Critical dimensions for polyharmonic operators: The Pucci-Serrin conjecture for solutions of bounded energy

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CRITICAL DIMENSIONS FOR POLYHARMONIC OPERATORS: THE PUCCI-SERRIN CONJECTURE FOR SOLUTIONS OF BOUNDED ENERGY

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ABSTRACT. We prove a Pucci-Serrin conjecture on critical dimensions under a uniform bound on the energy. The method is based on the analysis of the Green's function of polyharmonic operators with "almost" Hardy potential

1. INTRODUCTION

Let B be the unit ball of $\mathbb{R}^n$ and let $k \in \mathbb{N}$ be such that $n > 2k \geq 2$. Consider $\lambda \in \mathbb{R}$ and $u \in C^{2k}(\overline{B})$ such that

$$\left\{ \begin{array}{cc} \Delta^k u - \lambda u = |u|^{2^\star - 2} u & \text{in } B \\ u = \partial_\nu u = ... = \partial_\nu^{k-1} u = 0 & \text{on } \partial B \end{array} \right\} \tag{1}$$

where $2^\star := \frac{2n}{n-2k}$. A very interesting conjecture of Pucci and Serrin ([20], p58) is stated as follows:

Conjecture 1.1. *Let B be the unit ball of $\mathbb{R}^n$ and let $k \in \mathbb{N}$ be such that $n > 2k \geq 2$. Assume that*

$$2k < n < 4k.$$

Then there exists $\lambda_0(n,k) > 0$ such that for all $0 < \lambda < \lambda_0(n,k)$, any radial solution to (1) is identically null.

Edmunds-Fortunato-Janelli [7] and Grunau [10] proved that there exists a positive radial solution to (1) for all $\lambda \in (0, \lambda_1)$ when $n > 4k$, where $\lambda_1 > 0$ is the first eigenvalue of Δ^k on B with Dirichlet boundary condition. In particular, the expected range $(2k, 4k)$ is optimal. In this paper, we prove the following:

Theorem 1.1. *Let B be the unit ball of $\mathbb{R}^n$ and let $k \in \mathbb{N}$ be such that $n > 2k \geq 2$. Assume that*

$$2k < n < 4k.$$

Then, for any $M > 0$, there exists $\lambda_0(n,k,M) > 0$ such that for all $0 < \lambda < \lambda_0(n,k,M)$, any radial solution to (1) satisfying that $\|u\|_{2^\star} \leq M$ is identically null.

Concerning terminology, Pucci-Serrin defined that a dimension $n > 2k$ is *critical* if there exists $\lambda_0(n,k) > 0$ such that any radial solution of (1) is identically null when $0 < \lambda < \lambda_0(n,k)$. Theorem 1.1 proves the conjecture under any arbitrary fixed bound on the Lebesgue's norm.

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Here is a brief history of the problem. The conjecture turns to be true in the following situations:

- $k = 1$ (Brézis-Nirenberg [4]);
- $k = 2$ (Pucci-Serrin [20]);
- $k \geq 2$ et $n = 2k + 1$ (Pucci-Serrin [20]);
- $k \geq 2$ et $2k < n < 2k + 6$ (Bernis-Grunau [2] and Grunau [11]).

In these situations, the proofs are based on Pohozaev-type identities for radial functions. The larger k is, the trickier and longer the computations are and achieving $n < 2k+6$ is a true "tour de force". Moreover, beside the computational difficulties, the methods in these papers do not seem enough to tackle the full conjecture (see Grunau [11] for discussions on this issue).

The case of positive functions is interesting in itself. Grunau [12] proved the validity of the conjecture when restricted to positive functions (*weakly critical dimensions*). In this situation, the key is to test a solution u to (1) against a carefully chosen positive polyharmonic function on $\overline{B}$. The case of arbitrary sign-changing solutions involved in the original conjecture, the one we address here, is much more involved.

As a final remark, we mention that Jannelli [14] has formalized the notion of *critical dimensions* in a more general setting by connecting it to the L^2-integrability of the Green's function.

In the present paper, we adopt a new approach that is based on the concentration analysis of families of solutions to (1): this permits to develop a method that is uniform and independent of the value of the power k. This approach is particularly relevant due to the critical exponent $2^\star$ that may tolerate an unbounded family of solutions as $\lambda \to 0$: in this situation, this family should concentrate along explicit profiles referred to as bubbles. The general theory for second-order problems ($k = 1$) has been performed in Druet-Hebey-Robert [5] for positive solutions and was based on the comparison principle, see also Hebey [13] for a modern point of view on such issues. We refer also to Druet-Laurain [6] regarding a method for positive solutions and to Premoselli [18] for a more recent and promising approach for sign-changing solutions.

Due to the sign-change and to the lack of comparison principle when $k \geq 2$, we develop tools based on Green's representation formula for a linear equations. More precisely, we rewrite (1) as $Pu = 0 + \{\text{bdy conditions}\}$ where $P = \Delta^k - \lambda - |u|^{2^\star - 2}$ and we express u in terms of the Green's function of P. The core and the bulk of our analysis is to get a sharp pointwise control of this Green's function, which is the object of Theorem 5.2. This control is based on the regularity Lemma 6.1 for solutions to linear equations with "almost" Hardy-type potential.

Most of the analysis is valid for any elliptic operator like $\Delta^k + ...$: the restriction $n < 4k$ and the specificity of $\Delta^k - \lambda$ are used only for the final argument involving the Pohozaev-Pucci-Serrin identity. We will make an intensive use of the elliptic regularity of the reference Agmon-Douglis-Nirenberg [1]. For the convenience of the reader, the last section 7 is a collection of results contained in [1].

Notations: $C(a, b, ...)$ will denote any constant depending only on $a, b, ...$ The same notation might refer different constants from line to line, and even in the same line.

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2. Preliminary analysis

We prove Theorem 1.1 by contradiction. We fix $M > 0$. If Theorem 1.1 is not true, then there exists a sequence $(\lambda_i)_{i\in\mathbb{N}} \in \mathbb{R}_{>0}$ and $(u_i)_{i\in\mathbb{N}} \in C^{2k}(\overline{B})$ radially symmetrical such that

$$\left\{\begin{array}{cc} \Delta^k u_i - \lambda_i u = |u_i|^{2^\star-2}u_i & \text{in } B \\ u_i = \partial_\nu u_i = ... = \partial_\nu^{k-1} u_i = 0 & \text{on } \partial B \\ u_i \not\equiv 0 & \\ \|u_i\|_{2^\star} \leq M & \\ \lim_{i\to\infty} \lambda_i = 0 & \end{array}\right\} \tag{2}$$

In order to simplify the exposition, we assume that there exists $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$, there exists $u_\lambda \in C^{2k}(\overline{B})$ radially symmetrical such that

$$\left\{\begin{array}{cc} \Delta^k u_\lambda - \lambda u_\lambda = |u_\lambda|^{2^\star-2}u_\lambda & \text{in } B \\ u_\lambda = \partial_\nu u_\lambda = ... = \partial_\nu^{k-1} u_\lambda = 0 & \text{on } \partial B \\ u_\lambda \not\equiv 0 & \\ \|u_\lambda\|_{2^\star} \leq M & \end{array}\right\} \tag{3}$$

We are performing an analysis of u_λ as $\lambda \to 0$. All the results and statements will be up to the extraction of subfamilies, although we will always refer to u_λ. A preliminary remark is that $u_\lambda \in C^{2k+1,\theta}(B)$, $0 < \theta < 1$, due to elliptic regularity.

