Light Scattering by an Ellipsoid according to A.F. Stevenson's method

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Résumé : ce papier reprend le calcul du champ électromagnétique diffusé par un ellipsoïde, dont le grand axe est plus petit que la longueur d'onde du champ incident, selon la méthode de A.F. Stevenson (1953). Contrairement à la publication originale de Stevenson, les calculs y sont détaillés et complétés. Ceci devrait autoriser et faciliter d'autres applications que celles proposées par Stevenson.

Abstract: This paper computes the electromagnetic field scattered by an ellipsoid, whose major axis is smaller than the wavelength of the incident wave. Calculations have been achieved according to the method of A.F. Stevenson (1953). Unlike Stevenson's original publication, the calculations are detailed and completed. This should allow and facilitate other applications than those proposed by Stevenson.

Content

1. Introduction

The interaction between an electromagnetic wave and an object results in the existence of an electromagnetic field inside and outside the object (the scattered field). The electromagnetic field is obtained by solving Maxwell's equations associated with boundary conditions (the surface of the object and the surface of a sphere of infinite radius having the same center as the object). The corresponding exact solutions exist for very few objects: the sphere (Mie theory, 1908), the doublet of spheres (Trinks, 1935). Consequently, a large number of numerical methods were developed from 1960 onwards (Kahnert, 2003).

Recently, these latter have been extended to compute not only the fields, but the normalized (Quasi-Normal) Modes of a scattering system (one or several nanoparticles, possibly above a multi-layered substrate) (Bai et al. ,2013 ; Yan et al., 2018).

Despite this impressive progress on the numerics side, it is still an issue to derive analytical expressions at least in some physical regimes, to describe the field inside and outside a (non-spherical) particle. At the single particle level, this would be helpful to compute accurately e.g. the torques exerted by light on a trapped anisotropic nanoobject (Bellando et al., 2022), what is difficult to do with numerical methods, e.g. Finite Element Methods, when the particle is lossless. This could also permit to answer fundamental questions on the Rayleigh Hypothesis in the case of non-spherical particles (Augiué et al., 2016), comparing the sought expansions to an analytical expression instead of numerical calculations. At a higher level of complexity, when many coupled anisotropic nanoparticles are involved (Bertrand et al., 2020) , the scattering system can be modeled by an ensemble of coupled multipoles, each of them being computed from the sole knowledge of the internal field only (Radescu et al., 2002). These multipoles could be derived in a convenient way, if one could obtain an analytical expression for the internal field. From the very start of the search for a solution to this problem, and in view of its difficulty, investigators proposed approximate solutions applicable in restricted domains of the plane defined by

- the ratio of object size to the wavelength of the incident electromagnetic wave
- the optical contrast between the material of the object and the medium in which it is immersed.

We thus speak of the Rayleigh approximation for objects that are very small compared to the wavelength, the Rayleigh-Debye-Gans approximation for small objects with low optical contrast, Fraunhofer diffraction for very large objects with high optical contrast, etc....

These approximations, which result in much simpler computations, are applicable to objects of any shape.

An intermediate approach goes beyond the Rayleigh approximation by expanding the electromagnetic field, which is a function of the wave vector, as an $(ik)^n$ series, where *k* is the modulus of the wave vector. This approach was formally discussed by Stevenson (1953) and later criticized by Kleinman (1965). Stevenson applied it to the simplest non-spherical object: the ellipsoid (1953). The topic was subsequently abandoned; however, it has been recently revived by the possibility of synthesizing nanometric ellipsoidal particles and their use in nanooptics. , in the field of vacuum levito-dynamics (Gonzalez-Ballestero, 2022).

Stevenson's article on ellipsoids is difficult to read, as various researchers have mentioned:

- no intermediate calculation is provided, only the final result, with this indication: “The calculations are somewhat long and involved, but are fairly straightforward applications of ellipsoidal harmonics”.

- The calculation shows the result to 0th and 1st orders ($n = 0,1$) and very partially to 2nd order, bearing in mind that the higher the order, the more complex the computations. Stevenson has strictly limited himself to calculations enabling the far-field scattering to be obtained.

The aim of this paper is to present, based on Stevenson's paper, the detailed calculations leading to the expansion of the electromagnetic field to 2nd order (complete). This will enable the reader to calculate the electromagnetic field at any point (inside, outside the ellipsoid), to continue the calculation at higher orders, to compute, for instance, the polarizability of the object as a function of the modulus of the wave vector.

This paper is based on the analysis of three documents:

- Stratton's well-known book on electromagnetism (1941)
- Stevenson's papers (1953)
- Dassios’ book on ellipsoidal harmonics (2012).

For reasons of anteriority of published work, I have approached 0th and 1st orders from Stratton and Stevenson's ideas without explicitly using ellipsoidal harmonics, then using them; on the other hand, for 2nd order requiring more complex calculations, the use of ellipsoidal harmonics proved necessary. The notations used in this paper will, if possible, be those of Stevenson and Dassios.

The following table lists, for each order, the main equations required to calculate the Stevenson functions and hence the electromagnetic field. Note that the harmonic potentials $V_0, V_0^{(i)}, U_1, U_1^{(i)}, V_2, V_2^{(i)}$ appear in the field expressions as their gradients:

$$E_0 = \nabla V_0 \quad E_0^{(i)} = \nabla V_0^{(i)}$$
$$H_1 = G_1 + \nabla U_1 \quad H_1^{(i)} = G_1^{(i)} + \nabla U_1^{(i)}$$
$$E_2 = F_2 + \nabla V_2 \quad E_2^{(i)} = F_2^{(i)} + \nabla V_2^{(i)}$$

Maxwell's equations are invariant under the substitutions $E \to H, \quad H \to -E, \quad \varepsilon \leftrightarrow \mu$. So, $H_0, H_0^{(i)}, -E_1, -E_1^{(i)}, H_2, H_2^{(i)}$ are directly obtained from $E_0, E_0^{(i)}, H_1, H_1^{(i)}, E_2, E_2^{(i)}$ if we make the substitutions for the direction cosines (of the incident electric and magnetic fields) and the material properties:

$$(l_1 \quad m_1 \quad n_1) \to (l_2 \quad m_2 \quad n_2)$$
$$(l_2 \quad m_2 \quad n_2) \to -(l_1 \quad m_1 \quad n_1)$$
$$\varepsilon \leftrightarrow \mu$$

Main results :

The table below describes the main results of the paper. Note that both the cartesian and ellipsoidal coordinates appear in the field expressions (which are valid for any aspect ratio of the ellipsoid). This is at the difference with the theory developed in (Boyde et al., 2009), which was focused on larger objects, close to spheres.

0th order	4.25, 5.1	$V_0, V_0^{(i)}$
1st order	5.12 5.71, 5.72 5.110, 5.112, 5.113	$G_1, G_1^{(i)}$ $U_1, U_1^{(i)}$
2nd order	6.3 6.14, 6.17 Tables 1 and 2 6.45, 6.56 6.58 Table 3 6.89, 6.90 6.167 6.186, 6.189, 6.191 6.196, 6.198, 6.199	F_2 V_2 " $F_2^{(i)}$ $V_2, V_2^{(i)}$

The corresponding software may be required to the author

Biographical details: Arthur.F. Stevenson (1899-1968), a mathematician by training, was a researcher in mathematical physics who worked in atomic physics from the early 1930s. While at Wayne State University, he turned his attention to the study of the interaction between electromagnetic waves (microwaves and light) and various objects. His last publication dates from 1968.

2. Fundamentals on Stevenson's method

Using Stevenson's notations, the incident polarized plane electromagnetic wave is characterized by the three sets of direction cosines :

(l,m,n) for the vector k

(l_1,m_1,n_1) for the vector $E^{(0)}$

(l_2,m_2,n_2) for the vector $H^{(0)}$

then,

$$\begin{aligned} E^{(0)} &= \begin{pmatrix} l_1 & m_1 & n_1 \end{pmatrix} e^{ik(lx+my+nz)} \\ H^{(0)} &= \begin{pmatrix} l_2 & m_2 & n_2 \end{pmatrix} e^{ik(lx+my+nz)} \end{aligned} \tag{2.1}$$

Incident fields and scattered fields can be expressed as a power series :

$$\begin{aligned} E^{(0)} &= \sum_{p=0} E_p^{(0)} (ik)^p \qquad E = \sum_{p=0} E_p (ik)^p \\ H^{(0)} &= \sum_{p=0} H_p^{(0)} (ik)^p \qquad H = \sum_{p=0} H_p (ik)^p \end{aligned} \tag{2.2}$$

The power series truncated at the order 2 for the incident wave is:

$$\begin{aligned} E_0^{(0)} &= \begin{pmatrix} l_1 & m_1 & n_1 \end{pmatrix} \quad E_1^{(0)} = \begin{pmatrix} l_1 & m_1 & n_1 \end{pmatrix} (lx+my+nz) \quad E_2^{(0)} = \frac{1}{2} \begin{pmatrix} l_1 & m_1 & n_1 \end{pmatrix} (lx+my+nz)^2 \\ H_0^{(0)} &= \begin{pmatrix} l_2 & m_2 & n_2 \end{pmatrix} \quad H_1^{(0)} = \begin{pmatrix} l_2 & m_2 & n_2 \end{pmatrix} (lx+my+nz) \quad H_2^{(0)} = \frac{1}{2} \begin{pmatrix} l_2 & m_2 & n_2 \end{pmatrix} (lx+my+nz)^2 \end{aligned} \tag{2.3}$$

According to Stevenson, we distinguish the field inside the ellipsoid, denoted ($E^{(i)}$,$H^{(i)}$), and the scattered field (E,H). They obey the equations (coming directly from the Maxwell's equations) :

$$
\begin{aligned}
&E_0 = \nabla V_0 \quad E_0^{(i)} = \nabla V_0^{(i)} \\
&H_1 = G_1 + \nabla U_1 \quad H_1^{(i)} = G_1^{(i)} + \nabla U_1^{(i)} \\
&E_2 = F_2 + \nabla V_2 \quad E_2^{(i)} = F_2^{(i)} + \nabla V_2^{(i)} \\
&\nabla \wedge G_1 = -E_0 \quad \nabla \wedge G_1^{(i)} = -\varepsilon E_0^{(i)} \\
&\nabla \wedge F_2 = H_1 \quad \nabla \wedge F_2^{(i)} = \mu H_1^{(i)} \\
&\nabla \bullet G_1 = \nabla \bullet G_1^{(i)} = \nabla \bullet F_2 = \nabla \bullet F_2^{(i)} = 0
\end{aligned}
\tag{2.4}
$$

$\wedge$ is the cross product, $\bullet$ is the scalar product.

with the boundary conditions (ellipsoid surface) :

$$
\begin{aligned}
&n \wedge \nabla \left(V_0 - V_0^{(i)} \right) = -n \wedge E_0^{(0)} \\
&n \bullet \nabla \left(V_0 - \varepsilon V_0^{(i)} \right) = -n \bullet E_0^{(0)} \\
&n \wedge \nabla \left(U_1 - U_1^{(i)} \right) = -n \wedge \left(G_1 - G_1^{(i)} + H_1^{(0)} \right) \\
&n \bullet \nabla \left(U_1 - \mu U_1^{(i)} \right) = -n \bullet \left(G_1 - \mu G_1^{(i)} + H_1^{(0)} \right) \\
&n \wedge \nabla \left(V_2 - V_2^{(i)} \right) = -n \wedge \left(F_2 - F_2^{(i)} + E_2^{(0)} \right) \\
&n \bullet \nabla \left(V_2 - \varepsilon V_2^{(i)} \right) = -n \bullet \left(F_2 - \varepsilon F_2^{(i)} + E_2^{(0)} \right)
\end{aligned}
\tag{2.5}
$$

n is the outward pointing unit normal.

with

$$
\int n \bullet E_0 dS = \int n \bullet H_1 dS = \int n \bullet E_2 dS = 0 \tag{2.6}
$$

It should be mentioned that, according to Stevenson, the functions V_0, U_1, V_2 and $V_0^{(i)}, U_1^{(i)}, V_2^{(i)}$ are harmonic functions in the mathematical sense and not electrical potentials ($E_0 = \nabla V_0$ and not $E_0 = -\nabla V_0$ in physics).

3. Scatterer: the ellipsoid

A point of the 3D space is specified by it cartesian coordinates x_1,x_2,x_3.

The ellipsoid is defined by it three semi-axes a_1,a_2,a_3 (denoted a,b and c by Stevenson). The equation of the surface of the ellipsoid is :

$$\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}=1 \text{ with } a_1>a_2>a_3 \tag{3.1}$$

The following notation will be used :

$$h_1=\left(a_2^2-a_3^2\right)^{\frac{1}{2}} \quad h_2=\left(a_1^2-a_3^2\right)^{\frac{1}{2}} \quad h_3=\left(a_1^2-a_2^2\right)^{\frac{1}{2}} \tag{3.2}$$

In the future, it will be useful to use ellipsoidal coordinates (ξ,η,ζ homogeneous to squared lengths) defined using the following expressions:

$$\begin{aligned}
&\frac{x_1^2}{a_1^2+\xi}+\frac{x_2^2}{a_2^2+\xi}+\frac{x_3^2}{a_3^2+\xi}=1 \quad ;\xi>-a_3^2\\
&\frac{x_1^2}{a_1^2+\eta}+\frac{x_2^2}{a_2^2+\eta}+\frac{x_3^2}{a_3^2+\eta}=1 \quad ;-a_3^2>\eta>-a_2^2\\
&\frac{x_1^2}{a_1^2+\zeta}+\frac{x_2^2}{a_2^2+\zeta}+\frac{x_3^2}{a_3^2+\zeta}=1 \quad ;-a_2^2>\zeta>-a_1^2
\end{aligned} \tag{3.3}$$

These coordinates cover the whole of three-dimensional space and are linked to Cartesian coordinates by the relationships :

$$\begin{aligned}
x_1^2&=\frac{\left(\xi+a_1^2\right)\left(\eta+a_1^2\right)\left(\zeta+a_1^2\right)}{\left(a_2^2-a_1^2\right)\left(a_3^2-a_1^2\right)}\\
x_2^2&=\frac{\left(\xi+a_2^2\right)\left(\eta+a_2^2\right)\left(\zeta+a_2^2\right)}{\left(a_3^2-a_2^2\right)\left(a_1^2-a_2^2\right)}\\
x_3^2&=\frac{\left(\xi+a_3^2\right)\left(\eta+a_3^2\right)\left(\zeta+a_3^2\right)}{\left(a_1^2-a_3^2\right)\left(a_2^2-a_3^2\right)}
\end{aligned} \tag{3.4}$$

Since we are interested in harmonic functions, we will need to write the Laplacian in ellipsoidal coordinates:

$$\Delta u=\frac{4}{(\xi-\eta)(\xi-\zeta)(\eta-\zeta)}\left[(\eta-\zeta)R_\xi\partial_\xi\left(R_\xi\partial_\xi u\right)+(\zeta-\xi)R_\eta\partial_\eta\left(R_\eta\partial_\eta u\right)+(\xi-\eta)R_\zeta\partial_\zeta\left(R_\zeta\partial_\zeta u\right)\right] \tag{3.5}$$

with

$$R_s = \sqrt{\left(s + a_1^2\right)\left(s + a_2^2\right)\left(s + a_3^2\right)}$$

The ellipsoidal coordinate ξ is a function of x_1, x_2, x_3. The gradient of ξ is obtained by taking the derivative of the equation 3.3 :

$$\nabla\xi = \frac{2}{h^2}\left(x_1 / \left(\xi + a_1^2\right), x_2 / \left(\xi + a_2^2\right), x_3 / \left(\xi + a_3^2\right)\right) \tag{3.6}$$

with $$h = \left(x_1^2 / \left(\xi + a_1^2\right)^2 + x_2^2 / \left(\xi + a_2^2\right)^2 + x_3^2 / \left(\xi + a_3^2\right)^2\right)^{\frac{1}{2}} \tag{3.7}$$

The Laplacian of ξ is :

$$\Delta\xi = \frac{2}{h^2}\left(\frac{1}{\xi + a_1^2} + \frac{1}{\xi + a_2^2} + \frac{1}{\xi + a_3^2}\right) \tag{3.8}$$

The surface of the ellipsoid corresponds to $\xi = 0$.

The normal at a surface point of the ellipsoid is:

$$n = \left(x_1 / a_1^2, x_2 / a_2^2, x_3 / a_3^2\right) / h_0 \tag{3.9}$$

with $$h_0 = \left(x_1^2 / a_1^4 + x_2^2 / a_2^4 + x_3^2 / a_3^4\right)^{\frac{1}{2}} \tag{3.10}$$

4. Zeroth order term

The zeroth order, which corresponds to k=0, is the so-called “static” solution of the Maxwell equations :

$$\Delta V_0 = 0 \quad \Delta V_0^{(i)} = 0 \tag{4.1}$$

with, on the surface of the ellipsoid (corresponding to $\xi = 0, \forall(\eta,\zeta)$) ,

$$\begin{aligned} n \wedge \nabla\left(V_0 - V_0^{(i)}\right) &= -n \wedge E_0^{(0)} \\ n \bullet \nabla\left(V_0 - \varepsilon V_0^{(i)}\right) &= -n \bullet E_0^{(0)} \end{aligned} \tag{4.2}$$

and

$$V_0^{(0)} = l_1 x_1 + m_1 x_2 + n_1 x_3 \tag{4.3}$$

Following Stratton's ‘intuitive’ approach, we propose a solution with separable variables in ellipsoidal coordinates for the external field (idem for the internal field):

$$V_0 = V_{0\xi}(\xi) V_{0\eta}(\eta) V_{0\zeta}(\zeta) \tag{4.4}$$

Given the linearity of the expression of $V_0^{(0)}$ (Eq.4.3), it is sufficient to look for a solution for the case of $l_1 = 1\ m_1 = 0\ n_1 = 0$ and then $V_0^{(0)} = x_1 = \sqrt{\frac{\left(\xi + a_1^2\right)\left(\eta + a_1^2\right)\left(\zeta + a_1^2\right)}{\left(a_2^2 - a_1^2\right)\left(a_3^2 - a_1^2\right)}}$. This expression suggests a solution for V_0 as :

$$V_0 = V_{0\xi}(\xi)\sqrt{\left(\eta + a_1^2\right)\left(\zeta + a_1^2\right)} \tag{4.5}$$

Hence,

$$\Delta V_0 = 0 \quad \Rightarrow \quad (\eta - \zeta) R_\xi \partial_\xi \left(R_\xi \partial_\xi V_{0\xi}\right) + \left((\zeta - \xi)\left(\frac{\eta}{2} + \frac{a_2^2 + a_3^2}{4}\right) + (\xi - \eta)\left(\frac{\zeta}{2} + \frac{a_2^2 + a_3^2}{4}\right)\right) V_{0\xi} = 0 \tag{4.6}$$

which, surprisingly, leads to the ordinary differential equation:

$$R_\xi \left(R_\xi V_{0\xi}'\right)' + f(\xi) V_{0\xi} = 0 \quad \text{with} \quad f(\xi) = -\frac{1}{4}\left(2\xi + a_2^2 + a_3^2\right) \tag{4.7}$$

a particular solution of which is :

$$V_{0\xi,1} = \left(\xi + a_1^2\right)^{\frac{1}{2}} \tag{4.8}$$

Another particular solution can be obtained using the latter:

$$V_{0\xi,2} = V_{0\xi,1} Z\left(\xi\right) \tag{4.9}$$

This leads to

$$Z = C_1 \int_0^{\xi} \frac{du}{R_u V_{0u,1}^2} + C_2 \tag{4.10}$$

C_1 and C_2 are two integration constants (the integration constants, to be determined later, will be referred to generically as C_1 and C_2),

and then to the general solution:

$$V_{0\xi} = \left[C_1 \int_0^{\xi} \frac{du}{R_u V_{0u,1}^2} + C_2 \right] V_{0\xi,1} \tag{4.11}$$

finally

$$V_0 = \left[C_1 \int_0^{\xi} \frac{du}{R_u \left(u + a_1^2\right)} + C_2 \right] \left(\xi + a_1^2\right)^{\frac{1}{2}} \left(\eta + a_1^2\right)^{\frac{1}{2}} \left(\zeta + a_1^2\right)^{\frac{1}{2}} \tag{4.12}$$

or

$$V_0 = \left[C_1 \int_0^{\xi} \frac{du}{R_u \left(u + a_1^2\right)} + C_2 \right] x_1 \tag{4.13}$$

Expression 4.13 is also valid for $V_0^{(i)}$ with $C_1^{(i)}, C_2^{(i)}$.

The boundary conditions can be rewritten as :

i) $\partial_{\xi} \left(V_0 - \varepsilon V_0^{(i)} + l_1 x_1\right) = 0$ on $\xi = 0$ (4.14)

thus $\left(C_1 - \varepsilon C_1^{(i)}\right) \frac{1}{a_1 a_2 a_3} + \frac{1}{2} \left(C_2 - \varepsilon C_2^{(i)} + l_1\right) = 0$ (4.15)

ii)

$$\begin{cases} \partial_\eta \left(V_0 - V_0^{(i)} + l_1 x_1 \right) = 0 \\ \partial_\zeta \left(V_0 - V_0^{(i)} + l_1 x_1 \right) = 0 \end{cases} \quad \text{on} \quad \xi = 0 \quad \forall \eta, \zeta \tag{4.16}$$

thus $C_2 - C_2^{(i)} + l_1 = 0$ (4.17)

iii)

$\xi \to \infty \quad V_0 \to 0$ (4.18)

thus $C_1 \int_0^\infty \frac{du}{R_u \left(u + a_1^2 \right)} + C_2 = 0$ (4.19)

iv)

$\xi \to -a_3^2 \quad \partial_\xi V_0^{(i)} \not\to \infty$ (4.20)

thus $C_1^{(i)} = 0$ (4.21)

Finally,

$$\begin{aligned} C_1 &= \left(\varepsilon - 1 \right) f_1 \left(\varepsilon \right) l_1 \\ C_2 &= -C_1 I_1 \\ C_2^{(i)} &= \frac{2 l_1}{a_1 a_2 a_3} f_1 \left(\varepsilon \right) \end{aligned} \tag{4.22}$$

by calling,

$$I_1 = \int_0^\infty \frac{du}{R_u \left(u + a_1^2 \right)} \text{ and } f_1 \left(\varepsilon \right) = 1 / \left(\frac{2}{a_1 a_2 a_3} + \left(\varepsilon - 1 \right) I_1 \right)$$

The harmonic solutions are :

external : $V_0 = -C_1 x_1 I_1 \left(\xi \right)$ (4.23)

and

internal : $V_0^{(i)} = C_2^{(i)} x_1$ (4.24)

with $I_1 \left(\xi \right) = \int_\xi^\infty \frac{du}{R_u \left(u + a_1^2 \right)}$

that are the Stevenson's solutions, by adopting its notations : $A_1 = -C_1$, $A_1^{(i)} = C_2^{(i)}$.

In the expressions of $V_0^{(i)}$ and V_0 , we find the ellipsoidal harmonic functions (see, Dassios) of degree 1 (to within one multiplicative constant) : internal (inside the ellipsoid) x_1 and external $x_1 I_1(\xi)$.

Given the linearity of the treated PDE's, for any polarization of the incident field (by performing the appropriate index permutations to obtain the expressions for A_j and $A_j^{(i)}$) we have :

$$\begin{aligned} V_0 &= \sum_{j=1}^{3} A_j x_j I_j(\xi) \\ V_0^{(i)} &= \sum_{j=1}^{3} A_j^{(i)} x_j \end{aligned} \qquad (4.25)$$

In the previous development, equation 2.6, which reduces to $\int n \bullet E_0 dS = 0$ to order 0, was not used. We can check this equality by directly computing $\int n \bullet \nabla V_0 dS$ (using 4.25). Another method would be to use Green-Ostrogradsky's theorem :

$$\int_{\partial E} n \bullet E_0 dS = \int_E \nabla E_0 dV = \int_E \Delta V_0 dV = 0 .$$

However, the application of this theorem requires a few precautions: its classical use is limited to a closed surface ∂E and the volume E inside this surface. A priori, therefore, it does not apply to the complementary of this volume in $\mathbb{R}^3$, which is our case since we are talking about the external potential and external electric field. In order to apply Green-Ostrogradsky's theorem, the external field must have additional properties :

- decrease monotonically with distance from the body
- decrease faster than r^{-2} or ξ^{-1}

but

$$\lim_{\xi \to \infty} \left| x_j I_j(\xi) \right| = \left| x_j \int_\xi^\infty \frac{du}{R_u\left(u + a_j^2\right)} \right| < \left| x_j \right| \left| \int_\xi^\infty \frac{du}{R_u\left(u + a_j^2\right)} \right| < \left| x_j \right| \left| \int_\xi^\infty \frac{du}{u^{5/2}} \right| < \frac{2}{3} \left| x_j \right| \xi^{-3/2} = \frac{2}{3} \left(\left| x_j \right| \xi^{-1/2} \right) \xi^{-1}$$

which does not guarantee sufficiently rapid decay. We will therefore settle for the less elegant direct method:

$$\int_{\partial E} n \bullet E_0 dS = \int_{\partial E} n \bullet \sum_{j=1}^{3} A_j \nabla\left(x_j I_j\left(\xi\right)\right) dS = \int_{\partial E} n \bullet \sum_{j=1}^{3} A_j \left(I_j\left(\xi\right)\nabla\left(x_j\right) + x_j I_j{}'\left(\xi\right)\nabla\xi\right) dS$$
$$= \sum_{j=1}^{3} A_j \left(I_j\left(0\right)\frac{1}{a_j^2} + 2I_j{}'\left(0\right)\right)\int_{\partial E} \frac{x_j}{h_0} dS \tag{4.26}$$

given,

$$\int_{\partial E} \frac{x_j}{h_0} dS = 0 \tag{4.27}$$

we therefore have $\int_{\partial E} n \bullet E_0 dS = 0$.

5. First order term

Stevenson's idea was to introduce vector functions $G_1, G_1^{(i)}$ and to base (apply) the boundary conditions on $U_1, U_1^{(i)}$. So first we look for a particular solution $G_1, G_1^{(i)}$ to the problem.

5.1. Calculation of $G_1, G_1^{(i)}$

In the following we will call A the vector with components A_j and P (respectively $P^{(i)}$) the vector with components $x_j I_j(\xi)$ (respectively x_j). We then have (2.4 and 4.25) :

$$\begin{aligned} V_0 &= A \bullet P \\ V_0^{(i)} &= A^{(i)} \bullet P^{(i)} \end{aligned} \tag{5.1}$$

We need to find solutions for:

$$\begin{aligned} \nabla \wedge G_1 &= -\nabla\left(A \bullet P\right) \\ \nabla \wedge G_1^{(i)} &= -\varepsilon \nabla\left(A^{(i)} \bullet P^{(i)}\right) \end{aligned} \tag{5.2}$$

with
$$\begin{aligned} \nabla \bullet G_1 &= 0 \\ \nabla \bullet G_1^{(i)} &= 0 \end{aligned} \tag{5.3}$$

What are the properties of $A, P, A^{(i)}, P^{(i)}$ vectors when the nabla operator is applied to them?

$A, A^{(i)}$ being constant, we obviously have :

$$\begin{aligned} \nabla \bullet A &= \nabla \bullet A^{(i)} = 0 \\ \nabla \wedge A &= \nabla \wedge A^{(i)} = 0 \end{aligned} \tag{5.4}$$

$P^{(i)}$ being the position vector, we have :

$$\nabla \wedge P^{(i)} = 0 \quad \nabla \bullet P^{(i)} = 3 \tag{5.5}$$

P vector requires more attention (using 3.6) :

$$\nabla \wedge P = \left(....,\partial_j P_i - \partial_i P_j,....\right) = \left(....,\frac{1}{R_\xi}\left(\frac{x_j}{\xi + a_j^2}\partial_i \xi - \frac{x_i}{\xi + a_i^2}\partial_j \xi\right),.....\right) = (....,0,....) = 0 \tag{5.6}$$

$$\nabla \bullet P = \sum_j I_j(\xi) + \sum_j x_j \partial_j I_j(\xi) = \sum_j I_j(\xi) + \sum_j x_j \left(I_j(\xi)\right)' \partial_j \xi = \sum_j I_j(\xi) - \sum_j \frac{x_j}{R_\xi\left(\xi + a_j^2\right)} \partial_j \xi$$

$$= \sum_j I_j(\xi) - \sum_j \frac{2x_j^2}{h^2 R_\xi\left(\xi + a_j^2\right)^2} = \sum_j I_j(\xi) - \frac{2}{R_\xi}$$

Considering the equality :

$$\sum_j I_j(\xi) = \int_\xi^\infty \frac{du}{R_u} \sum_j \frac{1}{u + a_j^2} = \int_\xi^\infty \frac{du}{R_u} 2\left(\ln R_u\right)' = 2\int_\xi^\infty \frac{R_u' du}{R_u^2} = \frac{2}{R_\xi}$$

One deduces

$$\nabla \bullet P = 0 \tag{5.7}$$

Using the two canonical vectorial equations, involving any two vectors V_1, V_2 :

$$\begin{aligned} &\nabla \bullet \left(V_1 \wedge V_2\right) = V_2 \bullet \left(\nabla \wedge V_1\right) - V_1 \bullet \left(\nabla \wedge V_2\right) \\ &\nabla \wedge \left(V_1 \wedge V_2\right) = V_1\left(\nabla \bullet V_2\right) - V_2\left(\nabla \bullet V_1\right) + \left(V_2 \bullet \nabla\right) V_1 - \left(V_1 \bullet \nabla\right) V_2 \end{aligned} \tag{5.8}$$

and applying them to the pairs $\left(V_1, V_2\right) = \left(A, P\right)$ and $\left(V_1, V_2\right) = \left(A^{(i)}, P^{(i)}\right)$ and using the above-mentioned properties, we check that :

$$\begin{aligned} &\nabla \bullet \left(A \wedge P\right) = 0 \\ &\nabla \bullet \left(A^{(i)} \wedge P^{(i)}\right) = 0 \end{aligned} \tag{5.9}$$

and (reminding that the A vector is a constant)

$$\nabla \wedge \left(A \wedge P\right) = -\left(A \bullet \nabla\right) P = -\begin{cases} \sum_j A_j\left(\partial_j P_1\right) \\ \sum_j A_j\left(\partial_j P_2\right) \\ \sum_j A_j\left(\partial_j P_3\right) \end{cases} = -\begin{cases} \sum_j A_j\left(\partial_1 P_j\right) \\ \sum_j A_j\left(\partial_2 P_j\right) \\ \sum_j A_j\left(\partial_3 P_j\right) \end{cases} = -\nabla\left(A \bullet P\right) \tag{5.10}$$

$$\nabla \wedge \left(A^{(i)} \wedge P^{(i)}\right) = 3A^{(i)} - \left(A^{(i)} \bullet \nabla\right) r = 3A^{(i)} - \begin{cases} A_1^{(i)} \\ A_2^{(i)} \\ A_3^{(i)} \end{cases} = 2A^{(i)} = 2\nabla\left(A^{(i)} \bullet P^{(i)}\right) \tag{5.11}$$

Finally, by comparison (between 5.2 and 5.10, 5.11), we find the Stevenson's result :

$$G_1 = A \wedge P$$
$$G_1^{(i)} = -\frac{\varepsilon}{2} A^{(i)} \wedge P^{(i)} \tag{5.12}$$

5.2 Calculation of $U_1, U_1^{(i)}$

5.2.1 Harmonic functions for ellipsoid

We search for the harmonic solutions of the PDE's,

$$\Delta U_1 = 0 \quad \Delta U_1^{(i)} = 0 \tag{5.13}$$

with, on the ellipsoid surface (corresponding to $\xi = 0, \forall(\eta, \zeta)$) ,

$$\begin{aligned} n \wedge \nabla\left(U_1 - U_1^{(i)}\right) &= -n \wedge \left(G_1 - G_1^{(i)} + H_1^{(0)}\right) \\ n \bullet \nabla\left(U_1 - \mu U_1^{(i)}\right) &= -n \bullet \left(G_1 - \mu G_1^{(i)} + H_1^{(0)}\right) \end{aligned} \tag{5.14}$$

with $H_1^{(0)} = \begin{pmatrix} l_2 & m_2 & n_2 \end{pmatrix}\begin{pmatrix} lx + my + nz \end{pmatrix}$

Here again, let's use the intuitive approach: the vector of the right-hand side of the boundary conditions is linear in x_i, so we expect a quadratic solution ($Cx_i x_j$), more precisely a linear combination of quadratic terms like (using 3.4) :

$$W_{i,j}(\xi, \eta, \zeta) = W_{\xi,i,j}(\xi)\left(\eta + a_i^2\right)^{\frac{1}{2}}\left(\zeta + a_i^2\right)^{\frac{1}{2}}\left(\eta + a_j^2\right)^{\frac{1}{2}}\left(\zeta + a_j^2\right)^{\frac{1}{2}} \text{ avec } i, j = 1, 2, 3 \tag{5.15}$$

the calculation of $\Delta W_{i,j}(\xi, \eta, \zeta) = 0$ leads to

$$R_\xi\left(R_\xi W_{\xi,i,j}'\right)' + f(\xi) W_{\xi,i,j} = 0 \tag{5.16}$$

with $f(\xi) = -\frac{1}{4}\left(6\xi + a_i^2 + a_j^2 + 4a_l^2\right)$ where $l \neq i, j$ et $i \neq j$ (5.17)

a particular solution of which is,

$$W_{\xi,i,j,1} = \left(\xi + a_i^2\right)^{\frac{1}{2}}\left(\xi + a_j^2\right)^{\frac{1}{2}}. \tag{5.18}$$

Another particular solution can be obtained using the latter:

$$W_{\xi,i,j,2} = W_{\xi,i,j,1} Z(\xi) \tag{5.19}$$

This leads to

$$Z = C_1 \int_0^{\xi} \frac{du}{R_u W_{u,i,j,1}^2} + C_2 \tag{5.20}$$

And therefore to the general solution :

$$U_{1,i,j} = \left[C_1 \int_0^{\xi} \frac{du}{R_u \left(u + a_i^2\right)\left(u + a_j^2\right)} + C_2 \right] W_{\xi,i,j,1} \left(\eta + a_i^2\right)^{\frac{1}{2}} \left(\zeta + a_i^2\right)^{\frac{1}{2}} \left(\eta + a_j^2\right)^{\frac{1}{2}} \left(\zeta + a_j^2\right)^{\frac{1}{2}} \tag{5.21}$$

or

$$U_{1,i,j} = \left[C_1 \int_0^{\xi} \frac{du}{R_u \left(u + a_i^2\right)\left(u + a_j^2\right)} + C_2 \right] x_i x_j \tag{5.22}$$

Expression 5.22 is also valid for $U_1^{(i)}$ with $C_1^{(i)}, C_2^{(i)}$.

