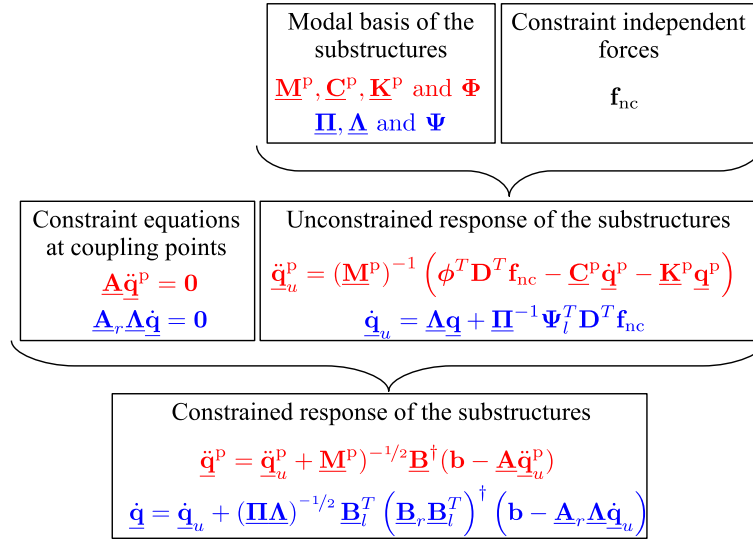


Figure 1: Summary of the UK formalisms based on real modes (red) and complex modes (blue). See sections 2.2.1 and 2.2.2 for details regarding mathematical notations.

### 277 3. Application cases

278 Now that the modal U-K formulation based on complex modes of the  
 279 dissipative subsystems is developed, its validity is first assessed in the case  
 280 of a simple academic system. Then two practical examples are presented

281 and consist in the coupling of the analytical theory of vibrating strings, first,  
 282 with an experimental 2 dofs fluid-structure guitar model to demonstrate the  
 283 relevance of this new formulation for internal vibro-acoustic problems, and  
 284 second, with an experimental 32 dofs structural model of a harp from Central  
 285 Africa showing high modal complexity.

286 In order to assess the benefit of taking into account the non-proportionality  
 287 of the damping, two simulations are compared in all following examples:

288 Case 1 Truly complex modes of the subsystems are used, leading to a fully  
 289 populated modal damping matrix  $\underline{\mathbf{C}}^P = \underline{\Phi}^T \mathbf{C} \underline{\Phi}$  in principal coordi-  
 290 nates.

291 Case 2 Falsely complex modes are used to render the modal damping ma-  
 292 trix  $\underline{\mathbf{C}}^P$  diagonal and equal to  $-2\text{Re}(\underline{\Lambda})$ , obtained from the usual  
 293 approximation[34] of real mode shapes  $\tilde{\Phi} = \text{Re}(\Psi \sqrt{2j\text{Im}(\underline{\Lambda})})$  then  
 294 converted back by  $\tilde{\Psi} = \tilde{\Phi} / \sqrt{2j\text{Im}(\underline{\Lambda})}$  with  $\Psi$  normalized such that  
 295  $\underline{\mathbf{I}}$  is the identity matrix.

296 Following simulations are obtained using a three-step Adams-Bashforth  
 297 time integration[35, 36] scheme to solve for complex modal coordinates  $\underline{\mathbf{q}}$   
 298 from the knowledge of their derivative.

### 299 3.1. Academic system

300 The academic example consists in a 4 dof subsystem  $S_1$  with fixed-free  
 301 boundary conditions coupled to a 5 dofs subsystem  $S_2$  with free-fixed bound-  
 302 ary conditions as depicted in Fig. 2. The coupling constraint is prescribed  
 303 as a continuity of acceleration between both free ends.