2.1. **Sobolev spaces and inequalities.** For any $\Omega \subset \mathbb{R}^n$ a smooth domain, $p \geq 1$ and $l \in \mathbb{N}$, we define $H^p_l(\Omega)$ (resp. $H^p_{l,0}(\Omega)$) as the completion of $\{u \in C^\infty(\Omega) \text{ s.t. } \|u\|_{H^p_l} < \infty\}$ (resp. $C^\infty_c(\Omega)$) for the norm $u \mapsto \|u\|_{H^p_l} := \sum_{i\leq l} \|\nabla^i u\|_p$. Given a finite set $S \subset \Omega$, we define $L^p_{loc}(\Omega\backslash S) = \{u : \Omega \to \mathbb{R} \text{ s.t. } \eta u \in L^p(\Omega) \text{ for all } \eta \in C^\infty_c(\mathbb{R}^n \setminus S)\}$, $H^p_{l,loc}(\Omega \setminus S) = \{u : \Omega \to \mathbb{R} \text{ s.t. } \eta u \in H^p_l(\Omega) \text{ for all } \eta \in C^\infty_c(\mathbb{R}^n \setminus S)\}$ and $H^p_{l,0,loc}(\Omega \setminus S) = \{u : \Omega \to \mathbb{R}/ \text{ s.t. } \eta u \in H^p_{l,0}(\Omega) \text{ for all } \eta \in C^\infty_c(\mathbb{R}^n \setminus S)\}$. This notation is a bit abusive since $\Omega \setminus S$ is open, but there will be no ambiguity in this paper. In the specific case $p = 2$ and Ω is bounded, note that on $H^2_{k,0}(\Omega)$, $\|\cdot\|_{H^2_k}$ is equivalent to $u \mapsto \|\Delta^{k/2}u\|_2$. Here and in the sequel, $\Delta^{\frac{i}{2}} = \nabla\Delta^{\frac{i-1}{2}}$ when i is odd. Note that for $u \in C^{2k}(\overline{\Omega})$ and Ω a smooth bounded domain of $\mathbb{R}^n$ or Ω is a half-space, then $\{u \in H^2_{k,0}(\Omega)\} \Leftrightarrow \{u = \partial_\nu u = ... = \partial_\nu^{k-1} u = 0 \text{ on } \partial\Omega\}$.

We let $D^2_k(\mathbb{R}^n)$ be the completion of $C^\infty_c(\mathbb{R}^n)$ for the norm $u \mapsto \|\Delta^{k/2}u\|_2$. It follows from Sobolev's theorem that there exists $K(n,k) > 0$ such that

$$\left(\int_{\mathbb{R}^n} |u|^{2^\star}\,dx\right)^{\frac{2}{2^\star}} \leq K(n,k)\int_{\mathbb{R}^n} (\Delta^{\frac{k}{2}}u)^2\,dx \text{ for all } u \in D^2_k(\mathbb{R}^n). \tag{4}$$

As one checks, this inequality is a also valid for all $u \in H^2_{k,0}(\Omega)$.

Lemma 2.1. *Let $(u_\lambda)_{\lambda>0} \in C^{2k}(\overline{B})$ be a family radially symmetrical solution to (3). Then $\lim_{\lambda\to 0} \|u_\lambda\|_\infty = +\infty$.*

Proof. We argue by contradiction. If the conclusion does not hold, then there exists $C > 0$ such that $\|u_\lambda\|_\infty \leq C$ for all $\lambda > 0$. It follows from elliptic theory (Theorems 7.1 and 7.2) that $\|u_\lambda\|_{C^{2k,1/2}} \leq C$ for all $\lambda > 0$. It then follows from Ascoli's theorem that there exists $u_0 \in C^{2k}(\overline{B})$ such that $\lim_{\lambda\to 0} u_\lambda = u_0$ in $C^{2k}(\overline{B})$. Passing to the

limit in (3) yields

$$\left\{ \begin{array}{cc} \Delta^k u_0 = |u_0|^{2^\star-2}u_0 & \text{in } B \\ u_0 = \partial_\nu u_0 = ... = \partial_\nu^{k-1} u_0 = 0 & \text{on } \partial B \end{array} \right\} \tag{5}$$

It then follows from Lazzo-Schmidt (point (a) of Corollary 3.10 of [15]) that $u_0 \equiv 0$. Multiplying (3) by u_λ, integrating by parts and using Hölder's inequality yield

$$\int_B (\Delta^{k/2} u_\lambda)^2 \, dx = \int_B u_\lambda \Delta^k u_\lambda \, dx = \lambda \int_B u_\lambda^2 \, dx + \int_B |u_\lambda|^{2^\star} \, dx \leq C\lambda \|u_\lambda\|_{2^\star}^2 + \|u_\lambda\|_{2^\star}^{2^\star}.$$

With the Sobolev inequality (4) and using that $u_\lambda \not\equiv 0$ and $u_\lambda \in H^2_{k,0}(B)$, we get that $K(n,k)^{-1} \leq C\lambda + \|u_\lambda\|_{2^\star}^{2^\star-2}$. Passing to the limit $\lambda \to 0$ and using that $u_0 \equiv 0$, we get a contradiction. This proves the Lemma. □

Note that as a consequence of the preceding argument, $(u_\lambda)_\lambda$ is bounded in $H^2_{k,0}(B)$, that is there exists $C(M) > 0$ such that $\|u_\lambda\|_{H^2_k} \leq C(M)$ for all $\lambda > 0$.

Lemma 2.2. *Let $(y_\lambda)_\lambda \in B$ and $(r_\lambda)_{\lambda>0} \in \mathbb{R}_{>0}$ be such that $\lim_{\lambda\to 0} r_\lambda^{-1}|y_\lambda| = +\infty$. Then*

$$\lim_{\lambda\to 0} \int_{B_{r_\lambda}(y_\lambda)\cap B} |u_\lambda|^{2^\star} \, dx = 0.$$

Proof. Let us fix $N \in \mathbb{N}$. There exists a group of isometries of $\mathbb{R}^n$, say G, such that $\sharp G \geq N$ and there exists $\epsilon_N > 0$ such that $d(\sigma(e_1), \tau(e_1)) \geq \epsilon_N$ for all $\sigma, \tau \in G$, $\sigma \neq \tau$. Here, e_1 is the first vector of the canonical basis of $\mathbb{R}^n$. Therefore, as one checks, $B_{r_\lambda}(\sigma(y_\lambda)) \cap B_{r_\lambda}(\tau(y_\lambda)) = \emptyset$ for all $\sigma, \tau \in G$, $\sigma \neq \tau$ and $\lambda > 0$ is small enough. With the invariance of u_λ under the action of the group G, we get that

$$\begin{aligned} M^{2^\star} &\geq \int_B |u_\lambda|^{2^\star} \, dx \geq \int_{\bigcup_{\sigma\in G} B_{r_\lambda}(\sigma(y_\lambda))\cap B} |u_\lambda|^{2^\star} \, dx \\ &\geq \sum_{\sigma\in G} \int_{B_{r_\lambda}(\sigma(y_\lambda))\cap B} |u_\lambda|^{2^\star} \, dx = \sharp G \int_{B_{r_\lambda}(y_\lambda)\cap B} |u_\lambda|^{2^\star} \, dx \end{aligned}$$

and therefore

$$\int_{B_{r_\lambda}(y_\lambda)\cap B} |u_\lambda|^{2^\star} \, dx \leq \frac{M^{2^\star}}{N} \text{ as } \lambda \to 0.$$

Since this is valid for all N, the conclusion follows. □

Lemma 2.3. *Let $(u_\lambda)_{\lambda>0} \in C^{2k}(\overline{B})$ be a family radially symmetrical solution to (3). Then there exists $C > 0$ such that $|x|^{\frac{n-2k}{2}}|u_\lambda(x)| \leq C$ for all $x \in B$ and $\lambda \to 0$.*

Proof. We prove the lemma by contradiction. We set $w_\lambda(x) := |x|^{\frac{n-2k}{2}}|u_\lambda(x)|$ for all $x \in B$ and $\lambda > 0$. Let us assume that

$$w_\lambda(y_\lambda) := \sup_{x\in B} w_\lambda(x) \to +\infty \text{ as } \lambda \to 0.$$

We define $r_\lambda := |u_\lambda(y_\lambda)|^{-\frac{2}{n-2k}}$. We have that

$$\frac{|y_\lambda|}{r_\lambda} = w_\lambda(y_\lambda)^{\frac{2}{n-2k}} \to \infty \text{ and } r_\lambda \to 0 \text{ as } \lambda \to 0. \tag{6}$$

