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Abstract

We introduce higher-order Massey products for algebras over algebraic operads. This extends the work of Fernando Muro on secondary ones. We study their basic properties and behavior with respect to morphisms of algebras and operads and give some connections to formality. We prove that these higher-order operations represent the differentials in a naturally associated operadic Eilenberg–Moore spectral sequence. We also study the interplay between particular choices of higher-order Massey products and quasi-isomorphic $\mathscr{P}_\infty$-structures on the homology of a $\mathscr{P}$-algebra. We focus on Koszul operads over a characteristic zero field and explain how our results generalize to the non-Koszul case.

1 Introduction

In [18], reprinted as [19], W. S. Massey introduced the classical triple Massey product, a secondary operation on the (co)homology of differential graded associative algebras. He used this new operation to show that the Borromean rings are non-trivially linked. Similar secondary operations were defined independently by Allday and Retah on the homology of differential graded Lie algebras, see [1, 2, 26]. The existence of these higher-order products is due to the vanishing of certain equations that follow from the associativity and Jacobi relations at the chain level, respectively. Recently, F. Muro has shown that secondary operations analogous to Massey's in the case of associative algebras on the homology of differential graded algebraic structures are not ad-hoc at all [22]. Indeed, the theory of algebraic operads explains and organizes the existence and construction of these operations. An algebraic operad is an operad in the symmetric monoidal category of $\mathbb{Z}$-graded vector spaces over a characteristic zero field, and will be assumed to be Koszul. In *loc. cit.*, Muro defines secondary Massey products for algebras over algebraic operads. Given an algebraic operad $\mathscr{P}$, each quadratic relation in the presentation of $\mathscr{P}$ defines a secondary Massey-product-like operation on the homology of the $\mathscr{P}$-algebras. This secondary operation takes as many inputs as the arity of the relation. In this way, the associativity relation of the associative operad yields the classical triple Massey products, while the Jacobi identity relation of the Lie operad yields the Lie–Massey brackets. Under this new point of view, Muro uncovered secondary Massey-product-like operations for many distinct types of algebras for the first time, and gave applications to hyper-commutative and Gerstenhaber algebras.

In Muro's paradigm, there is no restriction as to the arity of the relation. Thus, a relation Γ of arity r in a presentation of an operad $\mathscr{P}$ produces a Massey-product-like operation with r inputs $\langle -,\dots,-\rangle_\Gamma$ on the homology of the $\mathscr{P}$-algebras. However, this still left the definition of *higher-order* Massey product operations unclear. This is where our work enters the picture. It is well-known that the triple Massey product is just the first in an infinite series of higher-order operations on the homology of differential graded associative algebras, roughly witnessing the different ways in which an n-fold product in homology vanishes as a consequence of associativity. These higher-order products have been shown many times to be essential in a wide range of topics where triple Massey products are not enough, see for example the survey [14]. In particular, they are concrete tools for computations when a fully-fledged A_∞-structure is not available.

In this work, we introduce and study *higher-order Massey products for algebras over algebraic operads*. These higher operations include Muro's secondary ones, and gather together to form the

hierarchy of higher operations on the homology of algebras over algebraic operads mentioned before. Our approach generalizes the fruitful framework of higher-order Massey products for differential graded associative algebras to algebras over any algebraic operad, producing a new tool to perform computations in many kinds of differential graded algebras.

The importance of these higher-order operations seems to have been neglected due to a widespread misconception. This misconception consists of thinking that, whenever a higher-order Massey product set $\langle x_1, ..., x_r \rangle$ on the homology of a differential graded associative algebra is defined, then any transferred A_∞-structure $\{m_r\}$ on the homology of this differential algebra via the homotopy transfer theorem satisfies

$$\pm m_r(x_1, ..., x_r) \in \langle x_1, ..., x_r \rangle. \tag{1}$$

This is true only for the triple Massey product, but fails in general [5]. Algebras over operads other than the associative one behave in the same manner (Theorem 4.2). This fact makes the higher-order operations defined in this paper important, filling a fundamental gap in the understanding of the homology of differential graded algebraic structures. Being slightly more precise, we show that if the homology of an algebra over a Koszul operad $\mathscr{P}$ is endowed with a $\mathscr{P}_\infty$-algebra structure quasi-isomorphic to the original structure, then the $\mathscr{P}_\infty$-algebra structure maps recover higher-order Massey products only up to lower-arity $\mathscr{P}_\infty$-algebra structure maps. We also prove, however, a positive result in this direction: for any choice of class in a higher-order Massey product set, under a linearly independence hypothesis, one can make appropriate choices in the homotopy transfer theorem so that the induced $\mathscr{P}_\infty$ structure on the homology of the $\mathscr{P}$-algebra recovers this choice exactly by Formula (1).

Let us briefly explain how these higher-order Massey products arise. Let $\mathscr{P}$ be a Koszul operad with Koszul dual cooperad $\mathscr{P}^{\text{¡}}$ (we explain in Remark 2.16 how to deal with the non-Koszul case). Each weight-homogeneous cooperation Γ^c of $\mathscr{P}^{\text{¡}}$ gives rise to a partially defined higher operation $\langle -, ..., - \rangle_{\Gamma^c}$ on the homology of any $\mathscr{P}$-algebra. The number of inputs of this operation is the arity r of Γ^c. If A is a $\mathscr{P}$-algebra, then out of homogeneous elements $x_1, ..., x_r \in H_*(A)$, the operation gives a (possibly empty) set of homology classes

$$\langle x_1, ..., x_r \rangle_{\Gamma^c} \subseteq H_*(A).$$

The non-emptiness depends on the vanishing, in a precise sense, of higher operations of the same kind that arise from Γ^c and have strictly lower weight-degree. We call $\langle x_1, ..., x_r \rangle_{\Gamma^c}$ the Γ^c-*Massey product* of the classes $x_1, ..., x_r$. The process to construct the Γ^c-Massey product operation $\langle -, ..., - \rangle_{\Gamma^c}$ is done by a non-trivial analogy with the case of differential graded associative algebras. To wit, the cooperation Γ^c determines a set of indices $I(\Gamma^c)$ which is then used to form *defining systems*. Fixed a $\mathscr{P}$-algebra A and homogeneous elements $x_1, ..., x_r \in H_*(A)$, where r is the arity of Γ^c, a defining system for the Γ^c-Massey product $\langle x_1, ..., x_r \rangle_{\Gamma^c}$ is a coherent choice of elements $\{a_\alpha\}$ of A indexed by $I(\Gamma^c)$ that conspire together to create a cycle. Running over all possible choices of defining systems for $x_1, ..., x_r$, we obtain all possible representatives of the homology classes in the set $\langle x_1, ..., x_r \rangle_{\Gamma^c}$. This construction is the core of the paper, and it is performed in Section 2. Since the details are quite technical, we skip them for the moment and refer the reader to the mentioned section. There, we give explicit examples, including the case of the associative, commutative, Lie, and dual numbers operads. We prove that our framework generalizes Muro's in Proposition 2.9. In Section 2.1, we study the basic properties enjoyed by these new operations. For example, we prove that morphisms of $\mathscr{P}$-algebras preserve higher-order Massey products, and that quasi-isomorphisms induce a bijective correspondence between them. This makes the higher-order Massey products a useful tool in the study of homotopy types of algebras over operads; in particular, they can be used to study formality-type results. In Section 2.2, we explain how higher-order Massey products behave with respect to morphisms of operads. Under mild assumptions, higher-order Massey products can be pulled back and forward along morphisms of operads. This allows one to relate the formality (or more generally, the quasi-isomorphism class) of an algebra of a certain type to the formality (or quasi-isomorphism class) of a functorially associated algebra of a distinct type. The reader can have in mind the adjoint pair between taking

the universal enveloping differential graded associative algeba of a differential graded Lie algebra, and forming the commutator bracket of a dg associative algebra. Under some hypotheses, one can relate formality and quasi-isomorphism classes in both directions.

We prove some further results related to higher-order Massey products. It is a well-known and celebrated result that higher-order Massey products for associative algebras provide a concrete description of the differentials in the Eilenberg–Moore spectral sequence. In Section 1.1.1, we explain how to construct an Eilenberg–Moore-type spectral sequence for any algebra over an algebraic operad. Under mild hypotheses, this spectral sequence computes the Quillen homology of the algebras over this operad. The spectral sequence is then exploited in Section 3. Our main result in this direction is Theorem 3.2, which proves that the higher-order Massey products defined in this paper provide concrete representatives for the differentials in this Eilenberg–Moore-type spectral sequence. To finish the paper, we give in Section 4 a precise relationship between the higher-order Massey products on the homology of a $\mathscr{P}$-algebra, and transferred $\mathscr{P}_\infty$-structures on it.

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Notation and conventions

In this paper, all algebraic structures are taken over a base field $\Bbbk$ of characteristic zero. We work on the category of unbounded chain complexes over $\Bbbk$ with homological convention. That is, the differential $d : A_* \to A_{*-1}$ of a chain complex (A, d) is of degree -1. The degree of a homogeneous element x is denoted by $|x|$. The suspension of a chain complex (A, d_A) is the chain complex $(sA, d_{sA}) = (\Bbbk s \otimes A, 1 \otimes d)$, where s is a formal variable of degree 1. For a homogeneous element $a \in A$, we denote $sa = s \otimes a \in sA$. Thus, $(sA)_* \cong A_{*-1}$, and $d_{sA}(sa) = -sd_A(a)$ for every such $a \in A$. The symmetric group on n elements is denoted $\mathbb{S}_n$. The operads in this paper are taken in the symmetric monoidal category of $\mathbb{Z}$-graded vector spaces, and therefore have zero differential. In this monoidal category, we follow the Koszul sign rule. That is, the symmetry isomorphism $U \otimes V \xrightarrow{\cong} V \otimes U$ that identifies two graded vector spaces is given on homogeneous elements by $u \otimes v \mapsto (-1)^{|u||v|} v \otimes u$. Algebras over operads are always differential graded (dg) and homological. We will frequently omit the adjective "dg" and assume it is implicitly understood. The reason for choosing the operads to have trivial differential is that in this case, the homology of any dg $\mathscr{P}$-algebra is a graded (non-dg) $\mathscr{P}$-algebra again. If $f : A \to B$ is a morphism of differential graded algebras over an operad, then we denote by $f_* : H_*(A) \to H_*(B)$ the induced map in homology.

1.1 Preliminaries

In this section, we collect some of the prerequisites for understanding this paper. We start in Section 1.1.1 by giving a brief recollection of the results of operad theory that we will make use of, mainly to establish our notation. We borrow most of the notation from [16], which is an excellent reference for algebraic operads. A non-standard topic explained in this section is the construction of the Eilenberg–Moore-type spectral sequence mentioned in the introduction. In Section 1.1.2, we recall the higher-order Massey products for differential graded associative algebras. To finish, we briefly summarize in Section 1.1.3 the construction of the secondary Massey products for algebras over algebraic operads as defined by Muro in [22].

1.1.1 Operadic background

In this paper, we work with operads in the symmetric monoidal category of graded vector spaces. Our generic operad $\mathscr{P}$ is therefore arity-wise made up of $\mathbb{Z}$-graded vector spaces, but it has no

differential. That is, we work with non-dg operads. The reason is that if A is a $\mathscr{P}$-algebra, we will need A and its homology $H_*(A)$ to be algebras over the same operad. Our operads will always satisfy $\mathscr{P}(0) = 0$, except for theorems 1.2 and 4.2, where they need to be reduced. Recall that an operad $\mathscr{P}$ is *reduced* if $\mathscr{P}(0) = 0$ and $\mathscr{P}(1) = \Bbbk$.

This paper will assume familiarity with the results and notation from [16], and we will adopt its notation for most of the objects used in this paper (infinitesimal compositions, twisting morphisms, weight gradings, and Koszul duality). We shall briefly sketch only those results that will be essential to understand this paper.

Quadratic and Koszul operads. A symmetric sequence E is *reduced* if $E(0) = E(1) = 0$. An operad $\mathscr{P}$ is *quadratic* if it is given by a presentation $\mathscr{F}(E,R)$, that is, if it is given as the quotient $\mathscr{F}(E)/(R)$ of the free operad $\mathscr{F}(E)$ on the reduced symmetric sequence E by the operadic ideal of relations generated by a sub $\mathbb{S}$-module of relations $R \subseteq \mathscr{F}(E)^{(2)}$. Here, $\mathscr{F}(E)^{(n)}$ is the sub $\mathbb{S}$-module of $\mathscr{F}(E)$ formed by elements of *weight n*, that is, formed by combining exactly n generating operations from E. The free operad $\mathscr{F}(E)$ comes equipped with a *weight grading* concentrated in non-negative degrees. Since the operadic ideal (R) is homogeneous with respect to the weight grading of $\mathscr{F}(E)$ and $\mathscr{P}$ is a quotient of $\mathscr{F}(E)$, the weight grading of $\mathscr{F}(E)$ naturally descends to $\mathscr{P}$. The degree n component of this weight grading on $\mathscr{P}$ will be denoted $\mathscr{P}^{(n)}$. Similarly, one can construct the cofree conilpotent cooperad $\mathscr{F}^c(E)$. To do so, consider the same underlying symmetric sequence $\mathscr{F}(E)$, endowed with the same weight-grading. Dually, we can consider the conilpotent sub-cooperad $\mathscr{F}^c(E,R)$ of $\mathscr{F}^c(E)$ which is final among the conilpotent sub-cooperads $\mathscr{C}$ of $\mathscr{F}^c(E)$ equipped with a morphism of $\mathbb{S}$-modules $\mathscr{C} \to E$ such that the composite

$$\mathscr{C} \hookrightarrow \mathscr{F}^c(E) \twoheadrightarrow \mathscr{F}^c(E)^{(2)}/R$$

is 0. The weight grading of $\mathscr{F}^c(E)$ restricts to the sub-cooperad $\mathscr{F}^c(E,R)$, and the degree n component of this weight grading on $\mathscr{F}^c(E,R)$ will be denoted $\mathscr{F}^c(E,R)^{(n)}$. In particular, and this will be important later, the weight 2 component of $\mathscr{F}^c(E,R)$ is precisely the submodule of co-relations R,

$$\mathscr{F}^c(E,R)^{(2)} = R.$$

We call $\mathscr{F}^c(E,R)$ the cofree conilpotent cooperad cogenerated by E with corelations R. A cooperad $\mathscr{C}$ is *quadratic* if it is given by a presentation $\mathscr{F}^c(E,R)$ as above, that is, if is is given as the subcooperad of $\mathscr{F}^c(E)$ just described. Let $\mathscr{P} = \mathscr{F}(E,R)$ be a quadratic operad. Its *Koszul dual cooperad* is defined as

$$\mathscr{P}^{\text{¡}} = \mathscr{F}^c\left(sE, s^2R\right).$$

The *canonical twisting morphism* is the degree -1 morphism of $\mathbb{S}$-modules $\kappa : \mathscr{P}^{\text{¡}} \to \mathscr{P}$ given by the composite

$$\kappa : \mathscr{F}^c\left(sE, s^2R\right) \twoheadrightarrow sE \xrightarrow{s^{-1}} E \to \mathscr{F}(E,R).$$

If $\mathscr{P}$ is augmented, then we can functorially associate to it a quasi-free differential graded conilpotent cooperad $B\mathscr{P}$, called the *bar construction* of $\mathscr{P}$. If $\mathscr{P}$ is quadratic, then it is naturally augmented, and the Koszul dual cooperad $\mathscr{P}^{\text{¡}}$ is a subcooperad of $B\mathscr{P}$ with trivial differential. The operad $\mathscr{P}$ is *Koszul* if the inclusion $\mathscr{P}^{\text{¡}} \hookrightarrow B\mathscr{P}$ is a quasi-isomorphism. The cooperad $B\mathscr{P}$, being differential graded, has a homology cooperad $H_*(B\mathscr{P})$. This homology admits an extra cohomological degree called the syzygy degree. It can be seen that $\mathscr{P}$ is Koszul if, and only if, $H^0(B\mathscr{P}) \cong \mathscr{P}^{\text{¡}}$. The assignment of a Koszul dual cooperad is functorial on weighted operads as long as the morphisms of operads preserve the weight.

$\mathscr{P}_\infty$-structures and Quillen homology. In this section, we discuss several ways to present a $\mathscr{P}_\infty$-structure on a chain complex A for a given Koszul operad $\mathscr{P}$, and define the Quillen homology of a $\mathscr{P}$-algebra. A convenient choice of model for $\mathscr{P}_\infty$ is the cobar construction $\Omega\mathscr{P}^{\text{¡}}$, where $\mathscr{P}^{\text{¡}}$ is the Koszul dual cooperad of $\mathscr{P}$. Recall that the cobar construction is the left adjoint of the bar construction B, mapping onto the category of augmented differential graded operads. A $\mathscr{P}_\infty$ structure on A is therefore a morphism of differential graded operads $\Omega\mathscr{P}^{\text{¡}} \to \mathrm{End}_A$, where End_A is the endomorphism operad of A. Under this point of view, we can think of a $\mathscr{P}_\infty$-algebra structure on A as a family of operations $\{A^{\otimes n} \to A\}$ parametrized by the operad $\Omega\mathscr{P}^{\text{¡}}$.