The case $i = j$ requires special treatment. The solution proposed for $i \neq j$ by taking $i = j$, i.e. $W_{i,i}\left(\xi,\eta,\zeta\right) = W_{\xi,i,i}\left(\xi\right)\left(\eta + a_i^2\right)\left(\zeta + a_i^2\right)$, leads to $\Delta W_{\xi,i,i} \neq 0$; so we propose to generalize the latter expression by replacing a_i^2 with $-\nu$ (parameter to be determined later and not imposed) :

$$W_{\nu}\left(\xi,\eta,\zeta\right) = W_{\xi,\nu}\left(\xi\right)\left(\eta - \nu\right)\left(\zeta - \nu\right) \tag{5.23}$$

We therefore need to determine ν (if this solution makes sense). So,

$$\Delta W_{\nu} = 0 \Rightarrow \left(\eta - \zeta\right)\left(\eta - \nu\right)\left(\zeta - \nu\right) R_{\xi} \partial_{\xi} \left(R_{\xi} \partial_{\xi} W_{\xi,\nu} \right) + Z W_{\xi,\nu} = 0 \tag{5.24}$$

with

$$Z = \frac{1}{2}\left(\zeta - \xi\right)\left(\zeta - \nu\right) g\left(\eta\right) + \frac{1}{2}\left(\xi - \eta\right)\left(\eta - \nu\right) g\left(\zeta\right) \tag{5.25}$$

and

$$g\left(x\right) = \left(x + a_1^2\right)\left(x + a_2^2\right) + \left(x + a_1^2\right)\left(x + a_3^2\right) + \left(x + a_2^2\right)\left(x + a_3^2\right) \tag{5.26}$$

This polynomial of degree 2, which always has two real roots, can be expressed as :

$$g\left(x\right) = 3\left(x - \nu\right)\left(x - \nu'\right) \tag{5.27}$$

ν and ν' are also solutions of the equation :

$$\left(x+a_1^2\right)^{-1}+\left(x+a_2^2\right)^{-1}+\left(x+a_3^2\right)^{-1}=0 \tag{5.28}$$

Thus,

$$Z=\frac{3}{2}(\zeta-\nu)(\eta-\nu)\left[(\zeta-\xi)(\eta-\nu')+(\xi-\eta)(\zeta-\nu')\right]=\frac{3}{2}(\zeta-\nu)(\eta-\nu)(\zeta-\eta)(\xi-\nu') \tag{5.29}$$

Hence (things are working out wonderfully)

$$\Delta W_\nu=0\Rightarrow R_\xi\left(R_\xi W_{\xi,\nu}'\right)'-\frac{3}{2}(\xi-\nu')W_{\xi,\nu}=0 \tag{5.30}$$

A particular solution of this equation is: $\xi-\nu$

Finally,

$$U_{1,\nu}=\left[C_1\int_0^\xi\frac{du}{R_u(u-\nu)^2}+C_2\right](\xi-\nu)(\eta-\nu)(\zeta-\nu) \tag{5.31}$$

At this point, we need to express Eq.5.31 in the same form as Eq.5.22, i.e. to express $(\xi-\nu)(\eta-\nu)(\zeta-\nu)$ as a function of x_1,x_2,x_3. To do this, let's rewrite Eq.3.4 :

$$\begin{aligned}
\left(a_2^2-a_1^2\right)\left(a_3^2-a_1^2\right)x_1^2&=\left(\xi'+\nu+a_1^2\right)\left(\eta'+\nu+a_1^2\right)\left(\zeta'+\nu+a_1^2\right)\\
\left(a_3^2-a_2^2\right)\left(a_1^2-a_2^2\right)x_2^2&=\left(\xi'+\nu+a_2^2\right)\left(\eta'+\nu+a_2^2\right)\left(\zeta'+\nu+a_2^2\right)\\
\left(a_1^2-a_3^2\right)\left(a_2^2-a_3^2\right)x_3^2&=\left(\xi'+\nu+a_3^2\right)\left(\eta'+\nu+a_3^2\right)\left(\zeta'+\nu+a_3^2\right)
\end{aligned} \tag{5.32}$$

$$\xi'=\xi-\nu\quad\eta'=\eta-\nu\quad\zeta'=\zeta-\nu$$

This gives us a system of 3 equations with 3 unknowns ($\xi'\eta'\zeta'$, $\xi'\eta'+\xi'\zeta'+\eta'\zeta'$, $\xi'+\eta'+\zeta'$), whose solution for the unknown $\xi'\eta'\zeta'=(\xi-\nu)(\eta-\nu)(\zeta-\nu)$, called X_ν by Stevenson, is :

$$\xi'\eta'\zeta'=(\xi-\nu)(\eta-\nu)(\zeta-\nu)=\delta\left[\sum_{j=1}^{3}\frac{x_j^2}{\nu+a_j^2}-1\right] \tag{5.33}$$

with

$$\delta=\prod_{j=1}^{3}\left(\nu+a_j^2\right)$$

Finally, the general solution for $i=j$ is written as :

$$U_{1,\nu} = \left[C_1 \int_0^{\xi} \frac{du}{R_u \left(u-\nu\right)^2} + C_2 \right] X_\nu \tag{5.34}$$

Obviously, the roles of ν et ν' are interchangeable. So we have $U_{1,\nu'}$.

The boundary conditions, away from the ellipsoid and at its centre, specify the expressions for $U_{1,i,j}, U^{(i)}_{1,i,j}, U_{1,\nu}, U_{1,\nu'}, U^{(i)}_{1,\nu}, U^{(i)}_{1,\nu'}$:

i) $\xi \to \infty \quad U_{1,i,j}, U_{1\nu}, U_{1\nu'} \to 0$ (5.35)

ii) $\xi \to -a_3^2 \quad \partial_\xi U^{(i)}_{1,i,j}, \partial_\xi U^{(i)}_{1,\nu}, \partial_\xi U^{(i)}_{1,\nu'} \not\to \infty$ (5.36)

Hence, using Stevenson's notations (except for the integration constants) :

$$\begin{aligned} U_{1,i,j} &= A_{i,j}\, I_{i,j}\left(\xi\right) x_i x_j \\ U_{1,\nu} &= A_\nu\, T_\nu\left(\xi\right) X_\nu \\ U_{1,\nu'} &= A_{\nu'}\, T_{\nu'}\left(\xi\right) X_{\nu'} \end{aligned} \tag{5.37}$$

$$\begin{aligned} U^{(i)}_{1,i,j} &= A^{(i)}_{i,j}\; x_i x_j \\ U^{(i)}_{1,\nu} &= A^{(i)}_{\nu}\; X_\nu \\ U^{(i)}_{1,\nu'} &= A^{(i)}_{\nu'}\; X_{\nu'} \end{aligned} \tag{5.38}$$

with

$$I_{i,j}\left(\xi\right) = \int_{\xi}^{\infty} \frac{du}{R_u \left(u+a_i^2\right)\left(u+a_j^2\right)} \qquad \text{and} \qquad T_\nu\left(\xi\right) = \int_{\xi}^{\infty} \frac{du}{R_u \left(u-\nu\right)^2}$$

Here again, we find in the expressions of $U_{1,i,j}, U^{(i)}_{1,i,j}, U_{1,\nu}, U_{1,\nu'}, U^{(i)}_{1,\nu}, U^{(i)}_{1,\nu'}$ the ellipsoidal harmonic functions (see Dassios) of degree 2 (to within one multiplicative constant), of which there are 10.

Internal harmonics :

$$\mathrm{E}^1_2 = X_\nu \; ; \mathrm{E}^2_2 = X_{\nu'} \; ; \mathrm{E}^3_2 = \left(h_1 h_2 h_3\right) h_3 x_1 x_2 \; ; \mathrm{E}^4_2 = \left(h_1 h_2 h_3\right) h_2 x_1 x_3 \; ; \mathrm{E}^5_2 = \left(h_1 h_2 h_3\right) h_1 x_2 x_3 \tag{5.39}$$

External harmonics :

$$\Upsilon^1_2 = \frac{5}{2}\mathrm{E}^1_2 T_\nu\left(\xi\right); \Upsilon^2_2 = \frac{5}{2}\mathrm{E}^2_2 T_{\nu'}\left(\xi\right); \Upsilon^3_2 = \frac{5}{2}\mathrm{E}^3_2 I_{1,2}\left(\xi\right); \Upsilon^4_2 = \frac{5}{2}\mathrm{E}^4_2 I_{1,3}\left(\xi\right); \Upsilon^5_2 = \frac{5}{2}\mathrm{E}^5_2 I_{2,3}\left(\xi\right) \tag{5.40}$$

The solution of the PDE's (5.13) are linear combinations of the set of harmonic functions 5.37 (or 5.40) for U_1 and linear combinations of 5.38 (or 5.39) for $U_1^{(i)}$.

The integration constants (Eqs. 5.37 and 5.38) will be determined using the boundary conditions:

$$\begin{aligned} n \wedge \nabla\left(U_1 - U_1^{(i)}\right) &= -n \wedge \left(G_1 - G_1^{(i)} + H_1^{(0)}\right) \\ n \bullet \nabla\left(U_1 - \mu U_1^{(i)}\right) &= -n \bullet \left(G_1 - \mu G_1^{(i)} + H_1^{(0)}\right) \end{aligned} \tag{5.41}$$

i.e. four scalar equations (that depends on the surface coordinates) for 2x5 unknowns.

If $\mu = 1$, then the boundary condition (surface of the ellipsoid) reduces to the vector equation :

$$\nabla\left(U_1 - U_1^{(i)}\right) + G_1 - G_1^{(i)} + H_1^{(0)} = 0 \tag{5.42}$$

5.2.2 Simple case : the disk

Expecting complex calculations, we will first study a simpler case: the disc replacing the ellipsoid**. The difficulty arises from the mixing of Cartesian and ellipsoidal coordinates (here polar, with the radial coordinate r playing the role of ξ) in the expressions of the general solutions and, above all, of the boundary conditions**. We want to solve :

$$\begin{aligned} &\Delta U = r^{-1}\partial_r\left(r\partial_r U\right) + r^{-2}\partial_{\theta\theta} U = 0 \\ &\Delta U^{(i)} = 0 \\ &\nabla U\big|_R + A = \nabla U^{(i)}\big|_R + A^{(i)} \quad \left(r = R\right) \end{aligned} \tag{5.43}$$

with

$$\begin{aligned} A &= a x_1 + b x_2 \\ A^{(i)} &= a^{(i)} x_1 + b^{(i)} x_2 \end{aligned} \tag{5.44}$$

$A, A^{(i)} a, b, a^{(i)}, b^{(i)}$ are vectors of the plane. $a, b, a^{(i)}, b^{(i)}$ are constant vectors.

Several approaches will be used.

a. Traditional approach

We begin with the classical method, which consists of a separation of variables method (r, θ) and the use of an appropriate orthogonal functions set :

$$U = \sum_{n=0} \left(C_n \cos n\theta + D_n \sin n\theta \right) / r^n$$
$$U^{(i)} = \sum_{n=0} \left(C_n^{(i)} \cos n\theta + D_n^{(i)} \sin n\theta \right) r^n \tag{5.45}$$

The boundary condition, obtained by projecting onto the radial and orthoradial coordinates, leads to non-zero terms of degree 2 :

$$\begin{aligned}
2R^{-3}C_2 + 2RC_2^{(i)} &= \frac{R}{2}\left(b_2^{(i)} - b_2 - a_1^{(i)} + a_1 \right) \\
-2R^{-3}C_2 + 2RC_2^{(i)} &= \frac{R}{2}\left(b_2^{(i)} - b_2 - a_1^{(i)} + a_1 \right) \\
2R^{-3}D_2 + 2RD_2^{(i)} &= \frac{R}{2}\left(b_1 - b_1^{(i)} + a_2 - a_2^{(i)} \right) \\
-2R^{-3}D_2 + 2RD_2^{(i)} &= \frac{R}{2}\left(b_1 - b_1^{(i)} + a_2 - a_2^{(i)} \right)
\end{aligned} \tag{5.46}$$

hence $C_2 = D_2 = 0$ (5.47)

and

$$\begin{aligned}
C_2^{(i)} &= \left(b_2^{(i)} - b_2 - a_1^{(i)} + a_1 \right) / 4 \\
D_2^{(i)} &= \left(b_1 - b_1^{(i)} + a_2 - a_2^{(i)} \right) / 4
\end{aligned} \tag{5.48}$$

The form of the boundary condition imposes the following relations, derived from the absence of a term of order 0 :

$$\begin{aligned}
b_2^{(i)} - b_2 + a_1^{(i)} - a_1 &= 0 \\
b_1 - b_1^{(i)} + a_2^{(i)} - a_2 &= 0
\end{aligned} \tag{5.49}$$

These latter conditions are normally satisfied by the coefficients *a* and *b* calculated at lower orders. The set of coefficients *a,b* cannot therefore be arbitrary.

Finally,

$$\begin{aligned}
U &= 0 \\
U^{(i)} &= \frac{1}{4} r^2 \left[\left(b_2^{(i)} - b_2 - a_1^{(i)} + a_1 \right) \cos 2\theta + \left(b_1 - b_1^{(i)} + a_2 - a_2^{(i)} \right) \sin 2\theta \right] \\
&= \frac{1}{4} \left[\left(b_2^{(i)} - b_2 - a_1^{(i)} + a_1 \right) \left(x_1^2 - x_2^2 \right) + 2 \left(b_1 - b_1^{(i)} + a_2 - a_2^{(i)} \right) x_1 x_2 \right]
\end{aligned} \tag{5.50}$$

b. Intuitive way

The general solution compatible with the linear form of $A, A^{(i)}$ in x_1, x_2 is easily obtained by a method of separation of variables, but by formulating it with **Cartesian coordinates**:

$$\begin{aligned} U_{1,2} &= C_{1,2} r^{-4} x_1 x_2 \\ U_{1,1} &= U_{2,2} = C_{j,j} r^{-4} \left(x_1^2 - x_2^2 \right) \\ U_{1,2}^{(i)} &= C_{1,2}^{(i)} x_1 x_2 \\ U_{1,1}^{(i)} &= U_{2,2}^{(i)} = C_{j,j}^{(i)} \left(x_1^2 - x_2^2 \right) \end{aligned} \tag{5.51}$$

or

$$\begin{aligned} U &= r^{-4} \left(C_{1,2} x_1 x_2 + C_{j,j} \left(x_1^2 - x_2^2 \right) \right) \\ U^{(i)} &= C_{1,2}^{(i)} x_1 x_2 + C_{j,j}^{(i)} \left(x_1^2 - x_2^2 \right) \end{aligned} \tag{5.52}$$

The integration constants will be determined using the boundary conditions projected onto the Cartesian coordinates:

$$\nabla U \big|_R + A = \nabla U^{(i)} \big|_R + A^{(i)} \tag{5.53}$$

denoting,

$$\begin{aligned} H_{j,j} &= 2R^{-4} C_{j,j} - 2C_{j,j}^{(i)} \\ H_{1,2} &= R^{-4} C_{1,2} - C_{1,2}^{(i)} \end{aligned} \tag{5.54}$$

The boundary conditions become :

$$\begin{aligned} &\forall \left(x_1, x_2 \right) \in \partial S \\ &\left(H_{j,j} + a_1 - a_1^{(i)} \right) x_1 + \left(H_{1,2} + b_1 - b_1^{(i)} \right) x_2 + \left(-\frac{4x_1}{R^2} \right) U_{r=R} = 0 \\ &\left(H_{1,2} + a_2 - a_2^{(i)} \right) x_1 + \left(-H_{j,j} + b_2 - b_2^{(i)} \right) x_2 + \left(-\frac{4x_2}{R^2} \right) U_{r=R} = 0 \end{aligned} \tag{5.55}$$

$U_{r=R}$ being a priori a bilinear form, these equations will be verified if :

$$
\begin{aligned}
&H_{1,2} + b_1 - b_1^{(i)} = 0 \\
&H_{1,2} + a_2 - a_2^{(i)} = 0 \\
&H_{j,j} + a_1 - a_1^{(i)} = -H_{j,j} + b_2 - b_2^{(i)} \\
&a_1 - a_1^{(i)} + b_2 - b_2^{(i)} - \frac{8}{R^2} U_{r=R} = 0
\end{aligned}
\tag{5.56}
$$

hence

$$
\begin{aligned}
&b_1 - b_1^{(i)} + a_2^{(i)} - a_2 = 0 \\
&b_2^{(i)} - b_2 + a_1^{(i)} - a_1 = 0 \\
&U_{r=R} = 0
\end{aligned}
\tag{5.57}
$$

but, also

$$
H_{j,j} + a_1 - a_1^{(i)} = -H_{j,j} + b_2 - b_2^{(i)} = 0 \tag{5.58}
$$

and then

$$
\begin{aligned}
&U = 0 \\
&U^{(i)} = \frac{1}{2}\left(b_1 - b_1^{(i)} + a_2 - a_2^{(i)}\right) x_1 x_2 + \frac{1}{4}\left(a_1 - a_1^{(i)} - b_2 + b_2^{(i)}\right)\left(x_1^2 - x_2^2\right)
\end{aligned}
\tag{5.59}
$$

The result is the same as that obtained using the classic method.

5.2.3 Intuitive way for ellipsoid

Let's return to the case of the ellipsoid with $\mu = 1$; the boundary condition (5.42) translates into :

along the $x\left(i.e\, x_1\right)$ coordinate :

$$
\begin{aligned}
&\left(-\frac{\varepsilon}{2} A_3^{(i)} - A_3 I_2 + A_{1,2} I_{1,2} - A_{1,2}^{(i)} + l_2 m\right) x_2 + \left(\frac{\varepsilon}{2} A_2^{(i)} + A_2 I_3 + A_{1,3} I_{1,3} - A_{1,3}^{(i)} + l_2 n\right) x_3 + \\
&2\left(\prod_{j=1}^{3}\left(\nu + a_j^2\right)\frac{A_\nu T_\nu - A_\nu^{(i)}}{\nu + a_1^2} + \prod_{j=1}^{3}\left(\nu' + a_j^2\right)\frac{A_{\nu'} T_{\nu'} - A_{\nu'}^{(i)}}{\nu' + a_1^2} + \frac{l_2 l}{2}\right) x_1 = \frac{Z}{a_1 a_2 a_3}\partial_1 \xi = \frac{Z}{a_1 a_2 a_3 h_0^2}\frac{2x_1}{a_1^2}
\end{aligned}
\tag{5.60}
$$

The components along $y\left(i.e\, x_2\right)$ and $z\left(i.e\, x_3\right)$ are easily obtained.

with the Z factor common to all 3 components,

$$
Z = A_{1,2}\frac{x_1 x_2}{a_1^2 a_2^2} + A_{1,3}\frac{x_1 x_3}{a_1^2 a_3^2} + A_{2,3}\frac{x_2 x_3}{a_2^2 a_3^2} + A_\nu \frac{X_\nu}{\nu^2} + A_{\nu'}\frac{X_{\nu'}}{\nu'^2} \tag{5.61}
$$

By analogy with Eqs. 5.55-5.58,

$$
\begin{aligned}
&-\frac{\varepsilon}{2}A_3^{(i)} - A_3 I_2 + A_{1,2} I_{1,2} - A_{1,2}^{(i)} + l_2 m = 0 \\
&\frac{\varepsilon}{2}A_2^{(i)} + A_2 I_3 + A_{1,3} I_{1,3} - A_{1,3}^{(i)} + l_2 n = 0 \\
&\prod_{j=1}^{3}\left(\nu + a_j^2\right)\frac{A_\nu T_\nu - A_\nu^{(i)}}{\nu + a_1^2} + \prod_{j=1}^{3}\left(\nu' + a_j^2\right)\frac{A_{\nu'} T_{\nu'} - A_{\nu'}^{(i)}}{\nu' + a_1^2} + \frac{l_2 l}{2} = 0 \\
&Z = 0
\end{aligned}
\tag{5.62}
$$

The same applies to the other two components.

$Z = 0$ leads to $A_{1,2} = A_{1,3} = A_{2,3} = A_\nu = A_{\nu'} = 0$ (5.63)

Finally, we have to solve the two systems :

unknowns : $A_{1,2}^{(i)}, A_{1,3}^{(i)}, A_{2,3}^{(i)}$

$$
\begin{aligned}
&-\frac{\varepsilon}{2}A_3^{(i)} - A_3 I_2 - A_{1,2}^{(i)} + l_2 m = 0 \\
&\frac{\varepsilon}{2}A_2^{(i)} + A_2 I_3 - A_{1,3}^{(i)} + l_2 n = 0 \\
&\frac{\varepsilon}{2}A_3^{(i)} + A_3 I_1 - A_{1,2}^{(i)} + m_2 l = 0 \\
&-\frac{\varepsilon}{2}A_1^{(i)} - A_1 I_3 - A_{2,3}^{(i)} + m_2 n = 0 \\
&-\frac{\varepsilon}{2}A_2^{(i)} - A_2 I_1 - A_{1,3}^{(i)} + n_2 l = 0 \\
&\frac{\varepsilon}{2}A_1^{(i)} + A_1 I_2 - A_{2,3}^{(i)} + n_2 m = 0
\end{aligned}
\tag{5.64}
$$

and

unknowns : $A_\nu^{(i)}, A_{\nu'}^{(i)}$

$$
\begin{aligned}
&-\prod_{j=1}^{3}\left(\nu + a_j^2\right)\frac{A_\nu^{(i)}}{\nu + a_1^2} - \prod_{j=1}^{3}\left(\nu' + a_j^2\right)\frac{A_{\nu'}^{(i)}}{\nu' + a_1^2} + \frac{l_2 l}{2} = 0 \\
&-\prod_{j=1}^{3}\left(\nu + a_j^2\right)\frac{A_\nu^{(i)}}{\nu + a_2^2} - \prod_{j=1}^{3}\left(\nu' + a_j^2\right)\frac{A_{\nu'}^{(i)}}{\nu' + a_2^2} + \frac{m_2 m}{2} = 0 \\
&-\prod_{j=1}^{3}\left(\nu + a_j^2\right)\frac{A_\nu^{(i)}}{\nu + a_3^2} - \prod_{j=1}^{3}\left(\nu' + a_j^2\right)\frac{A_{\nu'}^{(i)}}{\nu' + a_3^2} + \frac{n_2 n}{2} = 0
\end{aligned}
\tag{5.65}
$$

These two systems appear to be overdetermined in relation to the number of unknowns. In fact, this is not the case, as there are relationships between coefficients such as those shown in 5.57: for instance, take the two equations containing the unknown $A_{1,2}^{(i)}$; we can check that there is the following relationship between coefficients:

$$-\frac{\varepsilon}{2}A_3^{(i)} - A_3 I_2 + l_2 m = \frac{\varepsilon}{2}A_3^{(i)} + A_3 I_1 + m_2 l \tag{5.66}$$

leading to

$$A_{1,2}^{(i)} = -\frac{\varepsilon}{2}A_3^{(i)} - A_3 I_2 + l_2 m \tag{5.67}$$

The same applies to 5.65: the sum of the left-hand side of the three equations is zero, because the vectors $(l\ m\ n),(l_2\ m_2\ n_2)$ are orthogonal and relation 5.28 is used; two equations for two unknowns are therefore sufficient.

$\mu = 1$ leads to simplifications in Stevenson's expressions: it is easy to check that $B_1 = A_{2,3} = 0$ (B_1 is the Stevenson's notation , idem for B_2, B_3), that $B = A_\nu = 0; B' = A_{\nu'} = 0$ (B and B' are the Stevenson's notations). In the same way, non-zero terms are

$A_{2,3}^{(i)} = \frac{1}{2}\left(m_2 n + n_2 m + A_1\left(I_2 - I_3\right)\right) = B_1^{(i)}$, where $B_1^{(i)}$ is the Stevenson's notation.

Here again, we find Stevenson's expression, expressed in a different way.

likewise,

$A_\nu^{(i)} = \frac{1}{6(\nu'-\nu)}\left[\frac{ll_2}{\nu + a_1^2} + \frac{mm_2}{\nu + a_2^2} + \frac{nn_2}{\nu + a_3^2}\right] = B$, where B is the Stevenson's notation,

which is less straightforward to prove:

We multiply the i[th] equation of the system 5.65 by $\left(\nu + a_i^2\right)^{-1}$, then we add the new equations of the system 5.65 :

The coefficient of $A_{\nu'}^{(i)}$ vanishes due to the expression (using 5.28) :

$$\sum_{i=1}^{3}\frac{1}{\left(\nu + a_i^2\right)\left(\nu' + a_i^2\right)} = \frac{1}{(\nu'-\nu)}\left[\sum_{i=1}^{3}\frac{1}{\left(\nu + a_i^2\right)} - \sum_{i=1}^{3}\frac{1}{\left(\nu' + a_i^2\right)}\right] = 0 \tag{5.68}$$

The coefficient of $A_\nu^{(i)}$ is :

$$-\prod_{j=1}^{3}\left(\nu+a_j^2\right)\sum_{i=1}^{3}\frac{1}{\left(\nu+a_i^2\right)^2}=-\left(\prod_{j=1}^{3}\left(\nu+a_j^2\right)\right)^{-1}\sum_{i>j}\left(\left(\nu+a_i^2\right)\left(\nu+a_j^2\right)\right)^2$$

$$=-\left(\prod_{j=1}^{3}\left(\nu+a_j^2\right)\right)^{-1}\left[\left(\sum_{i>j}\left(\left(\nu+a_i^2\right)\left(\nu+a_j^2\right)\right)\right)^2-2\prod_{j=1}^{3}\left(\nu+a_j^2\right)\sum_{j=1}^{3}\left(\nu+a_j^2\right)\right] \tag{5.69}$$

$$=-\left(\prod_{j=1}^{3}\left(\nu+a_j^2\right)\right)^{-1}\left[0-2\prod_{j=1}^{3}\left(\nu+a_j^2\right)\sum_{j=1}^{3}\left(\nu+a_j^2\right)\right]=2\sum_{j=1}^{3}\left(\nu+a_j^2\right)=3\left(\nu-\nu'\right)$$

because, following 5.28 : $\nu+\nu'=-\frac{2}{3}\sum_{j=1}^{3}a_j^2$

Finally,

$$3\left(\nu-\nu'\right)A_\nu^{(i)}+\frac{1}{2}\left[\frac{ll_2}{\nu+a_1^2}+\frac{mm_2}{\nu+a_2^2}+\frac{nn_2}{\nu+a_3^2}\right]=0 \tag{5.70}$$

and then,

$$A_\nu^{(i)}=\frac{1}{6\left(\nu'-\nu\right)}\left[\frac{ll_2}{\nu+a_1^2}+\frac{mm_2}{\nu+a_2^2}+\frac{nn_2}{\nu+a_3^2}\right]$$

5.2.4 Traditional approach for ellipsoid

We consider the general case : $\mu\neq 1$.

The function set on which the solution of the PDE will be expanded is the ellipsoidal harmonics set (Eqs. 5.39 and 5.40).

Generally speaking, the solution of the PDE is :

$$U_1^{(i)}=K_{0,1}E_0^1+\sum_{m=1}^{5}K_{2,m}E_2^m \tag{5.71}$$

$$U_1=L_{0,1}\Upsilon_0^1+\sum_{m=1}^{5}L_{2,m}\Upsilon_2^m \tag{5.72}$$

The integration constants $K_{n,m},L_{n,m}$ will be determined using boundary conditions 5.14. We cannot ignore terms of degree 0 $\left(E_0^1=1\right)$ in equations 5.71 and 5.72, given the form of the right-hand side of

equations 5.14. We will now try to determine these constants, starting with the contribution of terms of degree 2, which is more difficult to deal with (by assuming $K_{0,1} = L_{0,1} = 0$).