Case 1: assume that

$$\lim_{\lambda\to 0} \frac{d(y_\lambda, \partial B)}{r_\lambda} = +\infty. \tag{7}$$

We define

$$v_\lambda(x) := r_\lambda^{\frac{n-2k}{2}} u_\lambda(y_\lambda + r_\lambda x) \text{ for } x \in \frac{B - y_\lambda}{r_\lambda}.$$

A change of variable in (3) yields

$$\Delta^k v_\lambda - \lambda r_\lambda^{2k} v_\lambda = |v_\lambda|^{2^\star - 2} v_\lambda \text{ in } \frac{B - y_\lambda}{r_\lambda}. \tag{8}$$

It follows from the definition of y_λ that

$$|y_\lambda + r_\lambda x|^{\frac{n-2k}{2}} |u_\lambda(y_\lambda + r_\lambda x)| \le |y_\lambda|^{\frac{n-2k}{2}} |u_\lambda(y_\lambda)| \text{ for } x \in \frac{B - y_\lambda}{r_\lambda},$$

and then

$$\left| \frac{y_\lambda}{|y_\lambda|} + \frac{r_\lambda}{|y_\lambda|} x \right|^{\frac{n-2k}{2}} |v_\lambda(x)| \le 1 \text{ for } x \in \frac{B - y_\lambda}{r_\lambda}.$$

We fix $R > 0$. It follows from (7) and the above inequality that there exists $\lambda_R > 0$ such that

$$B_R(0) \subset \frac{B - y_\lambda}{r_\lambda} \text{ and } |v_\lambda(x)| \le 2 \text{ for all } x \in B_R(0) \text{ and } 0 < \lambda < \lambda_R.$$

With (8), it then follows from elliptic theory (Theorems 7.1 and 7.2) and Ascoli's theorem that there exists $v \in C^{2k}(\mathbb{R}^n)$ such that $\lim_{\lambda \to 0} v_\lambda = v$ in $C^{2k}_{loc}(\mathbb{R}^n)$. Given $R > 0$, with a change of variable, we get that

$$\int_{B_R(0)} |v_\lambda|^{2^\star} \, dx = \int_{B_{Rr_\lambda}(y_\lambda)} |u_\lambda|^{2^\star} \, dx.$$

It follows from Lemma 2.2 and (6) that passing to the limit yields $\int_{B_R(0)} |v|^{2^\star} \, dx = 0$ for all $R > 0$, so that $v \equiv 0$ since it is continuous. However, since $|v_\lambda(0)| = 1$, we get that $|v(0)| = 1$, which contradicts $v \equiv 0$. This ends Case 1.

Case 2:

$$\lim_{\lambda \to 0} \frac{d(y_\lambda, \partial B)}{r_\lambda} = \rho \in [0, +\infty).$$

Up to a rotation, we then get that

$$\lim_{\lambda \to 0} \frac{B - y_\lambda}{r_\lambda} = (-\infty, \rho) \times \mathbb{R}^{n-1}.$$

The proof is then similar to Case 1 by working on this half-space. We leave the details to the reader. This yields also to a contradiction.

In both cases, we have gotten a contradiction, which proves the Lemma. □

Lemma 2.4. *Let $(u_\lambda)_{\lambda > 0} \in C^{2k}(\overline{B})$ be a family radially symmetrical solution to (3). Then $\lim_{\lambda \to 0} u_\lambda = 0$ in $C^{2k}_{loc}(\overline{B} \setminus \{0\})$.*

Proof. It follows from Lemma 2.3 that for all $\delta > 0$, there exists $C(\delta) > 0$ such that $|u_\lambda(x)| \le C(\delta)$ for all $\lambda > 0$ and $x \in B \setminus B_\delta(0)$. It follows from elliptic theory (Theorems 7.1 and 7.2) and Ascoli's theorem that there exists $u_0 \in C^{2k}(\overline{B} \setminus \{0\})$ such that $\lim_{\lambda \to 0} u_\lambda = u_0$ in $C^{2k}_{loc}(\overline{B} \setminus \{0\})$. Since $\|u_\lambda\|_{H^2_k} \le C(M)$ for all $\lambda > 0$, we also get that $u_0 \in H^2_{k,0}(B)$ and $u_\lambda \rightharpoonup u_0$ weakly in $H^2_{k,0}(B)$. Passing to the limit $\lambda \to 0$ in (3), we get that u_0 is a weak solution to (5). Regularity theory (see Van der Vorst [23] and Theorems 7.1 and 7.2) yields $u_0 \in C^{2k}(\overline{B})$ is a strong solution to (5), and then $u_0 \equiv 0$ by [15] since it is radial. This proves the Lemma. □

We will make use of the following classification:

Theorem 2.1 (Swanson [22]). *Let $k, n \in \mathbb{N}$ be such $2 \leq 2k < n$. Let $u \in D^2_k(\mathbb{R}^n)$ be a distributional solution to $\Delta^k u = |u|^{2^\star-2}u$ in $\mathbb{R}^n$. Assume that u is radially symmetric. Then there exists $\mu > 0$ and $\epsilon \in \{-1, 0, +1\}$ such that*

$$u(x) = \epsilon \left(\frac{\mu}{\mu^2 + a_{n,k}|x|^2} \right)^{\frac{n-2k}{2}}, \text{ where } a_{n,k} := \left(\Pi_{j=-k}^{k-1}(n+2j) \right)^{-\frac{1}{k}}.$$

Proof. Although Swanson's Theorem 4 in [22] is only stated for positive functions, the proof is working for any functions. More precisely, if $u(0) \neq 0$, we follow exactly Swanson's proof. If $u(0) = 0$, the arguments of Swanson (Lemma 7) yield $u \equiv 0$. □

Lemma 2.5. *Let $(y_\lambda)_\lambda \in B$ be such that $\lim_{\lambda\to 0} |y_\lambda|^{\frac{n-2k}{2}} |u_\lambda(y_\lambda)| = c \in (0, +\infty)$. Then there exists $(r_\lambda)_\lambda \in (0, +\infty)$ such $\lim_{\lambda\to 0} r_\lambda = 0$, $\lim_{\lambda\to 0} r_\lambda^{-1}|y_\lambda| = c' \in (0, +\infty)$ and*

$$\lim_{\lambda\to 0} r_\lambda^{\frac{n-2k}{2}} u_\lambda(r_\lambda \cdot) = \epsilon U \text{ in } C^{2k}_{loc}(\mathbb{R}^n \setminus \{0\}), \tag{9}$$

for some $\epsilon \in \{-1, +1\}$ where

$$U(x) = \left(\frac{1}{1 + a_{n,k}|x|^2} \right)^{\frac{n-2k}{2}} \text{ for all } x \in \mathbb{R}^n. \tag{10}$$

Proof. It follows from Lemma 2.4 that $y_\lambda \to 0$ as $\lambda \to 0$. We set $s_\lambda := |y_\lambda|$ and we define $W_\lambda(x) := s_\lambda^{\frac{n-2k}{2}} u_\lambda(s_\lambda x)$ for $x \in B_{1/s_\lambda}(0)$ and $\lambda > 0$. Lemma 2.3 yields

$$|W_\lambda(x)| \leq C|x|^{-\frac{n-2k}{2}} \text{ for all } x \in B_{1/s_\lambda}(0) \text{ and } \lambda > 0. \tag{11}$$

A change of variable in (3) yields

$$\Delta^k W_\lambda - \lambda s_\lambda^{2k} W_\lambda = |W_\lambda|^{2^\star-2} W_\lambda \text{ in } B_{1/s_\lambda}(0). \tag{12}$$

Due to elliptic theory (Theorems 7.1 and 7.2) and Ascoli's theorem, (11) and (12) yield the existence of $W \in C^{2k}(\mathbb{R}^n \setminus \{0\})$ such that

$$\lim_{\lambda\to 0} W_\lambda = W \text{ in } C^{2k}_{loc}(\mathbb{R}^n \setminus \{0\}).$$

Since $W_\lambda\left(\frac{y_\lambda}{|y_\lambda|}\right) = |y_\lambda|^{\frac{n-2k}{2}} u_\lambda(y_\lambda)$, passing to the limit $\lambda \to 0$ yields $|W(Y_0)| = c > 0$ where $Y_0 := \lim_{\lambda\to 0} \frac{y_\lambda}{|y_\lambda|}$. Therefore $W \not\equiv 0$.