By the *Rosetta Stone Theorem* [16, Theorem 10.1.13], an equivalent approach, and the one which we shall use in the rest of this document, is to define a $\mathscr{P}_\infty$*-algebra* to be a chain complex A along with a degree -1 square zero coderivation

$$\delta : \mathscr{P}^{\text{¡}}(A) \to \mathscr{P}^{\text{¡}}(A).$$

Briefly recall that if A is a $\mathscr{P}$-algebra, then $\mathscr{P}^{\text{¡}}(A)$ is a quasi-free $\mathscr{P}^{\text{¡}}$-coalgebra whose coderivation codifies the internal differential of A as well as its $\mathscr{P}$-algebra structure. The coderivation is meant as a $\mathscr{P}^{\text{¡}}$-coalgebra, and we explain next how to understand this. Since it squares to zero, we might call it the codifferential of $\mathscr{P}^{\text{¡}}(A)$. It will often be convenient to present δ in two different ways. Firstly, as a collection of linear maps $\delta_r : \mathscr{P}^{\text{¡}}(r) \otimes A^{\otimes r} \to A$, for $r \geq 1$, where each δ_r is the composition

$$\mathscr{P}^{\text{¡}}(r) \otimes A^{\otimes r} \hookrightarrow \bigoplus_{k\geq 1} \mathscr{P}^{\text{¡}}(k) \otimes A^{\otimes k} = \mathscr{P}^{\text{¡}}(A) \xrightarrow{\delta} \mathscr{P}^{\text{¡}}(A) \xrightarrow{\epsilon_A} A.$$

Here, ϵ is the counit of the $\mathscr{P}^{\text{¡}}$ comonad. The coderivation δ can be reconstructed from the family $\{\delta_r\}_{r\geq 1}$ as the map

$$\mathscr{P}^{\text{¡}}(A) \xrightarrow{\triangle_{(1)}} \left(\mathscr{P}^{\text{¡}} \circ_{(1)} \mathscr{P}^{\text{¡}}\right)(A) = \mathscr{P}^{\text{¡}} \circ \left(A; \mathscr{P}^{\text{¡}}(A)\right) \xrightarrow{\mathrm{id}\circ(\mathrm{id};m)} \mathscr{P}^{\text{¡}} \circ (A; A) \to \mathscr{P}^{\text{¡}}(A).$$

Here, $\triangle_{(1)}$ is the infinitesimal decomposition coproduct of $\mathscr{P}^{\text{¡}}$, see [16, §6.1.4], and m is the map $(\delta_r)_{r\geq 1} : \bigoplus_{r\geq 1} \mathscr{P}^{\text{¡}}(r) \otimes A^{\otimes r} \to A$ induced by the universal property of the coproduct of the underlying graded vector spaces. Secondly, we can present δ as a collection of degree $n-2$ linear maps $\delta^{(n)} : \mathscr{P}^{\text{¡}}(A)^{(n)} \to A$, for $n \geq 1$, where each $\delta^{(n)}$ is the composition

$$\mathscr{P}^{\text{¡}}(A)^{(n)} \hookrightarrow \mathscr{P}^{\text{¡}}(A) \xrightarrow{\delta} \mathscr{P}^{\text{¡}}(A) \xrightarrow{\epsilon_A} A,$$

and where $\mathscr{P}^{\text{¡}}(A)^{(n)}$ consists of the weight n part of $\mathscr{P}^{\text{¡}}(A)$,

$$\mathscr{P}^{\text{¡}}(A)^{(n)} = \bigoplus_{r\geq n} \left(\mathscr{P}^{(n)}(r) \otimes_{S_r} A^{\otimes r}\right).$$

To reconstruct δ from the family $\left\{\delta^{(n)}\right\}_{n\geq 1}$, one proceeds *mutatis mutandis* as in the case of $\{\delta_r\}_{r\geq 1}$.

The object $(\mathscr{P}^{\text{¡}}(A), \delta)$ is called the *operadic chain complex*. The *Quillen homology* of a $\mathscr{P}$-algebra A is the homology $H_*\left(\mathscr{P}^{\text{¡}}(A), \delta\right)$ of this operadic chain complex. It forms a (non-differential) graded $\mathscr{P}^{\text{¡}}$-coalgebra.

A $\mathscr{P}_\infty$-algebra A is a *strict* $\mathscr{P}$-algebra if the map m factors through the canonical twisting morphism $\kappa : \mathscr{P}^{\text{¡}} \to \mathscr{P}$. Conversely, any $\mathscr{P}$-algebra A can be seen as a $\mathscr{P}_\infty$-algebra by pulling back its algebra structure along the morphism of operads $\Omega\mathscr{P}^{\text{¡}} \to \mathscr{P}$.

A $\mathscr{P}_\infty$*-morphism* is a map of (dg) $\mathscr{P}^{\text{¡}}$-coalgebras $F : \left(\mathscr{P}^{\text{¡}}(A), \delta\right) \to \left(\mathscr{P}^{\text{¡}}(B), \delta'\right)$. As in the case of a codifferential on a $\mathscr{P}^{\text{¡}}$-coalgebra, it will often be convenient to present F as a collection of linear maps $F_n : \mathscr{P}^{\text{¡}}(n) \otimes A^{\otimes n} \to B$, for $n \geq 1$, where each F_n is the composition

$$\mathscr{P}^{\text{¡}}(n) \otimes A^{\otimes n} \hookrightarrow \bigoplus_{k\geq 1} \mathscr{P}^{\text{¡}}(k) \otimes A^{\otimes k} = \mathscr{P}^{\text{¡}}(A) \xrightarrow{F} \mathscr{P}^{\text{¡}}(B) \xrightarrow{\epsilon_B} B.$$

The map F can be reconstructed from the family $\{F_n\}_{n\geq 1}$ as the map

$$\mathscr{P}^{\text{¡}}(A) \xrightarrow{\triangle} \mathscr{P}^{\text{¡}} \circ \mathscr{P}^{\text{¡}}(A) \xrightarrow{\mathscr{P}^{\text{¡}}(f)} \mathscr{P}^{\text{¡}}(B),$$

where f is the map $(F_i)_{i\geq 1} : \bigoplus_{i\geq 1} \mathscr{P}^{\text{¡}}(n) \otimes A^{\otimes n} \to B$ induced by the universal property of the coproduct. Similarly, we can decompose by weight instead of arity to produce a collection of degree $n-1$ linear maps $F^{(n)} : \mathscr{P}^{\text{¡}}(A)^{(n)} \to B$.

The $\mathscr{P}$-Eilenberg–Moore spectral sequence. Let A be an algebra over a Koszul operad $\mathscr{P}$ and $H = H_*(A)$ be its homology. There is a spectral sequence, which we call the $\mathscr{P}$*-Eilenberg–Moore spectral sequence*, that computes the Quillen homology of A as long as A is positively graded of

finite type (which is implicitly assumed whenever we speak of convergence). It is constructed as follows. The operadic chain complex $\mathscr{P}^{\mathrm{i}}(A)$ admits the ascending filtration

$$F_p\mathscr{P}^{\mathrm{i}}(A) = \bigoplus_{n=1}^{p} \mathscr{P}^{\mathrm{i}}(A)^{(n)}.$$

This filtration is bounded below and exhaustive. Therefore, the associated spectral sequence converges to the operadic homology of A as a graded module. The complex $\mathscr{P}^{\mathrm{i}}(A)$ also has the structure of a conilpotent cofree $\mathscr{P}^{\mathrm{i}}$-coalgebra with comultiplication Δ, which respects the filtration in the sense that

$$\Delta\left(F_p\mathscr{P}^{\mathrm{i}}(A)\right) \subseteq \bigoplus_{k=1}^{p} \bigoplus_{i_1+\cdots+i_k=p} \mathscr{P}^{\mathrm{i}}(k) \otimes \left(F_{i_1}\mathscr{P}^{\mathrm{i}}(A) \otimes \cdots \otimes F_{i_k}\mathscr{P}^{\mathrm{i}}(A)\right).$$

This further implies that each page of the spectral sequence inherits a $\mathscr{P}^{\mathrm{i}}$-coalgebra structure, and furthermore, the spectral sequence converges as a $\mathscr{P}^{\mathrm{i}}$-coalgebra. A morphism of $\mathscr{P}^{\mathrm{i}}$-coalgebras naturally induces a morphism of the corresponding spectral sequences. The E^0-page of this spectral sequence is explicitly given by

$$E^0_{p,q} = \left(\mathscr{P}^{\mathrm{i}}(A)^{(p)}\right)_{p+q} \cong \left(\bigoplus_{r\geq 1} \left(\mathscr{P}^{\mathrm{i}}\right)^{(p)}(r) \otimes_{S_r} A^{\otimes r}\right)_{p+q}$$

where the $p+q$ grading is induced from the internal grading of A. Under the isomorphism above, the differential d^0 is determined by the differential d of A, and there is an isomorphism of differential bigraded modules

$$\left(E^0, d^0\right) \cong \left(\mathscr{P}^{\mathrm{i}}(A), \delta^{(1)}\right),$$

where abusing the notation, $\delta^{(1)}$ stands for the coderivation of $\mathscr{P}^{\mathrm{i}}(A)$ induced by the weight 1 component of the codifferential δ. Taking homology of $\left(E^0, d^0\right)$, it follows that the E^1-page of the spectral sequence is

$$E^1_{p,q} = \left(\mathscr{P}^{\mathrm{i}}\left(H_*\left(A\right)\right)^{(p)}\right)_{p+q}$$

and the differential on this page is therefore entirely determined by the weight 2 component of the codifferential. In other words, we have that $d^1 = H_*\left(\delta^{(2)}\right)$. Taking homology again, we finally have

$$E^2_{p,q} = H_{p+q}\left(\mathscr{P}^{\mathrm{i}}(H)^{(p)}\right) \overset{p}{\Longrightarrow} H_*\left(\mathscr{P}^{\mathrm{i}}(A), \delta\right).$$

While this definition seems to be original to this paper for general operads, it has some very well-known special cases. When $\mathscr{P}$ is binary, that is, generated by operations of arity 2, the weight grading coincides with the arity grading up to a shift. So, for example, when $\mathscr{P} = \mathsf{Ass}$ is the associative operad, the $\mathscr{P}$-Eilenberg–Moore spectral sequence is exactly the classical Eilenberg–Moore spectral sequence [8]. When $\mathscr{P} = \mathsf{Lie}$ is the Lie operad, the $\mathscr{P}$-Eilenberg–Moore spectral sequence is exactly a classical Quillen spectral sequence that appears in [24, (6.9) p. 262].

Remarks 1.1.

1. If A is an algebra over a Koszul operad $\mathscr{P}$, there are several spectral sequences closely related to the one defined above. First, we can filter $\mathscr{P}^{\mathrm{i}}$ by weight. This gives the spectral sequence we studied above. Second, we can filter $\mathscr{P}^{\mathrm{i}}$ by arity. This produces a spectral sequence that coincides with the previous one up to a shift when the operad is binary generated, or more generally, when the generators of the operad are concentrated in a single arity. However, in general, these two spectral sequences differ. Third, one can replace $\mathscr{P}^{\mathrm{i}}$ with the bar construction $B\mathscr{P}$ and filter similarly. Since not every operad is Koszul, this spectral sequence will be useful in those situations.
2. If A is a $\mathscr{P}_\infty$-algebra, then the construction of the spectral sequence above goes through with straightforward adjustments.

A version of the homotopy transfer theorem. In [23, Theorem 2], D. Petersen gave what probably is the most general form of T. Kadeishvili's version of the classical homotopy transfer theorem [11] for algebras over binary algebraic operads. Adapted to our needs, it reads as follows. In the statement, $\mathscr{P}$ is a reduced Koszul operad.

Theorem 1.2. *Let (A,d) be a $\mathscr{P}$-algebra, H its homology, and $f : H \to A$ a cycle-choosing (and therefore necessarily degree 0) linear map. Let δ_A be the degree -1 square-zero coderivation of $\mathscr{P}^{\text{¡}}(A)$ representing the $\mathscr{P}$-algebra structure on A whose arity 1 term equals the given differential d. Then there exists noncanonically a square-zero degree -1 coderivation δ of $\mathscr{P}^{\text{¡}}(H)$ whose arity 1 term vanishes, and a morphism of $\mathscr{P}^{\text{¡}}$-coalgebras $F : \mathscr{P}^{\text{¡}}(H) \to \mathscr{P}^{\text{¡}}(A)$ whose linear term F_1 is f and which is a chain map with respect to the differentials defined by δ_A and δ.*

Sketch of the proof. The homology H is equipped with the structure of a $\mathscr{P}$-algebra descending from the $\mathscr{P}$-algebra structure on A. This induces a degree -1 coderivation $\delta^1 : \mathscr{P}^{\text{¡}}(H) \to \mathscr{P}^{\text{¡}}(H)$ whose arity 1 component δ^1_1 is identically 0. Now, by induction, assume that for some $n \geq 2$, we have a degree -1 coderivation $\delta^{n-1} : \mathscr{P}^{\text{¡}}(H) \to \mathscr{P}^{\text{¡}}(H)$ and a $\mathscr{P}^{\text{¡}}$-coalgebra morphism $F^{n-1} : \mathscr{P}^{\text{¡}}(H) \to \mathscr{P}^{\text{¡}}(A)$ with $F_1 = f$, such that the restrictions of δ^{n-1} and F^{n-1} to $F_{n-1}\mathscr{P}^{\text{¡}}(H)$ satisfy

$$\begin{cases} \delta^{n-1} \circ \delta^{n-1} = 0 \\ F^{n-1} \circ \delta^{n-1} - \delta_A \circ F^{n-1} = 0. \end{cases}$$

Above, $\circ$ denotes the usual composition of maps, not the operadic circle product. Write F^1 for the coalgebra map determined by f in arity 1 and vanishing in higher arities. Then δ^1 and F^1 satisfy the identities above, providing the base case in the induction. The idea now is to modify only the arity n terms of δ^{n-1} and F^{n-1} to produce new δ^n and F^n such that the equations above are satisfied on $F_n\mathscr{P}^{\text{¡}}(A)$. One can show that there are e and e' such that

$$\left(F^{n-1} \circ \delta^{n-1} - \delta_A \circ F^{n-1}\right)_n = f \circ e + d e'$$

where $e \in \mathrm{Hom}(\mathscr{P}^{\text{¡}}(n) \otimes H^{\otimes n}, H)$ and $e' \in \mathrm{Hom}(\mathscr{P}^{\text{¡}}(n) \otimes H^{\otimes n}, A)$. Therefore, we can define

$$\delta^n_i = \begin{cases} \delta^{n-1}_i & \text{for } i \neq n. \\ \delta^{n-1}_n - e & \text{for } i = n. \end{cases}$$

In fact, e may be computed as the projection of $\left(F^{n-1} \circ \delta^{n-1} - \delta_A \circ F^{n-1}\right)_n$ onto H. Similarly, we can define F^n_n to be

$$F^n_i = \begin{cases} F^{n-1}_i & \text{if } i \neq n. \\ F^n_n & \text{for any } F^n_n \text{ such that } d F^n_n = F^{n-1}_n - e' \text{ when } i = n. \end{cases}$$

So defined, the coderivation δ^n and the coalgebra map F^n satisfy the required conditions, and the proof is complete. □

1.1.2 Higher-order Massey products for associative algebras

The triple Massey product for differential graded associative algebras was introduced in the fifties, see [29] and [18] (reprinted as [19]). Massey himself soon realized that the triple product could be extended to n-fold Massey products [17], see also [21]. Our generalization of the higher-order Massey products to algebras over algebraic operads has its roots in this definition. Therefore, we find it convenient to devote this section to recall the higher-order Massey products for differential graded associative algebras. Excellent references for this topic include [12, 21, 25].

Let (A,d) be a differential graded associative algebra, and $x_1, x_2 \in H_*(A)$ homogeneous elements. The Massey product $\langle x_1, x_2 \rangle$ is defined as the singleton $\{x_1 x_2\}$ formed by the product of the two classes in $H_*(A)$. It is also possible to identify the set $\{x_1 x_2\}$ with the product $x_1 x_2$ itself and define the Massey product of two homogeneous elements in homology as their ordinary product. Let us define next the triple and higher-order Massey products. First, we introduce the auxiliary notion of a defining system. A defining system in the case of the Massey product of two homology classes $\langle x_1, x_2 \rangle$ is just a choice $\{b_1, b_2\}$ of cycle representatives of x_1 and x_2.

Definition 1.3. Let (A,d) be a differential graded associative algebra, and $x_1,\dots,x_n$ be $n\geq 3$ homogeneous elements in $H_*(A)$. A *defining system for the n^{th}-order Massey product of the classes $x_1,\dots,x_n$* is a set of homogeneous elements

$$\{b_{ij}\}\subseteq A,\quad \text{for}\quad 0\leq i<j\leq n\quad \text{and}\quad 1\leq j-i\leq n-1,$$

defined as follows.

- (Initial step) For $i=1,\dots,n$ the element $b_{i-1,i}$ is a cycle representative of x_i.
- (Inductive relation) For each $0\leq i<j\leq n$ and $1\leq j-i\leq n-1$, the element $b_{ij}\in A$ satisfies

$$d\left(b_{ij}\right)=\sum_{0\leq i<k<j\leq n}(-1)^{|b_{ik}|+1}b_{ik}b_{kj}. \tag{2}$$

The *n^{th}-order Massey product of the classes $x_1,\dots,x_n$* is the set

$$\langle x_1,\dots,x_n\rangle=\left\{\left[\sum_{0\leq i<k<j\leq n}(-1)^{|b_{ik}|+1}b_{ik}b_{kn}\right]\;\middle|\;\{b_{ij}\}\text{ is a defining system}\right\}\subseteq H_{s+2+n}(A),$$

where $s=\sum_{i=1}^{n}|x_i|$, and the bracket $[-]$ denotes taking homology class.

The elements b_{ij} of Equation (2) might not exist at all, in which case the Massey product set is empty. The necessary and sufficient condition for $\langle x_1,\dots,x_n\rangle$ to be non-empty is that for all $1\leq i<j\leq n$ and $1\leq j-i\leq n-2$, the Massey product sets $\langle x_i,\dots,x_j\rangle$ are non-empty and furthermore contain the zero class in a coherent manner.

The fact that for a fixed defining system the sum

$$\sum_{0\leq i<k<j\leq n}(-1)^{|b_{ik}|+1}b_{ik}b_{kn}$$

defines a cycle is a straightforward check by applying d and using the inductive relations. If there are no defining systems for the classes $x_1,\dots,x_n$, their Massey product $\langle x_1,\dots,x_n\rangle$ is defined as the empty set, or it is said to be undefined.

A similar definition for higher Lie–Massey brackets on the homology of a differential graded Lie algebra exists, see [1, 2, 26, 4, 28]. The main purpose of this paper is to provide a suitable generalization of Definition 1.3 to algebras over Koszul operads, see Section 2.

1.1.3 Secondary Massey products for algebras over algebraic operads

In this section, we briefly outline Muro's definition of secondary Massey products for algebras over algebraic operads. Our eventual definition of Massey products for algebras over operads, Def. 2.7, is shown to extend the one below in Proposition 2.9.

Definition 1.4. ([22, Def. 2.1]) Let $\mathscr{P}=\mathscr{F}(E,R)$ be a Koszul operad generated by the reduced symmetric sequence E with quadratic relations $R\subseteq\mathscr{F}(E)^{(2)}$. Fix

$$\Gamma=\sum\left(\mu^{(1)}\circ_k\mu^{(2)}\right)\cdot\sigma$$

a relation of arity r of R. Here, $\mu^{(i)}\in E(r_i)$, with $r_1+r_2=r+1$, the symbol $\circ_k$ denotes the k-th partial composition product, $1\leq k\leq r_1$, and $\sigma\in\mathbb{S}_r$. Let A be a $\mathscr{P}$-algebra and let $x_1,\dots,x_r\in H_*(A)$ be homogeneous elements such that

$$\mu^{(2)}\left(x_{\sigma^{-1}(k)},\dots,x_{\sigma^{-1}(k+r_2-1)}\right)=0 \tag{3}$$

in $H_*(A)$ for each term in the relation. For each $1\leq i\leq r$, fix $y_i\in A$ a cycle representative of x_i and, for each summand in the relation, let $\rho^{(2)}\in A$ be an element such that

$$d\rho^{(2)}=\mu^{(2)}\left(y_{\sigma^{-1}(k)},\dots,y_{\sigma^{-1}(k+r_2-1)}\right) \tag{4}$$

in A. Such an element exists by Equation (3). The Γ-*Massey product* set $\langle x_1,\dots,x_r\rangle_\Gamma$ is the set of homology classes represented by cycles of the form

$$\sum(-1)^\gamma \mu^{(1)}\left(y_{\sigma^{-1}(1)},\dots,y_{\sigma^{-1}(k-1)},\rho^{(2)},y_{\sigma^{-1}(k+r_2)},\dots,y_{\sigma^{-1}(r)}\right),$$

where

$$\gamma=\alpha+|\mu^{(1)}|+\left(|\mu^{(2)}|-1\right)\sum_{m=1}^{k-1}|x_{\sigma^{-1}(m)}|,\qquad \alpha=\sum_{\substack{i<j\\ \sigma(i)>\sigma(j)}}|x_i||x_j|.$$

for all possible coherent choices of elements $\rho^{(2)}$.

