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Identification of deformable droplets from boundary measurements: the case of non-stationary Stokes problem

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Abstract

In this paper we use asymptotic expansion of the velocity field to reconstruct small deformable droplets (i.e., their forms and locations) immersed in an incompressible Newtonian fluid. Here the fluid motion is assumed to be governed by the non-stationary linear Stokes system. Taking advantage of the smallness of the droplets, our asymptotic formula and identification methods extend those already derived for rigid inhomogeneity and for stationary Stokes system. Our derivations, based on dynamical boundary measurements, are rigorous and proved by involving the notion of viscous moment tensor VMT. The viability of our reconstruction approach is documented by numerical results.

Key words. Non-stationary Stokes system, asymptotic expansion, viscous moment tensor, deformable droplet.

2010 AMS subject classifications. 35R30, 35Q30, 35B30, 76D55.

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1 Introduction

In this paper we derive an asymptotic expansion of a weighted boundary measurements, expressed in term of velocity field, in the presence of multiple small deformable droplets (modeled by deformable inhomogeneities) immersed in an incompressible Newtonian fluid. This asymptotic expansion, through a convenient application system, will allow us to identify these droplets having kinematic viscosities different from the background medium. The system is modeled as a non-stationary Stokes flow, in an open bounded domain $\Omega \subset \mathbb{R}^d$ that contains deformable inhomogeneities (droplets) of small size, say α . The non-stationary Stokes equations are a standard system of PDE's governing the flow of continuum matter in fluid form, such as liquid or gas, occupying the domain Ω . These equations describe the change with respect to time $t \in [0, T]$ of the velocity and pressure of the fluid.

The detection and the reconstruction of an object immersed in a fluid is a source of many investigations [1, 2, 5, 6, 10, 11, 14]. This kind of inverse problems arises, for example, in moulds filling during which small gas bubbles can be created and trapped inside the material (as it is mentioned in [10, 14]) or in the detection of mines.

In this paper, we consider a dilute suspension of deformable droplets in a matrix fluid. We assume that the droplets and matrix fluid to be Newtonian and neglect inertia and body forces. As it is mentioned in [11], the evolution of droplet shape only depends on the viscosity difference between the fluids and on surface tension. Then, we make the assumption of small size of these inhomogeneities to use asymptotic formula which allows us to solve a given inverse problem. This assumption was used before, for different kinds of problems, in [4, 3, 5, 14, 18, 17] and in others. We further assume that the small droplets are at a distance $\gg \alpha$ away from each other, and are also away from the boundary, to be able to neglect droplet interactions and their interaction with the boundary $\partial\Omega$.

The goal of this work is to expand, in a mathematically rigorous way, a numerical approach to identify the locations of the droplets from boundary measurements of the velocity field. We get an asymptotic expansion of solutions to the transmission problem for the non-stationary Stokes equations in terms of the geometry of the droplets. This procedure describes the perturbation of the solution caused by the presence of deformable obstacles with small diameters. Based on this results, we derive an asymptotic expansion for a weighted boundary measurements $\Lambda_\alpha(T)$ (as $\alpha \rightarrow 0$) in terms of viscous moment tensor VMT. This is partially inspired from the general idea developed, for a heat equation, in [3] to locate multiple inclusions and from the fact that the Stokes system can be viewed as the incompressible limit (the compression modulus is infinite)[5].

The formula that is developed **is considerably different from the topological sensitivity analysis method** and allows us to find the locations of the inhomogeneities with significant accuracy. It is explicit and can be easily performed numerically. The results and the general approach that we will adopt here extend more those for rigid inclusions [2, 20, 6, 10, 21, 25] and for stationary stokes cases [11, 14, 18, 19, 5] among others. However, they remain a little close to the common ideas of that developed for reconstructing small conductivity inclusions from boundary measurements, see [20, 12, 4, 17, 23] and the references therein. In addition, readers are recommended to consult important works on other types of inverse problems [7, 8, 9, 16], and do not forget that we can refer to the books (for example, [13, 24, 26]) to get acquainted with the main notations,

the calculation techniques and the most well-known theorems used naturally for Stokes system.

The paper is organized as follows. In the next section we formulate our model problem for droplets in a non-stationary Stokes flow. We state our main results in Section 3. The rigorous derivation of the asymptotic expansion, for the pattern $\Lambda_\alpha(T)$, is detailed in Section 4 in terms of the notion of viscous moment tensor VMT. In Section 5 we use our asymptotic expansion to make up a reconstruction method by introducing a so called identifier of interest which, by some $T > 0$, recover the discrete locations $z_j; j = 1, \dots, m$ of the droplets. We then perform numerical experiments to test the viability of the method.

2 Problem formulation

Let Ω be a bounded, open **connected** subdomain of \mathbb{R}^d , $d = 2, 3$ with $C^{1,1}$ - boundary $\partial\Omega$. Let ν denote the outward unit normal to $\partial\Omega$. We suppose that Ω contains a finite number of droplets, each of the form $z_j + \alpha B_j$, where $B_j \subset \mathbb{R}^d$ is a bounded, smooth domain containing the origin. The total collection of deformable inhomogeneities thus takes the form $\mathcal{B}_\alpha = \cup_{j=1}^m (z_j + \alpha B_j)$. The points $z_j \in \Omega, j = 1, \dots, m$, that determine the location of the droplets are assumed to satisfy

$$|z_j - z_l| \geq d_0 > 0, \forall j \neq l \quad \text{and} \quad \text{dist}(z_j, \partial\Omega) \geq d_0 > 0. \quad (2.1)$$

We also assume that $\alpha > 0$, the common order of magnitude of the diameters of the inhomogeneities, is sufficiently small that these are disjoint and their distance to $\mathbb{R}^d \setminus \overline{\Omega}$ is large than $d_0/2$. Let μ_0 denote the viscosity of the background medium, for simplicity we shall assume in this paper that it is constant. Let μ_j denote the constant viscosity of the j th inhomogeneity $z_j + \alpha B_j$. Using this notation we introduce the piecewise constant kinematic viscosity

$$\mu_\alpha(x) = \begin{cases} \mu_0 & x \in \Omega \setminus \overline{\mathcal{B}_\alpha}, \\ \mu_j & x \in z_j + \alpha B_j, j = 1, \dots, m. \end{cases}$$

Before formulating the model problem in the presence of droplets and via a non-stationary Stokes flow, we introduce the following notations.

Notation: I_d denotes the unit matrix in $\mathbb{R}^{d \times d}$, and \mathbf{I} denotes the identity fourth-order tensor. The scalar product on $\mathbb{R}^{d \times d}$ is defined by $A : B = \text{trace}(A^{tr} B)$ where the superscript tr denotes the transpose of a matrix. $L_0^2(\Omega)$ denotes the space of the functions of $L^2(\Omega)$ which differ by a constant and throughout this paper, the inner product between two vectors u and w is denoted by $u \cdot w$. The strain rate tensor \mathbf{e} for the flow is defined as follows:

$$\mathbf{e}(u) := \frac{1}{2}(\nabla u + (\nabla u)^{tr}) = \left(\frac{1}{2} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) \right)_{1 \leq k, l \leq d}; \quad u = (u_1, \dots, u_d) \in \mathbb{R}^d. \quad (2.2)$$

The divergence of a matrix-valued function A is denoted $\text{Div}(A)$, while the divergence of a vector-valued function u is denoted $\text{div}(u)$. **From now we denote by $L^2(\Omega)^d$ the space of vector-valued functions whose components of its elements belong to the space $L^2(\Omega)$ of square integrable functions. Likewise, we denote by $H^s(\Omega)^d$ the space of**

vector-valued functions where the notation H^s is used to denote those functions who along with all their derivatives of order less than or equal to s are in $L^2(\Omega)$. We denote by $H^s(\partial\Omega)^d$ the trace space.