Equations 5.14 show the gradients of the ellipsoidal harmonics expressed in the Cartesian basis $\{e_i\}$:

$$\nabla E_2^1 = 2\delta \begin{vmatrix} x_1/\left(\nu + a_1^2\right) \\ x_2/\left(\nu + a_2^2\right) \\ x_3/\left(\nu + a_3^2\right) \end{vmatrix} \quad \nabla E_2^2 = 2\delta' \begin{vmatrix} x_1/\left(\nu' + a_1^2\right) \\ x_2/\left(\nu' + a_2^2\right) \\ x_3/\left(\nu' + a_3^2\right) \end{vmatrix}$$
$$\nabla E_2^3 = \left(h_1h_2h_3\right)h_3 \begin{vmatrix} x_2 \\ x_1 \\ 0 \end{vmatrix} \quad \nabla E_2^4 = \left(h_1h_2h_3\right)h_2 \begin{vmatrix} x_3 \\ 0 \\ x_1 \end{vmatrix} \quad \nabla E_2^5 = \left(h_1h_2h_3\right)h_1 \begin{vmatrix} 0 \\ x_3 \\ x_2 \end{vmatrix} \tag{5.73}$$

and

$$\begin{aligned}
\nabla \Upsilon_2^1 &= \frac{5}{2} T_\nu(\xi) \nabla \mathrm{E}_2^1 - \frac{5}{2} \mathrm{E}_2^1 \frac{\nabla \xi}{R_\xi \left(\xi - \nu\right)^2} \\
\nabla \Upsilon_2^2 &= \frac{5}{2} T_{\nu'}(\xi) \nabla \mathrm{E}_2^2 - \frac{5}{2} \mathrm{E}_2^2 \frac{\nabla \xi}{R_\xi \left(\xi - \nu'\right)^2} \\
\nabla \Upsilon_2^3 &= \frac{5}{2} I_{1,2}(\xi) \nabla \mathrm{E}_2^3 - \frac{5}{2} \mathrm{E}_2^3 \frac{\nabla \xi}{R_\xi \left(\xi + a_1^2\right)\left(\xi + a_2^2\right)} \\
\nabla \Upsilon_2^4 &= \frac{5}{2} I_{1,3}(\xi) \nabla \mathrm{E}_2^4 - \frac{5}{2} \mathrm{E}_2^4 \frac{\nabla \xi}{R_\xi \left(\xi + a_1^2\right)\left(\xi + a_3^2\right)} \\
\nabla \Upsilon_2^5 &= \frac{5}{2} I_{2,3}(\xi) \nabla \mathrm{E}_2^5 - \frac{5}{2} \mathrm{E}_2^5 \frac{\nabla \xi}{R_\xi \left(\xi + a_2^2\right)\left(\xi + a_3^2\right)}
\end{aligned} \tag{5.74}$$

We deduce,

$$n \wedge \nabla \left(U - U_1^{(i)}\right)\Big|_{\xi=0} = \sum_{m=1}^{5} P_m \left(n \wedge \nabla E_2^m\right) \tag{5.75}$$

with

$$\begin{aligned}
&P_1 = \frac{5}{2} L_{2,1} T_\nu - K_{2,1} \quad P_2 = \frac{5}{2} L_{2,2} T_{\nu'} - K_{2,2} \\
&P_3 = \frac{5}{2} L_{2,3} I_{1,2} - K_{2,3} \quad P_4 = \frac{5}{2} L_{2,4} I_{1,3} - K_{2,4} \quad P_5 = \frac{5}{2} L_{2,5} I_{2,3} - K_{2,5}
\end{aligned} \tag{5.76}$$

and reminding that (3.6 and 3.9) $\nabla\xi\big|_{\xi=0} = \frac{2}{h_0^2}\left(x_1 / a_1^2, x_2 / a_2^2, x_3 / a_3^2\right) = \frac{2}{h_0} n$,

hence,

$$
\begin{aligned}
n \wedge \nabla \Upsilon_2^1\big|_{\xi=0} &= \frac{5}{2} T_\nu(\xi)\left(n \wedge \nabla \mathrm{E}_2^1\big|_{\xi=0}\right) \\
n \wedge \nabla \Upsilon_2^2\big|_{\xi=0} &= \frac{5}{2} T_{\nu'}(\xi)\left(n \wedge \nabla \mathrm{E}_2^2\big|_{\xi=0}\right) \\
n \wedge \nabla \Upsilon_2^3\big|_{\xi=0} &= \frac{5}{2} I_{1,2}(\xi)\left(n \wedge \nabla \mathrm{E}_2^3\big|_{\xi=0}\right) \\
n \wedge \nabla \Upsilon_2^4\big|_{\xi=0} &= \frac{5}{2} I_{1,3}(\xi)\left(n \wedge \nabla \mathrm{E}_2^4\big|_{\xi=0}\right) \\
n \wedge \nabla \Upsilon_2^5\big|_{\xi=0} &= \frac{5}{2} I_{2,3}(\xi)\left(n \wedge \nabla \mathrm{E}_2^5\big|_{\xi=0}\right)
\end{aligned}
\qquad (5.77)
$$

As for the scalar product $n \bullet \nabla\left(U_1 - \mu U_1^{(i)}\right)$, it contains (starting from 5.74) :

$$
\begin{aligned}
n \bullet \nabla \Upsilon_2^1\big|_{\xi=0} &= \frac{5}{2} T_\nu(\xi)\, n \bullet \nabla \mathrm{E}_2^1\big|_{\xi=0} - 5\,\mathrm{E}_2^1\big|_{\xi=0} \frac{1}{h_0 a_1 a_2 a_3 \nu^2} \\
n \bullet \nabla \Upsilon_2^2\big|_{\xi=0} &= \frac{5}{2} T_{\nu'}(\xi)\, n \bullet \nabla \mathrm{E}_2^2\big|_{\xi=0} - 5\,\mathrm{E}_2^2\big|_{\xi=0} \frac{1}{h_0 a_1 a_2 a_3 \nu'^2} \\
n \bullet \nabla \Upsilon_2^3\big|_{\xi=0} &= \frac{5}{2} I_{1,2}(\xi)\, n \bullet \nabla \mathrm{E}_2^3\big|_{\xi=0} - 5\,\mathrm{E}_2^3\big|_{\xi=0} \frac{1}{h_0 \left(a_1 a_2 a_3\right)^3 \left(a_1 a_2\right)^2} \\
n \bullet \nabla \Upsilon_2^4\big|_{\xi=0} &= \frac{5}{2} I_{1,3}(\xi)\, n \bullet \nabla \mathrm{E}_2^4\big|_{\xi=0} - 5\,\mathrm{E}_2^4\big|_{\xi=0} \frac{1}{h_0 \left(a_1 a_2 a_3\right)^3 \left(a_1 a_3\right)^2} \\
n \bullet \nabla \Upsilon_2^5\big|_{\xi=0} &= \frac{5}{2} I_{2,3}(\xi)\, n \bullet \nabla \mathrm{E}_2^5\big|_{\xi=0} - 5\,\mathrm{E}_2^5\big|_{\xi=0} \frac{1}{h_0 \left(a_1 a_2 a_3\right)^3 \left(a_2 a_3\right)^2}
\end{aligned}
\qquad (5.78)
$$

hence,

$$
n \bullet \nabla\left(U_1 - \mu U_1^{(i)}\right) = \sum_{m=1}^{5} P_m^\mu\, n \bullet \nabla E_2^m\big|_{\xi=0} - \frac{1}{h_0}\left[\begin{array}{l}
L_{2,1} \dfrac{5\,\mathrm{E}_2^1\big|_{\xi=0}}{a_1 a_2 a_3 \nu^2} + L_{2,2} \dfrac{5\,\mathrm{E}_2^2\big|_{\xi=0}}{a_1 a_2 a_3 \nu'^2} \\
+ L_{2,3} \dfrac{5\,\mathrm{E}_2^3\big|_{\xi=0}}{a_1 a_2 a_3 \left(a_1 a_2\right)^2} + L_{2,4} \dfrac{5\,\mathrm{E}_2^4\big|_{\xi=0}}{a_1 a_2 a_3 \left(a_1 a_3\right)^2} + L_{2,5} \dfrac{5\,\mathrm{E}_2^5\big|_{\xi=0}}{a_1 a_2 a_3 \left(a_2 a_3\right)^2}
\end{array}\right]
\qquad (5.79)
$$

with

$$P_1^\mu = \frac{5}{2} L_{2,1} T_\nu - \mu K_{2,1} \quad P_2^\mu = \frac{5}{2} L_{2,2} T_{\nu'} - \mu K_{2,2}$$
$$P_3^\mu = \frac{5}{2} L_{2,3} I_{1,2} - \mu K_{2,3} \quad P_4^\mu = \frac{5}{2} L_{2,4} I_{1,3} - \mu K_{2,4} \quad P_5^\mu = \frac{5}{2} L_{2,5} I_{2,3} - \mu K_{2,5} \tag{5.80}$$

Let us consider the cross product $n \wedge \nabla\left(U_1 - U_1^{(i)}\right)$ and let us compute its components :

$$h_0\, n \wedge \nabla\left(U_1 - U_1^{(i)}\right)\Big|_{l=1} = \left[2\delta \left(\frac{1}{a_2^2\left(\nu + a_3^2\right)} - \frac{1}{a_3^2\left(\nu + a_2^2\right)} \right) P_1 + (') P_2 \right] x_2 x_3$$
$$-\left(h_1 h_2 h_3\right) \frac{h_3}{a_3^2} P_3\, x_1 x_3 + \left(h_1 h_2 h_3\right) \frac{h_2}{a_2^2} P_4\, x_1 x_2 + \left(h_1 h_2 h_3\right) h_1 P_5 \left(\frac{x_2^2}{a_2^2} - \frac{x_3^2}{a_3^2} \right)$$
$$h_0\, n \wedge \nabla\left(U_1 - U_1^{(i)}\right)\Big|_{l=2} = \left[2\delta \left(\frac{1}{a_3^2\left(\nu + a_1^2\right)} - \frac{1}{a_1^2\left(\nu + a_3^2\right)} \right) P_1 + (') P_2 \right] x_1 x_3$$
$$+\left(h_1 h_2 h_3\right) \frac{h_3}{a_3^2} P_3\, x_2 x_3 + \left(h_1 h_2 h_3\right) h_2 P_4 \left(\frac{x_3^2}{a_3^2} - \frac{x_1^2}{a_1^2} \right) - \left(h_1 h_2 h_3\right) \frac{h_1}{a_1^2} P_5\, x_1 x_2 \tag{5.81}$$
$$h_0\, n \wedge \nabla\left(U_1 - U_1^{(i)}\right)\Big|_{l=3} = \left[2\delta \left(\frac{1}{a_1^2\left(\nu + a_2^2\right)} - \frac{1}{a_2^2\left(\nu + a_1^2\right)} \right) P_1 + (') P_2 \right] x_1 x_2$$
$$+\left(h_1 h_2 h_3\right) h_3 P_3 \left(\frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} \right) - \left(h_1 h_2 h_3\right) \frac{h_2}{a_2^2} P_4\, x_2 x_3 + \left(h_1 h_2 h_3\right) \frac{h_1}{a_1^2} P_5\, x_1 x_3$$

Before continuing, we will return to the theory of ellipsoidal harmonics.

The location of the studied boundary condition corresponds to the surface of the ellipsoid, so it makes more sense to work with the family of ellipsoidal surface harmonics, which are directly related to the ellipsoidal (volume) harmonics used so far. The following lines are inspired by the book by Dassios.

Ellipsoidal (volume) harmonics are classically defined from Lamé functions $La_n^m(x)$ (called $E_n^m(x)$ by Dassios) where x is one of the three ellipsoidal coordinates (which form an orthogonal system) $\rho = \sqrt{\xi + a_1^2}, \mu, \nu$. So we have :

$$E_n^m(\rho, \mu, \nu) = La_n^m(\rho)\, La_n^m(\mu)\, La_n^m(\nu) \tag{5.82}$$

We previously saw that $E_n^m(\rho, \mu, \nu)$ can be expressed in terms of Cartesian coordinates.

However, since the surface of the ellipsoid corresponds to $\xi = 0$ or $\rho = a_1$, the ellipsoidal surface harmonics are defined as :

$$S_n^m(\mu,\nu) = La_n^m(\mu)La_n^m(\nu) \tag{5.83}$$

then,

$$S_n^m(\mu,\nu) = E_n^m(x,y,z;\xi=0) / La_n^m(a_1) \tag{5.84}$$

$S_n^m(\mu,\nu)$ also form a set of orthogonal functions on which any function associating a real value with a point on the surface of the ellipsoid can be theoretically expanded.

Dassios showed that the second order terms $x_i x_j$ can be expressed using the ellipsoidal surface harmonics (on the surface of the ellipsoid $\rho = a_1$),

$$\begin{aligned} x_n^2 &= \frac{a_n^2}{3}\left[1 - \frac{S_2^1}{(\nu + a_n^2)(\nu - \nu')} + \frac{S_2^2}{(\nu' + a_n^2)(\nu - \nu')}\right] \\ \frac{x_1 x_2 x_3}{x_n} &= \beta \frac{1}{h_n a_n} S_2^{6-n} \quad ; \quad \beta = \frac{a_1 a_2 a_3}{h_1 h_2 h_3} \end{aligned} \tag{5.85}$$

Introducing 5.85 into 5.81 leads to :

$$h_0\, n \wedge \nabla\left(U_1 - U_1^{(i)}\right)\Big|_{l=1} = \left[2\delta\left(\frac{1}{a_2^2\left(\nu + a_3^2\right)} - \frac{1}{a_3^2\left(\nu + a_2^2\right)} \right) P_1 + (') P_2 \right] \beta \frac{1}{h_1 a_1} S_2^5$$

$$-\left(h_1 h_2 h_3\right) \frac{h_3}{a_3^2} P_3\, \beta \frac{1}{h_2 a_2} S_2^4 + \left(h_1 h_2 h_3\right) \frac{h_2}{a_2^2} P_4\, \beta \frac{1}{h_3 a_3} S_2^3$$

$$+\left(h_1 h_2 h_3\right) h_1 P_5 \frac{1}{3\left(\nu - \nu'\right)} \left(\frac{1}{\nu' + a_2^2} - \frac{1}{\nu' + a_3^2} \right) S_2^2 - \left(h_1 h_2 h_3\right) h_1 P_5 \frac{1}{3\left(\nu - \nu'\right)} \left(\frac{1}{\nu + a_2^2} - \frac{1}{\nu + a_3^2} \right) S_2^1$$

$$h_0\, n \wedge \nabla\left(U_1 - U_1^{(i)}\right)\Big|_{l=2} = \left[2\delta\left(\frac{1}{a_3^2\left(\nu + a_1^2\right)} - \frac{1}{a_1^2\left(\nu + a_3^2\right)} \right) P_1 + (') P_2 \right] \beta \frac{1}{h_2 a_2} S_2^4$$

$$+\left(h_1 h_2 h_3\right) \frac{h_3}{a_3^2} P_3\, \beta \frac{1}{h_1 a_1} S_2^5 - \left(h_1 h_2 h_3\right) \frac{h_1}{a_1^2} P_5 \beta \frac{1}{h_3 a_3} S_2^3$$

$$+\left(h_1 h_2 h_3\right) h_2 P_4 \frac{1}{3\left(\nu - \nu'\right)} \left(\frac{1}{\nu' + a_3^2} - \frac{1}{\nu' + a_1^2} \right) S_2^2 - \left(h_1 h_2 h_3\right) h_2 P_4 \frac{1}{3\left(\nu - \nu'\right)} \left(\frac{1}{\nu + a_3^2} - \frac{1}{\nu + a_1^2} \right) S_2^1$$

$$h_0\, n \wedge \nabla\left(U_1 - U_1^{(i)}\right)\Big|_{l=3} = \left[2\delta\left(\frac{1}{a_1^2\left(\nu + a_2^2\right)} - \frac{1}{a_2^2\left(\nu + a_1^2\right)} \right) P_1 + (') P_2 \right] \beta \frac{1}{h_3 a_3} S_2^3$$

$$-\left(h_1 h_2 h_3\right) \frac{h_2}{a_2^2} P_4 \beta \frac{1}{h_1 a_1} S_2^5 + \left(h_1 h_2 h_3\right) \frac{h_1}{a_1^2} P_5 \beta \frac{1}{h_2 a_2} S_2^4$$

$$+\left(h_1 h_2 h_3\right) h_3 P_3 \frac{1}{3\left(\nu - \nu'\right)} \left(\frac{1}{\nu' + a_1^2} - \frac{1}{\nu' + a_2^2} \right) S_2^2 - \left(h_1 h_2 h_3\right) h_3 P_3 \frac{1}{3\left(\nu - \nu'\right)} \left(\frac{1}{\nu + a_1^2} - \frac{1}{\nu + a_2^2} \right) S_2^1$$

(5.86)

Let us now consider the scalar product $n \bullet \nabla\left(U_1 - \mu U_1^{(i)}\right)$, starting from 5.79 :

$$h_0 n \bullet \nabla\left(U_1 - \mu U_1^{(i)}\right) = 2\delta \sum_{i=1}^{3} \frac{x_i^2}{a_i^2\left(\nu + a_i^2\right)} P_1^{\mu} + 2\delta' \sum_{i=1}^{3} \frac{x_i^2}{a_i^2\left(\nu' + a_i^2\right)} P_2^{\mu}$$

$$+\left(h_1 h_2 h_3\right) h_3 \left(\frac{1}{a_1^2} + \frac{1}{a_2^2} \right) x_1 x_2 P_3^{\mu} + \left(h_1 h_2 h_3\right) h_2 \left(\frac{1}{a_1^2} + \frac{1}{a_3^2} \right) x_1 x_3 P_4^{\mu} + \left(h_1 h_2 h_3\right) h_1 \left(\frac{1}{a_2^2} + \frac{1}{a_3^2} \right) x_2 x_3 P_5^{\mu}$$

$$-L_{2,1} \frac{5\mathrm{E}_2^1\big|_{\xi=0}}{a_1 a_2 a_3 \nu^2} - L_{2,2} \frac{5\mathrm{E}_2^2\big|_{\xi=0}}{a_1 a_2 a_3 \nu'^2} - L_{2,3} \frac{5\mathrm{E}_2^3\big|_{\xi=0}}{a_1 a_2 a_3 \left(a_1 a_2\right)^2} - L_{2,4} \frac{5\mathrm{E}_2^4\big|_{\xi=0}}{a_1 a_2 a_3 \left(a_1 a_3\right)^2} - L_{2,5} \frac{5\mathrm{E}_2^5\big|_{\xi=0}}{a_1 a_2 a_3 \left(a_2 a_3\right)^2}$$

(5.87)

Using the equations 5.85 :

$$h_0 n \bullet \nabla\left(U_1 - \mu U_1^{(i)}\right) = 2\delta \frac{1}{3(\nu - \nu')}\left[\sum_{i=1}^{3} \frac{1}{\left(\nu + a_i^2\right)\left(\nu' + a_i^2\right)} S_2^2 - \sum_{i=1}^{3} \frac{1}{\left(\nu + a_i^2\right)^2} S_2^1\right] P_1^\mu$$

$$+2\delta' \frac{1}{3(\nu - \nu')}\left[\sum_{i=1}^{3} \frac{1}{\left(\nu' + a_i^2\right)^2} S_2^2 - \sum_{i=1}^{3} \frac{1}{\left(\nu + a_i^2\right)\left(\nu' + a_i^2\right)} S_2^1\right] P_2^\mu$$

$$+\left(h_1 h_2 h_3\right) h_3 \left(\frac{1}{a_1^2} + \frac{1}{a_2^2}\right) \beta \frac{1}{h_3 a_3} S_2^3 P_3^\mu + \left(h_1 h_2 h_3\right) h_2 \left(\frac{1}{a_1^2} + \frac{1}{a_3^2}\right) \beta \frac{1}{h_2 a_2} S_2^4 P_4^\mu + \left(h_1 h_2 h_3\right) h_1 \left(\frac{1}{a_2^2} + \frac{1}{a_3^2}\right) \beta \frac{1}{h_1 a_1} S_2^5 P_5^\mu$$

$$-L_{2,1} \frac{5\mathrm{E}_2^1\big|_{\xi=0}}{a_1 a_2 a_3 \nu^2} - L_{2,2} \frac{5\mathrm{E}_2^2\big|_{\xi=0}}{a_1 a_2 a_3 \nu'^2} - L_{2,3} \frac{5\mathrm{E}_2^3\big|_{\xi=0}}{a_1 a_2 a_3 \left(a_1 a_2\right)^2} - L_{2,4} \frac{5\mathrm{E}_2^4\big|_{\xi=0}}{a_1 a_2 a_3 \left(a_1 a_3\right)^2} - L_{2,5} \frac{5\mathrm{E}_2^5\big|_{\xi=0}}{a_1 a_2 a_3 \left(a_2 a_3\right)^2}$$

(5.88)

applying 5.84,

$$E_2^m\left(x, y, z; \xi = 0\right) = La_2^m\left(a_1\right) S_2^m\left(\mu, \nu\right) \tag{5.89}$$

with

$$La_2^1\left(a_1\right) = -\nu \quad La_2^2\left(a_1\right) = -\nu' \quad La_2^3\left(a_1\right) = a_1 a_2 \quad La_2^4\left(a_1\right) = a_1 a_3 \quad La_2^5\left(a_1\right) = a_2 a_3 \tag{5.90}$$

Finally 5.88 becomes, using 5.68 and 5.69 :

$$h_0 n \bullet \nabla\left(U_1 - \mu U_1^{(i)}\right)$$

$$= \left[2P_1^\mu + L_{2,1} \frac{5}{a_1 a_2 a_3 \nu}\right] S_2^1 + \left[2P_2^\mu + L_{2,2} \frac{5}{a_1 a_2 a_3 \nu'}\right] S_2^2$$

$$+\left[\left(h_1 h_2 h_3\right) h_3 \left(\frac{1}{a_1^2} + \frac{1}{a_2^2}\right) \beta \frac{1}{h_3 a_3} P_3^\mu - L_{2,3} \frac{5}{a_1 a_2 a_3 \left(a_1 a_2\right)}\right] S_2^3 \tag{5.91}$$

$$+\left[\left(h_1 h_2 h_3\right) h_2 \left(\frac{1}{a_1^2} + \frac{1}{a_3^2}\right) \beta \frac{1}{h_2 a_2} P_4^\mu - L_{2,4} \frac{5}{a_1 a_2 a_3 \left(a_1 a_3\right)}\right] S_2^4$$

$$+\left[\left(h_1 h_2 h_3\right) h_1 \left(\frac{1}{a_2^2} + \frac{1}{a_3^2}\right) \beta \frac{1}{h_1 a_1} P_5^\mu - L_{2,5} \frac{5}{a_1 a_2 a_3 \left(a_2 a_3\right)}\right] S_2^5$$

Let's now look at the right-hand side of equation 5.14. We may write :

$$G_1 - \mu^e G_1^{(i)} + H_1^{(0)} = \left[B\right] x \quad x =^T \left(x_1, x_2, x_3\right) \quad e = 0,1 \tag{5.92}$$

with

$$B=\begin{bmatrix} l_2 l & ml_2 - A_3 I_2 - \frac{1}{2}\varepsilon\mu^e A_3^{(i)} & nl_2 + A_2 I_3 + \frac{1}{2}\varepsilon\mu^e A_2^{(i)} \\ m_2 l + A_3 I_1 + \frac{1}{2}\varepsilon\mu^e A_3^{(i)} & m_2 m & nm_2 - A_1 I_3 - \frac{1}{2}\varepsilon\mu^e A_1^{(i)} \\ n_2 l - A_2 I_1 - \frac{1}{2}\varepsilon\mu^e A_2^{(i)} & n_2 m + A_1 I_2 + \frac{1}{2}\varepsilon\mu^e A_1^{(i)} & n_2 n \end{bmatrix} \qquad (5.93)$$

This matrix has a zero trace (due to the orthogonality of the unit vectors carrying wave vector and fields). We can also check that the matrix is symmetrical. The matrix will be noted $\left[B\right]$ if $e=0$ and $\left[B^{\mu}\right]$ if $e=1$.

Let us consider the cross product $n\wedge\nabla\left(G_1-G_1^{(i)}+H_1^{(0)}\right)$ and let us compute its components. So, for the first component ($l=1$) :

$$\begin{aligned} & h_0 n\wedge\nabla\left(G_1-G_1^{(i)}+H_1^{(0)}\right)\Big|_{l=1}=\frac{x_2}{a_2^2}\sum_{i=1}^{3}B_{3,i}x_i-\frac{x_3}{a_3^2}\sum_{i=1}^{3}B_{2,i}x_i= \\ & -\frac{1}{3(\nu-\nu')}\left(\frac{B_{3,2}}{\nu+a_2^2}-\frac{B_{2,3}}{\nu+a_3^2}\right)S_2^1+\frac{1}{3(\nu-\nu')}\left(\frac{B_{3,2}}{\nu'+a_2^2}-\frac{B_{2,3}}{\nu'+a_3^2}\right)S_2^2 \\ & +\beta\frac{B_{3,1}}{a_2^2}\frac{1}{h_3 a_3}S_2^3-\beta\frac{B_{2,1}}{a_3^2}\frac{1}{h_2 a_2}S_2^4+\beta\left(\frac{B_{3,3}}{a_2^2}-\frac{B_{2,2}}{a_3^2}\right)\frac{1}{h_1 a_1}S_2^5 \\ & +\frac{1}{3}\left(B_{3,2}-B_{2,3}\right) \end{aligned} \qquad (5.94)$$

The same applies to the two other components:

$$\begin{aligned} & h_0 n\wedge\nabla\left(G_1-G_1^{(i)}+H_1^{(0)}\right)\Big|_{l=2}= \\ & -\frac{1}{3(\nu-\nu')}\left(-\frac{B_{3,1}}{\nu+a_1^2}+\frac{B_{1,3}}{\nu+a_3^2}\right)S_2^1+\frac{1}{3(\nu-\nu')}\left(-\frac{B_{3,1}}{\nu'+a_1^2}+\frac{B_{1,3}}{\nu'+a_3^2}\right)S_2^2 \\ & -\beta\frac{B_{3,2}}{a_1^2}\frac{1}{h_3 a_3}S_2^3+\beta\left(\frac{B_{1,1}}{a_3^2}-\frac{B_{3,3}}{a_1^2}\right)\frac{1}{h_2 a_2}S_2^4+\beta\frac{B_{1,2}}{a_3^2}\frac{1}{h_1 a_1}S_2^5 \\ & +\frac{1}{3}\left(B_{1,3}-B_{3,1}\right) \end{aligned} \qquad (5.95)$$

$$
\begin{aligned}
&h_0 n \wedge \nabla\left(G_1 - G_1^{(i)} + H_1^{(0)}\right)\Big|_{l=3} = \\
&-\frac{1}{3(\nu-\nu')}\left(\frac{B_{2,1}}{\nu+a_1^2}-\frac{B_{1,2}}{\nu+a_2^2}\right)S_2^1 + \frac{1}{3(\nu-\nu')}\left(\frac{B_{2,1}}{\nu'+a_1^2}-\frac{B_{1,2}}{\nu'+a_2^2}\right)S_2^2 \\
&+\beta\left(\frac{B_{2,2}}{a_1^2}-\frac{B_{1,1}}{a_2^2}\right)\frac{1}{h_3 a_3}S_2^3 + \beta\frac{B_{2,3}}{a_1^2}\frac{1}{h_2 a_2}S_2^4 - \beta\frac{B_{1,3}}{a_2^2}\frac{1}{h_1 a_1}S_2^5 \\
&+\frac{1}{3}\left(B_{2,1}-B_{1,2}\right)
\end{aligned}
\tag{5.96}
$$

Let us now consider the scalar product $n \bullet \nabla\left(G_1 - \mu G_1^{(i)} + H_1^{(0)}\right)$. Using the same procedure, we find :

$$
\begin{aligned}
&h_0 n \bullet \nabla\left(G_1 - \mu G_1^{(i)} + H_1^{(0)}\right) = -\frac{1}{3(\nu-\nu')}\left(\sum_{i=1}^{3}\frac{B_{i,i}^{\mu}}{\nu+a_i^2}\right)S_2^1 + \frac{1}{3(\nu-\nu')}\left(\sum_{i=1}^{3}\frac{B_{i,i}^{\mu}}{\nu'+a_i^2}\right)S_2^2 \\
&+\beta\sum_{\substack{i>j \\ n\neq i,j}}\left(\frac{B_{j,i}^{\mu}}{a_j^2}+\frac{B_{i,j}^{\mu}}{a_i^2}\right)\frac{1}{h_n a_n}S_2^{6-n} + \frac{1}{3}\sum_{i=1}^{3}B_{i,i}^{\mu}
\end{aligned}
\tag{5.97}
$$

Given the properties of the matrices $\left[B\right]$ et $\left[B^{\mu}\right]$, the constant terms in 5.94-5.97 vanish. $n \wedge \nabla\left(G_1 - G_1^{(i)} + H_1^{(0)}\right)$ and $n \bullet \nabla\left(G_1 - \mu G_1^{(i)} + H_1^{(0)}\right)$ therefore contain only ellipsoidal surface harmonics of degree 2. It must therefore be the same for $n \wedge \nabla\left(U_1 - U_1^{(i)}\right)$ and $n \bullet \nabla\left(U_1 - \mu U_1^{(i)}\right)$; It is therefore not necessary to take into account ellipsoidal surface harmonics of degree 0.

We have therefore just expressed the various terms appearing in the boundary conditions as linear combinations of ellipsoidal surface harmonics. Boundary conditions 5.14 are therefore expressed as linear combinations of ellipsoidal surface harmonics equal to 0. As these form an orthonormal basis, this implies that the coefficients (which contain the unknowns of the problem) of the harmonics are identically zero.

From the scalar product (5.91 and 5.97) we can deduce :

$$2P_1^\mu + L_{2,1}\frac{5}{a_1a_2a_3\nu} = \frac{1}{3(\nu-\nu')}\left(\sum_{i=1}^{3}\frac{B_{i,i}^\mu}{\nu+a_i^2}\right)$$

$$2P_2^\mu + L_{2,2}\frac{5}{a_1a_2a_3\nu'} = -\frac{1}{3(\nu-\nu')}\left(\sum_{i=1}^{3}\frac{B_{i,i}^\mu}{\nu'+a_i^2}\right)$$

$$n=1,2,3$$

$$(a_1a_2a_3)\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{1}{a_3^2}-\frac{1}{a_n^2}\right)P_{6-n}^\mu - L_{2,6-n}\frac{5a_n^2}{(a_1a_2a_3)^2} = -\frac{a_1a_2a_3}{h_1h_2h_3}\left(\frac{B_{j,i}^\mu}{a_j^2}+\frac{B_{i,j}^\mu}{a_i^2}\right)\frac{1}{h_n} \quad i>j \quad n\neq i,j$$

(5.98)

Let us now consider the cross product (Eqs. 5.86, 5.94-5.96) :

Let us take the example of the unknown P_3 ; it is involved in 3 equations:

$$l=1 \quad S_2^4:$$

$$P_3 = -\frac{B_{2,1}}{(h_1h_2h_3)h_3}$$

$$l=2 \quad S_2^5:$$

$$P_3 = -\frac{B_{1,2}}{(h_1h_2h_3)h_3}$$

$$l=3 \quad S_2^2:$$

$$P_3\left(\frac{1}{\nu'+a_1^2}-\frac{1}{\nu'+a_2^2}\right) = -\frac{1}{(h_1h_2h_3)h_3}\left(\frac{B_{2,1}}{\nu'+a_1^2}-\frac{B_{1,2}}{\nu'+a_2^2}\right)$$

(5.99)

$$l=3 \quad S_2^1:$$

$$P_3\left(\frac{1}{\nu+a_1^2}-\frac{1}{\nu+a_2^2}\right) = -\frac{1}{(h_1h_2h_3)h_3}\left(\frac{B_{2,1}}{\nu+a_1^2}-\frac{B_{1,2}}{\nu+a_2^2}\right)$$

As the matrix $[B]$ is symmetrical, these 4 equations can fortunately be reduced to a single one:

$$P_3 = -\frac{B_{2,1}}{(h_1h_2h_3)h_3} \tag{5.100}$$

It is the same for P_4 and P_5, that obey the equations :

$$P_4 = -\frac{B_{3,1}}{(h_1h_2h_3)h_2} \tag{5.101}$$

$$P_5 = -\frac{B_{2,3}}{\left(h_1 h_2 h_3\right) h_1} \tag{5.102}$$

In the same way , P_1 and P_2 obey :

$$\begin{aligned}
&l=1 \quad S_2^5 : \\
&2\delta\left(\frac{1}{a_2^2\left(\nu+a_3^2\right)}-\frac{1}{a_3^2\left(\nu+a_2^2\right)}\right)P_1+(')P_2=-\left(\frac{B_{3,3}}{a_2^2}-\frac{B_{2,2}}{a_3^2}\right) \\
&l=2 \quad S_2^4 : \\
&2\delta\left(\frac{1}{a_3^2\left(\nu+a_1^2\right)}-\frac{1}{a_1^2\left(\nu+a_3^2\right)}\right)P_1+(')P_2=-\left(\frac{B_{1,1}}{a_3^2}-\frac{B_{3,3}}{a_1^2}\right) \\
&l=3 \quad S_2^3 : \\
&2\delta\left(\frac{1}{a_1^2\left(\nu+a_2^2\right)}-\frac{1}{a_2^2\left(\nu+a_1^2\right)}\right)P_1+(')P_2=-\left(\frac{B_{2,2}}{a_1^2}-\frac{B_{1,1}}{a_2^2}\right)
\end{aligned} \tag{5.103}$$

These three equations are not independent, because the weighted sum (by $a_i^2 a_j^2$) of the three previous equations (5.103) :

$$\begin{aligned}
&2\delta\nu\left(\frac{1}{\left(\nu+a_3^2\right)}-\frac{1}{\left(\nu+a_2^2\right)}\right)P_1+(')P_2=B_{3,3}a_3^2-B_{2,2}a_2^2 \\
&2\delta\nu\left(\frac{1}{\left(\nu+a_1^2\right)}-\frac{1}{\left(\nu+a_3^2\right)}\right)P_1+(')P_2=B_{1,1}a_1^2-B_{3,3}a_3^2 \\
&2\delta\nu\left(\frac{1}{\left(\nu+a_2^2\right)}-\frac{1}{\left(\nu+a_1^2\right)}\right)P_1+(')P_2=B_{2,2}a_2^2-B_{1,1}a_1^2
\end{aligned} \tag{5.104}$$

leads to 0=0.