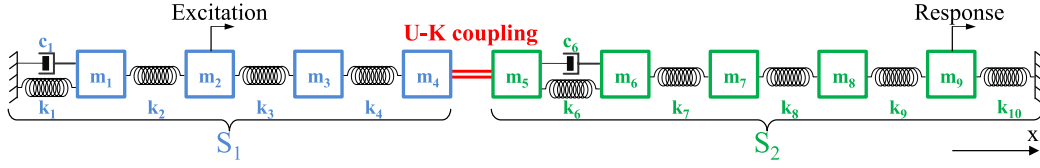


Figure 2: 4 dofs subsystem coupled to 5 dofs subsystem

304 The global system (after coupling) verifies the following differential equa-  
 305 tion

$$\mathbf{M}_A \ddot{\mathbf{x}}_A + \mathbf{C}_A \dot{\mathbf{x}}_A + \mathbf{K}_A \mathbf{x}_A = \mathbf{f}_e \quad (33)$$

306 whose solution in the frequency domain is

$$\tilde{\mathbf{x}}_A = \left( -\omega^2 \mathbf{M}_A + j\omega \mathbf{C}_A + \mathbf{K}_A \right)^{-1} \mathbf{f}_e \quad (34)$$

307 with  $\mathbf{x}_A$  the vector regrouping the displacement of masses  $[m_1, m_2, m_3,$   
 308  $m_4+m_5, m_6, m_7, m_8, m_9]$  and  $\tilde{\mathbf{x}}_A$  its Fourier transform.

309 Physical matrices of both subsystems are presented in Eqs. (35) and (36).  
 310 Physical matrices of the global system (after coupling) used in the reference  
 311 solution are presented in Eq. (37).



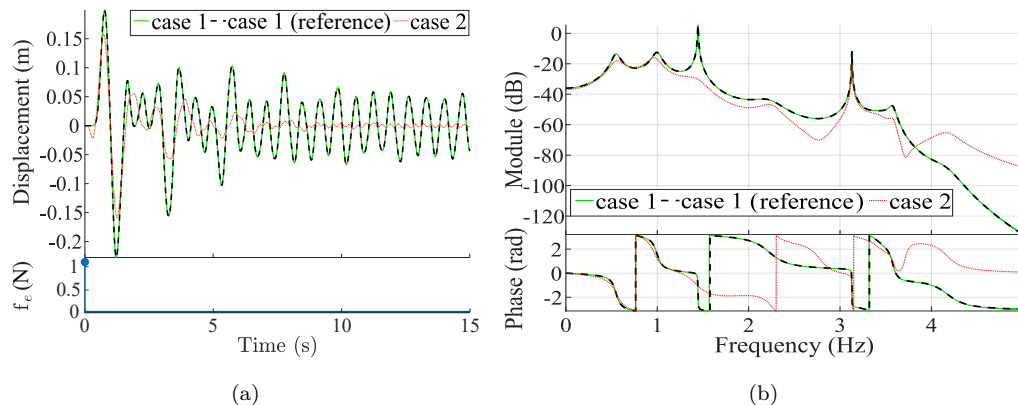


Figure 3: Simulated displacement waveform and spectrum of the discrete mass  $m_9$  of  $S_2$  for damping cases 1 and 2 together with the reference solution for case 1. (a) waveforms and excitation force, (b) spectra.

321 First, the complex modal U-K formulation perfectly overlaps with the  
 322 reference solution when considering the damping case 1 thus validating the  
 323 inclusion of non-proportional damping. Second, the influence of the propor-  
 324 tional damping assumption is clearly visible. Indeed, the simulation relying  
 325 on the assumption of proportional damping presents globally lower resonance  
 326 amplitudes. More importantly, the third mode at 1.45 Hz appears to be much  
 327 more damped than it really is in the reference solution. These differences  
 328 come from the fact that the modal bases of  $S_1$  and  $S_2$  contain truly complex  
 329 modes (not proportional to modes of the conservative subsystems up to a  
 330 complex multiplicative constant) as visible on Figs. (4a-4b) and indicated  
 331 by the Modal Phase Colinearity [37, 38] (MPC) in Tab. 1, corresponding to  
 332 the Modal Assurance Criterion [39, 40] (MAC) between a mode shape and  
 333 its complex conjugate

$$\text{MPC}(\mathbf{u}) = \text{MAC}(\mathbf{u}, \bar{\mathbf{u}}) = \left( \frac{|\mathbf{u}^H \bar{\mathbf{u}}|}{\|\mathbf{u}\| \|\bar{\mathbf{u}}\|} \right)^2.$$