Let $(v_\alpha, p_\alpha) \in L^2(0, T; H^1(\Omega)^d) \times L^2(0, T; H^1(\Omega))$ be the solution to the following nonstationary linearized Navier-Stokes system:

$$\begin{cases} \frac{\partial v_\alpha}{\partial t} - \text{Div} (2\mu_\alpha \mathbf{e}(v_\alpha)) + \nabla p_\alpha = 0 & \text{in } \Omega \times (0, T), \\ \text{div} (v_\alpha) = 0 & \text{in } \Omega, \\ v_\alpha|_{t=0} = \varphi & \text{in } \Omega, \\ v_\alpha = g & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (2.3)$$

Here $T > 0$ is a final observation time, the initial condition $\varphi \in \mathcal{C}^\infty(\overline{\Omega})^d$ and, $g \in L^2(0, T; H^{1/2}(\partial\Omega)^d)$ is a nonhomogeneous Dirichlet boundary data satisfying the standard flux compatibility condition:

$$\int_{\partial\Omega} g(x, t) \cdot \nu \, ds(x) = 0, \quad a.e. \, t \in (0, T). \quad (2.4)$$

We denote by $(v, p) \in L^2(0, T; H^1(\Omega)^d) \times L^2(0, T; H^1(\Omega))$ the solution of the non-stationary incompressible Stokes problem in the absence of any deformable inhomogeneities (the background solution):

$$\begin{cases} \frac{\partial v}{\partial t} - \text{Div} (2\mu_0 \mathbf{e}(v)) + \nabla p = 0 & \text{in } \Omega \times (0, T), \\ \text{div} (v) = 0 & \text{in } \Omega, \\ v|_{t=0} = \varphi & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega \times (0, T), \end{cases} \quad (2.5)$$

where φ and g are given in (2.3).

Let us now recall the notion of viscous moment tensor (VMT) which appears naturally as a limit of the elastic moment tensor (EMT) corresponds to the Lamé system and introduced firstly in [4]. We still denote by μ the viscosity in the whole space containing a rescaled inhomogeneity, i.e.

$$\mu(x) := \begin{cases} \mu_0, & x \in \mathbb{R}^d \setminus \overline{B_j}, \\ \tilde{\mu}_0, & x \in B_j, \end{cases}$$

for $j = 1, \dots, m$.

According to [4, 5] we define the symmetric, fourth-order, viscous moment tensor (VMT) denoted by $\mathcal{V}^{(j)} = (V_{klpq}^{(j)})_{k,l,p,q=1,\dots,d}$ as follows:

$$V_{klpq}^{(j)} = (\tilde{\mu}_0 - \mu_0) \int_{B_j} \nabla \hat{\mathbf{v}}_{pq}^{(j)} : (\nabla(\zeta_k e_l) + \nabla(\zeta_k e_l)^{tr}) d\zeta \quad \text{for } j = 1, \dots, m, \quad (2.6)$$

where, for $p, q = 1, \dots, d$, $\hat{\mathbf{v}}_{pq}^{(j)} \in H^1(\Omega)^d$ is the solution to

$$\begin{cases} \mu_0 \Delta \hat{\mathbf{v}}_{pq}^{(j)} + \nabla \hat{p}^{(j)} = 0 & \text{in } \mathbb{R}^d \setminus \bar{B}_j, \\ \tilde{\mu}_0 \Delta \hat{\mathbf{v}}_{pq}^{(j)} + \nabla \hat{p}^{(j)} = 0 & \text{in } B_j, \\ \hat{\mathbf{v}}_{pq}^{(j)}|_- = \hat{\mathbf{v}}_{pq}^{(j)}|_+ & \text{on } \partial B_j, \\ (\hat{p}^{(j)}\nu + \tilde{\mu}_0 \frac{\partial \hat{\mathbf{v}}_{pq}^{(j)}}{\partial \nu})|_- = (\hat{p}^{(j)}\nu + \mu_0 \frac{\partial \hat{\mathbf{v}}_{pq}^{(j)}}{\partial \nu})|_+ & \text{on } \partial B_j, \\ \operatorname{div}(\hat{\mathbf{v}}_{pq}^{(j)}) = 0 & \text{in } \mathbb{R}^d, \\ \hat{\mathbf{v}}_{pq}^{(j)}(\zeta) - \zeta_p e_q + \delta_{pq} \tilde{d}(\zeta) = O(|\zeta|^{1-d}) & \text{as } |\zeta| \rightarrow \infty, \\ \hat{p}^{(j)}(\zeta) = O(|\zeta|^{-d}) & \text{as } |\zeta| \rightarrow \infty. \end{cases} \quad (2.7)$$

The subscripts $+$ and $-$ indicate the limits from outside and inside of B_j , respectively, and throughout this paper (e_1, \dots, e_d) denotes the standard basis for \mathbb{R}^d . Moreover, for $\zeta \in \mathbb{R}^d$ we denote

$$\tilde{d}(\zeta) := 1/d \sum_{k=1}^d \zeta_k e_k,$$

the Kronecker's index is denoted by δ_{pq} .

3 The main results

The main objective of this article is to find several small deformable droplets modeled by;

$$B_\alpha^j := z_j + \alpha B_j, \quad (3.1)$$

included in Ω and located at points $z_j \in \Omega$, $j = 1, \dots, m$.

Let v be the solution to (2.5), then it is convenient to define the conormal derivative $\frac{\partial v}{\partial \nu}(x, t)$ on $\partial\Omega \times (0, T)$ by:

$$\frac{\partial v}{\partial \nu}|_{\partial\Omega \times (0, T)} := (2\mu_0 \mathbf{e}(v) - p I_d)\nu.$$

Similarly, for the solution v_α of (2.3), we may introduce the conormal derivative;

$$\frac{\partial v_\alpha}{\partial \nu}|_{\partial\Omega \times (0, T)} := (2\mu_\alpha \mathbf{e}(v_\alpha) - p_\alpha I_d)\nu = (2\mu_0 \mathbf{e}(v_\alpha) - p_\alpha I_d)\nu$$

because $\mu_\alpha = \mu_0$ on $\partial\Omega$ (outside of B_α^j , $j = 1, \dots, m$). Consequently,

$$\frac{\partial(v_\alpha - v)}{\partial \nu}|_{\partial\Omega \times (0, T)} = (2\mu_0 \mathbf{e}(v_\alpha - v) - (p_\alpha - p) I_d)\nu.$$

Then we have,

$$\frac{\partial(v_\alpha - v)}{\partial \nu} = \sigma(v_\alpha - v, p_\alpha - p)\nu \text{ on } \partial\Omega \times (0, T), \quad (3.2)$$

where generally $\sigma(w, \pi) := 2\mu_0 \mathbf{e}(w) - \pi I_d$ denotes the stress tensor, and $\sigma(w, \pi)\nu$ the Cauchy force on the boundary $\partial\Omega \times (0, T)$.

To give the main results of this paper, we need firstly to define in terms of the Cauchy force $\sigma(v_\alpha - v, p_\alpha - p)\nu$, given by (3.2), the following weighted boundary measurements:

$$\Lambda_\alpha(T) := \int_0^T \int_{\partial\Omega} \psi \cdot \sigma(v_\alpha - v, p_\alpha - p)\nu_x ds(x)dt, \quad (3.3)$$

where v_α, v are the solutions to (2.3) and (2.5) respectively, and the vector-valued function $\psi \in C^2(\bar{\Omega} \times [0, T])^d$ satisfies $\partial_t \psi - \text{Div}(2\mu_0 \mathbf{e}(\psi)) + \nabla p = 0$ in $\Omega \times [0, T]$ with $\psi(x, T) = 0$ for $x \in \Omega$, $\text{div}(\psi) = 0$.

According to (3.3), we propose a resolution of the inverse problem of reconstructing B_α^j based on the following main result.

Theorem 3.1 *Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$, be a bounded domain with $C^{1,1}$ -boundary. Suppose that we have all hypothesis (2.1) and (2.4). Let v_α, v be solutions to (2.3) and (2.5) respectively. Then, the following asymptotic expansion for $\Lambda_\alpha(T)$ holds as $\alpha \rightarrow 0$:*

$$\Lambda_\alpha(T) = \alpha^d \sum_{j=1}^m (\mu_j - \mu_0) \int_0^T \mathbf{e}(v)(z_j, t) : \mathcal{V}^{(j)} \mathbf{e}(\psi)(z_j, t) dt + O(\alpha^{d+1} |\log \alpha|^{3-d}). \quad (3.4)$$

The proof of the above theorem will be given later. On the other hand, by making appropriate choice of test function ψ and background solution v , we will develop from the asymptotic formula given by Theorem 3.1 an efficient location search algorithm for detecting the droplets $B_\alpha^j, j = 1, \dots, m$. This problem of reconstruction will be based on finite measurements of the Cauchy force $\sigma(v_\alpha - v, p_\alpha - p)\nu$ defined by (3.2) on $\partial\Omega \times (0, T)$.

Before proving Theorem 3.1, we suggest the following estimate of $v_\alpha - v$ as follows.

Theorem 3.2 *Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$, be a bounded domain with $C^{1,1}$ -boundary. Suppose that we have all hypothesis (2.1) and (2.4). Let v_α, v be solutions to (2.3) and (2.5) respectively. Then, there exist a constant C such that*

$$\|v_\alpha - v\|_{L^2(\Omega \times [0, T])^d} \leq C \alpha^{\frac{d}{2}}. \quad (3.5)$$

Here C can be given explicitly and dependant on T, v, B_j, μ_0 and μ_j .

