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Andrea Mattioni, Christophe Prieur, Sophie Tarbouriech. Control design for beam stabilization with self-sensing piezoelectric actuators: managing presence and absence of hysteresis. Mathematics of Control, Signals, and Systems, In press, 10.1007/s00498-024-00393-6 . hal-04659186

**HAL Id: hal-04659186**

**<https://hal.science/hal-04659186>**

Submitted on 22 Jul 2024

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# Control Design for Beam Stabilization with Self-Sensing Piezoelectric Actuators: Managing Presence and Absence of Hysteresis

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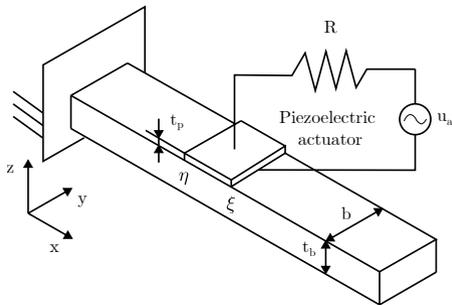
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## Abstract

This paper deals with the modeling and stabilization of a flexible clamped beam controlled with a piezoelectric actuator in the self-sensing configuration. We derive the model starting from general principles, using the general laws of piezoelectricity. The obtained model is composed by a PDE, describing the flexible deformations dynamics, interconnected with an ODE describing the electric charge dynamics. Firstly, we show that the derived linear model is well-posed and the origin is globally asymptotically stable when a voltage control law, containing the terms estimated in the self-sensing configuration, is applied. Secondly, we make the more realistic assumption of the presence of hysteresis in the electrical domain. Applying a passive control law, we show the well-posedness and the origin's global asymptotic stability of the nonlinear closed-loop system.

**Keywords:** Piezoelectric actuator, Nonlinear control, Partial Differential Equations, Asymptotic stability



**Fig. 1** Clamped flexible beam with a piezoelectric actuation.

## 1 Introduction

Piezoelectricity is a reversible process that consists of the property of certain materials of cumulating electric charge in response to mechanical stress, and the generation of a mechanical strain resulting from an electrical field (Yang et al, 2005). Therefore, piezoelectric materials can be used for different applications ranging from the stabilization of flexible structures (Choi and Han, 2016) to self-sensing actuation for nanopositioning applications (Liseli et al, 2019) or for energy harvesting (Erturk and Inman, 2008; Homayouni-Amlashi et al, 2020).

In this paper, we consider the stabilization problem of a clamped flexible beam controlled by a piezoelectric actuator in the self-sensing configuration. Many studies have already been devoted to the modeling procedure and characterization of this kind of system (Meirovitch, 1975; Destuynder, 1999). In these works, the electrical dynamics are not taken into account in the modeling procedure, while more attention is devoted to the characterization of the mechanical dynamics. The well-posedness and stabilization problems for this system without the electrical dynamics have already been investigated in the linear case (Le Gall et al, 2007b,a). Moreover, it is known that the system is exactly controllable with respect to the moment generated on the beam by the piezoelectric actuator (see *e.g.*, (Tucsnak, 1996; Crépeau and Prieur, 2006)). More recently, in (Prieur and Tarbouriech, 2019) the authors investigated the case in which the input moment is subject to amplitude saturation.

In this paper, we extend the work of (Prieur and Tarbouriech, 2019) detailing the modeling procedure (Section 2) when taking into account the electrical dynamics of the piezoelectric actuator in the self-sensing actuation configuration (Dosch et al, 1992), as shown in Fig. 1. The beam is assumed to be clamped on one side and free on the other. The resulting dynamics are modeled by the interconnection of an Euler-Bernoulli PDE describing the flexible

dynamics with an ODE describing the electrical charge dynamics

$$\begin{cases} \rho w_{tt}(x, t) = -EI w_{xxxx}(x, t) + \frac{\alpha}{C_p} q_a(t) \frac{d}{dx} (\delta_\eta(x) - \delta_\xi(x)) \\ w(0, t) = w_x(0, t) = w_{xx}(L, t) = w_{xxx}(L, t) = 0 \\ \dot{q}_a(t) = -\frac{1}{RC_p} q_a(t) - \alpha [w_{xt}(x, t)]_\eta^\xi + \frac{1}{R} u_a(t) \end{cases} \quad (1)$$

where  $w(x, t)$  corresponds to the beam deformation at point  $x \in [0, L]$  of the beam and a time  $t \in \mathbb{R}^+$ ,  $q_a$  is the free electrical charge accumulated in the piezoelectric actuator (with extremities  $\eta < \xi$ ) minus the equivalent charge due to polarization, while  $u_a$  is the control input corresponding to the voltage applied to the electric circuit. Here,  $\delta_{x^*}$  is the delta function, while  $\rho, EI, \alpha, R, C_p \in \mathbb{R}^+$  are physical parameters of the beam and they correspond to the density of the beam, the mechanical elasticity constant, the coupling coefficient between the mechanical and electrical dynamics, the electrical resistance and the electrical capacitance, respectively. Since the influence of the piezoelectric actuator on the elastic beam results in an in-domain actuation, the beam state  $w(x, t)$  should meet some compatibility conditions (defined in Section 3.1) to obtain a well-posed PDE

$$[w_{xx}]_\eta = \frac{\alpha}{EIC_p} q_a(t) = -[w_{xx}]_\xi, \quad [w_{3x}]_\eta = [w_{3x}]_\xi = 0, \quad (2)$$

where  $[w]_\eta = w(\eta^+) - w(\eta^-)$ . In this paper, we make use of the so-called *strong dissipation* feedback to stabilize the system in closed loop. Different works have shown the effectiveness of using this control term, in particular for boundary stabilization of hyperbolic equation (Mattioni et al, 2020, 2022). In (d' Andréa-Novel and Coron, 2000) and (Prieur and Tarbouriech, 2019), the control strategy containing this term has been obtained using backstepping techniques. Next, we study the well-posedness and global asymptotic stability of the origin in closed loop with the control law

$$u_a(t) = -\gamma \left( R[w_{xtt}]_\eta^\xi + \frac{1 + \beta}{C_p} [w_{xt}]_\eta^\xi \right) - \frac{\beta}{C_p} q_a \quad (3)$$

where  $\gamma, \beta \in \mathbb{R}$ . It is worth to remark that the term  $[w_{xt}]_\eta^\xi$  is available thanks to the self-sensing actuator configuration (Liseli et al, 2019), while the term  $[w_{xtt}]_\eta^\xi$  can be obtained by time differentiation of the previous term. The term  $\frac{q_a}{C_p}$  corresponds to the voltage across the piezoelectric actuator that is indeed available. There exists abundant literature regarding the control design of flexible beams controlled by piezoelectric actuators in the self-sensing configuration. A portion of these papers approaches the topic from a practical standpoint: the control laws are directly applied to real benchmarks and results are subsequently discussed as in (Dosch et al, 1992; Simmers Jr et al, 2004; Rakotondrabe et al, 2014; Liseli et al, 2019) and references therein. Another group of these papers focuses on control design and closed-loop analysis, but

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they do not account for the electrical dynamics introduced by the electrical circuit required in the self-sensing configuration (Prieur and Tarbouriech, 2019; Rittenschober and Schlacher, 2012; Aldraihem et al, 1996; Dadfarnia et al, 2004). In this paper, we perform control design and stability analysis on a model that includes a PDE for the mechanical vibrations and an ODE for the electrical circuit dynamics.

The asymptotic stability result stated in the next theorem is obtained considering the *non-critical* positioning (defined at the beginning of Section 3) of the piezoelectric actuator of length  $L_p = \xi - \eta$ , designing a weak Lyapunov function and making use of the LaSalle's invariance principle (see *e.g.* (Luo et al, 2012) for an introduction on the LaSalle's invariance principle). The proof of Theorem 1 is given in Section 3.

**Theorem 1** *For all non-critical positioning  $\eta \notin \mathbb{S}(L_p)$ , for  $\gamma > 0, \beta > -1$  and for all initial conditions  $(w(0), w_t(0), q_a(0))$  in  $(H^2(0, L) \cap H^4(0, \eta) \cap H^4(\eta, \xi) \cap H^4(\xi, L)) \times H^2(0, L) \times \mathbb{R}$  satisfying the boundary conditions in (1) together with the compatibility conditions in (2), the solutions of (1) in closed-loop with (3) satisfy the following stability property for all  $t \geq 0$*

$$\|w(t)\|_{H^2}^2 + \|w_t(t)\|_{L^2}^2 + (q_a(t) - [w_{xt}(t)]_{\eta}^{\xi})^2 \leq \|w(0)\|_{H^2}^2 + \|w_t(0)\|_{L^2}^2 + (q_a(0) - [w_{xt}(0)]_{\eta}^{\xi})^2 \quad (4)$$

together with the attractivity property

$$\|w(t)\|_{H^2}^2 + \|w_t(t)\|_{L^2}^2 + q_a(t)^2 \xrightarrow[t \rightarrow \infty]{} 0. \quad (5)$$

Then, we consider the more realistic scenario where a hysteresis behaviour is present in the piezoelectric actuator. The hysteresis can be modeled using the Maxwell slip model, composed of a series of  $N$  linear capacitors in parallel with as many nonlinear resistors (Yeh et al, 2008; Goldfarb and Celanovic, 1997). The resulting dynamical equations can be written as follows

$$\begin{cases} \rho w_{tt}(x, t) = EI w_{xxxx}(x, t) + \frac{\alpha}{C_p} q_a(t) \frac{d}{dx} (\delta_{\eta}(x) - \delta_{\xi}(x)) \\ w(0, t) = w_x(0, t) = w_{xx}(L, t) = w_{xxx}(L, t) = 0 \\ \dot{q}_a(t) = -\frac{1}{R} \left( \frac{q_a(t)}{C_p} + \sum_{j=1}^N \frac{q_{H,j}(t)}{C_{H,j}} \right) - \alpha [w_{xt}(x, t)]_{\eta}^{\xi} + \frac{1}{R} u_a(t). \\ \dot{q}_{H,j}(t) = -\frac{1}{R} \left( \frac{q_a(t)}{C_p} + \sum_{j=1}^N \frac{q_{H,j}(t)}{C_{H,j}} \right) - i_{RH,j} \left( \frac{q_{H,j}(t)}{C_{H,j}} \right) + \frac{1}{R} u_a(t), \end{cases} \quad (6)$$

where the last equation has to be intended as a set of  $N$  equations, each of which describes the charge  $q_{H,j}$  dynamics of the  $j$ -th hysteron.  $C_{H,j}$  corresponds to the linear capacitance of the  $j$ -th hysteron, while  $i_{RH,j}$  is the nonlinear current passing through the nonlinear resistor of the  $j$ -th hysteron. The modeling procedure to obtain the above equations, together with the explicit expression of  $i_{RH,j}$ , is detailed in Section 2.3.

The introduction of the nonlinear function makes the stability study more challenging since it is known that limit cycles and new equilibrium points may be introduced (see for example (Teel and Zaccarian, 2011; Tarbouriech et al,

2011)). The results of this work can be related to (Prieur et al, 2016) where the wave equation with cone-bounded nonlinearities in the case of bounded and unbounded control operators has been considered. Similarly, in (Singh et al, 2022) the authors proved the well-posedness of a nonlinear infinite dimensional system obtained as the perturbation of a scattering passive linear system. We refer to (Curtain and Zwart, 2020, Chapter 11) for a collection of results regarding the well-posedness and the stability of semilinear differential equations with Lipschitz nonlinearity. In this work, we study the well-posedness of system (6) in closed-loop with the control law

$$u_a = -\beta \left( \frac{q_a(t)}{C_p} + \sum_{j=1}^N \frac{q_{H,j}(t)}{C_{H,j}} \right). \quad (7)$$

It is worth mentioning that the term in the parenthesis corresponds to the voltage across the piezoelectric actuator when considering the model with the hysteresis. This means that this term is directly available from measurements. Differently from the linear control law proposed in (3), in (7) are missing the terms depending on  $[w_{xtt}]_{\eta}^{\xi}$ ,  $[w_{xt}]_{\eta}^{\xi}$ . This is because the insertion of the *strong dissipation* terms would require the elimination of the hysteresis contribution  $\sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}}$  to obtain a stable closed-loop system. This could be done, for example, through the use of a state observer and can be a possible topic of future research. In this work, the well-posedness of system (6) in closed-loop with (7) is investigated using the theory developed in (Miyadera, 1992), while the closed-loop stability has been assessed using Lyapunov arguments. We refer the interested reader to (Bastin and Coron, 2016; Krstic and Smyshlyaev, 2008), for works using Lyapunov techniques for the stability analysis of nonlinear PDE.

**Theorem 2** *For all non-critical positioning  $\eta \notin \mathbb{S}(L_p)$ ,  $\beta > -1$  and for all initial conditions  $(w(0), w_t(0), q_a(0), q_{H,j}(0))$  in  $(H^2(0, L) \cap H^4(0, \eta) \cap H^4(\eta, \xi) \cap H^4(\xi, L)) \times H^2(0, L) \times \mathbb{R}^{N+1}$  satisfying the boundary conditions in (6) together with the compatibility conditions in (2), the solutions of system (6) in closed loop with (7) satisfy the following properties:*

- *Stability:*

$$\| \overline{w(t)} \|_{H^2}^2 + \| w_t(t) \|_{L^2}^2 + q_a(t)^2 \leq \| w(0) \|_{H^2}^2 + \| w_t(0) \|_{L^2}^2 + q_a(0)^2 \quad (8)$$

- *Attractivity:*

$$\| w(t) \|_{H^2}^2 + \| w_t(t) \|_{L^2}^2 + q_a(t)^2 \xrightarrow[t \rightarrow \infty]{} 0. \quad (9)$$

This paper is organised as follows. In Section 2, we detail the modeling procedure of the system under analysis. In Section 3, we introduce some necessary notions for the non-controlled Euler-Bernoulli beam equation as well as a result regarding the exact observability of the system. Then, we present

the well-posedness result for the closed-loop linear system (1) as well as its asymptotic stability proof. Similarly, in Section 4 is given the well-posedness result as well as the asymptotic stability proof for the closed-loop nonlinear system (6). Finally, some concluding remarks and comments on future works are given in Section 5.