334 Thus the dynamic responses of  $S_1$  and  $S_2$  cannot be exactly represented by  
 335 a set of uncoupled equations in principal coordinates.

$f_n$ (Hz)	0.71	2.48	3.66	4.21	$f_n$ (Hz)	0.42	1.33	1.61	3.13	3.59
MPC	1.00	0.97	0.85	0.69	MPC	0.98	0.66	0.42	1.00	0.99

(a) (b)

Table 1: Modal phase colinearity criteria of  $S_1$  and  $S_2$  for the damping case 1. (a) subsystem  $S_1$ , (b) subsystem  $S_2$ .

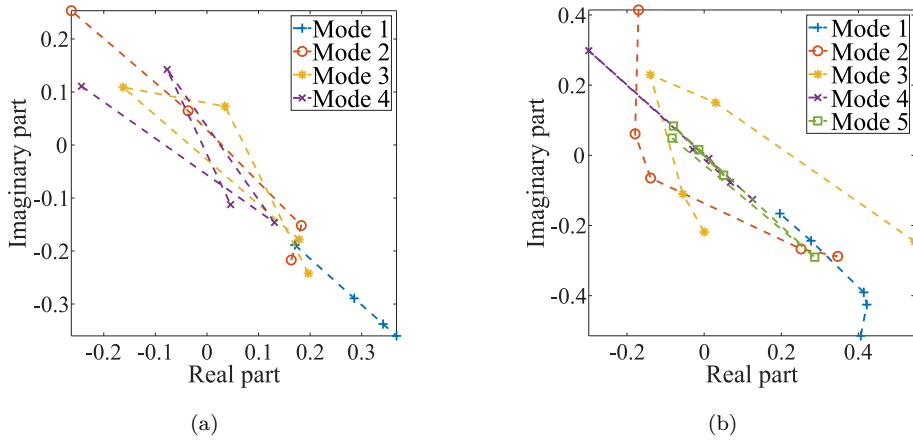


Figure 4: Complex mode shapes of  $S_1$  and  $S_2$  for the damping case 1. (a) subsystem  $S_1$ , (b) subsystem  $S_2$ .

### 336 3.2. Experimental data

#### 337 3.2.1. Guitar model

338 As a second example of application, a free-fixed stiff string is coupled to  
 339 a 2 dofs guitar model, experimentally identified in [41]. The 2 dofs corre-



340 spond respectively to the structural transverse displacement of a point on  
 341 the soundboard, and the acoustic pressure in the middle of the sound hole.  
 342 Physical mass, damping and stiffness matrices corresponding to the matrix  
 343 formulation of Eq. (10) (using Eq. (8 - 9)) are

$$\mathbf{M} = \begin{bmatrix} 0.031 & 0 \\ 0 & 2.7 \cdot 10^{-7} \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 2.2 \cdot 10^4 & 0 \\ 0 & 0.12 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1.4 & -0.036 \\ 0.036 & 3.1 \cdot 10^{-6} \end{bmatrix}. \quad (38)$$

344 Note that physical matrices of Eq. 38 could be prescribed on a theoretical  
 345 basis, however the choice is made to use experimental data to ensure their  
 346 order of magnitude.