Muro shows that the definition above recovers the usual triple Massey products for differential graded associative algebras when Γ is the associativity relation of the associative operad, and the triple Lie–Massey brackets for differential graded Lie algebras when Γ is the Jacobi relation of the Lie operad.

The perspective we take to construct higher-order Massey products for algebras over algebraic operads differs significantly from the construction of Muro just explained. Muro uses the form of relations defined using partial composition. The definition does not depend exclusively on the relation Γ, but also on a specific choice of expansion of Γ in terms of the partial compositions. This choice is not unique. Our approach is also affected by a choice in the explicit form of the higher relations. To generalize, we prefer to see such relations as the weight 2 cooperations in the Koszul dual cooperad of $\mathscr{P}$, and work with defining systems in a similar way as in Definition 1.3. This makes our formulas easier to write in the usual language of algebraic operads and Koszul duality theory. To take into account the dependency of the higher relations on a presentation, we will assume all through that a $\Bbbk$-linear basis of the symmetric sequence E has been fixed, and then there is an induced basis on $\mathscr{F}^c(sE)$ given by symmetric tree monomials. This will be recalled in the corresponding section. We show in Proposition 2.9 that the secondary case of our definition coincides with Muro's definition.

2 Higher-order operadic Massey products

In this section, we define higher-order Massey products for algebras over algebraic operads. We focus on the case of Koszul operads and explain in Remark 2.16 how to deal with the non-Koszul case. We recommend familiarity with the classical higher-order Massey products for differential graded associative algebras recalled in Section 1.1.2.

Let $\mathscr{P}=\mathscr{F}(E,R)$ be a Koszul operad with Koszul dual cooperad $\mathscr{P}^{\text{¡}}=\mathscr{F}^c(sE,s^2R)$. We will assume all through the paper that a $\Bbbk$-linear basis of E has been fixed. Then, there are induced bases on $\mathscr{F}(E)$ and on $\mathscr{F}^c(sE)$ given by appropriate symmetric tree monomials, see [7, Section 2.4]. These bases will also be fixed once and for all. Since $\mathscr{P}^{\text{¡}}\subseteq\mathscr{F}^c(sE)$, we will use this basis to linearly expand the elements of $\mathscr{P}^{\text{¡}}$ in our results.

As mentioned in the introduction, each weight-homogeneous cooperation Γ^c of $\mathscr{P}^{\text{¡}}$ creates a partially defined higher-order operation $\langle -,\dots,-\rangle_{\Gamma^c}$ on the homology of any $\mathscr{P}$-algebra, with as many inputs as the arity r of Γ^c. Out of homogeneous elements $x_1,\dots,x_r\in H_*(A)$ on the homology of a $\mathscr{P}$-algebra A, this operation creates a (possibly empty) set of homology classes

$$\langle x_1,\dots,x_r\rangle_{\Gamma^c}\subseteq H_*(A).$$

The non-emptiness depends on the vanishing, in a precise sense, of strictly lower-order operations of the same kind that depend on Γ^c. The set $\langle x_1,\dots,x_r\rangle_{\Gamma^c}$ is called the Γ^c-*Massey product* of the classes $x_1,\dots,x_r$.

To construct the Γ^c-Massey product operation $\langle -,\dots,-\rangle_{\Gamma^c}$, we proceed as follows. First, the cooperation Γ^c determines a set of indices $I(\Gamma^c)$ which is then used to form *defining systems*. A defining system for the concrete Γ^c-Massey product set $\langle x_1,\dots,x_r\rangle_{\Gamma^c}$ is a coherent choice of elements $\{a_\alpha\}$ of A indexed by $I(\Gamma^c)$ that are combined to create a cycle. The homology classes contained in $\langle x_1,\dots,x_r\rangle_{\Gamma^c}$ are obtained by running over all possible choices of defining systems for $x_1,\dots,x_r$ and taking the homology class of the associated cycle.

The section is organized as follows. First, we introduce the *Massey inductive map*. This map depends on the coproduct of $\mathscr{P}^{\mathrm{i}}$ and a fixed twisting morphism $\kappa : \mathscr{P}^{\mathrm{i}} \to \mathscr{P}$. It is an essential ingredient when dealing with the inductive definitions that follow. Then, we define the indexing set $I(\Gamma^c)$ associated to an arbitrary cooperation Γ^c and compute some examples. Once the concept of indexing sets is established, we proceed to explain what a defining system is and give examples of them. Then, we define the higher-order Γ^c-Massey products, and compute examples including the associative, commutative, Lie, Poisson, and dual numbers operads. Later on, we show that our higher-order Massey products framework includes Muro's [22] (Prop. 2.9). We study the elementary properties of these higher-order products in Section 2.1. These include the behavior along morphisms of $\mathscr{P}$-algebras, quasi-isomorphisms, and some connections to formality. Some further properties are explored in Section 2.2. There, we focus on the behavior of the higher-order Massey products along morphisms of operads and give some applications to formality.

Recall that the decomposition map $\Delta : \mathscr{C} \to \mathscr{C} \circ \mathscr{C}$ of any counital cooperad $\mathscr{C}$ can be uniquely written as

$$\Delta(c) = \Delta^+(c) + (\mathrm{id}; c)$$

for every arity-homogeneous $c \in \mathscr{C}$. Here, $\mathrm{id} \in \mathscr{C}(1)$ is the element that corresponds to the identity element 1 of the ground field $\Bbbk$ under the linear isomorphism $\mathscr{C}(1) \to \Bbbk$ induced by the counit. We call Δ^+ the half-reduced decomposition map of $\mathscr{C}$.

Definition 2.1. The *Massey inductive map* is the degree -1 map

$$D : \mathscr{F}^c(sE) \xrightarrow{\Delta^+} \mathscr{F}^c(sE) \circ \mathscr{F}^c(sE) \xrightarrow{\kappa \circ \mathrm{id}} E \circ \mathscr{F}^c(sE).$$

Applied to some cooperation μ, we shall write

$$D(\mu) = \sum (\zeta; \zeta_1, \ldots, \zeta_m; \sigma), \tag{5}$$

where $\zeta \in E(m)$, $\zeta_i \in \mathscr{F}^c(sE)(v_i)$, $\sigma \in \mathbb{S}_m$ and $v_1 + \cdots + v_m$ is equal to the arity of μ.

The sum in Equation (5) is indexed over all ζ along the chosen basis of E, and each term may have a $\Bbbk$-coefficient (possibly 0). The map D is inductive in the sense that, for any cooperation μ, the cooperations $\zeta_1, \ldots, \zeta_m$ appearing on the terms of $D(\mu)$ will each always have weight strictly less than that of μ. This will allow us to establish the inductive relations of our defining systems later on. If $\mathscr{P}$ is a Koszul operad, then the fact that $\mathscr{P}^{\mathrm{i}}$ is a subcooperad of $\mathscr{F}^c(sE)$ allows us to restrict the Massey inductive map to a map

$$D : \mathscr{P}^{\mathrm{i}} \xrightarrow{\Delta^+} \mathscr{P}^{\mathrm{i}} \circ \mathscr{P}^{\mathrm{i}} \xrightarrow{\kappa \circ \mathrm{id}} E \circ \mathscr{P}^{\mathrm{i}}.$$

Abusing the notation, we call this restriction the Massey inductive map too, and use the same symbols to denote the maps that constitute it.

As mentioned before, the cofree conilpotent cooperad $\mathscr{F}^c(sE)$ has a fixed combinatorial description in terms of rooted tree monomials whose internal vertices are labeled by elements of sE. Each such tree monomial has a *first vertex*, which is the unique child of the root and corresponds to the first generating cooperation to be applied. The action of D is determined by sending any tree monomial T to $\left(s^{-1}x; T_1, \ldots T_m\right)$, where $x \in (sE)(m)$ is the label of the first vertex of T, and $T_1, \ldots, T_m$ are the tree monomials attached to this first vertex of T. Intuitively, the Massey inductive map is trimming level 1 edges. See figures 1 and 2.

Next, we introduce the set associated with a cooperation of $\mathscr{P}^{\mathrm{i}}$ that will provide the indices for our defining systems. It is defined by induction on the weight of arity-homogeneous cooperations of $\mathscr{P}^{\mathrm{i}}$, with the Massey inductive map providing the necessary inductive step.

Definition 2.2. Let $\Gamma^c \in \mathscr{P}^{\mathrm{i}}(r)$ be a weight-homogeneous cooperation. For each permutation $(k_1, \ldots, k_r) \in \mathbb{S}_r$, we define the Γ^c*-indexing set* $I(\Gamma^c, (k_1, \ldots, k_r))$ by induction on the weight $w(\Gamma^c)$ of Γ^c as follows.

- If $w(\Gamma^c) = 0$, then $I(\Gamma^c) = \emptyset$.
- If $w(\Gamma^c) = 1$, then $I(\Gamma^c, (1)) = \{(\mathrm{id}, (1)), \ldots, (\mathrm{id}, (r))\}$.

Assume next that $I(\Gamma^c,(k_1,...,k_r))$ has been defined for cooperations up to weight n, and suppose Γ^c is of weight $n+1$. If

$$D\left(\Gamma^c\right)=\sum(\zeta;\zeta_1,\ldots,\zeta_m;\sigma)$$

as in Equation (5), and the leaves on top of each ζ_i are labeled $l_1,...,l_{v_i}$, then

$$I\left(\Gamma^c,(k_1,...,k_r)\right):=\bigcup_{i=1}^{m} I\left(\zeta_i,\left(k_{l_1},...,k_{l_{v_i}}\right)\right)\cup\left\{\left(\zeta_i,\left(k_{l_1},...,k_{l_{v_i}}\right)\right)\right\}.$$

The super index c in Γ^c indicates that we are seeing the corresponding element in the Koszul dual *cooperad* of $\mathscr{P}$. At a later place, we will see this same element as a relation Γ in the free operad $\mathscr{F}(E)$. Since we will need to distinguish between these two elements, we keep the super index in the notation.

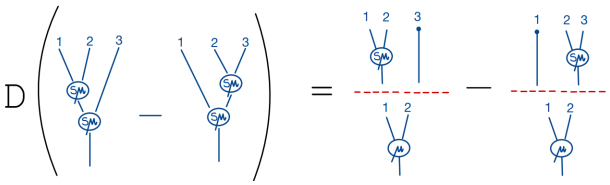

Figure 1: The Massey inductive map for Ass

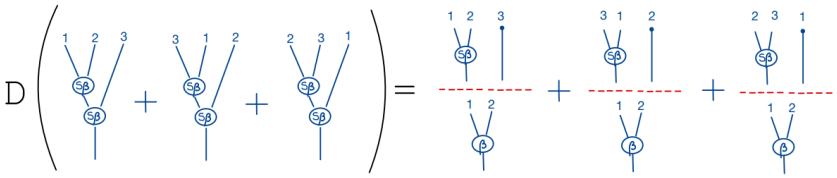

Figure 2: The Massey inductive map for Lie

The following elementary observation will be the base case of the inductive definition of defining systems below. We record this fact before giving some explicit examples.

Remark 2.3. If $\mathscr{P}$ is any Koszul operad and $\Gamma^c\in\left(\mathscr{P}^{\text{¡}}\right)^{(1)}(r)=(sE)(r)$ is any cogenerator of arity r, then the Γ^c-indexing set is always given by

$$I\left(\Gamma^c\right)=\{(\mathrm{id},(1)),...,(\mathrm{id},(r))\}.$$

Let us illustrate the definition of indexing sets with some examples.

Example 2.4. Let $\mathscr{P}=\mathrm{Ass}$. Then the weight n component of $\mathscr{P}^{\text{¡}}$ is freely generated as an $\mathbb{S}_{n+1}$-module by a single generator $\mu^c_{n+1}\in\mathrm{Ass}^{\text{¡}}(n+1)$. Recall that

$$\Delta\left(\mu^c_n\right)=\sum_{i_1+\cdots+i_k=n}(-1)^{\sum(i_j+1)(k-j)}\left(\mu^c_k;\mu^c_{i_1},\ldots\mu^c_{i_k};\mathrm{id}\right).$$

Here, we denote $\mu^c_1=\mathrm{id}\in\mathrm{Ass}^{\text{¡}}(1)$. Since $\kappa\left(\mu^c_2\right)=\mu_2$ and $\kappa\left(\mu^c_k\right)=0$ for $k\geq 3$, this implies that

$$D\left(\mu^c_n\right)=\sum_{i_1+i_2=n}(-1)^{i_1+1}\left(\mu_2;\mu^c_{i_1},\mu^c_{i_2};\mathrm{id}\right).$$

This means that the defining system $I\left(\mu^c_n\right)$ contains the elements

$$\left(\mu^c_{i_1},(1,2,\ldots,i_1)\right)\quad\text{and}\quad\left(\mu^c_{i_2},(n-i_2,n-i_2+1,\ldots,n)\right),$$

where $i_1+i_2=n$. By iterating this process, we see that

$$I\left(\mu^c_n\right)=\left\{\left(\mu^c_k,(i,i+1,\ldots,i+k-1)\right)\mid k<n\text{ and }i\in\{1,2,\ldots,n-k-1\}\right\}.$$

□

Example 2.5. Let $\mathscr{P} = \mathsf{Lie}$. Then the weight n part of $\mathscr{P}^!$ is one-dimensional and generated by $\tau^c_{n+1} \in \mathsf{Lie}^!(n+1)$. Recall that

$$\Delta\left(\tau^c_n\right) = \sum_{\substack{i_1+\cdots+i_k=n \\ \sigma\in\overline{Sh}^{-1}(i_1,\ldots,i_k)}} (-1)^{\sum(i_j+1)(k-j)} \operatorname{sgn}(\sigma)\left(\tau^c_k;\tau^c_{i_1},\ldots\tau^c_{i_k};\sigma\right),$$

where $\overline{Sh}^{-1}(i_1,\ldots,i_k)$ is the set of reduced unshuffles. Here, an unshuffle is the inverse of a shuffle, and *reduced* signifies that we are considering only those shuffles that fix the position of the first element, i.e. $\sigma(1) = 1$. Since $\kappa(\tau^c_2) = \tau_2$ and $\kappa(\tau^c_k) = 0$ for $k \geq 3$, this implies that

$$D\left(\tau^c_n\right) = \sum_{\substack{i+j=n \\ \sigma\in\overline{Sh}^{-1}(i,j)}} (-1)^{i+1} \operatorname{sgn}(\sigma)\left(\tau_2;\tau^c_i,\tau^c_j;\sigma\right).$$

This means that the defining system $I\left(\tau^c_n\right)$ contains the elements

$$\left(\tau^c_i,(\sigma(1),\sigma(2),\ldots,\sigma(i))\right) \quad \text{and} \quad \left(\tau^c_j,\left(\sigma(n-j),\sigma(n-j+1),\ldots,\sigma(n)\right)\right)$$

for each reduced shuffle $\sigma \in \overline{Sh}(i,j)$ with $i+j=n$. In this step, we changed from using unshuffles to shuffles, because there is an inversion involved. By iterating this process, we find that

$$I\left(\tau^c_n\right) = \left\{\left(\tau^c_k,(i_1,\ldots i_k)\right) \mid k < n \text{ and } 1 \leq i_1 \leq \cdots \leq i_l \leq n\right\}.$$

□

As mentioned before, each cooperation Γ^c of weight n in the Koszul dual cooperad $\mathscr{P}^!$ of $\mathscr{P}$ produces a partially defined n-th order operation $\langle -,\ldots,-\rangle_{\Gamma^c}$ on the homology $H_*(A)$ of a $\mathscr{P}$-algebra A. This higher operation has r inputs, where r is the arity of Γ^c, and the output is the set of homology classes created from all possible choices of *defining systems*, generalizing the case of associative algebras of Section 1.1.2. Our next task is to explain what the defining systems are. Each defining system will depend on a weight-homogeneous cooperation Γ^c of arity r and r homogeneous homology classes $x_1,\ldots,x_r \in H_*(A)$. Their definition is given by induction on the weight of the cooperation.

Definition 2.6. Let $\Gamma^c \in \left(\mathscr{P}^!\right)^{(n)}(r)$ for some $n \geq 1$, A a $\mathscr{P}$-algebra, and $x_1,\ldots,x_r \in H_*(A)$ homogeneous elements. A Γ^c*-defining system (associated to* $x_1,\ldots,x_r$*)* is a collection $\{a_\alpha\}_{\alpha\in I(\Gamma^c)}$ of elements of A indexed by $I(\Gamma^c)$ such that:

1. Each $a_{(\mathrm{id},(i))} \in A$ is a cycle representative of x_i.
2. For each index $\left(\mu,(k_1,\cdots,k_i)\right) \in I(\Gamma^c)$ with $\mu \neq \mathrm{id}$, the corresponding element $a_{(\mu,(k_1,\cdots,k_i))}$ is such that

$$d\left(a_{\mu,(k_1,\cdots,k_i)}\right) = \sum \zeta\left(a_{\left(\zeta_1,\left(k_{\sigma^{-1}(1)},\ldots,k_{\sigma^{-1}(\nu_1)}\right)\right)},\ldots,a_{\left(\zeta_m,\left(k_{\sigma^{-1}(\nu_1+\cdots+\nu_{m-1}+1)},\ldots,k_{\sigma^{-1}(i)}\right)\right)}\right),$$

   where $D\left(\mu\right) = \sum(\zeta;\zeta_1,\ldots,\zeta_m;\sigma)$.

Next, we use the defining systems explained above to define the Γ^c-Massey products. If the cooperation Γ^c is of weight 1 and arity r, that is, a cogenerator, then $\Gamma = \kappa(\Gamma^c)$ is a generator of $\mathscr{P}$. For any homogeneous elements $x_1,\ldots,x_r \in H_*(A)$, we define their Γ^c*-Massey product* as the set

$$\langle x_1,\ldots,x_r\rangle_{\Gamma^c} := \{\Gamma(x_1,\ldots,x_r)\}.$$

We may also identify this set with its unique element $\Gamma(x_1,\ldots,x_r) \in H_*(A)$. Let us define the Γ^c-Massey products for elements of weight ≥ 2.