## 2 Modeling

### 2.1 Physical assumptions and energy definition

We consider the modeling problem of the system depicted in Figure 1. The goal is to obtain distributed parameter dynamics for the mechanical part while lumped parameter dynamics for the piezoelectric actuator (PA) and the electrical actuation circuit. To do so, we consider distributed parameters constitutive equations for the mechanical part while lumped parameters constitutive equations for the PA (Dosch et al, 1992, eq. (10),(19))

$$\begin{cases} s_1(x) = Y_p \varepsilon_1(x) - Y_p d_{31} V(x) \\ v_p = \frac{q_f}{C_p} - \frac{q_p}{C_p} \end{cases} \quad (10)$$

where  $s_1$  and  $\varepsilon_1$  are the stress and strain along the  $x$  direction, respectively;  $V$  and  $v_p$  are the electrical field in the PA and the voltage applied to the PA, respectively;  $q_f$  and  $q_p$  are the free charge on the PA and the piezoelectric polarization charge due to stress, respectively. Finally,  $Y_p$  corresponds to Young's modulus while  $d_{31}$  is the piezoelectric constant. The electric field is assumed to be proportional to the voltage  $v_p$  and different to zero only in the interval  $[\eta, \xi]$  where the PA is placed

$$V(x) = \frac{v_p}{t_b + t_p} (H_\eta(x) - H_\xi(x)). \quad (11)$$

According to Euler-Bernoulli assumptions, the normal strain corresponds to

$$\varepsilon = -yw_{xx} \quad (12)$$

where  $w$  corresponds to the beam deformation. Using the Euler-Bernoulli assumptions, the polarization charge  $q_p$  is proportional to the beam slope difference between  $\eta$  and  $\xi$  (Dosch et al, 1992, eq. (32))

$$q_p = \alpha [w_x]_\eta^\xi \quad \alpha = Y_p d_{31} b \frac{t_b + t_p}{2}. \quad (13)$$

The total potential energy of the system is defined as

$$E_p = \frac{1}{2} \int_0^L \int_0^{t_b+t_p} \int_0^b \{\varepsilon_1 s_1\} dz dy dx + \frac{1}{2} v_p q_f. \quad (14)$$

Substituting (10)-(13) into (14) we obtain

$$\begin{aligned}
E_p &= \frac{1}{2} \int_0^L \int_0^{t_b+t_p} \int_0^b \left\{ -y w_{xx} (-Y_p y w_{xx} \right. \\
&\quad \left. - Y_p d_{31} \frac{q_f - q_p}{C_p(t_a+t_p)} (H_\eta(x) - H_\xi(x)) \right\} dz dy dx + \frac{q_f - q_p}{2C_p} q_f \\
&= \frac{1}{2} \int_0^L \int_0^{t_b+t_p} \left\{ b Y_p y^2 w_{xx}^2 \right. \\
&\quad \left. + Y_p d_{31} b y \frac{q_f - q_p}{C_p(t_b+t_p)} (H_\eta(x) - H_\xi(x)) w_{xx} \right\} dy dx + \frac{q_f - q_p}{2C_p} q_f \\
&= \frac{1}{2} \int_0^L \left\{ EI w_{xx}^2 + \alpha \frac{q_f - q_p}{C_p} (H_\eta(x) - H_\xi(x)) w_{xx} \right\} dx + \frac{q_f - q_p}{2C_p} q_f
\end{aligned} \tag{15}$$

where in the last equality we have used the  $\alpha$  definition in (13) and defined  $EI = bY_p \int_0^{t_b} y^2 dy$ . Using the definition of the polarization charge in (13) we obtain

$$E_p = \frac{1}{2} \int_0^L EI w_{xx}^2 dx + \frac{(q_f - \alpha[w_x]_\eta^\xi)^2}{2C_p}. \tag{16}$$

The kinetic energy of the system can be described by

$$E_K = \frac{1}{2} \int_0^L \rho w_t^2 dx. \tag{17}$$

The overall energy of the system is defined as the sum of the kinetic  $E_K$  and potential  $E_P$  energies  $E_{tot} = E_K + E_P$ .

## 2.2 Derivation of governing equations

To derive the dynamic equations we first have to define the *action*

$$A = \int_{t_1}^{t_2} \{E_K - E_P + W_{nc}\} dt \tag{18}$$

where  $t_1, t_2$  are two arbitrary instants of time such that  $t_2 > t_1$  while  $W_{nc}$  is the work of non-conservative forces applied on the system. The principle of least actions states that the true evolution of the system described by the action (18) satisfies  $dAh = 0$  for any  $h$  such that  $h(t_1) = h(t_2) = 0$  and where  $dA$  is the Fréchet derivative of the action functional and  $h$  is the *variation function* (in the mechanics' literature are usually referred to as *virtual displacements*). According to the variational principle, we have that

$$\int_{t_1}^{t_2} \{dE_K h - dE_P h + dW_{nc} h\} dt = 0. \tag{19}$$

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We start by computing the Fréchet derivative of the Kinetic energy

$$dE_K h = \int_0^L \rho w_t h_{w_t} dx. \quad (20)$$

The Fréchet derivative of the Potential energy writes

$$dE_P h = \int_0^L \left\{ \left( EI w_{xx} - \frac{\alpha}{C_p} (q_f - \alpha [w_x]_{\eta}^{\xi}) (H_{\eta} - H_{\xi}) \right) h_{w_{xx}} \right. \\ \left. + \frac{1}{C_p} (q_f - \alpha [w_x]_{\eta}^{\xi}) h_{q_f} \right\} dx \quad (21)$$

We plug equations (20) and (21) in (19) to obtain

$$\int_{t_1}^{t_2} \int_0^L \left\{ \rho w_t h_{w_t} - \left( EI w_{xx} - \frac{\alpha}{C_p} (q_f - \alpha [w_x]_{\eta}^{\xi}) (H_{\eta} - H_{\xi}) \right) h_{w_{xx}} \right\} dx \\ + \left( \frac{1}{C_p} (q_f - \alpha [w_x]_{\eta}^{\xi}) \right) h_{q_f} + dW_{nc} h dt = 0. \quad (22)$$

The Fréchet derivative of the work of the non-conservative forces is defined such to obtain the electrical circuit dynamical equation represented in Figure 1. To do so, we apply 2<sup>nd</sup> Kirchhoff's law to be able to write

$$0 = R\dot{q}_f + v_{pa} - u_a. \quad (23)$$

Therefore, to obtain the same voltage balance in the Action's Fréchet derivative, we define

$$dW_{nc} h = (R\dot{q}_f - u_a) h_{q_f}. \quad (24)$$

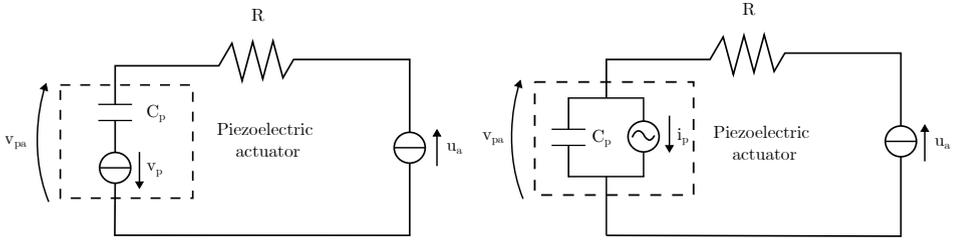
We plug (24) in (22) and assuming, without loss of generality, that  $h_{w_t} = \frac{\partial h_w}{\partial t}$  and that  $h_{w_{xx}} = \frac{\partial^2 h_w}{\partial x^2}$ , we perform two integrations by parts with respect to  $x$  and one with respect to  $t$  on (22) to obtain

$$\int_{t_1}^{t_2} \int_0^L \left\{ -\rho w_{tt} - EI w_{x4} + \frac{\alpha}{C_p} (q_f - \alpha [w_x]_{\eta}^{\xi}) \frac{d}{dx} (\delta_{\eta} - \delta_{\xi}) \right\} h_w dx - [EI w_{xx} h_{w_x}]_0^L \\ + [EI w_{x3} h_w]_0^L - \frac{\alpha}{C_p} (q_f - \alpha [w_x]_{\eta}^{\xi}) \left( [(\delta_{\eta} - \delta_{\xi}) h_w]_0^L - [(H_{\eta} - H_{\xi}) h_{w_x}]_0^L \right) \\ + \left( \frac{1}{C_p} (q_f - \alpha [w_x]_{\eta}^{\xi}) + R\dot{q}_f - u_a \right) h_{q_f} dt + \int_0^L [\rho w_t h_w]_{t_1}^{t_2} dx = 0. \quad (25)$$

We remark that the terms computed at the boundaries involving the Dirac and Heaviside functions are zero by definition. The remaining boundary terms are equal to zero as soon as we guarantee the boundary conditions

$$w(0, t) = w_x(0, t) = w_{xx}(L, t) = w_{x3}(L, t) = 0. \quad (26)$$

The last term in (22) is equal to zero because of the vanishing condition of functions  $h_w$  at  $t_1, t_2$ , *i.e.*  $h_w(t_2) = h_w(t_1) = 0$ . From the remaining terms, we can extract the equations of motion and boundary conditions corresponding to the



**Fig. 2** Two equivalent lumped parameter electrical representations of the piezoelectric actuation.

series between a capacitor and voltage generator piezoelectric representation, represented in Figure 2,

$$\begin{cases} \rho w_{tt} = -EIw_{4x} + \frac{\alpha}{C_p}(q_f - \alpha[w_x]_{\eta}^{\xi}) \frac{d}{dx}(\delta_{\eta} - \delta_{\xi}) \\ w(0, t) = w_x(0, t) = w_{xx}(L, t) = w_{x3}(L, t) = 0 \\ R\dot{q}_f = -\frac{1}{C_p}(q_f - \alpha[w_x]_{\eta}^{\xi}) + u_a. \end{cases} \quad (27)$$

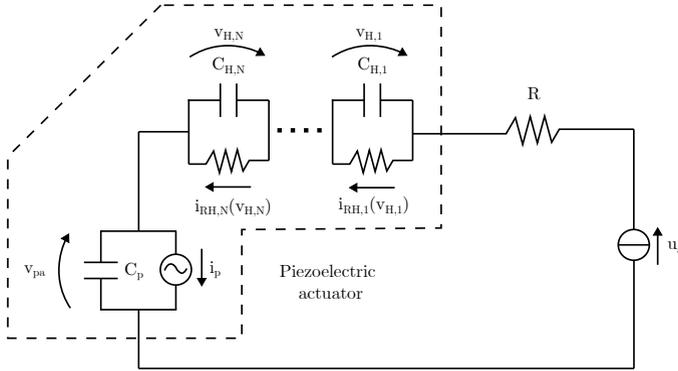
The second PA equivalent representation of a capacitor in parallel with a current generator can be obtained through the following change of variable

$$q_a = q_f - \alpha[w_x]_{\eta}^{\xi}. \quad (28)$$

The corresponding equations of motions are the ones given in (1). From a physical point of view,  $\dot{q}_a(t)$  is the current passing through the capacitor  $C_p$  in the right electrical circuit of Figure 2. The term  $\frac{1}{R}\left(u_a - \frac{q_a}{C_p}\right)$  is the total current in the circuit, while  $i_p = \alpha[w_{xt}]_{\xi}^{\eta}$  is the current generated by the direct piezoelectric effect.