From the first and second equations (5.104), we straightforwardly get for P_1 :

$$\begin{aligned}
&4\nu\nu'\left(h_1 h_2 h_3\right)^2\left(\nu-\nu'\right)P_1= \\
&\quad -2\nu'\left[B_{1,1}a_1^2\left(a_2^2-a_3^2\right)\left(\nu'+a_1^2\right)+B_{2,2}a_2^2\left(a_3^2-a_1^2\right)\left(\nu'+a_2^2\right)+B_{3,3}a_3^2\left(a_1^2-a_2^2\right)\left(\nu'+a_3^2\right)\right]
\end{aligned} \tag{5.105}$$

or

$$P_1 = -\frac{1}{2\nu(\nu-\nu')}\left[B_{1,1}a_1^2 \frac{h_1^2}{(h_1h_2h_3)^2}(\nu'+a_1^2) - B_{2,2}a_2^2 \frac{h_2^2}{(h_1h_2h_3)^2}(\nu'+a_2^2) + B_{3,3}a_3^2 \frac{h_3^2}{(h_1h_2h_3)^2}(\nu'+a_3^2) \right] \tag{5.106}$$

Now, according to Dassios,

$$(\nu + a_n^2)(\nu'+a_n^2) = (-1)^{n+1}\frac{h_1^2h_2^2h_3^2}{3h_n^2} \tag{5.107}$$

and given the zero trace of $[B]$, we deduce :

$$\begin{aligned} P_1 &= -\frac{1}{6(\nu-\nu')}\left[B_{1,1}\frac{a_1^2}{\nu(\nu+a_1^2)} + B_{2,2}\frac{a_2^2}{\nu(\nu+a_2^2)} + B_{3,3}\frac{a_3^2}{\nu(\nu+a_3^2)} \right] \\ &= -\frac{1}{6(\nu-\nu')}\left[\frac{1}{\nu}(B_{1,1}+B_{2,2}+B_{3,3}) - \frac{B_{1,1}}{(\nu+a_1^2)} - \frac{B_{2,2}}{(\nu+a_2^2)} - \frac{B_{3,3}}{(\nu+a_3^2)} \right] \\ &= \frac{1}{6(\nu-\nu')}\left[\sum_{i=1}^{3}\frac{B_{i,i}}{(\nu+a_i^2)} \right] \end{aligned} \tag{5.108}$$

The same applies to P_2.

Finally, the unknown coefficients $L_{2,1}$ and $K_{2,1}$ are obtained from 5.98 (scalar product), 5.108 (cross product), 5.76 and 5.80 (definitions of P_1, P_1^μ) and from the equality $B_{i,i}^\mu = B_{i,i}$:

$$\begin{aligned} 2P_1^\mu + L_{2,1}\frac{5}{a_1a_2a_3\nu} &= \frac{1}{3(\nu-\nu')}\left[\sum_{i=1}^{3}\frac{B_{i,i}^\mu}{\nu+a_i^2} \right] \\ P_1 &= \frac{1}{6(\nu-\nu')}\left[\sum_{i=1}^{3}\frac{B_{i,i}}{(\nu+a_i^2)} \right] \\ P_1 &= \frac{5}{2}L_{2,1}T_\nu - K_{2,1} \\ P_1^\mu &= \frac{5}{2}L_{2,1}T_\nu - \mu K_{2,1} \end{aligned} \tag{5.109}$$

whose solution is :

$$\frac{5}{2}L_{2,1}=\frac{\mu-1}{T_\nu(\mu-1)-1/(a_1a_2a_3\nu)}\frac{1}{6(\nu-\nu')}\left[\sum_{i=1}^{3}\frac{B_{i,i}}{\nu+a_i^2}\right]$$
$$K_{2,1}=\left[\frac{1/(a_1a_2a_3\nu)}{T_\nu(\mu-1)-1/(a_1a_2a_3\nu)}\right]\frac{1}{6(\nu-\nu')}\left[\sum_{i=1}^{3}\frac{B_{i,i}}{\nu+a_i^2}\right] \qquad (5.110)$$

The result for $L_{2,2}$ and $K_{2,2}$ is obtained by carrying out the exchange $\nu\leftrightarrow\nu'$.

$L_{2,m}$ and $K_{2,m}$ with $m=3,4,5\ \ (m=6-n)$ are obtained in the same way (5.98, 5.100-102):

$i>j \quad n\neq i,j$

$$(a_1a_2a_3)\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{1}{a_3^2}-\frac{1}{a_n^2}\right)P_{6-n}^{\mu}-L_{2,6-n}\frac{5a_n^2}{(a_1a_2a_3)^2}=-\frac{a_1a_2a_3}{h_1h_2h_3}\left(\frac{B_{j,i}^{\mu}}{a_j^2}+\frac{B_{i,j}^{\mu}}{a_i^2}\right)\frac{1}{h_n}$$
$$P_{6-n}=-\frac{(B_{j,i}+B_{i,j})}{2(h_1h_2h_3)h_n} \qquad (5.111)$$
$$P_{6-n}=\frac{5}{2}I_{i,j}L_{2,6-n}-K_{2,6-n}$$
$$P_{6-n}^{\mu}=\frac{5}{2}I_{i,j}L_{2,6-n}-\mu K_{2,6-n}$$

then,

$i>j \quad n\neq i,j$

$$L_{2,6-n}=\frac{a_1a_2a_3}{h_1h_2h_3h_n}\times$$
$$\left[-\frac{B_{j,i}^{\mu}}{a_j^2}-\frac{B_{i,j}^{\mu}}{a_i^2}+\frac{(B_{j,i}+B_{i,j})}{2}\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{1}{a_3^2}-\frac{1}{a_n^2}\right)\mu\right]/\left[(a_1a_2a_3)\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{1}{a_3^2}-\frac{1}{a_n^2}\right)\frac{5}{2}I_{i,j}(1-\mu)-\frac{5a_n^2}{(a_1a_2a_3)^2}\right]$$
$$K_{2,6-n}=\frac{5}{2}I_{i,j}L_{2,6-n}+\frac{(B_{j,i}+B_{i,j})}{2(h_1h_2h_3)h_n}$$

(5.112)

These results (5.110 and 5.112) are exactly the same as Stevenson's. To make the comparison, it should be noted that :

$$
\begin{aligned}
&B\left(Stevenson\right)=\frac{5}{2}L_{2,1}\\
&B^{(i)}\left(Stevenson\right)=K_{2,1}\\
&B_n\left(Stevenson\right)=\frac{5}{2}\left(h_1h_2h_3\right)h_nL_{2,6-n}\\
&B_n^{(i)}\left(Stevenson\right)=\left(h_1h_2h_3\right)h_nK_{2,6-n}
\end{aligned}
\tag{5.113}
$$

To conclude this section, let us compute $\int_{\partial E} n\bullet H_1 dS$:

$$\int_{\partial E} n\bullet H_1 dS=\int_{\partial E} n\bullet G_1 dS+\int_{\partial E} n\bullet \nabla U_1 dS$$

with

$$\int_{\partial E} n\bullet G_1 dS=\int_{\partial E} n\bullet\left(A\wedge P\right)dS$$

This last integral is the sum of integrals of the type $\int_{\partial E}\frac{x_ix_j}{h_0}dS \quad i\neq j$, that all vanish.

$$\int_{\partial E} n\bullet \nabla U_1 dS=\sum_{m=1}^{5}L_{2,m}\int_{\partial E} n\bullet\nabla\Upsilon_2^m$$

Using 5.39, 5.73 et 5.78, we show easily that the terms corresponding to $m=3,4,5$ vanish, because they contain the integral $\int_{\partial E}\frac{x_ix_j}{h_0}dS \quad i\neq j$. So we have :

$$
\begin{aligned}
&\int_{\partial E} n\bullet \nabla U_1 dS=\sum_{m=1}^{2}L_{2,m}\int_{\partial E} n\bullet\nabla\Upsilon_2^m\\
&=5L_{2,1}\delta\left(T_\nu+\frac{1}{a_1a_2a_3\nu}\right)\sum_{i=1}^{3}\frac{1}{\left(\nu+a_i^2\right)}\int_{\partial E}\frac{x_i^2}{h_0a_i^2}dS+5L_{2,2}\delta'\left(T_{\nu'}+\frac{1}{a_1a_2a_3\nu'}\right)\sum_{i=1}^{3}\frac{1}{\left(\nu'+a_i^2\right)}\int_{\partial E}\frac{x_i^2}{h_0a_i^2}dS
\end{aligned}
\tag{5.114}
$$

So using 3.1,

$$\sum_{i=1}^{3}\int_{\partial E}\frac{x_i^2}{h_0a_i^2}dS=\int_{\partial E}\frac{1}{h_0}dS \tag{5.115}$$

and by symmetry,

$$\int_{\partial E} \frac{x_i^2}{h_0 a_i^2} dS = \frac{1}{3} \int_{\partial E} \frac{1}{h_0} dS \qquad (5.116)$$

We deduce :

$$\int_{\partial E} n \bullet \nabla U_1 dS = \frac{5}{3} \left(\int_{\partial E} \frac{1}{h_0} dS \right) \left\{ L_{2,1} \delta \left(T_\nu + \frac{1}{a_1 a_2 a_3 \nu} \right) \sum_{i=1}^{3} \frac{1}{\left(\nu + a_i^2 \right)} + L_{2,2} \delta' \left(T_{\nu'} + \frac{1}{a_1 a_2 a_3 \nu'} \right) \sum_{i=1}^{3} \frac{1}{\left(\nu' + a_i^2 \right)} \right\} = 0$$

(5.117)

Using the definitions of ν, ν' (5.28), we have :

$$\int_{\partial E} n \bullet H_1 dS = \int_{\partial E} n \bullet G_1 dS + \int_{\partial E} n \bullet \nabla U_1 dS = 0 .$$

Here again, as with 0th order, the proposed solution verifies this equation, but does not need it to get it.

6. Second order term

6.1 Calculation of $F_2, F_2^{(i)}$

The notations will be as close as possible to Stevenson's notations.

Adopting the same approach as for order 1, we first look for a (particular) solution of :

$$\begin{aligned} &\nabla \wedge F_2 = H_1 \quad \nabla \wedge F_2^{(i)} = \mu H_1^{(i)} \\ &\nabla \bullet F_2 = \nabla \bullet F_2^{(i)} = 0 \end{aligned} \tag{6.1}$$

with

$$H_1 = G_1 + \nabla U_1 \quad H_1^{(i)} = G_1^{(i)} + \nabla U_1^{(i)} \tag{6.2}$$

F_2 therefore contains two contributions, such that :

$$F_2 = F_{2,G} + F_{2,U} \tag{6.3}$$

with

$$\nabla \wedge F_{2,G} = G_1 \tag{6.4}$$

$$\nabla \wedge F_{2,U} = \nabla U_1 \tag{6.5}$$

We have the same,

$$F_2^{(i)} = F_{2,G}^{(i)} + F_{2,U}^{(i)} \tag{6.6}$$

with

- $\nabla \wedge F_{2,G}^{(i)} = \mu G_1^{(i)}$ (6.7)
- $\nabla \wedge F_{2,U}^{(i)} = \mu \nabla U_1^{(i)}$ (6.8)

6.1.1. Calculation of F_2

Let us start with equation 6.4 :

$$\nabla \wedge F_{2,G} = A \wedge P \tag{6.9}$$

Any component of this vector equation can be written as :

$$\partial_i F_{2,G,j} - \partial_j F_{2,G,i} = A_i x_j I_j(\xi) - A_j x_i I_i(\xi) \tag{6.10}$$

The idea is to express the right-hand side as a derivative with respect to x_i or x_j . So, let's start from :

$$\partial_i\left(x_k^2 I_k\left(\xi\right)\right) = x_k^2 I_k\,'\left(\xi\right)\partial_i\xi + 2x_k I_k\left(\xi\right)\delta_{i,k} \tag{6.11}$$

Hence, using 3.3

$$\partial_i\left(\sum_k x_k^2 I_k\left(\xi\right)\right) = \sum_k x_k^2 I_k\,'\left(\xi\right)\partial_i\xi + 2x_i I_i\left(\xi\right) = -\frac{1}{R_\xi}\partial_i\xi + 2x_i I_i\left(\xi\right) \tag{6.12}$$

thus,

$$\partial_i F_{2,G,j} - \partial_j F_{2,G,i} = \partial_j\left(\frac{A_i}{2}\sum_k x_k^2 I_k\left(\xi\right)\right) + \frac{A_i}{2R_\xi}\partial_j\xi - \partial_i\left(\frac{A_j}{2}\sum_k x_k^2 I_k\left(\xi\right)\right) - \frac{A_j}{2R_\xi}\partial_i\xi \tag{6.13}$$

We deduce that :

$$F_{2,G,i} = -\frac{A_i}{2}\left[\sum_k x_k^2 I_k\left(\xi\right) - \int_\xi^\infty \frac{du}{R_u}\right] \tag{6.14}$$

The solution to equation 6.5 is more difficult to find. The expression of ∇U_1 (herein, the i-component with $j \neq i, l \neq i, j \neq l$)

$$\begin{aligned}\partial_i U_1 = {} & 2B\delta T_\nu\left(\xi\right)\frac{x_i}{\nu + a_i^2} + 2B'\delta' T_{\nu'}\left(\xi\right)\frac{x_i}{\nu' + a_i^2} + B_j I_{i,l} x_l + B_l I_{i,j} x_j + \\ & \left(BX_\nu T_\nu\,' + B'X_{\nu'} T_{\nu'}\,' + \sum_l B_l I'_{i,j}\, x_i x_j\right)\partial_i\xi\end{aligned} \tag{6.15}$$

or

$$\begin{aligned}\partial_i U_1 = {} & 2B\delta T_\nu\left(\xi\right)\frac{x_i}{\nu + a_i^2} + 2B'\delta' T_{\nu'}\left(\xi\right)\frac{x_i}{\nu' + a_i^2} + B_j I_{i,l} x_l + B_l I_{i,j} x_j \\ & -\left(BX_\nu\frac{1}{\left(\xi - \nu\right)^2} + B'X_{\nu'}\frac{1}{\left(\xi - \nu'\right)^2} + \sum_l B_l y_i y_j\right)\frac{2}{R_\xi h^2}\, y_i\end{aligned} \tag{6.16}$$

(Let us remind that $\partial_i\xi = \frac{2}{h^2}\, y_i$ with $y_i = x_i / \left(\xi + a_i^2\right)$)

leads us to look for an expression for the k-component of $F_{2,U}$ as :

$$F_{2,U,k} = D_0^k(\xi) + \sum_{i \geq j} D_{i,j}^k(\xi) x_i x_j \tag{6.17}$$

So, for the component 1,

$$\begin{aligned} \partial_2 F_{2,U,3} - \partial_3 F_{2,U,2} &= \left[D_0^3{}' y_2 - D_0^2{}' y_3 + \sum_{i \geq j} \left(D_{i,j}^3{}' y_2 - D_{i,j}^2{}' y_3 \right) x_i x_j \right] \frac{2}{h^2} \\ &+ \left(D_{1,2}^3 - D_{1,3}^2 \right) x_1 + \left(2D_{2,2}^3 - D_{2,3}^2 \right) x_2 + \left(D_{2,3}^3 - 2D_{3,3}^2 \right) x_3 \end{aligned} \tag{6.18}$$

or

$$\begin{aligned} \partial_2 F_{2,U,3} - \partial_3 F_{2,U,2} &= -\left[\alpha_0^3 y_2 - \alpha_0^2 y_3 + \sum_{i \geq j} \left(\alpha_{i,j}^3 y_2 - \alpha_{i,j}^2 y_3 \right) y_i y_j \right] \frac{2}{R_\xi h^2} \\ &+ \left(D_{1,2}^3 - D_{1,3}^2 \right) x_1 + \left(2D_{2,2}^3 - D_{2,3}^2 \right) x_2 + \left(D_{2,3}^3 - 2D_{3,3}^2 \right) x_3 \end{aligned} \tag{6.19}$$

with

$$\alpha_0^k = -D_0^k{}' R_\xi \qquad \alpha_{i,j}^k = -D_{i,j}^k{}' \left(\xi + a_i^2 \right) \left(\xi + a_j^2 \right) R_\xi \tag{6.20}$$

The aim is therefore to identify expressions 6.16 (with i=1) and 6.19 and deduce the D coefficients. In fact, we have the impression that several sets of D coefficients would be appropriate.

Eq. 6.19 can be rewritten :

$$\begin{aligned} \partial_2 F_{2,U,3} - \partial_3 F_{2,U,2} &= -R_1 \frac{2}{R_\xi h^2} - \sum_{i \geq 1} \left(\alpha_{i,1}^3 y_2 - \alpha_{i,1}^2 y_3 \right) y_i \frac{2}{R_\xi h^2} y_1 \\ &+ \left(D_{1,2}^3 - D_{1,3}^2 \right) x_1 + \left(2D_{2,2}^3 - D_{2,3}^2 \right) x_2 + \left(D_{2,3}^3 - 2D_{3,3}^2 \right) x_3 \end{aligned} \tag{6.21a}$$

with

$$R_1 = \alpha_0^3 y_2 - \alpha_0^2 y_3 + \sum_{i \geq j > 1} \left(\alpha_{i,j}^3 y_2 - \alpha_{i,j}^2 y_3 \right) y_i y_j \tag{6.21b}$$

Expression 6.22 contains only y_2 and y_3. However, taking equations 3.3 and 3.7, we can easily find :

$$\begin{aligned} y_2^2 &= \left[h^2 \left(\xi + a_3^2 \right) - 1 + y_1^2 \left(a_1^2 - a_3^2 \right) \right] / \left(a_3^2 - a_2^2 \right) \\ y_3^2 &= \left[-h^2 \left(\xi + a_2^2 \right) + 1 + y_1^2 \left(a_2^2 - a_1^2 \right) \right] / \left(a_3^2 - a_2^2 \right) \end{aligned} \tag{6.22}$$

We deduce that :

$$R_1 = y_2 p_2 - y_3 p_3 + y_2 h^2 q_2 - y_3 h^2 q_3 + y_2 y_1^2 r_2 - y_3 y_1^2 r_3 \tag{6.23}$$

with

$$\begin{aligned}
p_2 &= \left(\alpha_0^3 \left(s_3 - s_2\right) - \alpha_{2,2}^3 + \alpha_{3,3}^3 - \alpha_{2,3}^2\right) / \left(s_3 - s_2\right) \\
p_3 &= \left(\alpha_0^2 \left(s_3 - s_2\right) - \alpha_{2,2}^2 + \alpha_{3,3}^2 + \alpha_{2,3}^3\right) / \left(s_3 - s_2\right) \\
q_2 &= \left(\alpha_{2,2}^3 s_3 - \left(\alpha_{3,3}^3 - \alpha_{2,3}^2\right) s_2\right) / \left(s_3 - s_2\right) \\
q_3 &= \left(\left(\alpha_{2,2}^2 - \alpha_{2,3}^3\right) s_3 - \alpha_{3,3}^2 s_2\right) / \left(s_3 - s_2\right) \\
r_2 &= \left(\alpha_{2,2}^3 \left(s_1 - s_3\right) + \left(\alpha_{3,3}^3 - \alpha_{2,3}^2\right)\left(s_2 - s_1\right)\right) / \left(s_3 - s_2\right) \\
r_3 &= \left(\left(\alpha_{2,2}^2 - \alpha_{2,3}^3\right)\left(s_1 - s_3\right) + \alpha_{3,3}^2 \left(s_2 - s_1\right)\right) / \left(s_3 - s_2\right)
\end{aligned} \tag{6.24}$$

and $s_j = a_j^2 + \xi$

by injecting 6.23 into 6.21,

$$\begin{aligned}
&\partial_2 F_{2,U,3} - \partial_3 F_{2,U,2} = \\
&-\left(y_2 p_2 - y_3 p_3\right) \frac{2}{R_\xi h^2} - \left(y_2 q_2 - y_3 q_3\right) \frac{2}{R_\xi} - \left[y_2 y_1 r_2 - y_3 y_1 r_3 + \sum_{i \geq 1} \left(\alpha_{i,1}^3 y_2 - \alpha_{i,1}^2 y_3\right) y_i \right] \frac{2}{R_\xi h^2} y_1 \\
&+ \left(D_{1,2}^3 - D_{1,3}^2\right) x_1 + \left(2D_{2,2}^3 - D_{2,3}^2\right) x_2 + \left(D_{2,3}^3 - 2D_{3,3}^2\right) x_3
\end{aligned} \tag{6.25}$$

thus

$$\begin{aligned}
&\partial_2 F_{2,U,3} - \partial_3 F_{2,U,2} = \\
&-\left(y_2 p_2 - y_3 p_3\right) \frac{2}{R_\xi h^2} - \left[y_2 y_1 \left(r_2 + \alpha_{1,1}^3\right) - y_3 y_1 \left(r_3 + \alpha_{1,1}^2\right) + y_2 y_3 \left(\alpha_{3,1}^3 - \alpha_{2,1}^2\right) + \alpha_{2,1}^3 y_2^2 - \alpha_{3,1}^2 y_3^2 \right] \frac{2}{R_\xi h^2} y_1 \\
&+ \left(D_{1,2}^3 - D_{1,3}^2\right) x_1 + \left(2D_{2,2}^3 - D_{2,3}^2 - q_2 \frac{2}{s_2 R_\xi} \right) x_2 + \left(D_{2,3}^3 - 2D_{3,3}^2 + q_3 \frac{2}{s_3 R_\xi} \right) x_3
\end{aligned} \tag{6.26}$$

By comparing term by term with 6.16 (i=1), we deduce :

$$p_2 = p_3 = 0$$

$$D^3_{1,2} - D^2_{1,3} = 2B\delta T_\nu(\xi)\frac{1}{\nu + a_1^2} + 2B'\delta' T_{\nu'}(\xi)\frac{1}{\nu' + a_1^2}$$

$$B_3 I_{1,2} = 2D^3_{2,2} - D^2_{2,3} - q_2\frac{2}{s_2 R_\xi}$$

$$B_2 I_{1,3} = D^3_{2,3} - 2D^2_{3,3} + q_3\frac{2}{s_3 R_\xi} \qquad (6.27)$$

$$B_1 = \alpha^3_{3,1} - \alpha^2_{2,1}$$

$$B_2 = -\left(r_3 + \alpha^2_{1,1}\right)$$

$$B_3 = r_2 + \alpha^3_{1,1}$$

The comparison (and identification) $BX_\nu\frac{1}{(\xi-\nu)^2} + B'X_{\nu'}\frac{1}{(\xi-\nu')^2}$ to $\alpha^3_{2,1}y_2^2 - \alpha^2_{3,1}y_3^2$ poses a problem and requires special treatment:

indeed, using 3.3 and 3.7,

$$BX_\nu\frac{1}{(\xi-\nu)^2} + B'X_{\nu'}\frac{1}{(\xi-\nu')^2} = B\delta\left[\sum_{j=1}^{3}\frac{x_j^2}{\nu + a_j^2} - 1\right]\frac{1}{(\xi-\nu)^2} + (')$$

$$= B\delta\left[\sum_{j=1}^{3}\left(\frac{x_j^2}{\nu + a_j^2} - \frac{x_j^2}{s_j}\right)\right]\frac{1}{(\xi-\nu)^2} + (') = B\delta\left[\sum_{j=1}^{3}\left(\frac{s_j}{\left(\nu + a_j^2\right)}y_j^2\right)\right]\frac{1}{(\xi-\nu)} + (') \qquad (6.28)$$

$$= B\delta\left[\frac{s_1}{\nu + a_1^2}h^2 + \left(\frac{s_2}{\nu + a_2^2} - \frac{s_1}{\nu + a_1^2}\right)y_2^2 + \left(\frac{s_3}{\nu + a_3^2} - \frac{s_1}{\nu + a_1^2}\right)y_3^2\right]\frac{1}{(\xi-\nu)} + (')$$

noting $\delta = \prod_{j=1}^{3}\left(\nu + a_j^2\right)$

We see that the identification/comparison is thwarted by the presence of the term in h^2 in 6.28. This term appears in $\partial_1 U_1$ (Eq. 6.16) as $= -B\delta\frac{s_1}{\nu + a_1^2}\frac{2}{R_\xi(\xi-\nu)}y_1 + (')$, , i.e. associated with $\frac{1}{R_\xi}y_1 = \frac{x_1}{R_\xi s_1}$. By identifying/comparing the terms in x_1,

$$D^3_{1,2} - D^2_{1,3} = 2B\delta T_\nu(\xi)\frac{1}{\nu + a_1^2} - 2B\delta\frac{1}{\left(\nu + a_1^2\right)R_\xi(\xi-\nu)} + (') \qquad (6.29)$$

This expression corrects the corresponding expression in 6.27.

As for the terms in y_2^2, y_3^2 , they lead to :

$$\alpha_{2,1}^3 = B\delta\left(\frac{s_2}{\nu+a_2^2}-\frac{s_1}{\nu+a_1^2}\right)\frac{1}{(\xi-\nu)}+(') \tag{6.30}$$

$$\alpha_{3,1}^2 = -B\delta\left(\frac{s_3}{\nu+a_3^2}-\frac{s_1}{\nu+a_1^2}\right)\frac{1}{(\xi-\nu)}+(') \tag{6.31}$$

note : taking the derivative of equation 6.29 with respect to ξ , one gets :

$$\begin{aligned} D_{1,2}^3{}' - D_{1,3}^2{}' &= 2B\delta\frac{1}{(\xi-\nu)\left(\nu+a_1^2\right)}\left(\frac{R_\xi{}'}{R_\xi^2}\right)+(') = 2B\delta\frac{1}{(\xi-\nu)\left(\nu+a_1^2\right)}\frac{1}{R_\xi}\left(\frac{R_\xi{}'}{R_\xi}\right)+(') \\ &= B\delta\frac{1}{(\xi-\nu)\left(\nu+a_1^2\right)}\frac{1}{R_\xi}\left(\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}\right)+(') \end{aligned} \tag{6.32}$$

On the other hand, using the definitions 6.20 and the expressions 6.30 and 6.31, we have :

$$\begin{aligned} D_{1,2}^3{}' - D_{1,3}^2{}' &= -\frac{\alpha_{1,2}^3}{s_1 s_2 R_\xi}+\frac{\alpha_{1,3}^2}{s_1 s_3 R_\xi} \\ &= \frac{B\delta}{(\xi-\nu)s_1 R_\xi}\left[-\frac{1}{\nu+a_2^2}+\frac{s_1}{s_2\left(\nu+a_1^2\right)}-\frac{1}{\nu+a_3^2}+\frac{s_1}{s_3\left(\nu+a_1^2\right)}\right]+(') \\ &= \frac{B\delta}{(\xi-\nu)s_1 R_\xi}\left[\frac{1}{\nu+a_1^2}+\frac{s_1}{s_2\left(\nu+a_1^2\right)}+\frac{s_1}{s_3\left(\nu+a_1^2\right)}\right]+(') \\ &= \frac{B\delta}{(\xi-\nu)\left(\nu+a_1^2\right)R_\xi}\left[\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}\right]+(') \end{aligned} \tag{6.33}$$

From this we can deduce that 6.29 on the one hand and 6.30 and 6.31 on the other are equivalent (apart from one constant, which turns out to be zero). We will keep only 6.30 and 6.31. So we have nine equations:

$$\alpha_0^3\left(s_3-s_2\right)-\alpha_{2,2}^3+\alpha_{3,3}^3-\alpha_{2,3}^2=0$$
$$\alpha_0^2\left(s_3-s_2\right)-\alpha_{2,2}^2+\alpha_{3,3}^2+\alpha_{2,3}^3=0$$
$$B_3I_{1,2}=2D_{2,2}^3-D_{2,3}^2-\frac{2}{s_2R_\xi}\frac{\left(\alpha_{2,2}^3s_3-\left(\alpha_{3,3}^3-\alpha_{2,3}^2\right)s_2\right)}{\left(s_3-s_2\right)}$$
$$B_2I_{1,3}=D_{2,3}^3-2D_{3,3}^2+\frac{2}{s_3R_\xi}\frac{\left(\left(\alpha_{2,2}^2-\alpha_{2,3}^3\right)s_3-\alpha_{3,3}^2s_2\right)}{\left(s_3-s_2\right)}$$
$$B_1=\alpha_{3,1}^3-\alpha_{2,1}^2$$
$$B_2=-\alpha_{1,1}^2-\left(\left(\alpha_{2,2}^2-\alpha_{2,3}^3\right)\left(s_1-s_3\right)+\alpha_{3,3}^2\left(s_2-s_1\right)\right)/\left(s_3-s_2\right)$$
$$B_3=\alpha_{1,1}^3+\left(\alpha_{2,2}^3\left(s_1-s_3\right)+\left(\alpha_{3,3}^3-\alpha_{2,3}^2\right)\left(s_2-s_1\right)\right)/\left(s_3-s_2\right)$$
$$\alpha_{2,1}^3=B\delta\left(\frac{s_2}{\nu+a_2^2}-\frac{s_1}{\nu+a_1^2}\right)\frac{1}{\left(\xi-\nu\right)}+(')$$
$$\alpha_{3,1}^2=-B\delta\left(\frac{s_3}{\nu+a_3^2}-\frac{s_1}{\nu+a_1^2}\right)\frac{1}{\left(\xi-\nu\right)}+(') \qquad (6.34)$$

The same calculations have been carried out for the components 2 and 3.