We prove that $W \in D^2_k(\mathbb{R}^n)$. Let us fix $l \in \{0, ..., k\}$. It follows from Sobolev's embedding that there exists $C(l, k, n) > 0$ such that

$$\left(\int_B |\nabla^l \varphi|^{2^\star(l)} \, dx \right)^{\frac{2}{2^\star(l)}} \leq C(l, k, n) \int_B (\Delta^{k/2}\varphi)^2 \, dx \tag{13}$$

for all $\varphi \in H^2_{k,0}(B)$, where $2^\star(l) := \frac{2n}{n-2(k-l)}$. Given $R > 0$, with a change of variable, we get

$$\begin{aligned} \left(\int_{B_R(0)\setminus B_{R^{-1}}(0)} |\nabla^l W_\lambda|^{2^\star(l)} \, dx \right)^{\frac{2}{2^\star(l)}} &= \left(\int_{B_{Rr_\lambda}(0)\setminus B_{R^{-1}r_\lambda}(0)} |\nabla^l u_\lambda|^{2^\star(l)} \, dx \right)^{\frac{2}{2^\star(l)}} \\ &\leq C(l, k, n) \int_B (\Delta^{k/2} u_\lambda)^2 \, dx \leq C \end{aligned}$$

since $\|u_\lambda\|_{H^2_k}$ is uniformly bounded. Letting $\lambda \to 0$ and $R \to +\infty$ yields $|\nabla^l W| \in L^{2^\star(l)}(\mathbb{R}^n)$. We now let $\eta \in C^\infty_c(\mathbb{R}^n)$ be such that $\eta(x) = 1$ for $x \in B_1(0)$ and $\eta(x) = 0$ for $x \in \mathbb{R}^n \setminus B_2(0)$. For $R > 0$, we define $W_R(x) := (1 - \eta(Rx))\,\eta(\frac{x}{R})W(x)$ for all $x \in \mathbb{R}^n$. Since $|\nabla^l W| \in L^{2^\star(l)}(\mathbb{R}^n)$ for all $l \in \{0, ..., k\}$, one gets that $(W_R)_R$ is a Cauchy family in $D^2_k(\mathbb{R}^n)$ as $R \to +\infty$, so it has a limit in $D^2_k(\mathbb{R}^n)$ as $R \to +\infty$, and then $W \in D^2_k(\mathbb{R}^n)$. So Theorem 2.1 yields the existence of $t > 0$ and $\epsilon \in \{-1, +1\}$ such that

$$W(x) = \epsilon \left(\frac{t}{t^2 + a_{n,k}|x|^2} \right)^{\frac{n-2k}{2}} \text{ for all } x \in \mathbb{R}^n.$$

Therefore, setting $r_\lambda := ts_\lambda$, we get the conclusion of the Lemma. □

3. Sharp analysis at the furthest scale

Proposition 3.1. *Let $(u_\lambda)_\lambda \in C^{2k}(\overline{B})$ be a family of solutions to (3). Then there exists $(\nu_\lambda)_\lambda \in (0, +\infty)$ and $\epsilon_0 \in \{-1, +1\}$ such that*

$$\begin{aligned} &\lim_{\lambda \to 0} \nu_\lambda = 0; \\ &\lim_{\lambda \to 0} \nu_\lambda^{\frac{n-2k}{2}} u_\lambda(\nu_\lambda \cdot) = \epsilon_0 U \text{ in } C^{2k}_{loc}(\mathbb{R}^n \setminus \{0\}); \\ &\lim_{R \to +\infty} \lim_{\lambda \to 0} \sup_{x \in B \setminus B_{R\nu_\lambda}(0)} |x|^{\frac{n-2k}{2}} |u_\lambda(x)| = 0. \end{aligned} \tag{14}$$

Proof. Given $N \geq 1$, we say that $(\mathcal{H}_N)$ holds if there exists $(\mu_{\lambda,1})_\lambda, ..., (\mu_{\lambda,N})_\lambda \in (0, +\infty)$ such that

$$\lim_{\lambda \to 0} \frac{\mu_{\lambda,i}}{\mu_{\lambda,i+1}} = 0 \text{ for all } i = 1, ..., N-1 \text{ and } \lim_{\lambda \to 0} \mu_{\lambda,N} = 0,$$

and that for all $i \in \{1, ..., N\}$, there exists $\epsilon_i \in \{-1, +1\}$ such that

$$\lim_{\lambda \to 0} v_{\lambda,i} = \epsilon_i U \text{ in } C^{2k}_{loc}(\mathbb{R}^n \setminus \{0\}) \text{ where } v_{\lambda,i}(x) := \mu_{\lambda,i}^{\frac{n-2k}{2}} u_\lambda(\mu_{\lambda,i} x) \text{ for all } x \in B_{1/\mu_{\lambda,i}}(0),$$

while for $i = 1$, this convergence holds in $C^{2k}_{loc}(\mathbb{R}^n)$.

Step 1: We claim that $(\mathcal{H}_1)$ holds.

We prove the claim. We define $x_\lambda \in B$ and $\mu_{\lambda,1} := \mu_\lambda := |u_\lambda(x_\lambda)|^{-\frac{2}{n-2k}}$ where $|u_\lambda(x_\lambda)| = \sup_B |u_\lambda|$. We define

$$U_\lambda(x) := \mu_\lambda^{\frac{n-2k}{2}} u_\lambda(\mu_\lambda x) \text{ for all } x \in B_{1/\mu_\lambda}(0). \tag{15}$$

It then follows from elliptic theory (Theorems 7.1 and 7.2) that there exists $\tilde{U} \in C^{2k}(\mathbb{R}^n)$ such that $\lim_{\lambda \to 0} U_\lambda = \tilde{U}$ in $C^{2k}_{loc}(\mathbb{R}^n)$ and $\Delta^k \tilde{U} = |\tilde{U}|^{2^\star - 2}\tilde{U}$. The definiton of μ_λ and Lemma 2.3 yield $|x_\lambda| \leq C\mu_\lambda$, so there exists $X_0 \in \mathbb{R}^n$ such that $\lim_{\lambda \to 0} \frac{x_\lambda}{\mu_\lambda} = X_0$. We have that $|U_\lambda(\frac{x_\lambda}{\mu_\lambda})| = 1$, so that, letting $\lambda \to 0$ yields $|\tilde{U}(X_0)| = 1$. Therefore $\tilde{U} \not\equiv 0$ and $|\tilde{U}| \leq |\tilde{U}(x_0)| = 1$. As in Lemma 2.5, we get that $\tilde{U} \in D^2_k(\mathbb{R}^n)$ and Theorem 2.1 yields the conclusion.

Step 2: Assume that $(\mathcal{H}_N)$ holds for some $N \geq 1$ and that

$$\lim_{R \to +\infty} \lim_{\lambda \to 0} \sup_{x \in B \setminus B_{R\mu_{\lambda,N}}(0)} |x|^{\frac{n-2k}{2}} |u_\lambda(x)| > 0. \tag{16}$$

Then $(\mathcal{H}_{N+1})$ holds.

We prove the claim. It follows from (16) that there exists $(y_\lambda)_\lambda \in B$ such that $\lim_{\lambda\to 0} \frac{|y_\lambda|}{\mu_{\lambda,N}} = +\infty$ and $\lim_{\lambda\to 0} |y_\lambda|^{\frac{n-2k}{2}} |u_\lambda(y_\lambda)| = c > 0$. We define $\mu_{\lambda,N+1} := r_\lambda$, where $r_\lambda > 0$ is given by Lemma 2.5. As one checks, we get that $(\mathcal{H}_{N+1})$ holds. The claim is proved.