### 2.3 Hysteresis model insertion in the equations of motion

Experimental observations indicate that the PA hysteresis behaviour lies in the electrical domain between the applied actuator voltage and the resulting charge  $q_a$  responsible for the mechanical stress (Goldfarb and Celanovic, 1997). The hysteresis behaviour can be modeled using the Maxwell slip model which, from an electrical point of view, can be represented by a series of  $N$  linear capacitors in parallel with as many nonlinear resistors (Yeh et al, 2008; Goldfarb and Celanovic, 1997). An equivalent electrical representation consists of the parallel of  $N$  inductors in series with as many nonlinear resistors (Caballeria et al, 2021). In this work, we consider the capacitor-resistor representation of the Maxwell-slip model depicted in Figure 3. From now on, we denote with *hysterion* the element composed by a capacitor in parallel with a nonlinear resistor. To obtain the electrical dynamical equations we write the



**Fig. 3** Electrical representation of the PA embedded with the Hysteresis Maxwell-slip model.

1<sup>st</sup> Kirchhoff's law for  $\dot{q}_a$  and  $\dot{q}_{H,j}$  for  $j \in \{1, \dots, N\}$ .

$$\begin{cases} \dot{q}_a = i_a - i_p \\ \dot{q}_{H,j} = i_a - i_{RH,j}(v_{H,j}). \end{cases} \quad (29)$$

The current  $i_a$  corresponds to the current flowing in the resistor  $R$

$$\begin{aligned} i_a &= \frac{1}{R} \left( u_a - v_{pa} - \sum_{j=1}^N v_{H,j} \right) \\ &= \frac{1}{R} \left( u_a - \frac{q_a}{C_p} - \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}} \right). \end{aligned} \quad (30)$$

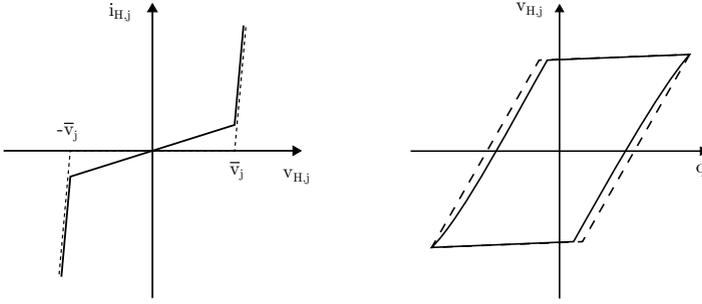
We embed the last equation in the 1<sup>st</sup> Kirchhoff's law and  $i_p = \alpha[w_{xt}]_\xi^\eta$  in (29) to obtain

$$\begin{cases} \dot{q}_a = -\frac{1}{R} \left( \frac{q_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}} \right) - \alpha[w_{xt}]_\xi^\eta + \frac{1}{R} u_a \\ \dot{q}_{H,j} = -\frac{1}{R} \left( \frac{q_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}} \right) - i_{RH,j} \left( \frac{q_{H,j}}{C_{H,j}} \right) + \frac{1}{R} u_a. \end{cases} \quad (31)$$

The relation between  $i_{RH,j}$  and  $v_{H,j}$  proposed in (Yeh et al, 2008; Goldfarb and Celanovic, 1997) can be formalized by the following function

$$i_{H,j}(v_{H,j}) = \begin{cases} 0 & \text{if } |v_{H,j}| < \bar{v}_j \\ \bar{G}_H(v_{H,j} + \bar{v}_j) & \text{if } v_{H,j} \leq -\bar{v}_j \\ \bar{G}_H(v_{H,j} - \bar{v}_j) & \text{if } v_{H,j} \geq \bar{v}_j. \end{cases} \quad (32)$$

where  $\bar{G}_H \gg 1$  is the common conductance to all the hysteresions, while  $\bar{v}_j$  are the limit voltages different between hysteresions. In this paper, we consider that each hysteresion lets through leakage current when the voltage is in the interval  $[-\bar{v}_j, \bar{v}_j]$ . Therefore, introducing the conductance  $\underline{G}_H \ll 1$ , we consider the



**Fig. 4** Graphical representation of the proposed current as nonlinear function of the voltage, as well as the resulting hysteresis relation between voltage and charge. The dashed lines correspond to the function in (32) and the corresponding hysteresis, while the solid lines correspond to (33).

following nonlinear current function

$$i_{H,j}(v_{H,j}) = \begin{cases} \underline{G}_H v_{H,j} & \text{if } |v_{H,j}| < \bar{v}_j \\ \bar{G}_H v_{H,j} + \bar{v}_j(\bar{G}_H - \underline{G}_H) & \text{if } v_{H,j} \leq -\bar{v}_j \\ \bar{G}_H v_{H,j} - \bar{v}_j(\bar{G}_H - \underline{G}_H) & \text{if } v_{H,j} \geq \bar{v}_j. \end{cases} \quad (33)$$

In Figure 4 we plot the comparison between the two different current nonlinear functions, together with the resulting hysteresis between the voltage  $v_{H,j}$  and the charge  $q = \int_0^t i_a(\tau) d\tau$ . It is easy to show that the current nonlinear function in (33) has the following properties.

**Proposition 3** For all  $j \in \{1, \dots, N\}$ , the nonlinear function  $i_{H,j} : \mathbb{R} \rightarrow \mathbb{R}$  in (33) is

- continuous, monotonically increasing (i.e.  $(i_{H,j}(v_{H,j1}) - i_{H,j}(v_{H,j2}))(v_{H,j1} - v_{H,j2}) \geq 0$ ) and such that  $i_{H,j}(0) = 0$
- such that  $i_{H,j}(v_{H,j1})v_{H,j1} \geq \underline{G}_H v_{H,j1}^2$  for all  $v_{H,j1} \in \mathbb{R}$ .

### 3 Well-posedness and asymptotic stability: linear system

#### 3.1 PA critical positioning and $L^2$ beam equation analysis

In this subsection, we provide some useful preliminaries for studying the in-domain stabilization of the Euler-Bernoulli beam equation. As already shown in previous publications on stabilization of piezoelectric controlled flexible beams (Priour and Tarbouriech, 2019; Crépeau and Priour, 2006), stabilization is possible only in case the actuator is non-critical positioned. To understand what are non-critical points, it is necessary to recall some useful facts about the free evolution of the Euler-Bernoulli equations. To do that, we consider

the homogeneous Cauchy problem

$$\begin{aligned} \phi_{tt}(x, t) + \frac{EI}{\rho} \phi_{x4}(x, t) &= 0, & \phi(\cdot, 0) &= \phi^0, & \phi_t(\cdot, 0) &= v^0, \\ \phi(0, t) = \phi_x(0, t) &= \phi_{xx}(L, t) = \phi_{x3}(L, t) &= 0. \end{aligned} \quad (34)$$

For the sake of compactness, we shall not explicit the time and space dependency of the variables, unless it is not clear from the context. We define the state  $z = (\phi, \nu)$  where  $\nu = \phi_t$  and let  $A_0 : D(A_0) \rightarrow L^2(0, L)^2$  be the open-loop operator with domain

$$\begin{aligned} D(A_0) := \{(\phi, \nu) \in H^4(0, L) \times H^2(0, L) \mid \nu(0) = \nu_x(0) = 0, \\ \phi(0) = \phi_x(0) = \phi_{xx}(L) = \phi_{x3}(L) = 0\} \end{aligned} \quad (35)$$

and defined by  $A_0 z = (\nu, -\frac{EI}{\rho} \phi_{x4})$ . We define the set  $\mathcal{H} = H^2(0, L) \times L^2(0, L)$  with inner product  $\langle z_1, z_2 \rangle_{\mathcal{H}} = \int_0^L \phi_{1,xx} \phi_{2,xx} + \phi_{t,1} \phi_{t,2} dx$ . The operator  $A_0^{-1}$  is compact and symmetric on  $\mathcal{H}^{-1}$ , hence there exists a countable orthonormal basis of  $\mathcal{H}$  consisting of eigenvectors of  $A_0^{-1}$ . In the following proposition, we provide a result on the characterization of the  $A_0$  eigenvectors.

**Proposition 4** (Lemma 2.1 in (Crépeau and Prieur, 2006)) *The  $L^2(0, L)$ -normalized eigenfunctions of  $A_0$  are the functions  $(\psi_n)_{n \geq 1}$ , defined for all  $x$  in  $(0, L)$  by*

$$\psi_n(x) = \gamma_n (\cos(\alpha_n x) - \cosh(\alpha_n x) + \mu_n (\sinh(\alpha_n x) - \sin(\alpha_n x))) \quad (36)$$

where  $\alpha_n$  is the  $n$ -th positive root of

$$1 + \cos(\alpha_n L) \cosh(\alpha_n L) = 0 \quad (37)$$

with  $\mu_n = \frac{\cos(\alpha_n x) + \cosh(\alpha_n x)}{\sin(\alpha_n x) + \sinh(\alpha_n x)}$  and  $\gamma_n = \frac{1}{\sqrt{L}}$ .

We are now in the position to define the *critical set*. To do that, let  $L_p = \xi - \eta$  be the length of the PA. For any  $n \geq 1$  and any  $L_p \in (0, L)$ , let

$$\mathbb{S}_n(L_p) := \{\eta \in [0, L - L_p], \psi_{n,x}(\eta) - \psi_{n,x}(\eta + L_p) = 0\}, \quad (38)$$

where  $\psi_{n,x}$  corresponds to the spatial derivative of the  $n$ -th eigenfunction  $\psi_n$ , therefore the *critical set* can be defined as

$$\mathbb{S}(L_p) = \cup_{n \geq 1} \mathbb{S}_n(L_p). \quad (39)$$

The exact observability of (34) with respect to the output  $[\nu_x]_{\xi}^{\eta}$  is shown in the following lemma for any non-critical positioning  $(\eta, \xi)$  of the PA.

**Lemma 5** *If  $\eta \notin \mathbb{S}(L_p)$ , the system (34) is exactly observable with respect to  $[\nu_x]_{\xi}^{\eta}$ .*

*Proof* According to (Curtain and Zwart, 2020, Corollary 6.2.15), a system is exactly observable on  $[0, \tau]$  if and only if  $\int_0^\tau \|y(t)\|^2 dt \geq \gamma \|z\|_{\mathcal{H}}^2$  for some  $\gamma > 0$  and for all  $z \in \mathcal{H}$ . It is easily seen that if  $\phi^0 = \sum_{n \geq 1} \phi_n^0 \psi_n$  and  $\nu^0 = \sum_{n \geq 1} \nu_n^0 \psi_n$ , then the solution  $\phi = \phi(x, t)$  to (34) reads

$$\phi(x, t) = \sum_{n=1}^{\infty} \left( \phi_n^0 \cos(\alpha_n^2 t) + \frac{\nu_n^0}{\alpha_n^2} \sin(\alpha_n^2 t) \right) \psi_n(x). \quad (40)$$

Using the former equation in the output definitions we obtain

$$[\nu_x(x, t)]_\xi^\eta = \sum_{n=1}^{+\infty} \left( -\phi_n^0 \alpha_n^2 \sin(\alpha_n^2 t) + \nu_n^0 \cos(\alpha_n^2 t) \right) (\psi_{n,x}(\eta) - \psi_{n,x}(\xi)). \quad (41)$$

Since  $\alpha_{n+1}^2 - \alpha_n^2 \rightarrow \infty$  we infer from a generalization of Ingham's inequality that for any  $\tau > 0$

$$\begin{aligned} \int_0^\tau \|[\nu_x(x, t)]_\xi^\eta\|^2 dt &= \int_0^\tau \left( \sum_{n=1}^{+\infty} \left( -\phi_n^0 \alpha_n^2 \sin(\alpha_n^2 t) + \nu_n^0 \cos(\alpha_n^2 t) \right) (\psi_{n,x}(\eta) - \psi_{n,x}(\xi)) \right)^2 dt \\ &\geq C_\tau \sum_{n=1}^{+\infty} \left( (\phi_n^0 \alpha_n^2)^2 + (\nu_n^0)^2 \right) (\psi_{n,x}(\eta) - \psi_{n,x}(\xi))^2. \end{aligned} \quad (42)$$

Therefore, since  $\eta \notin \mathcal{S}$  then  $\psi_{n,x}(\eta) - \psi_{n,x}(\xi) \neq 0$  and from the previous inequalities we obtain that  $\int_0^\tau \|\sigma(t)\|^2 dt \geq \gamma \|z\|_{\mathcal{H}}^2$ , that indeed concludes the proof.  $\square$

Now, we introduce the necessary tools to obtain that both hands of the PDE in (6) belong to  $L^2(0, L)$ . If  $w$  is any function in  $H^1(0, \eta) \cap H^1(\eta, \xi) \cap H^1(\xi, L)$ , we define  $\{w_x\} \in L^2(0, L)$  by

$$\{w_x\}(x) := \begin{cases} w_x^{\mathcal{D}'(0, \eta)}(x) & \text{if } x \in (0, \eta) \\ w_x^{\mathcal{D}'(\eta, \xi)}(x) & \text{if } x \in (\eta, \xi) \\ w_x^{\mathcal{D}'(\xi, \pi)}(x) & \text{if } x \in (\xi, L). \end{cases} \quad (43)$$

We set also  $[w]_\eta := w(\eta^+) - w(\eta^-)$ , and  $[w]_\xi := w(\xi^+) - w(\xi^-)$ . Then it follows that

$$w_x = \{w_x\} + [w]_\eta \delta_\eta + [w]_\xi \delta_\xi \quad \text{in } \mathcal{D}'(0, L). \quad (44)$$

Assume now that  $w \in H^2(0, L)$  and that  $v \in H^2(0, L)$ , and define  $\varpi = -\frac{EI}{\rho} w_{x4} + \varepsilon \frac{d}{dx}(\delta_\eta - \delta_\xi)$ . If  $\varpi \in L^2(0, L)$ , then the restriction of  $\varpi$  to each of the intervals  $(0, \eta)$ ,  $(\eta, \xi)$  and  $(\xi, L)$  has also to be a square integrable function. The same conclusion holds for  $w_{x4}$ , hence  $w \in H^4(0, \eta) \cap H^4(\eta, \xi) \cap H^4(\xi, L)$ . We may then compute the first space derivatives of  $w$  and then  $\varpi$ . We obtain

$$\begin{aligned} w_x &= \{w_x\} + [w]_\eta \delta_\eta + [w]_\xi \delta_\xi = \{w_x\} \\ w_{xx} &= \{w_{xx}\} + [w_x]_\eta \delta_\eta + [w_x]_\xi \delta_\xi = \{w_{xx}\} \\ w_{x3} &= \{w_{x3}\} + [w_{xx}]_\eta \delta_\eta + [w_{xx}]_\xi \delta_\xi \\ w_{x4} &= \{w_{x4}\} + [w_{x3}]_\eta \delta_\eta + [w_{x3}]_\xi \delta_\xi + [w_{xx}]_\eta \frac{d}{dx} \delta_\eta + [w_{xx}]_\xi \frac{d}{dx} \delta_\xi \end{aligned} \quad (45)$$

and

$$\begin{aligned} \varpi &= -\frac{EI}{\rho} (\{w_{x4}\} + [w_{x3}]_\eta \delta_\eta + [w_{x3}]_\xi \delta_\xi + [w_{xx}]_\eta \frac{d}{dx} \delta_\eta + [w_{xx}]_\xi \frac{d}{dx} \delta_\xi) \\ &\quad + \varepsilon \frac{d}{dx} (\delta_\eta - \delta_\xi). \end{aligned}$$

Then,  $\varpi$  is in  $L^2(0, L)$  provided that all the coefficients in front of the Dirac functions vanish, i.e.  $[w_{xx}]_\eta = \frac{\rho}{EI}\varepsilon = -[w_{xx}]_\xi$  and  $[w_{x3}]_\eta = [w_{x3}]_\xi = 0$ .