347 Following simulations are obtained using a sampling frequency of 1200  
 348 kHz and a ramp of force of 1 s, with a maximum amplitude of 1.1 N,  
 349 applied on the string at three-fifths of its length, represented by a vec-  
 350 tor  $\mathbf{f}_e[n] = \left( H[n] - H \left[ n - \frac{1}{dt} \right] \right) 1.1 n dt$  with  $H[n]$  being the Heaviside step  
 351 function (see the bottom graph of Fig. 5a). The string has a radius of 0.48  
 352 mm, a length of 64 cm, a density of 1100 kg m<sup>-3</sup>, a Young's modulus of  
 353 7.4 GPa and a tuning frequency of 82.4 Hz. Its transverse displacement is  
 354 described by 150 modes.

355 Fig. 5 presents the dynamic response of the structural dof of the guitar  
 356 model in terms of waveform and spectrum for the damping cases 1 and 2.  
 357 The MPCs of the two vibro-acoustic complex modes of the system described  
 358 by physical matrices of Eq. (38) are equal to 1, therefore there mode shapes  
 359 are proportional to the eigenvectors of the associated conservative system.  
 360 This means that the latter are orthogonal to the damping matrix  $\mathbf{C}$  and no  
 361 difference should be visible between damping cases 1 and 2. Indeed, the two

362 simulations perfectly overlap in Figs. 5a - 5c which confirms the assumption  
 363 of proportional damping for this system.

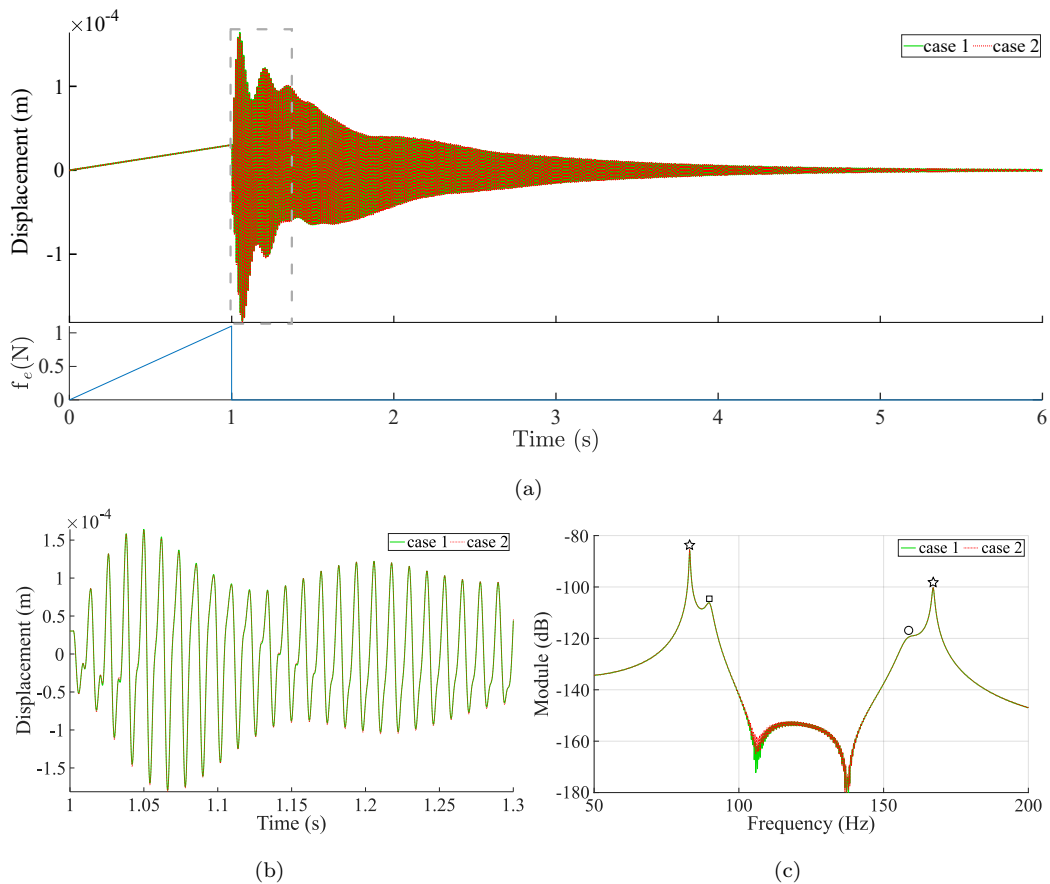


Figure 5: Simulated displacement waveform and spectrum of the structural dof of the guitar for damping cases 1 and 2. (a) waveforms and excitation force, (b) zoom on waveforms in the time frame [1 - 1.3] s, (c) spectra (stars indicate string partials, the square indicates the air cavity resonance inside the soundbox and the circle indicates the soundboard resonance).