Step 3: We claim that there exists $C(M,n,k) > 0$ such that if $(\mathcal{H}_N)$ holds, then $N \le C(M,n,k)$.

We prove the claim. For any $i \in \{1,...,N\}$, we get that

$$\begin{aligned}\lim_{R\to+\infty}\lim_{\lambda\to 0}\int_{B_{R\mu_{\lambda,i}}(0)\setminus B_{R^{-1}\mu_{\lambda,i}}(0)} |u_\lambda|^{2^\star}\,dx &= \lim_{R\to+\infty}\lim_{\lambda\to 0}\int_{B_R(0)\setminus B_{R^{-1}}(0)} |v_{\lambda,i}|^{2^\star}\,dx \\ &= \int_{\mathbb{R}^n} U^{2^\star}\,dx\end{aligned}$$

Since the N domains $B_{R\mu_{\lambda,i}}(0)\setminus B_{R^{-1}\mu_{\lambda,i}}(0)$ are distinct for $\lambda\to 0$, we get that

$$\sum_{i=1}^N \int_{B_{R\mu_{\lambda,i}}(0)\setminus B_{R^{-1}\mu_{\lambda,i}}(0)} |u_\lambda|^{2^\star}\,dx = \int_{\bigcup_i B_{R\mu_{\lambda,i}}(0)\setminus B_{R^{-1}\mu_{\lambda,i}}(0)} |u_\lambda|^{2^\star}\,dx \le \int_B |u_\lambda|^{2^\star}\,dx \le M^{2^\star}.$$

And then $N \le C(M,n,k)$ with $C(M,n,k) := \frac{M^{2^\star}}{\int_{\mathbb{R}^n} U^{2^\star}\,dx}$. This proves the claim.

Step 4: We conclude the proof of the Proposition. We let $N \ge 1$ be maximal such that $(\mathcal{H}_N)$ holds: the existence follows from Step 2. It follows from Step 1 that

$$\lim_{R\to+\infty}\lim_{\lambda\to 0}\sup_{x\in B\setminus B_{R\mu_{\lambda,N}}(0)} |x|^{\frac{n-2k}{2}}|u_\lambda(x)| = 0.$$

Therefore Proposition 3.1 follows by taking $\nu_\lambda := \mu_{\lambda,N}$. □

Proposition 3.2. *Let $(u_\lambda)_\lambda \in C^{2k}(\overline{B})$ be a family of solutions to (3), and let $(\nu_\lambda)_\lambda$ be as in Proposition 3.1. Then for any $\gamma \in (0, n-2k)$, there exists $C > 0$ such that*

$$|u_\lambda(x)| \le C\frac{\nu_\lambda^{\frac{n-2k}{2}-\gamma}}{|x|^{n-2k-\gamma}} \text{ for all } x \in B\setminus B_{\nu_\lambda}(0) \text{ and } \lambda\to 0. \tag{17}$$

Proof. We fix $R > 0$ and we define $V_\lambda := \mathbf{1}_{B\setminus B_{R\nu_\lambda}(0)}|u_\lambda|^{2^\star-2}$ so that

$$(\Delta^k - \lambda - V_\lambda)u_\lambda = 0 \text{ in } B\setminus B_{2R\nu_\lambda}(0).$$

Let $\mu_\gamma > 0$ be as in the statement of Theorem 5.2. It follows from Proposition 3.1 that there exists $R = R_\gamma > 0$ such that

$$|X|^{2k}|\lambda + V_\lambda(x)| \le \mu_\gamma \text{ for all } x \in B \text{ and } \lambda > 0 \text{ small enough.}$$

We let G_λ be the Green's function for $\Delta^k - \lambda - V_\lambda$ on B with Dirichlet boundary condition given by Theorem 5.1 with the pointwise controls of Theorem 5.2. We choose $x \in B$ such that $|x| > 3R\nu_\lambda$. Since $(\Delta^k - \lambda - V_\lambda)u_\lambda = 0$, we get that

$$\begin{aligned} u_\lambda(x) &= \int_{B\setminus B_{2R\nu_\lambda}(0)} G_\lambda(x,\cdot)(\Delta^k-\lambda-V_\lambda)u_\lambda\,dy \\ &\quad+\int_{\partial(B\setminus B_{2R\nu_\lambda}(0))}\sum_{i=0}^{k-1}\left(\partial_\nu\Delta^i u_\lambda\Delta^{k-1-i}G_\lambda(x,\cdot)-\Delta^i u_\lambda\partial_\nu\Delta^{k-1-i}G_\lambda(x,\cdot)\right)d\sigma \\ &= \sum_{i=0}^{k-1}\int_{\partial B_{2R\nu_\lambda}(0)}\nabla^{1+2i}u_\lambda\star\nabla_y^{2(k-1-i)}G_\lambda(x,\cdot)+\nabla^{2i}u_\lambda\star\nabla_y^{1+2(k-1-i)}G_\lambda(x,\cdot) \end{aligned}$$

where $T\star S$ denotes any linear combination of contractions of the tensors T and S. For all $j=0,...,2k-1$, it follows from the convergence (14) that

$$|\nabla^j u_\lambda(y)|\le C\nu_\lambda^{-\frac{n-2k}{2}-j}\text{ for }y\in\partial B_{2R\nu_\lambda}(0).$$

The pointwise controls of Theorem 5.2 and (51) yield

$$|\nabla_y^j G(x,y)|\le C|y|^{-\gamma-j}|x|^{2k-n+\gamma}\text{ for all }x\in B\setminus B_{3R\nu_\lambda}(0)\text{ and }y\in\partial B_{2R\nu_\lambda}(0).$$

Therefore, we get that

$$u_\lambda(x)\le C\nu_\lambda^{\frac{n-2k}{2}-\gamma}|x|^{2k-n+\gamma}\text{ for all }x\in B\setminus B_{3R\nu_\lambda}(0).$$

The validity of this inequality on $B_{3R\nu_\lambda}(0)\setminus B_{\nu_\lambda}(0)$ is a consequence of (14). This proves Proposition 3.2. □

Proposition 3.3. *Let $(u_\lambda)_\lambda\in C^{2k}(\overline{B})$ be a family of solutions to (3), and let $(\nu_\lambda)_\lambda$ be as in Proposition 3.1. Then for any $\omega\subset\subset B$, there exists $C>0$ such that*

$$|u_\lambda(x)|\le C\frac{\nu_\lambda^{\frac{n-2k}{2}}}{|x|^{n-2k}}\text{ for all }x\in\omega\setminus B_{\nu_\lambda}(0)\text{ and }\lambda\to 0,\tag{18}$$

and

$$\lim_{\lambda\to 0}\frac{u_\lambda}{\nu_\lambda^{\frac{n-2k}{2}}}=H:=\epsilon_0\left(\int_{\mathbb{R}^n}U^{2^\star-1}\,dx\right)G_0(0,\cdot)\text{ in }C^{2k}_{loc}(B\setminus\{0\})\tag{19}$$

where $G_0(0,\cdot)$ is the Green's function for Δ^k on B with Dirichlet boundary condition. In particular $\Delta^k H=0$ in $B\setminus\{0\}$.