### 3.2 Well-posedness and stability analysis

In this subsection, we show the well-posedness and the asymptotic stability of solutions generated by the linear system (1) in closed loop with the control law (3). System (1) in closed loop with (3) writes:

$$\begin{cases} \rho w_{tt}(x, t) = -EIw_{xxxx}(x, t) + \frac{\alpha}{C_p}q_a(t)\frac{d}{dx}(\delta_\eta(x) - \delta_\xi(x)) \\ w(0, t) = w_x(0, t) = w_{xx}(L, t) = w_{xxx}(L, t) = 0 \\ \dot{q}_a(t) = -\frac{1+\beta}{RC_p}q_a(t) - \alpha[w_{xt}(x, t)]_\eta^\xi - \gamma[w_{xtt}]_\eta^\xi - \frac{\gamma(1+\beta)}{RC_p}[w_{xt}]_\eta^\xi \end{cases} \quad (46)$$

that, after the change of variables  $\bar{q}_a = q_a + \gamma[w_{xt}]_\eta^\xi$ , can be rewritten as

$$\begin{cases} \rho w_{tt}(x, t) = -EIw_{xxxx}(x, t) + \frac{\alpha}{C_p}(\bar{q}_a(t) - \gamma[w_{xt}(x, t)]_\eta^\xi)\frac{d}{dx}(\delta_\eta(x) - \delta_\xi(x)) \\ w(0, t) = w_x(0, t) = w_{xx}(L, t) = w_{xxx}(L, t) = 0 \\ \dot{\bar{q}}_a(t) = -\frac{1+\beta}{RC_p}\bar{q}_a(t) - \alpha[w_{xt}(x, t)]_\eta^\xi. \end{cases} \quad (47)$$

Therefore, we define the space

$$V = \{v \in H^2(0, L) \mid v(0) = v_x(0) = 0\} \quad (48)$$

with inner product  $\langle v_1, v_2 \rangle_V = \langle v_{1,xx}, v_{2,xx} \rangle_{L^2}$  and associated norm  $\|v\|_V = \sqrt{\langle v_1, v_2 \rangle_V}$ . Then, we can define the operator

$$A_1 z = \begin{bmatrix} v \\ -\frac{EI}{\rho}w_{x4} + \frac{\alpha}{\rho C_p} \left( \bar{q}_a - \gamma[v_x]_\eta^\xi \right) \frac{d}{dx}(\delta_\eta - \delta_\xi) \\ -\frac{1+\beta}{RC_p}\bar{q}_a - \alpha[v_x]_\eta^\xi \end{bmatrix} \quad (49)$$

with state  $z = (w, v, \bar{q}_a) \in Z_1 = V \times L^2(0, L) \times \mathbb{R}$  and domain

$$\begin{aligned} D(A_1) &= \{z \in Z \mid w \in H^4(0, \eta) \cap H^4(\eta, \xi) \cap H^4(\xi, L), v \in H^2(0, L) \\ &w(0) = w_x(0) = w_{xx}(L) = w_{x3}(L) = 0, v(0) = v_x(0) = 0, \\ &[w_{xx}]_\eta = \frac{\alpha}{EIC_p} \left( \bar{q}_a - \gamma[v_x]_\eta^\xi \right) = -[w_{xx}]_\xi, [w_{x3}]_\eta = [w_{x3}]_\xi = 0\}. \end{aligned} \quad (50)$$

We remark that the compatibility conditions necessary to obtain an  $L^2(0, L)$  right-hand side of the beam PDE obtained in Section 3.1 are present in  $D(A_1)$ . For the linear system's analysis, we equip the state space  $Z_1$  with the energy inner product

$$\langle z_1, z_2 \rangle_1 = \int_0^L \{EIw_{1,xx}w_{2,xx} + \rho v_1 v_2\} dx + \frac{1}{C_p}\bar{q}_{1,a}\bar{q}_{2,a} \quad (51)$$

and the associated norm  $\|z\|_1 = \sqrt{\langle z, z \rangle_1}$ . The energy inner product (51) corresponds to the standard inner product on  $Z_1$  and therefore is well-posed. Here we present the well-posedness result for operator  $A_1$  with domain  $D(A_1)$ .

**Theorem 6** *The closed-loop operator  $A_1$  defined in (49), with domain  $D(A_1)$  in (50), generates a contraction  $C_0$ -semigroup in the state space  $Z$ .*

*Proof* By the Lumer-Phillips Theorem, we have to show that the operator  $A_1$  is dissipative and that  $\text{Ran}(\lambda I - A_1) = Z_1$ . We start with the dissipativity of operator  $A_1$ , and applying two times integration by parts we obtain

$$\begin{aligned} \langle A_1 z, z \rangle_1 &= \int_0^L \left\{ EI v_{xx} w_{xx} + \rho \left( -\frac{EI}{\rho} w_{x4} + \frac{\alpha}{\rho C_p} (\bar{q}_a \right. \right. \\ &\quad \left. \left. - \gamma [v_x]_\xi^\eta) \frac{d}{dx} (\delta_\eta - \delta_\xi) \right) v \right\} dx \\ &\quad + \frac{1}{C_p} \left( -\frac{1+\beta}{RC_p} \bar{q}_a - \alpha [v_x]_\xi^\eta \right) \bar{q}_a \\ &= \int_0^L \{ EI v_{xx} w_{xx} - EI w_{xx} v_{xx} \} dx - [w_{x3} v]_0^L + [w_{xx} v_x]_0^L \\ &\quad - \frac{\alpha \bar{q}_a}{\rho C_p} \int_0^L v \frac{d}{dx} (\delta_\eta - \delta_\xi) dx - \frac{\alpha \gamma [v_x]_\xi^\eta}{\rho C_p} \int_0^L v \frac{d}{dx} (\delta_\eta - \delta_\xi) dx \\ &\quad + \frac{1}{C_p} \left( -\frac{1+\beta}{RC_p} \bar{q}_a - \alpha [v_x]_\xi^\eta \right) \bar{q}_a. \end{aligned} \quad (52)$$

Using the boundary conditions in (50) together with the definition of the Dirac and Heaviside functions, we obtain

$$\langle A_1 z, z \rangle_1 = -\frac{\alpha \gamma}{\rho C_p} \left( [v_x]_\eta^\xi \right)^2 - \frac{1+\beta}{RC_p^2} \bar{q}_a^2 \leq 0 \quad (53)$$

The range condition  $\text{Ran}(\lambda I - A_1) = Z_1$  consists on finding for a certain  $\lambda > 0$ ,  $\tilde{z} = (\tilde{w}, \tilde{v}, \tilde{q}_a) \in D(A_1)$  such that  $(\lambda I - A_1)\tilde{z} = f$  for all  $f \in Z_1$ . Let us first note that the latter equation is equivalent to

$$\begin{cases} \lambda \tilde{w} - \tilde{v} = f_w \\ \lambda \tilde{v} - \left( -\frac{EI}{\rho} \tilde{w}_{x4} + \frac{\alpha}{\rho C_p} \left( \tilde{q}_a - \gamma [\tilde{v}_x]_\eta^\xi \right) \frac{d}{dx} (\delta_\eta - \delta_\xi) \right) = f_v \\ \lambda \tilde{q}_a - \left( -\frac{1+\beta}{RC_p} \tilde{q}_a - \alpha [\tilde{v}_x]_\eta^\xi \right) = f_{q_a} \end{cases} \quad (54)$$

that can be rewritten as

$$\begin{cases} \tilde{v} = \lambda \tilde{w} + f_w \\ \lambda^2 \tilde{w} + \frac{EI}{\rho} \tilde{w}_{x4} - \frac{\alpha}{\rho C_p} \left( \tilde{q}_a - \gamma [\tilde{v}_x]_\eta^\xi \right) \frac{d}{dx} (\delta_\eta - \delta_\xi) = f_v - \lambda f_w \\ \tilde{q}_a = \frac{1}{\lambda + \frac{1+\beta}{RC_p}} f_{q_a} - \frac{\alpha}{\lambda + \frac{1+\beta}{RC_p}} [\tilde{v}_x]_\eta^\xi \end{cases} \quad (55)$$

and therefore  $\tilde{v}$  and  $\tilde{q}_a$  are uniquely determined by  $\tilde{w}$ . Then, we replace the term  $\tilde{w}_{x4}$  by the expression in (45), and using the conditions in  $D(A_1)$  we obtain that the second equation of (55) is equivalent to

$$\lambda^2 \tilde{w} + \frac{EI}{\rho} \{ \tilde{w}_{x4} \} = f_v - \lambda f_w. \quad (56)$$

We write the weak form of (56) for all  $\phi \in V$  (with  $V$  defined in (48)), that since  $\{ \tilde{w}_{x4} \} \in L^2(0, L)$  we can split the integral in three different parts

$$\int_0^L \lambda^2 \tilde{w} \phi dx + \frac{EI}{\rho} \sum_{i=1}^3 \int_{I_i} \tilde{w}_{x4} \phi dx = \int_0^L \tilde{f} \phi dx \quad (57)$$

with  $I_1 = (0, \eta^-)$ ,  $I_1 = (\eta^+, \xi^-)$  and  $I_1 = (\xi^+, L)$  and  $\tilde{f} = f_v - \lambda f_w$ . Using two times integration by parts on the second term and the boundary and compatibility conditions in  $D(A_1)$  we obtain

$$\begin{aligned} \int_0^L \tilde{f} \phi dx &= \int_0^L \lambda^2 \tilde{w} \phi dx + \frac{EI}{\rho} \int_0^L \{\tilde{w}_{xx}\} \phi_{xx} dx \\ &\quad + \frac{\alpha}{\rho C_p} \left( \tilde{q}_a - \gamma [\tilde{v}_x]_{\eta}^{\xi} \right) (\phi_x(\eta) - \phi_x(\xi)) \\ \int_0^L \tilde{f} \phi dx - \kappa_1 f_{qa}[\phi_x]_{\xi}^{\eta} &= \int_0^L \lambda^2 \tilde{w} \phi dx + \frac{EI}{\rho} \int_0^L \{\tilde{w}_{xx}\} \phi_{xx} dx \\ &\quad - \kappa_2 [\tilde{v}_x]_{\eta}^{\xi} [\phi_x]_{\xi}^{\eta} \end{aligned} \quad (58)$$

where  $\kappa_1 = \frac{R\alpha}{\rho(R\lambda C_p + 1 + \beta)}$  and  $\kappa_2 = \frac{R\alpha(\alpha + \gamma(\lambda + \frac{1+\beta}{RC_p}))}{\rho(RC_p\lambda + 1 + \beta)}$ . Using the last relation in (55) together with the fact that  $\{\tilde{w}_{xx}\} = \tilde{w}_{xx}$  we get

$$\begin{aligned} \int_0^L \tilde{f} \phi dx - \kappa_1 f_{qa}[\phi_x]_{\xi}^{\eta} - \kappa_2 [f_{w,x}]_{\eta}^{\xi} [\phi_x]_{\xi}^{\eta} &= \int_0^L \{\lambda^2 \tilde{w} \phi + \frac{EI}{\rho} \tilde{w}_{xx} \phi_{xx}\} dx \\ &\quad + \kappa_2 \lambda [\tilde{w}_x]_{\xi}^{\eta} [\phi_x]_{\xi}^{\eta}. \end{aligned} \quad (59)$$

We define the bilinear form

$$a(\tilde{w}, \phi) = \int_0^L \{\lambda^2 \tilde{w} \phi + \frac{EI}{\rho} \tilde{w}_{xx} \phi_{xx}\} dx + \kappa_2 \lambda [\tilde{w}_x]_{\xi}^{\eta} [\phi_x]_{\xi}^{\eta} \quad (60)$$

and since  $\kappa_2 > 2$ , we remark that this bilinear form is coercive in  $V$ . Indeed, we have that

$$\begin{aligned} a(\tilde{w}, \tilde{w}) &= \int_0^L \left\{ \lambda^2 \tilde{w}^2 + \frac{EI}{\rho} \tilde{w}_{xx}^2 \right\} dx + \kappa_2 \lambda \left( [\tilde{w}_x]_{\xi}^{\eta} \right)^2 \\ &\geq \frac{EI}{\rho} \int_0^L \tilde{w}_{xx}^2 dx = \frac{EI}{\rho} \|\tilde{w}\|_V^2. \end{aligned} \quad (61)$$