Definition 2.7. Let A be a $\mathscr{P}$-algebra, $\Gamma^c \in \left(\mathscr{P}^!\right)^{(n)}(r)$ with $n \geq 2$, and $x_1,\ldots,x_r$ homogeneous elements of $H_*(A)$. Then:

1. The Γ^c-*Massey product* associated to a Γ^c-defining system $\{a_\alpha\}$ and $x_1,...,x_r$ is the homology class of the cycle

$$a_{\Gamma^c,(1,...,r)} := \sum (-1)^\gamma \zeta \left(a_{\zeta_1,(\sigma^{-1}(1),\sigma^{-1}(2),...,\sigma^{-1}(\nu_1))}, \dots, a_{\zeta_m,(\sigma^{-1}(\nu_1+\cdots+\nu_{m-1}+1),...,\sigma^{-1}(r))} \right), \tag{6}$$

where $D(\Gamma^c) = \sum(\zeta;\zeta_1,\dots,\zeta_m;\sigma)$, and the sign is given by

$$\gamma = \alpha + \sum_{i=2}^{m} \left(|\zeta_i| - \mathrm{wgt}(\zeta_i)\right) \left(\sum_{k=1}^{\nu_1+\cdots+\nu_{i-1}} |x_{\sigma^{-1}(k)}| \right) + 1, \qquad \alpha = \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} |x_i||x_j|,$$

where $\mathrm{wgt}(\zeta_i)$ is the weight of ζ_i.

2. The Γ^c-*Massey product set* $\langle x_1,\dots,x_r\rangle_{\Gamma^c}$ is the (possibly empty) subset of $H_*(A)$ formed by the homology classes arising from all possible choices of Γ^c-defining systems $\{a_\alpha\}$ associated to $x_1,...,x_r$.

The next result shows that the proposed definition is correct. As a consequence of it, we readily see from the definition of defining systems that the Γ^c-Massey product set $\langle x_1,...,x_r\rangle_{\Gamma^c}$ is non-empty if, and only if, for all $(\mu;k_1,\dots k_i) \in I(\Gamma^c)$, the Massey product set $\langle x_{k_1},\dots,x_{k_i}\rangle_\mu$ is defined and contains the zero class in a coherent manner.

Proposition 2.8. *Let A be a $\mathscr{P}$-algebra, $\Gamma^c \in \left(\mathscr{P}^!\right)^{(n)}(r)$ for some $n \geq 2$, and $x_1,...,x_r$ homogeneous elements of $H_*(A)$. Then the Γ^c-Massey product x associated to any Γ^c-defining system for $x_1,...,x_r$ is a cycle.*

Proof. Let $\{a_\alpha\}$ be a defining system, and denote by x the associated cycle given by formula (6),

$$x = \sum (-1)^\gamma \zeta \left(a_{\zeta_1,(\sigma^{-1}(1),\sigma^{-1}(2),...,\sigma^{-1}(\nu_1))}, \dots, a_{\zeta_m,(\sigma^{-1}(\nu_1+\cdots+\nu_{m-1}+1),...,\sigma^{-1}(r))} \right).$$

Let us compute dx in terms of the Massey inductive map D and terms of the form $a_{\mu,(k_1,...k_i)}$. Recall that the differential of A fits into the commutative diagram

$$\begin{array}{ccc} \mathscr{P}\circ A & \xrightarrow{\gamma_A} & A \\ {\scriptstyle \mathrm{id}\circ' d}\downarrow & & \downarrow{\scriptstyle d} \\ \mathscr{P}\circ A & \xrightarrow{\gamma_A} & A \end{array}$$

where $\circ'$ is the infinitesimal composite. From here, it follows that

$$dx = d\left(\sum \zeta \left(a_{\zeta_1,(\sigma^{-1}(1),\sigma^{-1}(2),...,\sigma^{-1}(\nu_1))}, \dots, a_{\zeta_m,(\sigma^{-1}(\nu_1+\cdots+\nu_{m-1}+1),...,\sigma^{-1}(r))} \right) \right)$$

is equal to

$$\sum \sum_{i=1}^{m} (-1)^{\epsilon_i} \zeta \left(a_{\zeta_1,(\sigma^{-1}(1),\sigma^{-1}(2),...,\sigma^{-1}(\nu_1))}, \dots, d\left(a_{\zeta_i,(\sigma^{-1}(\nu_1+\cdots+\nu_{i-1}+1),...,\sigma^{-1}(\nu_1+\cdots+\nu_i))} \right), \dots a_{\zeta_m,(\sigma^{-1}(\nu_1+\cdots+\nu_{m-1}+1),...,\sigma^{-1}(r))} \right),$$

where

$$\epsilon_i = |\zeta| + |a_{\zeta_1,(\sigma^{-1}(1),\sigma^{-1}(2),...,\sigma^{-1}(\nu_1))}| + \cdots + |a_{\zeta_{i-1},(\sigma^{-1}(\nu_1+\cdots+\nu_{i-2}+1),...,\sigma^{-1}(\nu_1+\cdots+\nu_{i-1}))}|.$$

Each term $d\left(a_{\zeta_i,(\sigma^{-1}(\nu_1+\cdots+\nu_{i-1}+1)),...,\sigma^{-1}(\nu_1+\cdots+\nu_i))}\right)$ appearing in the sum above can be rewritten in terms of $a_{\mu,(k_1,...k_i)}$ of lower order, by using the inductive relation of the defining system (Def 2.6, item 2). In particular, if we consider the composite

$$G : \mathscr{P}^! \xrightarrow{\Delta^+} \mathscr{P}^! \circ \mathscr{P}^! \xrightarrow{\kappa\circ\mathrm{id}} \mathscr{P}\circ\mathscr{P}^! \xrightarrow{f} \mathscr{P}\circ\left(\mathscr{P}^!;\mathscr{P}^!\right) \xrightarrow{\mathrm{id}\circ(\mathrm{id};\Delta^+)} \mathscr{P}\circ(\mathscr{P}^!;\mathscr{P}^!\circ\mathscr{P}^!) \xrightarrow{\mathrm{id}\circ(\mathrm{id};\kappa\circ\mathrm{id})}$$
$$\mathscr{P}\circ\left(\mathscr{P}^!;\mathscr{P}\circ\mathscr{P}^!\right) \xrightarrow{p} \mathscr{P}\circ\left(\mathscr{P}^!;\mathscr{P}^!\right) \xrightarrow{q} \mathscr{P}\circ\mathscr{P}^!,$$

where f is the natural inclusion, p is induced by the partial composition in $\mathscr{P}$, and q is the forgetful map, then the element dx is given by

$$\sum \xi\left(a_{\xi_1,(\sigma^{-1}(1),\dots,\sigma^{-1}(\nu_1))},\dots,a_{\xi_m,(\sigma^{-1}(\nu_1+\dots+\nu_{m-1}+1),\dots,\sigma^{-1}(r))}\right),$$

where $G(\Gamma^c)=\sum(\xi;\xi_1,\dots,\xi_m;\sigma)$. So to prove the result, it suffices to show that G is identically 0. We shall do this by showing that $\operatorname{Im} G \subseteq R\circ\mathscr{P}^{\mathsf{i}}$, where R is the sub-module of relations in the fixed presentation $\mathscr{P}=\mathscr{F}(E,R)$. Recall that $\mathscr{P}^{\mathsf{i}}$ can be thought of as a subset of the tree module and all the maps defining G descend from maps on or between the free operad on E and the cofree conilpotent cooperad on sE. It follows that we may describe G combinatorially by giving its action on individual basis tree monomials T of $\mathscr{F}^c(sE)$. This action is as follows.

1. First, apply the Massey inductive map D. This is sending the tree monomial T to a sum of tree monomials of the form $\left(s^{-1}e;T_1,\dots T_m\right)$, where $e\in(sE)(m)$ is the label of the first vertex and $T_1,\dots,T_m$ are its children.
2. Now, repeat this procedure on each T_i individually, thereby obtaining sums of tree monomials of the form $\left(s^{-1}e_i;T_{i,1},\dots T_{i,m_i}\right)$, and take for each individual tree monomial T_i the sum over the results to obtain

$$(-1)^{\epsilon_i}\sum_{i=1}^{m}\left(s^{-1}e;T_1,\dots,\left(s^{-1}e_i;T_{i,1},\dots T_{i,m_i}\right),\dots T_m\right).$$

   Here, each e_i is the first vertex of the corresponding T_i.
3. The final step is to apply the partial composition in the free operad $\mathscr{F}(E)$ in order to obtain

$$\sum_{i=1}^{m}\left(s^{-1}e\circ_i s^{-1}e_i;T_1,\dots,T_{i-1},T_{i,1},\dots T_{i,m_i},T_{i+1},\dots T_m\right).$$

   This time, without the suspension.

From this description, it follows that there is another equivalent way to describe G:

- First, directly apply the cooperadic reduced decomposition map Δ^+ to T to obtain

$$\Delta^+(T)=\sum(S;S_1,\dots S_k).$$

- Then, project the first component of $\mathscr{F}^c(sE)\circ\mathscr{F}^c(sE)$ into weight 2. That is, map S to itself if it is in weight 2, and map it to 0 otherwise. This produces

$$\sum_{i=1}^{m}\left(e\circ_i e_i;T_1,\dots,T_{i-1},T_{i,1},\dots T_{i,m_i},T_{i+1},\dots T_m\right).$$

- Desuspend the tree monomial $e\circ_i e_i$ twice.

From this description, it follows that

$$\sum_{i=1}^{m}\left(e\circ_i e_i;T_1,\dots,T_{i-1},T_{i,1},\dots T_{i,m_i},T_{i+1},\dots T_m\right)\in\mathscr{P}^{\mathsf{i}(2)}\circ\mathscr{P}^{\mathsf{i}},$$

and thus that

$$\sum_{i=1}^{m}\left(s^{-1}e\circ_i s^{-1}e_i;T_1,\dots,T_{i-1},T_{i,1},\dots T_{i,m_i},T_{i+1},\dots T_m\right)\in R\circ\mathscr{P}^{\mathsf{i}}.$$

This is exactly what we wanted to prove. □

In the next result, we show that our higher-order Massey products recover the secondary Massey products defined by Muro in [22] when restricting to cooperations of weight 2, up to a sign. The construction of Muro is recalled in Section 1.1.3, and we stick to the notation used there.

Proposition 2.9. *Let $\mathscr{P}$ be a Koszul operad with fixed presentation $\mathscr{F}(E,R)$. Let*

$$\Gamma = \sum \left(\mu^{(1)} \circ_l \mu^{(2)}\right) \cdot \sigma \in R(r)$$

be a quadratic relation of arity r, and denote the corresponding weight 2 element of the Koszul dual cooperad $\mathscr{P}^{\text{¡}}$ by Γ^c, so that

$$\Gamma^c := s^2(\Gamma) = \sum (-1)^{|\mu^{(1)}|} \left(s\mu^{(1)} \circ_l s\mu^{(2)}\right) \cdot \sigma.$$

Let A be a $\mathscr{P}$-algebra, and let $x_1, ..., x_r \in H_*(A)$ be homogeneous elements. Then the Γ-Massey product set $\langle x_1, \dots, x_r\rangle_\Gamma$ of Def. 1.4 and the Γ^c-Massey product set $\langle x_1, \dots, x_r\rangle_{\Gamma^c}$ of Def. 2.7 are the same up to a sign,

$$\langle x_1, \dots, x_r\rangle_\Gamma = \pm \langle x_1, \dots, x_r\rangle_{\Gamma^c}.$$

In particular, the Massey product set $\langle x_1, \dots, x_r\rangle_\Gamma$ is non-empty if, and only if, the Γ^c-Massey product set $\langle x_1, \dots, x_r\rangle_{\Gamma^c}$ is non-empty.

Proof. One can directly verify that

$$\Delta^+\left(\Gamma^c\right) = \sum (-1)^{|\mu^{(1)}|} (s\mu^{(1)}; \mathrm{id}, \dots, \mathrm{id}, s\mu^{(2)}, \mathrm{id}, \dots, \mathrm{id}; \sigma).$$

Since $\mu^{(1)}$ has weight 1, it follows that $\kappa(s\mu^{(1)}) = \mu^{(1)}$. Therefore, a cycle representing the Γ^c-Massey product associated to the elements $x_1, \dots x_r$ is of the form

$$\sum (-1)^\gamma \mu^{(1)} \left(a_{\mathrm{id},\sigma^{-1}(1)}, \dots, a_{\mathrm{id},\sigma^{-1}(l-1)}, a_{s\mu^{(2)},\sigma^{-1}(l)}, a_{\mathrm{id},\sigma^{-1}(l+r_1)}, \dots, a_{\mathrm{id},\sigma^{-1}(r)}\right),$$

as in Equation (6). Now, the $a_{\mathrm{id},(i)}$ are just cycle representatives of the x_i. To finish, we will check that the element $a_{s\mu^{(2)},(l)}$ satisfies exactly Condition (4) in Muro's construction (Def. 1.4), so it corresponds to the element $\rho^{(2)}$ there. Indeed, since $s\mu^{(2)}$ has weight 1, it follows that $\Delta^+\left(s\mu^{(2)}\right) = \left(s\mu^{(2)}; \mathrm{id}, \mathrm{id} \dots, ; \mathrm{id}\right)$, and so $D\left(s\mu^{(2)}\right) = \left(\mu^{(2)}; \mathrm{id}, \mathrm{id} \dots, ; \mathrm{id}\right)$. Therefore,

$$d\, a_{s\mu^{(2)},(l)} = \mu^{(2)}\left(a_{\mathrm{id},l}, \dots a_{\mathrm{id},l+r_1-1}\right).$$

The sign $(-1)^\gamma$ that appears in Equation (6) gives exactly Muro's sign plus one because for binary operads, the weight equals the arity degree minus one. This completes the proof. □

In the following examples, we explain how our operadic framework for defining systems recovers the classical framework in the associative and Lie cases, and then explain how it creates completely new higher-order operations.

Example 2.10. When $\mathscr{P} = \mathrm{Ass}$ is the associative operad, our framework recovers the classical definition of higher-order Massey products as in Def. 1.3. To see this, recall from Example 2.4 that the weight n component of $\mathrm{Ass}^{\text{¡}}$ is freely generated as an $\mathbb{S}_{n+1}$-module by a single generator μ^c_{n+1}, and that the μ^c_n-indexing set attached to a cooperation is given by

$$\left\{\left(\mu^c_k, (i, i+1, \dots, i+i-1)\right) \mid 1 \le k < n \text{ and } i \in \{1, 2, \dots, n-k+1\}\right\}.$$

We show next that fixing a particular differential graded associative algebra (A, d) and homogeneous homology classes $x_1, ..., x_n \in H_*(A)$, there is a bijective correspondence between the classical defining systems $\{b_{ij}\}$ for the higher-order Massey product $\langle x_1, ..., x_n\rangle$, and the defining systems $\{a_\alpha\}$ for the μ^c_n-Massey product $\langle x_1, ..., x_n\rangle_{\mu^c_n}$ as defined in this paper. Indeed, the correspondence is given by

$$b_{i,j} = a_{\mu^c_{j-i},(i+1,i+2,\dots,i+(j-i)=j)} \quad \text{for all} \quad 0 \le i < j \le n \quad \text{and} \quad 1 \le j-i \le n-1.$$

To finish, it suffices to compare the boundaries of the elements in these sets. Recall that

$$D\left(\mu^c_r\right) = \sum_{l_1+l_2=r} (-1)^{l_1+1} \left(\mu_2; \mu^c_{l_1}, \mu^c_{l_2}; \mathrm{id}\right).$$

Therefore, by directly applying Definition 2.6, we see that

$$db_{ij} = \sum_{k=i+1}^{j-1} \bar{b}_{ik} b_{kj} = \sum_{k=i+1}^{j-1} (-1)^{|b_{ik}|+1} b_{ik} b_{kj} = \sum_{k=i+1}^{j-1} (-1)^{|b_{ik}|+1} a_{\mu^c_{k-i},(i+1,i+2,\dots,k)} \cdot a_{\mu^c_{j-k},(k+1,k+2,\dots,j)}$$

$$= \sum_{k=i+1}^{j-1} (-1)^{|a_{\mu^c_{k-i},(i+1,i+2,\dots,k)}|+1} a_{\mu^c_{k-i},(i+1,i+2,\dots,k)} \cdot a_{\mu^c_{j-k},(k+1,k+2,\dots,j)}$$

$$= \sum_{l_1+l_2=j-i} (-1)^{l_1+1}(-1)^{\gamma} a_{\mu^c_{l_1},(k-l_1+1,k-l_2+2,\dots,k)} \cdot a_{\mu^c_{l_2},(k+1,k+2,\dots,k+l_2)} = d a_{\mu^c_{j-i},(i+1,i+2,\dots,i+(j-i)=j)},$$

where $\gamma = |x_{i+1}| + |x_{i+2}| + \cdots + |x_{i+l_1}|$ and the change of sign from the second to the third line follows from the equality

$$|a_{\mu^c_{k-i},(i+1,i+2,\dots,k)}| = |x_{i+1}| + |x_{i+2}| + \cdots + |x_k| + |\mu^c_{k-i}| + 1.$$

□

The observant reader will likely have spotted that the above is just one of the several linearly independent Massey products that Ass possesses. In fact, there are different, linearly independent Massey products for each permutation $\sigma \in \mathbb{S}_n$, since $\tau^c_n \cdot \sigma$ is linearly independent of τ^c_n. Up to a sign, these are related by $\langle x_1, \dots x_n \rangle_{\tau^c_n \cdot \sigma} = \langle x_{\sigma^{-1}(1)}, \dots x_{\sigma^{-1}(n)} \rangle_{\tau^c_n}$, see Prop. 2.21. Similarly, different presentations of an operad (in the associative case, one could take for example the Livernet–Loday presentation [16, Prop. 9.1.1]) give rise to seemingly distinct Massey products, which are just the same expressed with respect to a different basis.