Proof. Let us fix $x\in\omega$ such that $|x|>4\nu_\lambda$. Let G_λ be the Green's function of $\Delta^k-\lambda$ in B with Dirichlet boundary condition. The existence follows from Theorem 5.1. Green's representation formula yields

$$u_\lambda(x)=\int_B G_\lambda(x,y)|u_\lambda(y)|^{2^\star-2}u_\lambda(y)\,dy=\int_{|x-y|>|x|/2}+\int_{|x-y|<|x|/2}\tag{20}$$

We estimate these terms separately. Regarding the second term of (20), for $y\in B$ such that $|x-y|<|x|/2$, we have that $|y|>|x|/2>2\nu_\lambda$, and we apply (17) and we use (96) to get

$$\begin{aligned}&\left|\int_{|x-y|<|x|/2}G_\lambda(x,y)|u_\lambda(y)|^{2^\star-2}u_\lambda(y)\,dy\right| \\ &\le C\int_{|x-y|<|x|/2}|x-y|^{2k-n}\frac{\nu_\lambda^{(\frac{n-2k}{2}-\gamma)(2^\star-1)}}{|x|^{(n-2k-\gamma)(2^\star-1)}}\,dy\le C\frac{\nu_\lambda^{\frac{n-2k}{2}}}{|x|^{n-2k}}\cdot\left(\frac{\nu_\lambda}{|x|}\right)^{2k-\gamma(2^\star-1)}\end{aligned}\tag{21}$$

We split the first term of (20) in three parts. First, using the pointwise control (96), Hölder's inequality and $\|u_\lambda\|_{2^\star} \leq M$, we get

$$\begin{aligned}
&\left|\int_{\{|x-y|<|x|/2\}\cap\{|y|<R^{-1}\nu_\lambda\}} G_\lambda(x,y)|u_\lambda(y)|^{2^\star-2}u_\lambda(y)\,dy\right| \\
&\leq \int_{\{|x-y|<|x|/2\}\cap\{|y|<R^{-1}\nu_\lambda\}} |x-y|^{2k-n}|u_\lambda(y)|^{2^\star-1}\,dy \\
&\leq C|x|^{2k-n}\int_{B_{R^{-1}\nu_\lambda}(0)} |u_\lambda(y)|^{2^\star-1}\,dy \\
&\leq C|x|^{2k-n}\left(\int_{B_{R^{-1}\nu_\lambda}(0)} dy\right)^{\frac{1}{2^\star}}\left(\int_{B_{R^{-1}\nu_\lambda}(0)} |u_\lambda(y)|^{2^\star}\,dy\right)^{\frac{2^\star-1}{2^\star}} \\
&\leq C|x|^{2k-n}\left(R^{-1}\nu_\lambda\right)^{\frac{n-2k}{2}} = CR^{-\frac{n-2k}{2}}\frac{\nu_\lambda^{\frac{n-2k}{2}}}{|x|^{n-2k}} \qquad (22)
\end{aligned}$$

Now, using (96) similarly and the pointwise control (17), we get

$$\begin{aligned}
&\left|\int_{\{|x-y|<|x|/2\}\cap\{|y|\geq R\nu_\lambda\}} G_\lambda(x,y)|u_\lambda(y)|^{2^\star-2}u_\lambda(y)\,dy\right| \\
&\leq \int_{\{|x-y|<|x|/2\}\cap\{|y|\geq R\nu_\lambda\}} |x-y|^{2k-n}|u_\lambda(y)|^{2^\star-1}\,dy \\
&\leq C|x|^{2k-n}\int_{B\setminus B_{R\nu_\lambda}(0)} \frac{\nu_\lambda^{(\frac{n-2k}{2}-\gamma)(2^\star-1)}}{|y|^{(n-2k-\gamma)(2^\star-1)}}\,dy \leq C\frac{|x|^{2k-n}\nu_\lambda^{\frac{n-2k}{2}}}{R^{2k-\gamma(2^\star-1)}} \qquad (23)
\end{aligned}$$

for $\gamma < \frac{2k}{2^\star-1}$. Taking $R=1$ and plugging (21), (22), (23) in (20), we get (18).

We fix $x \in B$ such that $x \neq 0$, so that all the preceding estimates hold. For any $R>0$, with a change of variable, we have that

$$\begin{aligned}
&\int_{\{|x-y|<|x|/2\}\cap B_{R\nu_\lambda}\setminus B_{R^{-1}\nu_\lambda}(0)} G_\lambda(x,y)|u_\lambda(y)|^{2^\star-2}u_\lambda(y)\,dy \\
&= \int_{B_{R\nu_\lambda}\setminus B_{R^{-1}\nu_\lambda}(0)} G_\lambda(x,y)|u_\lambda(y)|^{2^\star-2}u_\lambda(y)\,dy \\
&= \nu_\lambda^{\frac{n-2k}{2}}\int_{B_R\setminus B_{R^{-1}}(0)} G_\lambda(x,\nu_\lambda z)|U_\lambda(z)|^{2^\star-2}U_\lambda(z)\,dz.
\end{aligned}$$

Independently, given $x \in B\setminus\{0\}$, the definition and uniqueness of Green's functions of Theorem 6.1 combined with the integral bound (38) yields the convergence of $(G_\lambda(x,\cdot))_\lambda$ to $G_0(x,\cdot)$ uniformly in $C^0_{loc}(B\setminus\{x\})$ as $\lambda\to 0$. Therefore, (14) yields

$$\begin{aligned}
&\lim_{R\to+\infty}\lim_{\lambda\to 0}\nu_\lambda^{-\frac{n-2k}{2}}\int_{\{|x-y|<|x|/2\}\cap B_{R\nu_\lambda}\setminus B_{R^{-1}\nu_\lambda}(0)} G_\lambda(x,y)|u_\lambda(y)|^{2^\star-2}u_\lambda(y)\,dy \\
&= \epsilon_0\left(\int_{\mathbb{R}^n} U^{2^\star-1}\,dx\right)G_0(0,x).
\end{aligned}$$

Combining this latest limit with (21), (22), (23) and (20), we get the pointwise limit in (19). The convergence in C^{2k} is consequence of elliptic theory (Theorems 7.1 and 7.2). This ends the proof of the Proposition. □

4. Conclusion via the Pohozaev-Pucci-Serrin identity

The following identities are essentially in Pucci-Serrin [19] and are generalizations of the historical Pohozaev identity [17]. We recall them for the sake of completeness. The first lemma is a straightforward iteration:

Lemma 4.1. *For any $v \in C^\infty(\Omega)$, where Ω is a domain of $\mathbb{R}^n$, we have that*

$$\left\{ \begin{array}{cc} \Delta^p(x^i\partial_i v) = 2p\Delta^p v + x^i\partial_i\Delta^p v & \text{for all } p\in\mathbb{N} \text{ and} \\ \partial_j\Delta^p(x^i\partial_i v) = (2p+1)\partial_j\Delta^p v + x^i\partial_i(\partial_j\Delta^p v) & \text{for all } p\in\mathbb{N} \text{ and } j=1,...,n. \end{array} \right\}.$$

These identities rewrite $\Delta^{\frac{l}{2}}(x^i\partial_i v) = l\Delta^{\frac{l}{2}}v + x^i\partial_i(\Delta^{\frac{l}{2}}v)$ for all $l\in\mathbb{N}$.

Proposition 4.1. *Let $\Omega\subset\mathbb{R}^n$ be a smooth bounded domain with $2\le 2k<n$. Then for all $u\in C^{2k+1}(\mathbb{R}^n)$ and $c\in\mathbb{R}$, we have that*

$$\int_\Omega \left(\Delta^k u - c|u|^{2^\star-2}u\right) T(u)\,dx = \int_{\partial\Omega}\left((x,\nu)\left(\frac{|\Delta^{\frac{k}{2}}u|^2}{2} - \frac{c|u|^{2^\star}}{2^\star}\right) + S(u)\right) d\sigma$$

where $T(u) := \frac{n-2k}{2}u + x^i\partial_i u$ and

$$\begin{aligned} S(u) \quad := \quad & \sum_{i=0}^{E(k/2)-1} \left(-\partial_\nu\Delta^{k-i-1}u\Delta^i T(u) + \Delta^{k-i-1}u\partial_\nu\Delta^i T(u)\right) \\ & -\mathbf{1}_{\{k \text{ odd}\}}\partial_\nu(\Delta^{\frac{k-1}{2}}u)\Delta^{\frac{k-1}{2}}T(u) \end{aligned} \tag{24}$$

