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# Separation of the orbits in representations of $SO_2$ and $O_2$ over $\mathbb{R}$ and $\mathbb{C}$

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## Abstract

We provide a minimal set of invariant polynomials separating all the orbits for representations of  $SO_2$  and  $O_2$  over  $\mathbb{C}$  and  $\mathbb{R}$ . We obtain very small cardinalities by selecting only polynomials of support of bounded dimension. Although they suffice for a lot of applications, the obtained separating sets are much smaller than the already known generating set. We also remark that real separating sets are smaller than the complex ones. We next stratify entirely the orbit space through the evaluation of the invariant polynomials.

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## 1 Introduction

We present minimal separating set for arbitrary representations of  $\mathrm{SO}_2$  and  $\mathrm{O}_2$  over  $\mathbb{C}$  (Theorems 3.4 and 6.1), and over  $\mathbb{R}$  (Theorems 4.2 and 5.1). Over the complex numbers, it means that our set separate closed orbits. Over the reals, it means more naturally that they separate all the orbits (Definition 4.1). We obtain significantly low cardinality: in  $\mathcal{O}(n^2)$  for  $\mathrm{SO}_2$  and  $\mathcal{O}(n^4)$  for  $\mathrm{O}_2$ , where  $n$  is the dimension of the representation. Weyl already provided a minimal separating set for a direct summand of standard representations [Wey46]. We thus generalize this result to arbitrary representations. The irreducible representations of  $\mathrm{O}_2(\mathbb{R})$  are indexed by  $\mathbb{N}$  (Appendix A) and induce a lot of arithmetic difficulties.

The generators of the invariant algebra possess these separation property. Yet they are of very large cardinality. For  $\mathrm{SO}_2(\mathbb{C}) \cong \mathbb{C}^*$ , these generators consist of monomials of large support in  $\{1, \dots, n\}$  and thus their numbers is up to  $2^n$ . Yet for separation we show that one can choose monomials with bounded support. We prove that monomials of support of cardinality three separate the orbits. This comes from the arithmetical key Lemma 2.4. We next deduce the same for the action of  $\mathrm{O}_2(\mathbb{C})$ , where invariants of support of cardinality at most four suffice. We thus find much smaller separating set contained in the generating set. Furthermore, we provide explicit expressions of the separating set whereas generating set are computed by heavy algorithms [Stu08; Des+23].

Despite of their smaller cardinalities, separating set can be used instead of generating set for many applications. For instance, a separating set induces an epimorphism onto the orbit space, and thus can be used to stratify the orbit space. A complete stratification of representations of  $\mathrm{SO}_2(\mathbb{R})$  and  $\mathrm{O}_2(\mathbb{R})$  is provided in Section 7.

Since the definition of separation is weaker over  $\mathbb{R}$ , the real separating set have significantly smaller cardinalities than the complex ones (see Example 4). Although few studied in invariant theory, this difference is important for many applications concerning only the real numbers. As an example, real separating set are needed to compute universal invariant or equivariant networks converging on any compact [DG24; BV23]. The size of the separating set corresponds to the dimension of the first layer, thus the cardinality heavily reflects on the complexity.

We also highlight a great advantage to use rational invariants. Rational monomials of support of cardinality two suffice to separate the orbits of  $\mathrm{SO}_2(\mathbb{C})$ , providing a rational separating set much smaller than the polynomial separating set (see Example 3.7). Although less numerous, it separates many more orbits which are not closed. [Bür+21] yet study rational invariants to separate orbits of toric actions as the action of  $\mathrm{SO}_2(\mathbb{C})$ . But the algorithms in [Bür+21] compute invariants for any support in  $\llbracket 1, n \rrbracket$ . A total set of separating invariants would be much too large to be computed by that way.

We recall that if one has a separating set for a reductive group, almost all set of  $2 \dim(\mathcal{V}) + 1$  linear combinations of the polynomials is a separating set [DK15]. But as discussed in [Kem24, Remark 6.2] it still is relevant to compute minimal separating set of higher cardinal. Indeed the linear combinations remove useful properties of the polynomials (the homogeneity or the low dimension of the supports).

We start giving some technical lemmas in Section 2. Separating sets are presented in Sections 3, 4, 5 and 5. We finish by the stratification in Section 7. Essential statements about linear representations are provided in Appendix A. For a real vector space  $\mathcal{V}$ , we note  $\hat{\mathcal{V}} = \mathcal{V} \oplus i\mathcal{V}$  its complexification.

## 2 Arithmetics and wheels

The proofs of next sections rely on some technical lemmas of arithmetic. They describe how complex wheels intersects. In particular, the idea to keep only invariants of support two results from Lemma 2.4 and its Corollary.

**Definition 2.1.** For some  $c \in \mathbb{C}^*$  and  $a \in \mathbb{Z} \setminus \{0\}$ , we define  $W_c^a$  the wheel of  $a^{\text{th}}$  roots of  $c$ :  $W_c^a := \{z \in \mathbb{C}^* \mid z^a = c\}$ .

Let  $u, v$  be two non zero vectors in the representation  $\mathcal{L}_a$  for some  $a \in \mathbb{Z}^*$ . Then,  $W_{\frac{v}{u}}^a$  is the set of complex numbers  $z$  such that  $\phi_a(u) = v$ . For two integers  $v, v' \in \mathbb{Z}$ , we note  $v \wedge v'$  their greatest common divisor and  $v \vee v'$  their least positive common multiple.

**Lemma 2.2.** Let  $u, u', v, v'$  be integers and take the two set  $U := \{u + kv, k \in \mathbb{Z}\}$  and  $U' := \{u' + kv', k \in \mathbb{Z}\}$ . There is the equivalence:

$$U \cap U' \neq \emptyset \Leftrightarrow (v \wedge v') \text{ divides } (u - u')$$

*Proof.* The Bezout identity ensures that  $v \wedge v'$  is the least positive integer which is a linear combination of  $v$  and  $v'$ . Hence, for any integer  $u, \exists k, k' \in \mathbb{Z}, u = kv + k'v' \Leftrightarrow (v \wedge v')$  divides  $u$ . Then,  $(v \wedge v')$  divides  $(u - u')$  iff  $\exists k, k' \in \mathbb{Z}, u - u' = kv + k'v'$ . This is equivalently  $\underbrace{u - kv}_{\in U} = \underbrace{u' + k'v'}_{\in U'}$ .  $\square$

**Corollary 2.3.** Let  $c, c'$  be two complex numbers in  $\mathbb{C}^*$  and  $a, a'$  non zero integers. Then we have

$$W_c^a \cap W_{c'}^{a'} \neq \emptyset \Leftrightarrow c^{\frac{a \vee a'}{a}} = c'^{\frac{a \vee a'}{a'}}$$

In that case  $W_c^a \cap W_{c'}^{a'} = W_{c^{\frac{a \vee a'}{a}} c'^{\frac{a \vee a'}{a'}}}^{a \vee a'}$ , where  $q, q'$  are Bezout coefficients satisfying  $aq + a'q' = a \wedge a'$ .

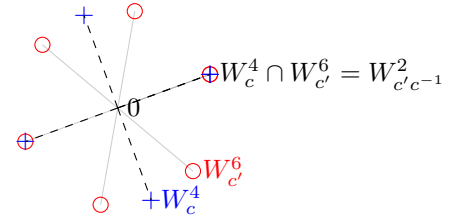


Figure 1: Intersection of wheels

*Proof.* Let  $w$  be a root in  $W_{c'}^{a'}$ . Recall that  $aa' = (a \vee a')(a \wedge a')$ . Then,  $w^{a \vee a'} = w^{\frac{aa'}{a \wedge a'}} = c^{\frac{a}{a \wedge a'}} = c^{\frac{a \vee a'}{a'}}$ . Thus,

$$c^{\frac{a \vee a'}{a}} = c'^{\frac{a \vee a'}{a'}} \Leftrightarrow W_c^a \subset W_{c^{\frac{a \vee a'}{a}} c'^{\frac{a \vee a'}{a'}}}^{a \vee a'} \text{ and } W_{c'}^{a'} \subset W_{c^{\frac{a \vee a'}{a}} c'^{\frac{a \vee a'}{a'}}}^{a \vee a'}$$

Note  $\varphi : \mathbb{Z} \rightarrow W_{c^{\frac{a \vee a'}{a}} c'^{\frac{a \vee a'}{a'}}}^{a \vee a'}$  a surjective map with, for any integer  $k, \varphi(k) = \varphi(0) * \exp(i \frac{2\pi k}{a \vee a'})$ . Then the preimage  $\varphi^{-1}(W_c^a)$  is some  $U = \{u + \frac{a \vee a'}{a} k, k \in \mathbb{Z}\}$  meanwhile the preimage  $\varphi^{-1}(W_{c'}^{a'})$  is some  $U' =$

$\{u' + \frac{a \vee a'}{a'} k, k \in \mathbb{Z}\}$ . However,  $\left(\frac{a \vee a'}{a}\right) \wedge \left(\frac{a \vee a'}{a'}\right) = 1$ . According to lemma 2.2, one has  $U' \cap U \neq \emptyset$ . This gives  $W_c^a \cap W_{c'}^{a'} \neq \emptyset$ , and one deduces that  $W_c^a \cap W_{c'}^{a'} = W_z^{a \wedge a'}$  for some unknown complex  $z \in \mathbb{C}^*$ .

Nevertheless, for any  $w \in W_c^a \cap W_{c'}^{a'} = W_z^{a \wedge a'}$  one has  $c = w^a = w^{\frac{a a'}{a'}} = w^{(a \wedge a') \frac{a \vee a'}{a'}} = z^{\frac{a \vee a'}{a'}}$  and  $c' = w^{a'} = w^{\frac{a a'}{a}} = w^{(a \wedge a') \frac{a \vee a'}{a}} = z^{\frac{a \vee a'}{a}}$ . If one choose, thanks to Bezout identity,  $q$  and  $q'$  such that  $a q + a' q' = a \wedge a'$ , then  $q \frac{a \vee a'}{a'} + q' \frac{a \vee a'}{a} = 1$ . That is,  $z = z^{q \frac{a \vee a'}{a'} + q' \frac{a \vee a'}{a}} = c^q c'^{q'}$ .  $\square$

The following lemma is key in selecting invariants of low support:

**Lemma 2.4.** *Let  $U_1, \dots, U_n$  be set as in Lemma 2.2 such that  $\forall 1 \leq i < j \leq n, U_i \cap U_j \neq \emptyset$ . Then,  $\bigcap_{i=1}^n U_i \neq \emptyset$ .*

*Proof.* We proceed by induction. The lemma holds for  $n = 2$ . Take  $n > 2$  such that the lemma holds for  $n - 1$ . Then,  $\bigcap_{i=1}^{n-1} U_i \neq \emptyset$  and one can rewrite  $u_i = 0$  for  $i \leq n - 1$ . Then,  $\bigcap_{i=1}^{n-1} U_i = \{kM, k \in \mathbb{Z}\}$  with  $M := \text{lcm}(r_1, \dots, r_{n-1})$ . By Lemma 2.2 we have  $U_i \cap U_n \neq \emptyset \Leftrightarrow (r_i \wedge r_n)$  divides  $(u_i - u_n) \Leftrightarrow (r_i \wedge r_n)$  divides  $u_n$ . For all prime number  $p$  and integer  $u$  we note  $\nu_p(u) = \max\{k \in \mathbb{N} \mid p^k \text{ divides } u\}$  the  $p$ -adic valuation of  $u$ .

$$\nu_p(u_n) \geq \max_{i \leq n-1} (\nu_p(r_i \wedge r_n)) \geq \max_{i \leq n-1} [\min(\nu_p(r_i), \nu_p(r_n))] \geq \min[\max_{i \leq n-1} (\nu_p(r_i)), \nu_p(r_n)] = \nu_p(M \wedge r_n)$$

This is,  $M \wedge r_n$  divides  $u_n$ . Hence, apply the Lemma 2.2 again to have  $U_n \cap \bigcap_{i=1}^{n-1} U_i \neq \emptyset$ .  $\square$

**Corollary 2.5.** *Let  $\{W_i = W_{c_i}^{a_i}, i \in I\}$  be a finite set of wheels such that any pair of them has a non empty intersection:  $\forall i, j \in I, W_i \cap W_j \neq \emptyset$ . Then,  $\bigcap_{i \in I} W_i \neq \emptyset$ .*

*Proof.* Take  $\{W_i = W_{c_i}^{a_i}, i \in I\}$  such a set. The union  $\bigcup_{i \in I} W_i$  is contained in a big wheel  $W$  of cardinal  $M = \text{lcm}(a_i, i \in I)$ , and one defines a surjective map  $\varphi : \mathbb{Z} \rightarrow W$  with, for any integer  $k$ ,  $\varphi(k) = \varphi(0) \exp\left(i \frac{2\pi k}{M}\right)$ . For  $i \in I$ , one has the preimage  $U_i := \varphi^{-1}(W_i) = \left\{u_i + k \frac{M}{a_i}, k \in \mathbb{Z}\right\}$ . For any  $i, j \in I, W_i \cap W_j \neq \emptyset \Rightarrow U_i \cap U_j \neq \emptyset$ . According to lemma 2.4 then,  $\bigcap_{i \in I} U_i \neq \emptyset$ . This conversely gives  $\bigcap_{i \in I} W_i \neq \emptyset$ .  $\square$

Not all pairwise intersections are necessary to get an overall intersection. It depends on the values of the integers  $a_i$ , as it appears on figure 2 with 2, 3, -6.