364 *3.2.2. Central Africa harp model*

365 As a last example, experimentally identified modal parameters of a harp  
366 from Central Africa, in the frequency range [100-1700] Hz, are used to demon-  
367 strate the interest of including the non-proportionality of the damping. In-  
368 deed, the making process of this harp is artisanal and involves natural ma-  
369 terials such as animal skin (for the soundboard). It is thus probable for  
370 such materials that the assumption of a uniformly distributed damping is  
371 not verified.

372 The experimental setup, presented in Fig. 6, consists in the harp hanging  
373 by means of bungee cords to approximate free boundary conditions. A strip  
374 of felt is intertwined with the strings to dampen, at least, their transverse  
375 vibrations. Frequency Response Functions are then obtained using a roving  
376 automatic hammer (force sensor PCB 086E80) at string attachments points  
377 on the neck (in two perpendicular directions) and the tailpiece for three  
378 reference positions (accelerometers: PCB M352C65 and 356A03 (triaxial))  
379 placed at the 5<sup>th</sup> string/neck attachment point and at the 7<sup>th</sup> string/tailpiece  
380 attachment point.

381 The MPCs of the 32 experimentally estimated complex modes (see Fig.  
382 7a) show that several modes cannot be approximated by real ones and are  
383 actually strongly coupled either by damping or frequency. To illustrate this,  
384 Fig. 7b shows the separation criterion proposed by Hasselman [42] to char-  
385 acterize the decoupling between modes

$$\frac{2\xi_j\omega_j}{|\omega_j - \omega_k|} \ll 1 \quad (39)$$

386 with  $\xi_j$  the loss factor of mode  $j$ ,  $\omega_j$  and  $\omega_k$  the undamped natural fre-  
387 quencies of modes  $j$  and  $k$ . This criterion signifies that the coupling due

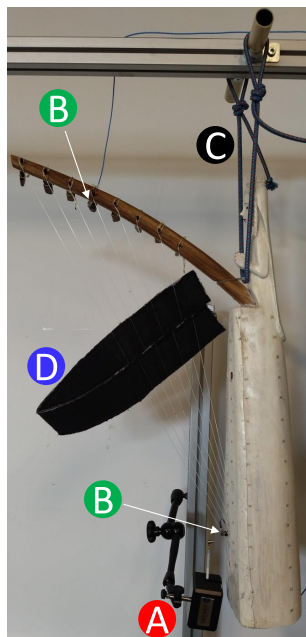


Figure 6: Experimental setup used for the modal analysis of the harp. (A): Automatic hammer; (B): Accelerometers; (C): Bungee cord; (D): Strip of felt.

388 to non-proportional damping can be neglected if the cross-modal impedance  
 389 is high. As suggested by Balmès [33], this criterion can be used to identify  
 390 groups of modes verifying the assumption of block proportional damping and  
 391 then, for each group, a proper basis is approximated by resolution of an alge-  
 392 braic Riccati equation [43]. For example, Fig. 7c shows mode combinations  
 393 (colored in black) for which this criterion crosses the threshold of 0.2 as well  
 394 as four groups of modes (red dashed squares) whose coupling by damping  
 395 can be considered rather negligible.