Example 2.11. When $\mathscr{P} = \mathsf{Lie}$ is the Lie operad, our framework recovers the classical definition of higher Lie–Massey brackets as in [2, 26] (see also [28, 3]). To see this, recall that the weight n part of $\mathsf{Lie}^{\text{¡}}$ is one-dimensional and generated by $\tau^c_{n+1} \in \mathsf{Lie}^{\text{¡}}(n+1)$. Recall also from Example 2.5 that in this case, the τ^c_n-indexing set is

$$I(\tau^c_n) := \{(\tau^c_k, (i_1, \dots i_k)) \mid \ k \leq n \text{ and } 1 \leq i_1 \leq \cdots \leq i_l \leq n\}.$$

We show next that fixed a particular differential graded Lie algebra (L, d) and homogeneous elements $x_1, \dots, x_n \in H_*(L)$, there is a bijective correspondence between the classical defining systems $\{x_{j_1,\dots,j_l}\}$ of [2] for the higher-order Whitehead product $[x_1, \dots, x_n]$, and the defining systems $\{a_\alpha\}$ for the τ^c_n-Massey product as defined in this paper. Indeed, the correspondence is given by

$$x_{j_1,\dots,j_l} = a_{\tau^c_l,(j_1,\dots,j_l)} \quad \text{for all } \ 1 \leq j_1 \leq \cdots \leq j_l \leq n.$$

Recall from Example 2.5 that

$$D(\tau^c_n) = \sum_{\substack{r_1+r_2=n \\ \sigma \in \overline{Sh}^{-1}(r_1,r_2)}} (-1)^{r_1+1} \operatorname{sgn}(\sigma) (\tau_2; \tau^c_{r_1}, \tau^c_{r_2}; \sigma).$$

Therefore, by directly applying Definition 2.6, we see that

$$dx_{j_1,\dots,j_l} = \sum_{p=1}^{l} \sum_{\sigma \in \overline{Sh}(p,l-p)} \epsilon(\sigma) \left[x_{j_{\sigma(1)},\dots,\sigma(p)}, x_{j_{\sigma(p+1),\dots,\sigma(l)}} \right]$$

$$= \sum_{\substack{r_1+r_2=l \\ \sigma \in \overline{Sh}^{-1}(r_1,r_2)}} (-1)^{r_1+1} \operatorname{sgn}(\sigma) \tau_2 \left(a_{\tau^c_{r_1},\left(j_{\sigma^{-1}(1)}, j_{\sigma^{-1}(2)},\dots,j_{\sigma^{-1}(r_1)}\right)}, a_{\tau^c_{r_2},\left(j_{\sigma^{-1}(r_1+1)},\dots,j_{\sigma^{-1}(l)}\right)} \right)$$

$$= d a_{\tau^c_l,(j_1,\dots,j_l)}.$$

□

As likely expected, the higher-order Massey products for commutative differential graded associative algebras coincide with those formed by forgetting that the structure is commutative. This can be seen as a consequence of the theory developed in the next section, see Example 2.25.

In [22], Muro contributed a new kind of triple Massey-product operation for Gerstenhaber and/or Poisson algebras. Our framework recovers this triple operation as a consequence of Proposition 2.9. It follows from our results that all the higher-order analogs of this new operation also exist. Although we will not give closed formulas, we hope these higher products will be successfully applied in the future in cases where the triple-product operation defined by Muro does not suffice.

Example 2.12. Recall that the Poisson operad Pois is self-Koszul dual, generated by a commutative associative product $\wedge$ and a Lie bracket $[-,-]$, both of degree zero, which are compatible via the Poisson relation,

$$[x \wedge y, z] = x \wedge [y,z] + (-1)^{|y||z|}[x,z] \wedge y.$$

The inner combinatorics of this operad are complex, and multiplying base elements frequently involves complicated rewriting procedures. Therefore, we cannot hope to write down formulas that are quite as clean as in examples 2.10 and 2.11. Nonetheless, it is possible to compute Poisson Massey products inductively in low weight.

For example, if one considers $[-,-] \wedge - \in \mathsf{Pois}^{\mathsf{i}}(3)$, where we are taking the Koszul suspensions to be implicit, one has

$$D([-,-] \wedge -) = (\wedge; [-,-], \mathrm{id}) - ([-,-]; \mathrm{id}, \wedge,) - ([-,-]; \mathrm{id}, \wedge,) \cdot (2,3).$$

Recall that $\kappa[-,-] = \wedge$ and $\kappa(\wedge) = [-,-]$. A defining system for the Massey product $\langle x_1, x_2, x_3 \rangle$ associated to the cooperation above in a Poisson algebra is therefore a set of elements $\{z_1, z_2, z_3, y_1, y_2, y_3\}$ where each z_i is a cycle representative of x_i for $i = 1,2,3$, and

$$dy_1 = z_1 \wedge z_2, \qquad dy_2 = [z_2, z_3] \qquad dy_3 = [z_1, z_3].$$

The cycle representative associated to this defining system is

$$[y_1, z_3] - (-1)^{|z_1|} z_1 \wedge y_2 - (-1)^{|z_2|+|z_1||z_2|} z_2 \wedge y_3.$$

Similarly, if we consider $[-, - \wedge -] \in \mathsf{Pois}^{\mathsf{i}}(3)$,

$$D([-, - \wedge -]) = ([-,-]; \mathrm{id}, \wedge) + (\wedge; [-,-], \mathrm{id}) + (\wedge; \mathrm{id}, [-,-]) \cdot (1,2)$$

A defining system for the Massey product $\langle x_1, x_2, x_3 \rangle$ associated to this cooperation in a Poisson algebra is therefore a set of elements $\{z_1, z_2, z_3, y_1, y_2, y_3\}$ where each z_i is a cycle representative of x_i for $i = 1,2,3$, and

$$dy_1 = [z_2, z_3], \qquad dy_2 = z_1 \wedge z_2 \qquad dy_3 = z_1 \wedge z_3.$$

The cycle representative associated to this defining system is

$$z_1 \wedge y_1 - (-1)^{|z_1|}[y_2, z_3] - (-1)^{|z_2|+|z_1||z_2|}[z_2, y_3].$$

□

Example 2.13. We continue the previous example by computing the higher Massey product corresponding to $\wedge \circ ([-,-],[-,-]) \in \mathsf{Pois}^{\mathsf{i}}(4)$. If one takes the Koszul suspensions and Koszul signs to be implicit, one has:

$$\begin{aligned}
&\Delta^+(\wedge \circ ([-,-],[-,-])) \\
&= (\wedge; [-,-],[-,-]) + (\wedge \circ (-,[-,-]); [-,-], \mathrm{id}, \mathrm{id}) + (\wedge \circ ([-,-],-); \mathrm{id}, \mathrm{id}, [-,-],) \\
&\quad + (\wedge; \mathrm{id}, [-, - \wedge -]) + (- \wedge [-,-]; \mathrm{id}, \mathrm{id}, \wedge) + (\wedge; [-, - \wedge -], \mathrm{id}) \cdot (2,4,3) \\
&\quad + ([-,-] \wedge -; \mathrm{id}, - \wedge -, \mathrm{id}) \cdot (2,4,3) + 2(\wedge; \mathrm{id}, - \wedge [-,-],) \cdot (2,4,3) + 2(- \wedge - \wedge -; \mathrm{id}, \mathrm{id}, [-,-]) \cdot (2,4,3) \\
&\quad + 2(\wedge; - \wedge -, [-,-]) \cdot (2,4,3) + 2(\wedge; \mathrm{id}, [-,-] \wedge -) + 2(- \wedge - \wedge -; \mathrm{id}, [-,-], \mathrm{id}) \\
&\quad + 2(\wedge; - \wedge [-,-], \mathrm{id}) + 2(\wedge; \mathrm{id}, [-,-] \wedge -) \cdot (1,2,4,3) + 2(- \wedge - \wedge -; \mathrm{id}, [-,-], \mathrm{id}) \cdot (1,2,4,3) \\
&\quad + 2(\wedge; - \wedge [-,-], \mathrm{id}) \cdot (1,2,4,3) + (\wedge; \mathrm{id}, [- \wedge -, -], \mathrm{id}) \cdot (1,2,3) + (- \wedge [-,-]; \mathrm{id}, - \wedge -, \mathrm{id}) \cdot (1,2,3) \\
&\quad + (\wedge; [- \wedge -, -], \mathrm{id}) + ([-,-] \wedge -; \wedge, \mathrm{id})
\end{aligned}$$

This means that

$$D(\wedge\circ([-,-],[-,-]))$$

$$\begin{aligned}
&= (\wedge;[-,-],[-,-]) + (\wedge;\mathrm{id},[-,-\wedge-]) + (\wedge;[-,-\wedge-],\mathrm{id})\cdot(2,4,3) + 2\,(\wedge;\mathrm{id},-\wedge[-,-],)\cdot(2,4,3)\\
&\quad + 2\,(\wedge;-\wedge-,[-,-])\cdot(2,4,3) + 2\,(\wedge;\mathrm{id},[-,-]\wedge-) + 2\,(\wedge;-\wedge[-,-],\mathrm{id}) + 2\,(\wedge;\mathrm{id},[-,-]\wedge-)\cdot(1,2,4,3)\\
&\quad + 2\,(\wedge;-\wedge[-,-],\mathrm{id})\cdot(1,2,4,3) + (\wedge;\mathrm{id},[-\wedge-,-],\mathrm{id})\cdot(1,2,3) + (\wedge;[-\wedge-,-],\mathrm{id})
\end{aligned}$$

Now we compute the product corresponding to the equation above. This means that the Massey product may be computed as being, up to Koszul sign

$$\begin{aligned}
&= [y_{\wedge,(1,2)}, y_{\wedge,(3,4)}] + [z_1, x_{b,(2,3,4)}] + [x_{b,(1,3,4)}, z_2] + 2[z_1, x_{a,(3,4,2)}]\\
&\quad + 2[y_{[-],(1,3)}, y_{\wedge,(4,2)}] + 2[z_1, x_{a,(2,3,4)}] + 2[x_{a,(1,2,3)}, z_4] + 2[z_3, x_{a,(1,4,2)}]\\
&\quad + 2[x_{a,(1,4,3)}, z_2] + [x_{b,(2,3,1)}, z_4] + [x_{b,(3,1,2)}, z_4]
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
dy_{\wedge,(i,j)} &= z_i \wedge z_j, \qquad y_{[-],(i,j)} = [z_i, z_j]\\
dx_{a,(i,j,k)} &= [y_{\wedge,(i,j)}, z_k] - (-1)^{|z_i|} z_j \wedge y_{[-],(j,k)} - (-1)^{|z_j|+|z_i||z_j|} z_j \wedge y_{[-],(i,k)}\\
dx_{b,(i,j,k)} &= z_i \wedge y_{[-],(j,k)} - (-1)^{|z_i|}[y_{\wedge,(i,j)}, z_k] - (-1)^{|z_j|+|z_i||z_k|}[z_j, y_{\wedge,(i,k)}]
\end{aligned}$$

□

The signs missing in each term of (7) can be computed as follows. These signs arise in three ways:

- Firstly, the products in Δ^+ come with the usual Koszul signs.
- Secondly, one has those signs corresponding to γ in Equation (6).
- Thirdly, to simplify the expression, for the three terms on the final line, we use the (anti)commutativity of the generating cooperations. This introduces signs coming from the (signed) identities

$$[x,y] = -(-1)^{|x||y|}[y,x] \qquad \text{and} \qquad x\wedge y = (-1)^{|x||y|} y\wedge x.$$

Our final example will illustrate the close connection of Massey products with spectral sequences. We point the reader to [16, 10.3.7] for some of the basic background on this example.

Definition 2.14. The *dual numbers operad* is the quadratic operad $\mathscr{D}$ presented as

$$\mathscr{D} := \mathscr{F}(\Bbbk\Delta, \Delta\circ\Delta),$$

where Δ is an arity 1 element of homological degree 1.

Algebras over this operad are precisely the bicomplexes, i.e., chain complexes (A,d) equipped with an operation $\Delta : A \to A$ such that $\Delta^2 = 0$ and $d\Delta + \Delta d = 0$. The dual numbers operad is Koszul, and its Koszul dual cooperad is cofree conilpotent on a single generator,

$$\mathscr{D}^{\mathrm{i}} = \mathscr{F}^c(s\Delta).$$

In particular, this cooperad has no corelations and is concentrated in degree 1.

Example 2.15. We shall compute the Massey products of the dual numbers operad. The arity 1 component of $\mathscr{D}^{\mathrm{i}}$ is

$$\mathscr{D}^{\mathrm{i}}(1) = \bigoplus \Bbbk\delta_n,$$

where δ_n has weight n and degree $2n$. Since

$$\Delta^+(\delta_n) = \sum_{k+l=n} (\delta_k;\delta_l),$$

where $l \geq 1$ and $k \geq 0$, it follows that

$$D(\delta_n) = (\Delta;\delta_{n-1}) \quad \text{for all } n \geq 2.$$

Therefore, the δ_n-indexing system is given by $\{a_{\delta_i} : 0 < i < n\}$, with the relation $d a_{\delta_i} = \Delta(a_{\delta_{i-1}})$. This is almost the definition of the d_{n-1}-differential in the spectral sequence associated to the bicomplex (A,d,Δ). More precisely, one can check that if $x \in \langle y\rangle_{\delta_n}$ is defined in $H_*(A)$, then y survives to the E_{n-1}-page of the associated spectral sequence and $d_{n-1}(y) = [x]$. □

Remark 2.16. In our higher-order Massey products framework for Koszul operads, there is nothing special about the Koszul dual cooperad $\mathscr{P}^{\text{¡}}$ aside from it being a very useful resolution. In principle, starting with any conilpotent cooperad $\mathscr{C}$ together with a choice of twisting morphism $\tau : \mathscr{C} \to \mathscr{P}$, it is possible to define a *relative Massey inductive map*

$$D : \mathscr{C} \xrightarrow{\Delta^+} \mathscr{C} \circ \mathscr{C} \xrightarrow{\tau \circ \mathrm{id}} \mathscr{P} \circ \mathscr{C}.$$

From this, one defines *relative Massey products* following, *mutatis mutandis*, the same recipe we gave in the Koszul case. Taking $\mathscr{C} = B\mathscr{P}$ to be the bar construction of $\mathscr{P}$ and $\tau : B\mathscr{P} \to \mathscr{P}$ the canonical twisting morphism, this allows for defining Massey products for non-Koszul operads.

2.1 Elementary properties of the operadic Massey products

In this section, we collect some elementary properties of the operadic Massey products. First, we show that Massey product sets do not depend on the initial choice of cycles in the defining system (Prop. 2.17). Then, that morphisms of $\mathscr{P}$-algebras preserve Massey products (Prop. 2.18). In particular, quasi-isomorphisms induce bijections of the corresponding Massey product sets. This provides an obstruction for two $\mathscr{P}$-algebras to be weakly equivalent. In particular, a nontrivial Massey product provides an obstruction to formality. At the end of the section, we collect a few elementary properties of the Massey products that might be useful elsewhere (Prop. 2.21).

For the next few results, we fix a cooperation $\Gamma^c \in \left(\mathscr{P}^{\text{¡}}\right)^{(n)}(k)$. We say that a Massey product set $\langle x_1, \ldots, x_k \rangle_{\Gamma^c}$ is *defined* if it is non-empty, that is, if there is some defining system for the Massey product; *trivial* if it contains the zero homology class; and *non-trivial* if it is defined and does not contain the zero homology class.

First, we shall show that Massey product sets do not depend on the initial choice of cycles in the defining system.

Proposition 2.17. *Let A be a $\mathscr{P}$-algebra. Suppose that $x_1, \ldots, x_k \in H_*(A)$ are homogeneous elements such that the Massey product set $\langle x_1, \ldots, x_k \rangle_{\Gamma^c}$ is defined. For each $x \in \langle x_1, \ldots, x_k \rangle_{\Gamma^c}$ and each choice of cycle representative $\overline{x_i}$ for x_i, one has a defining system $\{a_\beta\}$ for x such that $a_{\mathrm{id},(i)} = \overline{x_i}$*

Proof. Let $\{b_\beta\}$ be a defining system for a Massey product $x \in \langle x_1, \ldots, x_k \rangle_{\Gamma^c}$. We shall construct, by induction on the weight of the elements of the defining system, a defining system $\{a_\beta\}$ for a Massey product $x \in \langle x_1, \ldots, x_k \rangle_{\Gamma^c}$ such that $a_{\mathrm{id},(i)} = \overline{x_i}$ and that $a_{\Gamma^c,(1,\ldots,k)}$ is homologous to $b_{\Gamma^c,(1,\ldots,k)}$.

For the first step, simply fix $a_{\mathrm{id},(i)} = \overline{x_i}$. Since $a_{\mathrm{id},(i)}$ and $b_{\mathrm{id},(i)}$ are both choices of representative for x_i, it follows that $a_{\mathrm{id},(i)} - b_{\mathrm{id},(i)}$ is nullhomologous, which means that there is a $c_{\mathrm{id},(i)} \in A$ such that

$$d c_{\mathrm{id},(i)} = a_{\mathrm{id},(i)} - b_{\mathrm{id},(i)}.$$

The family $\{a_{\mathrm{id},(i)}\}$ gives the first inductive step. Now, suppose that for all indexes $(\mu, (i_1, \ldots, i_k)) \in I(\Gamma^c)$ with the weight of μ strictly less than N, with $1 < N < n$, where n is the weight of Γ^c, we have constructed $a_{\mu,(i_1,\ldots,i_k)}, c_{\mu,(i_1,\ldots,i_k)} \in A$ such that

$$d\left(a_{\mu,(i_1,\cdots,i_k)}\right) = \sum \zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)}\right)}, \ldots, a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(k)}\right)}\right),$$

where $D(\mu) = \sum(\zeta; \zeta_1, \ldots, \zeta_m; \sigma)$, and

$$d c_{\mu,(i_1,\ldots,i_k)} = a_{\mu,(i_1,\ldots,i_k)} - b_{\mu,(i_1,\ldots,i_k)} + Q_{\mu,(i_1,\ldots,i_k)},$$

where $Q_{\mu,(i_1,\ldots,i_k)}$ is the sum:

$$\sum\sum \zeta\left(x_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)}\right)}, \ldots, x_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)} \ldots, x_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(k)}\right)}\right).$$

Here, the outer summation is indexed by

$$D(\mu) = \sum(\zeta; \zeta_1, \ldots, \zeta_m; \sigma),$$

and for each term $(\zeta;\zeta_1,\dots,\zeta_m;\sigma)$, the inner sum is taken over every possible choice of tuple

$$\left(x_{\zeta_1,\left(i_{\sigma^{-1}(1)},\dots,i_{\sigma^{-1}(v_1)}\right)},\dots,x_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\dots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}\cdots,x_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\dots,i_{\sigma^{-1}(k)}\right)}\right),$$

where one of the

$$x_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\dots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}$$

is precisely

$$c_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\dots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)},$$

anything to the left of it in the tuple is

$$a_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\dots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)},$$

and anything to its right is

$$b_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\dots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}.$$

Now, let $\left(\mu,(i_1,\dots,i_k)\right)\in I(\Gamma^c)$ have μ of weight N. Then, $Q_{\mu,(i_1,\dots,i_k)}$ is well defined, because the cooperations appearing in its defining tuple have weight strictly less than μ. Its boundary is as follows:

$$dQ_{\mu,(i_1,\dots,i_k)}=db_{\mu,(i_1,\dots,i_k)}+\sum\zeta\left(a_{\zeta_1,(i_{\sigma^{-1}(1)},\dots i_{\sigma^{-1}(v_1)}},\dots,a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\dots,i_{\sigma^{-1}(k)}\right)}\right). \tag{8}$$