<u>Component 2</u> :

$$\alpha_0^3\left(s_3-s_1\right)-\alpha_{1,1}^3+\alpha_{3,3}^3-\alpha_{1,3}^1=0$$
$$\alpha_0^1\left(s_3-s_1\right)-\alpha_{1,1}^1+\alpha_{3,3}^1+\alpha_{1,3}^3=0$$
$$B_3I_{1,2}=-2D_{1,1}^3+D_{1,3}^1+\frac{2}{s_1R_\xi}\frac{\left(\alpha_{1,1}^3s_3-\left(\alpha_{3,3}^3-\alpha_{1,3}^1\right)s_1\right)}{\left(s_3-s_1\right)}$$
$$B_1I_{2,3}=-D_{1,3}^3+2D_{3,3}^1+\frac{2}{s_3R_\xi}\frac{\left(-\left(\alpha_{1,1}^1-\alpha_{1,3}^3\right)s_3+\alpha_{3,3}^1s_1\right)}{\left(s_3-s_1\right)}$$
$$B_2=\alpha_{1,2}^1-\alpha_{2,3}^3$$
$$B_3=-\alpha_{2,2}^3-\left(\left(\alpha_{3,3}^3-\alpha_{1,3}^1\right)\left(s_1-s_2\right)+\alpha_{1,1}^3\left(s_2-s_3\right)\right)/\left(s_3-s_1\right)$$
$$B_1=\alpha_{2,2}^1+\left(\alpha_{3,3}^1\left(s_1-s_2\right)+\left(\alpha_{1,1}^1-\alpha_{1,3}^3\right)\left(s_2-s_3\right)\right)/\left(s_3-s_1\right)$$
$$\alpha_{2,1}^3=B\delta\left(\frac{s_2}{\nu+a_2^2}-\frac{s_1}{\nu+a_1^2}\right)\frac{1}{\left(\xi-\nu\right)}+(')$$
$$\alpha_{2,3}^1=B\delta\left(\frac{s_3}{\nu+a_3^2}-\frac{s_2}{\nu+a_2^2}\right)\frac{1}{\left(\xi-\nu\right)}+(') \qquad (6.35)$$

<u>Component 3</u> :

$$\alpha_0^1\left(s_1-s_2\right)-\alpha_{2,2}^1+\alpha_{1,1}^1-\alpha_{1,2}^2=0$$
$$\alpha_0^2\left(s_1-s_2\right)-\alpha_{2,2}^2+\alpha_{1,1}^2+\alpha_{1,2}^1=0$$
$$B_1I_{2,3}=-2D_{2,2}^1+D_{1,2}^2-\frac{2}{s_2R_\xi}\frac{\left(\alpha_{2,2}^1s_1-\left(\alpha_{1,1}^1-\alpha_{1,2}^2\right)s_2\right)}{\left(s_2-s_1\right)}$$
$$B_2I_{1,3}=-D_{1,2}^1+2D_{1,1}^2+\frac{2}{s_1R_\xi}\frac{\left(\left(\alpha_{2,2}^2-\alpha_{1,2}^1\right)s_1-\alpha_{1,1}^2s_2\right)}{\left(s_2-s_1\right)}$$
$$B_3=\alpha_{2,3}^2-\alpha_{1,3}^1$$
$$B_2=\alpha_{3,3}^2+\left(\left(\alpha_{2,2}^2-\alpha_{1,2}^1\right)\left(s_1-s_3\right)+\alpha_{1,1}^2\left(s_3-s_2\right)\right)/\left(s_2-s_1\right)$$
$$B_1=-\alpha_{3,3}^1-\left(\alpha_{2,2}^1\left(s_1-s_3\right)+\left(\alpha_{1,1}^1-\alpha_{1,2}^2\right)\left(s_3-s_2\right)\right)/\left(s_2-s_1\right)$$
$$\alpha_{2,3}^1=B\delta\left(\frac{s_3}{\nu+a_3^2}-\frac{s_2}{\nu+a_2^2}\right)\frac{1}{\left(\xi-\nu\right)}+(')$$
$$\alpha_{3,1}^2=-B\delta\left(\frac{s_3}{\nu+a_3^2}-\frac{s_1}{\nu+a_1^2}\right)\frac{1}{\left(\xi-\nu\right)}+(') \qquad (6.36)$$

We therefore have 27 equations by treating the 3 components for 21 unknowns $\alpha_0^k,\alpha_{i,j}^k$. In fact, some equations are linear combinations of other equations. For example, consider the 3 equations containing only B_1 and see that only 2 are independent. If we consider the 7 equations containing $B_1,B_1I_{2,3}$ and α_0^1, we can reduce the system to 5 equations for 6 unknowns $\alpha_{1,1}^1,\alpha_{2,2}^1,\alpha_{3,3}^1,\alpha_0^1,\alpha_{1,3}^3,\alpha_{1,2}^2$. The core of this system is made up of the 3 equations with 4 unknowns (the two other are easily deduced):

$$-2D_{2,2}^1+D_{1,2}^2=-\frac{2s_3}{R_\xi\left(s_3-s_2\right)}\left(\frac{\alpha_{2,2}^1}{s_2}-\frac{\alpha_{3,3}^1}{s_3}-\frac{B_1}{s_3}\right)+B_1I_{2,3}$$
$$2D_{3,3}^1-D_{1,3}^3=-\frac{2s_2}{R_\xi\left(s_3-s_2\right)}\left(-\frac{\alpha_{2,2}^1}{s_2}+\frac{\alpha_{3,3}^1}{s_3}+\frac{B_1}{s_2}\right)+B_1I_{2,3} \qquad (6.37)$$
$$B_1=\alpha_{1,3}^3-\alpha_{1,2}^2$$

Let us take a look at this system.

An equation is needed to determine the 4 unknowns $\alpha_{2,2}^1,\alpha_{3,3}^1,\alpha_{1,2}^2,\alpha_{1,3}^3$.

Assuming,

$$\frac{\alpha_{2,2}^1}{s_2}-\frac{\alpha_{3,3}^1}{s_3}-\frac{B_1}{s_3}=0 \qquad (6.38)$$

, then :

$$\begin{aligned} &D^2_{1,2} = 0 \\ &D^1_{2,2} = -B_1 I_{2,3} / 2 \\ &\alpha^3_{1,3} = B_1 \quad \Rightarrow D^3_{1,3} = B_1 I_{1,3} \\ &\alpha^1_{3,3} = -3B_1 / 2 \quad \Rightarrow D^1_{3,3} = -3B_1 I_{3,3} / 2 \end{aligned} \tag{6.39}$$

It is the Stevenson's solution.

Symmetrically, we could have assumed that

$$-\frac{\alpha^1_{2,2}}{s_2} + \frac{\alpha^1_{3,3}}{s_3} + \frac{B_1}{s_2} = 0 \tag{6.40}$$

which leads us to another equally valid solution, not mentioned by Stevenson. This procedure, carried out on the system of equations containing B_1 , can be repeated on the systems of equations containing B_2 and B_3. The final result is presented in the form of two tables, one corresponding to Stevenson's solution, the other to the so-called symmetrical solution. Each row corresponds to the result of the calculation detailed above.

$D^1_{1,1}$	$D^1_{2,2}$	$D^1_{3,3}$	$D^2_{1,2}$	$D^3_{1,3}$	D^1_0	$D^1_{2,3}$
$-B_1 I_{1,3} / 2$	$-B_1 I_{2,3} / 2$	$-3B_1 I_{3,3} / 2$	0	$B_1 I_{1,3}$	$B_1 I_3 / 2$	$B\left(\nu + a_1^2\right)\left(a_2^2 - a_3^2\right) I_{2,3}$
$D^2_{1,1}$	$D^2_{2,2}$	$D^2_{3,3}$	$D^1_{1,2}$	$D^3_{2,3}$	D^2_0	$D^2_{1,3}$
$-3B_2 I_{1,1} / 2$	$-B_2 I_{1,2} / 2$	$-B_2 I_{1,3} / 2$	$B_2 I_{1,2}$	0	$B_2 I_1 / 2$	$B\left(\nu + a_2^2\right)\left(a_3^2 - a_1^2\right) I_{1,3}$
$D^3_{1,1}$	$D^3_{2,2}$	$D^3_{3,3}$	$D^1_{1,3}$	$D^2_{2,3}$	D^3_0	$D^3_{1,2}$
$-B_3 I_{1,2} / 2$	$-3B_3 I_{2,2} / 2$	$-B_3 I_{2,3} / 2$	0	$B_3 I_{2,3}$	$B_3 I_2 / 2$	$B\left(\nu + a_3^2\right)\left(a_1^2 - a_2^2\right) I_{1,2}$

Table 1 : Stevenson's solution : coefficients in 6.17 : $F_{2,U,k} = D^k_0\left(\xi\right) + \sum_{i \le j} D^k_{i,j}\left(\xi\right) x_i x_j$

$D_{1,1}^1$	$D_{2,2}^1$	$D_{3,3}^1$	$D_{1,2}^2$	$D_{1,3}^3$	D_0^1	$D_{2,3}^1$
$B_1 I_{1,2}/2$	$3B_1 I_{2,2}/2$	$B_1 I_{2,3}/2$	$-B_1 I_{1,2}$	0	$-B_1 I_2/2$	$B\left(\nu+a_1^2\right)\left(a_2^2-a_3^2\right)I_{2,3}$
$D_{1,1}^2$	$D_{2,2}^2$	$D_{3,3}^2$	$D_{1,2}^1$	$D_{2,3}^3$	D_0^2	$D_{1,3}^2$
$B_2 I_{1,3}/2$	$B_2 I_{2,3}/2$	$3B_2 I_{3,3}/2$	0	$-B_2 I_{2,3}$	$-B_2 I_3/2$	$B\left(\nu+a_2^2\right)\left(a_3^2-a_1^2\right)I_{1,3}$
$D_{1,1}^3$	$D_{2,2}^3$	$D_{3,3}^3$	$D_{1,3}^1$	$D_{2,3}^2$	D_0^3	$D_{1,2}^3$
$3B_3 I_{1,1}/2$	$B_3 I_{1,2}/2$	$B_3 I_{1,3}/2$	$-B_3 I_{1,3}$	0	$-B_3 I_1/2$	$B\left(\nu+a_3^2\right)\left(a_1^2-a_2^2\right)I_{1,2}$

Table 2 : Symmetrical solution : coefficients in 6.17 : $F_{2,U,k}=D_0^k(\xi)+\sum_{i\le j}D_{i,j}^k(\xi)x_i x_j$

Important note :

The expression of $F_{2,U,1}$, that corresponds to the second part of $\left(F_2'\right)_x$ (Stevenson (Eq.2.21)), is an asymmetrical expression : for example, there is a term missing in xz (x_1x_3) while xy (x_1x_2) is present ! The opposite is true for the symmetrical solution, where there is a term in xz, but no term in xy.

Conjecture: *Would not a linear combination of the two solutions be the right one?*

The proposed solution for $F_2=F_{2,G}+F_{2,U}$ verifies $\nabla\wedge F_2=H_1$; however, the condition $\nabla\bullet F_2=0$ was not included in the previous calculation.

Let us start by computing $\nabla\bullet F_{2,G}$:

An elementary calculation leads to (from 6.14) :

$$\nabla\bullet F_{2,G}=-\sum_j A_j I_j(\xi)x_j \tag{6.41}$$

while, from 6.17 and with the coefficients given in the tables 1 and 2,

$$\nabla\bullet F_{2,U}=0 \tag{6.42}$$

for the Stevenson's solution and the symmetrical solution.

As a result,

$$\nabla \bullet F_2 = -\sum_j A_j I_j(\xi) x_j \tag{6.43}$$

does not satisfy the zero divergence condition. To overcome this difficulty, we search a scalar function (called V_2" by Stevenson), such as $F_{2,new} = F_{2,old} + \nabla V_2$", with $F_{2,old} = F_{2,G} + F_{2,U}$. We have $\nabla \wedge F_{2,new} = 0$ (and always $\nabla \wedge F_{2,old} = 0$), thanks to $\nabla \wedge \nabla V_2" = 0$.

In order for $\nabla \bullet F_{2,new} = 0$, V_2" must obey :

$$\Delta V_2" = \sum_j A_j I_j(\xi) x_j \tag{6.44}$$

V_2" can be written as :

$$V_2" = \sum_j A_j \phi_j \tag{6.45}$$

with

$$\Delta \phi_j = I_j(\xi) x_j \tag{6.46}$$

We look for a solution ϕ_j^0 for 6.46 as :

$$\phi_j^0 = Z_j f_j(\xi) \tag{6.47}$$

with

$$Z_j = x_j \left(x_1^2 + x_2^2 + x_3^2\right) \tag{6.48}$$

where f_j is the unknown function.

Note that :

$$\Delta Z_j = 10 x_j \tag{6.49}$$

Using the equation 3.8,

$$\Delta\phi_j^0 = 10x_j f_j(\xi) + 4\frac{f_j'(\xi)}{h^2}\left(2x_j + \frac{Z_j}{\xi + a_j^2}\right) + \frac{2Z_j}{h^2}\left[\left(\frac{1}{\xi + a_1^2} + \frac{1}{\xi + a_2^2} + \frac{1}{\xi + a_3^2}\right) f_j'(\xi) + 2f_j''(\xi)\right] \tag{6.50}$$

if $f_j = I_j$

then

$$\Delta\left(Z_j I_j\right) = 10x_j I_j(\xi) - x_j \frac{8}{R_\xi\left(\xi + a_j^2\right)h^2} \tag{6.51}$$

We want to convert the last term of the right-hand side into a Laplacian :

$$\Delta\left(x_j G_j(\xi)\right) = x_j \frac{8}{R_\xi\left(\xi + a_j^2\right)h^2} \tag{6.52}$$

where G_j is a function to be determined :

$$\Delta\left(x_j G_j(\xi)\right) = 2\partial_{x_j} G_j + x_j \Delta G_j =$$
$$2G_j' \frac{2x_j}{\left(\xi + a_j^2\right)h^2} + \frac{2x_j}{h^2}\left[\left(\frac{1}{\xi + a_1^2} + \frac{1}{\xi + a_2^2} + \frac{1}{\xi + a_3^2}\right) G_j'(\xi) + 2G_j''(\xi)\right] = x_j \frac{8}{R_\xi\left(\xi + a_j^2\right)h^2} \tag{6.53}$$

hence

$$\left(\frac{1}{\xi + a_1^2} + \frac{1}{\xi + a_2^2} + \frac{1}{\xi + a_3^2} + \frac{2}{\left(\xi + a_j^2\right)}\right) G_j'(\xi) + 2G_j''(\xi) = \frac{4}{R_\xi\left(\xi + a_j^2\right)} \tag{6.54}$$

The general solution for 6.54 is :

$$G_j(\xi) = -2I + \left(2a_j^2 - J_j\right) I_j \tag{6.55}$$

where J_j is an integration constant. Bear in mind that F_2, which contains ∇V_2'' , is a particular solution, by definition not unique. We will not attempt to determine this constant at this stage.

$$\Delta\left(Z_j I_j + x_j G_j(\xi)\right) = 10x_j I_j(\xi)$$
$$\Rightarrow \phi_j^0 = \frac{1}{10}\left(Z_j I_j + x_j G_j(\xi)\right) = \frac{1}{10} x_j\left(x_1^2 + x_2^2 + x_3^2\right) I_j + \frac{1}{10} x_j\left(-2I + \left(2a_j^2 - J_j\right) I_j\right) \tag{6.56}$$

6.1.2 Calculation of $F_2^{(i)}$

Taking into account the linearity of $\nabla \wedge F_2^{(i)} = \mu H_1^{(i)}$, the calculation will be carried out for $\nabla \wedge F_2^{(i)} = H_1^{(i)}$. The final result will be obtained by multiplying by μ the result obtained for $\nabla \wedge F_2^{(i)} = H_1^{(i)}$.

The internal field $F_2^{(i)}$ obeys :

$$\begin{cases} \nabla \wedge F_2^{(i)} = G_1^{(i)} + \nabla U_1^{(i)} = -\dfrac{\varepsilon}{2} A^{(i)} \wedge r + \nabla \left(B^{(i)} X_\nu + B'^{(i)} X_{\nu'} + \sum_{i<j} B_l^{(i)} x_i x_j \right) \\ \nabla \bullet F_2^{(i)} = 0 \end{cases} \tag{6.57}$$

The coefficients appearing in the right-hand side do not depend on ξ; the search for a solution is therefore much easier. This takes the form of equation 6.17 :

$$F_{2,k}^{(i)} = D_0^k + \sum_{i \le j} D_{i,j}^k x_i x_j \tag{6.58}$$

In fact, $F_2^{(i)}$ is defined to within one constant.

The injection of 6.58 into 6.57 leads to the system :

$$D_{1,2}^3 - D_{1,3}^2 = B^{(i)} \frac{2\delta}{a_1^2 + \nu} + (')$$

$$2D_{2,2}^3 - D_{2,3}^2 = B_3^{(i)} + \frac{\varepsilon}{2} A_3^{(i)}$$

$$-2D_{3,3}^2 + D_{2,3}^3 = B_2^{(i)} - \frac{\varepsilon}{2} A_2^{(i)}$$

$$-2D_{1,1}^3 + D_{1,3}^1 = B_3^{(i)} - \frac{\varepsilon}{2} A_3^{(i)}$$

$$D_{2,3}^1 - D_{1,2}^3 = B^{(i)} \frac{2\delta}{a_2^2 + \nu} + (')$$

$$2D_{3,3}^1 - D_{1,3}^3 = B_1^{(i)} + \frac{\varepsilon}{2} A_1^{(i)}$$

$$2D_{1,1}^2 - D_{1,2}^1 = B_2^{(i)} + \frac{\varepsilon}{2} A_2^{(i)}$$

$$-2D_{2,2}^1 + D_{1,2}^2 = B_1^{(i)} - \frac{\varepsilon}{2} A_1^{(i)}$$

$$D_{1,3}^2 - D_{2,3}^1 = B^{(i)} \frac{2\delta}{a_3^2 + \nu} + (') \tag{6.59 1-9}$$

$$2D^1_{1,1} + D^2_{1,2} + D^3_{1,3} = 0$$
$$2D^2_{2,2} + D^1_{1,2} + D^3_{2,3} = 0 \qquad \text{(6.60 1-3)}$$
$$2D^3_{3,3} + D^1_{1,3} + D^2_{2,3} = 0$$

This system has 12 equations and 18 unknowns.

A particular solution is obtained with the additional trivial condition (which corresponds to 6 equations):

$$D^i_{i,j} = 0 \quad i \neq j \qquad (6.61)$$

For example, these conditions lead directly to :

$$D^1_{3,3} = \frac{B^{(i)}_1}{2} + \frac{\varepsilon}{4} A^{(i)}_1$$
$$D^1_{2,2} = -\frac{B^{(i)}_1}{2} + \frac{\varepsilon}{4} A^{(i)}_1 \qquad (6.62)$$
$$D^1_{1,1} = 0$$

The treatment of equations (6.59 1,5 and 9) is less straightforward: these three equations can be expressed as :

$$D^j - D^i = J_{i,j} = 2B^{(i)}\left(a_i^2 + \nu\right)\left(a_j^2 + \nu\right) + (') \qquad j > i \quad (if\ i = 3, then\ j = 1) \qquad (6.63)$$

We can propose a solution with β to be determined :

$$D^1 = \left(J_{1,2} - J_{3,1}\right)\beta$$
$$D^2 = \left(-J_{1,2} + J_{2,3}\right)\beta \qquad (6.64)$$
$$J_{1,2} + J_{3,1} + J_{2,3} = 0$$

hence

$$D^2 - D^1 = -3\beta J_{1,2} = J_{1,2}$$

then $\beta = -1/3$

finally

$$D^1 = D^1_{2,3} = \left(a_1^2 + \nu\right)\left(a_2^2 - a_3^2\right)\left(-2B^{(i)}/3\right) + (') \qquad (6.65)$$

We easily deduce D^2, D^3.

To summarize, the coefficients appearing in the expression of $F_2^{(i)}$ (to be multiplied by μ) are reported in table 3.

$D_{1,1}^1$	$D_{2,2}^1$	$D_{3,3}^1$	$D_{1,2}^2$	$D_{1,3}^3$	$D_{2,3}^1$
0	$-\frac{B_1^{(i)}}{2}+\frac{\varepsilon}{4}A_1^{(i)}$	$\frac{B_1^{(i)}}{2}+\frac{\varepsilon}{4}A_1^{(i)}$	0	0	$\frac{2}{3}B^{(i)}\left(\nu+a_1^2\right)\left(a_3^2-a_2^2\right)$
$D_{1,1}^2$	$D_{2,2}^2$	$D_{3,3}^2$	$D_{1,2}^1$	$D_{2,3}^3$	$D_{1,3}^2$
$\frac{B_2^{(i)}}{2}+\frac{\varepsilon}{4}A_2^{(i)}$	0	$-\frac{B_2^{(i)}}{2}+\frac{\varepsilon}{4}A_2^{(i)}$	0	0	$\frac{2}{3}B^{(i)}\left(\nu+a_2^2\right)\left(a_1^2-a_3^2\right)$
$D_{1,1}^3$	$D_{2,2}^3$	$D_{3,3}^3$	$D_{1,3}^1$	$D_{2,3}^2$	$D_{1,2}^3$
$-\frac{B_3^{(i)}}{2}+\frac{\varepsilon}{4}A_3^{(i)}$	$\frac{B_3^{(i)}}{2}+\frac{\varepsilon}{4}A_3^{(i)}$	0	0	0	$\frac{2}{3}B^{(i)}\left(\nu+a_3^2\right)\left(a_2^2-a_1^2\right)$

Table 3 : Stevenson's solution : coefficients into 6.58 : $F_{2,k}^{(i)}=\sum_{i\le j}D_{i,j}^k x_i x_j$

6.2 Calculation of $V_2, V_2^{(i)}$

6.2.1 Harmonic functions for ellipsoid

We look for the harmonic solutions for the following PDE's :

$$\Delta V_2 = 0 \quad \Delta V_2^{(i)} = 0 \tag{6.66}$$

with, on the ellipsoid surface (corresponding to $\xi = 0, \forall(\eta,\zeta)$) ,

$$\begin{aligned} n \wedge \nabla\left(V_2 - V_2^{(i)}\right) &= -n \wedge \left(F_2 - F_2^{(i)} + E_2^{(0)}\right) \\ n \bullet \nabla\left(V_2 - \varepsilon V_2^{(i)}\right) &= -n \bullet \left(F_2 - \varepsilon F_2^{(i)} + E_2^{(0)}\right) \end{aligned} \tag{6.67}$$

with

$$E_2^{(0)} = \frac{1}{2}\begin{pmatrix} l_1 & m_1 & n_1 \end{pmatrix}\left(lx+my+nz\right)^2$$

Here again, let us use the intuitive approach: the vector of the right-hand side of the boundary conditions (6.67) is quadratic, so we expect a solution of order 3 ($Cx_i x_j x_l$), more precisely a linear combination of terms like :

$$W_{i,j,l}(\xi,\eta,\zeta)=W_{\xi,i,j,l}(\xi)\left(\eta+a_i^2\right)^{\frac{1}{2}}\left(\zeta+a_i^2\right)^{\frac{1}{2}}\left(\eta+a_j^2\right)^{\frac{1}{2}}\left(\zeta+a_j^2\right)^{\frac{1}{2}}\left(\eta+a_l^2\right)^{\frac{1}{2}}\left(\zeta+a_l^2\right)^{\frac{1}{2}} \text{ with}$$

$$i,j,l=1,2,3 \tag{6.68}$$

The calculation of $\Delta W_{i,j,l}(\xi,\eta,\zeta)=0$ leads to

$$R_\xi\left(R_\xi W_{\xi,i,j,l}'\right)'+f(\xi)W_{\xi,i,j,l}=0 \tag{6.69}$$

with $\quad f(\xi)=-\left(3\xi+a_i^2+a_j^2+a_l^2\right)$ where $l\neq i,j$ and $i\neq j$ (6.70)

of which a particular solution is,

$$W_{\xi,i,j,l,1}=\left(\xi+a_i^2\right)^{\frac{1}{2}}\left(\xi+a_j^2\right)^{\frac{1}{2}}\left(\xi+a_l^2\right)^{\frac{1}{2}}. \tag{6.71}$$

Thus, the general solution is :

$$\begin{aligned}V_{2,i,j,l}&=\left[C_1\int_0^\xi\frac{du}{R_u\left(u+a_i^2\right)\left(u+a_j^2\right)\left(u+a_l^2\right)}+C_2\right]x_ix_jx_l\\&=\left[C_1\int_0^\xi\frac{du}{R_u\left(u+a_1^2\right)\left(u+a_2^2\right)\left(u+a_3^2\right)}+C_2\right]xyz\end{aligned} \tag{6.72}$$

This last expression is also valid for $V_2^{(i)}$ with the constants $C_1^{(i)},C_2^{(i)}$.

The case $i=j$ requires special treatment. The solution proposed for $i\neq j$, i.e. $W_{i,i,l}(\xi,\eta,\zeta)=W_{\xi,i,i,l}(\xi)\left(\eta+a_i^2\right)\left(\zeta+a_i^2\right)\left(\eta+a_l^2\right)^{\frac{1}{2}}\left(\zeta+a_l^2\right)^{\frac{1}{2}}$, leads to $\Delta W_{\xi,i,i,l}\neq 0$. We then propose to generalize this last expression by replacing a_i^2 with $-\nu$ (parameter to be determined and not imposed):

$$W_\nu(\xi,\eta,\zeta)=W_{\xi,\nu}(\xi)W_{T,\nu,l} \tag{6.73}$$

with

$$W_{T,\nu,l}=(\eta-\nu)(\zeta-\nu)\left(\eta+a_l^2\right)^{\frac{1}{2}}\left(\zeta+a_l^2\right)^{\frac{1}{2}} \tag{6.74}$$

On the other hand,

$$R_\eta \partial_\eta \left(R_\eta \partial_\eta W_{T,\nu,l} \right)$$
$$= \frac{1}{2}(\zeta - \nu)\left(\zeta + a_l^2\right)^{\frac{1}{2}} \left[6\eta^2 + \left(\frac{9}{2}a_m^2 + \frac{9}{2}a_n^2 + 2a_l^2 - \nu \right)\eta + 3a_m^2 a_n^2 + a_m^2 a_l^2 + a_n^2 a_l^2 - \frac{1}{2}\nu\left(a_m^2 + a_n^2\right) \right] \left(\eta + a_l^2\right)^{\frac{1}{2}}$$

(6.75)

with $m \neq n \quad m \neq l \quad n \neq l$

We therefore need to determine ν (if this solution makes sense). So

$$\Delta W_\nu = 0 \Rightarrow (\eta - \zeta)(\eta - \nu)(\zeta - \nu)\left(\eta + a_l^2\right)^{\frac{1}{2}} \left(\zeta + a_l^2\right)^{\frac{1}{2}} R_\xi \partial_\xi \left(R_\xi \partial_\xi W_{\xi,\nu} \right) + Z W_{\xi,\nu} = 0 \qquad (6.76)$$

with

$$Z = \frac{1}{2}(\zeta - \xi)(\zeta - \nu)\left(\zeta + a_l^2\right)^{\frac{1}{2}} \left(\eta + a_l^2\right)^{\frac{1}{2}} g(\eta) + \frac{1}{2}(\xi - \eta)(\eta - \nu)\left(\zeta + a_l^2\right)^{\frac{1}{2}} \left(\eta + a_l^2\right)^{\frac{1}{2}} g(\zeta)$$

(6.77)

with

$$g(x) = 6x^2 + \left(\frac{9}{2}a_m^2 + \frac{9}{2}a_n^2 + 2a_l^2 - \nu \right)x + 3a_m^2 a_n^2 + a_m^2 a_l^2 + a_n^2 a_l^2 - \frac{1}{2}\nu\left(a_m^2 + a_n^2\right) \qquad (6.78)$$

This polynomial of degree 2 can be written as :

$$g(x) = 6(x - \nu)(x - \nu') \qquad (6.79)$$

ν and ν’ are solutions of :

$$x^2 + \frac{2}{5}\left(2a_m^2 + 2a_n^2 + a_l^2\right)x + \frac{1}{5}\left(3a_m^2 a_n^2 + a_m^2 a_l^2 + a_n^2 a_l^2\right) = 0 \qquad (6.80)$$

which can be rewritten as

$$\sum_{k=1}^{3} \frac{1 + 2\delta_{l,k}}{x + a_k^2} = 0 \qquad (6.81)$$

The two roots of the polynomial only depend on *l* (for a given *l*-value, *m* and *n* are known, which play a symmetrical role in the previous expression). We will therefore call the two roots, according to Dassios $\nu_l = -\Lambda_l, \nu_l' = -\Lambda_l'$.

Then,

$$\Delta W_{\nu_l} = 0 \Rightarrow R_\xi \left(R_\xi W_{\xi,\nu_l}{}' \right)' - 3\left(\xi - \nu_l{}'\right) W_{\xi,\nu_l} = 0 \tag{6.82}$$

A particular solution of 6.82 is : $\left(\xi - \nu_l\right)\left(\xi + a_l^2\right)^{\frac{1}{2}}$

Finally,

$$V_{2,\nu_l} = \left[C_1 \int_0^\xi \frac{du}{R_u \left(u - \nu_l\right)^2 \left(u + a_l^2\right)} + C_2 \right] \left(\xi - \nu_l\right)\left(\eta - \nu_l\right)\left(\zeta - \nu_l\right)\left(\xi + a_l^2\right)^{\frac{1}{2}} \left(\eta + a_l^2\right)^{\frac{1}{2}} \left(\zeta + a_l^2\right)^{\frac{1}{2}} \tag{6.83}$$

We may express $\left(\xi - \nu_l\right)\left(\eta - \nu_l\right)\left(\zeta - \nu_l\right)$ as a function of x_1, x_2, x_3 : expression 5.33 is yet valid, with ν_l instead of ν . So we have :

$$V_{2,\nu_l} = \left[C_1 \int_0^\xi \frac{du}{R_u \left(u - \nu_l\right)^2 \left(u + a_l^2\right)} + C_2 \right] X_{\nu_l} x_l \tag{6.84}$$

Taking into account the boundary conditions, identical to those in equations 5.35-5.38, leads to a similar result:

Outside the ellipsoid, a linear combination of the functions $I_{1,2,3} xyz$

with $I_{1,2,3} = \int_\xi^\infty \frac{du}{R_u \left(u + a_1^2\right)\left(u + a_2^2\right)\left(u + a_3^2\right)}$ (6.85)

and $T_{\nu_l} X_{\nu_l} x_l$ (as well as $\nu_l{}'$)

with $T_{\nu_l} = \int_\xi^\infty \frac{du}{R_u \left(u - \nu_l\right)^2 \left(u + a_l^2\right)}$, (6.86)

of which there are 7.

Inside the ellipsoid, a linear combination of the functions xyz and $X_{\nu_l} x_l$ (as well as $\nu_l{}'$), also 7 in number.

We find the set of ellipsoidal harmonics (to within one constant) of degree 3 (according to Dassios), of which there are 14 in all:

Internal harmonics :

$$
\begin{aligned}
&\mathrm{E}_3^1 = \frac{h_1 h_2 h_3}{h_1} x_1 X_{\nu_1} \; ; \mathrm{E}_3^2 = \frac{h_1 h_2 h_3}{h_1} x_1 X_{\nu_1'} \\
&\mathrm{E}_3^3 = \frac{h_1 h_2 h_3}{h_2} x_2 X_{\nu_2} \; ; \mathrm{E}_3^4 = \frac{h_1 h_2 h_3}{h_2} x_2 X_{\nu_2'} \\
&\mathrm{E}_3^5 = \frac{h_1 h_2 h_3}{h_3} x_3 X_{\nu_3} \; ; \mathrm{E}_3^6 = \frac{h_1 h_2 h_3}{h_3} x_3 X_{\nu_3'} \\
&\mathrm{E}_3^7 = \left(h_1 h_2 h_3\right)^2 x_1 x_2 x_3
\end{aligned}
\tag{6.87}
$$

External harmonics :

$$
\begin{aligned}
&\Upsilon_3^1 = \frac{7}{2} \mathrm{E}_3^1 T_{\nu_1}(\xi) \; ; \Upsilon_3^2 = \frac{7}{2} \mathrm{E}_3^2 T_{\nu_1'}(\xi) \\
&\Upsilon_3^3 = \frac{7}{2} \mathrm{E}_3^3 T_{\nu_2}(\xi) \; ; \Upsilon_3^4 = \frac{7}{2} \mathrm{E}_3^4 T_{\nu_2'}(\xi) \\
&\Upsilon_3^5 = \frac{7}{2} \mathrm{E}_3^5 T_{\nu_3}(\xi) \; ; \Upsilon_3^6 = \frac{7}{2} \mathrm{E}_3^6 T_{\nu_3'}(\xi) \\
&\Upsilon_3^7 = \frac{7}{2} \mathrm{E}_3^7 I_{1,2,3}(\xi)
\end{aligned}
\tag{6.88}
$$

6.2.2 Traditional approach for ellipsoid

The solution of the PDE's will be written as :

$$
V_2^{(i)} = \sum_{m=1}^{3} K_{1,m} E_1^m + \sum_{m=1}^{7} K_{3,m} E_3^m \tag{6.89}
$$

$$
V_2 = \sum_{m=1}^{3} L_{1,m} \Upsilon_1^m + \sum_{m=1}^{7} L_{3,m} \Upsilon_3^m \tag{6.90}
$$

The integration constants $K_{n,m}, L_{n,m}$ will be determined using the boundary conditions 6.67. We cannot exclude terms of degree 1 in equations 6.89 and 6.90, given the form of the right-hand side of equations 6.67. We will now try to determine these constants, starting with the contribution of the degree 3 terms ($K_{1,m} = L_{1,m} = 0$), which is more difficult to deal with.