396 Fig. 8 presents simulations of waveforms and spectra of the acceleration  
 397 of the 4<sup>th</sup> string/tailpiece coupling point for damping cases 1 and 2. The  
 398 simulations are obtained using a sampling frequency of 800 kHz and a ramp

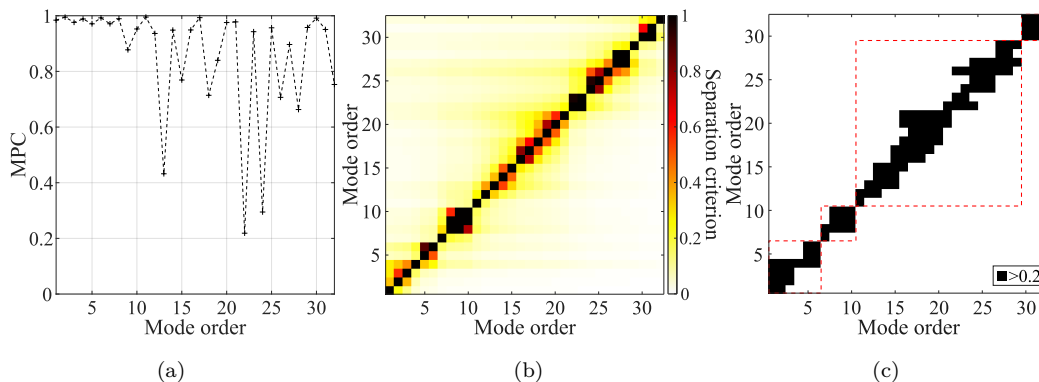


Figure 7: Mode complexity and coupling quantification of experimental modal basis. (a) Modal Phase Colinearity (b) Separation criterion of Eq. (39), (c) Binarized separation criterion with a threshold of 0.2 (red dashed squares indicate uncoupled groups of modes).

399 of force of 1 s, with a maximum amplitude of 1.1 N, applied on the string at  
 400 three-fifths of its length (as in section 3.2.1). The string has a radius of 0.4  
 401 mm, a length of 52.3 cm, a density of  $1100 \text{ kg m}^{-3}$ , a Young's modulus of  
 402 7.4 GPa and a tuning frequency of 247 Hz. It makes an angle of  $23.6^\circ$  with  
 403 the soundboard and its transverse displacement is described by 150 modes.

404 Noticeable differences appear on waveforms of Fig. 8a between the pro-  
 405 portionally and non-proportionally damped models. Looking at the fre-  
 406 quency domain representation on Fig. 8b, two explanations are found. Firstly,  
 407 amplitude differences are visible in the vicinity of harp body modes respond-  
 408 ing at the string attachment point and whose MPC is below 0.95 (indicated by  
 409 colored circles placed above the curves) showing that neglecting the damping  
 410 coupling between mode combinations not verifying the separation criterion  
 411 of Eq. (39) leads to an erroneous dynamic response of the system. Secondly,  
 412 and most importantly, the  $3^{\text{rd}}$  and  $4^{\text{th}}$  string partials almost coincide with  
 413 two strongly complex body modes of the non-proportionally damped system