Indeed, the sum $dQ_{\mu,(i_1,\dots,i_k)}$ can be separated into two parts: a telescoping part that converges to the right-hand side of the equation above, and a second part that can be divided into subsums each vanishing by arguments similar to the proof of Theorem 2.8. Now, from Equation (8), we deduce that the element

$$\sum\zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\dots i_{\sigma^{-1}(v_1)}\right)},\dots,a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\dots,i_{\sigma^{-1}(k)}\right)}\right)$$

where the sum ranges over $D\left(\mu\right)=\sum(\zeta;\zeta_1,\dots,\zeta_m;\sigma)$, is a cycle. Therefore, there is an element $a'_{\mu,(i_1,\dots,i_k)}\in A$ such that

$$da'_{\mu,(i_1,\dots,i_k)}=\sum\zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\dots i_{\sigma^{-1}(v_1)}\right)},\dots,a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\dots,i_{\sigma^{-1}(k)}\right)}\right).$$

Define a cycle

$$e_{\mu,(i_1,\dots,i_k)}=a'_{\mu,(i_1,\dots,i_k)}-b_{\mu,(i_1,\dots,i_k)}+Q_{\mu,(i_1,\dots,i_k)}.$$

Then there is an element $e'_{\mu,(i_1,\dots,i_k)}\in A$ such that $e'_{\mu,(i_1,\dots,i_k)}$ is homologous to $e_{\mu,(i_1,\dots,i_k)}$, that is, such that

$$dc_{\mu,(i_1,\dots,i_k)}=e'_{\mu,(i_1,\dots,i_k)}-e_{\mu,(i_1,\dots,i_k)},$$

and

$$a_{\mu,(i_1,\dots,i_k)}=a'_{\mu,(i_1,\dots,i_k)}-e'_{\mu,(i_1,\dots,i_k)}.$$

It follows that

$$d\left(a_{\mu,(i_1,\cdots,i_k)}\right)=\sum\zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\dots i_{\sigma^{-1}(v_1)}\right)},\dots,a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\dots,i_{\sigma^{-1}(k)}\right)}\right),$$

where the sum ranges over $D\left(\mu\right)=\sum(\zeta;\zeta_1,\dots,\zeta_m;\sigma)$, and furthermore, that

$$dc_{\mu,(i_1,\dots,i_k)}=a_{\mu,(i_1,\dots,i_k)}-b_{\mu,(i_1,\dots,i_k)}+Q_{\mu,(i_1,\dots,i_k)}.$$

This concludes the induction step. To finish, consider the element $Q_{\Gamma^c,(i_1,\ldots,i_k)}$. This is defined by the same logic as above, and its boundary satisfies

$$dQ_{\Gamma^c,(i_1,\ldots,i_k)} = a_{\Gamma^c,(i_1,\ldots,i_k)} - b_{\Gamma^c,(i_1,\ldots,i_k)}.$$

Therefore, the elements $a_{\Gamma^c,(i_1,\ldots,i_k)}$ and $b_{\Gamma^c,(i_1,\ldots,i_k)}$ are homologous, as we wanted to prove. □

Proposition 2.18. *A morphism of $\mathscr{P}$-algebras $f : A \to B$ preserves Massey products. That is, if $x_1,\ldots,x_k \in H_*(A)$ are homogeneous elements such that the Massey product set $\langle x_1,\ldots,x_k\rangle_{\Gamma^c}$ is defined, then $\langle f_*(x_1),\ldots,f_*(x_k)\rangle_{\Gamma^c}$ is also defined, and moreover*

$$f_*\langle x_1,\ldots,x_k\rangle_{\Gamma^c} \subseteq \langle f_*(x_1),\ldots,f_*(x_k)\rangle_{\Gamma^c}.$$

If furthermore f is a quasi-isomorphism, then f_* induces a bijection between the corresponding Massey product sets.

Proof. Assume that the Massey product set $\langle x_1,\ldots,x_k\rangle_{\Gamma^c}$ is defined. Then, any defining system $\{a_\alpha\}$ for $\langle x_1,\ldots,x_k\rangle_{\Gamma^c}$ produces a defining system $\{f(a_\alpha)\}$ for $\langle f_*(x_1),\ldots,f_*(x_k)\rangle_{\Gamma^c}$, because f commutes with the operadic structure maps and the differentials. Therefore, if $\langle x_1,\ldots,x_k\rangle_{\Gamma^c}$ is defined, then $\langle f_*(x_1),\ldots,f_*(x_k)\rangle_{\Gamma^c}$ is also defined, and the containment $f_*\langle x_1,\ldots,x_k\rangle_{\Gamma^c} \subseteq \langle f_*(x_1),\ldots,f_*(x_k)\rangle_{\Gamma^c}$ follows.

Next, assume that f is a quasi-isomorphism and let us prove that the corresponding Massey product sets are in bijective correspondence. Let $\{b_\beta\}$ be a defining system for a Massey product $y \in \langle f_*(x_1),\ldots,f_*(x_k)\rangle_{\Gamma^c}$. We shall construct, by induction on the weight of the elements of the defining system, a defining system $\{a_\beta\}$ for a Massey product $x \in \langle x_1,\ldots,x_k\rangle_{\Gamma^c}$ such that $f\left(a_{\Gamma^c,(1,\ldots,k)}\right)$ is homologous to $b_{\Gamma^c,(1,\ldots,k)}$, and therefore $f_*(x) = y$.

Let $a_{\mathrm{id},(i)}$ be any representative for x_i. This means that $f(a_{\mathrm{id},(i)}) - b_{\mathrm{id},(i)}$ is nullhomologous, which means that there is a $c_{\mathrm{id},(i)} \in B$ such that

$$dc_{\mathrm{id},(i)} = f\left(a_{\mathrm{id},(i)}\right) - b_{\mathrm{id},(i)}.$$

The family $\{a_{\mathrm{id},(i)}\}$ gives the first inductive step. Now, suppose that for all indexes $(\mu,(i_1,\ldots,i_k)) \in I(\Gamma^c)$ with the weight of μ strictly less than N, with $1 < N < n$, where n is the weight of Γ^c, we have constructed $a_{\mu,(i_1,\ldots,i_k)} \in A$ and $c_{\mu,(i_1,\ldots,i_k)} \in B$ such that

$$d\left(a_{\mu,(i_1,\cdots,i_k)}\right) = \sum \zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)}\right)},\ldots,a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(k)}\right)}\right),$$

where $D(\mu) = \sum(\zeta;\zeta_1,\ldots,\zeta_m;\sigma)$, and

$$dc_{\mu,(i_1,\ldots,i_k)} = f\left(a_{\mu,(i_1,\ldots,i_k)}\right) - b_{\mu,(i_1,\ldots,i_k)} + Q_{\mu,(i_1,\ldots,i_k)},$$

where $Q_{\mu,(i_1,\ldots,i_k)}$ is the sum:

$$\sum\sum\zeta\left(x_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)}\right)},\ldots,x_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}\ldots,x_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(k)}\right)}\right).$$

Here, the outer summation is indexed by

$$D(\mu) = \sum(\zeta;\zeta_1,\ldots,\zeta_m;\sigma),$$

and for each term $(\zeta;\zeta_1,\ldots,\zeta_m;\sigma)$, the inner sum is taken over every possible choice of tuple

$$\left(x_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots,i_{\sigma^{-1}(v_1)}\right)},\ldots,x_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}\ldots,x_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(k)}\right)}\right),$$

where one of the

$$x_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}$$

is precisely

$$c_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)},$$

anything to the left of it in the tuple is

$$f\left(a_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}\right),$$

and anything to its right is

$$b_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}.$$

Now, let $\left(\mu,(i_1,\ldots,i_k)\right)\in I(\Gamma^c)$ have μ of weight N. Then, $Q_{\mu,(i_1,\ldots,i_k)}$ is well defined, because the cooperations appearing in its defining tuple have weight strictly less than μ. Its boundary is as follows:

$$dQ_{\mu,(i_1,\ldots,i_k)}=db_{\mu,(i_1,\ldots,i_k)}+\sum\zeta\left(f(a_{\zeta_1,(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)})},\ldots,f(a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(k)}\right)})\right). \quad (9)$$

Indeed, the sum $dQ_{\mu,(i_1,\ldots,i_k)}$ can be separated into two parts: a telescoping part that converges to the right-hand side of the equation above, and a second part that can be divided into subsums each vanishing by arguments similar to the proof of Proposition 2.8. Now, from Equation (9), we deduce that the element

$$\sum\zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)}\right)},\ldots,a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(k)}\right)}\right)$$

where the sum ranges over $D(\mu)=\sum(\zeta;\zeta_1,\ldots,\zeta_m;\sigma)$, is a cycle. Therefore, there is an element $a'_{\mu,(i_1,\ldots,i_k)}\in A$ such that

$$da'_{\mu,(i_1,\ldots,i_k)}=\sum\zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)}\right)},\ldots,a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(k)}\right)}\right).$$

Define a cycle

$$e_{\mu,(i_1,\ldots,i_k)}=f\left(a'_{\mu,(i_1,\ldots,i_k)}\right)-b_{\mu,(i_1,\ldots,i_k)}+Q_{\mu,(i_1,\ldots,i_k)}.$$

Then there is an element $e'_{\mu,(i_1,\ldots,i_k)}\in A$ such that $f\left(e'_{\mu,(i_1,\ldots,i_k)}\right)$ is homologous to $e_{\mu,(i_1,\ldots,i_k)}$, that is, such that

$$dc_{\mu,(i_1,\ldots,i_k)}=f\left(e'_{\mu,(i_1,\ldots,i_k)}\right)-e_{\mu,(i_1,\ldots,i_k)},$$

and

$$a_{\mu,(i_1,\ldots,i_k)}=a'_{\mu,(i_1,\ldots,i_k)}-e'_{\mu,(i_1,\ldots,i_k)}.$$

It follows that

$$d\left(a_{\mu,(i_1,\cdots,i_k)}\right)=\sum\zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)}\right)},\ldots,a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(k)}\right)}\right),$$

where the sum ranges over $D(\mu)=\sum(\zeta;\zeta_1,\ldots,\zeta_m;\sigma)$, and furthermore, that

$$dc_{\mu,(i_1,\ldots,i_k)}=f\left(a_{\mu,(i_1,\ldots,i_k)}\right)-b_{\mu,(i_1,\ldots,i_k)}+Q_{\mu,(i_1,\ldots,i_k)}.$$

This concludes the induction step. To finish, consider the element $Q_{\Gamma^c,(i_1,\ldots,i_k)}$. This is defined by the same logic as above, and its boundary satisfies

$$dQ_{\Gamma^c,(i_1,\ldots,i_k)}=f\left(a_{\Gamma^c,(i_1,\ldots,i_k)}\right)-b_{\Gamma^c,(i_1,\ldots,i_k)}.$$

Therefore, the elements $f\left(a_{\Gamma^c,(i_1,\ldots,i_k)}\right)$ and $b_{\Gamma^c,(i_1,\ldots,i_k)}$ are homologous, as we wanted to prove. □

Recall that two $\mathscr{P}$-algebras are *weakly-equivalent*, or *quasi-isomorphic*, if there is a zig-zag of $\mathscr{P}$-algebra quasi-isomorphisms between them. From the two previous results, we can deduce the following.

Corollary 2.19. *There is a bijection between the Massey product sets of weakly-equivalent $\mathscr{P}$-algebras.*

Proof. Suppose one has a zig-zag of quasi-isomorphisms of $\mathscr{P}$-algebras

$$A \xrightarrow{f} B \xleftarrow{g} C.$$

By Proposition 2.18, there is a bijection between a Massey product set $\langle x_1, ..., x_k\rangle_{\Gamma^c}$ in A and the corresponding Massey product set $\langle f_*(x_1), ..., f_*(x_k)\rangle_{\Gamma^c}$ at B. Since g is a quasi-isomorphism, there exists y_i such that $g_*(y_i) = f_*(x_i)$. Therefore, a second application of Proposition 2.18 yields that the Massey product set $\langle f_*(y_1), ..., f_*(y_k)\rangle_{\Gamma^c}$ is in bijection with $\langle x_1, ..., x_k\rangle_{\Gamma^c}$. □

Recall that, for an operad $\mathscr{P}$ without differential, the homology of a $\mathscr{P}$-algebra is also a $\mathscr{P}$-algebra. We say that a $\mathscr{P}$-algebra is *formal* if it is weakly equivalent to its homology endowed with the induced $\mathscr{P}$-algebra structure (and trivial differential). In general, the Massey products of $\mathscr{P}$-algebras with trivial differential are always trivial because they have no relations that exist at the chain level but not at the homological level. In particular, the Massey products of the homology of a $\mathscr{P}$-algebra are all trivial whenever they are defined. From this, we immediately deduce the following result.

Corollary 2.20. *If a $\mathscr{P}$-algebra has a nontrivial Massey product, then it is not formal.*

Proof. Assume that a $\mathscr{P}$-algebra A has a nontrivial Massey product. Since the homology of A has a zero differential, it must be that all of its Massey products are trivial. Therefore, by Corollary 2.19 it cannot be quasi-isomorphic to A. □

Next, we collect some elementary properties satisfied by the operadic Massey products. These are similar to some of those explained in [13] and, more recently, in [25, p. 325]. The proofs follow from the definitions, and are left to the reader.

Proposition 2.21. *Let A be a $\mathscr{P}$-algebra, $\Gamma^c \in \left(\mathscr{P}^i\right)^{(n)}(r)$, and $x_1, \dots, x_r \in H_*(A)$ be homogeneous elements such that $\langle x_1, ..., x_r\rangle_{\Gamma^c}$ is defined. Then the following assertions hold.*

1. *(Homological linearity) If $k \in \Bbbk$ is a scalar, then for all $1 \le i \le r$,*

$$k\langle x_1, \dots, x_r\rangle_{\Gamma^c} \subseteq \langle x_1, \dots, kx_i, \dots, x_r\rangle_{\Gamma^c}.$$

2. *(Equivariance) For every permutation $\sigma \in \mathbb{S}_r$, there is a bijection*

$$\langle x_1, \dots, x_r\rangle_{\Gamma^c_n \cdot \sigma} = (-1)^{\varepsilon(\sigma^{-1})}\langle x_{\sigma^{-1}(1)}, \dots x_{\sigma^{-1}(r)}\rangle_{\Gamma^c_n},$$

where $(-1)^{\varepsilon(\sigma^{-1})}$ is the Koszul sign appearing by permuting the variables according to σ^{-1}.

Proof. 1. Let $\{a_\alpha\}$ be a defining system for a Massey product $x \in \langle x_1, \dots, x_r\rangle_{\Gamma^c}$. Consider a new defining system

$$b_{(\zeta, j_1, \dots j_s)} = \begin{cases} ka_{(\zeta, j_1, \dots j_s)} & \text{if } j_l = i \text{for some } l. \\ a_{(\zeta, j_1, \dots j_s)} & \text{otherwise.} \end{cases}$$

In particular, one has

$$b_{(\mathrm{id}, j)} = \begin{cases} ka_{(\mathrm{id}, i)} & \text{for } i = j. \\ a_{(\mathrm{id}, j)} & \text{otherwise.} \end{cases}$$

so $\{b_\alpha\}$ is a defining system for a Massey product in $\langle x_1, \dots, kx_i, \dots, x_r\rangle_{\Gamma^c}$. Furthermore, $b_{(\Gamma^c, 1, 2, \dots, r)} = ka_{(\Gamma^c, 1, 2, \dots, r)}$ so the corresponding Massey product is kx.

2. Let $\{a_\alpha\}$ be a defining system for a Massey product $x \in \langle x_1, \dots, x_r \rangle_{\Gamma^c}$. Consider a new defining system

$$b_{(\zeta, j_1, \dots j_s)} := a_{(\zeta, \sigma^{-1}(j_1), \dots \sigma^{-1}(j_s)}.$$

This is then a defining system for $(-1)^{\varepsilon(\sigma^{-1})}\langle x_{\sigma^{-1}(1)}, \dots x_{\sigma^{-1}(r)} \rangle_{\Gamma^c_n}$ and the result follows.

□

2.2 Massey products along morphisms of operads and formality

In this section, we shall discuss pullbacks and pushforwards of Massey products along morphisms of operads and give some applications to formality.

Before we begin, it will be helpful to remark some observations. Let $f : \mathscr{P} \to \mathscr{Q}$ be a morphism of weighted operads. In this case, taking Koszul dual cooperad is functorial, and therefore there is an induced map $f^{\text{¡}} : \mathscr{F}(E,R) = \mathscr{P}^{\text{¡}} \to \mathscr{Q}^{\text{¡}} = \mathscr{F}(F,S)$. Moreover, there is a commutative diagram

$$\begin{array}{ccccc} \mathscr{P}^{\text{¡}} & \xrightarrow{\Delta^+} & \mathscr{P}^{\text{¡}} \circ \mathscr{P}^{\text{¡}} & \xrightarrow{\kappa \circ \mathrm{id}} & \mathscr{P} \circ \mathscr{P}^{\text{¡}} \\ \downarrow{\scriptstyle f^{\text{¡}}} & & \downarrow{\scriptstyle f^{\text{¡}} \circ f^{\text{¡}}} & & \downarrow{\scriptstyle f \circ f^{\text{¡}}} \\ \mathscr{Q}^{\text{¡}} & \xrightarrow{\Delta^+} & \mathscr{Q}^{\text{¡}} \circ \mathscr{Q}^{\text{¡}} & \xrightarrow{\kappa' \circ \mathrm{id}} & \mathscr{Q} \circ \mathscr{Q}^{\text{¡}} \end{array}$$

From this, we conclude that the Massey inductive map D commutes with $f^{\text{¡}}$. Secondly, because the category of graded vector spaces admits finite colimits, on the level of algebras, f descends to an adjoint pair

$$f_! : \mathscr{P} - \mathsf{Alg} \leftrightarrows \mathscr{Q} - \mathsf{Alg} : f^*.$$

The functor f^* preserves the underlying chain complex of the $\mathscr{Q}$-algebras, and therefore there is a chain map $f^*(A) \to A$ which is just the identity morphism. We define next another chain map $h : A \to f_!(A)$. Given a $\mathscr{P}$-algebra A, the unit of the adjunction above is a morphism of $\mathscr{P}$-algebras

$$A \to f^* f_!(A).$$

Forgetting the $\mathscr{P}$-algebra structure and recalling that f^* preserves the underlying chain complex, there is a chain map

$$h : A \to f_!(A).$$

Pullbacks of Massey products. For any $\mathscr{Q}$-algebra B, the $\mathscr{P}$-Massey products on $f^*(B)$ induce $\mathscr{Q}$-Massey products on B. Since the underlying chain complex of both algebras is the same, we can prove the following result.