For showing that  $a(\tilde{w}, \phi)$  is continuous we begin by using Cauchy-Swartz and Poincaré inequalities

$$\begin{aligned} a(\tilde{w}, \phi) &\leq \lambda^2 \left( \int_0^L \tilde{w}^2 dx \right)^{\frac{1}{2}} \left( \int_0^L \phi^2 dx \right)^{\frac{1}{2}} + \frac{EI}{\rho} \left( \int_0^L \tilde{w}_{xx}^2 dx \right)^{\frac{1}{2}} \left( \int_0^L \phi_{xx}^2 dx \right)^{\frac{1}{2}} \\ &\quad + \kappa_2 \lambda \int_{\xi}^{\eta} w_{xx} dx \int_{\xi}^{\eta} \phi_{xx} dx \\ &\leq \left( \lambda^2 + \frac{EI}{\rho} \right) \left( \int_0^L \tilde{w}_{xx}^2 dx \right)^{\frac{1}{2}} \left( \int_0^L \phi_{xx}^2 dx \right)^{\frac{1}{2}} + \kappa_2 \lambda \left( \int_{\xi}^{\eta} w_{xx}^2 dx \right)^{\frac{1}{2}} \left( \int_{\xi}^{\eta} \phi_{xx}^2 dx \right)^{\frac{1}{2}}. \end{aligned} \quad (62)$$

Then, by increasing the integration intervals of the last terms, we obtain  $a(\tilde{w}, \phi) \leq (\lambda^2 + EI\rho^{-1} + \kappa) \|\tilde{w}\|_V \|\phi\|_V$ , that indeed proves that the bilinear form  $a(\tilde{w}, \phi)$  is continuous. Therefore, we define the linear functional  $\int_0^L \tilde{f} \phi dx - \kappa_1 f_{qa}[\phi_x]_{\xi}^{\eta} - \kappa_2 [f_{w,x}]_{\eta}^{\xi} [\phi_x]_{\xi}^{\eta}$ , and we apply the Lax-Milgram Theorem to conclude that there exists a unique solution  $\tilde{w} \in V$  to the weak formulation of (55). Now we show that  $\tilde{w} \in H^4(0, \eta) \cap H^4(\eta, \xi) \cap H^4(\xi, L)$ . To do that we start by showing that  $\tilde{w} \in H^4(0, \eta)$ . Therefore we select the test function such that  $\phi \in C_c^{\infty}(0, \eta) \subset V$  and from (56) we obtain

$$\int_0^{\eta} \lambda^2 \tilde{w} \phi dx + \frac{EI}{\rho} \int_0^{\eta} \tilde{w}_{xx} \phi dx = \int_0^{\eta} \tilde{f} \phi dx \quad (63)$$

from which we know that there exists a unique solution  $\tilde{w} \in V$ . Since  $\phi \in C_c^{\infty}(0, \eta) \subset V$ , using two integrations by parts, the previous equation can be rewritten as

$$\int_0^{\eta} \tilde{w}_{xx} \phi_{xx} dx = \frac{\rho}{EI} \int_0^{\eta} (\tilde{f} - \lambda^2 \tilde{w}) \phi dx \quad (64)$$

that indeed proves that  $\tilde{w}_{xx} \in H^2(0, \eta)$  and therefore  $\tilde{w}_{x4} \in L^2(0, \eta)$  according to the  $H^2$  definition (Brezis, 2011, page 216). Similar arguments can be used to prove that  $\tilde{w}_{x4} \in L^2(\eta, \xi)$  and  $\tilde{w}_{x4} \in L^2(\xi, L)$ . To summarize we have proven that there exists

a  $\tilde{w} \in L^2(0, \eta) \cap L^2(\eta, \xi) \cap L^2(\xi, L)$  that satisfies the boundary and compatibility conditions unique solution to (56). From the first and last equations of (55),  $\tilde{v}$  and  $\tilde{q}$  can be uniquely determined from  $\tilde{w}$ . Hence, we proved the range condition and using the Lumer-Phillips Theorem we can conclude that  $A_1$  generates a contraction  $C_0$ -semigroup in  $Z_1$ .  $\square$

In order to conclude about the global asymptotic stability of the origin of system (1) with (3), we first introduce the two following technical lemmas.

**Lemma 7** *The canonical embedding from  $D(A_1)$  to  $Z_1$  is compact.*

*Proof* The statement is equivalent to the fact that for each sequence in  $D(A_1)$ , which is bounded with the graph norm, there exists a subsequence that (strongly) converges in  $Z_1$ . Recalling the definition of the graph norm  $\|z\|_{D(A_1)}^2 := \|z\|_1^2 + \|A_1 z\|_1^2$

$$\begin{aligned} \|z\|_{D(A_1)}^2 = \int_0^L \left\{ EI(w_{xx})^2 + \rho v^2 + EI(v_{xx})^2 + \rho \left( -\frac{EI}{\rho} w_{x4} \right. \right. \\ \left. \left. + \frac{\alpha}{\rho C_p} \left( \bar{q}_a - \gamma [v_x]_{\eta}^{\xi} \right) \frac{d}{dx} (\delta_{\eta} - \delta_{\xi}) \right)^2 \right\} dx \\ + \frac{1}{C_p} \bar{q}_a^2 + \frac{1}{C_p} \left( -\frac{1+\beta}{RC_p} \bar{q}_a - \alpha [v_x]_{\eta}^{\xi} \right)^2 \end{aligned} \quad (65)$$

We substitute the  $w_{x4}$  expression in (45) to obtain

$$\begin{aligned} \|z\|_{D(A_1)}^2 \geq \int_0^L EI(w_{xx})^2 + \rho v^2 + EI(v_{xx})^2 + \frac{EI^2}{\rho} \{w_{x4}\}^2 dx \\ + \frac{1}{C_p} \bar{q}_a^2 + \frac{1}{C_p} \left( -\frac{1}{RC_p} \bar{q}_a - \alpha [v_x]_{\eta}^{\xi} \right)^2, \end{aligned} \quad (66)$$

From the previous inequality, we get

$$\begin{aligned} \|z\|_{D(A_1)}^2 &\geq \min \left\{ EI, \frac{EI^2}{\rho} \right\} \int_0^L (w_{xx})^2 + \{w_{x4}\}^2 dx, \\ \|z\|_{D(A_1)}^2 &\geq \min \{ EI, \rho \} \int_0^L v^2 + (v_{xx})^2 dx, \\ \|z\|_{D(A_1)}^2 &\geq \frac{1}{C_p} \bar{q}_a^2. \end{aligned} \quad (67)$$

Consider now a sequence  $(z_n)_{n \in \mathbb{N}}$  in  $D(A_1)$  bounded for the graph norm  $D(A_1)$ . From (67) we deduce that the sequence  $(w_n)_{n \in \mathbb{N}}$  is bounded in  $H^4(0, \eta) \cap H^4(\eta, \xi) \cap H^4(\xi, L)$ , the sequence  $(v_n)_{n \in \mathbb{N}}$  is bounded in  $H^2(0, L)$  and the sequence  $(\bar{q}_{a,n})_{n \in \mathbb{N}}$  is bounded in  $\mathbb{R}$ . It follows that there exist  $v \in L^2(0, L)$  and  $q \in \mathbb{R}$  such that extracting a subsequence from each sequence we obtain that  $(v_{n'})_{n' \in \mathbb{N}} \rightarrow v$  in  $L^2(0, L)$  and  $(\bar{q}_{a,n'})_{n' \in \mathbb{N}} \rightarrow \bar{q}_a$  in  $\mathbb{R}$ . Then, we extract a subsequence from  $(w_{n'})_{n' \in \mathbb{N}}$ , calling it  $(w_{n'})_{n' \in \mathbb{N}}$ , such that  $(w_{n'})_{n' \in \mathbb{N}} \rightarrow w_1$  in  $H^2(0, \eta)$ . We successively extract a sequence such that  $(w_{n'})_{n' \in \mathbb{N}} \rightarrow w_2$  in  $H^2(\eta, \xi)$ , and a final subsequence such that  $(w_{n'})_{n' \in \mathbb{N}} \rightarrow w_3$  in  $H^2(\xi, L)$ . Then, we define  $w \in H^2(0, L)$

$$w = \begin{cases} w_1 & \text{for } x \in (0, \eta) \\ w_2 & \text{for } x \in (\eta, \xi) \\ w_3 & \text{for } x \in (\xi, L). \end{cases} \quad (68)$$

Therefore we conclude that there exists a subsequence  $(w_{n'})_{n' \in \mathbb{N}} \rightarrow w$  in  $V$ . It follows that there exists a subsequence  $(z_{n'})_{n' \in \mathbb{N}}$  that strongly converges in  $Z_1$ .  $\square$

**Lemma 8** (Lemma 3.1.1 in (Mattioni, 2021)) *For a linear system*

$$\Sigma = \begin{cases} z_t(t) = Az(t) \\ y = Cz(t) \\ z(0) = z_0 \end{cases} \quad (69)$$

where  $A$  is the generator of a  $C_0$ -semigroup and  $C$  is an admissible output operator, the following two statements are equivalent

- if  $y(t)$  is constant then  $z_0 = 0$ .
- $\Sigma$  is approximately observable and its only equilibrium point is zero.

We are now in the position to prove Theorem 1.

*Proof of Theorem 1.* The stability property follows from the contraction semigroup generation of Theorem 6. For concluding the attractivity property, we define the candidate Lyapunov functional  $\mathcal{V}(z) = \frac{1}{2}\langle z, z \rangle_1$  for all  $z \in Z_1$ . Since the defined Lyapunov function corresponds to half the square of the norm in the space  $Z_1$ , its time-derivative along the solutions to system (1) are given by

$$\dot{\mathcal{V}} = \langle A_1 z, z \rangle_1 \leq -\frac{\alpha\gamma}{\rho C_p} \left( [v_x]_\eta^\xi \right)^2 - \frac{1 + \beta}{RC_p^2} \bar{q}_a^2. \quad (70)$$

To prove the asymptotic stability we make use of LaSalle's invariance principle. We notice that according to Lemma 7 the embedding of  $D(A_1)$  in  $Z_1$  is compact and using (Miyadera, 1992, Corollary 3.7) it follows that  $\|A_1 z(t)\|$  is decreasing. Therefore the trajectory set  $\{z(t) \mid t \geq 0\}$  is pre-compact in  $Z_1$ . Therefore, according to LaSalle's invariance principle, the solutions converge to the largest invariant set  $\Omega_0$  of  $\Omega = \{z \in Z_1 \mid \dot{\mathcal{V}} = 0\}$ . According to (70), we have that  $\Omega = \{z \in Z_1 \mid \bar{q}_a = 0, [v_x]_\eta^\xi = 0\}$ . To find its largest invariant set  $\Omega_0$ , we apply the conditions  $\bar{q}_a = 0, [v_x]_\eta^\xi = 0$  to (47) to obtain that the problem of finding  $\Omega_0$  consists of finding the solution to the over-constrained system

$$\begin{cases} w_t = v, \quad v_t = -\frac{EI}{\rho} w_{x4} \\ [v_x]_\eta^\xi = 0 \\ w(0) = w_x(0) = w_{xx}(L) = w_{x3}(L) = 0, \quad v(0) = v_x(0) = 0, \\ [w_{xx}]_\eta = [w_{xx}]_\xi = 0, \quad [w_{x3}]_\eta = [w_{x3}]_\xi = 0. \end{cases} \quad (71)$$

This corresponds to the over-constrained Euler-Bernoulli beam equation with homogeneous boundary conditions and output defined as  $y = [v_x]_\eta^\xi$ . First of all, we notice that the output  $y$  is a linear bounded operator from  $D(A_1)$  to  $\mathbb{R}$  and therefore is admissible. Then, it is possible to prove that the operator in (71) generates a contraction  $C_0$ -semigroup and that its only equilibrium is the origin. Since  $\eta \notin \mathcal{S}$ , according to Lemma 5, system (71) is exactly observable with respect to  $[v_x]_\eta^\xi$  and therefore it is also approximately observable with respect to the same output. Hence, since we know that the output  $y(t)$  is constrained to be constant, thanks to Lemma 8 we can conclude that the only solution to the over-constrained problem (71) is the origin. Hence, we obtain that  $\Omega_0 = \{0\}$  and by LaSalle's invariance principle we can conclude that the

origin of  $Z_1$  is a globally asymptotically stable equilibrium, and therefore we obtain the following attractivity property

$$w(t)_{H^2}^2 + v(t)_{L^2}^2 + (q_a(t) + \gamma[v_x(t)]_\eta^\xi)^2 \xrightarrow{t \rightarrow \infty} 0. \quad (72)$$

We assume that  $z(0) = z_0 \in D(A_1)$  and therefore  $z(t) \in D(A_1)$  for all  $t$ . Defining the operator  $C^e z(t) = [v_x]_\eta^\xi$ , we remark that the operator  $C^e A_1^{-1}$  is bounded. Therefore, we can rewrite

$$[v_x(t)]_\eta^\xi = C^e A_1^{-1} A_1 z(t). \quad (73)$$

Since in Theorem 6 we have shown that  $A_1$  is the generator of a contraction semigroup, we can write  $z(t) = T(t)z_0$ , where  $T(t)$  is the semigroup generated by  $A_1$ . If  $z_0 \in D(A_1)$ , we can use linearity to have  $A_1 T(t)z_0 = T(t)A_1 z_0$ , and therefore

$$[v_x(t)]_\eta^\xi = C^e A_1^{-1} A_1 T(t)z_0 = C^e A_1^{-1} T(t)A_1 z_0. \quad (74)$$