**Proposition 2.6.** *Let  $W_1 = W_{c_1}^{a_1}, W_2 = W_{c_2}^{a_2}$  and  $W_3 = W_{c_3}^{a_3}$  be three wheels. Suppose that  $W_1 \cap W_2$  and  $W_1 \cap W_3$  are non empty. If  $a_1$  divides  $a_2 \vee a_3$ , then  $W_2 \cap W_3 \neq \emptyset$ .*

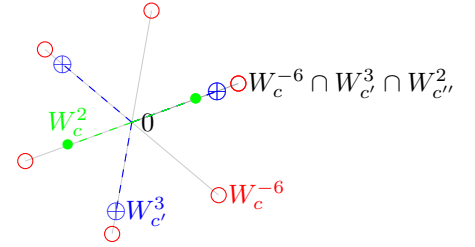


Figure 2: Automatic intersection

*Proof.* Define a surjective map  $\varphi : \mathbb{Z} \rightarrow W_1$  with, for any integer  $k$ ,  $\varphi(k) = \varphi(0) \exp\left(i \frac{2\pi k}{a_1}\right)$ . One has the preimages  $U_2 := \varphi^{-1}(W_1 \cap W_2) = \left\{u_2 + k \frac{a_1}{a_1 \wedge a_2}, k \in \mathbb{Z}\right\}$  and  $U_3 := \varphi^{-1}(W_1 \cap W_3) = \left\{u_3 + k \frac{a_1}{a_1 \wedge a_3}, k \in \mathbb{Z}\right\}$ . Lemma 2.3 claims that  $(U_2 \cap U_3 \neq \emptyset) \Leftrightarrow (M \text{ divides } u_2 - u_3)$ , where  $M = \left(\frac{a_1}{a_1 \wedge a_2} \wedge \frac{a_1}{a_1 \wedge a_3}\right)$ . A piece of  $p$ -adic calculus provides  $M = \frac{a_1}{a_1 \wedge (a_2 \vee a_3)}$ . Then  $a_1 \mid (a_2 \vee a_3) \Leftrightarrow M = 1$ , and  $M$  divides  $u_2 - u_3$ . Otherwise we choose  $c_1 = c_2 = 1$  and  $c_3 = \exp\left(\frac{2i\pi a_3}{a_1}\right)$ . Then,  $c_2^{\frac{a_2 \vee a_3}{a_2}} c_3^{\frac{a_2 \vee a_3}{a_3}} = \exp\left(\frac{2i\pi a_3}{a_1} \frac{a_2 \vee a_3}{a_3}\right) = \exp\left(\frac{2i\pi a_2 \vee a_3}{a_1}\right) \neq 1$ . Then lemma 2.3 claims that  $W_2 \cap W_3 = \emptyset$ .  $\square$

### 3 Separating invariants for $\mathrm{SO}_2(\mathbb{C})$

In this section we provide minimal separating set for arbitrary representations of  $\mathrm{SO}_2(\mathbb{C})$ . Let  $a_1, \dots, a_n$  be positive integers and  $a_{-1}, \dots, a_{-m}$  negative integers. We consider the representation of  $\mathrm{SO}_2(\mathbb{C})$  on

$$\hat{\mathcal{V}} = \bigoplus_{i=1}^m \mathcal{L}_{a_{-i}} \oplus \bigoplus_{i=1}^n \mathcal{L}_{a_i}$$

(the irreducible representations  $\mathcal{L}_k$  are defined in Appendix A). With the coordinates  $(v_{-m}, \dots, v_{-1}, v_1, \dots, v_n)$ , the rational monomial  $\mathbf{m}_e = \prod_{i=-m}^n v_i^{e_i}$  corresponds to the vector  $e = (e_{-m}, \dots, e_n)$  in  $\mathbb{Z}^{n+m}$ . Consider the row matrix  $A = (a_{-m}, \dots, a_n)$ . Then the module  $M_A = \{e \in \mathbb{Z}^{n+m} \mid Ae = 0\}$  corresponds to the Laurent monomials in  $\mathbb{C}[\hat{\mathcal{V}}]^{\mathrm{SO}_2(\mathbb{C})}$ . The monoid  $M_A^+ = \{e \in \mathbb{N}^{n+m} \mid Ae = 0\}$  corresponds to the monomials of positive degree in  $\mathbb{C}[\hat{\mathcal{V}}]^{\mathrm{SO}_2(\mathbb{C})}$  [Stu08, Lemma 1.4.2]. One naturally defines the support of a monomial  $\mathbf{m}_e$  as the set of indices  $1 \leq i \leq n$  such that  $e_i \neq 0$ .

We first exhibit a set of cardinality  $\mathcal{O}(n^2)$  composed of rational monomials of support of size two which separates any orbits of support of size more than two. (Theorem 3.1). Next we deduce a polynomial set of cardinality  $\mathcal{O}(n^3)$ , separating any closed orbits, by adding some monomials of support of size three (Theorem 3.4). The latter needs not be minimal for inclusion. We thus give a criterion to remove redundant polynomials (Proposition 3.5).

#### 3.1 With rational invariants

As pointed out in [Bür+21], rational invariants of tori separate many not closed orbits, something that is not possible with any set of polynomial invariants. Furthermore, a very low number of rational invariants allows to separate orbits.

Let  $u, v$  be two points in  $\hat{\mathcal{V}}$ . We note  $W_i$  the set of complex numbers  $z$  such that  $\phi_i(g_z^+)(u_i) = v_i$ . For some subscripts  $i, j$ , we note  $W_{ij} = W_i \cap W_j$ . When neither  $u_i$  nor  $v_i$  vanish, we note  $c_i = \frac{v_i}{u_i}$ , and then  $W_i = W_{c_i}^{a_i}$  is the wheel of  $a_i^{\mathrm{th}}$  roots of  $c_i$ .

**Theorem 3.1.** *For any  $-m \leq i < j \leq n$ , we note  $\varepsilon_{ij} = \frac{a_i \vee a_j}{a_i}$ . Consider the set of rational invariant*

$$f^* = \left\{ p_{ij} = v_i^{-\varepsilon_{ij}} \times v_j^{\varepsilon_{ji}}, \quad -m \leq i < j \leq n \right\}$$

and take two point  $u$  and  $v$  with at least two non zero coordinates. Then,

$$\exists P \in f^*, P(u) \neq P(v) \Leftrightarrow u, v \text{ are not in the same orbit.}$$

*Proof.* Firstly for any pair of integers  $-m \leq i < j \leq n$ , one has  $\frac{a_i \vee a_j}{a_i} a_i - \frac{a_i \vee a_j}{a_j} a_j = 0$ . Then  $p_{ij}$  is invariant.

Take some  $v \in \mathcal{C} \setminus \{0\}$ , and  $0 \leq i \leq n$  such that  $a_i \neq 0$ . Then, there exists some  $1 \leq j \leq m$  such that  $a_{-j} \neq 0$  and  $p_{-ij}(v) \neq 0$ . Hence the polynomials  $\{p_{i-j}, 1 \leq j \leq m\}$  separate orbits with  $v_i = 0$  from orbits with  $v_i \neq 0$ . We can suppose then that none of the coordinates of  $v$  vanishes.

Let's introduce  $u$  another point in  $\mathcal{C}$  without zero coordinate such that for any  $P \in f^*$ ,  $P(u) = P(v)$ . For all  $-m \leq i < j \leq n$ , one has  $p_{ij}(v) = p_{ij}(u) \Rightarrow \frac{p_{ij}(v)}{p_{ij}(u)} = c_i^{\varepsilon_{ij}} c_j^{-\varepsilon_{ji}} = 1$ . Then, lemma 2.3 claims that  $W_{ij} \neq \emptyset$ .

Corollary 2.4 ensures the existence of some non zero complex number  $z$  in the intersection  $\bigcap_{i=-m}^n W_i$ . This number verifies  $\rho(g_z^+)(u) = v$ .  $\square$

Only  $\frac{N(N+1)}{2}$  invariant rational functions suffice to separate almost all the orbits for  $\mathrm{SO}_2(\mathbb{C})$ . Note that rational invariants of support of size 1 do not exist. Hence, it is not possible to separate more orbits with rational invariants. As in [Bür+21], we separate the last orbits (of support 0 and 1) by their support: Two points of support of size less than 1 lie on the same orbit iff they have the same support.

## 3.2 With polynomial invariants

Functions  $p_{ij}$  of Theorem 3.1 are not polynomial whenever  $i$  and  $j$  have the same sign. One may need a set of invariant polynomials. Because they are smooth, invariant polynomials cannot separate non closed orbits from their closure. There is thus the following definition of separation:

**Definition 3.2.** [DK15, Definition 2.3.8] Let  $\hat{\mathcal{V}}$  be a complex  $G$ -variety. A subset  $S \subset \mathbb{C}[\mathcal{W}]^G$  is said to be separating if for any  $u, v \in \mathcal{W}$ ,

$$\exists P \in \mathbb{C}[\mathcal{W}]^G, P(u) \neq P(v) \Leftrightarrow \exists Q \in S, Q(u) \neq Q(v).$$

Since  $\mathrm{SO}_2(\mathbb{C})$  is the complexification of the compact Lie group  $\mathrm{SO}_2(\mathbb{R})$ , it is a reductive group [Ser66, Theorem 5]. We then need to find a set separating closed orbits [MFK94]. The following lemma determines the closed orbits of  $\mathrm{SO}_2(\mathbb{C})$  in the representation  $\hat{\mathcal{V}} = \bigoplus_{i=1}^m \mathcal{L}_{a_{-i}} \oplus \bigoplus_{i=1}^n \mathcal{L}_{a_i}$ .

**Lemma 3.3.** Note  $\mathcal{C} \subset \hat{\mathcal{V}}$  the subset of points in closed orbits. Then

$$v \in \mathcal{C} \Leftrightarrow (\exists i \leq n, v_i \neq 0 \Leftrightarrow \exists j \leq m, v_{-j} \neq 0)$$

*Proof.* Choose  $v \in \mathcal{V}$  with some  $1 \leq i \leq n$  satisfying  $v_i \neq 0$ . Assume that there exists a sequence  $z_n \in \mathbb{C}^{\mathbb{N}}$  such that  $\rho(g_{z_n}^+)(v)$  converges in  $\hat{\mathcal{V}}$ . Since  $\mathrm{SO}_2(\mathbb{C})$  acts regularly it implies that  $z_n$  converges toward some  $z \in \mathbb{C}$ . If  $z \neq 0$ ,  $\rho(g_{z_n}^+)(v)$  converges toward  $\rho(g_z^+)(v)$ , and this limit belongs to the orbit of  $v$ . If  $z = 0$ , suppose that  $\exists j \leq n$  such that  $v_j \neq 0$ . Hence,  $\frac{1}{z_n} v_j$  converges, and that is not possible.

Otherwise, if  $\forall j \leq m, v_{-j} = 0$ , then  $\rho(g_{z_n}^+)(v)$  converges to 0, and this limit does not belong to the orbit of  $v$ . Hence  $v$  does not belong to a closed orbit.  $\square$

Thanks to Bezout identity, for any integers  $i < j < k$ , we choose  $q, q'$  and  $r, r'$  integers such that  $a_i q + a_k q' = a_i \wedge a_k$  and  $a_j r + a_k r' = a_j \wedge a_k$ . With respect to this choice, we construct the function

$$h_{ijk} = v_i^{q \varepsilon_{ijk}} \times v_j^{-r \varepsilon_{jik}} \times v_k^{q' \varepsilon_{ijk} - r' \varepsilon_{jik}}$$

where  $\varepsilon_{ijk} = \frac{(a_i \wedge a_k) \vee (a_j \wedge a_k)}{a_i \wedge a_k}$ . When  $-m \leq k < 0 < i < j \leq n$  or  $-m \leq i < j < 0 < k \leq n$ , we can choose  $q, q'$  positive and  $r, r'$  negative. Then,  $h_{ijk}$  is a polynomial.