Proposition 2.22. *Let $f : \mathscr{P} \to \mathscr{Q}$ be a morphism of weighted operads, B a $\mathscr{Q}$-algebra, and $\Gamma^c \in \left(\mathscr{P}^{\text{¡}}\right)^{(n)}(r)$. Suppose that $x_1, \dots, x_r \in H_*\left(f^*(B)\right)$ are homogeneous elements such that the Massey product set $\langle x_1, \dots, x_r \rangle_{\Gamma^c}$ is defined. If $\mathscr{P}$ is finite type arity-wise and $f^{\text{¡}}(\Gamma^c) \neq 0$, then under the identification of $f^*(B)$ and B as chain complexes, we have*

$$\langle x_1, \dots, x_r \rangle_{\Gamma^c} \subseteq \langle x_1, \dots, x_r \rangle_{f^{\text{¡}}(\Gamma^c)}.$$

If $f^{\text{¡}}$ is injective, this is an equality.

Proof. Let $x \in \langle x_1, \dots, x_r \rangle_{\Gamma^c}$ have a defining system $\left\{b_{\mu,(i_1,\dots,i_r)}\right\}$. We shall prove the statement by constructing a defining system for x as a $f^{\text{¡}}(\Gamma^c)$-Massey product. If $f^{\text{¡}}$ is injective, then the statement is trivial. Indeed, since D commutes with $f^{\text{¡}}$, we may obtain the desired defining set $\left\{b_{f^{\text{¡}}(\mu),(i_1,\dots,i_r))}\right\}$ by setting $b_{f^{\text{¡}}(\mu),(i_1,\dots,i_r))} := b_{\mu,(i_1,\dots,i_r))}$. The converse is also true; each defining set $\left\{b_{f^{\text{¡}}(\mu),(i_1,\dots,i_r))}\right\}$ is a defining set for a Γ^c-Massey product.

If $f^{\text{¡}}$ is not injective, then the set before may fail to be a defining system. Two problems may arise. Firstly, $f^{\text{¡}}(\mu)$ may be zero. In this case, however, any term coming from D in which $f^{\text{¡}}(\mu)$ plays a role will also vanish, so we may safely remove any term of the form $b_{f^{\text{¡}}(\mu),(i_1,\dots,i_r))}$ from the

defining system altogether. Secondly, $f^!$ may fail to preserve linear independence. We circumvent this problem as follows. To fix notation, write $\mathscr{P} = \mathscr{F}(E,R)$ and $\mathscr{Q} = \mathscr{F}(F,S)$. The map $f^!$ is a map of weighted quadratic cooperads and, in particular, it sends cogenerators to cogenerators, $\left(f^!\right)^{(1)} : E \to F$. We shall assume that $\Bbbk$-linear bases of E and F are chosen such that the image of the basis elements of E are precisely the first m basis vectors $\{u_i\}$ of F, and further that the other basis elements of F are not in the image of $\left(f^!\right)^{(1)}$, and that the rest of the elements of the basis of E are in the kernel of $\left(f^!\right)^{(1)}$. These bases now, as explained in the second paragraph of Section 2, extend to bases of the operads $\mathscr{P}$ and $\mathscr{Q}$ using appropriate symmetric tree monomials.

The image of $f^!$ now entirely lies in the span of tree monomials labeled by the first m basis elements of F. This means that there is now a canonical (with respect to this choice of basis) linear section s of $f^!$ defined only on this codomain that preserves the cocomposition. This section is given by sending sums of tree monomials labeled by the first m basis elements of F to sums of tree monomials of the same shape labeled by the corresponding first m basis elements of E.

The section s induces a bijection between the indexing sets $I\left(f^!(\Gamma^c),(1,...,r)\right)$ and $I\left(\Gamma^c,(1,...,r)\right)$. Define $b_{\mu,(i_1,...i_n)}$ to be $b_{s(\mu),(i_1,...i_n)}$. This provides a defining system for x. □

Remark 2.23. This means that if $f^!$ is injective and $f^*(B)$ has nontrivial Massey products, then so does B.

Example 2.24. Consider the natural weighted operad morphism $f : \mathsf{Lie} \to \mathsf{Ass}$. For any differential graded associative algebra A, the differential graded Lie algebra $f^*(A)$ is the chain complex A equipped with the bracket $[a,b] = ab - (-1)^{|a||b|}ba$ for all homogeneous $a,b \in A$. Now, recall from Example 2.11 that $\mathsf{Lie}^!(n)^{(n-1)}$ is generated by an element denoted τ_n^c. Since on the level of Koszul dual cooperads, the map $f^! : \mathsf{Lie}^! \to \mathsf{Ass}^!$ is the linear dual of the canonical operad map $\mathsf{Ass} \to \mathsf{Com}$, one can verify that $f^!(\tau_n^c) = \sum_{\sigma\in\mathbb{S}_n} \mu_n^c \cdot \sigma$, where μ_n^c is the generator of $\left(\mathsf{Ass}^!\right)^{(n-1)}(n)$ as an $\mathbb{S}_n$-module. This map is injective and therefore, it follows from Proposition 2.22 that

$$\langle x_1,...,x_n\rangle_{\tau_n^c} = \langle x_1,...,x_n\rangle_{\sum_{\sigma\in\mathbb{S}_n}\mu_n^c\cdot\sigma}.$$

This can be used in two ways. Firstly, we can deduce that if $f^*(A)$ admits a nontrivial Lie–Massey bracket, then A admits a (nonclassical) associative bracket and so is not formal. In general, on the other hand, most of the time if A has a nontrivial (classical) Massey product, we cannot deduce the existence of a Massey product on $f^*(A)$ or its formality. However, if A admits a Massey product of the form $\langle x,x,...,x\rangle_{\mu_n^c}$, referred to in the literature as *Kraine's* $\langle x\rangle^n$ *product*, or *iterated* Massey product, then it follows that it admits a product of the form $\langle x,...,x\rangle_{\sum_{\sigma\in\mathbb{S}_n}\mu_n^c\cdot\sigma}$, and so we can deduce that $f^*(A)$ is not formal. □

Example 2.25. We can use Prop 2.22 to compute the Massey products for the commutative operad Com. Consider the canonical weighted operad map $f : \mathsf{Ass} \to \mathsf{Com}$. As mentioned in the example before, the map $f^! : \mathsf{Ass}^! \to \mathsf{Com}^!$ is the linear dual of the natural operad morphism $g : \mathsf{Lie} \to \mathsf{Ass}$. This last operad map is an embedding, so it follows that $f^!$ is surjective. Thus, for any $\tau \in \left(\mathsf{Com}^!\right)^{(n)}(r)$ there exists $\mu \in \left(\mathsf{Ass}^!\right)^{(n)}(r)$ such that $f^!(\mu) = \tau$, and it follows from Proposition 2.22 that

$$\langle x_1,...,x_k\rangle_\tau \subseteq \langle x_1,...,x_k\rangle_\mu,$$

whenever the products above make sense. □

Pushforwards of Massey products. For any $\mathscr{P}$-algebra A, the $\mathscr{P}$-Massey products on A induce $\mathscr{Q}$-Massey products on $f_!(A)$.

Proposition 2.26. *Let $f : \mathscr{P} \to \mathscr{Q}$ be a morphism of weighted operads, A a $\mathscr{P}$-algebra, and $\Gamma^c \in \left(\mathscr{P}^!\right)^{(n)}(r)$. Suppose that $x_1,...,x_r \in H_*(A)$ are homogeneous elements such that the Massey product set $\langle x_1,...,x_r\rangle_{\Gamma^c}$ is defined. Then, the $\mathscr{Q}$-Massey product set $\langle h_*(x_1),...,h_*(x_r)\rangle_{f^!(\Gamma^c)}$ is also defined, and*

$$h_*\langle x_1,...,x_r\rangle_{\Gamma^c} \subseteq \langle h_*(x_1),...,h_*(x_r)\rangle_{f^!(\Gamma^c)}.$$

Proof. One constructs a defining system for a $f^!(\Gamma^c)$-Massey product in essentially the same manner as in the proof of Proposition 2.22.

Let $x \in \langle x_1, \dots, x_r \rangle_{\Gamma^c}$ have a defining system $\{b_{\mu,(i_1,\dots,i_r)}\}$. We shall prove the statement by constructing a defining system for x as a $f^!(\Gamma^c)$-Massey product. If $f^!$ is injective, then the statement is trivial. Indeed, since D commutes with $f^!$, we may obtain the desired defining set $\left\{b_{f^!(\mu),(i_1,\dots,i_r))}\right\}$ by setting $b_{f^!(\mu),(i_1,\dots,i_r))} := h\left(b_{\mu,(i_1,\dots,i_r))}\right)$. If $f^!$ is not injective, then the set before may fail to be a defining system. Two problems may arise. Firstly, $f^!(\mu)$ may be zero. In this case, however, any term coming from D in which $f^!(\mu)$ plays a role will also vanish, so we may safely remove any term of the form $b_{f^!(\mu),(i_1,\dots,i_r))}$ from the defining system altogether. Secondly, $f^!$ may fail to preserve linear independence. We fix this problem as follows.

To fix notation, write $\mathscr{P} = \mathscr{F}(E,R)$ and $\mathscr{Q} = \mathscr{F}(F,S)$. The map $f^!$ is a map of weighted quadratic cooperads and, in particular, it sends cogenerators to cogenerators, $\left(f^!\right)^{(1)} : E \to F$. We shall assume that $\Bbbk$-linear bases of E and F are chosen such that the image of the basis elements of E are precisely the first m basis vectors $\{u_i\}$ of F, and further that the other basis elements of F are not in the image of $\left(f^!\right)^{(1)}$, and that the rest of the elements of the basis of E are in the kernel of $\left(f^!\right)^{(1)}$. These bases now, as explained in the second paragraph of Section 2, extend to bases of the operads $\mathscr{P}$ and $\mathscr{Q}$ using appropriate symmetric tree monomials.

The image of $f^!$ now entirely lies in the span of tree monomials labeled by the first m basis elements of F. This means that there is now a canonical (with respect to this choice of basis) linear section s of $f^!$ defined only on this codomain that preserves the cocomposition. This section is given by sending sums of tree monomials labeled by the first m basis elements of F to sums of tree monomials of the same shape labeled by the corresponding first m basis elements of E.

The section s induces a bijection between the indexing sets $I\left(f^!(\Gamma^c),(1,\dots,r)\right)$ and $I\left(\Gamma^c,(1,\dots,r)\right)$. Define $b_{\mu,(i_1,\dots i_n)}$ to be $\left(b_{s(\mu),(i_1,\dots i_n)}\right)$. This provides a defining system for x. □

Example 2.27. Consider the natural operad map $f : \mathsf{Lie} \to \mathsf{Ass}$. For any differential graded Lie algebra $\mathfrak{g}$, the differential graded associative algebra $f_!(\mathfrak{g})$ is the universal enveloping algebra of A. Recall that there is an embedding of graded vector spaces (in fact, graded Lie algebras) $h : \mathfrak{g} \to f_!(\mathfrak{g})$. Since on the level of Koszul dual cooperads, the map $f^! : \mathsf{Lie}^! \to \mathsf{Ass}^!$ is the linear dual of the forgetful functor $\mathsf{Ass} \to \mathsf{Com}$, one can verify that $f^!(\tau_n^c) = \sum_{\sigma\in\mathbb{S}_n} \mu_n^c \cdot \sigma$, where μ_n^c is the generator of $\left(\mathsf{Ass}^!\right)^{(n-1)}(n)$ as an $\mathbb{S}$-module. Therefore,

$$h_*\langle x_1,\dots,x_k\rangle_{\tau_n^c} \subseteq \langle h_*(x_1),\dots,h_*(x_k)\rangle_{\sum_{\sigma\in\mathbb{S}_n}\mu_n^c\cdot\sigma}.$$

□

A criterion for formality. In this section, we characterize the formality of a $\mathscr{P}$-algebra in terms of its Sullivan model, whenever it makes sense (Prop. 2.29 below). Although the result is presumably well-known to experts, we could not find a precise statement in the literature. The connection of the characterization with this paper is that it gives us a method to construct non-formal algebras with vanishing higher operadic Massey products of all orders. We leave the task of finding explicit examples to the interested reader.

The Sullivan model of a $\mathscr{P}$-algebra exists after imposing some connectivity assumptions on the operad and the algebra itself. To our knowledge, the first work in this direction is [15], where $\mathscr{P}$ is assumed to be Koszul and concentrated in degree 0, while the most general results are achieved in [6], were $\mathscr{P}$ is not required to be Koszul, but satisfy a mild connectivity requirement called being *tame*. We stick to the setting of [6], but will also require $\mathscr{P}$ to be Koszul to make use of infinity structures. An operad $\mathscr{P}$ is *r-tame* for a fixed integer $r \geq 0$ if for every $n \geq 2$,

$$\mathscr{P}(n)_q = 0 \text{ for all } q \geq (n-1)(1+r).$$

The operads $\mathsf{Ass}, \mathsf{Com}$ and Lie are examples of 0-tame operads, as well as their minimal models. The Gerstenhaber operad Gerst is 1-tame. The main results of [6] combine to read as follows.

Theorem 2.28. [6] *Every r-connected algebra over an r-tame operad has a Sullivan minimal model, unique to isomorphism.*

Now, suppose that $\mathscr{P}$ is an r-tame Koszul operad and A is an r-connected finite type algebra for some $r \geq 0$. Furthermore, suppose that A is $\mathscr{P}_\infty$-quasi-isomorphic to a minimal $\mathscr{P}_\infty$-algebra H with differential δ whose components $\delta^{(n)}$ vanish for all $n \geq 2$. Then there is a quasi-isomorphism of $\mathscr{P}^{\text{¡}}$-coalgebras

$$(\mathscr{P}^{\text{¡}}(A), \delta) \xrightarrow{\simeq} (\mathscr{P}^{\text{¡}}(H), \delta').$$

Taking the linear dual, one obtains a quasi-isomorphism of $\mathscr{P}^!$-algebras

$$(\mathscr{P}^!(H), d') \xrightarrow{\simeq} (\mathscr{P}^!(A), d).$$

The differential d' is decomposable and concentrated in weight 2. Therefore, $(\mathscr{P}^!(H), d)$ is a minimal Sullivan model for $(\mathscr{P}^!(A), d)$, which is the dual of the bar construction on A. This model is unique up to isomorphism, as mentioned before. We sum this discussion up in the following characterization.

Proposition 2.29. *Let $\mathscr{P}$ be an r-tame Koszul operad for some $r \geq 0$, and A an r-connected finite type $\mathscr{P}$-algebra. Then A is formal if, and only if, the Sullivan minimal model of the dual of the bar construction on A admits a differential concentrated in weight 2.*

3 Differentials in the $\mathscr{P}$-Eilenberg–Moore spectral sequence

Aside from providing obstructions to formality, one of the major uses of higher Massey products is in providing a concrete description of the differentials in the classical Eilenberg–Moore spectral sequence. The following is a classical result of May [20], compare also [27], but adapted to the notation of this paper.

Theorem 3.1. *Let A be a differential graded associative algebra, and let $x_1, \dots x_n$ be homology classes such that the Massey product set $\langle x_1, ..., x_n \rangle$ is non-empty. Then, the element $[sx_1 \mid \cdots \mid sx_n]$ survives to the E^{n-1}-page of the Eilenberg–Moore spectral sequence of A, and furthermore, the suspension sx of any representative of $x \in \langle x_1, \dots x_n \rangle$ is a representative for $d^{n-1}[sx_1 \mid \cdots \mid sx_n]$.*

An analogous statement for differential graded Lie algebras appears in [2]. Our following result generalizes these statements to all algebras over a Koszul (in fact, quadratic) operad. Recall from Section 1.1.1 the construction of the spectral sequence. We will sometimes confuse homology classes with representatives to lighten the notation.

Theorem 3.2. *Let A be a $\mathscr{P}$-algebra, and $x_1, \dots x_r$ homology classes such that the Massey product set $\langle x_1, ..., x_r \rangle_{\Gamma^c}$ is defined for a cooperation $\Gamma^c \in \mathscr{P}^{\text{¡}}(r)^{(n)}$. Then the element*

$$\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r \in \left(\mathscr{P}^{\text{¡}}\right)^{(n)}(r) \otimes H_*(A)^{\otimes r}$$

survives to the E^{n-1} page in the $\mathscr{P}$-Eilenberg–Moore spectral sequence, and for $x \in \langle x_1, \dots x_n \rangle$

$$d^{n-1}\left(\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r\right) \in (-1)^{n-2}\,[\mathrm{id} \otimes x]\,.$$

Our proof of this is an adaption of the classical one, so we shall therefore make use of the Staircase Lemma [13, Lemma 2.1], which we briefly recall next.

Lemma 3.3. *Let $A = \left(A_{*,*}, d', d''\right)$ be a bicomplex, denote by d the differential on its total complex, and fix $c_1, \dots, c_n$ homogeneous elements in A. Suppose that $d'c_s = d''c_{s+1}$ for $1 \leq s \leq n-1$, and define $c := c_1 - c_2 + \cdots + (-1)^{n-1}c_n$. Then, $dc = d'c + d''c = d''c_1 + (-1)^{n-1}d'c_n$, and furthermore, in the spectral sequence $\left\{(E^r, d^r)\right\}$ associated to the bicomplex, if $d''c_1 = 0$ then c_1 survives to E^n, and $d^n[c_1] = (-1)^{n-1}[d'c_n]$.*

Our approach to proving Theorem 3.2 is therefore to construct a sequence $c_1, \dots c_{r-1}$ satisfying the conditions of the Staircase Lemma.

Proof of Theorem 3.2. First, fix a defining system $\{a_{\mu,(k_1,\dots,k_i)}\}$ for the element $x \in \langle x_1,\dots,x_r\rangle_{\Gamma^c}$. For each s between 1 and $n-1$, we will define c_s in terms of this defining system and the auxiliary maps

$$\triangle_s : \mathscr{P}^{\mathrm{i}} \xrightarrow{\triangle} \mathscr{P}^{\mathrm{i}} \circ \mathscr{P}^{\mathrm{i}} \xrightarrow{p_s \circ \mathrm{id}} \left(\mathscr{P}^{\mathrm{i}}\right)^{(s)} \circ \mathscr{P}^{\mathrm{i}},$$

where p_s is the projection onto the weight s component. More precisely, the element c_s is defined as

$$c_s := \sum \left[\zeta^{(n-s)} \otimes a_{\mu_1,(\sigma^{-1}(1),\dots\sigma^{-1}(i_1))} \otimes \cdots \otimes a_{\mu_m,(\sigma^{-1}(i_1+\cdots+i_{m-1}+1),\dots\sigma^{-1}(r))}\right],$$

where $\triangle_{n-s}(\Gamma^c) = \sum\left(\zeta^{(s)};\mu_1,\dots,\mu_m;\sigma\right)$. In particular, $c_1 = [\Gamma^c \otimes a_{\mathrm{id},(1)} \otimes \cdots \otimes a_{\mathrm{id},(r)}]$, and

$$c_{n-1} = \sum\left(s\zeta^{(1)}\right) \otimes a_{\mu_1,(\sigma^{-1}(1),\dots,\sigma^{-1}(\nu_1))} \otimes \cdots \otimes a_{\mu_m,(\sigma^{-1}(\nu_1+\cdots+\nu_{m-1}+1),\dots,\sigma^{-1}(r))},$$

where $D(\Gamma^c) = \sum\left(\zeta^{(1)};\mu_1,\dots,\mu_m;\sigma\right)$ with $\zeta^{(1)} \in E$ and thus $s\zeta^{(1)} \in sE \subset \left(\mathscr{P}^{\mathrm{i}}\right)^{(1)}$. To finish, we must verify that the conditions of the Staircase Lemma 3.3 are met. Denote by ∂ the external differential on $\mathscr{P}^{\mathrm{i}}(A)$, and by $d^\bullet$ its internal differential. Then, since $d(a_{\mathrm{id},(i)}) = 0$ for each i, it follows that $d^\bullet c_1 = 0$. A routine calculation shows that $d^\bullet c_{s+1} = \partial c_s$ for each s. It follows from the Staircase Lemma that

$$d_{n-1}[c_1] = (-1)^n[\partial c_{n-1}] = (-1)^n[x].$$

In the expression of c_{n-1}, the element ζ is in the image of the twisting morphism κ. In particular, this implies that it is of weight 1, and so $\partial c_{n-1} \in \mathrm{id} \otimes \langle x_1,\dots,x_r\rangle_{\Gamma^c}$. This finishes the proof. □

The formality of a dg algebra of some type is well-known to be related to the collapse of the associated Eilenberg–Moore-type spectral sequence, see for instance [10] for the commutative case, and [9] for the Lie case. The following statement generalizes these results.