Proof. Let v_α be the solution to (2.5) and v the solution to (2.3) respectively. Expanding the following

$$\begin{aligned} \int_0^T \int_\Omega 2\mu_\alpha \mathbf{e}(v) : \mathbf{e}(v_\alpha - v) dxdt &= \int_0^T \int_\Omega 2\mu_0 \mathbf{e}(v) : \mathbf{e}(v_\alpha - v) dxdt \\ &\quad - \int_0^T \int_\Omega 2(\mu_0 - \mu_\alpha) \mathbf{e}(v) : \mathbf{e}(v_\alpha - v) dxdt. \end{aligned} \quad (3.6)$$

Choosing $v_\alpha - v$ as a test function in (2.3), integration by parts over Ω yields

$$\begin{aligned} \int_0^T \int_\Omega \frac{\partial v}{\partial t} \cdot (v_\alpha - v) dxdt + \int_0^T \int_\Omega 2\mu_0 \mathbf{e}(v) : \mathbf{e}(v_\alpha - v) dxdt - \int_0^T \int_{\partial\Omega} (2\mu_0 \mathbf{e}(v) - pI_d)\nu_x \cdot (v_\alpha - v) ds(x)dt \\ = \int_0^T \int_\Omega \left(\frac{\partial v}{\partial t} - \text{Div}(2\mu_0 \mathbf{e}(v)) + \nabla p \right) \cdot (v_\alpha - v) dxdt = 0, \end{aligned}$$

where we have used $\operatorname{div}(v) = 0$. Then by considering the Dirichelet conditions on $\partial\Omega$, for both vector-valued functions v and v_α , the following holds

$$\int_0^T \int_\Omega 2\mu_0 \mathbf{e}(v) : \mathbf{e}(v_\alpha - v) dxdt = - \int_0^T \int_\Omega \frac{\partial v}{\partial t} \cdot (v_\alpha - v) dxdt. \quad (3.7)$$

Inserting (3.7) into relation (3.6), we obtain that,

$$\begin{aligned} \int_0^T \int_\Omega 2\mu_\alpha \mathbf{e}(v) : \mathbf{e}(v_\alpha - v) dxdt &= - \int_0^T \int_\Omega \frac{\partial v}{\partial t} \cdot (v_\alpha - v) dxdt \\ &\quad - \int_0^T \int_\Omega 2(\mu_0 - \mu_\alpha) \mathbf{e}(v) : \mathbf{e}(v_\alpha - v) dxdt \\ &= - \int_0^T \int_\Omega \frac{\partial v}{\partial t} \cdot (v_\alpha - v) dxdt \\ &\quad + 2 \sum_{j=1}^m \int_0^T \int_{B_\alpha^j} (\mu_j - \mu_0) \mathbf{e}(v) : \mathbf{e}(v_\alpha - v) dxdt, \end{aligned} \quad (3.8)$$

where B_α^j is given by (3.1). On the other hand, choosing $v_\alpha - v$ as a test function in (2.5), then by integrating by parts over Ω and as done for (3.7), we may obtain that

$$\int_0^T \int_\Omega 2\mu_\alpha \mathbf{e}(v_\alpha) : \mathbf{e}(v_\alpha - v) dxdt = - \int_0^T \int_\Omega \frac{\partial v_\alpha}{\partial t} \cdot (v_\alpha - v) dxdt. \quad (3.9)$$

Subtracting relation (3.8) from (3.9), we immediately obtain

$$\begin{aligned} \int_0^T \int_\Omega 2\mu_\alpha |\mathbf{e}(v_\alpha - v)|^2 dxdt &= - \int_0^T \int_\Omega 2\mu_\alpha \mathbf{e}(v) : \mathbf{e}(v_\alpha - v) dxdt + \int_0^T \int_\Omega 2\mu_\alpha \mathbf{e}(v_\alpha) : \mathbf{e}(v_\alpha - v) dxdt \\ &= \int_0^T \int_\Omega \frac{\partial v}{\partial t} \cdot (v_\alpha - v) dxdt - \int_0^T \int_\Omega \frac{\partial v_\alpha}{\partial t} \cdot (v_\alpha - v) dxdt \\ &\quad + 2 \sum_{j=1}^m \int_0^T \int_{B_\alpha^j} (\mu_0 - \mu_j) \mathbf{e}(v) : \mathbf{e}(v_\alpha - v) dxdt. \end{aligned}$$

That is,

$$\begin{aligned} \int_0^T \int_\Omega \frac{\partial v_\alpha}{\partial t} \cdot (v_\alpha - v) dxdt - \int_0^T \int_\Omega \frac{\partial v}{\partial t} \cdot (v_\alpha - v) dxdt + \int_0^T \int_\Omega 2\mu_\alpha |\mathbf{e}(v_\alpha - v)|^2 dxdt \\ = 2 \sum_{j=1}^m \int_0^T \int_{B_\alpha^j} (\mu_0 - \mu_j) \mathbf{e}(v) : \mathbf{e}(v_\alpha - v) dxdt. \end{aligned} \quad (3.10)$$

Now inserting (3.9) and (3.7) into (3.10), the following holds

$$\begin{aligned} - \int_0^T \int_\Omega 2\mu_\alpha \mathbf{e}(v_\alpha) : \mathbf{e}(v_\alpha - v) dxdt + \int_0^T \int_\Omega 2\mu_0 \mathbf{e}(v) : \mathbf{e}(v_\alpha - v) dxdt + \int_0^T \int_\Omega 2\mu_\alpha |\mathbf{e}(v_\alpha - v)|^2 dxdt \\ = 2 \sum_{j=1}^m \int_0^T \int_{B_\alpha^j} (\mu_0 - \mu_j) \mathbf{e}(v) : \mathbf{e}(v_\alpha - v) dxdt. \end{aligned} \quad (3.11)$$

Let $\underline{\mu} := \inf_{x \in \Omega} \mu_\alpha(x)$, then by definition of μ_α , we have $\underline{\mu} > 0$ and $\mu_0 \geq \underline{\mu}$. Moreover, the following holds,

$$-2\underline{\mu} \int_0^T \int_\Omega |\mathbf{e}(v_\alpha - v)|^2 dx dt \leq - \int_0^T \int_\Omega 2\mu_\alpha \mathbf{e}(v_\alpha) : \mathbf{e}(v_\alpha - v) dx dt + \int_0^T \int_\Omega 2\mu_0 \mathbf{e}(v) : \mathbf{e}(v_\alpha - v) dx dt.$$

Therefore relation (3.11) becomes,

$$\int_0^T \int_\Omega (\mu_\alpha - \underline{\mu}) |\mathbf{e}(v_\alpha - v)|^2 dx dt \leq \sum_{j=1}^m \int_0^T \int_{B_\alpha^j} (\mu_0 - \mu_j) \mathbf{e}(v) : \mathbf{e}(v_\alpha - v) dx dt,$$

and this together with Cauchy-Schwarz inequality and the fact that $\mathbf{e}(v)$ is bounded in $\Omega \times (0, T)$ (by assumption of regularity of v) enables one to get the following estimates,

$$\begin{aligned} \int_0^T \int_\Omega (\mu_\alpha - \underline{\mu}) |\mathbf{e}(v_\alpha - v)|^2 dx dt &\leq \sum_{j=1}^m \int_0^T |\mu_0 - \mu_j| \|\mathbf{e}(v)(\cdot, t)\|_{L^2(B_\alpha^j)^{d^2}} \|\mathbf{e}(v_\alpha - v)(\cdot, t)\|_{L^2(B_\alpha^j)^{d^2}} dt \\ &\leq \sum_{j=1}^m |\mu_0 - \mu_j| |B_\alpha^j|^{\frac{d}{2}} T^{\frac{1}{2}} \sup_{(x,t) \in \Omega \times (0,T)} |\mathbf{e}(v)(x,t)| \|\mathbf{e}(v_\alpha - v)\|_{L^2(\Omega \times (0,T))^{d^2}} = C\alpha^{\frac{d}{2}} \|\mathbf{e}(v_\alpha - v)\|_{L^2(\Omega \times (0,T))^{d^2}}. \end{aligned}$$

Thus,

$$\|\mathbf{e}(v_\alpha - v)\|_{L^2(\Omega \times (0,T))^{d^2}} \leq C\alpha^{\frac{d}{2}},$$

and the desired result follows immediately by invoking the Korn inequality. \square

4 Proof of the asymptotic formula

In this section, we focus our attention on proving rigorously Theorem 3.1. Let us, firstly, introducing the following vector-valued function

$$V(x, t) = v_\alpha(x, t) - v(x, t) + \alpha \sum_{j=1}^m \sum_{k,l=1}^d \partial_l v(z_j; t)_k \widehat{\mathbf{v}}_{kl}^{(j)} \left(\frac{x - z_j}{\alpha} \right), \quad (4.1)$$

where v_α , v and $\widehat{\mathbf{v}}_{kl}^{(j)}$ are solutions to (2.3), (2.5) and (2.7) respectively. It is clearly that $V \in L^2(0, T; H^1(\Omega)^d)$ since $v, v_\alpha \in L^2(0, T; H^1(\Omega)^d)$, the function $\partial_l v(z_j; t)_k$ does not depend on space variable x and the solution $\widehat{\mathbf{v}}_{kl}^{(j)}$ of (2.7) belongs to $H^1(\Omega)^d$.