Since we have already shown that  $T(t)$  is an asymptotically stable semigroup, we have that  $T(t)A_1 z_0 \xrightarrow{t \rightarrow \infty} 0$  and therefore

$$[v_x(t)]_\eta^\xi = C^e A_1^{-1} T(t)A_1 z_0 \xrightarrow{t \rightarrow \infty} 0 \quad (75)$$

because of the boundedness of  $C^e A_1^{-1}$ . Since  $q_a(t) = \bar{q}_a(t) + \gamma[v_x(t)]_\eta^\xi$  and since  $\bar{q}_a(t) \xrightarrow{t \rightarrow \infty} 0$  we obtain that  $q_a(t) \xrightarrow{t \rightarrow \infty} 0$  for  $z_0 \in D(A_1)$ .  $\square$

## 4 Well-posedness and asymptotic stability: nonlinear system

In this section, we first give the proof of the well-posedness of the nonlinear system (6) in closed loop with (7). The equations of system (6) in closed-loop with (7) corresponds to

$$\begin{cases} \rho w_{tt}(x, t) = EI w_{xxxx}(x, t) + \frac{\alpha}{C_p} q_a(t) \frac{d}{dx} (\delta_\eta(x) - \delta_\xi(x)) \\ w(0, t) = w_x(0, t) = w_{xx}(L, t) = w_{xxx}(L, t) = 0 \\ \dot{q}_a(t) = -\frac{1+\beta}{R} \left( \frac{q_a(t)}{C_p} + \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}} \right) - \alpha [w_{xt}(x, t)]_\eta^\xi \\ \dot{q}_{H,1}(t) = -\frac{1+\beta}{R} \left( \frac{q_a(t)}{C_p} + \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}} \right) - i_{RH,1} \left( \frac{q_{H,1}}{C_{H,1}} \right) \\ \vdots \\ \dot{q}_{H,N}(t) = -\frac{1+\beta}{R} \left( \frac{q_a(t)}{C_p} + \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}} \right) - i_{RH,N} \left( \frac{q_{H,N}}{C_{H,N}} \right). \end{cases} \quad (76)$$

We now define the state  $z = (w, v, q_a, q_{H,1}, \dots, q_{H,N}) \in Z_2 = V \times L^2(0, L) \times \mathbb{R}^{N+1}$  where  $V = \{v \in H^2(0, L) \mid v(0) = v_x(0) = 0\}$ ,  $v = w_t$  and the

corresponding nonlinear operator is defined as

$$\dot{z} = A_2 z = \begin{bmatrix} v \\ -\frac{EI}{\rho} w_{x4} + \frac{\alpha}{\rho C_p} q_a \frac{d}{dx} (\delta_\eta - \delta_\xi) \\ -\frac{1+\beta}{R} \left( \frac{q_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}} \right) - \alpha [v_x]_\eta^\xi \\ -\frac{1+\beta}{R} \left( \frac{q_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}} \right) - i_{RH,1} \left( \frac{q_{H,1}}{C_{H,1}} \right) \\ \vdots \\ -\frac{1+\beta}{R} \left( \frac{q_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}} \right) - i_{RH,N} \left( \frac{q_{H,N}}{C_{H,N}} \right) \end{bmatrix} \quad (77)$$

with domain

$$\begin{aligned} D(A_2) = \{z \in Z_2 \mid w \in H^4(0, \eta) \cup H^4(\eta, \xi) \cup H^4(\xi, L), v \in H^2(0, L) \\ w(0) = w_x(0) = w_{xx}(L) = w_{x3}(L) = 0, v(0) = v_x(0) = 0, \\ [w_{xx}]_\eta = \frac{\alpha}{EIC_p} q_a = -[w_{xx}]_\xi, [w_{x3}]_\eta = [w_{x3}]_\xi = 0\}. \end{aligned} \quad (78)$$

We remark that, similarly to the linear case, the compatibility conditions of Section 3.1 are present in  $D(A_2)$ . We equip the state space  $Z_2$  with the inner product

$$\langle z_1, z_2 \rangle_2 = \int_0^L \{EI w_{xx,1} w_{xx,2} + \rho v_1 v_2\} dx + \frac{1}{C_p} q_{a,1} q_{a,2} + \sum_{j=1}^N \frac{q_{H,j,1} q_{H,j,2}}{C_{H,j}} \quad (79)$$

and the corresponding norm  $\|z\|_2 = \sqrt{\langle z, z \rangle_2}$ . Therefore, we rewrite (77) as an abstract Cauchy problem

$$\begin{cases} z_t(t) = A_2 z(t) \\ z(0) = z_0. \end{cases} \quad (80)$$

In the next theorem, we show that there exists a unique solution for the former Cauchy problem, characterized by some non-increasing properties, for all initial conditions taken in  $z_0 \in D(A_2)$ . We would like to point out that in order to demonstrate the range condition required to establish the m-dissipativity of the operator  $A_2$ , we employ a novel approach inspired by the methodology utilized in (Vanspranghe et al, 2022).

**Theorem 9** *For each  $z_0 \in D(A_2)$  there exists a unique absolutely continuous  $z : [0, \infty) \rightarrow Z_2$  such that  $z(0) = z_0$  and (80) holds at a.e  $t > 0$ . Moreover, if  $z(t) \in D(A_2)$  then  $\|z(t)\|_2, \|A_2 z(t)\|_2$  are non-increasing.*

*Proof* According to (Showalter, 2013, Proposition 3.1 p.174), the Cauchy problem (80) has a unique solution with the properties listed in the statements if the operator  $-A_2$  is m-accretive or equivalently if  $A_2$  is m-dissipative. To be m-dissipative, an

operator needs to satisfy the dissipativity condition  $\langle A_2 z - A_2 \tilde{z}, z - \tilde{z} \rangle_2 \leq 0$  together with the fact that  $\text{Ran}(I - \lambda A_2) = Z_2$  for some  $\lambda > 0$ . We start by computing the dissipativity inner product

$$\begin{aligned} \langle A_2 z - A_2 \tilde{z}, z - \tilde{z} \rangle_2 &= \int_0^L \left\{ EI \frac{\partial^2}{\partial x^2} (v - \tilde{v}) \frac{\partial^2}{\partial x^2} (w - \tilde{w}) - EI \frac{\partial^4}{\partial x^4} (w - \tilde{w}) (v - \tilde{v}) \right. \\ &\quad \left. + \frac{\alpha}{C_p} (q_a - \tilde{q}_a) \frac{d}{dx} (\delta_\eta - \delta_\xi) (v - \tilde{v}) \right\} dx \\ &\quad - \frac{1+\beta}{R} \left( \frac{q_a - \tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j} - \tilde{q}_{H,j}}{C_{H,j}} \right) \frac{q_a - \tilde{q}_a}{C_p} \\ &\quad - \frac{\alpha}{C_p} [v_x - \tilde{v}_x]_\eta^\xi (q_a - \tilde{q}_a) \\ &\quad - \sum_{j=1}^N \frac{1+\beta}{R} \left( \frac{q_a - \tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j} - \tilde{q}_{H,j}}{C_{H,j}} \right) \frac{q_{H,j} - \tilde{q}_{H,j}}{C_{H,j}} \\ &\quad - \sum_{j=1}^N \left( i_{RH,j} \left( \frac{q_{H,j}}{C_{H,j}} \right) - i_{RH,j} \left( \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) \right) \frac{q_{H,j} - \tilde{q}_{H,j}}{C_{H,j}} \end{aligned} \quad (81)$$

Using integration by parts together with the homogeneous boundary conditions makes the first two terms eliminate each other. Therefore we obtain

$$\begin{aligned} \langle A_2 z - A_2 \tilde{z}, z - \tilde{z} \rangle_2 &= -\frac{\alpha}{C_p} (q_a - \tilde{q}_a) \int_0^L (\delta_\eta - \delta_\xi) (v_x - \tilde{v}_x) dx \\ &\quad - \frac{\alpha}{C_p} [v_x - \tilde{v}_x]_\eta^\xi (q_a - \tilde{q}_a) \\ &\quad - \frac{1+\beta}{R} \left( \frac{q_a - \tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j} - \tilde{q}_{H,j}}{C_{H,j}} \right) \frac{q_a - \tilde{q}_a}{C_p} \\ &\quad - \frac{1+\beta}{R} \left( \frac{q_a - \tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j} - \tilde{q}_{H,j}}{C_{H,j}} \right) \sum_{j=1}^N \frac{q_{H,j} - \tilde{q}_{H,j}}{C_{H,j}} \\ &\quad - \sum_{j=1}^N \left( i_{RH,j} \left( \frac{q_{H,j}}{C_{H,j}} \right) - i_{RH,j} \left( \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) \right) \frac{q_{H,j} - \tilde{q}_{H,j}}{C_{H,j}}. \end{aligned} \quad (82)$$

Using the Dirac function properties and collecting the third and fourth terms we obtain

$$\begin{aligned} \langle A_2 z - A_2 \tilde{z}, z - \tilde{z} \rangle_2 &= -\frac{1+\beta}{R} \left( \frac{q_a - \tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j} - \tilde{q}_{H,j}}{C_{H,j}} \right)^2 \\ &\quad - \sum_{j=1}^N \left( i_{RH,j} \left( \frac{q_{H,j}}{C_{H,j}} \right) - i_{RH,j} \left( \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) \right) \frac{q_{H,j} - \tilde{q}_{H,j}}{C_{H,j}}. \end{aligned} \quad (83)$$

Using the monotonically increasing property of the nonlinear functions  $i_{H,j}$  we get Using the Dirac function properties and collecting the third and fourth terms we obtain

$$\langle A_2 z - A_2 \tilde{z}, z - \tilde{z} \rangle_2 \leq -\frac{1+\beta}{R} \left( \frac{q_a - \tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j} - \tilde{q}_{H,j}}{C_{H,j}} \right)^2, \quad (84)$$

and therefore  $\langle A_2 z - A_2 \tilde{z}, z - \tilde{z} \rangle_2 \leq 0$ . Let us now show the range condition, *i.e.* for some  $\lambda > 0$  we must find  $\tilde{z} \in D(A_2)$  such that  $(\lambda I - A_2)\tilde{z} = g$  for all  $g \in Z_2$ . The latter equation is equivalent to

$$\begin{aligned} \lambda \tilde{w} - \tilde{v} &= g_w && \text{in } V \\ \lambda \tilde{v} + \frac{EI}{\rho} \tilde{w}_{x4} - \frac{\alpha}{\rho C_p} \tilde{q}_a \frac{d}{dx} (\delta_\eta - \delta_\xi) &= g_v && \text{in } L^2(0, L) \\ \lambda \tilde{q}_a + \frac{1+\beta}{R} \left( \frac{\tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) + \alpha [\tilde{v}_x]_\eta^\xi &= g_a && \text{in } \mathbb{R} \\ \lambda \tilde{q}_{H,j} + \frac{1+\beta}{R} \left( \frac{\tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) + i_{RH,j} \left( \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) &= g_{H,j} && \text{in } \mathbb{R} \end{aligned} \quad (85)$$

where the last equation has to be intended as a set of  $N$  equations from  $j = 1$  to  $j = N$ . We take the weak form of the second equation, with  $v \in V$  (with  $V$  defined in (48))

$$\lambda \langle \tilde{v}, v \rangle_{L^2} + \frac{EI}{\rho} \langle \tilde{w}_{x4}, v \rangle_{L^2} - \frac{\alpha}{\rho C_p} \langle \tilde{q}_a \frac{d}{dx} (\delta_\eta - \delta_\xi), v \rangle_{L^2} = \langle g_v, v \rangle_{L^2}. \quad (86)$$

Using two times integration by parts on the second term, one time on the third together with the boundary conditions in  $D(A_2)$ , the former equation is equivalent to

$$\lambda \langle \tilde{v}, v \rangle_{L^2} + \frac{EI}{\rho} \langle \tilde{w}_{xx}, v_{xx} \rangle_{L^2} - \frac{\alpha}{\rho C_p} \tilde{q}_a [\tilde{v}_x]_{\eta}^{\xi} = \langle g_v, v \rangle_{L^2}. \quad (87)$$

We obtain  $\tilde{w} = \frac{1}{\lambda}(\tilde{v} + g_w)$  from the first equation in (85) and we substitute it in the former equation to obtain

$$\lambda \langle \tilde{v}, v \rangle_{L^2} + \frac{EI}{\lambda \rho} \langle \tilde{v}_{xx}, v_{xx} \rangle_{L^2} - \frac{\alpha}{\rho C_p} \tilde{q}_a [\tilde{v}_x]_{\eta}^{\xi} = \langle g_v, v \rangle_{L^2} + \frac{1}{\lambda} \langle g_w, v_{xx}, v_{xx} \rangle_{L^2}. \quad (88)$$

Taking the inner product in  $\mathbb{R}$  of the last  $N + 1$  equations with the test-elements  $q_a, q_{H,1}, \dots, q_{H,N} \in \mathbb{R}$ , the range condition transforms on finding  $\tilde{z} \in D(A_2)$  such that for all  $v \in V$  and  $q_a, q_{H,1}, \dots, q_{H,N} \in \mathbb{R}$

$$\begin{aligned} \tilde{w} &= \frac{1}{\lambda}(\tilde{v} + g_w) \\ \lambda \langle \tilde{v}, v \rangle_{L^2} + \frac{EI}{\lambda \rho} \langle \tilde{v}_{xx}, v_{xx} \rangle_{L^2} - \frac{\alpha}{\rho C_p} \tilde{q}_a [v_x]_{\eta}^{\xi} &= \langle g_v, v \rangle_{L^2} + \frac{1}{\lambda} \langle g_w, v_{xx}, v_{xx} \rangle_{L^2} \\ \lambda \langle \tilde{q}_a, q_a \rangle_{\mathbb{R}} + \frac{1+\beta}{R} \left\langle \left( \frac{\tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{H,j}}{C_{H,j}} \right), q_a \right\rangle_{\mathbb{R}} + \alpha \langle [\tilde{v}_x]_{\eta}^{\xi}, q_a \rangle_{\mathbb{R}} &= \langle g_a, q_a \rangle_{\mathbb{R}} \\ \lambda \langle \tilde{q}_{H,j}, q_{H,j} \rangle_{\mathbb{R}} + \frac{1+\beta}{R} \left\langle \left( \frac{\tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{H,j}}{C_{H,j}} \right), q_{H,j} \right\rangle_{\mathbb{R}} + \langle i_{RH,j} \left( \frac{\tilde{q}_{H,j}}{C_{H,j}} \right), q_{H,j} \rangle_{\mathbb{R}} &= \langle g_{H,j}, q_{H,j} \rangle_{\mathbb{R}} \end{aligned} \quad (89)$$