Equations 6.67 show the gradients of the ellipsoidal harmonics expressed in the Cartesian basis $\{e_i\}$:

$$
\nabla E_3^m = \alpha_m \, e_{f(m)} + \sum_{i=1}^{3} M_{i,m} \, e_i \quad 1 \le m \le 7 \tag{6.91}
$$

with

$$\alpha_m = -\frac{h_1 h_2 h_3}{h_{f(m)}} \prod_{j=1}^{3} \left(N_m + a_j^2 \right) \quad 1 \le m \le 6$$
$$\alpha_7 = 0 \tag{6.92}$$

and

$$M_{i,m} = -\alpha_m \left[\frac{2 x_{f(m)} x_i}{N_m + a_i^2} + \delta_{i,f(m)} \sum_{j=1}^{3} \frac{x_j^2}{N_m + a_j^2} \right] \quad 1 \le m \le 6$$
$$M_{i,7} = h_1^2 h_2^2 h_3^2 \, x_1 x_2 x_3 / x_i \tag{6.93}$$

$e_i \ \left(i = 1,2,3 \right)$ are the unit vectors in the Cartesian coordinate system. $f\left(m\right)$ and N_m are defined in the table 4:

m	1	2	3	4	5	6	7
$f\left(m\right)$	1	1	2	2	3	3	0
N_m	ν_1	ν_1 '	ν_2	ν_2 '	ν_3	ν_3 '	

Table 4 : $f\left(m\right)$ and N_m values

Similarly :

$$\nabla \Upsilon_3^m = \frac{7}{2} T_{N_m}\left(\xi\right) \nabla \mathrm{E}_3^m - \frac{7}{2} \mathrm{E}_3^m \frac{\nabla \xi}{R_\xi \left(\xi - N_m \right)^2 \left(\xi + a_{f(m)}^2 \right)} \quad 1 \le m \le 6$$
$$\nabla \Upsilon_3^7 = \frac{7}{2} I_{1,2,3}\left(\xi\right) \nabla \mathrm{E}_3^7 - \frac{7}{2} \mathrm{E}_3^7 \frac{\nabla \xi}{R_\xi \left(\xi + a_1^2 \right)\left(\xi + a_2^2 \right)\left(\xi + a_3^2 \right)} \tag{6.94}$$

We deduce that,

$$n \wedge \nabla \left(V_2 - V_2^{(i)} \right)\Big|_{\xi=0} = \sum_{m=1}^{7} P_m \left(\alpha_m \, n \wedge e_{f(m)} + \sum_{i=1}^{3} M_{i,m} \, n \wedge e_i \right) \tag{6.95}$$

with

$$P_m = \frac{7}{2} L_{3,m} T_{N_m} - K_{3,m}$$
$$P_7 = \frac{7}{2} L_{3,7} I_{1,2,3} - K_{3,7}$$

and reminding that (3.6 and 3.9) $\nabla\xi\big|_{\xi=0} = \frac{2}{h_0^2}\left(x_1 / a_1^2, x_2 / a_2^2, x_3 / a_3^2\right) = \frac{2}{h_0} n$ and thus

$$n \wedge \nabla \Upsilon_3^m\big|_{\xi=0} = \frac{7}{2} T_{N_m}(\xi)\left(n \wedge \nabla \mathrm{E}_3^m\big|_{\xi=0}\right) \quad 1 \le m \le 6$$
$$n \wedge \nabla \Upsilon_3^7\big|_{\xi=0} = \frac{7}{2} I_{1,2,3}(\xi)\left(n \wedge \nabla \mathrm{E}_3^7\big|_{\xi=0}\right)$$

As $n = \frac{1}{h_0}\sum_{j=1}^{3} \frac{x_j}{a_j^2} e_j$, the *l*-component of the cross product will be :

$$n \wedge \nabla\left(V_2 - V_2^{(i)}\right)\big|_{\xi=0}\big|_l = \frac{1}{h_0}\sum_{m=1}^{7} P_m \left(\alpha_m \sum_{j=1}^{3} \frac{x_j}{a_j^2} e_l \bullet \left(e_j \wedge e_{f(m)}\right) + \sum_{i,j=1}^{3} \frac{M_{i,m} x_j}{a_j^2} e_l \bullet \left(e_j \wedge e_i\right) \right) \tag{6.96}$$

The mixed products appearing in the equation 6.96 have values -1, 0 or 1. The three components of the cross product are :

$$\begin{aligned}
&\xrightarrow{l=1} \frac{1}{h_0}\left[-\left(\alpha_3 P_3 + \alpha_4 P_4\right)\frac{x_3}{a_3^2} + \left(\alpha_5 P_5 + \alpha_6 P_6\right)\frac{x_2}{a_2^2} + \left(\sum_{m=1}^{7} P_m M_{3,m}\right)\frac{x_2}{a_2^2} - \left(\sum_{m=1}^{7} P_m M_{2,m}\right)\frac{x_3}{a_3^2} \right] \\
&\xrightarrow{l=2} \frac{1}{h_0}\left[\left(\alpha_1 P_1 + \alpha_2 P_2\right)\frac{x_3}{a_3^2} - \left(\alpha_5 P_5 + \alpha_6 P_6\right)\frac{x_1}{a_1^2} + \left(\sum_{m=1}^{7} P_m M_{1,m}\right)\frac{x_3}{a_3^2} - \left(\sum_{m=1}^{7} P_m M_{3,m}\right)\frac{x_1}{a_1^2} \right] \\
&\xrightarrow{l=3} \frac{1}{h_0}\left[-\left(\alpha_1 P_1 + \alpha_2 P_2\right)\frac{x_2}{a_2^2} + \left(\alpha_3 P_3 + \alpha_4 P_4\right)\frac{x_1}{a_1^2} + \left(\sum_{m=1}^{7} P_m M_{2,m}\right)\frac{x_1}{a_1^2} - \left(\sum_{m=1}^{7} P_m M_{1,m}\right)\frac{x_2}{a_2^2} \right]
\end{aligned} \tag{6.97}$$

As for the scalar product $n \bullet \nabla\left(V_2 - \varepsilon V_2^{(i)}\right)$, it shows (starting from 6.94) :

$$\begin{aligned}
n \bullet \nabla \Upsilon_3^m\big|_{\xi=0} &= \frac{7}{2} T_{N_m}(\xi)\, n \bullet \nabla \mathrm{E}_3^m\big|_{\xi=0} - 7\,\mathrm{E}_3^m\big|_{\xi=0} \frac{1}{h_0 a_1 a_2 a_3 N_m^2 a_{f(m)}^2} \\
n \bullet \nabla \Upsilon_3^7\big|_{\xi=0} &= \frac{7}{2} I_{1,2,3}(\xi)\, n \bullet \nabla \mathrm{E}_3^7\big|_{\xi=0} - 7\,\mathrm{E}_3^7\big|_{\xi=0} \frac{1}{h_0\left(a_1 a_2 a_3\right)^3}
\end{aligned} \tag{6.98}$$

$$n \bullet \nabla \left(V_2 - \varepsilon V_2^{(i)} \right) =$$

$$\frac{1}{h_0} \left[\begin{array}{l} \left(\alpha_1 P_1^{\varepsilon} + \alpha_2 P_2^{\varepsilon} \right) \frac{x_1}{a_1^2} + \left(\alpha_3 P_3^{\varepsilon} + \alpha_4 P_4^{\varepsilon} \right) \frac{x_2}{a_2^2} + \left(\alpha_5 P_5^{\varepsilon} + \alpha_6 P_6^{\varepsilon} \right) \frac{x_3}{a_3^2} + \ldots \\ + \sum_{i=1}^{3} \left(\sum_{m=1}^{7} P_m^{\varepsilon} M_{i,m} \right) \frac{x_i}{a_i^2} - \sum_{m=1}^{6} L_{3,m} \frac{7 \mathrm{E}_3^m \big|_{\xi=0}}{a_1 a_2 a_3 N_m^2 a_{f(m)}^2} - L_{3,7} \frac{7 \mathrm{E}_3^7 \big|_{\xi=0}}{\left(a_1 a_2 a_3 \right)^3} \end{array} \right] \tag{6.99}$$

with

$$\begin{aligned} P_m^{\varepsilon} &= \frac{7}{2} L_{3,m} T_{N_m} - \varepsilon K_{3,m} \\ P_7^{\varepsilon} &= \frac{7}{2} L_{3,7} I_{1,2,3} - \varepsilon K_{3,7} \end{aligned} \tag{6.100}$$

We can see that scalar and cross products have terms of order 1 in x_j and terms of order 3, the latter having the generic form: $Z_{i,j} = \left(\sum_{m=1}^{7} P_m M_{i,m} \right) x_j$. It is easy to show :

$i \neq j$

$$Z_{i,j} = C_{l,i} 2 x_1 x_2 x_3 + C_{j,i} 2 x_i x_j^2 + C_{i,i} 2 x_i^2 x_j + P_7 \left(h_1 h_2 h_3 \right)^2 \underbrace{x_1 x_2 x_3 x_j / x_i}_{x_j^2 x_l} + \sum_{k=1}^{3} C_{i,k} x_k^2 x_j \quad l \neq i \; l \neq j \tag{6.101}$$

$i = j$

$$\begin{aligned} Z_{i,i} &= C_{1,i} 2 x_1 x_i^2 + C_{2,i} 2 x_2 x_i^2 + C_{3,i} 2 x_3 x_i^2 + P_7 \left(h_1 h_2 h_3 \right)^2 x_1 x_2 x_3 + \sum_{k=1}^{3} C_{i,k} x_k^2 x_i \\ &= \sum_{k=1}^{3} \left(C_{k,i} 2 x_k x_i^2 + C_{i,k} x_k^2 x_i \right) + P_7 \left(h_1 h_2 h_3 \right)^2 x_1 x_2 x_3 \end{aligned} \tag{6.102}$$

with $C_{i,k} = -\frac{\alpha_{2i-1} P_{2i-1}}{N_{2i-1} + a_k^2} - \frac{\alpha_{2i} P_{2i}}{N_{2i} + a_k^2}$ (6.103)

Into the expression of the scalar product $n \bullet \nabla \left(V_2 - \varepsilon V_2^{(i)} \right)$, it appears :

$$\sum_{i=1}^{3} \left(\sum_{m=1}^{7} P_m^{\varepsilon} M_{i,m} \right) \frac{x_i}{a_i^2} = \sum_{i=1}^{3} \frac{Z_{i,i}^{\varepsilon}}{a_i^2} = \sum_{i,j} \left(\frac{2}{a_i^2} + \frac{1}{a_j^2} \right) C_{j,i}^{\varepsilon} x_j x_i^2 + P_7^{\varepsilon} \left(h_1 h_2 h_3 \right)^2 \left(\sum_i \frac{1}{a_i^2} \right) x_1 x_2 x_3 \tag{6.104}$$

Dassios showed that the third-order terms could be expressed using ellipsoidal surface harmonics:

$$x_k x_n^2 = \frac{\left(1+2\delta_{k,n}\right)h_k}{5h_1h_2h_3}\sqrt{\rho^2-a_1^2+a_k^2}\left(\rho^2-a_1^2+a_n^2\right)\left[S_1^k - \frac{S_3^{2k-1}}{\left(N_{2k-1}+a_n^2\right)\left(N_{2k-1}-N_{2k}\right)} + \frac{S_3^{2k}}{\left(N_{2k}+a_n^2\right)\left(N_{2k-1}-N_{2k}\right)}\right]$$

$$x_1 x_2 x_3 = \frac{\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}{\left(h_1h_2h_3\right)^2} S_3^7$$

(6.105)

We will also need the relationship between first-order terms and ellipsoidal surface harmonics :

$$x_n = \frac{h_n}{h_1h_2h_3}\sqrt{\rho^2-a_1^2+a_n^2}\; S_1^n \tag{6.106}$$

On the surface of the ellipsoid $\rho = a_1$,

$$x_k x_n^2 = \frac{\left(1+2\delta_{k,n}\right)h_k}{5h_1h_2h_3} a_k a_n^2 \left[S_1^k - \frac{S_3^{2k-1}}{\left(N_{2k-1}+a_n^2\right)\left(N_{2k-1}-N_{2k}\right)} + \frac{S_3^{2k}}{\left(N_{2k}+a_n^2\right)\left(N_{2k-1}-N_{2k}\right)}\right]$$

$$x_1 x_2 x_3 = \frac{a_1a_2a_3}{\left(h_1h_2h_3\right)^2} S_3^7 \tag{6.107}$$

$$x_n = \frac{h_n a_n}{h_1h_2h_3}\, S_1^n$$

Using 6.107 into 6.104, we may write :

$$\sum_{i=1}^{3}\left(\sum_{m=1}^{7} P_m^{\varepsilon} M_{i,m}\right)\frac{x_i}{a_i^2} = \sum_{m=1}^{3} A_m^S\, S_1^m + \sum_{m=1}^{7} B_m^S\, S_3^m \tag{6.108}$$

with

$$A_m^S = \frac{h_m a_m}{5h_1h_2h_3}\sum_i\left(\frac{2}{a_i^2}+\frac{1}{a_m^2}\right)C_{m,i}^{\varepsilon}\left(1+2\delta_{m,i}\right)a_i^2$$

$$B_m^S = \frac{h_{f(m)}a_{f(m)}}{5h_1h_2h_3}\sum_i\left(\frac{2}{a_i^2}+\frac{1}{a_{f(m)}^2}\right)C_{f(m),i}^{\varepsilon}\left(1+2\delta_{f(m),i}\right)a_i^2\left(\frac{1}{N_m+a_i^2}\right)\left(\frac{1}{\nu_{f(m)}-\nu'_{f(m)}}\right)(-1)^m \quad 1\le m\le 6$$

$$B_7^S = P_7^{\varepsilon} a_1a_2a_3\sum_i\frac{1}{a_i^2}$$

(6.109)

We easily show that (thanks to 6.116) :

$$A_1^S = -\frac{h_1a_1}{h_1h_2h_3}\frac{1}{a_1^2}\left(\alpha_1 P_1^{\varepsilon}+\alpha_2 P_2^{\varepsilon}\right) \tag{6.110}$$

Injecting A_m^S into 6.99,

$$n \bullet \nabla \left(V_2 - \varepsilon V_2^{(i)} \right) = \frac{1}{h_0} \left[\sum_{m=1}^{7} B_m^S S_3^m - \sum_{m=1}^{6} L_{3,m} \frac{7 \mathrm{E}_3^m \Big|_{\xi=0}}{a_1 a_2 a_3 N_m^2 a_{f(m)}^2} - L_{3,7} \frac{7 \mathrm{E}_3^7 \Big|_{\xi=0}}{\left(a_1 a_2 a_3 \right)^3} \right] \tag{6.111}$$

Knowing that $E_3^m \Big|_{\xi=0} = La_3^m \left(a_1 \right) S_3^m$ and $La_3^m \left(a_1 \right) = -N_m a_{f(m)} \quad 1 \le m \le 6 \quad La_3^7 \left(a_1 \right) = a_1 a_2 a_3$

$$n \bullet \nabla \left(V_2 - \varepsilon V_2^{(i)} \right) = \frac{1}{h_0} \left[\sum_{m=1}^{7} B_m^S S_3^m + \sum_{m=1}^{6} L_{3,m} \frac{7}{a_1 a_2 a_3 N_m a_{f(m)}} S_3^m - L_{3,7} \frac{7}{\left(a_1 a_2 a_3 \right)^2} S_3^7 \right] \tag{6.112}$$

We observe that the terms in S_1^m vanish.

Let us now consider the cross product (6.97), by starting with the equivalent expression of 6.104 for the first component (*l*=1) of the cross product.

$$\left(\sum_{m=1}^{7} P_m M_{3,m} \right) \frac{x_2}{a_2^2} - \left(\sum_{m=1}^{7} P_m M_{2,m} \right) \frac{x_3}{a_3^2} = \left(\frac{C_{1,3}}{a_2^2} - \frac{C_{1,2}}{a_3^2} \right) 2 x_1 x_2 x_3 + \sum_k \left(\frac{C_{3,k}}{a_2^2} + 2\delta_{k,3} \left(\frac{C_{3,3}}{a_2^2} - \frac{C_{3,2}}{a_3^2} \right) \right) x_k^2 x_2$$
$$- \sum_k \left(\frac{C_{2,k}}{a_3^2} + 2\delta_{k,2} \left(\frac{C_{2,2}}{a_3^2} - \frac{C_{2,3}}{a_2^2} \right) \right) x_k^2 x_3 + \left(h_1 h_2 h_3 \right)^2 P_7 \left(\frac{x_2^2 x_1}{a_2^2} - \frac{x_3^2 x_1}{a_3^2} \right)$$

(6.113)

for *l*=2, we simply carry out the index permutation $\left(1,2,3\right) \to \left(2,3,1\right)$

, and for *l*=3 , $\left(1,2,3\right) \to \left(3,1,2\right)$

For example, if we consider *l*=3 and use the expressions 6.107, we obtain the equivalent of 6.108 :

$$\left(\sum_{m=1}^{7} P_m M_{2,m} \right) \frac{x_1}{a_1^2} - \left(\sum_{m=1}^{7} P_m M_{1,m} \right) \frac{x_2}{a_2^2} = \sum_{m=1}^{3} A_m^{S,3} S_1^m + \sum_{m=1}^{7} B_m^{S,3} S_3^m \tag{6.114}$$

with

$$A_1^{S,3} = \frac{1}{5 h_1 h_2 h_3} \frac{h_1 a_1}{a_1^2} \left[-5\alpha_3 P_3 - 5\alpha_4 P_4 \right]$$
$$A_2^{S,3} = \frac{1}{5 h_1 h_2 h_3} \frac{h_2 a_2}{a_2^2} \left[5\alpha_1 P_1 + 5\alpha_2 P_2 \right] \tag{6.115}$$
$$A_3^{S,3} = 0$$

where we used the relation G.40 by Dassios :

$$\sum_{k=1}^{3}\frac{1+2\delta_{i,k}}{\nu_i+a_k^2}a_k^2=\sum_{k=1}^{3}\frac{1+2\delta_{i,k}}{\nu_i'+a_k^2}a_k^2=5 \tag{6.116}$$

To better understand the expressions for $B_m^{S,3}$, let us take the case *m*=1, i.e. the term corresponding to S_3^1:

$$B_1^{S,3}=-\frac{h_1a_1}{5h_1h_2h_3\left(N_1-N_2\right)}\sum_k\left(\frac{C_{2,k}}{a_1^2}+2\delta_{k,2}\left(\frac{C_{2,2}}{a_1^2}-\frac{C_{2,1}}{a_2^2}\right)\right)\frac{\left(1+2\delta_{k,1}\right)a_k^2}{N_1+a_k^2}$$

$$=\frac{h_1a_1}{5h_1h_2h_3\left(N_1-N_2\right)}\left[\begin{array}{l}\alpha_3P_3\left(\begin{array}{l}\dfrac{3a_1^2}{\left(N_1+a_1^2\right)}\dfrac{1}{a_1^2\left(N_3+a_1^2\right)}+\dfrac{a_2^2}{\left(N_1+a_2^2\right)}\left(\dfrac{3}{a_1^2\left(N_3+a_2^2\right)}-\dfrac{2}{a_2^2\left(N_3+a_1^2\right)}\right)\\+\dfrac{a_3^2}{\left(N_1+a_3^2\right)}\dfrac{1}{a_1^2\left(N_3+a_3^2\right)}\end{array}\right)\\+\alpha_4P_4\left(.....N_3\to N_4.....\right)\end{array}\right] \tag{6.117}$$

which may be written as,

$$B_1^{S,3}=\frac{h_1a_1}{5h_1h_2h_3\left(N_1-N_2\right)}\left[\alpha_3P_3\,Y_3^1+\alpha_4P_4\,Y_4^1\right] \tag{6.118}$$

with

$$Y_p^q=\frac{1}{a_{f(q)}^2}\sum_{k=1}^{3}\frac{a_k^2\left(1+2\delta_{k,f(p)}+2\delta_{k,f(q)}\right)}{\left(N_p+a_k^2\right)\left(N_q+a_k^2\right)}-\frac{2}{\left(N_p+a_{f(q)}^2\right)\left(N_q+a_{f(p)}^2\right)} \tag{6.119}$$

We easily show that :

$$\sum_{m=1}^{7}B_m^{S,3}\,S_3^m=\frac{1}{5h_1h_2h_3}\left\{\begin{array}{l}\dfrac{h_1a_1}{\left(N_1-N_2\right)}\left[\left(\alpha_3P_3\,Y_3^1+\alpha_4P_4\,Y_4^1\right)S_3^1-\left(\alpha_3P_3\,Y_3^2+\alpha_4P_4\,Y_4^2\right)S_3^2\right]\\+\dfrac{h_2a_2}{\left(N_3-N_4\right)}\left[-\left(\alpha_1P_1\,Y_1^3+\alpha_2P_2\,Y_2^3\right)S_3^3+\left(\alpha_1P_1\,Y_1^4+\alpha_2P_2\,Y_2^4\right)S_3^4\right]\\-\dfrac{h_3a_3}{\left(N_5-N_6\right)}\left(h_1h_2h_3\right)^2\left[P_7z_5S_3^5-P_7z_6S_3^6\right]\\+10\dfrac{a_1a_2a_3}{h_1h_2h_3}\left[\alpha_5P_5\,Z_5+\alpha_6P_6\,Z_6\right]S_3^7\end{array}\right\} \tag{6.120}$$

with

$$Z_l = \frac{1}{a^2_{f(l)-1}\left(N_l + a^2_{f(l)+1}\right)} - \frac{1}{a^2_{f(l)+1}\left(N_l + a^2_{f(l)-1}\right)} \tag{6.121}$$

$$z_l = \frac{1}{\left(N_l + a^2_{f(l)+1}\right)} - \frac{1}{\left(N_l + a^2_{f(l)-1}\right)} \tag{6.122}$$

by convention (modulo 3) $a_0 = a_3$; $a_4 = a_1$

using 6.107 and 6.115,

$$\begin{aligned} n \wedge \nabla\left(V_2 - V_2^{(i)}\right)\Big|_{\xi=0}\Big|_{l=3} &= \frac{1}{h_0}\left[-\left(\alpha_1 P_1 + \alpha_2 P_2\right)\frac{x_2}{a_2^2} + \left(\alpha_3 P_3 + \alpha_4 P_4\right)\frac{x_1}{a_1^2} + \left(\sum_{m=1}^{7} P_m M_{2,m}\right)\frac{x_1}{a_1^2} - \left(\sum_{m=1}^{7} P_m M_{1,m}\right)\frac{x_2}{a_2^2}\right] \\ &= \frac{1}{h_0}\left[-\left(\alpha_1 P_1 + \alpha_2 P_2\right)\frac{x_2}{a_2^2} + \left(\alpha_3 P_3 + \alpha_4 P_4\right)\frac{x_1}{a_1^2} + \sum_{m=1}^{3} A_m^{S,3}\, S_1^m + \sum_{m=1}^{7} B_m^{S,3}\, S_3^m\right] \\ &= \frac{1}{h_0}\left[\sum_{m=1}^{7} B_m^{S,3}\, S_3^m\right] \end{aligned} \tag{6.123}$$

For the two other components :

$$n \wedge \nabla\left(V_2 - V_2^{(i)}\right)\Big|_{\xi=0}\Big|_{l=1} = \frac{1}{h_0}\left[\sum_{m=1}^{7} B_m^{S,1}\, S_3^m\right]$$

$$= \frac{1}{h_0}\frac{1}{5h_1h_2h_3}\left\{\begin{array}{l} \dfrac{h_2a_2}{\left(N_3 - N_4\right)}\left[\left(\alpha_5 P_5\, Y_5^3 + \alpha_6 P_6\, Y_6^3\right)S_3^3 - \left(\alpha_5 P_5\, Y_5^4 + \alpha_6 P_6\, Y_6^4\right)S_3^4\right] \\ + \dfrac{h_3a_3}{\left(N_5 - N_6\right)}\left[-\left(\alpha_3 P_3\, Y_3^5 + \alpha_4 P_4\, Y_4^5\right)S_3^5 + \left(\alpha_3 P_3\, Y_3^6 + \alpha_4 P_4\, Y_4^6\right)S_3^6\right] \\ - \dfrac{h_1a_1}{\left(N_1 - N_2\right)}\left(h_1h_2h_3\right)^2\left[P_7 z_1 S_3^1 - P_7 z_2 S_3^2\right] \\ +10\dfrac{a_1a_2a_3}{h_1h_2h_3}\left[\alpha_1 P_1\, Z_1 + \alpha_2 P_2\, Z_2\right]S_3^7 \end{array}\right\} \tag{6.124}$$

$$n \wedge \nabla \left(V_2 - V_2^{(i)} \right) \Big|_{\xi=0} \Big|_{l=2} = \frac{1}{h_0} \left[\sum_{m=1}^{7} B_m^{S,2} \, S_3^m \right]$$

$$= \frac{1}{h_0} \frac{1}{5 h_1 h_2 h_3} \left\{ \begin{array}{l} \frac{h_3 a_3}{\left(N_5 - N_6 \right)} \left[\left(\alpha_1 P_1 \, Y_1^5 + \alpha_2 P_2 \, Y_2^5 \right) S_3^5 - \left(\alpha_1 P_1 \, Y_1^6 + \alpha_2 P_2 \, Y_2^6 \right) S_3^6 \right] \\ + \frac{h_1 a_1}{\left(N_1 - N_2 \right)} \left[- \left(\alpha_5 P_5 \, Y_5^1 + \alpha_6 P_6 \, Y_6^1 \right) S_3^1 + \left(\alpha_5 P_5 \, Y_5^2 + \alpha_6 P_6 \, Y_6^2 \right) S_3^2 \right] \\ - \frac{h_2 a_2}{\left(N_3 - N_4 \right)} \left(h_1 h_2 h_3 \right)^2 \left[P_7 z_3 S_3^3 - P_7 z_4 S_3^4 \right] \\ + 10 \frac{a_1 a_2 a_3}{h_1 h_2 h_3} \left[\alpha_3 P_3 \, Z_3 + \alpha_4 P_4 \, Z_4 \right] S_3^7 \end{array} \right\} \qquad (6.125)$$

More generally,

$$l < m < n \quad (l,m,n) \to (1,2,3) \to (2,3,1) \to (3,1,2)$$

$$n \wedge \nabla \left(V_2 - V_2^{(i)} \right) \Big|_{\xi=0} \Big|_{l} = \frac{1}{h_0} \left[\sum_{i=1}^{7} B_i^{S,l} \, S_3^i \right]$$

$$= \frac{1}{h_0} \frac{1}{5 h_1 h_2 h_3} \left\{ \begin{array}{l} \frac{h_m a_m}{\left(N_{2m-1} - N_{2m} \right)} \left[\left(\alpha_{2n-1} P_{2n-1} \, Y_{2n-1}^{2m-1} + \alpha_{2n} P_{2n} \, Y_{2n}^{2m-1} \right) S_3^{2m-1} - \left(\alpha_{2n-1} P_{2n-1} \, Y_{2n-1}^{2m} + \alpha_{2n} P_{2n} \, Y_{2n}^{2m} \right) S_3^{2m} \right] \\ + \frac{h_n a_n}{\left(N_{2n-1} - N_{2n} \right)} \left[- \left(\alpha_{2m-1} P_{2m-1} \, Y_{2m-1}^{2n-1} + \alpha_{2m} P_{2m} \, Y_{2m}^{2n-1} \right) S_3^{2n-1} + \left(\alpha_{2m-1} P_{2m-1} \, Y_{2m-1}^{2n} + \alpha_{2m} P_{2m} \, Y_{2m}^{2n} \right) S_3^{2n} \right] \\ - \frac{h_l a_l}{\left(N_{2l-1} - N_{2l} \right)} \left(h_1 h_2 h_3 \right)^2 P_7 \left[z_{2l-1} S_3^{2l-1} - z_{2l} S_3^{2l} \right] \\ + 10 \frac{a_1 a_2 a_3}{h_1 h_2 h_3} \left[\alpha_{2l-1} P_{2l-1} \, Z_{2l-1} + \alpha_{2l} P_{2l} \, Z_{2l} \right] S_3^7 \end{array} \right\}$$

(6.126)

Let us now consider the contribution of terms of degree 1 (Eqs. 6.89-6.90, $K_{3,m} = L_{3,m} = 0$) :

The ellipsoidal harmonics of degree 1 are :

$$E_1^i = \frac{h_1 h_2 h_3}{h_i} x_i \qquad (6.127)$$

which are related to the ellipsoidal surface harmonics of degree 1 by

$$E_1^m \Big|_{\xi=0} = L a_1^m \left(a_1 \right) S_1^m = a_m S_1^m \qquad (6.128)$$

We easily deduce :

$$n \wedge \nabla \left(V_2 - V_2^{(i)} \right) \Big|_{\xi=0} = \frac{1}{h_0} \begin{vmatrix} -\frac{\gamma_2}{h_2}\frac{h_3}{a_3} S_1^3 + \frac{\gamma_3}{h_3}\frac{h_2}{a_2} S_1^2 \\ \frac{\gamma_1}{h_1}\frac{h_3}{a_3} S_1^3 - \frac{\gamma_3}{h_3}\frac{h_1}{a_1} S_1^1 \\ -\frac{\gamma_1}{h_1}\frac{h_2}{a_2} S_1^2 + \frac{\gamma_2}{h_2}\frac{h_1}{a_1} S_1^1 \end{vmatrix} \tag{6.129}$$

or

$$n \wedge \nabla \left(V_2 - V_2^{(i)} \right) \Big|_{\xi=0} \Big|_l = \frac{1}{h_0} \left[-\frac{\gamma_m}{h_m}\frac{h_n}{a_n} S_1^n + \frac{\gamma_n}{h_n}\frac{h_m}{a_m} S_1^m \right] \tag{6.130}$$

with

$$\gamma_m = 3L_{1,m} I_1^m \left(a_1 \right) - K_{1,m} = \frac{3}{2} L_{1,m} I_m - K_{1,m} \tag{6.131}$$

and

$$n \bullet \nabla \left(V_2 - \varepsilon V_2^{(i)} \right) \Big|_{\xi=0} = \frac{1}{h_0} \sum_{m=1}^{3} \frac{1}{a_m} \left[\left(\frac{3}{2} L_{1,m} I_m - \varepsilon K_{1,m} \right) - 3L_{1,m} / \left(a_1 a_2 a_3 \right) \right] S_1^m \tag{6.132}$$

The right-hand side of equation 6.67 reflecting the conditions at the surface of the ellipsoid contains the vector $F_2 - \varepsilon^e F_2^{(i)} + E_2^{(0)}$ with e=0 (cross product) or 1 (scalar product). It contains constant terms and quadratic terms in $x_i x_j$.