**Theorem 3.4.** The following set of invariant polynomials separates closed orbits:

$$f = \left\{ \begin{array}{l} p_{ij}, \quad -m \leq i < 0 \leq j \leq n \\ h_{ijk}, \quad -m \leq k < 0 < i < j \leq n \text{ or } -m \leq i < j < 0 < k \leq n \end{array} \right\}$$

*Proof.* Firstly take any  $1 \leq i < j \leq n$  and  $-m \leq k \leq -1$ . We compute the weighted sum of the exponents to prove that  $h_{ijk}$  is invariant:

$$\begin{aligned} a_i q \varepsilon_{ijk} - a_j r \varepsilon_{jik} + a_k (q' \varepsilon_{ijk} - r' \varepsilon_{jik}) &= (a_i \wedge a_k) \vee (a_j \wedge a_k) \left( \frac{q a_i + q' a_k}{a_i \wedge a_k} - \frac{r a_j + r' a_k}{a_j \wedge a_k} \right) \\ &= (a_i \wedge a_k) \vee (a_j \wedge a_k) (1 - 1) = 0 \end{aligned}$$

Let  $u$  and  $v$  be two points in  $\mathcal{C}$  such that for any  $P \in f$ ,  $P(u) = P(v)$ . Let  $1 \leq i < j \leq n$  be indices such that  $u_i, u_j \neq 0$ . According to lemma 3.3, there also exists some  $1 \leq k \leq m$  such that  $u_k \neq 0$ . Then,  $p_{ik}(u) = p_{ik}(v)$  implies that  $v_i, v_k \neq 0$  and  $W_{ik} \neq \emptyset$ . Following lemma 2.3 we have  $W_{ik} = W_{c_i^q c_k^{q'}}$ , and similarly  $W_{jk} = W_{c_j^r c_k^{r'}}$ . However,

$$\begin{aligned} \frac{h_{ijk}(v)}{h_{ijk}(u)} = 1 &\Rightarrow \left( \frac{v_i}{u_i} \right)^{q \varepsilon_{ijk}} \left( \frac{v_j}{u_j} \right)^{-r \varepsilon_{jik}} \left( \frac{v_k}{u_k} \right)^{q' \varepsilon_{ijk} - r' \varepsilon_{jik}} = 1 \\ &\Rightarrow c_i^{q \varepsilon_{ijk}} c_j^{-r \varepsilon_{jik}} c_k^{q' \varepsilon_{ijk} - r' \varepsilon_{jik}} = 1 \\ &\Rightarrow \left( c_i^q c_k^{q'} \right)^{\frac{(a_i \wedge a_k) \vee (a_j \wedge a_k)}{a_i \wedge a_k}} = \left( c_j^r c_k^{r'} \right)^{\frac{(a_i \wedge a_k) \vee (a_j \wedge a_k)}{a_j \wedge a_k}} \end{aligned}$$

Then, Lemma 2.3 claims that  $W_{ik} \cap W_{jk} \neq \emptyset$ . Equivalently this is  $W_{ij} \neq \emptyset$ . Similar computations holds for  $1 \leq -i < -j \leq m$ ,  $1 \leq k \leq n$ . That is, any pair of wheels in  $\{W_i, -m \leq i \leq n\}$  has a non empty intersection. According to lemma 2.4, one then find some  $z \in \bigcap_{i=-m}^n W_i$ . This non zero complex number verifies  $\rho(g_z^+)(u) = v$ .  $\square$

The separating set of Theorem 3.4 is not minimal in general. We provide a criterion to remove the redundant polynomials:

**Proposition 3.5.** *The set  $\mathcal{F}$  obtained removing  $h_{ijk}$  whenever  $a_k$  divides  $(a_i \vee a_j)$  is a minimal separating set for the inclusion.*

*Proof.* We try to remove a polynomial and determine if there exists two non separated orbits.

- Assume that we remove  $p_{ij}$  for some  $-m \leq i < 0 \leq j \leq n$ . We choose then  $u_i = u_j = 1$  and  $v_i = v_j = 2$  (other coordinates are supposed to be zero). Then, for any polynomial  $P \in \mathcal{F}$ ,  $P(u) = P(v) = 0$ . However, for any  $z \in \mathbb{C}^*$ ,  $\rho(g_z^+)(u) = v \Rightarrow \begin{cases} z^{a_i} = c_i = 2 \\ z^{a_j} = c_j = 2 \end{cases}$ . This is impossible since  $a_i$  and  $a_j$  have opposite sign.

- Assume that we remove  $h_{ijk}$  for some  $-m \leq k < 0 \leq i < j \leq n$ . If  $a_k$  divides  $(a_i \vee a_j)$  for some  $-m \leq i < j < 0 < k \leq n$ , recall that

$$\begin{cases} p_{ik}(u) = p_{jk}(v) \\ p_{jk}(u) = p_{jk}(v) \end{cases} \Leftrightarrow \begin{cases} W_i \cap W_k \neq \emptyset \\ W_j \cap W_k \neq \emptyset \end{cases}$$

But in that case, Lemma 2.6 ensures directly that

$$\begin{cases} W_i \cap W_k \neq \emptyset \\ W_j \cap W_k \neq \emptyset \end{cases} \Leftrightarrow W_{ik} \cap W_{jk} \neq \emptyset$$

Then,  $u$  and  $v$  are in the same orbit. On the opposite, suppose that  $a_k$  does not divide  $a_i \wedge a_j$ . We choose  $u_i = v_i = u_k = v_k = u_j = 1$  but  $v_j = \exp\left(\frac{ia_j\pi}{a_k}\right)$ . Then,  $c_i = c_k = 1$  while  $c_j = \exp\left(\frac{ia_j\pi}{a_k}\right)$ . Since  $W_{ik}$  and  $W_{jk}$  are non empty,  $p_{ik}(u) = p_{ik}(v)$  and  $p_{jk}(u) = p_{jk}(v)$ . All the other polynomials in  $f$  vanish at  $u$  and  $v$ . However, as discussed in the proof of lemma 2.6,  $W_i \cap W_j = \emptyset$  and  $u$  and  $v$  are not in the same orbit.  $\square$

**Remark 3.6.** *The cardinal of  $f$  is bounded by  $nm \leq \#f \leq nm + m\binom{n}{2} + n\binom{m}{2}$ . The lower bound is reached when all the integers  $a_i$  are the same up to sign. The upper bound is reached when the integers  $a_i$  are pairwise relatively prime.*

Our separating set is much smaller than a generating set. A minimal generating set is given by irreducible elements of  $M_A^+$ , that is elements which are not the sum of two non trivial elements. We note  $\mathcal{M}$  this set. Algorithm 1.4.5 in [Stu08] computes them. Irreducible solutions come with many supports in  $\llbracket 1, n \rrbracket$ . Hence the cardinal of  $\mathcal{M}$  is exponential in the dimension. The separating set provided in Theorem 3.4 consists of irreducible solutions of support of size at most three. It selects furthermore at most one monomial for each support. Thus the cardinal is dominated by  $\binom{n+m}{3} \sim (n+m)^3$ . The difference appears clearly in the following example:

**Example 3.7.** *Let  $(\rho, \hat{V})$  be the representation of  $\text{SO}_2(\mathbb{C})$  on  $\mathcal{L}_{-11} \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_5 \oplus \mathcal{L}_7 \oplus \mathcal{L}_{11}$ . A minimal generating set consists of the following 26 irreducible monomials:*

$$g := \left\{ \begin{array}{cccccccc} v_{-1}v_5 & v_{-1}v_2^2v_3 & v_{-1}v_1v_2^3 & v_{-1}v_1^2v_4 & v_{-1}v_1^3v_3 & v_{-1}v_1^4v_2 & v_{-1}v_3^3v_4 & v_{-1}v_2v_3v_4^2 \\ v_{-1}^2v_2^5v_4 & v_{-1}^2v_1v_3^4 & v_{-1}^2v_1v_2v_3^2v_4 & v_{-1}^2v_1v_2^2v_4^2 & v_{-1}^2v_1^2v_2v_3^3 & v_{-1}^2v_1^{11} & v_{-1}^3v_3v_4^4 & v_{-1}^3v_2v_3^6 \\ v_{-1}^3v_2^4v_4^3 & v_{-1}^3v_2^{11} & v_{-1}^3v_1v_2^2v_3^3 & v_{-1}^3v_1v_2v_4^4 & v_{-1}^4v_2^3v_4^5 & v_{-1}^4v_1v_4^6 & v_{-1}^5v_3^{11} & v_{-1}^5v_2^2v_4^7 \\ v_{-1}^6v_2v_4^9 & v_{-1}^7v_4^{11} & & & & & & \end{array} \right\}$$

Whereas, Theorem 3.4 gives the following polynomial separating set of cardinal 11 (where 4 among the 10 polynomials of type  $h_{ijk}$  have been removed):



$$f := \left\{ \begin{array}{ccccc} v_{-1}^2 v_1^{11} & v_{-1}^3 v_2^{11} & v_{-1}^5 v_3^{11} & v_{-1}^7 v_4^{11} & v_{-1} v_5 \\ h_{1,2,-1} = v_1^6 v_2^7 v_{-1}^3 & h_{1,3,-1} = v_1^6 v_3^2 v_{-1}^2 & h_{1,4,-1} = v_1^6 v_4^3 v_{-1}^3 & & \\ h_{2,3,-1} = v_2^4 v_3^2 v_{-1}^2 & h_{2,4,-1} = v_{-1}^3 v_2^4 v_3^3 & h_{3,4,-1} = v_3^9 v_4^3 v_{-1}^6 & & \end{array} \right\}$$

## 4 Separating invariants for $\mathrm{SO}_2(\mathbb{R})$

In this section we introduce another definition of separation over reals. We next give a minimal separating set for representations of  $\mathrm{SO}_2(\mathbb{R})$  of cardinality  $\left(\frac{\dim(\mathcal{V})}{2}\right)^2$  (Theorem 4.2). Since the definition of separation over  $\mathbb{R}$  is weaker, we obtain a much lower cardinality than its complex analogue provided by Theorem 3.4, where the cardinality is dominated by  $\dim(\hat{\mathcal{V}})^3$ . Example 4.4 exhibits clearly this difference.

Since  $\mathrm{SO}_2(\mathbb{R})$  and  $\mathrm{O}_2(\mathbb{R})$  are both compact, so do their orbits and polynomials can separate all the orbit space. This gives rise to a more natural definition of separation:

**Definition 4.1.** *Let  $G$  be a real compact algebraic group acting on a real variety  $\mathcal{V}$ . A subset  $S \subset \mathbb{R}[\mathcal{V}]^G$  is a separating set if for any  $u, v \in \mathcal{V}$ , polynomials in  $S$  agree on  $u$  and  $v$  iff  $u$  and  $v$  are in the same orbit.*

This is why separating set over  $\mathbb{R}$  are thus frequently smaller than over  $\mathbb{C}$ . For instance, the orbit space of a complex finite group can be separated by  $\dim(\hat{\mathcal{V}})$  polynomials iff the group is a reflexion group. Whereas, many non reflexion groups have this strong property over the reals [Duf09]. Thus the importance for many applications to provide specific separating set over the reals.

Consider the action of  $\mathrm{SO}_2(\mathbb{R})$  on  $\mathcal{V} = \bigoplus_{i=1}^n \mathcal{V}_{a_i}$  where  $a_1, \dots, a_n \in \mathbb{N} \setminus \{0\}$  (the irreducible representations  $\mathcal{V}_{a_i}$  of  $\mathrm{SO}_2(\mathbb{R})$  are defined in Appendix A). The complexification  $\hat{\mathcal{V}}$  decomposes into  $\bigoplus_{i=1}^n (\mathcal{L}_{a_i} \oplus \mathcal{L}_{-a_i})$ . We still endow  $\hat{\mathcal{V}}$  with the complex coordinates  $v_1, v_{-1}, \dots, v_n, v_{-n}$  as in Section 3. This is, for any  $v \in \mathcal{V}$  and  $1 \leq i \leq n$ ,  $v_i = \bar{v}_{-i}$ . Note that we have  $v_i \neq 0 \Leftrightarrow v_{-i} \neq 0$ . Hence the polynomials of type  $h_{ijk}$  can be omitted.

**Theorem 4.2.** *The following set of cardinality  $n^2$  is a minimal separating set for  $\mathrm{SO}_2(\mathbb{R}) \curvearrowright \mathcal{V}$ :*

$$f(\mathbb{R}) = \left\{ \begin{array}{l} 2\Re(p_{-ij}) = p_{-ij} + p_{-ji}, \quad 1 \leq i \leq j \leq n \\ -2\Im(p_{-ij}) = i(p_{-ij} - p_{-ji}), \quad 1 \leq i < j \leq n \end{array} \right\}$$

*Proof.* Let  $u$  and  $v$  be two points in  $\mathcal{V}$  such that for any  $1 \leq i \leq j \leq n$ ,  $p_{-ij}(u) = p_{-ij}(v)$ . For any integer  $1 \leq i \leq n$ , we note that  $u_i = 0 \Leftrightarrow u_{-i} = 0 \Leftrightarrow \Re(p_{-ii}) = 0$ . It remains then to separate the orbits in the subvariety  $\mathcal{V} = \bigoplus_{j \neq i} \mathcal{V}_{a_j}$ .

Hence we assume that none of the coordinates of  $u$  and  $v$  vanish. Then, quotients  $c_i$  and wheels  $W_i$  are well defined. For  $1 \leq i \leq n$ ,  $p_{ii}(u) = p_{ii}(v) \Rightarrow u_i u_{-i} = v_i v_{-i} \Rightarrow c_i = \frac{u_i}{v_i} = \frac{v_{-i}}{u_{-i}} = \frac{1}{\bar{c}_i}$ . Then,  $|c_i| = 1$  and the wheel  $W_i$  is included in the unit circle  $\mathcal{U}$ . As discussed in section A.3,  $z \in W_i \subset \mathcal{U} \Rightarrow g_z^+ \in \mathrm{SO}_2(\mathbb{R})$ . Furthermore as in Section 3, for  $1 \leq i < j \leq n$ ,  $p_{-ij}(u) = p_{-ij}(v) \Rightarrow W_i \cap W_j \neq \emptyset$ . Then lemma 2.4 ensures that  $\bigcap_{i=1}^n W_i$  is not empty. For some  $z \in \bigcap_{i=1}^n W_i$ , one has  $g_z^+ \in \mathrm{SO}_2(\mathbb{R})$  and  $\rho(g_z^+)(v) = u$ .