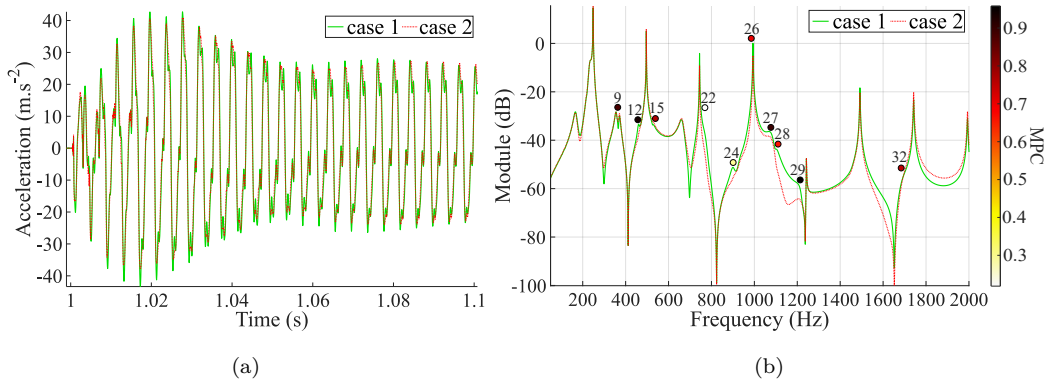


Figure 8: Simulated acceleration waveform and spectrum of the 4<sup>th</sup> string/tailpiece coupling point on a harp from Central Africa for damping cases 1 and 2. Colored circles indicate resonances of the harp body around which case 1 and case 2 show significant amplitude differences, numbers indicate mode orders corresponding to Fig. 7. (a) waveforms, (b) spectra.

414 (n°22 and 26) and are thus underestimated, due to previously mentioned  
 415 reasons, by approximately 2 to 5 dB in the damping case 2.

#### 416 4. Conclusion

417 An alternative modal Udwadia-Kalaba formulation based on complex  
 418 modes of the dissipative subsystems is proposed. This new formulation is  
 419 developed in the general case of vibro-acoustic substructures with an inter-  
 420 nal cavity, to which purely structural substructures previously addressed in  
 421 the literature is just a particular case. This new formulation is of particular  
 422 interest in the frame of experimental substructuring, where physical damping  
 423 matrices of substructures are usually not well estimated, since only a diago-  
 424 nal first order model based on eigenvectors of the state-space representation  
 425 of each subsystem is required instead of a second order model relying on a po-

426 tentially fully populated modal damping matrix in case of non-proportional  
427 damping. Moreover, the assumption of proportional damping is expected  
428 to be less and less reasonable with the increase of modal density so that  
429 the modal damping matrix should be more and more populated with the  
430 increase of frequency, as long as a modal description of the substructures  
431 remains consistent.

432 The new formulation developed in this paper is validated by comparing  
433 simulations of two coupled 4 and 5 dofs subsystems to the analytic solution  
434 of an equivalent academic 8 dofs system. Then an application based on  
435 the coupling of a string to an experimental vibro-acoustic guitar body model  
436 illustrates the advantage of adapting the Udwadia-Kalaba formulation to this  
437 type of problem. Finally, experimentally identified complex modes of a harp  
438 from Central Africa are coupled to a string in order to show the relevance of  
439 using a formulation directly based on complex modes of subsystems instead of  
440 relying on real modes usually obtained by assuming a proportional damping.

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#### 445 **Appendix A. Inverse of the modal mass matrix**

446 In the case of experimentally identified modal parameters, the physical  
447 mass matrix  $\mathbf{M}$  may be unknown and a direct expression of the modal mass

448 matrix  $\underline{\mathbf{M}}$  in term of the norm matrix of complex mode shapes  $\underline{\mathbf{\Pi}}$  would be  
 449 useful.

450 Both  $\underline{\mathbf{M}}$  and  $\underline{\mathbf{\Pi}}$  can be linked to  $\mathbf{M}$  as

$$\mathbf{M}^{-1} = \underline{\mathbf{\Psi}} \underline{\mathbf{M}}^{-1} \underline{\mathbf{\Psi}}^H \text{ and } \mathbf{M}^{-1} = \underline{\mathbf{\Psi}} \underline{\mathbf{\Pi}}^{-1} \underline{\mathbf{\Lambda}} \underline{\mathbf{\Psi}}^T \quad (\text{A.1})$$

451 Since  $\underline{\mathbf{\Psi}}$  is full column rank , the following property is verified

$$\underline{\mathbf{\Psi}}^\dagger \underline{\mathbf{\Psi}} = \mathbb{I}_{N_{\text{tot}}} \text{ with } \underline{\mathbf{\Psi}}^\dagger = (\underline{\mathbf{\Psi}}^H \underline{\mathbf{\Psi}})^{-1} \underline{\mathbf{\Psi}}^H. \quad (\text{A.2})$$