The following estimate holds.

Proposition 4.1 *Let $d = 2, 3$. Suppose that we have all hypothesis (2.1) and (2.4) and let V be given by (4.1). Then, there exist a constant C such that,*

$$\|\mathbf{e}(V)\|_{L^2(\Omega \times (0,T))^{d^2}} \leq C\alpha^{1+\frac{d}{2}} |\log \alpha|^{\frac{3-d}{2}}. \quad (4.2)$$

Here the constant C is independent of α .

Proof. We set,

$$\tilde{\Omega} := \left\{ \frac{x - z_*}{\alpha}; x \in \Omega \right\}; \quad z_* = z_1, \dots, z_m \quad (4.3)$$

and we observe that $\mathbf{e}(V)$ satisfies:

$$\|\mathbf{e}(V)\|_{L^2(\Omega \times (0, T))^{d^2}} = \left(\alpha^{2+d} \int_0^{T/\alpha^2} \|\mathbf{e}(\tilde{V})(\cdot, t)\|_{L^2(\tilde{\Omega})^{d^2}}^2 dt \right)^{1/2}; \quad \text{for } x \in \tilde{\Omega}, \quad 0 \leq t \leq T/\alpha^2, \quad (4.4)$$

where $\tilde{V}(x, t) := V(\alpha x, \alpha^2 t)$.

Hence to prove (4.2), it suffices to estimate the term $\int_0^{T/\alpha^2} \|\mathbf{e}(\tilde{V})(\cdot, t)\|_{L^2(\tilde{\Omega})^{d^2}}^2 dt$. To do this we may find out, firstly, the equations that the vector-valued function \tilde{V} can satisfy in the rescaled domain $\tilde{\Omega}$.

Let $P := p_\alpha - p$ where p_α, p are introduced in (2.3) and (2.5) respectively. One may use (4.1) to observe that V satisfies:

$$\frac{\partial V}{\partial t} - \text{Div} (2\mu_0 \mathbf{e}(V)) + \nabla P = \alpha \sum_{j=1}^m \sum_{k,l=1}^d \partial_t \partial_l v(z_*; t)_k \widehat{\mathbf{v}}_{kl}^{(j)} \left(\frac{x - z_*}{\alpha} \right) := F_1(x, t) \quad \text{in } (\Omega \setminus \bar{\mathcal{B}}_\alpha) \times (0, T),$$

$$\begin{aligned} \frac{\partial V}{\partial t} - \text{Div} (2\mu_\alpha \mathbf{e}(V)) + \nabla P &= (\mu_\alpha - \mu_0) \chi_{\mathcal{B}_\alpha} \text{Div} (2\mathbf{e}(v)) + \alpha \sum_{j=1}^m \sum_{k,l=1}^d \partial_t \partial_l v(z_*; t)_k \widehat{\mathbf{v}}_{kl}^{(j)} \left(\frac{x - z_*}{\alpha} \right) \\ &:= F_1(x, t) \quad \text{in } \mathcal{B}_\alpha \times (0, T), \end{aligned}$$

$$\text{div} (V) = 0 \quad \text{in } \Omega.$$

Moreover, we have

$$V|_+ = V|_- \quad \text{on } \partial(z_* + \alpha B) \times (0, T),$$

$$\mu_0 \frac{\partial V}{\partial \nu} |_+ - \mu_\alpha \frac{\partial V}{\partial \nu} |_- = (\mu_\alpha - \mu_0) \frac{\partial v}{\partial \nu} := F_2(x, t) \quad \text{on } \partial(z_* + \alpha B) \times (0, T)$$

and

$$V|_{t=0} = \alpha \sum_{j=1}^m \sum_{k,l=1}^d \partial_l \varphi(0)_k \widehat{\mathbf{v}}_{kl}^{(j)} \left(\frac{x - z_*}{\alpha} \right) \quad \text{in } \Omega,$$

$$\mu_0 V = \alpha \mu_0 \sum_{j=1}^m \sum_{k,l=1}^d \partial_l v(z_*; t)_k \widehat{\mathbf{v}}_{kl}^{(j)} \left(\frac{x - z_*}{\alpha} \right) := F_3(x, t) \quad \text{on } \partial\Omega \times (0, T).$$

Now let $f_1(x, t) := \alpha^2 F_1(\alpha x, \alpha^2 t)$ and $f_i(x, t) := \alpha F_i(\alpha x, \alpha^2 t)$ for $i = 2, 3$. Then it is easily seen that \tilde{V} satisfies:

$$\left\{ \begin{array}{l} \frac{\partial \tilde{V}}{\partial t} - \text{Div} (2\mu_0 \mathbf{e}(\tilde{V})) + \nabla \tilde{P} = f_1 \quad \text{in } (\tilde{\Omega} \setminus \bar{\mathcal{B}}_\alpha) \times (0, T/\alpha^2), \\ \frac{\partial \tilde{V}}{\partial t} - \text{Div} (2\mu_\alpha \mathbf{e}(\tilde{V})) + \nabla \tilde{P} = f_1 \quad \text{in } \mathcal{B}_\alpha \times (0, T/\alpha^2), \\ \text{div} (\tilde{V}) = 0 \quad \text{in } \tilde{\Omega}, \\ \tilde{V}|_+ = \tilde{V}|_- \quad \text{on } \partial(z_* + \alpha B) \times (0, T/\alpha^2), \\ \mu_0 \frac{\partial \tilde{V}}{\partial \nu}|_+ - \mu_\alpha \frac{\partial \tilde{V}}{\partial \nu}|_- = f_2 \quad \text{on } \partial(z_* + \alpha B) \times (0, T/\alpha^2), \\ \tilde{V}|_{t=0} = \alpha \sum_{j=1}^m \sum_{k,l=1}^d \partial_l \varphi(0)_k \hat{\mathbf{v}}_{kl}^{(j)}(x - z_*) \quad \text{in } \tilde{\Omega}, \\ \mu_0 \tilde{V} = f_3 \quad \text{on } \partial \tilde{\Omega} \times (0, T/\alpha^2). \end{array} \right. \quad (4.5)$$

Now, using the trace estimate and integrations by parts over $\Omega \times (0, t)$ for $0 \leq t \leq T$, straightforward calculations give

$$\begin{aligned} \sup_{0 \leq t \leq T} \|\tilde{V}(\cdot, t)\|_{L^2(\tilde{\Omega})^d} + \|\mathbf{e}(\tilde{V})\|_{L^2(\tilde{\Omega} \times (0, T))^{d^2}} &\leq C \|\tilde{V}|_{t=0}\|_{L^2(\tilde{\Omega})^d} + C' \left(\|f_1\|_{L^2(0, T; L^2(\tilde{\Omega})^d)} \right. \\ &\quad \left. + \|f_2\|_{L^2(0, T; L^2(\partial B)^d)} + \|f_3\|_{L^2(0, T; L^2(\partial \tilde{\Omega})^d)} \right), \end{aligned} \quad (4.6)$$

where the constants C, C' depend only on the domains $\Omega, B_j, j = 1, \dots, m$, and the constants $\mu_j, j = 0, \dots, m$.