Unfortunately the restrictions on  $\mathcal{V}$  of polynomials  $p_{-ij} \in \mathbb{C}[\hat{\mathcal{V}}]^{\mathrm{SO}_2(\mathbb{C})}$  are not real. We nevertheless notice that on  $\mathcal{V}$ ,  $p_{-ij} = \bar{p}_{-ji}$ . Then,  $p_{ij} = p_{-ij} + p_{-ji} = 2\Re(p_{-ij})$  and  $i(p_{-ij} - p_{-ji}) = -2\Im(p_{-ij})$  are real.  $\square$

**Remark 4.3.** *When  $a_1 = 1$  for any  $1 \leq i \leq n$ , the representation of  $\mathrm{SO}_2(\mathbb{R})$  on  $\mathcal{V}$  is isomorphic to the representation on  $2 \times n$  matrices  $\mathcal{M}_{2,n}(\mathbb{R})$ . Then,  $\Re(p_{-ij})$  is the standard scalar product between the  $i^{\mathrm{th}}$  and  $j^{\mathrm{th}}$  column, while  $\Im(p_{-ij})$  is their determinant. Thus, the separating set  $f(\mathbb{R})$  consist of the classical Weyl invariants [Wey46, Theorem 2.9.A].*

**Example 4.4.** Consider the representation of  $\mathrm{SO}_2(\mathbb{R})$  on  $\mathcal{V} = \mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \mathcal{V}_5 \oplus \mathcal{V}_7$ . Then, Theorem 4.2 provides a set of 16 invariant polynomials separating the orbits of  $\mathrm{SO}_2(\mathbb{R})$ :

$$f(\mathbb{R}) = \left\{ \begin{array}{ccccc} 2v_1v_{-1} & v_{-2}^2v_1^3 + v_{-1}^3v_2^2 & v_{-3}^2v_1^5 + v_{-1}^5v_3^2 & v_{-4}^2v_1^7 + v_{-1}^7v_4^2 & 2v_2v_{-2} \\ v_{-3}^3v_2^5 + v_{-2}^5v_3^3 & v_{-4}^3v_2^7 + v_{-2}^7v_4^3 & 2v_3v_{-3} & v_{-4}^5v_3^7 + v_{-3}^7v_4^5 & 2v_4v_{-4} \\ \mathbf{i} \left( v_{-2}^2v_1^3 - v_{-1}^3v_2^2 \right) & \mathbf{i} \left( v_{-3}^2v_1^5 - v_{-1}^5v_3^2 \right) & \mathbf{i} \left( v_{-4}^2v_1^7 - v_{-1}^7v_4^2 \right) & & \\ \mathbf{i} \left( v_{-3}^3v_2^5 - v_{-2}^5v_3^3 \right) & \mathbf{i} \left( v_{-4}^3v_2^7 - v_{-2}^7v_4^3 \right) & \mathbf{i} \left( v_{-4}^5v_3^7 - v_{-3}^7v_4^5 \right) & & \end{array} \right\}$$

Whereas, 24 monomials among the 48 of type  $h_{ijk}$  are required to separate closed orbits of  $\mathrm{SO}_2(\mathbb{C})$  in  $\hat{\mathcal{V}}$ . Thus Theorem 3.4 provides a separating set of cardinality  $16 + 24 = 40$ .

## 5 Separating invariants for $\mathrm{O}_2(\mathbb{R})$

We now consider actions of  $\mathrm{O}_2(\mathbb{R}) \cong \mathrm{SO}_2(\mathbb{R}) \times \mathbb{C}_2$ . The twist induces the existence, for each indice  $i$ , of two wheels  $W_i$  and  $W_i^-$ . It forces us to add invariants of support three or four. Thus the separating set has a cardinality in  $\mathcal{O}(n^4)$ . Consider the representation of  $\mathrm{O}_2(\mathbb{R})$  on

$$\mathcal{V} := \bigoplus_{i=1}^q \mathcal{V}_{-1} \oplus \bigoplus_{i=1}^n \mathcal{V}_{a_i}$$

endowed with the coordinates  $(d_1, \dots, d_q, v_1, v_{-1}, \dots, v_n, v_{-n})$ . We obtain a separating set in Theorem 5.1, which is not minimal in general for the inclusion. We then give a criterion to remove redundant polynomials and get a minimal separating set in Proposition 5.6. We finish comparing the obtained separating set to a generating set provided by the algorithm of [Des+23].

### 5.1 Computation of the set

In addition to the wheels  $W_i$  and  $W_{-i}$  introduced in Section 3, we note  $W_i^-$  and  $W_{-i}^-$  the set of complex numbers such that

$$\begin{cases} \forall z \in W_i^-, & \rho_{a_i}(g_z^-)(u_i, u_{-i}) = (*, v_{-i}) \\ \forall z \in W_{-i}^-, & \rho_{a_i}(g_z^-)(u_i, u_{-i}) = (v_i, *) \end{cases}$$

When the quotients  $c_i^- = \frac{v_{-i}}{u_i}$  and  $c_{-i}^- = \frac{v_i}{u_{-i}}$  are well defined we have  $W_i^- = W_{c_i^-}^{a_i}$  and  $W_{-i}^- = W_{c_{-i}^-}^{-a_i}$ . For any  $1 \leq i \leq n$ , we note  $W_I = W_i \cap W_{-i}$  (the set of complex numbers such that  $\rho_{a_i}(g_z^+)(u_i, u_{-i}) = (v_i, v_{-i})$ ) and  $W_I^- = W_i^- \cap W_{-i}^-$  (the set of complex numbers such that  $\rho_{a_i}(g_z^-)(u_i, u_{-i}) = (v_i, v_{-i})$ ).

We introduce  $\mathcal{T}(n) \subset \llbracket 1, n \rrbracket^3$  and  $\mathcal{Q}(n) \subset \llbracket 1, n \rrbracket^4$  the set of ordered triplets and quadruplets:

$$(i, j, k) \in \mathcal{T}(n) \Leftrightarrow \begin{cases} i < j \\ k \in \llbracket 1, n \rrbracket \setminus \{i, j\} \end{cases} \quad (i, j, k, l) \in \mathcal{Q}(n) \Leftrightarrow \begin{cases} i < j, k < l \\ \{i, j\} \cap \{k, l\} = \emptyset \\ i + j \leq k + l \\ \text{If } i + j = k + l, i < k \end{cases}$$

**Theorem 5.1.** The following set of polynomials separates orbits with respect to the action  $\mathrm{O}_2(\mathbb{R}) \curvearrowright \mathcal{V}$ :

$$\mathcal{F}(\mathbb{R}) = \left\{ \begin{array}{ll} P_{ij} = p_{-ij} + p_{-ji} & 1 \leq i \leq j \leq n \\ T_{ijk} = (p_{-ik} - p_{-ki})(p_{-jk} - p_{-kj}), & (i, j, k) \in \mathcal{T}(n) \\ Q_{ijkl} = (p_{-ij} - p_{-ji})(p_{-kl} - p_{-lk}), & i, j, k, l \in \mathcal{Q}(n) \\ D_{ij} = d_i d_j, & 1 \leq i \leq j \leq q \\ S_{ijk} = \mathbf{i}(p_{-ij} - p_{-ji}) d_k, & 1 \leq i < j \leq n, \quad 1 \leq k \leq q \end{array} \right\}$$

To prove it, we take  $u$  and  $v$  two points in  $\mathcal{V}$  such that for any  $P \in \mathcal{F}(\mathbb{R})$ ,  $P(u) = P(v)$  and check that there exists some  $g \in \text{O}_2(\mathbb{R})$  verifying  $\rho(g)(u) = v$ . As for Theorem 4.2 we assume that none of the coordinates of  $u$  and  $v$  vanish. Then for any  $1 \leq i \leq n$ , quotients  $c_i, c_i^-$  and wheels  $W_i, W_i^-$  are well defined and we notice that

$$P_{ii}(u) = P_{ii}(v) \Rightarrow u_i u_{-i} = v_i v_{-i} \Rightarrow \begin{cases} c_i = \frac{v_i}{u_i} = \frac{u_{-i}}{v_{-i}} = \frac{1}{c_i^-} \\ c_i^- = \frac{v_{-i}}{u_{-i}} = \frac{u_i}{v_i} = \frac{1}{c_i} \end{cases}$$

Then,  $|c_i| = |c_i^-| = 1$  and the wheels  $W_i, W_i^-$  are both included in the unit circle  $\mathcal{U}$ . As discussed in Appendix A.3,  $z \in \mathcal{U} \Rightarrow g_z^+, g_z^- \in \text{O}_2(\mathbb{R})$ . We next prove the following sequence of lemmas:

**Lemma 5.2.** *For any integer  $1 \leq i \leq n$ , we have  $W_i = W_{-i} = W_I$  and  $W_i^- = W_{-i}^- = W_I^-$ .*

*Proof.* By assumption we have  $P_{ii}(u) = P_{ii}(v) \Rightarrow u_i u_{-i} = v_i v_{-i}$ . Since no coordinate vanish,  $c_i = \frac{v_i}{u_i} = \frac{u_{-i}}{v_{-i}} = \frac{1}{c_i^-}$  and  $c_i^- = \frac{v_{-i}}{u_{-i}} = \frac{u_i}{v_i} = \frac{1}{c_i}$ . This implies  $W_i = W_{-i}$  and  $W_i^- = W_{-i}^-$ .  $\square$

**Lemma 5.3.** *For any integers  $1 \leq i < j \leq n$ , we have  $W_{IJ} \neq \emptyset$  or  $W_{IJ}^- \neq \emptyset$ .*

*Proof.* We notice that  $p_{-ij}(v) = v_{-i}^{\varepsilon_{ij}} v_j^{\varepsilon_{ji}} = u_{-i}^{\varepsilon_{ij}} c_i^{\varepsilon_{ij}} u_j^{\varepsilon_{ji}} c_j^{\varepsilon_{ji}} = c_i^{\varepsilon_{ij}} c_j^{\varepsilon_{ji}} p_{-ij}(u)$ . It provides

$$\begin{aligned} P_{-ij}(u) = P_{-ij}(v) &\Leftrightarrow p_{-ij}(u) + p_{-ji}(u) = p_{-ij}(v) + p_{-ji}(v) \\ &\Leftrightarrow p_{-ij}(u) + p_{-ji}(u) = c_i^{\varepsilon_{ij}} c_j^{\varepsilon_{ji}} p_{-ij}(u) + c_{-i}^{\varepsilon_{ij}} c_{-j}^{\varepsilon_{ji}} p_{-ji}(u) \\ &\Leftrightarrow p_{-ij}(u) + p_{-ji}(u) = c_i^{\varepsilon_{ij}} c_j^{\varepsilon_{ji}} p_{-ij}(u) + c_i^{-\varepsilon_{ij}} c_j^{-\varepsilon_{ji}} p_{-ji}(u) \end{aligned}$$

This can be viewed as a second order equation of unknown  $c_i^{\varepsilon_{ij}} c_j^{\varepsilon_{ji}}$ , which has the two solutions  $c_i^{\varepsilon_{ij}} c_j^{\varepsilon_{ji}} = 1$  or  $c_i^{\varepsilon_{ij}} c_j^{\varepsilon_{ji}} = \frac{p_{-ji}(u)}{p_{-ij}(u)}$ . The first solution implies  $W_{ij} \neq \emptyset$  by lemma 2.3, and lemma 5.2 completes in  $W_{IJ} \neq \emptyset$ .

The second solution gives  $p_{-ij}(u) = c_i^{-\varepsilon_{ij}} c_j^{-\varepsilon_{ji}} p_{-ji}(u) = p_{-ji}(v)$ , and

$$p_{-ij}(v) = p_{-ji}(u) \Rightarrow v_{-i}^{\varepsilon_{ij}} v_j^{\varepsilon_{ji}} = u_{-i}^{\varepsilon_{ij}} u_i^{\varepsilon_{ji}} \Rightarrow \left(\frac{v_{-i}}{u_i}\right)^{\varepsilon_{ij}} = \left(\frac{v_j}{u_{-j}}\right)^{\varepsilon_{ji}} \Rightarrow (c_i^-)^{\varepsilon_{ij}} = (c_{-j}^-)^{\varepsilon_{ji}}$$

Then lemma 2.3 claims that  $W_{-ij}^- \neq \emptyset$ , and lemma 5.2 completes in  $W_{IJ}^- \neq \emptyset$ .  $\square$

**Lemma 5.4.** *Let  $(i, j, k) \in \mathcal{T}(n)$ . Then, one has  $\begin{cases} W_{IK} \neq \emptyset \\ W_{JK} \neq \emptyset \end{cases}$  or  $\begin{cases} W_{IK}^- \neq \emptyset \\ W_{JK}^- \neq \emptyset \end{cases}$ .*

*Proof.* Following Lemma 5.3 we have  $p_{-ij}(u) = p_{-ij}(v)$  or  $p_{-ij}(u) = p_{-ji}(v)$ , and this is  $(p_{-ij} - p_{-ij})(u) = \pm(p_{-ij} - p_{-ij})(v)$ . Note furthermore that

$$\begin{cases} p_{-ij}(u) = p_{-ij}(v) \Rightarrow W_{IJ} \neq \emptyset \\ p_{-ij}(u) = p_{-ji}(v) \Rightarrow W_{IJ}^- \neq \emptyset \end{cases}$$

This holds also for  $j, k$ . Recall that we have  $T_{ijk}(u) = T_{ijk}(v)$ :

$$(p_{-ik} - p_{-ki})(-p_{-jk} - p_{-kj})(u) = (p_{-ik} - p_{-ki})(-p_{-jk} - p_{-kj})(v)$$