Proof. Integrating by parts, for any $l\in\mathbb{N}$, $l\ge 1$, $U,V\in C^{2l}(\mathbb{R}^n)$, we have that

$$\int_\Omega(\Delta^l U)V\,dX = \int_\Omega U(\Delta^l V)\,dX + \int_{\partial\Omega}\mathcal{B}^{(l)}(U,V)\,d\sigma \tag{25}$$

where

$$\mathcal{B}^{(l)}(U,V) := \sum_{i=0}^{l-1}\left(-\partial_\nu\Delta^{l-i-1}U\Delta^i V + \Delta^{l-i-1}U\partial_\nu\Delta^i V\right) \tag{26}$$

We first assume that $k=2p$ is even, with $p\in\mathbb{N}$. Using Lemma 4.1, we get

$$\begin{aligned} &\int_\Omega \left(\Delta^k u - c|u|^{2^\star-2}u\right) T(u)\,dx = \int_\Omega \Delta^p u\Delta^p T(u)\,dx + \int_{\partial\Omega}\mathcal{B}^{(p)}(\Delta^p u, T(u))\,d\sigma \\ &-\left(\frac{n-2k}{2}\int_\Omega c|u|^{2^\star}\,dx + \int_\Omega cx^i\frac{\partial_i|u|^{2^\star}}{2^\star}\,dx\right) \\ &= \int_\Omega \Delta^p\left(\frac{n}{2}\Delta^p u + x^i\partial_i(\Delta^p u)\right)\,dx + \int_{\partial\Omega}\mathcal{B}^{(p)}(\Delta^p u, T(u))\,d\sigma \\ &-\left(\frac{n-2k}{2}\int_\Omega c|u|^{2^\star}\,dx - \frac{n}{2^\star}\int_\Omega c|u|^{2^\star}\,dx + \int_{\partial\Omega}(x,\nu)\frac{c|u|^{2^\star}}{2^\star}\,dx\right) \\ &= \int_\Omega \partial_i\left(\frac{x^i(\Delta^p u)^2}{2}\right)\,dx + \int_{\partial\Omega}\mathcal{B}^{(p)}(\Delta^p u, T(u))\,d\sigma - \int_{\partial\Omega}(x,\nu)\frac{c|u|^{2^\star}}{2^\star}\,dx \\ &= \int_{\partial\Omega}\left((x,\nu)\left(\frac{(\Delta^p u)^2}{2} - \frac{c|u|^{2^\star}}{2^\star}\right) + \mathcal{B}^{(p)}(\Delta^p u, T(u))\right) d\sigma \end{aligned}$$

which proves Proposition 4.1 when k is even. When $k = 2q+1$ is odd, we get that

$$\begin{aligned}\int_\Omega \Delta^k u T(u)\,dx &= \int_\Omega \Delta^q(\Delta^{q+1}u)T(u)\,dx\\ &= \int_\Omega \Delta^{q+1}u\Delta^q T(u)\,dx + \int_{\partial\Omega}\mathcal{B}^{(q)}(\Delta^{q+1}u, T(u))\,d\sigma\\ &= \int_\Omega \sum_j \partial_j(\Delta^q u)\partial_j(\Delta^q T(u))\,dx\\ &\quad + \int_{\partial\Omega}\left(\mathcal{B}^{(q)}(\Delta^{q+1}u, T(u)) - \partial_\nu(\Delta^q u)\Delta^q T(u)\right)d\sigma\end{aligned}$$

Using Lemma 4.1, we get that

$$\begin{aligned}&\int_\Omega \Delta^k u T(u)\,dx = \int_\Omega \sum_j \partial_j(\Delta^q u)\left(\frac{n}{2}\partial_j\Delta^q u + x^i\partial_i\partial_j\Delta^q u\right)\\ &+ \int_{\partial\Omega}\left(\mathcal{B}^{(q)}(\Delta^{q+1}u, T(u)) - \partial_\nu(\Delta^q u)\Delta^q T(u)\right)d\sigma\\ &= \int_\Omega \partial_i\left(x^i\frac{(\partial_j\Delta^q u)^2}{2}\right)dx + \int_{\partial\Omega}\left(\mathcal{B}^{(q)}(\Delta^{q+1}u, T(u)) - \partial_\nu(\Delta^q u)\Delta^q T(u)\right)d\sigma\\ &= \int_{\partial\Omega}(x,\nu)\frac{|\nabla\Delta^q u|^2}{2}\,d\sigma + \int_{\partial\Omega}\left(\mathcal{B}^{(q)}(\Delta^{q+1}u, T(u)) - \partial_\nu(\Delta^q u)\Delta^q T(u)\right)d\sigma\end{aligned}$$

Using the same computations as in the case when k is even, we get the conclusion of Proposition 4.1. □

We fix $\delta \in (0,1)$. Since u_λ solves (3), Proposition 4.1 yields

$$\lambda\int_{B_\delta(0)} u_\lambda T(u_\lambda)\,dx = \int_{\partial B_\delta(0)}\left((x,\nu)\left(\frac{|\Delta^{\frac{k}{2}}u_\lambda|^2}{2} - \frac{|u_\lambda|^{2^\star}}{2^\star}\right) + S(u_\lambda)\right)d\sigma \quad (27)$$

where $T(u_\lambda)$ and $S(u_\lambda)$ are as in (24).

Proposition 4.2. *Let $(u_\lambda)_\lambda \in C^{2k}(\overline{B})$ be a family of solutions to (3), and let $(\nu_\lambda)_\lambda$ as in Proposition 3.1. Fix $0 < \delta < 1$. Then there exists $A \in \mathbb{R}$ such that*

$$\lim_{\lambda\to 0}\nu_\lambda^{2k-n}\int_{\partial B_\delta(0)}\left((x,\nu)\left(\frac{|\Delta^{\frac{k}{2}}u_\lambda|^2}{2} - \frac{|u_\lambda|^{2^\star}}{2^\star}\right) + S(u_\lambda)\right)d\sigma = A < 0. \quad (28)$$

Proof. Setting $\bar{u}_\lambda := \nu_\lambda^{-\frac{n-2k}{2}}u_\lambda$ and using (19), we get that

$$\begin{aligned}&\int_{\partial B_\delta(0)}\left((x,\nu)\left(\frac{|\Delta^{\frac{k}{2}}u_\lambda|^2}{2} - \frac{|u_\lambda|^{2^\star}}{2^\star}\right) + S(u_\lambda)\right)d\sigma\\ &= \nu_\lambda^{n-2k}\int_{\partial B_\delta(0)}\left((x,\nu)\left(\frac{|\Delta^{\frac{k}{2}}\bar{u}_\lambda|^2}{2} - \nu_\lambda^{2k}\frac{|\bar{u}_\lambda|^{2^\star}}{2^\star}\right) + S(\bar{u}_\lambda)\right)d\sigma\\ &= \nu_\lambda^{n-2k}\left(\int_{\partial B_\delta(0)}\left((x,\nu)\left(\frac{|\Delta^{\frac{k}{2}}H|^2}{2}\right) + S(H)\right)d\sigma + o(1)\right)\end{aligned}$$

It follows from Boggio's formula [3] (see also Lemma 2.27 in [8]) that there exists $A_{k,n} > 0$ such that

$$G_0(x,0) = A_{k,n}|x|^{2k-n}\int_1^{1/|x|}(v^2-1)^{k-1}v^{1-n}\,dv$$

for all $x \in B \setminus \{0\}$. Therefore, since $\epsilon_0 \int_{\mathbb{R}^n} U^{2^\star-1}\,dx \neq 0$, there exists $\beta \in C^{2k}(\overline{B})$ and $\alpha \neq 0$ such that

$$H(x) = \alpha\left(\Gamma(x) + \beta(x)\right) \text{ for all } x \in B \setminus \{0\},\ \Gamma(x) := |x|^{2k-n} \text{ and } \beta(0) < 0.$$