Now we need to estimate the following terms:

$$\int_0^{T/\alpha^2} \|f_1(\cdot, t)\|_{L^2(\tilde{\Omega})^d}^2 dt, \quad \int_0^{T/\alpha^2} \|f_2(\cdot, t)\|_{L^2(\partial B)^d}^2 dt, \quad \text{and} \quad \int_0^{T/\alpha^2} \|f_3(\cdot, t)\|_{L^2(\partial \tilde{\Omega})^d}^2 dt.$$

Before doing this we recall from (2.7) that,

$$\hat{\mathbf{v}}_{pq}(x) - x_p e_q + \delta_{pq} \tilde{d}(x) = O(|x|^{1-d}) \text{ as } |x| \rightarrow \infty, \quad (4.7)$$

where for $x \in \mathbb{R}^d$, $\tilde{d}(x) := \frac{1}{d} \sum_{k=1}^d x_k e_k$. Since $\text{diam}(\tilde{\Omega}) = O(1/\alpha)$, we have

$$\int_{\tilde{\Omega}} |\hat{\mathbf{v}}_{pq}(x) - x_p e_q + \delta_{pq} \tilde{d}(x)|^2 dx = \begin{cases} O(\log(1/\alpha)), & \text{if } d = 2, \\ O(1), & \text{if } d = 3. \end{cases} \quad (4.8)$$

Therefore,

$$\int_{\tilde{\Omega}} |\tilde{V}(x, 0)|^2 dx \leq C \begin{cases} \alpha^2 |\log(\alpha)|, & \text{if } d = 2, \\ \alpha^2, & \text{if } d = 3. \end{cases} \quad (4.9)$$

Moreover, by definition of f_1 we get that

$$\begin{aligned} \int_0^{T/\alpha^2} \|f_1(\cdot, t)\|_{L^2(\tilde{\Omega})^d}^2 dt &\leq C \alpha^4 \int_0^{T/\alpha^2} \int_B |\text{Div}(2\mathbf{e}(v))(\alpha x, \alpha^2 t)|^2 dx dt \\ &+ C \alpha^6 \sum_{j=1}^m \sum_{k,l=1}^d \int_0^{T/\alpha^2} |\partial_l \partial_l v(z_*; t)_k|^2 dt \int_{\tilde{\Omega}} |\hat{\mathbf{v}}_{pq}(x) - x_p e_q + \delta_{pq} \tilde{d}(x)|^2 dx. \end{aligned}$$

From (4.8) and the fact that v is a smooth vector-valued function, we obtain:

$$\int_0^{T/\alpha^2} \|f_1(\cdot, t)\|_{L^2(\tilde{\Omega})^d}^2 dt \leq C\alpha^2. \quad (4.10)$$

On the other hand, we have

$$(\mu_\alpha - \mu_0) \frac{\partial v}{\partial \nu}(x, t) = O(|x|) \quad \text{for all } (x, t) \in \partial(z_* + \alpha B) \times (0, T).$$

Consequently, $f_2(x, t) = \alpha F_2(\alpha x, \alpha^2 t) = O(\alpha^2)$, and hence

$$\int_0^{T/\alpha^2} \|f_2(\cdot, t)\|_{L^2(\partial B)^d}^2 dt \leq C\alpha^2. \quad (4.11)$$

Regarding (4.7) and (4.3), we can write $|\hat{v}_{pq}(x) - x_p e_q + \delta_{pq} \tilde{d}(x)| = O(\alpha^{d-1})$ for all $x \in \partial\tilde{\Omega}$. Thus,

$$\begin{aligned} \int_0^{T/\alpha^2} \|f_3(\cdot, t)\|_{L^2(\partial\tilde{\Omega})^d}^2 dt &\leq \mu_0 \alpha^2 \sum_{j=1}^m \sum_{k,l=1}^d \int_0^{T/\alpha^2} |\partial_l v(z_*; t)_k|^2 dt \int_{\partial\tilde{\Omega}} |\hat{v}_{kl}(x) - x_k e_l + \delta_{kl} \tilde{d}(x)|^2 dx \\ &\leq C\alpha^2. \end{aligned} \quad (4.12)$$

Now from (4.6) we have,

$$\begin{aligned} \|\mathbf{e}(\tilde{V})\|_{L^2(\tilde{\Omega} \times (0, T))^{d^2}} &\leq C \|\tilde{V}|_{t=0}\|_{L^2(\tilde{\Omega})^d} + C' \left(\|f_1\|_{L^2(0, T; L^2(\tilde{\Omega})^d)} \right. \\ &\quad \left. + \|f_2\|_{L^2(0, T; L^2(\partial B)^d)} + \|f_3\|_{L^2(0, T; L^2(\partial\tilde{\Omega})^d)} \right), \end{aligned} \quad (4.13)$$

Therefore by using (4.9), (4.10), (4.11) and (4.12) the inequality (4.13) becomes:

$$\|\mathbf{e}(\tilde{V})\|_{L^2(\tilde{\Omega} \times (0, T))^{d^2}} \leq C \begin{cases} \alpha |\log(\alpha)|^{1/2}, & \text{if } d = 2, \\ \alpha, & \text{if } d = 3. \end{cases} \quad (4.14)$$

On the other hand, by change of variable at t , we have:

$$\int_0^{T/\alpha^2} \|\mathbf{e}(\tilde{V})(\cdot, t)\|_{L^2(\tilde{\Omega})^{d^2}}^2 dt = 1/\alpha^2 \|\mathbf{e}(\tilde{V})\|_{L^2(\tilde{\Omega} \times (0, T))^{d^2}}^2,$$

which by (4.14) becomes:

$$\int_0^{T/\alpha^2} \|\mathbf{e}(\tilde{V})(\cdot, t)\|_{L^2(\tilde{\Omega})^{d^2}}^2 dt \leq C \begin{cases} |\log(\alpha)|, & \text{if } d = 2, \\ 1, & \text{if } d = 3. \end{cases} \quad (4.15)$$

To achieve the proof, we may insert (4.15) into relation (4.4). \square

We now proceed to prove Theorem 3.1.

Proof of Theorem 3.1. Let us start by simplifying the definition (3.3). So let setting $\mathbf{u} = v_\alpha - v$ where v_α, v are solutions to (2.3) and (2.5) respectively. Then

integrating by parts over $\Omega \times (0, T)$, and using both conditions $\mathbf{u}(x, 0) = v_\alpha(x, 0) - v(x, 0) = 0$ and $\psi(x, t = T) = 0$, we obtain that

$$\begin{aligned}
\int_0^T \int_\Omega \left(\frac{\partial \psi}{\partial t} \cdot \mathbf{u} - \mu_0 \mathbf{e}(\mathbf{u}) : \mathbf{e}(\psi) \right) dx dt &= \int_\Omega (\mathbf{u}(x, T) \cdot \psi(x, T) - \mathbf{u}(x, 0) \cdot \psi(x, 0)) dx \\
&\quad - \int_0^T \int_\Omega \left(\frac{\partial \mathbf{u}}{\partial t} \cdot \psi + \mu_0 \mathbf{e}(\mathbf{u}) : \mathbf{e}(\psi) \right) dx dt \\
&= - \int_0^T \int_\Omega \left(\frac{\partial (v_\alpha - v)}{\partial t} \cdot \psi + \mu_0 \mathbf{e}(v_\alpha - v) : \mathbf{e}(\psi) \right) dx dt \\
&= - \int_0^T \int_{\partial \Omega} \psi \cdot (2\mu_0 \mathbf{e}(v_\alpha - v) - (p_\alpha - p) I_d) \nu_x ds(x) dt.
\end{aligned}$$

Taking (3.2) into consideration, the following holds,

$$\int_0^T \int_\Omega \left(\frac{\partial \psi}{\partial t} \cdot \mathbf{u} - \mu_0 \mathbf{e}(\mathbf{u}) : \mathbf{e}(\psi) \right) dx dt = - \int_0^T \int_{\partial \Omega} \psi \cdot \sigma(v_\alpha - v, p_\alpha - p) \nu_x ds(x) dt. \quad (4.16)$$

On the other hand, by integrating by parts over $(0, T)$, we have

$$\begin{aligned}
\int_0^T \int_\Omega \left(\frac{\partial \psi}{\partial t} \cdot \mathbf{u} - \mu_0 \mathbf{e}(\mathbf{u}) : \mathbf{e}(\psi) \right) dx dt &= \int_\Omega (\mathbf{u}(x, T) \cdot \psi(x, T) - \mathbf{u}(x, 0) \cdot \psi(x, 0)) dx \\
&\quad - \int_0^T \int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \psi dx dt - \int_0^T \int_\Omega \mu_0 \mathbf{e}(\mathbf{u}) : \mathbf{e}(\psi) dx dt \\
&= - \int_0^T \int_\Omega \left(\frac{\partial \mathbf{u}}{\partial t} \cdot \psi + \mu_0 \mathbf{e}(\mathbf{u}) : \mathbf{e}(\psi) \right) dx dt \quad (4.17)
\end{aligned}$$