- If  $(p_{-ik} - p_{-ki})(u) = (p_{-ik} - p_{-ki})(v) \neq 0$  and  $(p_{-jk} - p_{-kj})(u) = (p_{-jk} - p_{-kj})(v) \neq 0$ . Then,  $p_{-ik}(u) = p_{-ik}(v) \neq 0$  and  $p_{-jk}(u) = p_{-jk}(v) \neq 0$ . This is,  $\begin{cases} W_{IK} \neq \emptyset \\ W_{JK} \neq \emptyset \end{cases}$

- If  $(p_{-ik} - p_{-ki})(u) = -(p_{-ik} - p_{-ki})(v) \neq 0$  and  $(p_{-jk} - p_{-kj})(u) = -(p_{-jk} - p_{-kj})(v) \neq 0$ . Then,  $p_{-ik}(u) = p_{-ki}(v) \neq 0$  and  $p_{-jk}(u) = p_{-kj}(v) \neq 0$ . This is,  $\begin{cases} W_{IK}^- \neq \emptyset \\ W_{JK}^- \neq \emptyset \end{cases}$

- If  $(p_{-ik} - p_{-ki})(u) = 0$ , then  $p_{-ik}(u) = p_{-ik}(v) = p_{-ki}(u) = p_{-ki}(v) \neq 0$ . We thus have both  $W_{IK} \neq \emptyset$  and  $W_{IK}^- \neq \emptyset$ . However Lemma 5.3 ensures that  $W_{KJ} \neq \emptyset$  or  $W_{KJ}^- \neq \emptyset$ . We deduce that  $\begin{cases} W_{IJ} \neq \emptyset \\ W_{KJ} \neq \emptyset \end{cases}$  or

$$\begin{cases} W_{IJ}^- \neq \emptyset \\ W_{KJ}^- \neq \emptyset \end{cases} . \quad \square$$

**Lemma 5.5.** *Let  $(i, j, k, l) \in \mathcal{Q}(n)$ . Then one has  $\left\{ \begin{array}{l} W_{IJ} \neq \emptyset \\ W_{KL} \neq \emptyset \end{array} \right\}$  or  $\left\{ \begin{array}{l} W_{IJ}^- \neq \emptyset \\ W_{KL}^- \neq \emptyset \end{array} \right\}$ .*

*Proof.* It is the same proof than for Lemma 5.4, but with the polynomial equality  $Q_{ijkl}(u) = Q_{ijkl}(v)$ .  $\square$

We are now able to prove Theorem 5.1. According to lemmas 5.4 and 5.5 we have

$$\forall 1 \leq i < j \leq n \text{ and } 1 \leq k < l \leq n, \left\{ \begin{array}{l} W_{IJ} \neq \emptyset \\ W_{KL} \neq \emptyset \end{array} \right\} \text{ or } \left\{ \begin{array}{l} W_{IJ}^- \neq \emptyset \\ W_{KL}^- \neq \emptyset \end{array} \right\}$$

This is equivalent to  $(\forall 1 \leq i < j \leq n, W_{IJ} \neq \emptyset)$  or  $(\forall 1 \leq i < j \leq n, W_{IJ}^- \neq \emptyset)$ . Then, according to Corollary 2.4, there exists  $z \in \bigcap_{i=1}^n W_I$  (+) or  $z \in \bigcap_{i=1}^n W_I^-$  (-).

Furthermore for any  $1 \leq i \leq q$ , the polynomial  $D_{ii}$  ensures that  $|d_i| = |d'_i|$ . This is,  $\rho_{-1}(g_z^+)(d_i) = d'_i$  or  $\rho_{-1}(g_z^-)(d_i) = d'_i$ . If one has (+) but not (-) then there is  $1 \leq i < j \leq n$  with  $W_{IJ}^- = \emptyset$ , and this is  $(p_{-ij} - p_{-ji})(u) = (p_{-ij} - p_{-ji})(v) \neq 0$ . One notices that for any  $1 \leq k \leq q$ ,  $S_{ijk}(u) = S_{ijk}(v) \Rightarrow d_k = d'_k$ . Then,  $\rho(g_z^+)(u) = v$ . Conversely, if one has (-) but not (+) then there is  $1 \leq i < j \leq n$  with  $W_{IJ} = \emptyset$ , and this is  $(p_{-ij} - p_{-ji})(u) = -(p_{-ij} - p_{-ji})(v) \neq 0$ . One notices that for any  $1 \leq k \leq q$ ,  $S_{ijk}(u) = S_{ijk}(v) \Rightarrow d_k = -d'_k$ . Then,  $\rho(g_z^-)(u) = v$ .

If one has both (+) and (-), then for any  $1 \leq i < j \leq n$ ,  $(p_{-ij} - p_{-ji})(u) = (p_{-ij} - p_{-ji})(v) = 0$ . Note that for any  $1 \leq i \leq j \leq q$  with  $u_i \neq 0$  and  $u_j \neq 0$ ,  $D_{ij}(u) = D_{ij}(v) \Rightarrow \frac{d_i}{d'_i} = \frac{d'_j}{d_j}$ . Suppose that all these quotients are positive. Then  $\rho(g_z^+)(u) = v$ . Conversely, if all these quotients are negative, then  $\rho(g_z^-)(u) = v$ . Anyway there exists some  $g \in \mathcal{O}_2(\mathbb{R})$  verifying  $\rho(g)(u) = v$  and Theorem 5.1 is proven.

## 5.2 Minimality

In general, the separating set of Theorem 5.1 is not minimal for inclusion. We identify the redundant polynomials:

**Proposition 5.6.** *Upon the following removals,  $\mathcal{F}(\mathbb{R})$  is a minimal separating set:*

- If  $(i, j, k) \in \mathcal{T}(n)$  is a triplet such that  $a_k \mid (a_i \vee a_j)$ ,  $a_i \mid (a_k \vee a_j)$  and  $a_j \mid (a_i \vee a_k)$ , we remove  $T_{ijk}$ ,  $T_{jki}$  and  $T_{ikj}$ .
- If  $(i, j, k) \in \mathcal{T}(n)$  is a triplet such that  $a_k$  divides  $a_i \vee a_j$ , we keep  $T_{ijk}$  but remove  $T_{jki}$  and  $T_{ikj}$ .
- If  $(i, j, k, l) \in \mathcal{Q}(n)$  is such that  $(a_i \wedge a_j \text{ divides } a_k \vee a_l)$  or  $(a_k \wedge a_l \text{ divides } a_i \vee a_j)$ , we remove  $Q_{ijkl}$ .

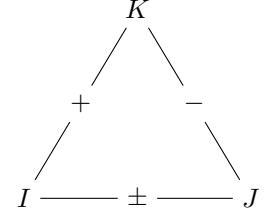
*Proof.* Again we try to remove a polynomial and determine if there exists two non separated orbits. Recall that we need points in the real trace, that is points verifying  $v_i = \bar{v}_i$  for all  $i$ . For the polynomials of type  $P$  we refer to section 3.

• Assume that we remove  $T_{jki}$  and  $T_{ikj}$  for some  $(i, j, k) \in \mathcal{T}(n)$ . By lemma 5.4,  $T_{ijk}(u) = T_{ijk}(v)$  ensures that  $\left\{ \begin{array}{l} W_{IK}^\delta \neq \emptyset \\ W_{JK}^\delta \neq \emptyset \end{array} \right\}$  for some  $\delta = \pm 1$ . Then, if  $a_k$  divides  $a_i \vee a_j$ , it results directly by Proposition 2.5 that  $W_{IJK}^\delta \neq \emptyset$ . This is  $u$  and  $v$  belong to the same orbit.

On the opposite, suppose that we remove the polynomial  $T_{ijk}$  although  $a_i$  does not divides  $a_k \vee a_j$  and  $a_j$  does not divides  $a_k \vee a_i$ . We choose  $u$  and  $v$  as in the following table.

	$i$	$-i$	$j$	$-j$	$k$	$-k$
$u$	1	1	1	1	$\exp\left(-ia_k\pi\left(\frac{1}{a_i} + \frac{1}{a_j}\right)\right)$	$\exp\left(ia_k\pi\left(\frac{1}{a_i} + \frac{1}{a_j}\right)\right)$
$v$	1	1	1	1	$\exp\left(ia_k\pi\left(\frac{1}{a_i} - \frac{1}{a_j}\right)\right)$	$\exp\left(ia_k\pi\left(\frac{1}{a_j} - \frac{1}{a_i}\right)\right)$

These choices clearly provide  $c_i = c_i^- = c_j = c_j^- = 1$ . We also compute  $c_k = \exp\left(\frac{2i\pi a_k}{a_i}\right)$  and  $c_k^- = \exp\left(\frac{2ia_k\pi}{a_j}\right)$ . Note for instance that that  $c_k^{a_j \vee a_k} c_j^{a_j \vee a_k} = \exp\left(\frac{2i\pi a_k}{a_i} \frac{a_j \vee a_k}{a_k}\right) = \exp\left(2i\pi \frac{a_j \vee a_k}{a_i}\right) \neq 1$ . Lemma 2.3 provides by that way the intersections described in the right figure. We deduce that polynomials of type  $P$  agree on  $u$  and  $v$ . So do  $T_{jki}$  and  $T_{ikj}$ . All the others polynomials in  $\mathcal{F}(\mathbb{R})$  vanish. Although,  $u$  and  $v$  are not in the same orbit.

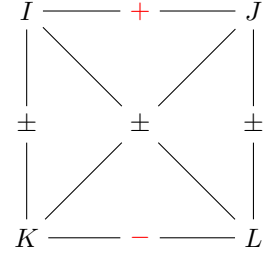


• Assume that we remove  $Q_{ijkl}$  for some  $(i, j, k, l) \in \mathcal{Q}(n)$  such that  $a_i \wedge a_j$  divides  $a_k \vee a_l$ . Take  $u$  and  $v$  such that  $\begin{cases} W_{IJ}^\delta \neq \emptyset \\ W_{IJ}^{-\delta} = \emptyset \end{cases}$  for some  $\delta = \pm 1$ . By Lemma 5.4, polynomials of type  $T$  ensure that  $W_{IJ}^\delta \cap W_K^\delta \neq \emptyset$  and  $W_{IJ}^\delta \cap W_L^\delta \neq \emptyset$ . However, the whell  $W_{IJ}^\delta$  is of cardinality  $a_i \wedge a_j$ . Then, the criterion of Proposition 2.6 gives  $W_{KL}^\delta \neq \emptyset$ , and  $u$  and  $v$  are in the same orbit.

On the opposite, assume that  $a_i \wedge a_j$  does not divides  $a_k \vee a_l$  and  $a_k \wedge a_l$  does not divides  $a_i \vee a_j$ . We choose  $u$  and  $v$  as in the following table.

	$i$	$-i$	$j$	$-j$	$k$	$-k$	$l$	$-l$
$u$	1	1	$\exp\left(-i \frac{a_j \pi}{a_k \wedge a_l}\right)$	$\exp\left(i \frac{a_j \pi}{a_k \wedge a_l}\right)$	1	1	$\exp\left(-i \frac{a_l \pi}{a_i \wedge a_j}\right)$	$\exp\left(i \frac{a_l \pi}{a_i \wedge a_j}\right)$
$v$	1	1	$\exp\left(-i \frac{a_j \pi}{a_k \wedge a_l}\right)$	$\exp\left(i \frac{a_j \pi}{a_k \wedge a_l}\right)$	1	1	$\exp\left(i \frac{a_l \pi}{a_i \wedge a_j}\right)$	$\exp\left(-i \frac{a_l \pi}{a_i \wedge a_j}\right)$

These choices clearly provide  $c_i = c_i^- = c_j = c_k = c_k^- = c_l^- = 1$ . We find also  $c_j^- = \exp\left(\frac{2i\pi a_j}{a_k \wedge a_l}\right)$  and  $c_l = \exp\left(\frac{2i\pi a_l}{a_i \wedge a_j}\right)$ . Note for instance that  $(c_i^-)^{\frac{a_i \vee a_j}{a_i}} (c_j^-)^{\frac{a_i \vee a_j}{a_j}} = \exp\left(\frac{2i\pi a_j}{a_k \wedge a_l} \frac{a_i \vee a_j}{a_j}\right) = \exp\left(\frac{2i\pi a_i \vee a_j}{a_k \wedge a_l}\right) \neq 1$ . Lemma 2.3 provides by that way the intersections described in the right figure. We deduce that polynomials of type  $P$  and  $T$  agree on  $u$  and  $v$ . So do  $Q_{ikjl}$  and  $Q_{iljk}$ . All the others polynomials in  $\mathcal{F}(\mathbb{R})$  vanish. Although,  $u$  and  $v$  are not in the same orbit.



• Assume that we remove  $D_{ii}$  for some  $1 \leq i \leq q$ . We choose then  $d_i = 1$  but  $d'_i = 2$ . Then all the polynomials in  $\mathcal{F}(\mathbb{R})$  vanish. Although,  $u$  and  $v$  are not in the same orbit.

• Assume that we remove  $D_{ij}$  for some  $1 \leq i < j \leq q$ . We choose then  $d_i = d'_i = d_j = 1$  but  $d'_j = -1$ . Then  $D_{ii}(u) = D_{ii}(v) = D_{jj}(u) = D_{jj}(v) = 1$  while all the others polynomials in  $\mathcal{F}(\mathbb{R})$  vanish. Although,  $u$  and  $v$  are not in the same orbit.