To simplify the notation we define  $r = (v, q_a, q_{H,1}, \dots, q_{H,N}) \in V \times \mathbb{R}^{N+1}$ . We now define  $\forall \tilde{r} \in V \times \mathbb{R}^{N+1}$  the function  $\phi(\tilde{r}) \in (V \times \mathbb{R}^{N+1})'$ , using the duality product and  $\forall r \in V \times \mathbb{R}^{N+1}$ :

$$\begin{aligned} \langle \phi(\tilde{r}), r \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})} &= \\ \rho \lambda \langle \tilde{v}, v \rangle_{L^2} + \frac{EI}{\lambda} \langle \tilde{v}_{xx}, v_{xx} \rangle_{L^2} - \frac{\alpha}{C_p} \langle \tilde{q}_a, [v_x]_{\eta}^{\xi} \rangle_{\mathbb{R}} + \frac{\lambda}{C_p} \langle \tilde{q}_a, q_a \rangle_{\mathbb{R}} & \\ + \frac{1+\beta}{RC_p} \left\langle \left( \frac{\tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{H,j}}{C_{H,j}} \right), q_a \right\rangle_{\mathbb{R}} + \frac{\alpha}{C_p} \langle [\tilde{v}_x]_{\eta}^{\xi}, q_a \rangle_{\mathbb{R}} + \lambda \sum_{j=1}^N \frac{\langle \tilde{q}_{H,j}, q_{H,j} \rangle_{\mathbb{R}}}{C_{H,j}} & \\ + \frac{1+\beta}{R} \left\langle \left( \frac{\tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{H,j}}{C_{H,j}} \right), \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}} \right\rangle_{\mathbb{R}} + \sum_{j=1}^N \langle i_{RH,j} \left( \frac{\tilde{q}_{H,j}}{C_{H,j}} \right), \frac{q_{H,j}}{C_{H,j}} \rangle_{\mathbb{R}} & \end{aligned} \quad (90)$$

and similarly, the function  $\xi(g)$  as

$$\langle \xi(g), r \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})} = \rho \langle g_v, v \rangle_{L^2} + \frac{EI}{\lambda} \langle g_w, v_{xx}, v_{xx} \rangle_{L^2} + \frac{1}{C_p} \langle g_a, q_a \rangle_{\mathbb{R}} + \sum_{j=1}^N \frac{1}{C_{H,j}} \langle g_{H,j}, q_{H,j} \rangle_{\mathbb{R}} \quad (91)$$

such that to rewrite the last two equations of (89) as  $\phi(\tilde{r}) = \xi(g)$  for all  $\tilde{r} \in V \times \mathbb{R}^{N+1}$ . It is worth noticing that  $\phi(\tilde{r}) \in (V \times \mathbb{R}^{N+1})'$  is a continuous linear function on  $V \times \mathbb{R}^{N+1}$ . To show the range condition, we have to prove that  $\phi$  is onto.

**Claim 10** *The function  $\phi$  is onto if the following conditions are satisfied*

1. *There exists  $C > 0$  such that*

$$\langle \phi(\tilde{r}_1) - \phi(\tilde{r}_2), \tilde{r}_1 - \tilde{r}_2 \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})} \geq C \|\tilde{r}_1 - \tilde{r}_2\|_{V \times \mathbb{R}^{N+1}}^2; \quad (92)$$

2. *For any  $\tilde{r}_1, \tilde{r}_2 \in V \times \mathbb{R}$ , the scalar function*

$$t \rightarrow \langle \phi(\tilde{r}_1 + t\tilde{r}_2), \tilde{r}_2 \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})} \quad (93)$$

*is continuous;*

3.  $\phi$  maps bounded sets in  $V \times \mathbb{R}^{N+1}$  into bounded sets in  $(V \times \mathbb{R}^{N+1})'$ .

*Proof of Claim 1.* According to Lemma 2.1 and Theorem 2.1 in Chapter 2 of (Showalter, 2013),  $\phi$  is onto if conditions 2 and 3 hold together with

1.  $\phi$  is monotone, i.e.  $\langle \phi(\tilde{r}_1) - \phi(\tilde{r}_2), \tilde{r}_1 - \tilde{r}_2 \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})} \geq 0$ .
2. there exists  $\alpha > 0$  such that

$$\langle \phi(\tilde{r}), \tilde{r} \rangle_{(V \times \mathbb{R})' \times (V \times \mathbb{R})} > \langle \xi(g), \tilde{r} \rangle_{(V \times \mathbb{R})' \times (V \times \mathbb{R})} \quad (94)$$

for all  $\tilde{r} \in V \times \mathbb{R}^{N+1} : \|\tilde{r}\| > \alpha$ .

Therefore, to prove Claim 10, it suffices to derive items 1 and 2. Item 1 follows trivially from condition 1. In order to get item 2 from condition 1, we remark that (92) implies  $\langle \phi(\tilde{r}), \tilde{r} \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})} \geq C\|\tilde{r}\|_{V \times \mathbb{R}^{N+1}}^2$ . Then, since  $\langle \xi(g), \tilde{r} \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})}$  is linear for  $\tilde{r} \in V \times \mathbb{R}^{N+1}$ , we obtain that

$$\langle \xi(g), \tilde{r} \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})} \leq \|\xi(g)\|_{(V \times \mathbb{R})'} \|\tilde{r}\|_{V \times \mathbb{R}}. \quad (95)$$

Using the previous two inequalities, it is possible to get that

$$\begin{aligned} & \langle \phi(\tilde{r}), \tilde{r} \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})} - \langle \xi(g), \tilde{r} \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})} \\ & \geq C\|\tilde{r}\|_{V \times \mathbb{R}^{N+1}}^2 - \|\xi(g)\|_{(V \times \mathbb{R}^{N+1})'} \|\tilde{r}\|_{V \times \mathbb{R}^{N+1}}. \end{aligned} \quad (96)$$

Selecting  $\|\tilde{r}\|_{V \times \mathbb{R}^{N+1}}$  sufficiently big, we can finally retrieve condition (94), concluding the proof of Claim 10.  $\square$

We now check the three conditions in the previous claim. From the  $\phi$  definition in (90), rearranging the nonlinear terms, it is possible to obtain

$$\begin{aligned} & \langle \phi(\tilde{r}_1) - \phi(\tilde{r}_2), \tilde{r}_1 - \tilde{r}_2 \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})} \\ & = \rho\lambda \langle \tilde{v}_1 - \tilde{v}_2, \tilde{v}_1 - \tilde{v}_2 \rangle_{L^2} + \frac{EI}{\lambda} \langle \tilde{v}_{1,xx} - \tilde{v}_{2,xx}, \tilde{v}_{1,xx} - \tilde{v}_{2,xx} \rangle_{L^2} \\ & \quad - \frac{\alpha}{C_p} \langle \tilde{q}_{1,a} - \tilde{q}_{2,a}, [v_{1,x}]_{\tilde{\eta}}^{\xi} - [v_{2,x}]_{\tilde{\eta}}^{\xi} \rangle_{\mathbb{R}} + \frac{\lambda}{C_p} \langle \tilde{q}_{1,a} - \tilde{q}_{2,a}, \tilde{q}_{1,a} - \tilde{q}_{2,a} \rangle_{\mathbb{R}} \\ & \quad + \frac{1+\beta}{R} \left\langle \left( \frac{\tilde{q}_{1,a} - \tilde{q}_{2,a}}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{1,H,j} - \tilde{q}_{2,H,j}}{C_{H,j}} \right), \frac{\tilde{q}_{1,a} - \tilde{q}_{2,a}}{C_p} \right\rangle_{\mathbb{R}} \\ & \quad + \frac{\alpha}{C_p} \langle [v_{1,x}]_{\tilde{\eta}}^{\xi} - [v_{2,x}]_{\tilde{\eta}}^{\xi}, \tilde{q}_{1,a} - \tilde{q}_{2,a} \rangle_{\mathbb{R}} + \lambda \sum_{j=1}^N \frac{\langle \tilde{q}_{1,H,j} - \tilde{q}_{2,H,j}, \tilde{q}_{1,H,j} - \tilde{q}_{2,H,j} \rangle}{C_{H,j}} \\ & \quad + \frac{1+\beta}{R} \left\langle \left( \frac{\tilde{q}_{1,a} - \tilde{q}_{2,a}}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{1,H,j} - \tilde{q}_{2,H,j}}{C_{H,j}} \right), \sum_{j=1}^N \frac{\tilde{q}_{1,H,j} - \tilde{q}_{2,H,j}}{C_{H,j}} \right\rangle_{\mathbb{R}} \\ & \quad \sum_{j=1}^N \langle i_{RH,j} \left( \frac{\tilde{q}_{1,H,j}}{C_{H,j}} \right) - i_{RH,j} \left( \frac{\tilde{q}_{2,H,j}}{C_{H,j}} \right), \frac{\tilde{q}_{1,H,j}}{C_{H,j}} - \frac{\tilde{q}_{2,H,j}}{C_{H,j}} \rangle_{\mathbb{R}} \end{aligned} \quad (97)$$

Simplifying the equal terms with opposite signs, grouping the fifth and eighth terms and using the monotonicity of the nonlinear current we get

$$\begin{aligned} & \langle \phi(\tilde{r}_1) - \phi(\tilde{r}_2), \tilde{r}_1 - \tilde{r}_2 \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})} \\ & \geq \rho\lambda \langle \tilde{v}_1 - \tilde{v}_2, \tilde{v}_1 - \tilde{v}_2 \rangle_{L^2} + \frac{EI}{\lambda} \langle \tilde{v}_{1,xx} - \tilde{v}_{2,xx}, \tilde{v}_{1,xx} - \tilde{v}_{2,xx} \rangle_{L^2} \\ & \quad + \frac{\lambda}{C_p} \langle \tilde{q}_{1,a} - \tilde{q}_{2,a}, \tilde{q}_{1,a} - \tilde{q}_{2,a} \rangle_{\mathbb{R}} + \frac{1+\beta}{R} \left( \frac{\tilde{q}_{1,a} - \tilde{q}_{2,a}}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{1,H,j} - \tilde{q}_{2,H,j}}{C_{H,j}} \right)^2 \\ & \quad + \lambda \sum_{j=1}^N \frac{\langle \tilde{q}_{1,H,j} - \tilde{q}_{2,H,j}, \tilde{q}_{1,H,j} - \tilde{q}_{2,H,j} \rangle}{C_{H,j}} \\ & \geq \frac{EI}{\lambda} \langle \tilde{v}_{1,xx} - \tilde{v}_{2,xx}, \tilde{v}_{1,xx} - \tilde{v}_{2,xx} \rangle_{L^2} + \frac{\lambda}{C_p} \langle \tilde{q}_{1,a} - \tilde{q}_{2,a}, \tilde{q}_{1,a} - \tilde{q}_{2,a} \rangle_{\mathbb{R}} \\ & \quad + \lambda \sum_{j=1}^N \frac{\langle \tilde{q}_{1,H,j} - \tilde{q}_{2,H,j}, \tilde{q}_{1,H,j} - \tilde{q}_{2,H,j} \rangle}{C_{H,j}} \end{aligned} \quad (98)$$

Defining  $C = \min(\frac{EI}{\lambda}, \frac{\lambda}{C_p}, \frac{\lambda}{C_{H,1}}, \dots, \frac{\lambda}{C_{H,N}})$  proves condition 1 of Claim 10. To prove condition 2 of Claim 10, we remark that the scalar function

$$\begin{aligned} & \langle \phi(\tilde{r} + t\tilde{r}, \tilde{r})_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})} = \rho\lambda \langle \tilde{v} + t\tilde{v}, \tilde{v} \rangle_{L^2} + \frac{EI}{\lambda} \langle \tilde{v}_{xx} + t\tilde{v}_{xx}, \tilde{v}_{xx} \rangle_{L^2} \\ & - \frac{\alpha}{C_p} \langle \tilde{q}_a + t\tilde{q}_a, [\tilde{v}_x]_{\eta}^{\xi} \rangle_{\mathbb{R}} + \frac{\lambda}{C_p} \langle \tilde{q}_a + t\tilde{q}_a, \tilde{q}_a \rangle_{\mathbb{R}} \\ & \frac{1+\beta}{R} \left\langle \left( \frac{\tilde{q}_a + t\tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{H,j} + t\tilde{q}_{H,j}}{C_{H,j}} \right), \frac{\tilde{q}_a}{C_p} \right\rangle_{\mathbb{R}} \\ & + \lambda \sum_{j=1}^N \left\langle \frac{\tilde{q}_{H,j} + t\tilde{q}_{H,j}}{C_{H,j}} + \frac{\tilde{q}_{H,j}}{C_p}, [\tilde{v}_x]_{\eta}^{\xi} + t[\tilde{v}_x]_{\eta}^{\xi}, \tilde{q}_a \right\rangle_{\mathbb{R}} \\ & \frac{1+\beta}{R} \left\langle \left( \frac{\tilde{q}_a + t\tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{H,j} + t\tilde{q}_{H,j}}{C_{H,j}} \right), \sum_{j=1}^N \frac{\tilde{q}_{H,j}}{C_{H,j}} \right\rangle_{\mathbb{R}} \\ & \sum_{j=1}^N \langle i_{RH,j} \left( \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) + t i_{RH,j} \left( \frac{\tilde{q}_{H,j}}{C_{H,j}} \right), \frac{\tilde{q}_{H,j}}{C_{H,j}} \rangle_{\mathbb{R}} \end{aligned} \quad (99)$$

is continuous because of the continuity of the nonlinear functions  $i_{RH,j}$ . To prove condition 3 of Claim 10, it is sufficient to show that for all bounded sets  $B$  of  $V \times \mathbb{R}$ , there exists  $M \in \mathbb{R}$  such that  $\|\phi(B)\| \leq M$  where

$$\|\phi(B)\| = \sup_{\tilde{r} \in B} \sup_{r \in V \times \mathbb{R}^{N+1}} \frac{|\langle \phi(\tilde{r}), r \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})}|}{\|r\|_{V \times \mathbb{R}^{N+1}}}. \quad (100)$$