Replacing $\mathbf{u} = v_\alpha - v$ in the right-hand side of (4.17) and integrating by parts over Ω , we obtain that

$$\begin{aligned}
\int_0^T \int_\Omega \left(\frac{\partial \mathbf{u}}{\partial t} \cdot \psi + \mu_0 \mathbf{e}(\mathbf{u}) : \mathbf{e}(\psi) \right) dx dt &= - \int_0^T \int_\Omega \left(\frac{\partial v}{\partial t} \cdot \psi + \mu_0 \mathbf{e}(v) : \mathbf{e}(\psi) \right) dx dt \\
&\quad + \int_0^T \int_\Omega \left(\frac{\partial v_\alpha}{\partial t} \cdot \psi + \mu_0 \mathbf{e}(v_\alpha) : \mathbf{e}(\psi) \right) dx dt \\
&= - \int_0^T \int_{\partial \Omega} \psi \cdot (2\mu_0 \mathbf{e}(v) - p I_d) \nu_x ds(x) dt \\
&\quad + \int_0^T \int_\Omega (\mu_0 - \mu_\alpha) \mathbf{e}(v_\alpha) : \mathbf{e}(\psi) dx dt \\
&\quad + \int_0^T \int_\Omega \left(\frac{\partial v_\alpha}{\partial t} \cdot \psi + \mu_\alpha \mathbf{e}(v_\alpha) : \mathbf{e}(\psi) \right) dx dt.
\end{aligned}$$

Therefore, by integrating by parts again, we obtain

$$\begin{aligned}
\int_0^T \int_\Omega \left(\frac{\partial \mathbf{u}}{\partial t} \cdot \psi + \mu_0 \mathbf{e}(\mathbf{u}) : \mathbf{e}(\psi) \right) dx dt &= - \int_0^T \int_{\partial \Omega} \psi \cdot (2\mu_0 \mathbf{e}(v) - p I_d) \nu_x ds(x) dt \\
&\quad + \int_0^T \int_{\partial(\Omega \setminus \mathcal{B}_\alpha)} \psi \cdot (2\mu_0 \mathbf{e}(v_\alpha) - p_\alpha I_d) \nu_x ds(x) dt
\end{aligned}$$

$$\begin{aligned}
& + \int_0^T \sum_{j=1}^m \int_{\partial B_\alpha^j} \psi \cdot (2\mu_j \mathbf{e}(v_\alpha) - p_\alpha I_d) |-\nu_x ds(x) dt + \int_0^T \sum_{j=1}^m \int_{B_\alpha^j} (\mu_0 - \mu_j) \mathbf{e}(v_\alpha) : \mathbf{e}(\psi) dx dt \\
& = \sum_{j=1}^m \int_0^T \int_{B_\alpha^j} (\mu_0 - \mu_j) \mathbf{e}(v_\alpha) : \mathbf{e}(\psi) dx dt,
\end{aligned}$$

where from Section 2 and (3.1) we have $\mathcal{B}_\alpha := \cup_{j=1}^m B_\alpha^j$.

Further, returning up to relation (4.17) and using the above relation, the following holds,

$$\int_0^T \int_\Omega \left(\frac{\partial \psi}{\partial t} \cdot \mathbf{u} - \mu_0 \mathbf{e}(\mathbf{u}) : \mathbf{e}(\psi) \right) dx dt = - \sum_{j=1}^m \int_0^T \int_{B_\alpha^j} (\mu_0 - \mu_j) \mathbf{e}(v_\alpha) : \mathbf{e}(\psi) dx dt.$$

Now one may compare relations (4.16) and (3.3) to find that,

$$\Lambda_\alpha(T) := \sum_{j=1}^m (\mu_0 - \mu_j) \int_0^T \int_{B_\alpha^j} \mathbf{e}(v_\alpha)(x, t) : \mathbf{e}(\psi)(x, t) dx dt. \quad (4.18)$$

Moreover, since $\mathbf{e}(\hat{\mathbf{v}}_{pq}(x) - x_p e_q + \delta_{pq} \tilde{d}(x)) = O(|x|^{-d})$ as $|x| \rightarrow \infty$ and the fact that the inclusions are well-separated, it follows from (3.1) and (4.18) that

$$\begin{aligned}
& |\Lambda_\alpha(T) - \sum_{j=1}^m (\mu_j - \mu_0) \int_0^T \int_{B_\alpha^j} [\mathbf{e}(v)(x, t) - \sum_{k,l=1}^d \partial_l v(z_j; t)_k \mathbf{e}(\hat{\mathbf{v}}_{kl}^{(j)}) \left(\frac{x - z_j}{\alpha} \right)] : \mathbf{e}(\psi)(x, t) dx dt| \\
& = \left| \sum_{j=1}^m (\mu_j - \mu_0) \int_0^T \int_{B_\alpha^j} |\mathbf{e}(V)(x, t) : \mathbf{e}(\psi)(x, t) dx dt \right| \leq C \alpha^{d/2} \|\mathbf{e}(V)\|_{L^2(\Omega \times (0, T))}^{d^2},
\end{aligned}$$

and hence, by using Proposition 4.1, the following inequality holds

$$\begin{aligned}
& |\Lambda_\alpha(T) - \sum_{j=1}^m (\mu_j - \mu_0) \int_0^T \int_{B_\alpha^j} [\mathbf{e}(v)(x, t) - \sum_{k,l=1}^d \partial_l v(z_j; t)_k \mathbf{e}(\hat{\mathbf{v}}_{kl}^{(j)}) \left(\frac{x - z_j}{\alpha} \right)] : \mathbf{e}(\psi)(x, t) dx dt| \\
& \leq C \alpha^{1+d} |\log \alpha|^{3-d}.
\end{aligned} \quad (4.19)$$

On the other hand, since both v and ψ are divergence free, we have for all $\zeta \in \mathbb{R}^d$:

$$\begin{aligned}
\mathbf{e}(v)(x, t) \zeta & = \sum_{k,l=1}^d \partial_l v(z_j; t)_k (\zeta_k e_l - \delta_{kl} \tilde{d}(\zeta)) + O(\alpha) \text{ and } \mathbf{e}(\psi)(x, t) \zeta \\
& = \sum_{p,q=1}^d \partial_q \psi(z_j; t)_q (\zeta_q e_p - \delta_{pq} \tilde{d}(\zeta)) + O(\alpha) \text{ for all } x \in B_j. \quad (4.20)
\end{aligned}$$

Now inserting (4.20) into the left-hand side of (4.19), we obtain that

$$\begin{aligned}
\Lambda_\alpha(T) & = \alpha^d \sum_{j=1}^m (\mu_j - \mu_0) \int_0^T \sum_{k,l=1}^d \sum_{p,q=1}^d \partial_l v(z_j; t)_k \partial_q \psi(z_j; t)_q \int_{B_j} \nabla \hat{v}_{pq}^{(j)} : (\nabla(\zeta_k e_l) + \nabla(\zeta_k e_l)^{tr}) d\zeta dt \\
& \quad + O(\alpha^{1+d} |\log \alpha|^{3-d}),
\end{aligned}$$

which by definition (2.6) gives that,

$$\Lambda_\alpha(T) = \alpha^d \sum_{j=1}^m (\mu_j - \mu_0) \int_0^T \mathbf{e}(v)(z_j; t) : \sum_{p,q=1}^d \mathbf{e}(\psi)_{pq}(z_j; t) \int_{B_j} \mathbf{e}(\widehat{\mathbf{v}}_{pq}^{(j)})(x) dx dt + O(\alpha^{1+d} |\log \alpha|^{3-d}).$$

Thus,

$$\Lambda_\alpha(T) = \alpha^d \sum_{j=1}^m (\mu_j - \mu_0) \int_0^T \mathbf{e}(v)(z_j; t) : \mathcal{V}^{(j)} \mathbf{e}(\psi)(z_j; t) dt + O(\alpha^{1+d} |\log \alpha|^{3-d})$$

which achieves the proof of Theorem 3.1.

5 The location method

In this section, we focus our attention on developing a simple localization method by formulating a location search algorithm. This method is deeply based on the formula (3.4) for the purpose of identifying locations of the deformable inhomogeneities $B_\alpha^j = z_j + \alpha B_j$, $j = 1, \dots, m$ modeling the small deformable droplets immersed in an incompressible Newtonian fluid. A suitable choice of test functions ψ and background solutions v may be needed to develop a real-time location search algorithm [23]. This algorithm is related to the location search algorithm associated to a class of parabolic system as in [3]. This approach is also related to the linear sampling method [15, 22]. Without loss of generality, we consider the two-dimensional problem along the rest of this section. The case of dimension three can be done in the same way.