• Assume that we remove  $S_{ijk}$  for some  $1 \leq i < j \leq r$  and  $1 \leq k \leq q$ . We choose then  $d_k = 1$ ,  $d'_k = -1$ ,  $(u_i, u_{-i}) = (v_i, v_{-i}) = (1, 1)$  and  $(u_j, u_{-j}) = (v_j, v_{-j}) = (\exp(i\sqrt{2}), \exp(-i\sqrt{2}))$ . Here  $W_{IJ}$  is not empty because it contains at least 1, while  $W_{IJ}^-$  is empty because  $\sqrt{2}$  is not rational. Then  $P_{ii}, P_{ij}$  and  $P_{jj}$  agree on  $u$  and  $v$ . All the other polynomials in  $\mathcal{F}(\mathbb{R})$  vanish on  $u$  and  $v$ . Although,  $W_{IJ}^- = \emptyset$ , thus  $u$  and  $v$  are not in the same orbit.  $\square$

**Remark 5.7.** *The obtained minimal separating set thus has a cardinality bounded by*

$$n^2 \sim \underbrace{\frac{n(n+1)}{2}}_{P_{ij}} + \underbrace{\frac{q(q+1)}{2}}_{D_{ij}} + \underbrace{q \binom{n}{2}}_{S_{ijk}} \leq \#\mathcal{F}(\mathbb{R}) \leq \underbrace{\frac{n(n+1)}{2}}_{P_{ij}} + \underbrace{3 \binom{n}{3}}_{T_{ijk}} + \underbrace{3 \binom{n}{4}}_{Q_{ijkl}} + \underbrace{\frac{q(q+1)}{2}}_{D_{ij}} + \underbrace{q \binom{n}{2}}_{S_{ijk}} \sim n^4$$

*The lower bound is reached when all the integers  $a_i$  are equal to 1.*

**Remark 5.8.** *When  $a_1 = 1$  for any  $1 \leq i \leq n$  and  $q = 0$ , the representation of  $O_2(\mathbb{R})$  on  $\mathcal{V}$  is isomorphic to the representation on  $2 \times n$  matrices  $M_{2,n}(\mathbb{R})$ . All the polynomials of type  $T$  and  $Q$  are removed. Only the polynomials  $P_{ij}$  for  $1 \leq i \leq j \leq n$  remain, and they are the scalar product between the  $i^{\text{th}}$  and  $j^{\text{th}}$  columns. Here again, the separating set  $\mathcal{F}(\mathbb{R})$  consist of the well known Weyl invariants [Wey46, Theorem 2.9.A].*



In some circumstances, one assumes that the coordinate  $d_1$  does not vanish. For instance, when we study the restriction of a representation of  $\mathrm{SO}_3(\mathbb{R})$  via a Seshadri slice [HJ24]. In that situation the separating set is drastically simplified:

**Corollary 5.11.** *Consider the representation of  $\mathrm{SO}_2(\mathbb{R})$  on  $\mathcal{V}^* := \mathcal{V}_{-1}^* \oplus \bigoplus_{i=2}^q \mathcal{V}_{-1} \oplus \bigoplus_{i=1}^n \mathcal{V}_{a_i}$ , where  $\mathcal{V}^* = \mathcal{V}_{-1} \setminus \{0\}$ . Then we have the following separating set of cardinality  $q + n^2$ :*

$$\mathcal{F}^* = \left\{ \begin{array}{ll} P_{ij} = \frac{1}{2}(p_{-ij} + p_{-ji}) & 1 \leq i \leq j \leq n \\ D_{1j} = d_1 d_j, & 1 \leq j \leq q \\ S_{ij1} = i(p_{-ij} - p_{-ji}) d_1, & 1 \leq i < j \leq n \end{array} \right\}$$

*Proof.* Consider  $u = (d_1, \dots, d_q, u_1, u_{-1}, \dots, u_n, u_{-n})$  and  $v = (d'_1, \dots, d'_q, v_1, v_{-1}, \dots, v_n, v_{-n})$  two points in  $\mathcal{V}^*$  where polynomials of  $\mathcal{F}^*$  agree. Since  $d'_1 \neq 0$  we can fix  $\delta = \frac{d_1}{d'_1} = \pm 1$ . Then,  $\forall 2 \leq j \leq q$ ,  $D_{1j}(u) = D_{1j}(v) \Rightarrow \frac{d_j}{d'_j} = \delta$ .

Furthermore, for all  $1 \leq i \leq j \leq n$ ,  $P_{ij}(u) = P_{ij}(v)$  ensures that  $W_{IJ} \neq \emptyset$  or  $W_{IJ}^- \neq \emptyset$ . Then,  $S_{ij1}(u) = S_{ij1}(v) \Rightarrow W_{IJ}^\delta \neq \emptyset$ . Corollary 2.5 claims that there exists some  $z \in \bigcap_{i=1}^n W_{IJ}^\delta$ . This complex verifies  $\rho(g_z^\delta)(u) = v$ .  $\square$

## 6 Separating invariants for $\mathrm{O}_2(\mathbb{C})$

In this section we give a polynomial separating set for an arbitrary representation of  $\mathrm{O}_2(\mathbb{C})$  (Theorem 6.1). As for  $\mathrm{SO}_2(\mathbb{R})$  we delete redundant polynomials to get a minimal separating set in Proposition 6.6.

We consider the representation of  $\mathrm{O}_2(\mathbb{C})$  on

$$\hat{\mathcal{V}} = \bigoplus_{i=1}^q \hat{\mathcal{V}}_{-1} \oplus \bigoplus_{i=1}^n \hat{\mathcal{V}}_{a_i}$$

where  $a_1, \dots, a_n \in \mathbb{N} \setminus \{0\}$ . The great difficulty over  $\mathbb{C}$  is the degenerated coordinates: it is possible to have, for some  $1 \leq i \leq n$ ,  $v_i = 0$  while  $v_{-i} \neq 0$ . It implies many disjoint cases in the proofs. It requires also to add some polynomials of type  $H_{ijk}$  in the separating set:

**Theorem 6.1.** *The following set of polynomials separates closed orbits with respect to the action  $\mathrm{O}_2(\mathbb{C}) \curvearrowright \hat{\mathcal{V}}$ :*

$$\mathcal{F} = \left\{ \begin{array}{ll} P_{ij} = p_{-ij} + p_{-ji} & 1 \leq i \leq j \leq n \\ H_{ijk} = n_{ij-k} + n_{-i-jk} & (i, j, k) \in \mathcal{T}(n) \\ T_{ijk} = (p_{-ik} - p_{-ki})(p_{-jk} - p_{-kj}), & (i, j, k) \in \mathcal{T}(n) \\ Q_{ijkl} = (p_{-ij} - p_{-ji})(p_{-kl} - p_{-lk}), & (i, j, k, l) \in \mathcal{Q}(n) \\ D_{ij} = d_i d_j, & 1 \leq i \leq j \leq q \\ S_{ijk} = i(p_{-ij} - p_{-ji}) d_k, & 1 \leq i < j \leq n \text{ and } 1 \leq k \leq q \end{array} \right\}$$

To prove it, we take  $u = (d_1, \dots, d_q, u_1, u_{-1}, \dots, u_n, u_{-n})$  and  $v = (d'_1, \dots, d'_q, v_1, v_{-1}, \dots, v_n, v_{-n})$  in  $\mathcal{C}$  such that for any  $P \in \mathcal{F}$ ,  $P(u) = P(v)$  and exhibit some  $g \in \mathrm{O}_2(\mathbb{C})$  verifying  $\rho(g)(u) = v$ . We first notice some facts about degenerated indices. Take  $i$  such that  $u_i$  or  $u_{-i}$  vanish. Then  $P_{ii}(u) = P_{ii}(v) = 0$  ensures that  $v_i$  or  $v_{-i}$  vanish as well. Moreover, assume that we have  $u_i \neq 0$  while  $u_{-i} = 0$ . By Lemma 3.3, there exists some  $1 \leq j \leq n$  such that  $u_{-j} \neq 0$ . Then  $P_{ij}(u) = P_{ij}(v) \neq 0$  and we necessarily have  $v_i \neq 0$  or  $v_j \neq 0$ . The pairs  $(u_i, u_{-i})$  and  $(v_i, v_{-i})$  have has many vanishing components. There is a common  $\delta = \pm 1$  such that for any degenerated indice  $i$ ,  $u_i = 0 \Leftrightarrow v_{\delta \times i} = 0$ . And then,  $W_I^\delta \neq \emptyset$  while  $W_I^{-\delta} = \emptyset$ .

The sequence of lemmas in Section 5 holds with non degenerated indices over  $\mathbb{C}$  with the same proofs. We thus prove similar statements when some involved coordinates vanish:

**Lemma 6.2.** For any degenerated indice  $1 \leq i \leq n$ , we have one of the rows of the following table:

Vanishing coordinates	Defined quotient	$W_I = W_i \cap W_{-i}$	$W_I^- = W_i^- \cap W_{-i}^-$
$u_i$ and $v_i$	$c_{-i}$	$W_{-i}$	$\emptyset$
$u_{-i}$ and $v_{-i}$	$c_i$	$W_i$	$\emptyset$
$u_i$ and $v_{-i}$	$c_{-i}^-$	$\emptyset$	$W_{-i}^-$
$u_{-i}$ and $v_i$	$c_i^-$	$\emptyset$	$W_i^-$

*Proof.* Suppose that  $u_i = v_i = 0$ . Then for any  $z \in \mathbb{C}^*$  one has  $z^{a_i} u_i = 0 = v_i$  while  $z^{-a_i} u_i = 0 \neq v_{-i}$ . This is  $W_i = \mathbb{C}^*$  and  $W_i^- = \emptyset$ . Other rows of the table are obtained by similar considerations.  $\square$

**Lemma 6.3.** For any integers  $1 \leq i < j \leq n$ , we have  $W_{IJ} \neq \emptyset$  or  $W_{IJ}^- \neq \emptyset$ .

*Proof.* Suppose that  $u_i = v_i = 0$ . By lemma 5.2,  $W_I = W_{-i}$  while  $W_J = W_j = W_{-j}$ .

$$\begin{aligned} P_{ij}(v) = P_{ij}(u) &\Leftrightarrow p_{-ij}(u) = u_{-i}^{\varepsilon_{ij}} u_j^{\varepsilon_{ji}} = v_{-i}^{\varepsilon_{ij}} v_j^{\varepsilon_{ji}} = p_{-ij}(v) \\ &\Leftrightarrow c_{-i}^{\varepsilon_{ij}} = c_j^{\varepsilon_{ji}} \end{aligned}$$

Then lemma 2.3 claims that  $W_{-ij} \neq 0$ , and lemma 5.2 completes in  $W_{IJ} \neq \emptyset$ . By similar computations, we can fill the following table:

Vanishing coordinates	$\delta$	Non empty intersection	Vanishing coordinates	$\delta$	Non empty intersection
$u_i, v_i$	+	$W_{IJ}$	$u_i, v_{-i}$	-	$W_{IJ}^-$
$u_{-i}, v_{-i}$	+	$W_{IJ}$	$u_{-i}, v_i$	-	$W_{IJ}^-$
$u_i, v_i, u_{-j}, v_{-j}$	+	$W_{IJ}$	$u_i, v_{-i}, u_{-j}, v_j$	-	$W_{IJ}^-$
$u_{-i}, v_{-i}, u_j, v_j$	+	$W_{IJ}$	$u_{-i}, v_i, u_j, v_{-j}$	-	$W_{IJ}^-$
$u_i, v_i, u_j, v_{-j}$	+	Unknown	$u_i, v_{-i}, u_j, v_{-j}$	-	Unknown
$u_{-i}, v_{-i}, u_{-j}, v_{-j}$	+	Unknown	$u_{-i}, v_i, u_{-j}, v_j$	-	Unknown

The unknown cases of the table need the polynomials of the type  $H_{ijk}$  to be solved. Assume that  $u_i = u_j = v_i = v_j = 0$ . Since  $u \in \mathcal{C} \setminus \{0\}$ , lemma 3.3 ensures that there exists  $1 \leq k \leq n$  with  $u_k \neq 0$ . The known cases of the table ensure that  $W_{IK}$  and  $W_{JK}$  are not empty. Then,

$$\begin{aligned} H_{ijk}(v) = H_{ijk}(u) &\Leftrightarrow u_{-i}^{q\varepsilon_{ijk}} u_{-j}^{-r\varepsilon_{jik}} u_k^{(q'\varepsilon_{ijk} - r'\varepsilon_{jik})} = v_{-i}^{q\varepsilon_{ijk}} v_{-j}^{-r\varepsilon_{jik}} v_k^{(q'\varepsilon_{ijk} - r'\varepsilon_{jik})} \\ &\Leftrightarrow \left( c_{-i}^q c_k^{q'} \right)^{\varepsilon_{ijk}} = \left( c_{-i}^r c_k^{r'} \right)^{\varepsilon_{jik}} \end{aligned}$$

and Lemma 2.3 ensures that  $W_{IJK} \neq \emptyset$ . By inclusion  $W_{IJ}$  is not empty. The other unknown rows of the table are solved by similar considerations.  $\square$

**Lemma 6.4.** Let  $(i, j, k) \in \mathcal{T}(n)$  be some triple. Then, one has  $\left\{ \begin{array}{l} W_{IK} \neq \emptyset \\ W_{JK} \neq \emptyset \end{array} \right\}$  or  $\left\{ \begin{array}{l} W_{IK}^- \neq \emptyset \\ W_{JK}^- \neq \emptyset \end{array} \right\}$ .