so, the constant vector, noted Ct, may be written as :

$$Ct = C + Y \nabla \xi \tag{6.133}$$

with

$$C = \begin{vmatrix} \frac{1}{2} B_1 I_3 - \frac{1}{5} A_1 I + \frac{1}{2} A_1 I + A_1 I_1 \left(2a_1^2 - J_1 \right) / 10 \\ \frac{1}{2} B_2 I_1 - \frac{1}{5} A_2 I + \frac{1}{2} A_2 I + A_2 I_2 \left(2a_2^2 - J_2 \right) / 10 \\ \frac{1}{2} B_3 I_2 - \frac{1}{5} A_3 I + \frac{1}{2} A_3 I + A_3 I_3 \left(2a_3^2 - J_3 \right) / 10 \end{vmatrix} \tag{6.134}$$

and

$$5a_1a_2a_3Y = \frac{1}{2}\sum_{m=1}^{3}\frac{J_m}{a_m^2}A_m x_m \tag{6.135}$$

as $n \bullet \nabla\xi\big|_{\xi=0} = 2/h_0$ and $n \wedge \nabla\xi\big|_{\xi=0} = 0$,

$$n \wedge Ct = n \wedge C = \frac{1}{h_0}\begin{vmatrix} \frac{x_2}{a_2^2}C_3 - \frac{x_3}{a_3^2}C_2 \\ \frac{x_3}{a_3^2}C_1 - \frac{x_1}{a_1^2}C_3 \\ \frac{x_1}{a_1^2}C_2 - \frac{x_2}{a_2^2}C_1 \end{vmatrix} \tag{6.136}$$

and

$$n \bullet Ct = n \bullet C + 2Y/h_0 = \frac{1}{h_0}\sum_{m=1}^{3}\left(\frac{C_m}{a_m^2} + \frac{1}{5a_1a_2a_3}\frac{J_m}{a_m^2}A_m\right)x_m = \frac{1}{h_1h_2h_3h_0}\sum_{m=1}^{3}h_m a_m\left(\frac{C_m}{a_m^2} + \frac{1}{5a_1a_2a_3}\frac{J_m}{a_m^2}A_m\right)S_1^m \tag{6.137}$$

The quadratic terms are defined by

$$F_2 - \varepsilon^e F_2^{(i)} + E_2^{(0)}\Big|_k = \sum_{i\le j}^{3} B_{i,j}^k x_i x_j + Z\partial_k\xi \tag{6.138}$$

with

$$
\begin{aligned}
B^1_{1,1} &= -\frac{1}{2}B_1 I_{1,3} + \frac{3}{10}A_1 I_1 + \frac{1}{2}l_1 l^2 - \frac{1}{2}A_1 I_1 \\
B^1_{2,2} &= -\frac{1}{2}B_1 I_{2,3} + \frac{1}{10}A_1 I_1 + \frac{1}{2}l_1 m^2 - \frac{1}{2}A_1 I_2 + \left(\frac{B_1^{(i)}}{2} - \frac{\varepsilon}{4}A_1^{(i)}\right)\varepsilon^e \mu \\
B^1_{3,3} &= -\frac{3}{2}B_1 I_{3,3} + \frac{1}{10}A_1 I_1 + \frac{1}{2}l_1 n^2 - \frac{1}{2}A_1 I_3 + \left(-\frac{B_1^{(i)}}{2} - \frac{\varepsilon}{4}A_1^{(i)}\right)\varepsilon^e \mu \\
B^1_{1,2} &= B_2 I_{1,2} + \frac{1}{5}A_2 I_2 + l_1 lm \\
B^1_{1,3} &= \frac{1}{5}A_3 I_3 + l_1 nl \\
B^1_{2,3} &= \left(B\left(\nu + a_1^2\right)\left(a_2^2 - a_3^2\right) + (')\right)I_{2,3} + \frac{2}{3}\left(B^{(i)}\left(\nu + a_1^2\right)\left(a_2^2 - a_3^2\right)\varepsilon^e + (')\right)\mu + l_1 mn
\end{aligned}
\tag{6.139}
$$

$$
\begin{aligned}
B^2_{1,1} &= -\frac{3}{2}B_2 I_{1,1} + \frac{1}{10}A_2 I_2 + \frac{1}{2}m_1 l^2 - \frac{1}{2}A_2 I_1 + \left(-\frac{B_2^{(i)}}{2} - \frac{\varepsilon}{4}A_2^{(i)}\right)\varepsilon^e \mu \\
B^2_{2,2} &= -\frac{1}{2}B_2 I_{1,2} + \frac{3}{10}A_2 I_2 + \frac{1}{2}m_1 m^2 - \frac{1}{2}A_2 I_2 \\
B^2_{3,3} &= -\frac{1}{2}B_2 I_{1,3} + \frac{1}{10}A_2 I_2 + \frac{1}{2}m_1 n^2 - \frac{1}{2}A_2 I_3 + \left(\frac{B_2^{(i)}}{2} - \frac{\varepsilon}{4}A_2^{(i)}\right)\varepsilon^e \mu \\
B^2_{1,2} &= \frac{1}{5}A_1 I_1 + m_1 ml \\
B^2_{1,3} &= \left(B\left(\nu + a_2^2\right)\left(a_3^2 - a_1^2\right) + (')\right)I_{1,3} + \frac{2}{3}\left(B^{(i)}\left(\nu + a_2^2\right)\left(a_3^2 - a_1^2\right)\varepsilon^e + (')\right)\mu + m_1 nl \\
B^2_{2,3} &= B_3 I_{2,3} + \frac{1}{5}A_3 I_3 + m_1 mn
\end{aligned}
\tag{6 .140}
$$

$$
\begin{aligned}
B^3_{1,1} &= -\frac{1}{2}B_3 I_{1,2} + \frac{1}{10}A_3 I_3 + \frac{1}{2}n_1 l^2 - \frac{1}{2}A_3 I_1 + \left(\frac{B_3^{(i)}}{2} - \frac{\varepsilon}{4}A_3^{(i)}\right)\varepsilon^e \mu \\
B^3_{2,2} &= -\frac{3}{2}B_3 I_{2,2} + \frac{1}{10}A_3 I_3 + \frac{1}{2}n_1 m^2 - \frac{1}{2}A_3 I_2 + \left(-\frac{B_3^{(i)}}{2} - \frac{\varepsilon}{4}A_3^{(i)}\right)\varepsilon^e \mu \\
B^3_{3,3} &= -\frac{1}{2}B_3 I_{2,3} + \frac{3}{10}A_3 I_3 + \frac{1}{2}n_1 n^2 - \frac{1}{2}A_3 I_3 \\
B^3_{1,2} &= \left(B\left(\nu + a_3^2\right)\left(a_1^2 - a_2^2\right) + (')\right)I_{1,2} + \frac{2}{3}\left(B^{(i)}\left(\nu + a_3^2\right)\left(a_1^2 - a_2^2\right)\varepsilon^e + (')\right)\mu + n_1 lm \\
B^3_{1,3} &= B_1 I_{1,3} + \frac{1}{5}A_1 I_1 + n_1 nl \\
B^3_{2,3} &= \frac{1}{5}A_2 I_2 + n_1 mn
\end{aligned}
\tag{6.141}
$$

$$Z = -\frac{1}{10\,a_1 a_2 a_3}\left(x_1^2 + x_2^2 + x_3^2\right)\left(A_1\frac{x_1}{a_1^2} + A_2\frac{x_2}{a_2^2} + A_3\frac{x_3}{a_3^2}\right) \tag{6.142}$$

To calculate the scalar and vector products, again use the relations :

$n \bullet \nabla\xi\big|_{\xi=0} = 2/h_0$ et $n \wedge \nabla\xi\big|_{\xi=0} = 0$.

thus,

$$n \wedge \left(F_2 - \varepsilon^e F_2^{(i)} + E_2^{(0)}\right) = n \wedge \left(F_2 - \varepsilon^e F_2^{(i)} + E_2^{(0)}\right)_{Z=0} \tag{6.143}$$

and

$$n \bullet \left(F_2 - \varepsilon^e F_2^{(i)} + E_2^{(0)}\right) = n \bullet \left(F_2 - \varepsilon^e F_2^{(i)} + E_2^{(0)}\right)_{Z=0} + 2Z/h_0 \tag{6.144}$$

The 1-component of the cross product (terms of order 3) is therefore :

$$h_0 n \wedge \left(F_2 - \varepsilon^e F_2^{(i)} + E_2^{(0)}\right)\Big|_{l=1} = 0x_1^3 + \frac{B_{1,2}^3}{a_2^2}x_1x_2^2 - \frac{B_{1,3}^2}{a_3^2}x_1x_3^2 + \frac{B_{1,1}^3}{a_2^2}x_1^2x_2 + \frac{B_{2,2}^3}{a_2^2}x_2x_2^2 + \left(\frac{B_{3,3}^3}{a_2^2} - \frac{B_{2,3}^2}{a_3^2}\right)x_2x_3^2$$
$$-\frac{B_{1,1}^2}{a_3^2}x_1^2x_3 + \left(\frac{B_{2,3}^3}{a_2^2} - \frac{B_{2,2}^2}{a_3^2}\right)x_2^2x_3 - \frac{B_{3,3}^2}{a_3^2}x_3x_3^2 + \left(\frac{B_{1,3}^3}{a_2^2} - \frac{B_{1,2}^2}{a_3^2}\right)x_1x_2x_3 \tag{6.145}$$

Using the expressions 6.107, the expression 6.145 (taking into account terms of order 1 (6.136)) can be written as :

$$h_0\, n \wedge \left(F_2 - \varepsilon^e F_2^{(i)} + E_2^{(0)}\right)\Big|_{l=1} = \sum_{m=1}^{3} A_m^{T,1} S_1^m + \sum_{m=1}^{7} B_m^{T,1} S_3^m \tag{6.146}$$

with

$$A_1^{T,1} = \frac{h_1 a_1}{5h_1h_2h_3}\left(B_{1,2}^3 - B_{1,3}^2\right)$$
$$A_2^{T,1} = \frac{h_2 a_2}{5h_1h_2h_3}\left(w_3^2 - B_{2,3}^2\right) \tag{6.147}$$
$$A_3^{T,1} = \frac{h_3 a_3}{5h_1h_2h_3}\left(-w_2^3 + B_{2,3}^3\right)$$

$$
\begin{aligned}
B_1^{T,1} &= -\frac{1}{5h_1h_2h_3}\frac{h_1a_1}{\left(N_1-N_2\right)}\left[\frac{B_{1,2}^3}{N_1+a_2^2}-\frac{B_{1,3}^2}{N_1+a_3^2}\right]\\
B_2^{T,1} &= \frac{1}{5h_1h_2h_3}\frac{h_1a_1}{\left(N_1-N_2\right)}\left[\frac{B_{1,2}^3}{N_2+a_2^2}-\frac{B_{1,3}^2}{N_2+a_3^2}\right]\\
B_3^{T,1} &= -\frac{1}{5h_1h_2h_3}\frac{h_2a_2}{\left(N_3-N_4\right)}W_3^3\\
B_4^{T,1} &= \frac{1}{5h_1h_2h_3}\frac{h_2a_2}{\left(N_3-N_4\right)}W_3^4\\
B_5^{T,1} &= \frac{1}{5h_1h_2h_3}\frac{h_3a_3}{\left(N_5-N_6\right)}W_2^5\\
B_6^{T,1} &= -\frac{1}{5h_1h_2h_3}\frac{h_3a_3}{\left(N_5-N_6\right)}W_2^6\\
B_7^{T,1} &= \frac{a_1a_2a_3}{\left(h_1h_2h_3\right)^2}\left[\frac{B_{1,3}^3}{a_2^2}-\frac{B_{1,2}^2}{a_3^2}\right]
\end{aligned}
\qquad (6.148)
$$

where w_i^j and W_i^j are defined by :

$$
\begin{aligned}
w_i^j &= \frac{1}{a_j^2}\left(B_{1,1}^i a_1^2+B_{2,2}^i a_2^2+B_{3,3}^i a_3^2+5C_i\right)+2B_{j,j}^i\\
&= \frac{1}{a_j^2}\sum_{k=1}^{3}a_k^2\left(1+2\delta_{k,j}\right)B_{k,k}^i+5\frac{C_i}{a_j^2} \qquad 1\le i,j\le 3
\end{aligned}
\qquad (6.149)
$$

$$
W_i^p = \frac{1}{a_{f(p)}^2}\sum_{k=1}^{3}\frac{a_k^2\left(1+2\delta_{k,f(p)}\right)B_{k,k}^i}{\left(N_p+a_k^2\right)}-\frac{B_{i,f(p)}^{f(p)}}{\left(N_p+a_i^2\right)} \quad 1\le i\le 3 \quad 1\le p\le 6 \qquad (6.150)
$$

The same applies to the components *l*=2 and *l*=3 :

$$
\begin{aligned}
A_1^{T,2} &= \frac{h_1a_1}{5h_1h_2h_3}\left(B_{1,3}^1-w_3^1\right)\\
A_2^{T,2} &= \frac{h_2a_2}{5h_1h_2h_3}\left(B_{2,3}^1-B_{1,2}^3\right)\\
A_3^{T,2} &= \frac{h_3a_3}{5h_1h_2h_3}\left(w_1^3-B_{1,3}^3\right)
\end{aligned}
\qquad (6.151)
$$

$$B_1^{T,2} = \frac{1}{5h_1h_2h_3}\frac{h_1a_1}{\left(N_1 - N_2\right)}W_3^1$$

$$B_2^{T,2} = -\frac{1}{5h_1h_2h_3}\frac{h_1a_1}{\left(N_1 - N_2\right)}W_3^2$$

$$B_3^{T,2} = -\frac{1}{5h_1h_2h_3}\frac{h_2a_2}{\left(N_3 - N_4\right)}\left[\frac{B_{2,3}^1}{N_3 + a_3^2} - \frac{B_{1,2}^3}{N_3 + a_1^2}\right]$$

$$B_4^{T,2} = \frac{1}{5h_1h_2h_3}\frac{h_2a_2}{\left(N_3 - N_4\right)}\left[\frac{B_{2,3}^1}{N_4 + a_3^2} - \frac{B_{1,2}^3}{N_4 + a_1^2}\right]$$

$$B_5^{T,2} = -\frac{1}{5h_1h_2h_3}\frac{h_3a_3}{\left(N_5 - N_6\right)}W_1^5$$

$$B_6^{T,2} = \frac{1}{5h_1h_2h_3}\frac{h_3a_3}{\left(N_5 - N_6\right)}W_1^6$$

$$B_7^{T,2} = \frac{a_1a_2a_3}{\left(h_1h_2h_3\right)^2}\left[\frac{B_{1,2}^1}{a_3^2} - \frac{B_{2,3}^3}{a_1^2}\right] \qquad (6.152)$$

$$A_1^{T,3} = \frac{h_1a_1}{5h_1h_2h_3}\left(w_2^1 - B_{1,2}^1\right)$$

$$A_2^{T,3} = \frac{h_2a_2}{5h_1h_2h_3}\left(B_{1,2}^2 - w_1^2\right) \qquad (6.153)$$

$$A_3^{T,3} = \frac{h_3a_3}{5h_1h_2h_3}\left(B_{1,3}^2 - B_{2,3}^1\right)$$

$$B_1^{T,3} = -\frac{1}{5h_1h_2h_3}\frac{h_1a_1}{\left(N_1 - N_2\right)}W_2^1$$

$$B_2^{T,3} = \frac{1}{5h_1h_2h_3}\frac{h_1a_1}{\left(N_1 - N_2\right)}W_2^2$$

$$B_3^{T,3} = \frac{1}{5h_1h_2h_3}\frac{h_2a_2}{\left(N_3 - N_4\right)}W_1^3$$

$$B_4^{T,3} = -\frac{1}{5h_1h_2h_3}\frac{h_2a_2}{\left(N_3 - N_4\right)}W_1^4$$

$$B_5^{T,3} = -\frac{1}{5h_1h_2h_3}\frac{h_3a_3}{\left(N_5 - N_6\right)}\left[\frac{B_{1,3}^2}{N_5 + a_1^2} - \frac{B_{2,3}^1}{N_5 + a_2^2}\right]$$

$$B_6^{T,3} = \frac{1}{5h_1h_2h_3}\frac{h_3a_3}{\left(N_5 - N_6\right)}\left[\frac{B_{1,3}^2}{N_6 + a_1^2} - \frac{B_{2,3}^1}{N_6 + a_2^2}\right]$$

$$B_7^{T,3} = \frac{a_1a_2a_3}{\left(h_1h_2h_3\right)^2}\left[\frac{B_{2,3}^2}{a_1^2} - \frac{B_{1,3}^1}{a_2^2}\right] \qquad (6.154)$$

There are, of course, regularities in the above expressions, which allow us to write :

$$
\begin{aligned}
&i = l+1 \quad j = l+2 \\
&A_l^{T,l} = \frac{h_l a_l}{5h_1h_2h_3}\left(B_{l,i}^{j} - B_{l,j}^{i}\right) \\
&A_i^{T,l} = \frac{h_i a_i}{5h_1h_2h_3}\left(w_j^{j} - B_{i,j}^{i}\right) \\
&A_j^{T,l} = \frac{h_j a_j}{5h_1h_2h_3}\left(-w_i^{j} + B_{i,j}^{j}\right)
\end{aligned}
\tag{6.155}
$$

$$
\begin{aligned}
&i = l+1 \quad j = l+2 \\
&B_{2i-1}^{T,l} = -\frac{1}{5h_1h_2h_3}\frac{h_i a_i}{N_{2i-1} - N_{2i}} W_j^{2i-1} \\
&B_{2i}^{T,l} = \frac{1}{5h_1h_2h_3}\frac{h_i a_i}{N_{2i-1} - N_{2i}} W_j^{2i} \\
&B_{2j-1}^{T,l} = \frac{1}{5h_1h_2h_3}\frac{h_j a_j}{N_{2j-1} - N_{2j}} W_i^{2j-1} \\
&B_{2j}^{T,l} = -\frac{1}{5h_1h_2h_3}\frac{h_j a_j}{N_{2j-1} - N_{2j}} W_i^{2j} \\
&B_{2l-1}^{T,l} = -\frac{1}{5h_1h_2h_3}\frac{h_l a_l}{\left(N_{2l-1} - N_{2l}\right)}\left[\frac{B_{i,l}^{j}}{N_{2l-1} + a_i^2} - \frac{B_{j,l}^{i}}{N_{2l-1} + a_j^2}\right] \\
&B_{2l}^{T,l} = \frac{1}{5h_1h_2h_3}\frac{h_l a_l}{\left(N_{2l-1} - N_{2l}\right)}\left[\frac{B_{i,l}^{j}}{N_{2l} + a_1^2} - \frac{B_{j,l}^{i}}{N_{2l} + a_j^2}\right] \\
&B_7^{T,l} = \frac{a_1a_2a_3}{\left(h_1h_2h_3\right)^2}\left[\frac{B_{j,l}^{j}}{a_i^2} - \frac{B_{i,l}^{i}}{a_j^2}\right]
\end{aligned}
\tag{6.156}
$$

Similarly, for the scalar product (e=1), the contribution of ellipsoidal surface harmonics of degree 3 is as follows:

$$
\begin{aligned}
&n \bullet \left(F_2 - \varepsilon^e F_2^{(i)} + E_2^{(0)}\right) = \\
&\frac{1}{5h_1h_2h_3h_0}\sum_{j=1}^{3}\left\{\begin{aligned}
&-\frac{h_j a_j}{N_{2j-1} - N_{2j}}\left[\frac{1}{a_j^2}\sum_{k=1}^{3}\frac{a_k^2\left(1+2\delta_{k,j}\right)B_{k,k}^{j}\left(e=1\right)}{\left(N_{2j-1} + a_k^2\right)} + \frac{B_{j,p}^{p}\left(1\right)}{N_{2j-1} + a_p^2} + \frac{B_{j,q}^{q}\left(1\right)}{N_{2j-1} + a_q^2} - \frac{A_j}{a_1a_2a_3a_j^2}\right] S_3^{2j-1} \\
&+\frac{h_j a_j}{N_{2j-1} - N_{2j}}\left[\frac{1}{a_j^2}\sum_{k=1}^{3}\frac{a_k^2\left(1+2\delta_{k,j}\right)B_{k,k}^{j}\left(1\right)}{\left(N_{2j} + a_k^2\right)} + \frac{B_{j,p}^{p}\left(1\right)}{N_{2j} + a_p^2} + \frac{B_{j,q}^{q}\left(1\right)}{N_{2j} + a_q^2} - \frac{A_j}{a_1a_2a_3a_j^2}\right] S_3^{2j}
\end{aligned}\right\} \\
&+\frac{1}{h_0}\left(\frac{B_{2,3}^{1}\left(1\right)}{a_1^2} + \frac{B_{1,3}^{2}\left(1\right)}{a_2^2} + \frac{B_{1,2}^{3}\left(1\right)}{a_3^2}\right)\frac{a_1a_2a_3}{h_1^2h_2^2h_3^2} S_3^7
\end{aligned}
$$

(6.157)

with $p \neq j \; q \neq j \; p \neq q$

to which is added the contribution of ellipsoidal surface harmonics of degree 1, derived from the linear terms in x_j (eq. 6.137) and the ternary terms $x_k x_n^2$ (Eq. 6.107):

$$n \bullet \left(F_2 - \varepsilon^e F_2^{(i)} + E_2^{(0)} \right) = \frac{1}{5 h_1 h_2 h_3 h_0} \sum_{m=1}^{3} h_m a_m \left(w_m^m (e=1) + \chi_m A_m + B_{m,p}^p (1) + B_{m,q}^q (1) \right) S_1^m$$

(6.158)

with

$$\chi_m = \frac{1}{a_1 a_2 a_3 a_m^2} \left(J_m - \frac{1}{5} \sum_{k=1}^{3} \left(1 + 2\delta_{k,m} \right) a_k^2 \right)$$

and $p \neq m \quad q \neq m \quad p \neq q$.

We have therefore just expressed the various terms appearing in the boundary conditions as linear combinations of ellipsoidal surface harmonics. Boundary conditions 6.67 are therefore linear combinations of ellipsoidal surface harmonics equal to 0. As these form an orthonormal basis of functions, this implies that the coefficients (which contain the unknowns of the problem) of the harmonics are identically zero.

so, for harmonics of degree 1 :

Scalar product (Eqs. 6.132 and 6.158):

$$\frac{1}{a_m} \left[\left(\frac{3}{2} L_{1,m} I_m - \varepsilon K_{1,m} \right) - 3 L_{1,m} / \left(a_1 a_2 a_3 \right) \right] = - \frac{1}{5 h_1 h_2 h_3} h_m a_m \left(w_m^m \left(e=1 \right) + \chi_m A_m + B_{m,p}^p + B_{m,q}^q \right)$$
$$m = 1, 2, 3$$

(6.159)

Cross product (Eqs. 6.129, 6.147, 6.151 and 6.153)

$$n \wedge \nabla\left(V_2 - V_2^{(i)}\right)\Big|_{\xi=0} = \frac{1}{h_0}\begin{vmatrix} -\frac{\gamma_2}{h_2}\frac{h_3}{a_3}S_1^3 + \frac{\gamma_3}{h_3}\frac{h_2}{a_2}S_1^2 \\ \frac{\gamma_1}{h_1}\frac{h_3}{a_3}S_1^3 - \frac{\gamma_3}{h_3}\frac{h_1}{a_1}S_1^1 \\ -\frac{\gamma_1}{h_1}\frac{h_2}{a_2}S_1^2 + \frac{\gamma_2}{h_2}\frac{h_1}{a_1}S_1^1 \end{vmatrix} =$$

$$-n \wedge \left(F_2 - F_2^{(i)} + E_2^{(0)}\right)\Big|_{\xi=0} = -\frac{1}{5h_0h_1h_2h_3}\begin{vmatrix} h_1a_1\left(B_{1,2}^3 - B_{1,3}^2\right)S_1^1 + h_2a_2\left(w_3^2 - B_{2,3}^2\right)S_1^2 + h_3a_3\left(-w_2^3 + B_{2,3}^3\right)S_1^3 \\ h_1a_1\left(B_{1,3}^1 - w_3^1\right)S_1^1 + h_2a_2\left(B_{2,3}^1 - B_{1,2}^3\right)S_1^2 + h_3a_3\left(w_1^3 - B_{1,3}^3\right)S_1^3 \\ h_1a_1\left(w_2^1 - B_{1,2}^1\right)S_1^1 + h_2a_2\left(B_{1,2}^2 - w_1^2\right)S_1^2 + h_3a_3\left(B_{1,3}^2 - B_{2,3}^1\right)S_1^3 \end{vmatrix}$$

(6.160)

thus

$$\gamma_1 = -\frac{h_1}{5h_1h_2h_3}a_3^2\left(w_1^3 - B_{1,3}^3\right) \qquad (e=0)$$
$$\gamma_1 = \frac{h_1}{5h_1h_2h_3}a_2^2\left(B_{1,2}^2 - w_1^2\right)$$
$$\gamma_2 = \frac{h_2}{5h_1h_2h_3}a_3^2\left(-w_2^3 + B_{2,3}^3\right)$$
$$\gamma_2 = -\frac{h_2}{5h_1h_2h_3}a_1^2\left(w_2^1 - B_{1,2}^1\right)$$
$$\gamma_3 = -\frac{h_3}{5h_1h_2h_3}a_2^2\left(w_3^2 - B_{2,3}^2\right)$$
$$\gamma_3 = \frac{h_3}{5h_1h_2h_3}a_1^2\left(B_{1,3}^1 - w_3^1\right)$$

(6.161)

$$0 = B_{1,2}^3 - B_{1,3}^2$$
$$0 = B_{2,3}^1 - B_{1,2}^3$$
$$0 = B_{1,3}^2 - B_{2,3}^1$$

(6.162)

with

$$\gamma_m = \frac{3}{2}L_{1,m}I_m - K_{1,m}$$

Equations 6.162 do not contain any unknowns; we can show that equations 6.162 are verified. Let us consider, for instance, $B_{2,3}^1 - B_{1,3}^2 = 0$:

Reminding that $e=0$ and assuming $\mu=1$, we have,

$$B = B' = 0$$

$$B^{(i)} = A_{\nu}^{(i)} = \frac{1}{6(\nu'-\nu)}\left[\frac{ll_2}{\nu+a_1^2}+\frac{mm_2}{\nu+a_2^2}+\frac{nn_2}{\nu+a_3^2}\right]$$

$$B^{(i)\,\prime} = -\frac{1}{6(\nu'-\nu)}\left[\frac{ll_2}{\nu'+a_1^2}+\frac{mm_2}{\nu'+a_2^2}+\frac{nn_2}{\nu'+a_3^2}\right]$$

from 6.139 and 6.140 :

$$B_{2,3}^1 - B_{1,3}^2 = \frac{2}{3}B^{(i)}\left(\left(\nu+a_1^2\right)\left(a_2^2-a_3^2\right)-\left(\nu+a_2^2\right)\left(a_3^2-a_1^2\right)\right)+(')+l_1mn-m_1nl$$

$$= 2B^{(i)}\left(\nu+a_1^2\right)\left(\nu+a_2^2\right)+(')+\left(l_1m-m_1l\right)n$$

$$= -\frac{1}{3(\nu-\nu')}\left[ll_2\left(\nu+a_2^2\right)+mm_2\left(\nu+a_1^2\right)+nn_2\frac{\left(\nu+a_1^2\right)\left(\nu+a_2^2\right)}{\left(\nu+a_3^2\right)}-(')\right]+\left(l_1m-m_1l\right)n$$

$$= -\frac{1}{3}\left[\underbrace{ll_2+mm_2}_{-nn_2}+nn_2\left(\frac{\left(\nu+a_1^2\right)\left(\nu+a_2^2\right)}{\left(\nu+a_3^2\right)}-\frac{\left(\nu'+a_1^2\right)\left(\nu'+a_2^2\right)}{\left(\nu'+a_3^2\right)}\right)\frac{1}{\nu-\nu'}\right]+\left(l_1m-m_1l\right)n$$

$$= -\frac{1}{3}nn_2\left[-3\right]+\left(l_1m-m_1l\right)n = 0$$

The definition of ν,ν' was used (eq. 5.28), as well as the properties of unit vectors, in particular :

$$\begin{vmatrix} l \\ m \\ n \end{vmatrix} \wedge \begin{vmatrix} l_1 \\ m_1 \\ n_1 \end{vmatrix} = \begin{vmatrix} l_2 \\ m_2 \\ n_2 \end{vmatrix}$$

It was numerically checked that $B_{2,3}^1 - B_{1,3}^2 = 0$ if $\mu \neq 1$.

We can also see that the unknowns γ_i appear twice in the system 6.161. Here again, we can show the redundancy of the two equations involving γ_i, which we will illustrate for γ_1 by checking (eq.6.161):

$$Y = a_3^2\left(w_1^3 - B_{1,3}^3\right)+a_2^2\left(B_{1,2}^2 - w_1^2\right) = 0 \qquad (6.163)$$

Indeed, using 6.139-141 and 6.149, as well as the orthogonality relations between unit vectors :

$$Y = -2\left(B^1_{2,2}a_2^2 - B^1_{3,3}a_3^2\right) - a_3^2\left(B_1 I_{1,3} + \frac{1}{5}A_1 I_1 + n_1 nl\right) + a_2^2\left(\frac{1}{5}A_1 I_1 + m_1 ml\right)$$

$$= -B_1\left(a_3^2 I_{1,3} + 3a_3^2 I_{3,3} - a_2^2 I_{2,3}\right) - n\left(n_1 l - l_1 n\right)a_3^2 + m\left(m_1 l - l_1 m\right)a_2^2 + A_1\left(a_2^2 I_2 - a_3^2 I_3\right)$$

$$-\mu B_1^{(i)}\left(a_3^2 + a_2^2\right) - \frac{\varepsilon\mu}{2}A_1^{(i)}\left(a_3^2 - a_2^2\right) \qquad (6.164)$$

$$= -B_1\left(\frac{2}{a_1 a_2 a_3} - \left(a_2^2 + a_3^2\right)I_{2,3}\right) + nm_2 a_3^2 + mn_2 a_2^2 + A_1\left(a_2^2 I_2 - a_3^2 I_3\right)$$

$$-\mu B_1^{(i)}\left(a_3^2 + a_2^2\right) - \frac{\varepsilon\mu}{2}A_1^{(i)}\left(a_3^2 - a_2^2\right)$$

$\mu = 1$ leads to simplifications in Stevenson's expressions: we have seen that $B_1 = 0$ and

$$B_1^{(i)} = \frac{1}{2}\left(m_2 n + n_2 m + A_1\left(I_2 - I_3\right)\right).$$

We then have :

$$Y = \frac{1}{2}\left(a_3^2 - a_2^2\right)\left(nm_2 - mn_2 - A_1\left(I_2 + I_3\right) - \varepsilon A_1^{(i)}\right) \qquad (6.165)$$

Knowing that,

$$A_1 = -\left(\varepsilon - 1\right)f_1\left(\varepsilon\right)l_1 \quad A_1^{(i)} = \frac{2}{a_1 a_2 a_3}f_1\left(\varepsilon\right)l_1$$

$$nm_2 - mn_2 = l_1 \quad f_1\left(\varepsilon\right) = 1/\left(\frac{2}{a_1 a_2 a_3} + \left(\varepsilon - 1\right)I_1\right)$$

(6.165) becomes :

$$Y = \frac{1}{2}\left(a_3^2 - a_2^2\right)f_1\left(\varepsilon\right)l_1\left(\varepsilon - 1\right)\left(I_1 + I_2 + I_3 - \frac{2}{a_1 a_2 a_3}\right) \qquad (6.166)$$

Knowing that $I_1 + I_2 + I_3 = \frac{2}{a_1 a_2 a_3}$, we deduce that $Y = 0$.

It was numerically checked that $Y = 0$ if $\mu \neq 1$.