452 Thus

$$\underline{\mathbf{\Psi}}^\dagger \underline{\mathbf{\Psi}} \underline{\mathbf{M}}^{-1} \underline{\mathbf{\Psi}}^H (\underline{\mathbf{\Psi}}^\dagger)^H = \underline{\mathbf{M}}^{-1} \quad (\text{A.3})$$

453 Finally

$$\underline{\mathbf{M}}^{-1} = \underline{\mathbf{\Psi}}^\dagger \underline{\mathbf{\Psi}} \underline{\mathbf{\Pi}}^{-1} \underline{\mathbf{\Lambda}} \underline{\mathbf{\Psi}}^T (\underline{\mathbf{\Psi}}^\dagger)^H = \begin{bmatrix} \mathbb{I}_{N_{\text{tot}}} & \underline{\mathbf{\Psi}}^\dagger \underline{\mathbf{\Psi}} \end{bmatrix} \underline{\mathbf{\Pi}}^{-1} \underline{\mathbf{\Lambda}} \begin{bmatrix} (\underline{\mathbf{\Psi}}^\dagger \underline{\mathbf{\Psi}})^T \\ \mathbb{I}_{N_{\text{tot}}} \end{bmatrix} \quad (\text{A.4})$$

## 454 Appendix B. Development of $\underline{\mathbf{\Xi}}$

$$\underline{\mathbf{\Xi}} = \underline{\mathbf{\Psi}}^T \underline{\mathbf{\Delta}} \underline{\mathbf{A}} \underline{\mathbf{M}}^{-1} \underline{\mathbf{K}} \underline{\mathbf{\Psi}} + \underline{\mathbf{\Psi}}^T \underline{\mathbf{\Delta}} \underline{\mathbf{A}} \underline{\mathbf{M}}^{-1} \underline{\mathbf{C}} \underline{\mathbf{\Psi}} \underline{\mathbf{\Lambda}} \quad (\text{B.1})$$

$$= \underline{\mathbf{\Psi}}^T \underline{\mathbf{\Delta}} \underline{\mathbf{A}} \underline{\mathbf{M}}^{-1} (\underline{\mathbf{K}} \underline{\mathbf{\Psi}} + \underline{\mathbf{C}} \underline{\mathbf{\Psi}} \underline{\mathbf{\Lambda}}) \quad \text{with } \underline{\mathbf{M}}^{-1} = \underline{\mathbf{\Psi}} \underline{\mathbf{\Pi}}^{-1} \underline{\mathbf{\Lambda}} \underline{\mathbf{\Psi}}^T \text{ [33]} \quad (\text{B.2})$$

$$= \underline{\mathbf{\Psi}}^T \underline{\mathbf{\Delta}} \underline{\mathbf{A}} \underline{\mathbf{\Psi}} \underline{\mathbf{\Pi}}^{-1} \underline{\mathbf{\Lambda}} (\underline{\mathbf{\Psi}}^T \underline{\mathbf{K}} \underline{\mathbf{\Psi}} + \underline{\mathbf{\Psi}}^T \underline{\mathbf{C}} \underline{\mathbf{\Psi}} \underline{\mathbf{\Lambda}}) \quad (\text{B.3})$$

$$= -\underline{\mathbf{\Psi}}^T \underline{\mathbf{\Delta}} \underline{\mathbf{A}} \underline{\mathbf{\Psi}} \underline{\mathbf{\Pi}}^{-1} \underline{\mathbf{\Lambda}} \underline{\mathbf{\Psi}}^T \underline{\mathbf{M}} \underline{\mathbf{\Psi}} \underline{\mathbf{\Lambda}}^2 \quad \text{from Eq. (20)} \quad (\text{B.4})$$

$$= -\underline{\mathbf{\Psi}}^T \underline{\mathbf{\Delta}} \underline{\mathbf{A}} \underline{\mathbf{\Psi}} \underline{\mathbf{\Lambda}}^2. \quad (\text{B.5})$$



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