*Proof.* If  $p_{-ij}(u) \neq 0$  or  $p_{-ji}(u) \neq 0$  and  $p_{-jk}(u) \neq 0$  or  $p_{-kj}(u) \neq 0$ , we refer to the proof of Lemma 5.4.

Suppose that  $p_{-ik}(u) = p_{-ki}(u) = 0$  while the indice  $j$  is not degenerated. Then,  $u_i = u_k = 0$  or  $u_{-i} = u_{-k} = 0$ . In the first case, since  $u \in \mathcal{C} \setminus \{0\}$ , we can find some  $1 \leq t \leq n$  such that  $u_{-t} \neq 0$  and  $p_{-ti}(u) \neq 0$ . In the second case we can find some  $1 \leq t \leq n$  such that  $u_t \neq 0$  and  $p_{-it}(u) \neq 0$ . Anyway then

$$T_{ijt}(u) = T_{ijt}(v) \Rightarrow \left\{ \begin{array}{l} W_{IT} \neq \emptyset \\ W_{JT} \neq \emptyset \end{array} \right\} \text{ or } \left\{ \begin{array}{l} W_{IT}^- \neq \emptyset \\ W_{JT}^- \neq \emptyset \end{array} \right\}$$

However, Lemma 6.2 ensures that  $W_I^\delta \neq \emptyset$  while  $W_I^{-\delta} = \emptyset$ . It follows by Lemma 6.3 that  $\left\{ \begin{array}{l} W_{IT}^\delta \neq \emptyset \\ W_{JT}^\delta \neq \emptyset \end{array} \right\}$ . We

deduce that  $\left\{ \begin{array}{l} W_{IK}^\delta \neq \emptyset \\ W_{KJ}^\delta \neq \emptyset \end{array} \right\}$ .



Else, suppose that  $p_{-ij}(u) = p_{-ji}(u) = p_{-kj}(u) = p_{-jk}(u) = 0$ . Here each pair  $(i, j)$  and  $(k, j)$  has a degenerated indice. Then Lemma 6.3 concludes that  $\begin{cases} W_{IJ}^\delta \neq \emptyset \\ W_{KJ}^\delta \neq \emptyset \end{cases}$  and the lemma holds.  $\square$

**Lemma 6.5.** *Let  $(i, j, k, l) \in \mathcal{Q}(n)$ . Then one has  $\begin{cases} W_{IJ} \neq \emptyset \\ W_{KL} \neq \emptyset \end{cases}$  or  $\begin{cases} W_{IJ}^- \neq \emptyset \\ W_{KL}^- \neq \emptyset \end{cases}$ .*

*Proof.* If  $p_{-ij}(u) \neq 0$  or  $p_{-ji}(u) \neq 0$  and  $p_{-kl}(u) \neq 0$  or  $p_{-lk}(u) \neq 0$ , we refer to the real proof of Lemma 5.5.

Suppose that  $p_{-ij}(u) = p_{-ji}(u) = 0$  while  $p_{-kl}(u) \neq 0$  or  $p_{-lk}(u) \neq 0$ . That is,  $u_i = u_j = 0$  or  $u_{-i} = u_{-j} = 0$ . In the first case, by Lemma 3.3, there exists some  $1 \leq t \leq n$  such that  $u_{-t} \neq 0$  and  $p_{-ti}(u) \neq 0$ . In the second case we can find some  $1 \leq t \leq n$  such that  $u_t \neq 0$  and  $p_{-it}(u) \neq 0$ . Anyway then

$Q_{itkl}(u) = Q_{itkl}(v) \Rightarrow \begin{cases} W_{IT} \neq \emptyset \\ W_{KL} \neq \emptyset \end{cases}$  or  $\begin{cases} W_{IT}^- \neq \emptyset \\ W_{KL}^- \neq \emptyset \end{cases}$ . However, Lemma 6.2 ensures that  $W_I^\delta \neq \emptyset$  while

$W_I^{-\delta} = \emptyset$ . It follows by Lemma 6.3 that  $\begin{cases} W_{IJ}^\delta \neq \emptyset \\ W_{KL}^\delta \neq \emptyset \end{cases}$ , and  $\begin{cases} W_{IJ}^\delta \neq \emptyset \\ W_{KL}^\delta \neq \emptyset \end{cases}$ .

Otherwise,  $p_{-ij}(u) = p_{-ji}(u) = p_{-kl}(u) = p_{-lk}(u) \neq 0$ , that is each pair  $(i, j)$  and  $(k, l)$  has a degenerated indice. Lemma 6.3 implies that  $\begin{cases} W_{IJ}^\delta \neq \emptyset \\ W_{KL}^\delta \neq \emptyset \end{cases}$  and the lemma holds.  $\square$

Once we have Lemmas 6.4 and 6.5, it is possible to conclude the proof of Theorem 6.1 as the real analogue in Section 5.

In general, the separating set of Theorem 6.1 is not minimal for the inclusion. We remove redundant polynomials to get a minimal separating set:

**Proposition 6.6.** *In addition to those of Proposition 5.6, we make the following removals:*

- For any  $(i, j, k) \in \mathcal{T}(n)$  such that  $a_k$  divides  $a_i \vee a_j$ , we remove  $H_{ijk}$ .

*Then the separating set  $\mathcal{F}$  is minimal for the inclusion.*

*Proof.* We try to remove a polynomial and find two non separated closed orbits. For the polynomials of type  $P, T, Q, D$  and  $S$  we refer to section 5.

Assume that we remove  $H_{ijk}$  for some  $(i, j, k) \in \mathcal{T}(n)$ . We take  $u$  and  $v$  satisfying one of the four unknown cases of Lemma 6.3 (otherwise, polynomials of type  $P, T, Q, D$  and  $S$  suffice to determine if  $u$  and  $v$  are in the same orbit). This is:

$$\begin{cases} W_I^\delta \neq \emptyset \\ W_J^\delta \neq \emptyset \end{cases} \quad \text{and} \quad \begin{cases} W_I^{-\delta} = \emptyset \\ W_J^{-\delta} = \emptyset \end{cases}$$

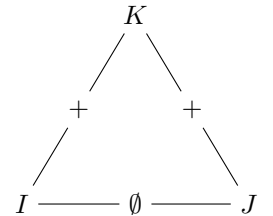
If there exists another integer  $l \neq k$  such that  $P_{il}(u) \neq 0$ , then by Lemma 6.3,  $W_{IJ}^\delta \neq \emptyset$  and  $u$  and  $v$  are in the same orbit. Otherwise, by Lemma 3.3,  $k$  is the only integer such that  $P_{ik}$  and  $P_{jk}$  do not vanish at

$u$  and  $v$ . Then, by Lemma 6.3 we have  $\begin{cases} W_{IK}^\delta \neq \emptyset \\ W_{JK}^\delta \neq \emptyset \end{cases}$ . If  $a_k$  divides  $a_i \vee a_j$ , Proposition 2.5 claims directly that  $W_{IJ}^\delta \neq \emptyset$  and  $u$  and  $v$  lies in the same orbit.

On the opposite, if  $a_k$  does not divides  $a_i \vee a_j$ , we choose  $u$  and  $v$  as in the following table:

	$i$	$-i$	$j$	$-j$	$k$	$-k$
$u$	1	0	1	0	1	1
$v$	1	0	$\exp\left(\frac{2i\pi a_j}{a_k}\right)$	0	1	1

These choices imply first that  $W_I^- = W_J^- = \emptyset$ . We next note  $c_i = c_k = 1$  while  $c_j = \exp\left(\frac{2i\pi a_j}{a_k}\right)$ . Then,  $c_i \frac{a_i \vee a_j}{a_i} c_j \frac{a_i \vee a_j}{a_j} = \exp\left(\frac{2i\pi a_j}{a_k} \frac{a_i \vee a_j}{a_j}\right) = \exp\left(2i\pi \frac{a_i \vee a_j}{a_k}\right) \neq 1$ . Lemma 2.3 provides by that way the intersections described in the right figure. We deduce that  $P_{ik}$  and  $P_{jk}$  agree at  $u$  and  $v$  while all the other polynomials in  $\mathcal{F}$  vanish. Although,  $u$  and  $v$  are not in the same orbit.



□

**Remark 6.7.** *The obtained minimal separating set thus has cardinality bounded by*

$$n^2 \sim \underbrace{\frac{n(n+1)}{2}}_{P_{ij}} + \underbrace{\frac{q(q+1)}{2}}_{D_{ij}} + \underbrace{q \binom{n}{2}}_{S_{ijk}} \leq \#\mathcal{F}(\mathbb{R}) \leq \underbrace{\frac{n(n+1)}{2}}_{P_{ij}} + \underbrace{3 \binom{n}{3}}_{H_{ijk}} + \underbrace{3 \binom{n}{3}}_{T_{ijk}} + \underbrace{3 \binom{n}{4}}_{Q_{ijkl}} + \underbrace{\frac{q(q+1)}{2}}_{D_{ij}} + \underbrace{q \binom{n}{2}}_{S_{ijk}} \sim n^4$$

The lower bound is reached when all the integers  $a_i$  are equal to 1.

**Example 6.8.** *Let us consider again the representation of  $O_2(\mathbb{C})$  on  $\hat{\mathcal{V}} := \hat{\mathcal{V}}_{-1}^3 \oplus \hat{\mathcal{V}}_2 \oplus \hat{\mathcal{V}}_3 \oplus \hat{\mathcal{V}}_5 \oplus \hat{\mathcal{V}}_7$ . To the real separating set of cardinality 46 introduced in Example 5.10 are added 12 polynomials of type  $H$ :*

$$\left\{ \begin{array}{cccccc} H_{2,3,1} & H_{1,2,3} & H_{1,3,2} & H_{2,4,1} & H_{1,2,4} & H_{1,4,2} \\ H_{3,4,1} & H_{1,3,4} & H_{1,4,3} & H_{3,4,2} & H_{2,3,4} & H_{2,4,3} \end{array} \right\}$$

We then obtain a separating set for  $SO_2(\mathbb{C}) \curvearrowright \hat{\mathcal{V}}$  of cardinality 58 by Theorem 6.1. The difference of cardinality between real and complex separating set still exists but is less remarkable than for the representations of  $SO_2(\mathbb{C})$  and  $SO_2(\mathbb{R})$  (see Example 4.4).

## 7 Stratification of the real orbit space

In this section we give a stratification of the orbit space through the real separating set of Sections 4 and 5. From the evaluation of the polynomials of  $f(\mathbb{R})$  or  $\mathcal{F}(\mathbb{R})$  at some point  $v \in \mathcal{V}$ , we provide inequalities determining the isotropy class of  $v$ .

**Definition 7.1.** *Let  $G$  be a compact group and  $\rho$  a continuous representation on a vector space  $\mathcal{V}$ . Take  $v$  a point in  $\mathcal{V}$ . The isotropy group of  $v$  is defined as  $G_v = \{g \in G \mid \rho(g)(v) = v\}$ . Two points are of same orbit type when their isotropy groups are conjugated. The isotropy stratum  $\Sigma_J$  is the subset of points whose isotropy group is conjugated to  $J < G$ :*

$$\Sigma_J = \{v \in V \text{ such that } G_v \in [J]\}$$

There is a partial order among isotropy classes. Namely, when  $J_1, J_2$  are two isotropy groups, one notes

$$[J_1] < [J_2] \Leftrightarrow \exists K \in [J_1] \text{ such that } K < J_2$$

This induces a partial order among isotropy conjugacy classes, and one defines the isotropy stratum closure which corresponds to the set of points stabilized by at least a member of  $[J]$ :

$$\overline{\Sigma_J} = \bigcup_{[J] < [K]} \Sigma_K$$

The possible isotropy classes in  $SO_2(\mathbb{R})$  are  $[\{I_2\}]$ ,  $[SO_2(\mathbb{R})]$  itself and  $[C_k]$  the cyclic groups of order  $k \in \mathbb{N} \setminus \{0\}$  (Figure 3). The possible isotropy classes in  $O_2(\mathbb{R})$  are  $[\{I_2\}]$ ,  $[SO_2(\mathbb{R})]$ ,  $[O_2(\mathbb{R})]$ ,  $[C_k]$  for  $k \in \mathbb{N} \setminus \{0\}$  and  $[D_k]$  the dihedral groups of order  $2k \in \mathbb{N}^*$  (Figure 4). The isotropy class  $[C_k]$  has a single representant: the subgroup of  $SO_2(\mathbb{R})$  generated by  $g_z^+$  where  $z$  is a  $k^{th}$  primitive root. However, the isotropy class  $[D_k]$  has an infinite number of conjugated representants. These are the subgroups of  $O_2(\mathbb{R})$  generated by  $C_k$  and a symmetry  $g_z^- \in O_2(\mathbb{R})$ .