Finally, for m=1 we have the system of 2 equations with 2 unknowns $L_{1,1}, K_{1,1}$:

$$\begin{cases}\left(\frac{3}{2}L_{1,1}I_1 - \varepsilon K_{1,1}\right) - 3L_{1,1}/\left(a_1a_2a_3\right) = -\frac{1}{5h_1h_2h_3}h_1a_1^2\left(w_1^1\left(e=1\right) + \chi_1 A_1 + B_{1,2}^2 + B_{1,3}^3\right) \\ \frac{3}{2}L_{1,1}I_1 - K_{1,1} = \frac{1}{5h_2h_3}a_2^2\left(B_{1,2}^2 - w_1^2\right)\end{cases} \quad (6.167)$$

whose solution has been successfully compared with the expression in D_1 (Stevenson), given that $D_1 = \frac{3}{2}h_2h_3L_{1,1}$. More generally, we have to solve 3 independent systems of 2 equations with 2 unknowns ($L_{1,m}, K_{1,m}$, $m = 1,2,3$).

At first glance, the solution of 6.167 depends on J_1 (see 6.55). We will show that not :

Let us rewrite 6.167 by explicitly showing J_1 :

$$\begin{cases}\left(\frac{3}{2}L_{1,1}I_1 - \varepsilon K_{1,1}\right) - 3L_{1,1}/\left(a_1a_2a_3\right) = \alpha_1 + \beta_1 J_1 \\ \frac{3}{2}L_{1,1}I_1 - K_{1,1} = \alpha_2 + \beta_2 J_1\end{cases} \quad (6.168)$$

$\alpha_1, \alpha_2, \beta_1, \beta_2$ do not depend on J_1 .

This gives,

$$L_{1,1} = \frac{1}{3}\frac{\beta_1 - \beta_2\varepsilon}{\frac{I_1}{2}(1-\varepsilon) - \frac{1}{a_1a_2a_3}}J_1 + Q = -\frac{1}{15h_2h_3}\frac{(\varepsilon-1)\frac{A_1I_1}{2} + \frac{A_1}{a_1a_2a_3}}{\frac{I_1}{2}(1-\varepsilon) - \frac{1}{a_1a_2a_3}}J_1 + Q = \frac{A_1}{15h_2h_3}J_1 + Q_L \quad (6.169)$$

$$K_{1,1} = 0J_1 + Q_K$$

where Q_L, Q_K are terms without J_1 .

It is the sum $V_2 + V_2$ " and not V_2 and V_2 " separately that has a physical meaning (intervening in E_2). Note then that the contribution of the term in J_1 to V_2 " is $-\frac{1}{10}A_1J_1I_1(\xi)x_1$ and that of J_1 to V_2 is $\frac{3}{2}L_{1,1}I_1(\xi)E_1^1 = \frac{1}{10}A_1J_1I_1(\xi)x_1$; these two contributions cancel each other out.

The same applies to $V_2^{(i)}$.

Let us do the same with harmonics of degree 3 :

Cross product (eq. 6.126 and 6.146) :

$$B_i^{S,l} = -B_i^{T,l} \quad 1 \le i \le 7 \quad 1 \le l \le 3 \tag{6.170}$$

So if we plot all the equations showing the pair of unknowns $\left(P_3, P_4\right)$:

$$\begin{aligned}
&\alpha_3 P_3 Y_3^5 + \alpha_4 P_4 Y_4^5 = W_2^5 \\
&\alpha_3 P_3 Y_3^6 + \alpha_4 P_4 Y_4^6 = W_2^6 \\
&\alpha_3 P_3 Y_3^1 + \alpha_4 P_4 Y_4^1 = W_2^1 \\
&\alpha_3 P_3 Y_3^2 + \alpha_4 P_4 Y_4^2 = W_2^2 \\
&\alpha_3 P_3 Z_3 + \alpha_4 P_4 Z_4 = X_2
\end{aligned} \tag{6.171}$$

with

$$X_j = \frac{1}{2}\left(\frac{B_{j,j+1}^{j+1}}{a_{j-1}^2} - \frac{B_{j-1,j}^{j-1}}{a_{j+1}^2} \right) \tag{6.172}$$

which can be generalized to the unknowns pair (P_{2n-1}, P_{2n}) :

with $n = 1,2,3$; $j \neq n$ ($\Rightarrow \delta_{n,j} = 0$)

$$\begin{aligned}
&\alpha_{2n-1} P_{2n-1} Y_{2n-1}^{2j-1} + \alpha_{2n} P_{2n} Y_{2n}^{2j-1} = W_n^{2j-1} \\
&\alpha_{2n-1} P_{2n-1} Y_{2n-1}^{2j} + \alpha_{2n} P_{2n} Y_{2n}^{2j} = W_n^{2j} \\
&\alpha_{2n-1} P_{2n-1} Z_{2n-1} + \alpha_{2n} P_{2n} Z_{2n} = X_n
\end{aligned} \tag{6.173}$$

The previous equations give us $P_i \quad 1 \le i \le 6$. However, the system of 5 equations 6.171 (or 6.173) has only 2 unknowns! In the following we will show that this system can be reduced to a system of 2 equations with 2 unknowns:

First let us rewrite Y_p^q and W_i^p (6.119 and 6.150),

$$\begin{aligned}
Y_p^q &= \frac{1}{a_{f(q)}^2} \sum_{k=1}^{3} \frac{a_k^2 \left(1 + 2\delta_{k,f(p)} + 2\delta_{k,f(q)}\right)}{\left(N_p + a_k^2\right)\left(N_q + a_k^2\right)} - \frac{2}{\left(N_p + a_{f(q)}^2\right)\left(N_q + a_{f(p)}^2\right)} \\
&= \frac{5}{a_{f(q)}^2 \left(h_1 h_2 h_3\right)^2} \sum_{k=1}^{3} \frac{(-1)^{k+1} a_k^2 h_k^2 \left(1 + 2\delta_{k,f(p)} + 2\delta_{k,f(q)}\right)}{\left(1 + 2\delta_{k,f(q)}\right)\left(N_p + a_k^2\right)} \left(N_{q'} + a_k^2\right) \\
&- \frac{10}{\left(h_1 h_2 h_3\right)^2} \frac{(-1)^{f(p)+1} h_{f(p)}^2}{\left(1 + 2\delta_{f(p),f(q)}\right)\left(N_p + a_{f(q)}^2\right)} \left(N_{q'} + a_{f(p)}^2\right)
\end{aligned} \tag{6.174}$$

q and q' are such as $f(q)=f(q')$.

$$W_i^p=\frac{1}{a_{f(p)}^2}\sum_{k=1}^{3}\frac{a_k^2\left(1+2\delta_{k,f(p)}\right)B_{k,k}^i}{\left(N_p+a_k^2\right)}-\frac{B_{i,f(p)}^{f(p)}}{\left(N_p+a_i^2\right)}$$
$$=\frac{5}{a_{f(p)}^2\left(h_1h_2h_3\right)^2}\sum_{k=1}^{3}(-1)^{k+1}a_k^2h_k^2B_{k,k}^i\left(N_{p'}+a_k^2\right)-\frac{5}{\left(h_1h_2h_3\right)^2}(-1)^{i+1}B_{i,f(p)}^{f(p)}\frac{1}{\left(1+2\delta_{i,f(p)}\right)}h_i^2\left(N_{p'}+a_i^2\right)$$

(6.175)

where we used the relation (from Dassios) :

$$\left(N_{2j-1}+a_k^2\right)\left(N_{2j}+a_k^2\right)=(-1)^{k+1}\frac{1+2\delta_{k,j}}{5}\frac{\left(h_1h_2h_3\right)^2}{h_k^2}\quad j,k=1,2,3 \tag{6.176}$$

The first equation of the system 6.173 becomes :

$$\alpha_{2n-1}P_{2n-1}Y_{2n-1}^{2j-1}+\alpha_{2n}P_{2n}Y_{2n}^{2j-1}=W_n^{2j-1}$$
$$\Downarrow$$
$$\alpha_{2n-1}P_{2n-1}\left[\frac{1}{a_j^2}\sum_{k=1}^{3}\frac{(-1)^{k+1}a_k^2h_k^2\left(1+2\delta_{k,n}+2\delta_{k,j}\right)}{\left(1+2\delta_{k,j}\right)\left(N_{2n-1}+a_k^2\right)}\left(N_{2j}+a_k^2\right)-\frac{2(-1)^{n+1}h_n^2}{\left(N_{2n-1}+a_j^2\right)}\left(N_{2j}+a_n^2\right)\right]$$
$$+\alpha_{2n}P_{2n}\left[\frac{1}{a_j^2}\sum_{k=1}^{3}\frac{(-1)^{k+1}a_k^2h_k^2\left(1+2\delta_{k,n}+2\delta_{k,j}\right)}{\left(1+2\delta_{k,j}\right)\left(N_{2n}+a_k^2\right)}\left(N_{2j}+a_k^2\right)-\frac{2(-1)^{n+1}h_n^2}{\left(N_{2n}+a_j^2\right)}\left(N_{2j}+a_n^2\right)\right]$$
$$=\frac{1}{a_j^2}\sum_{k=1}^{3}(-1)^{k+1}a_k^2h_k^2B_{k,k}^n\left(N_{2j}+a_k^2\right)-(-1)^{n+1}B_{n,j}^jh_n^2\left(N_{2j}+a_n^2\right)$$

(6.177)

or

$$\left(U_{n,j,0}+V_{n,j,0}\right)N_{2j}+U_{n,j,1}+V_{n,j,1}=0 \tag{6.178}$$

with

$$U_{n,j,u}=\frac{1}{a_j^2}\sum_{k=1}^{3}\left[\frac{\left(1+2\delta_{k,n}+2\delta_{k,j}\right)}{\left(1+2\delta_{k,j}\right)}\left(\frac{\alpha_{2n-1}P_{2n-1}}{\left(N_{2n-1}+a_k^2\right)}+\frac{\alpha_{2n}P_{2n}}{\left(N_{2n}+a_k^2\right)}\right)-B_{k,k}^n\right](-1)^{k+1}a_k^{2+2u}h_k^2$$
$$V_{n,j,u}=\left(B_{n,j}^j-2\frac{\alpha_{2n-1}P_{2n-1}}{\left(N_{2n-1}+a_j^2\right)}-2\frac{\alpha_{2n}P_{2n}}{\left(N_{2n}+a_j^2\right)}\right)(-1)^{n+1}a_n^{2u}h_n^2 \tag{6.179}$$

as a reminder $\delta_{n,j}=0$.

The second equation of the system 6.173 becomes :

$$\alpha_{2n-1}P_{2n-1}Y_{2n-1}^{2j}+\alpha_{2n}P_{2n}Y_{2n}^{2j}=W_n^{2j}$$
$$\Downarrow$$
$$\alpha_{2n-1}P_{2n-1}\left[\frac{1}{a_j^2}\sum_{k=1}^{3}\frac{(-1)^{k+1}a_k^2h_k^2\left(1+2\delta_{k,n}+2\delta_{k,j}\right)}{\left(1+2\delta_{k,j}\right)\left(N_{2n-1}+a_k^2\right)}\left(N_{2j-1}+a_k^2\right)-\frac{2(-1)^{n+1}h_n^2}{\left(N_{2n-1}+a_j^2\right)}\left(N_{2j-1}+a_n^2\right)\right]$$
$$+\alpha_{2n}P_{2n}\left[\frac{1}{a_j^2}\sum_{k=1}^{3}\frac{(-1)^{k+1}a_k^2h_k^2\left(1+2\delta_{k,n}+2\delta_{k,j}\right)}{\left(1+2\delta_{k,j}\right)\left(N_{2n}+a_k^2\right)}\left(N_{2j-1}+a_k^2\right)-\frac{2(-1)^{n+1}h_n^2}{\left(N_{2n}+a_j^2\right)}\left(N_{2j-1}+a_n^2\right)\right]$$
$$=\frac{1}{a_j^2}\sum_{k=1}^{3}(-1)^{k+1}a_k^2h_k^2B_{k,k}^n\left(N_{2j-1}+a_k^2\right)-(-1)^{n+1}B_{n,j}^jh_n^2\left(N_{2j-1}+a_n^2\right)$$

(6.180)

or

$$\left(U_{n,j,0}+V_{n,j,0}\right)N_{2j-1}+U_{n,j,1}+V_{n,j,1}=0 \qquad (6.181)$$

6.178 and 6.181 lead to :

$$\begin{aligned}U_{n,j,0}+V_{n,j,0}&=0\\U_{n,j,1}+V_{n,j,1}&=0\end{aligned} \qquad (6.182)$$

for the two j values different from n (for example, si n=2, j may have the values 1 or 3). We can check that, given n and k, $\frac{\left(1+2\delta_{k,n}+2\delta_{k,j}\right)}{\left(1+2\delta_{k,j}\right)}=1+2\delta_{k,n}$ does not depend on j. As a result, $a_j^2U_{n,j,u}=U_{n,u}$ does not depend on j. We also have $V_{n,j,u}=V_{n,j,0}a_n^{2u}$.

So for a given n and calling the 2 permitted values of j : n' and n'' ,

$$\begin{aligned}\frac{U_{n,0}}{a_{n'}^2}+V_{n,n',0}&=0\\\frac{U_{n,1}}{a_{n'}^2}+a_n^2V_{n,n',0}&=0\\\frac{U_{n,0}}{a_{n''}^2}+V_{n,n'',0}&=0\\\frac{U_{n,1}}{a_{n''}^2}+a_n^2V_{n,n'',0}&=0\end{aligned} \qquad (6.183)$$

which reduces to :

$$\begin{aligned} &U_{n,0} + a_{n'}^2 V_{n,n',0} = 0 \\ &a_{n'}^2 V_{n,n',0} = a_{n''}^2 V_{n,n'',0} \\ &U_{n,1} = a_n^2 U_{n,0} \end{aligned} \tag{6.184}$$

which leads to the system of 4 equations with 2 unknowns P_{2n-1}, P_{2n} (including the last equation of the system 6.173) :

$$T_n(j,1)\alpha_{2n-1}P_{2n-1} + T_n(j,2)\alpha_{2n}P_{2n} = Q_n(j) \quad j = 1,2,3,4 \tag{6.185}$$

with

$$\begin{aligned} T_n(1,1) &= \sum_{k=1}^{3} \frac{(1+2\delta_{k,n})}{(N_{2n-1}+a_k^2)}(-1)^{k+1} a_k^2 h_k^2 - \frac{2}{(N_{2n-1}+a_j^2)}(-1)^{n+1} a_j^2 h_n^2 \\ T_n(1,2) &= \sum_{k=1}^{3} \frac{(1+2\delta_{k,n})}{(N_{2n}+a_k^2)}(-1)^{k+1} a_k^2 h_k^2 - \frac{2}{(N_{2n}+a_j^2)}(-1)^{n+1} a_j^2 h_n^2 \\ T_n(2,1) &= \frac{a_{n'}^2}{(N_{2n-1}+a_{n'}^2)} - \frac{a_{n''}^2}{(N_{2n-1}+a_{n''}^2)} \\ T_n(2,2) &= \frac{a_{n'}^2}{(N_{2n}+a_{n'}^2)} - \frac{a_{n''}^2}{(N_{2n}+a_{n''}^2)} \\ T_n(3,1) &= \sum_{k=1}^{3} \frac{(1+2\delta_{k,n})}{(N_{2n-1}+a_k^2)}(-1)^{k+1} a_k^2 h_k^2 (a_k^2 - a_n^2) \\ T_n(3,2) &= \sum_{k=1}^{3} \frac{(1+2\delta_{k,n})}{(N_{2n}+a_k^2)}(-1)^{k+1} a_k^2 h_k^2 (a_k^2 - a_n^2) \\ Q_n(1) &= \sum_{k=1}^{3} B_{k,k}^n (-1)^{k+1} a_k^2 h_k^2 - B_{n,j}^j (-1)^{n+1} a_j^2 h_n^2 \\ Q_n(2) &= (B_{n,n'}^{n'} a_{n'}^2 - B_{n,n''}^{n''} a_{n''}^2)/2 \\ Q_n(3) &= \sum_{k=1}^{3} B_{k,k}^n (-1)^{k+1} a_k^2 h_k^2 (a_k^2 - a_n^2) \end{aligned} \tag{6.186}$$

and for the last equation of the system 6.173, taking into account 6.121 and 6.172 :

$$T_n(4,1)=\frac{1}{\left(N_{2n-1}+a_{n'}^2\right)a_{n'}^2}-\frac{1}{\left(N_{2n-1}+a_{n'}^2\right)a_{n''}^2}$$
$$T_n(4,2)=\frac{1}{\left(N_{2n}+a_{n''}^2\right)a_{n'}^2}-\frac{1}{\left(N_{2n}+a_{n'}^2\right)a_{n''}^2} \qquad (6.187)$$
$$Q_n(4)=\frac{1}{2}\left(\frac{B_{n,n''}^{n''}}{a_{n'}^2}-\frac{B_{n,n'}^{n'}}{a_{n''}^2}\right)$$

We can immediately see that the equations for j=2 and j=4 are identical (j=2 is obtained from j=4 by multiplying the latter by $-a_{n'}^2a_{n''}^2$). The same applies to j=2 and j=3 (j=3 is obtained from j=2 by multiplying the latter by $h_{n'}^2h_{n''}^2$). This last equivalence is based on the relationship checked numerically:

$$B_{n',n'}^{n}a_{n'}^2-B_{n'',n''}^{n}a_{n''}^2=\left(B_{n,n'}^{n'}a_{n'}^2-B_{n,n''}^{n''}a_{n''}^2\right)/2 \qquad (6.188)$$

Equation 6.188 is none other than 6.163 for which we have given a partial proof.

The system 6.185 therefore reduces to j=1 and j=2, whose solution is :

$$P_{2n-1}=\frac{1}{\alpha_{2n-1}}\frac{Q_n(1)T_n(2,2)-T_n(1,2)Q_n(2)}{T_n(1,1)T_n(2,2)-T_n(1,2)T_n(2,1)}$$
$$P_{2n}=\frac{1}{\alpha_{2n}}\frac{Q_n(2)T_n(1,1)-T_n(2,1)Q_n(1)}{T_n(1,1)T_n(2,2)-T_n(1,2)T_n(2,1)} \qquad (6.189)$$

as a reminder,

$$\frac{7}{2}L_{3,2n-1}T_{N_{2n-1}}-K_{3,2n-1}=P_{2n-1}$$
$$\frac{7}{2}L_{3,2n}T_{N_{2n}}-K_{3,2n}=P_{2n}$$

P_7 is treated separately: for the first component of the cross product $n\wedge\nabla\left(V_2-V_2^{(i)}\right)\Big|_{\xi=0}\Big|_1$ we have (Eqs. 6.124 and 6.146) :

$$n\wedge\nabla\left(V_2-V_2^{(i)}\right)\Big|_{\xi=0}\Big|_1=-\frac{1}{h_0}\frac{1}{5h_1h_2h_3}\frac{h_1a_1}{\left(N_1-N_2\right)}\left(h_1h_2h_3\right)^2P_7\left[z_1S_3^1-z_2S_3^2\right]=-B_1^{T,1}S_3^1-B_2^{T,1}S_3^2$$
$$=\frac{1}{5h_1h_2h_3}\frac{h_1a_1}{\left(N_1-N_2\right)}\left[\frac{B_{1,2}^3}{N_1+a_2^2}-\frac{B_{1,3}^2}{N_1+a_3^2}\right]S_3^1-\frac{1}{5h_1h_2h_3}\frac{h_1a_1}{\left(N_1-N_2\right)}\left[\frac{B_{1,2}^3}{N_2+a_2^2}-\frac{B_{1,3}^2}{N_2+a_3^2}\right]S_3^2 \qquad (6.190)$$

By comparing/identifying the terms in S_3^1, S_3^2 :

$$P_7 = -\frac{1}{\left(h_1 h_2 h_3\right)^2}\left[\frac{B_{1,2}^3}{N_1 + a_2^2} - \frac{B_{1,3}^2}{N_1 + a_3^2}\right]\left[\frac{1}{\left(N_1 + a_2^2\right)} - \frac{1}{\left(N_1 + a_3^2\right)}\right]^{-1}$$

$$P_7 = -\frac{1}{\left(h_1 h_2 h_3\right)^2}\left[\frac{B_{1,2}^3}{N_2 + a_2^2} - \frac{B_{1,3}^2}{N_2 + a_3^2}\right]\left[\frac{1}{\left(N_2 + a_2^2\right)} - \frac{1}{\left(N_2 + a_3^2\right)}\right]^{-1}$$

Taking into account 6.162, these two last equations may be reduced to :

$$P_7 = -\frac{1}{\left(h_1 h_2 h_3\right)^2} B_{1,2}^3 \tag{6.191}$$

Processing the two other components leads to the same expression.

Let us remind that,

$$\frac{7}{2} L_{3,7} I_{1,2,3} - K_{3,7} = P_7$$

Scalar product (Eqs. 6.109, 6.112, 6.157) :

By comparing/identifying the terms S_3^m :

$$B_{2n-1}^S + L_{3,2n-1}\frac{7}{a_1 a_2 a_3 N_{2n-1} a_n} \qquad 1 \le n \le 3$$

$$= \frac{1}{5 h_1 h_2 h_3}\frac{h_n a_n}{N_{2n-1} - N_{2n}}\left[\frac{1}{a_n^2}\sum_{k=1}^{3}\frac{a_k^2\left(1 + 2\delta_{k,n}\right)B_{k,k}^n\left(e=1\right)}{\left(N_{2n-1} + a_k^2\right)} + \frac{B_{n,p}^p\left(1\right)}{N_{2n-1} + a_p^2} + \frac{B_{n,q}^q\left(1\right)}{N_{2n-1} + a_q^2} - \frac{A_n}{a_1 a_2 a_3 a_n^2}\right]$$

$$B_{2n}^S + L_{3,2n}\frac{7}{a_1 a_2 a_3 N_{2n} a_n}$$

$$= -\frac{1}{5 h_1 h_2 h_3}\frac{h_n a_n}{N_{2n-1} - N_{2n}}\left[\frac{1}{a_n^2}\sum_{k=1}^{3}\frac{a_k^2\left(1 + 2\delta_{k,n}\right)B_{k,k}^n\left(1\right)}{\left(N_{2n} + a_k^2\right)} + \frac{B_{n,p}^p\left(1\right)}{N_{2n} + a_p^2} + \frac{B_{n,q}^q\left(1\right)}{N_{2n} + a_q^2} - \frac{A_n}{a_1 a_2 a_3 a_n^2}\right]$$

$$B_7^S - L_{3,7}\frac{7}{\left(a_1 a_2 a_3\right)^2} = -\left(\frac{B_{2,3}^1\left(1\right)}{a_1^2} + \frac{B_{1,3}^2\left(1\right)}{a_2^2} + \frac{B_{1,2}^3\left(1\right)}{a_3^2}\right)\frac{a_1 a_2 a_3}{h_1^2 h_2^2 h_3^2}$$

(6.192)

One significant simplification concerns B_m^S (Eqs 6.109 and 6.103) ; indeed, using 6.81 and 6.116 :

$$B_{2n-1}^{S}=\frac{h_n a_n}{5h_1h_2h_3}\left(\frac{1}{N_{2n-1}-N_{2n}}\right)\times$$

$$\sum_i\left(2+\frac{a_i^2}{a_n^2}\right)\left(\alpha_{2n-1}P_{2n-1}^{\varepsilon}\frac{\left(1+2\delta_{n,i}\right)}{\left(N_{2n-1}+a_i^2\right)^2}+\alpha_{2n}P_{2n}^{\varepsilon}\left(\frac{\left(1+2\delta_{n,i}\right)}{\left(N_{2n}+a_i^2\right)}-\frac{\left(1+2\delta_{n,i}\right)}{\left(N_{2n-1}+a_i^2\right)}\right)\frac{1}{\left(N_{2n-1}-N_{2n}\right)}\right) \quad (6.193)$$

$$=\frac{h_n a_n}{5h_1h_2h_3}\left(\frac{1}{N_{2n-1}-N_{2n}}\right)\sum_i\left(2+\frac{a_i^2}{a_n^2}\right)\left(\frac{\left(1+2\delta_{n,i}\right)}{\left(N_{2n-1}+a_i^2\right)^2}\right)\quad\alpha_{2n-1}P_{2n-1}^{\varepsilon}$$

likewise,

$$B_{2n}^{S}=-\frac{h_n a_n}{5h_1h_2h_3}\left(\frac{1}{N_{2n-1}-N_{2n}}\right)\sum_i\left(2+\frac{a_i^2}{a_n^2}\right)\left(\frac{\left(1+2\delta_{n,i}\right)}{\left(N_{2n}+a_i^2\right)^2}\right)\quad\alpha_{2n}P_{2n}^{\varepsilon} \quad (6.194)$$

Equations 6.192-4 lead to

$$\begin{aligned}T_{L,2n-1}L_{3,2n-1}-\varepsilon T_{K,2n-1}K_{3,2n-1}&=T_{KL,2n-1}\\ T_{L,2n}L_{3,2n}-\varepsilon T_{K,2n}K_{3,2n}&=T_{KL,2n}\end{aligned} \quad (6.195)$$

with

$$T_{L,2n-1}=\frac{7}{2}\alpha_{2n-1}T_{N_{2n-1}}\sum_i\left(2+\frac{a_i^2}{a_n^2}\right)\left(\frac{\left(1+2\delta_{n,i}\right)}{\left(N_{2n-1}+a_i^2\right)^2}\right)+\frac{35h_1h_2h_3}{a_1a_2a_3a_n^2h_n}\frac{N_{2n-1}-N_{2n}}{N_{2n-1}}$$

$$T_{L,2n}=\frac{7}{2}\alpha_{2n}T_{N_{2n}}\sum_i\left(2+\frac{a_i^2}{a_n^2}\right)\left(\frac{\left(1+2\delta_{n,i}\right)}{\left(N_{2n}+a_i^2\right)^2}\right)-\frac{35h_1h_2h_3}{a_1a_2a_3a_n^2h_n}\frac{N_{2n-1}-N_{2n}}{N_{2n}}$$

$$T_{K,2n-1}=\alpha_{2n-1}\sum_i\left(2+\frac{a_i^2}{a_n^2}\right)\left(\frac{\left(1+2\delta_{n,i}\right)}{\left(N_{2n-1}+a_i^2\right)^2}\right)$$

$$T_{K,2n}=\alpha_{2n}\sum_i\left(2+\frac{a_i^2}{a_n^2}\right)\left(\frac{\left(1+2\delta_{n,i}\right)}{\left(N_{2n}+a_i^2\right)^2}\right)$$

$$T_{KL,2n-1}=\frac{1}{a_n^2}\sum_{k=1}^{3}\frac{a_k^2\left(1+2\delta_{k,n}\right)B_{k,k}^{n}\left(e=1\right)}{\left(N_{2n-1}+a_k^2\right)}+\frac{B_{n,p}^{p}\left(1\right)}{N_{2n-1}+a_p^2}+\frac{B_{n,q}^{q}\left(1\right)}{N_{2n-1}+a_q^2}-\frac{A_n}{a_1a_2a_3a_n^2}$$

$$T_{KL,2n}=\frac{1}{a_n^2}\sum_{k=1}^{3}\frac{a_k^2\left(1+2\delta_{k,n}\right)B_{k,k}^{n}\left(1\right)}{\left(N_{2n}+a_k^2\right)}+\frac{B_{n,p}^{p}\left(1\right)}{N_{2n}+a_p^2}+\frac{B_{n,q}^{q}\left(1\right)}{N_{2n}+a_q^2}-\frac{A_n}{a_1a_2a_3a_n^2} \quad (6.196)$$

likewise,

$$T_{L,7}L_{3,7}-\varepsilon T_{K,7}K_{3,7}=T_{KL,7} \quad (6.197)$$

with

$$T_{L,7} = \frac{7}{2} I_{1,2,3} \sum_i \frac{1}{a_i^2} - \frac{7}{\left(a_1 a_2 a_3\right)^3}$$

$$T_{K,7} = \sum_i \frac{1}{a_i^2} \qquad (6.198)$$

$$T_{KL,7} = -\left(\frac{B_{2,3}^1(1)}{a_1^2} + \frac{B_{1,3}^2(1)}{a_2^2} + \frac{B_{1,2}^3(1)}{a_3^2} \right) \frac{1}{h_1^2 h_2^2 h_3^2}$$

From equations 6.186, 6.189, 6.191 (coming from the cross product) and 6.195-6.198 (coming from the scalar product), we easily deduce analytic formula for the pairs $\left(L_{3,2n-1}, K_{3,2n-1}\right), \left(L_{3,2n}, K_{3,2n}\right), \left(L_{3,7}, K_{3,7}\right)$:

$$L_{3,m} = \left(T_{KL,m} - \varepsilon T_{K,m} P_m\right) / \left(T_{L,m} - \frac{7}{2} \varepsilon T_{N_m} T_{K,m} \right)$$

$$K_{3,m} = \frac{\frac{7}{2} T_{N_m} T_{KL,m} - P_m T_{L,m}}{\left(T_{L,m} - \frac{7}{2} \varepsilon T_{N_m} T_{K,m} \right)} \qquad m = 2n-1, 2n$$

$$L_{3,7} = \left(T_{KL,7} - \varepsilon T_{K,7} P_7\right) / \left(T_{L,7} - \frac{7}{2} \varepsilon I_{1,2,3} T_{K,7} \right) \qquad (6.199)$$

$$K_{3,7} = \frac{\frac{7}{2} I_{1,2,3} T_{KL,7} - P_7 T_{L,7}}{\left(T_{L,7} - \frac{7}{2} \varepsilon I_{1,2,3} T_{K,7} \right)}$$

To conclude this paragraph, let us compute $\int_{\partial E} n \bullet E_2 dS$:

$$\int_{\partial E} n \bullet E_2 dS = \int_{\partial E} n \bullet F_{2,G} dS + \int_{\partial E} n \bullet F_{2,U} dS + \int_{\partial E} n \bullet \nabla V_2 " dS + \int_{\partial E} n \bullet \nabla U_2 dS$$

$\int_{\partial E} n \bullet F_{2,G} dS, \int_{\partial E} n \bullet F_{2,U} dS$ are linear combinations of integrals as $\int_{\partial E} \frac{x_i x_j x_k}{h_0} dS$, $\int_{\partial E} \frac{x_i}{h_0} dS$, which are equal to zero.

on the other hand,

$$n \bullet \nabla V_2 "\big|_{\xi=0} = \sum_j \frac{2}{h_0}\frac{1}{10} A_j x_j \left(\left(x_1^2 + x_2^2 + x_3^2 \right) I_j '(0) + \left(-2I'(0) + \left(2a_j^2 - J_j \right) I_j '(0) \right) \right)$$

$$+ \sum_j \frac{1}{10} A_j \frac{x_j}{h_0 a_j^2} \left(\left(x_1^2 + x_2^2 + x_3^2 + 2x_j^2 \right) I_j(0) - 2I(0) + \left(2a_j^2 - J_j \right) I_j(0) \right)$$

We deduce that $\int_{\partial E} n \bullet \nabla V_2 "\, dS = 0$ for the same reasons.

$$\int_{\partial E} n \bullet \nabla V_2 dS = \int_{\partial E} n \bullet \nabla \left(\sum_{m=1}^{3} L_{1,m} \Upsilon_1^m + \sum_{m=1}^{7} L_{3,m} \Upsilon_3^m \right) dS$$

We therefore need to evaluate the integrals $\int_{\partial E} n \bullet \nabla \Upsilon_m^n dS \quad n = 1,3$. Using the definitions 6.87 and 6.88, we can see that all these integrals are also linear combinations of integrals as $\int_{\partial E} \frac{x_i x_j x_k}{h_0} dS$, $\int_{\partial E} \frac{x_i}{h_0} dS$. We deduce that $\int_{\partial E} n \bullet \nabla V_2 dS = 0$.

so, $\int_{\partial E} n \bullet E_2 dS = 0$. Here again, as for orders 0 and 1, the proposed solution verifies this equation, but does not need it to obtain it.

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