Theorem 3.4. *The Eilenberg–Moore spectral sequence of a formal $\mathscr{P}$-algebra over a Koszul operad collapses at the E^2-page. The same is true for formal $\mathscr{P}_\infty$-algebras.*

Proof. Since every $\mathscr{P}$-algebra is a $\mathscr{P}_\infty$-algebra, we prove the result for $\mathscr{P}_\infty$-algebras. Let A be a formal $\mathscr{P}_\infty$-algebra, and denote by $H = H_*(A)$ its homology as a $\mathscr{P}$-algebra. Then there are $\mathscr{P}_\infty$-quasi-isomorphisms $A \leftrightarrows H$, or equivalently, $\mathscr{P}^{\mathrm{i}}$-coalgebra quasi-isomorphisms

$$\mathscr{P}^{\mathrm{i}}(A) \leftrightarrows \mathscr{P}^{\mathrm{i}}(H).$$

Recall that the codifferential $\delta_H(\mu,-)$ of $\mathscr{P}^{\mathrm{i}}(H)$ vanishes unless $\mu \in \mathscr{P}^{\mathrm{i}}$ has weight 1. By comparison, both Eilenberg–Moore spectral sequences are isomorphic from the first page. Therefore, it suffices to consider the case where A has no internal differential and $\delta_A^{(i)}$ vanishes when $i \neq 1$. We now check that the differential d^i in the Eilenberg–Moore spectral sequence vanishes for $i \geq 2$. To do so, we will use the standard relative cycles and boundaries spaces,

$$Z_p^r = F_p\mathscr{P}^{\mathrm{i}}(A) \cap \delta^{-1}\left(F_{p-r}\mathscr{P}^{\mathrm{i}}(A)\right) \quad \text{and} \quad D_p^r = F_p\mathscr{P}^{\mathrm{i}}(A) \cap \delta\left(F_{p+r}\mathscr{P}^{\mathrm{i}}(A)\right).$$

The differential d^r in the successive pages of the spectral sequence is induced by the restrictions of the differential δ of $\mathscr{P}^{\mathrm{i}}(A)$ to Z_p^r, as shown below:

$$\begin{array}{ccc} Z_p^r & \xrightarrow{\quad\delta\quad} & Z_{p-r}^r \\ \downarrow & & \downarrow \\ E_p^r = Z_p^r/Z_{p-1}^{r-1} + D_p^{r-1} & \xrightarrow{\quad d^r\quad} & E_{p-r}^k = Z_{p-r}^r/Z_{p-r-1}^{r-1} + D_{p-r}^{r-1} \end{array}$$

Fix some $r \geq n$. To check that $d^r = 0$, we will fix an element $x \in Z_p^r$ and find a representative y of the class $[x] \in E_p^r$ such that

$$\delta(y) \in Z_{p-r-1}^{r-1} + D_{p-r}^{r-1} = F_{p-r-1}\mathscr{P}^{\mathrm{i}}(A) \cap \delta^{-1}\left(F_{p-r-1}\mathscr{P}^{\mathrm{i}}(A)\right) + F_{p-r}\mathscr{P}^{\mathrm{i}}(A) \cap \delta\left(F_{p-1}\mathscr{P}^{\mathrm{i}}(A)\right).$$

Indeed, write $x = x_1 + \cdots + x_p$ where each $x_i \in \mathscr{P}^{\text{¡}}(i) \otimes A^{\otimes i}$. Now, since $\delta(x) \in F_{p-r}\mathscr{P}^{\text{¡}}(A)$, it follows that $\delta\left(x_{p-r+1} + \cdots + x_p\right) = 0$. Thus, we take $y = x - \left(x_{p-r+1} + \cdots + x_p\right)$ as a representative of the form we needed, finishing the proof. □

The converse to Theorem 3.4 in general is not true, it fails even in the associative case and some examples are computed in some of the references given before the statement of the theorem.

Remark 3.5. Massey products sometimes completely determine formality. For the case of the dual numbers operad, the Massey products are precisely the differentials in the spectral sequence associated to the bicomplex. So if the differentials all vanish, the spectral sequence must collapse on the E^2-page.

4 Higher-order operadic Massey products and $\mathscr{P}_\infty$-structures

In this section, we fix a Koszul operad $\mathscr{P}$. In this case, there is a natural relationship between the higher-order operadic Massey products and $\mathscr{P}_\infty$-structures on the homology of the $\mathscr{P}$-algebras.

Let A be a $\mathscr{P}$-algebra, and denote by H its homology. Since $\mathscr{P}$ has no operadic differential, H is a $\mathscr{P}$-algebra in a natural way. It is well-known that the homotopy transfer theorem (in its various forms) extend this $\mathscr{P}$-algebra structure on H to a $\mathscr{P}_\infty$-structure that retains the quasi-isomorphism class of A as a $\mathscr{P}_\infty$-algebra. In this paper, we mainly focus on D. Petersen's extension [23] of T. Kadeishvili's classical transfer theorem [11], which is recalled in Theorem 1.2. See also [16, Section 10.3]. It is a common misconception to expect that higher-order Massey products sets of the sort $\langle x_1, ..., x_r\rangle$ are related to $\mathscr{P}_\infty$-structure maps θ_r induced on the homology H via the homotopy transfer theorem by the clean formula

$$\pm\theta_r(x_1, ..., x_r) \in \langle x_1, ..., x_r\rangle.$$

At this level of generality, the assertion is incorrect. However, it is true for secondary Massey products, as shown in [22, Theorem 3.9].

Let us make the connection between infinity structures and higher-order Massey products more precise. First, recall that codifferentials on the cofree conilpotent $\mathscr{P}^{\text{¡}}$-coalgebra $\mathscr{P}^{\text{¡}}(A)$ are in bijective correspondence with $\mathscr{P}_\infty$-structures on the chain complex A [16, Theorem 10.1.13].

Definition 4.1. Let A be a $\mathscr{P}$-algebra, $\Gamma^c \in \left(\mathscr{P}^{\text{¡}}\right)^{(n)}(r)$, and $x_1, ..., x_r$ homogeneous elements of $H = H_*(A)$ for which the Γ^c-Massey product set $\langle x_1, \ldots, x_r\rangle_{\Gamma^c}$ is defined. A given $\mathscr{P}_\infty$ structure δ on H for which A and H are quasi-isomorphic is said to *recover* the Massey product element $x \in \langle x_1, \ldots, x_r\rangle_{\Gamma^c}$ if, up to sign,

$$\delta_r\left(\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r\right) = x.$$

We begin by showing below that given a higher-order Massey product $x \in \langle x_1, \ldots, x_r\rangle_{\Gamma^c}$, there is always a choice of $\mathscr{P}_\infty$ structure on H quasi-isomorphic to A which recovers x. In general, however, an arbitrary $\mathscr{P}_\infty$ structure on H quasi-isomorphic to A only recovers a given higher-order Massey product element up to multiplications of lower arity. Our proof strategy is very similar to the proof in [5], where the authors demonstrated this result in the associative case. In the result below, we require the operad to be reduced for Theorem 1.2 to apply.

Theorem 4.2. *Let A be an algebra over a reduced Koszul operad $\mathscr{P}$, and let H be its homology. Let $\Gamma^c \in \left(\mathscr{P}^{\text{¡}}\right)^{(n)}(r)$, and assume that $x_1, ..., x_r$ are $r \geq 3$ homogeneous elements of H for which the Γ^c-Massey product set $\langle x_1, \ldots x_r\rangle_{\Gamma^c}$ is defined. Let $x \in \langle x_1, \ldots x_r\rangle_{\Gamma^c}$. Then:*

(i) For any $\mathscr{P}_\infty$ structure δ on H quasi-isomorphic to A, we have

$$\delta^{(n)}\left(\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r\right) = x + \Phi,$$

where $\Phi \in \sum_{i=1}^{n-1} \operatorname{Im}\left(\delta^{(i)}\right)$.

(ii) If $\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_l}$ are linearly independent in the corresponding copy of $\mathscr{P}^{\mathsf{i}} \otimes_{\mathbb{S}_l} A^{\otimes l}$, where $(\mu, (i_1, ..., i_l)) \in I(\Gamma^c)$, then there is a choice of $\mathscr{P}_\infty$ structure δ on H which recovers x.

Proof. (i) We will construct a $\mathscr{P}_\infty$ structure on H recovering x via the procedure established in the proof of Theorem 1.2. We shall continue to use the notation of that proof. First, we choose a defining system $\{a_\alpha\}$ for the Massey product element $x \in \langle x_1, \ldots, x_r \rangle_{\Gamma^c}$. We proceed by induction on arity, starting in arity 1 with δ^1 initially defined as the coderivation corresponding to the strict $\mathscr{P}$-algebra structure on H induced from it being the homology of A, and defining f as any chain quasi-isomorphism $H \to A$ extending the choice $f(x_1) = a_{\mathrm{id},(1)}, f(x_2) = a_{\mathrm{id},(2)}, \ldots, f(x_r) = a_{\mathrm{id},(r)}$. This defines a map $F_1 : \mathscr{P}^{\mathsf{i}}(1) \otimes H \to A$, since $\mathscr{P}^{\mathsf{i}}(1) = \Bbbk$. We give next the arity 2 step. This step is not needed for the inductive procedure, but we include it because we think it sheds light on the general case. Recall that the algorithm of Theorem 1.2 automatically determines the multiplication on H, but there are choices for F_2. First, we make the following observation. If $\left(s\mu, (i, j)\right)$ appears in the Γ^c-indexing system, then $\overline{\gamma_A}(\mu; x_i, x_j) = 0$. This is because $D\left(s\mu\right) = \left(\mu, \mathrm{id}, \mathrm{id}\right)$, and therefore

$$d a_{s\mu,(i,j)} = \kappa\left(s\mu\right)\left(a_{\mathrm{id},(i)}, a_{\mathrm{id},(j)}\right),$$

which implies that $\overline{\gamma_A}(\mu; x_i, x_j) \in H$ admits a lift to A which is a coboundary, which implies that it is 0 on homology. It therefore follows that $\left(F^1 \circ \delta^1\right)_2$ is 0 when applied to $[s\mu \otimes x_i \otimes x_j]$. On the other hand,

$$\left(\delta^1 \circ F^1\right)_2 [s\mu \otimes x_i \otimes x_j] = \kappa\left(s\mu\right)\left(F_1\left(x_i\right), F_1\left(x_j\right)\right) = d a_{s\mu,(i,j)}.$$

So we choose $F_2 : \mathscr{P}^{\mathsf{i}}(2) \otimes H^{\otimes 2} \to A$ to extend the choice $F_2\left(\mu, x_i, x_j\right) = a_{\mu,(i,j)}$. The general case is similar. Our inductive hypothesis has the following two parts:

$$F_l\left[\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_l}\right] = a_{\mu,(i_1,\ldots,i_l)}, \text{where} \left(\mu, (i_1, \ldots i_l)\right) \in I\left(\Gamma^c\right) \text{and } l < n, \tag{10}$$

and

$$\delta_l^{k-1}\left[\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_l}\right] = 0, \text{ where } \left(\mu, (i_1, \ldots i_l)\right) \in I\left(\Gamma^c\right) \text{ for } l \leq n. \tag{11}$$

We verified these two items in the arity 2 case in the previous paragraph. Next, we shall compute $(\delta_A \circ F)_n\left[\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}\right]$. The map $(\delta_A \circ F)_n$ is precisely the composite

$$\mathscr{P}^{\mathsf{i}}(H) \xrightarrow{\Delta(H)} \mathscr{P}^{\mathsf{i}} \circ \mathscr{P}^{\mathsf{i}}(H) \xrightarrow{\mathscr{P}^{\mathsf{i}}(f)} \mathscr{P}^{\mathsf{i}}(A) \xrightarrow{\kappa(A)} \mathscr{P}(A) \xrightarrow{\gamma_A} A.$$

The arity n component of f is 0, and in particular $\mathscr{P}^{\mathsf{i}}\left(f\right)\left(\mathrm{id}; \mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}\right) = 0$. It follows that $(\delta_A \circ F)_n$ sends the class of $\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}$ to the same element as the following map does:

$$\mathscr{P}^{\mathsf{i}}(H) \xrightarrow{\Delta^+} \mathscr{P}^{\mathsf{i}} \circ \mathscr{P}^{\mathsf{i}}(H) \xrightarrow{\mathscr{P}^{\mathsf{i}}(f)} \mathscr{P}^{\mathsf{i}}(A) \xrightarrow{\kappa(A)} \mathscr{P}(A) \xrightarrow{\gamma_A} A.$$

The map above is tightly related to the Massey inductive map D. Indeed, the image of $\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}$ is given by

$$\sum \zeta\left(f\left(\zeta_1 \otimes x_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes x_{i_{\sigma^{-1}(v_1)}}\right), \ldots, f\left(\zeta_m \otimes x_{i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)}} \otimes \cdots \otimes x_{i_{\sigma^{-1}(n)}}\right)\right),$$

where

$$D\left(\mu\right) = \sum\left(\zeta; \zeta_1, \ldots, \zeta_m; \sigma\right).$$

By the first assumption of our inductive hypothesis (10), this is equal to

$$\sum \zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)}\right)}, \ldots, a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(n)}\right)}\right).$$

It follows from the definition of a defining system that this is equal to

$$d a_{\mu,(x_{i_1},\ldots,x_{i_n})}.$$

The second assumption of our inductive hypothesis (11) implies that $\left(F^{n-1}\circ\delta^{n-1}\right)_l\left[\mu\otimes x_{i_1}\otimes\cdots\otimes x_{i_l}\right]=0$, so we have that

$$\left(F\circ\delta^{n-1}-\delta_A\circ F\right)_n\left[\mu\otimes x_{i_1}\otimes\cdots\otimes x_{i_n}\right]=-d\,a_{\mu,(x_{i_1},\ldots,x_{i_n})}.$$

Therefore, there is no obstruction to obtaining a lift F_n such that $F_n(\mu\otimes x_{i_1}\otimes\cdots\otimes x_{i_n})=a_{\mu,(x_{i_1},\ldots,x_{i_n})}$. Notice that the algorithm also tells us that $\delta^n_n\left[\mu\otimes x_{i_1}\otimes\cdots\otimes x_{i_n}\right]=0$ (the projection of a boundary in homology).

Next, we shall verify that $\delta^n_{n+1}\left(\mu\otimes x_{i_1}\otimes\cdots\otimes x_{i_{n+1}}\right)=0$ when $(\mu,(i_1,\ldots,i_{n+1}))\in I\left(\Gamma^c\right)$. Because the arity $(n+1)$-component of δ^n comes from the $\mathscr{P}$-algebra structure induced on H from A, we have that

$$\delta^n\left(\mu\otimes x_{i_1}\otimes\cdots\otimes x_{i_{n+1}}\right)=\overline{\gamma_A}\left(\kappa\left(\mu\right);x_{i_1},\ldots,x_{i_{n+1}}\right).$$

But if $\kappa\left(\mu\right)$ is non-zero, then μ must be of weight 1. It then follows that $D\left(\mu\right)=(\mu;\mathrm{id},\ldots,\mathrm{id})$. So, by the same argument as in the arity 2 case, we conclude that $\overline{\gamma_A}\left(\kappa\left(\mu\right);x_{i_1},\ldots,x_{i_{n+1}}\right)=0$.

(ii) Consider any $\mathscr{P}_\infty$ quasi-isomorphism $H\xrightarrow{\simeq}A$ and the corresponding quasi-isomorphism of $\mathscr{P}^{\mathrm{i}}$-coalgebras

$$\mathscr{P}^{\mathrm{i}}(H)\xrightarrow{\simeq}\mathscr{P}^{\mathrm{i}}(A).$$

The induced morphism of $\mathscr{P}$-Eilenberg–Moore spectral sequences is, at the E_1 level, the identity on $\mathscr{P}^{\mathrm{i}}(H)$. By comparison, all the terms in both spectral sequences are also isomorphic. Now, it follows from Theorem 3.2 that if $\langle x_1,\ldots x_r\rangle_{\Gamma^c}$ is nonempty, then the element $[\Gamma^c\otimes x_1\otimes\cdots\otimes x_r]$ survives to the $(n-1)$-page (E^{n-1},d^{n-1}). Moreover, given any $x\in\langle x_1,\ldots x_r\rangle_{\Gamma^c}$, one has

$$d^{n-1}\overline{\Gamma^c\otimes x_1\otimes\cdots\otimes x_r}=(-1)^r\overline{x}.$$

Here, $\overline{\cdot}$ denotes the class in E^{n-1}. In other words, there exists $\Phi\in F_{n-1}\mathscr{P}^{\mathrm{i}}(H)$ such that

$$\delta_H\left(\Gamma^c\otimes x_1\otimes\cdots\otimes x_r+\Phi\right)=x.$$

Applying the counit $\epsilon_H:\mathscr{P}^{\mathrm{i}}(H)\to H$ to both sides, we obtain

$$m_H\left(\Gamma^c\otimes x_1\otimes\cdots\otimes x_r+\Phi\right)=x.$$

Write $m_H=\sum_{i\geq 2}\partial^{(i)}_H$, and decompose $\Phi=\sum_{i=2}^{r-1}\phi_i$ with $\phi_i\in\mathscr{P}^{\mathrm{i}}(H)^{(i)}$. By a word length argument,

$$\delta^{(n)}_H\left(\Gamma^c\otimes x_1\otimes\cdots\otimes x_r\right)+\sum_{i=2}^{r-1}\delta^{(i)}_H(\phi_i)=x.$$

This completes the proof. □

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