Using the  $\phi(\tilde{r})$  definition in (90), we get that

$$\begin{aligned} & |\langle \phi(\tilde{r}), r \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})}| \leq \\ & \left| \left( \rho\lambda + \frac{EI}{\lambda} \right) \langle \tilde{v}_{xx}, v_{xx} \rangle_{L^2} - \frac{\alpha}{C_p} \tilde{q}_a \langle v \rangle_{\eta}^{\xi} + \frac{\alpha}{C_p} [\tilde{v}]_{\eta}^{\xi} \tilde{q}_a + \frac{\lambda}{C_p} \tilde{q}_a q_a + \lambda \sum_{j=1}^N \frac{\tilde{q}_{H,j} q_{H,j}}{C_{H,j}} \right. \\ & \left. \frac{1+\beta}{R} \left( \frac{\tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) \left( \frac{q_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}} \right) + \sum_{j=1}^N i_{RH,j} \left( \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) \frac{q_{H,j}}{C_{H,j}} \right| \\ & = \left| \left( \rho\lambda + \frac{EI}{\lambda} \right) \langle \tilde{v}_{xx}, v_{xx} \rangle_{L^2} - \frac{\alpha}{C_p} \langle \tilde{q}_a, v_{xx} \rangle_{L^2} + \frac{\alpha}{C_p} \langle \tilde{v}_{xx}, q_a \rangle_{L^2} \right. \\ & \left. + \left( \lambda \tilde{q}_a + \frac{1+\beta}{R} \left( \frac{\tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) \right) \frac{q_a}{C_p} \right. \\ & \left. + \sum_{j=1}^N \left( \frac{q_{H,j}}{C_{H,j}} + \frac{1+\beta}{R} \left( \frac{\tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) + i_{RH,j} \left( \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) \right) \frac{q_{H,j}}{C_{H,j}} \right|. \end{aligned} \quad (101)$$

Then, we split the absolute value and apply Cauchy-Schwarz to obtain

$$\begin{aligned} & |\langle \phi(\tilde{r}), r \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})}| \leq \\ & \left( \rho\lambda + \frac{EI}{\lambda} \right) \|\tilde{v}_{xx}\|_{L^2} \|v_{xx}\|_{L^2} + \frac{\alpha}{C_p} |\tilde{q}_a| \|v_{xx}\|_{L^2} + \frac{\alpha}{C_p} \|\tilde{v}_{xx}\|_{L^2} |q_a| \\ & + \frac{1}{C_p} |\lambda \tilde{q}_a + \frac{1+\beta}{R} \left( \frac{\tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{H,j}}{C_{H,j}} \right)| \|q_a\| \\ & + \sum_{j=1}^N \left| \frac{q_{H,j}}{C_{H,j}} + \frac{1+\beta}{R} \left( \frac{\tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) + i_{RH,j} \left( \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) \right| \left| \frac{q_{H,j}}{C_{H,j}} \right|. \end{aligned} \quad (102)$$

Since  $\|v_{xx}\|_{L^2}, |q_a|, |q_{H,j}| \leq \|r\|_{V \times \mathbb{R}^{N+1}}$ , we obtain that

$$\begin{aligned} & |\langle \phi(\tilde{r}), r \rangle_{(V \times \mathbb{R}^{N+1})' \times (V \times \mathbb{R}^{N+1})}| \leq \\ & \left( \left( \rho\lambda + \frac{EI}{\lambda} + \frac{\alpha}{C_p} \right) \|\tilde{v}_{xx}\|_{L^2} + \frac{\alpha}{C_p} |\tilde{q}_a| + \frac{1}{C_p} \left| \lambda \tilde{q}_a + \frac{1+\beta}{R} \left( \frac{\tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) \right| \right. \\ & \left. + \sum_{j=1}^N \left| \frac{q_{H,j}}{C_{H,j}} + \frac{1+\beta}{R} \left( \frac{\tilde{q}_a}{C_p} + \sum_{j=1}^N \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) + i_{RH,j} \left( \frac{\tilde{q}_{H,j}}{C_{H,j}} \right) \right| \right) \|r\|_{V \times \mathbb{R}^{N+1}} \end{aligned} \quad (103)$$

Since  $\tilde{r} \in B$  and the continuous nonlinear currents  $i_{RH,j}$  map compact sets (in  $\mathbb{R}$ ) into bounded sets (in  $\mathbb{R}$ ), we can conclude that  $\|\phi(B)\|$  is bounded, and condition 3 of Claim 10 holds. Therefore, applying Claim 10, we deduce that  $\phi$  is onto, which implies that there exists  $\tilde{r} \in V \times \mathbb{R}^{N+1}$  such that the last  $N+2$  equations in (89) are satisfied. Using the first equation in (89) we obtain that  $\tilde{w}$  is uniquely determined by  $\tilde{v}$ . Then, using (86) and similar arguments as in the proof of Theorem 6, it is possible to prove that  $\tilde{w} \in H^4(0, \eta) \cap H^4(\eta, \xi) \cap H^4(\xi, L)$ . Finally, we use Proposition 3.1 in Chapter 3 of (Showalter, 2013) to conclude the proof of Theorem 9.  $\square$

As in the previous section, to conclude about the pre-compactness of trajectories in the state space  $Z_2$ , we introduce the following lemma.

**Lemma 11** *The canonical embedding from  $D(A_2)$  to  $Z_2$  is compact.*

*Proof* Similarly to Lemma 7, we have to prove that for each sequence  $(z_n)_{n \in \mathbb{N}}$  in  $D(A_2)$  which is bounded with the operator norm, there exists a subsequence that strongly converges in  $Z_2$ . We compute the operator norm  $\|z\|_{D(A_2)}^2 := \|z\|_2^2 + \|A_2 z\|_2^2$

$$\begin{aligned} \|z\|_{D(A_2)}^2 &= \int_0^L \{EI(w_{xx})^2 + \rho v^2 + EI(-\frac{EI}{\rho} w_{x4} + \frac{\alpha}{\rho C_p} q_a \frac{d}{dx}(\delta_\eta - \delta_\xi))^2 \\ &\quad + \rho(v_{xx})^2\} dx + \frac{1}{C_p} q_a^2 + \frac{1}{C_p} \left(-\frac{1+\beta}{R} \left(\frac{q_a(t)}{C_p} + \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}}\right) - \alpha [v_x]_\eta^\xi\right)^2 \\ &\quad + \sum_{j=1}^N \frac{q_{H,j}^2}{C_{H,j}} + \sum_{j=1}^N \frac{1}{C_{H,j}} \left(-\frac{1+\beta}{R} \left(\frac{q_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}}\right) - i_{RH,j} \left(\frac{q_{H,j}}{C_{H,j}}\right)\right)^2 \end{aligned} \quad (104)$$

that using the  $w_{x4}$  definition in (45) transforms into

$$\begin{aligned} \|z\|_{D(A_2)}^2 &= \int_0^L \{EI(w_{xx})^2 + \rho v^2 + \frac{EI^3}{\rho} w_{x4}^2 + \rho(v_{xx})^2\} dx + \frac{1}{C_p} q_a^2 \\ &\quad + \frac{1}{C_p} \left(-\frac{1+\beta}{R} \left(\frac{q_a(t)}{C_p} + \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}}\right) - \alpha [v_x]_\eta^\xi\right)^2 + \sum_{j=1}^N \frac{q_{H,j}^2}{C_{H,j}} \\ &\quad + \sum_{j=1}^N \frac{1}{C_{H,j}} \left(-\frac{1+\beta}{R} \left(\frac{q_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}}\right) - i_{RH,j} \left(\frac{q_{H,j}}{C_{H,j}}\right)\right)^2 \\ &\geq \int_0^L \{EI(w_{xx})^2 + \rho v^2 + \frac{EI^3}{\rho} w_{x4}^2 + \rho(v_{xx})^2\} dx + \frac{1}{C_p} q_a^2 + \sum_{j=1}^N \frac{q_{H,j}^2}{C_{H,j}} \end{aligned} \quad (105)$$

Then, similarly as in the proof of Lemma 7, we can conclude that for each subsequence  $(z_n)_{n \in \mathbb{N}}$  there exists a subsequence  $(z_{n'})_{n' \in \mathbb{N}}$  that strongly converges in  $Z_2$ .  $\square$

We can now prove the global asymptotic stability of the origin of the nonlinear closed-loop system (80).

*Proof of Theorem 2.* The stability condition directly follows from Theorem 9. To show the attractivity property we define the Lyapunov functional  $\mathcal{V} = \frac{1}{2} \langle z, z \rangle_2$  and computing its time-derivative along the system trajectories we obtain

$$\begin{aligned} \dot{\mathcal{V}} &= -\frac{1+\beta}{R} \left(\frac{q_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}}\right)^2 - \sum_{j=1}^N i_{RH,j} \left(\frac{q_{H,j}}{C_{H,j}}\right) \frac{q_{H,j}}{C_{H,j}} \\ &\leq -\frac{1+\beta}{R} \left(\frac{q_a}{C_p} + \sum_{j=1}^N \frac{q_{H,j}}{C_{H,j}}\right)^2 - \sum_{j=1}^N \underline{G}_j \frac{q_{H,j}^2}{C_{H,j}^2}. \end{aligned} \quad (106)$$

Similarly as in the proof of Theorem 1, to show the origin's global asymptotic stability, we make use of LaSalle's invariance principle. Firstly, we note that according to Lemma 11 and the fact that  $\|A_2 z(t)\|_2$  is decreasing, the trajectory set  $\{z(t) \mid t \geq 0\}$  is pre-compact in  $Z_2$ . Moreover, the  $\omega$ -limit set  $w(z_0) \in D(A_2)$  is not empty and invariant for the nonlinear semigroup generated by  $A_2$  (see (Slemrod, 1989, Theorem 3.1)). The set  $\Omega$  for which  $\Omega = \{z \in Z_2 \mid \dot{\mathcal{V}} = 0\}$  corresponds to  $\Omega = \{z \in Z_2 \mid q_a = 0, q_{H,j} = 0, j = \{1, \dots, N\}\}$ .

To search for its largest invariant subspace, we substitute the relations in  $\Omega$  in the closed-loop system to obtain

$$\begin{cases} \dot{w} = v \\ \dot{v} = -\frac{EI}{\rho} w_{x4} \\ [v_x]_{\eta}^{\xi} = 0 \\ \tilde{w}(0) = \tilde{w}_x(0) = \tilde{w}_{xx}(L) = \tilde{w}_{x3}(L) = 0, v(0) = v_x(0) = 0, \\ [\tilde{w}_{xx}]_{\eta} = [\tilde{w}_{xx}]_{\xi} = 0, [\tilde{w}_{x3}]_{\eta} = [\tilde{w}_{x3}]_{\xi} = 0. \end{cases} \quad (107)$$

Since  $\eta \notin \mathcal{S}$ , according to Lemma 5 the Euler Bernoulli equation is exactly observable for  $[v_x]_{\eta}^{\xi}$ . Therefore, according to Lemma 8, the only solution of the over-constrained system (107) is the origin. Then, the largest invariant subset of  $\Omega$  is the origin. Therefore, according to LaSalle's invariance principle, we can conclude that the origin is a globally asymptotically stable equilibrium of system (80).  $\square$

## 5 Conclusions and perspectives

This paper is devoted to the modeling and stability analysis of a flexible beam controlled by a collocated piezoelectric actuator in the self-sensing configuration. A linear model describing the system dynamics, composed of a set of PDE-ODE equations, has been obtained from general principles. The obtained linear model is well-posed and its origin is globally asymptotically stable when a voltage control law composed of the PA voltage and the available terms from the self-sensing configuration is applied. Furthermore, the more realistic case of a piezoelectric actuator affected by hysteresis has been considered. Finally, a passive control law that globally asymptotically stabilizes the origin of the closed-loop system has been proposed.

Possible future research can be focused on the design of a nonlinear observer capable of estimating the hysteresis effect. The observed state can then be used to cancel the hysteresis effect and the application of the strong feedback terms can become possible.

## 6 Declarations

**Funding** This work has been received funding from MIAI@Grenoble Alpes project (grant ANR-19-P3IA-0003) and by the ANR Labex CIMI (grant ANR-11-LABX- 0040).

**Conflict of interest** The authors have no conflicts of interest to declare that are relevant to the content of this article.

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