$SO_2(\mathbb{R})$



$C_k$

$\{1\}$

Figure 3

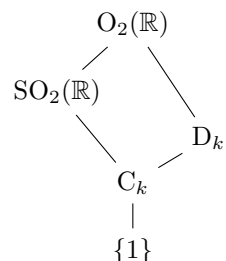


Figure 4

The decomposition into irreducible components  $\mathcal{V} = \bigoplus_{i=1}^q \mathcal{V}_{-1} \oplus \bigoplus_{i=1}^n \mathcal{V}_{a_i}$  helps to compute the stabilizer of a point. The isotropy group of  $v \in \mathcal{V}$  is the intersection of the isotropy groups of its projections onto each representation in the irreducible decomposition [Oli14]. This leads to the following Theorem:

**Theorem 7.2.** *Consider the action of  $\text{SO}_2(\mathbb{R})$  on  $\mathcal{V}$ . Let  $v$  be a point in  $\mathcal{V}$ .*

- For  $k \in \mathbb{N} \setminus \{0\}$ ,  $v \in \overline{\Sigma_{C_k}}$  iff for all  $i$  such that  $k$  does not divide  $a_i$ ,  $\begin{cases} \Re(p_{-ii})(v) = 0 \\ \Im(p_{-ii})(v) = 0 \end{cases}$
- $v \in \overline{\Sigma_{\text{SO}_2(\mathbb{R})}}$  iff for all  $i \leq n$ ,  $\begin{cases} \Re(p_{-ii})(v) = 0 \\ \Im(p_{-ii})(v) = 0 \end{cases}$

*Proof.* For any  $k \in \mathbb{N}^*$ , the possible isotropy classes in  $\mathcal{V}_k$  are  $\text{SO}_2(\mathbb{R})$  and  $C_k$ , and the only point of isotropy class  $\text{SO}_2(\mathbb{R})$  is  $(0, 0)$ . By intersection then, the isotropy group  $G_v$  is  $C_{\varphi(v)}$ , where  $\varphi(v)$  is the gcd of all  $a_i$  such that the projection of  $v$  onto  $\mathcal{V}_{a_m}$  is not zero, or equivalently  $p_{-ii}(v) \neq 0$ . If there is no such  $a_i$ , then  $G_v$  is  $\text{SO}_2(\mathbb{R})$ .  $\square$

**Theorem 7.3.** *Consider the action of  $\text{O}_2(\mathbb{R})$  on  $\mathcal{V}$ . Let  $v$  be a point in  $\mathcal{V}$ .*

- For any  $k \in \mathbb{N} \setminus \{0\}$ ,  $v \in \overline{\Sigma_{C_k}}$  iff  $\forall 1 \leq i \leq n$  such that  $a_i \neq 0 [k]$ ,  $P_{ii}(v) = 0$ .
- For any  $k \in \mathbb{N} \setminus \{0\}$ ,  $v \in \overline{\Sigma_{D_k}}$  iff  $\begin{cases} \forall 1 \leq i \leq n \mid a_i \neq 0 [k], P_{ii}(v) = 0. \\ \forall 1 \leq i < j \leq n, (P_{ij}^2 - 4P_{ii}^{\varepsilon_{ij}} P_{jj}^{\varepsilon_{ji}})(v) = 0. \\ \forall 1 \leq i \leq q, D_{ii}(v) = 0. \end{cases}$
- $v \in \overline{\Sigma_{\text{SO}_2(\mathbb{R})}}$  iff for all  $i \leq n$ ,  $P_{ii}(v) = 0$ .
- $v \in \overline{\Sigma_{\text{O}_2(\mathbb{R})}}$  iff for all  $i \leq n$ ,  $P_{ii}(v) = 0$  and for all  $1 \leq i \leq q$ ,  $D_{ii}(v) = 0$ .

*Proof.* We look at the possible isotropy classes of the projection on each irreducible representation:

- For any  $k \geq 2$ , the possible isotropy classes in  $\mathcal{V}_k$  are  $\text{O}_2(\mathbb{R})$  and  $D_k$ , and the only point of isotropy class  $\text{O}_2(\mathbb{R})$  is  $(0, 0)$ .
- The possible isotropy classes in  $\mathcal{V}_{-1}$  are  $\text{SO}_2(\mathbb{R})$  and  $\text{O}_2(\mathbb{R})$ , and the only point of stabilizer  $\text{O}_2(\mathbb{R})$  is 0.

We first identify the subset  $\text{Stab}(v)^+ = \text{Stab}(v) \cap \text{SO}_2(\mathbb{R})$ . As for Theorem 7.3,  $\text{Stab}(v)^+ = C_{\varphi(v)}$ , where  $\varphi(v)$  is the gcd of all  $a_i$  such that the projection of  $v$  onto  $\mathcal{V}_{a_i}$  is not zero, equivalently  $P_{ii}(v) \neq 0$ . If there is no such  $a_i$ , then  $\text{Stab}(v)^+ = \text{SO}_2(\mathbb{R})$ .

We next determine if there exists a symmetry  $g_z^- \in \text{Stab}(v)$ . If not,  $\text{Stab}(v) = \text{Stab}(v)^+$ . If yes,  $\text{Stab}(v) = \text{Stab}(v)^+ \rtimes C_2$ . There exists  $z \in \mathbb{C}^*$  such that  $\rho(g_z^-)(v) = v$  iff  $\bigcap_{i=1}^n W_I^- \neq \emptyset$  and  $\forall 1 \leq i \leq q$ ,  $d_i = 0$ .

We then check that for all  $1 \leq i < j \leq n$ ,

$$\begin{aligned} (P_{ij}^2 - 4P_{ii}^{\varepsilon_{ij}} P_{jj}^{\varepsilon_{ji}})(v) = 0 &\Leftrightarrow v_{-i}^{2\varepsilon_{ij}} v_j^{2\varepsilon_{ji}} + 2(v_{-i} v_i)^{\varepsilon_{ij}} (v_{-j} v_j)^{\varepsilon_{ji}} + v_i^{2\varepsilon_{ij}} v_{-j}^{2\varepsilon_{ji}} - 4(v_{-i} v_i)^{\varepsilon_{ij}} (v_{-j} v_j)^{\varepsilon_{ji}} = 0 \\ &\Leftrightarrow v_{-i}^{2\varepsilon_{ij}} v_j^{2\varepsilon_{ji}} - 2(v_{-i} v_i)^{\varepsilon_{ij}} (v_{-j} v_j)^{\varepsilon_{ji}} + v_i^{2\varepsilon_{ij}} v_{-j}^{2\varepsilon_{ji}} = 0 \\ &\Leftrightarrow (p_{-ij} - p_{-ji})(v)^2 = 0 \\ &\Leftrightarrow W_{I,J}^- \neq \emptyset \end{aligned}$$

Furthermore we have  $\forall 1 \leq i \leq q$ ,  $d_i = 0 \Leftrightarrow \forall 1 \leq i \leq q$ ,  $D_{ii}(v) = 0$ .  $\square$

**Remark 7.4.** *Although the groups  $C_2$  and  $D_1$  are isomorphic, they are not conjugated in  $\text{SO}_2(\mathbb{R})$ . They thus represent two distinct isotropy classes, as enumerated in Theorem 7.3. The corresponding strata form two distinct irreducible varieties in the orbit space.*

**Example 7.5.** Consider the representation of  $O_2(\mathbb{R})$  on  $\mathcal{S} = \mathcal{V}_{-1}^3 \oplus \mathcal{V}_0 \oplus \mathcal{V}_1^3 \oplus \mathcal{V}_2^2 \oplus \mathcal{V}_3$  endowed with the coordinates  $(d_1, d_2, d_3, t_1, v_1, v_{-1}, v_2, v_{-2}, v_3, v_{-3}, v_4, v_{-4}, v_5, v_{-5}, v_6, v_{-6})$ . This action corresponds to the reduction of the action of  $SO_3(\mathbb{R})$  on the space of piezoelectricity tensors  $\text{Piez}$  thanks to a Sechadri slice [HJ24]. The stratification of the orbit space  $\mathcal{S}/O_2(\mathbb{R})$  allows to deduce almost everywhere the isotropy group of tensors in  $\text{Piez}$ . Namely, for any  $h \in \mathcal{S}$  we have:

- $h \in \overline{\Sigma_{C_2}} \Leftrightarrow P_{11}(h) = P_{22}(h) = P_{33}(h) = P_{66}(h) = 0.$
- $h \in \overline{\Sigma_{C_3}} \Leftrightarrow P_{11}(h) = P_{22}(h) = P_{33}(h) = P_{44} = P_{55}(h) = 0.$
- $h \in \overline{\Sigma_{D_1}} \Leftrightarrow \begin{cases} \forall 1 \leq i < j \leq 6, (P_{ii}^{\varepsilon_{ij}} P_{jj}^{\varepsilon_{ji}} - P_{ij}^2)(v) = 0. \\ D_{11}(h) = D_{22}(h) = D_{33}(h) = 0 \end{cases}$
- $h \in \overline{\Sigma_{D_2}} \Leftrightarrow \begin{cases} P_{11}(h) = P_{22}(h) = P_{33}(h) = P_{66}(h) = 0 \\ (P_{44}P_{55} - P_{45}^2)(v) = 0 \\ D_{11}(h) = D_{22}(h) = D_{33}(h) = 0 \end{cases}$
- $h \in \overline{\Sigma_{D_3}} \Leftrightarrow \begin{cases} P_{11}(h) = P_{22}(h) = P_{33}(h) = P_{44}P_{55}(h) = 0 \\ D_{11}(h) = D_{22}(h) = D_{33}(h) = 0 \end{cases}$
- $h \in \overline{\Sigma_{SO_2(\mathbb{R})}} \Leftrightarrow P_{11}(h) = P_{22}(h) = P_{33}(h) = P_{44}(h) = P_{55}(h) = P_{66}(h) = 0.$
- $h \in \overline{\Sigma_{O_2(\mathbb{R})}} \Leftrightarrow \begin{cases} P_{11}(h) = P_{22}(h) = P_{33}(h) = P_{44} = P_{55}(h) = P_{66}(h) = 0 \\ D_{11}(h) = D_{22}(h) = D_{33}(h) = 0 \end{cases}$

## A Irreducible representations of $SO_2$ and $O_2$

We recall essential properties of representations of  $SO_2$  and  $O_2$ .  $SO_2(\mathbb{C})$  is in fact the torus  $\mathbb{C}^*$  and there is a semidirect product  $O_2(\mathbb{C}) \cong SO_2(\mathbb{C}) \rtimes C_2$ . For  $z \in \mathbb{C}^*$  we note  $g_z^+ = (z, 1) \in SO_2(\mathbb{C})$  and  $g_z^- = (z, -1) \in O_2(\mathbb{C})$ . As it is abelian,  $SO_2(\mathbb{C})$  has irreducible representations of dimension one. These are gathered pairwise to form the irreducible representations of  $O_2(\mathbb{C})$ :

**Theorem A.1.** *The irreducible representations of  $SO_2(\mathbb{C})$  consist of  $(\phi_j, \mathcal{L}_j)$ ,  $j \in \mathbb{Z}$  where  $\mathcal{L}_j = \mathbb{C}$  and for  $z \in \mathbb{C}^*$ ,  $\phi_j(g_z^+) = z^j$ .*

**Theorem A.2.** [GSS88] *The irreducible representations of  $O_2(\mathbb{C})$  consist of*

- $(\rho_i, \hat{\mathcal{V}}_i)$  for  $i \in \mathbb{N} \setminus \{0\}$ , where  $\hat{\mathcal{V}}_i = \mathbb{C}^2$  and for any  $z \in \mathbb{C}^*$ ,  $\rho_i(g_z^+) = \begin{pmatrix} z^i & 0 \\ 0 & z^{-i} \end{pmatrix}$  and  $\rho_i(g_z^-) = \begin{pmatrix} 0 & z^{-i} \\ z^i & 0 \end{pmatrix}$ .
- *The trivial representations  $(\rho_0, \hat{\mathcal{V}}_0)$ , and the determinant  $(\rho_{-1}, \hat{\mathcal{V}}_{-1})$ , both of dimension one.*

The standard parameterization of  $SO_2(\mathbb{R})$  is  $\left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \theta \in \mathbb{R} \right\}$ . The conjugation by  $P := \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$  recovers the parameterization

$$O_2(\mathbb{R}) \cong \left\{ g_z^+ = \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix}, z \in \mathcal{U} \right\} \cup \left\{ g_z^- = \begin{pmatrix} 0 & \frac{1}{z} \\ z & 0 \end{pmatrix}, z \in \mathcal{U} \right\},$$

where  $\mathcal{U} \subset \mathbb{C}$  is the unit circle. We obtain the irreducible representations restricting the complex representations to the real trace:

**Theorem A.3.** *The irreducible representations of  $SO_2(\mathbb{R})$  consist of*

- $(\rho_i, \mathcal{V}_i)$ , for  $i \in \mathbb{N} \setminus \{0\}$ .
- *The trivial representations  $(\rho_0, \mathcal{V}_0)$ .*

**Theorem A.4.** *The irreducible representations of  $O_2(\mathbb{R})$  are*

- $(\rho_i, \mathcal{V}_i)$ , for  $i \in \mathbb{N}^*$ .
- *The trivial representations  $(\rho_0, \mathcal{V}_0)$  and the determinant  $(\rho_{-1}, \mathcal{V}_{-1})$ .*

For each  $i$  we endow  $\hat{\mathcal{V}}_i$  with coordinates  $(v_i, v_{-i})$ . On the real trace  $\mathcal{V}_i \subset \hat{\mathcal{V}}_i$ , the coordinates satisfy  $v_i = \overline{v_{-i}}$ . Throughout the article we ignore the trivial representations. Indeed, to separate the orbits in a representation with an additional trivial part  $\mathcal{L}_0^k$ , it suffices to add the linear polynomials  $\{t_i, 1 \leq i \leq k\}$ .

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