For $y \in \mathbb{R}^2 \setminus \overline{\Omega}$ the function ψ , introduced in Section 3, satisfies $(\partial_t - \mu_0 \Delta) \psi + \nabla p = 0$ in $\Omega \times [0, T]$ with $\psi(x, T) = 0$ for $x \in \Omega$ and is given by:

$$\psi(x, t) = \psi_y(x, t) := \widetilde{\psi}(1 - \operatorname{erf}(\eta_y(x, T - t))) \quad (5.1)$$

where $\widetilde{\psi}$ is a constant vector (can be given explicitly), $\eta_y(x, t) = x / \sqrt{4\mu_0 t}$ and $\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\xi^2} d\xi$.
More precisely,

$$\psi(x, t) = \psi_y(x, t) := \widetilde{\psi} \left(1 - \frac{2}{\sqrt{4\mu_0 \pi (T - t)}} \int_0^x e^{-\frac{|\xi - y|^2}{4\mu_0 (T - t)}} d\xi \right). \quad (5.2)$$

Therefore, by using (2.2) we get

$$\mathbf{e}(\psi_y)(z_j, t) := -\frac{2}{\sqrt{4\mu_0 \pi (T - t)}} e^{-\frac{|z_j - y|^2}{4\mu_0 (T - t)}} \widetilde{\Psi}, \quad (5.3)$$

where $\widetilde{\Psi}$ is a matrix determined according to the components of $\widetilde{\psi}$.

On the other hand, for $y' \in \mathbb{R}^2 \setminus \overline{\Omega}$, we choose

$$v(x, t) = v_{y'}(x, t) := \frac{\widetilde{v}}{\sqrt{4\mu_0 \pi t}} \int_0^x e^{-\frac{|\xi - y'|^2}{4\mu_0 t}} d\xi, \quad (5.4)$$

where \tilde{v} is a known constant vector. Here the Dirichlet condition g corresponds to the point source y' with initial condition $\varphi(x) = 0$ in Ω . Consequently, in the presence of the deformable inhomogeneities $z_j + \alpha B_j$, $j = 1, \dots, m$ and using Theorem 3.1, we can obtain that

$$\begin{aligned} \Lambda_\alpha(T) &= \alpha^2 \sum_{j=1}^m \frac{\mu_0 - \mu_j}{\pi \mu_0} (\tilde{V} : \mathcal{V}^{(j)} \tilde{\Psi}) \int_0^T \frac{1}{\sqrt{t(T-t)}} e^{(-\frac{|z_j - y|^2}{4\mu_0(T-t)} - \frac{|z_j - y'|^2}{4\mu_0 t})} dt \\ &\quad + O(\alpha^3 |\log(\alpha)|) \end{aligned}$$

where the matrix \tilde{V} is determined according to the components of \tilde{v} .

Let P be the orthogonal projection from the space of symmetric matrices onto the space of symmetric matrices of trace zero. Let I_d and \mathbf{I} be defined as in Section 2. Then according to [4], the orthogonal projection $P = (P_{klpq})_{1 \leq k, l, p, q \leq d}$ is given by

$$P = \mathbf{I} - \frac{1}{d} I_d \otimes I_d,$$

and more explicitly

$$P_{klpq} = \frac{1}{2} (\delta_{kp} \delta_{lq} + \delta_{kq} \delta_{lp}) - \frac{1}{d} \delta_{kl} \delta_{pq}. \quad (5.5)$$

Now we recall that the (4-tensor) viscous moment tensor (VMT) $\mathcal{V}^{(j)}$ can be regarded as a linear transformation on the space of symmetric matrices because of its symmetry. Then, as the Stokes system appears as a limiting case of the Lamé system, all coefficients of $\mathcal{V}^{(j)}$ can be determined explicitly by using P and the (4-tensor) elasticity moment tensor (EMT) M . For more details one can refer to [4, 5].

Next, suppose for the sake of simplicity that all the domains B_j are disks. Then it follows from [5] that $\mathcal{V}^{(j)} = a^{(j)} P$, where P is given by (5.5) for $d = 2$, and let

$$a^{(j)} = 4\mu_0 \frac{\mu_0 - \mu_j}{\mu_0 + \mu_j} |B_j|.$$

Let the source points $y_l \in \mathbb{R}^2 \setminus \overline{\Omega}$ for $l \in \mathbb{N}$, then the proposed location method for detecting the droplets B_α^j is as follows. For $n \in \mathbb{N}$, define the matrix $A = (A_{ll'})_{1 \leq l, l' \leq n}$ by

$$A_{ll'} := \alpha^2 \sum_{j=1}^m \frac{\mu_0 - \mu_j}{\pi \mu_0} a^{(j)} (\tilde{V} : P \tilde{\Psi}) \int_0^T \frac{1}{t^2(t-T)^2} e^{(-\frac{|z_j - y_l|^2}{4\mu_0(T-t)} - \frac{|z_j - y_{l'}|^2}{4\mu_0 t})} dt.$$

On the other hand, we can define the symmetric real matrix $C = (C_{ll'})_{1 \leq l, l' \leq n}$ by

$$C_{ll'} := \int_0^T \frac{1}{t^2(t-T)^2} e^{(-\frac{|z - y_l|^2}{4\mu_0(T-t)} - \frac{|z - y_{l'}|^2}{4\mu_0 t})} dt, \quad \text{for all } z \in \Omega.$$

One can remark that the matrix C may be decomposed as follows

$$C = \sum_{l=1}^p w_l(z) w_l(z)^*$$

for some $p \leq n$, where $w_l \in \mathbb{R}^n$ and w_l^* denotes the transpose of w_l . Let w_{l1}, \dots, w_{ln} be the components of the vector w_l , for $l = 1, \dots, p$.

Then, for $l = 1, \dots, p$ and for $z \in \Omega$ we define the vector $h^{(l)} \in \mathbb{R}^{n \times 2}$ by

$$h^{(l)} = [\tilde{\psi}_1 w_{l1}(z), \dots, \tilde{\psi}_n w_{ln}(z)]^*.$$

Let $y_l = (y_1^l, y_2^l)$ for $l = 1, \dots, n$, $z = (z_1, z_2)$, and $z_j = (z_1^{(j)}, z_2^{(j)})$. For $l = 1, \dots, p$, we also introduce

$$h_1^{(l)} = [\tilde{\psi}_{11} w_{l1}(z), \dots, \tilde{\psi}_{n1} w_{ln}(z)]^* \quad \text{and} \quad h_2^{(l)} = [\tilde{\psi}_{12} w_{l1}(z), \dots, \tilde{\psi}_{n2} w_{ln}(z)]^*.$$

Then we can now characterize the location of the inclusions in terms of the range of the matrix A as follows

$$\text{For all } l = 1, \dots, p: \quad h_1^{(l)} \text{ and } h_2^{(l)} \in \text{Range}(A) \text{ iff } z \in \{z_1, \dots, z_m\}. \quad (5.6)$$

The above characterization can be justified by following, for example, the approach done in the proof of Lemma 5.2 in [3] for the case of heat problem.

Now we define the singular value decomposition (SVD) of the matrix A by

$$A = U \Sigma V^*. \quad (5.7)$$

Moreover, we assume that for $l = 1, \dots, p$, the vectors $\{h_1^{(l)}, h_2^{(l)}\}$ are chosen to be linearly independent. Consequently, the rank of A is fixed to be equal to $2p$ and the first $2p$ columns of U , $\{w_1, \dots, w_{2p}\}$, make a basis for column space of A that we note it by U_S . But the other columns, $\{w_{2p+1}, w_{2p+2}, \dots, w_n\}$, provide a basis for the left null space of A that we note it by U_N . Let $p_{roj} = I - (U_S U_S^*)$ be the orthogonal projection onto the null space of A . Within the above remark (5.6) we claim that a test point z coincides with one of the positions z_j if and only if $p_{roj}(h^{(l)} u_0) = 0$, for any $u_0 \in \mathbb{R}^2 \setminus \{0\}$. Thus we can form an image of the locations $\{z_j; \quad j = 1, \dots, m\}$ by plotting, at each point $z \in \Omega$, the identifier of interest

$$I_{u_0}^{(l)}(z) := \frac{\|h^{(l)} u_0\|}{\|p_{roj}(h^{(l)} u_0)\|} \text{ for } l = 1, \dots, p.$$

The obtained plot will have sharp peaks at the discrete points $z_j; j = 1, \dots, m$ which determine the locations of the droplets. Precisely, for some positive T and for small finite number of equidistantly distributed source points $y_l, \quad l = 1, \dots, d$, we perform together the calculation of the SVD (5.7) of the matrix A (defined by the pattern $\Lambda_\alpha(T)$) and the decomposition of the matrix C . Then we fix the arbitrary vector u_0 to be equal to an element of the canonical basis in \mathbb{R}^d or equal to a linear combination between elements of this basis. Therefore, it is expected that the plots of the identifier of interest $z \mapsto I_{u_0}^{(l)}(z)$ illustrate the result indicating that sharp peaks should occur at the droplets locations.

5.1 Numerical experiments

In this subsection we suppose that Ω is a unit disk in \mathbb{R}^2 centered at the origin. We present here a numerical experiments for locating two or three inclusions, but our

procedure remains valid to locate any finite number m of inclusions. To begin, we assume that we know the values $A_{ll'}$ of the pattern $\Lambda_\alpha(T)$ for some positive T and for a small finite number of equidistantly distributed source points y_l .

Let,

$$\delta := \min_{l \neq l'} \|y_l - y_{l'}\|.$$

Then, for $l = 1, \dots, n$ we may set

$$y_l = \delta \left(\cos\left(\frac{2\pi l}{n}\right), \sin\left(\frac{2\pi l}{n}\right) \right).$$

To proceed with this numerical method, we notice that there are two cases of interest: $\delta \leq T$ and $\delta > T$. For our numerical experiments, we see that the second case is sufficient to show the validity of the localization method described in Section 5.

In the first numerical experiment we take two homogeneous disks B_1 and B_2 of common diameter $\alpha = 0.1$ and respectively centered at $z_1 = (0.21, 0.32)$ and $z_2 = (-0.43, -0.35)$, to be retrieved using $n = 10$ source points. We assume that the viscosity of the background medium is $\mu_0 = 1$, while the viscosities μ_j , of B_j , $j = 1, 2$ are equated to 3.

Notice that, the retrieval of the inclusions involves the calculation of the SVD (with $A = U\Sigma V^*$) of the matrix $A = [A_{ll'}] \in \mathbb{R}^{d \times d}$ and the decomposition of the matrix C . Numerically, we can find that the rank of C is equal to one ($p = 1$), and we expect to see four or six nonzero singular values β_j of the matrix A . Moreover, for a sampling step $h = 0.03$ and according to different discrete locations $z \in \Omega$, we calculate the identifier of interest $I_{u_0}^{(l=1)}(z)$ where $u_0 \in \{e_1, e_2, e_1 + e_2\}$ and $\{e_i\}_{i=1}^2$ means the orthonormal basis in \mathbb{R}^2 . Let $I_{u_0}(z) := I_{u_0}^{(1)}(z)$. Then the plots of $z \mapsto I_{u_0}(z)$ illustrate the results, where the sharp peaks are expected to occur at the locations z_j ($j = 1, 2$) of the inclusions. Using a standard log scale, we display also other associated results about the singular values of the matrix A .

For $T = 1$ and $\delta = 2.5$, the singular values are displayed in Fig. 5.1 (a) while the identifier of interest $I_{u_0}(z)$, for $u_0 = e_1 + e_2$, is displayed in Fig. 5.1 (b).

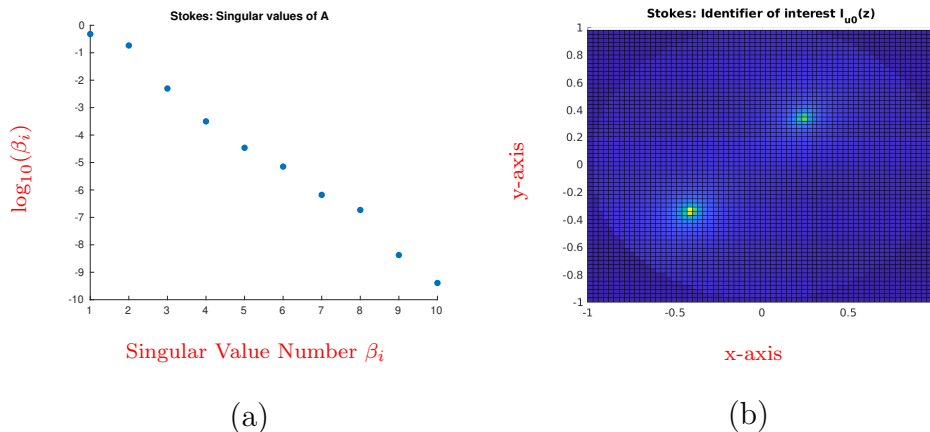


Fig. 5.1. $T = 1$, $\delta = 2.5$: (a) distribution of the singular values of A for $n = 10$ source points; (b) gloss level (or color) map of $I_{u_0}(z)$, $u_0 = e_1 + e_2$, for $z \in \Omega$.

Here the numerical results are not difficult to be interpreted. There are four singular values which move out from the noise, and two singular values are associated with a

specific inclusion. The inclusions, being that they are observed via different $I_{u_0}(z)$, are clearly discriminated from the background, the visual aspect depending on the choice of u_0 .

In Fig. 5.2 (a), we see the distribution of the singular values of matrix A for $T = 3$ and $\delta = 4$.

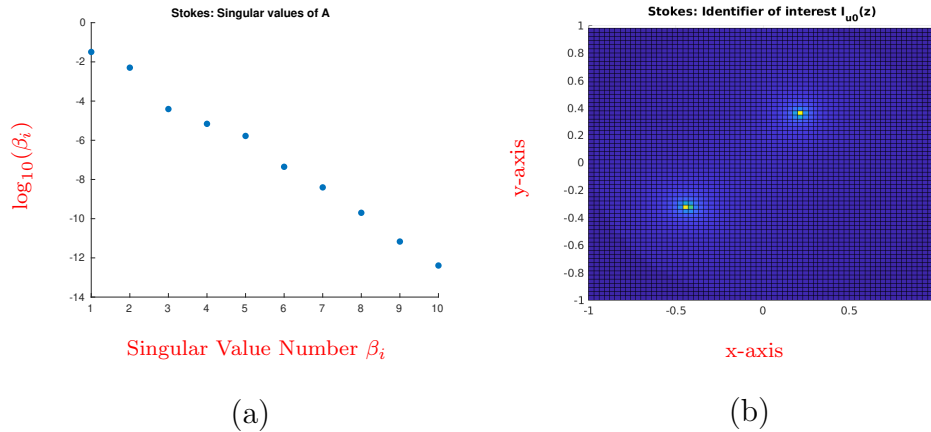


Fig. 5.2. $T = 3$, $\delta = 4$: (a) distribution of the singular values of A for $n = 10$ source points; (b) gloss level (or color) map of $I_{u_0}(z)$, $u_0 = e_1 + e_2$, for $z \in \Omega$.

Fig. 5.2 (b) shows also the identifier of interest $I_{u_0}(z)$ for $u_0 = e_1 + e_2$, where the images are obtained by using the first 5 largest singular vectors associated, of course, with the 5 largest singular values of the matrix A , respectively.

In the second numerical example, we add one more homogeneous disk B_3 with the same diameter $\alpha = 0.1$, centered at $z_3 = (0.4, 0.51)$ and having the viscosity $\mu_3 = 3$. We need to retrieve the three inclusions by using the same number of source points $n = 10$, but for different values of δ , T than the previous example.

Therefore, for $\delta = 4$, $T = 1$, the result is shown in Fig. 5.3 where we have the distribution of the singular values of matrix A for 10 source points and the identifier of interest $I_{u_0}(z)$ for $u_0 = e_1$.

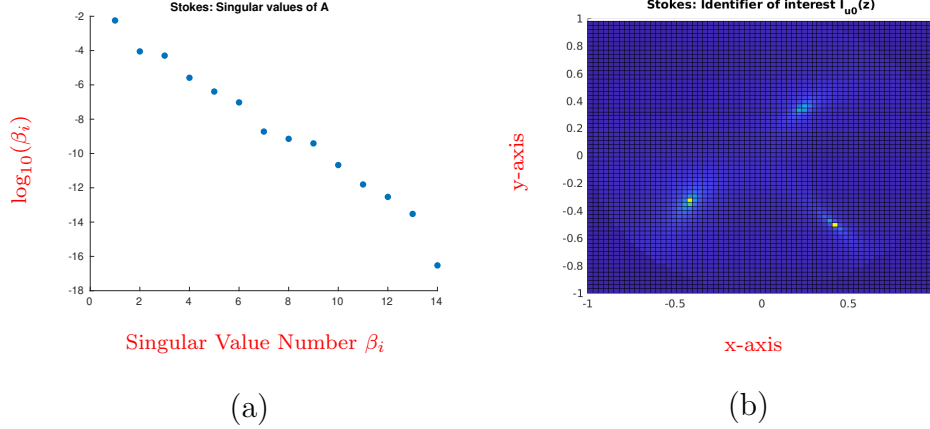


Fig. 5.3. $T = 1$, $\delta = 4$: (a) distribution of the singular values of A for $n = 10$ source points; (b) gloss level (or color) map of $I_{u_0}(z)$, $u_0 = e_1$, for $z \in \Omega$.

The interpretation here is slightly similar the first numerical example concerned with two disks. Then there are six singular values emerge from noise, and two singular values are considered to associate with one specific inclusion. The inclusions, being that they are observed via $I_{u_0}(z)$ for $u_0 = e_1$, are discriminated from the background medium.

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