We then get that

$$\lim_{\lambda\to 0} \nu_\lambda^{2k-n}\int_{\partial B_\delta(0)}\left((x,\nu)\left(\frac{|\Delta^{\frac{k}{2}}u_\lambda|^2}{2} - \frac{|u_\lambda|^{2^\star}}{2^\star}\right) + S(u_\lambda)\right)d\sigma = C_\delta \tag{29}$$

$$\text{where } C_\delta := \alpha^2\int_{\partial B_\delta(0)}\left((x,\nu)\left(\frac{|\Delta^{\frac{k}{2}}(\Gamma+\beta)|^2}{2}\right) + S(\Gamma+\beta)\right)d\sigma.$$

Applying Proposition 4.1 to $\Gamma+\beta$ on $B_\delta(0)\setminus B_r(0)$ for $0 < r < \delta$ with $\epsilon = 0$ and $f \equiv 0$, we get that C_δ is independent of the choice of $0 < \delta < 1$. We compute the different terms of C_δ separately. Using that β and all its derivatives are bounded in $\overline{B}$, we get that

$$\int_{\partial B_\delta(0)}(x,\nu)\frac{|\Delta^{\frac{k}{2}}(\Gamma+\beta)|^2}{2}\,d\sigma = \int_{\partial B_\delta(0)}(x,\nu)\frac{|\Delta^{\frac{k}{2}}\Gamma|^2}{2}\,d\sigma + O\left(\delta^k\right) + O(\delta^n)$$

We let S_P be the natural bilinear form such that $S(u) = S_P(u,u)$ for all u. With the expression (24) of S, we get that

$$\begin{aligned} S(\Gamma+\beta) &= S_P(\Gamma,\Gamma) + S_P(\Gamma,\beta) + S_P(\beta,\Gamma) + S_P(\beta,\beta) \\ &= S(\Gamma) - (\partial_\nu\Delta^{k-1}\Gamma)T(\beta) + O(|x|^{2-n}) \end{aligned}$$

With (89), we get that

$$S(\Gamma+\beta) = S(\Gamma) + \frac{n-2k}{2\omega_{n-1}}\beta(0)|x|^{1-n} + O(|x|^{2-n})$$

These identities yield

$$C_\delta = \alpha^2 D_\delta + \frac{(n-2k)\alpha^2}{2\omega_{n-1}}\beta(0) + O(\delta)$$

where

$$D_r := \int_{\partial B_r(0)}\left((x,\nu)\frac{|\Delta^{\frac{k}{2}}\Gamma|^2}{2} + S(\Gamma)\right)d\sigma$$

for all $r > 0$. Taking the identity of Proposition 4.1 for $c = 0$, $u \equiv \Gamma$ so that $\Delta^k u = 0$ and $\Omega = B_1(0) - B_r(0)$ for $0 < r < 1$, we get that $D_r = D_1$ for all $0 < r < 1$. A quick computation yields the existence of $D_{k,n} \in \mathbb{R}$ such that

$$\left(|x|\frac{|\Delta^{\frac{k}{2}}\Gamma|^2}{2} + S(\Gamma)\right) = D_{k,n}|x|^{2k+1-2n} \text{ for all } x \in \mathbb{R}^n \setminus \{0\},$$

so that $D_r = D_{k,n}\omega_{n-1}r^{2k-n}$ for all $0 < r < 1$. Since this quantity is independent of r, we get that $D_{k,n} = 0$, so that $D_\delta = 0$ for all $\delta > 0$ and then

$$C_\delta = \frac{(n-2k)\alpha^2}{2\omega_{n-1}}\beta(0) + O(\delta).$$

Since C_δ is independent of δ, using (29), we get (28) with $A = \frac{(n-2k)\alpha^2}{2\omega_{n-1}}\beta(0) < 0$. □

Proposition 4.3. *Let $(u_\lambda)_\lambda \in C^{2k}(\overline{B})$ be a family of solutions to (3), and let $(\nu_\lambda)_\lambda$ as in Proposition 3.1. Fix $0 < \delta < 1$. Then*

$$\int_{B_\delta(0)} u_\lambda T(u_\lambda)\,dx = O(\nu_\lambda^{n-2k}) \text{ if } 2k < n < 4k. \tag{30}$$

Proof. Integrating by parts, we get that

$$\begin{aligned}\int_{B_\delta(0)} u_\lambda T(u_\lambda)\,dx &= \int_{B_\delta(0)} u_\lambda\left(\frac{n-2k}{2}u_\lambda + x^i\partial_i u_\lambda\right)dx \\ &= -k\int_{B_\delta(0)} u_\lambda^2\,dx + \int_{\partial B_\delta(0)}(x,\nu)u_\lambda^2\,d\sigma\end{aligned}$$

With Hölder's inequality and the pointwise control (18), using that $n < 4k$, we get

$$\begin{aligned}\int_{B_\delta(0)} u_\lambda^2\,dx &\le \int_{B_{\nu_\lambda}(0)} u_\lambda^2\,dx + \int_{B_\delta(0)\setminus B_{\nu_\lambda}(0)} \nu_\lambda^{n-2k}|x|^{2(2k-n)}\,dx \\ &\le \left(\int_{B_{\nu_\lambda}(0)} dx\right)^{\frac{2^\star-2}{2^\star}}\left(\int_{B_{\nu_\lambda}(0)} u_\lambda^{2^\star}\,dx\right)^{\frac{2}{2^\star}} + \int_{B_\delta(0)\setminus B_{\nu_\lambda}(0)} \nu_\lambda^{n-2k}|x|^{2(2k-n)}\,dx \\ &\le C\nu_\lambda^{2k} + \int_{B_\delta(0)\setminus B_{\nu_\lambda}(0)} \nu_\lambda^{n-2k}|x|^{2(2k-n)}\,dx \le C\nu_\lambda^{n-2k}\end{aligned}$$

since $n < 4k$. The result then follows from these estimates and (19). □

Conclusion of the argument and proof of Theorem 1.1. Putting (28) and (30) into the identity (27), we get that $o(\nu_\lambda^{n-2k}) = (A+o(1))\nu_\lambda^{n-2k}$ as $\lambda \to 0$, which contradicts $A \neq 0$.

5. Green's function for an "almost" Hardy operator

Let Ω be a smooth domain of $\mathbb{R}^n$ and let $k \in \mathbb{N}$ be such that $2 \le 2k < n$. Given $h \in L^\infty(\Omega)$, we consider operators like $P = \Delta^k + h$. Integrating by parts yields $\int_\Omega uPu\,dx = \int_\Omega \left((\Delta^{\frac{k}{2}}u)^2 + hu^2\right)dx$ for all $u \in C_c^\infty(\Omega)$, so that this expression makes sense for $u \in H^2_{k,0}(\Omega)$. We say that P is coercive if there exists $c > 0$ such that $\int_\Omega uPu\,dx \ge c\|u\|^2_{H^2_k}$ for all $u \in H^2_{k,0}(\Omega)$. We prove the following theorems:

Theorem 5.1. *Let Ω be a smooth domain of $\mathbb{R}^n$ such that $0 \in \Omega$ is an interior point. Fix $k \in \mathbb{N}$ such that $2 \le 2k < n$. We consider an operator $P = \Delta^k + h$, where $h \in L^\infty(\Omega)$ and P is coercive. We let $V \in L^1(\Omega)$ such that for some $\mu > 0$, $|x|^{2k}|V(x)| \le \mu$ for all $x \in \Omega$.*

Then there is $\mu_0(P,h) > 0$ such that for $0 < \mu < \mu_0(P,h)$, there exists $G : (\Omega\setminus\{0\})\times(\Omega\setminus\{0\})\setminus\{(z,z)/z \in \Omega\setminus\{0\}\} \to \mathbb{R}$ such that:

- *For all $x \in \Omega\setminus\{0\}$, $G(x,\cdot) \in L^q(\Omega)$ for all $1 \le q < \frac{n}{n-2k}$*
- *For all $x \in \Omega\setminus\{0\}$, $G(x,\cdot) \in L^{\frac{2n}{n-2k}}_{loc}(\Omega\setminus\{x\})$*
- *For all $f \in L^{\frac{2n}{n+2k}}(\Omega) \cap L^p_{loc}(\Omega\setminus\{0\})$, $p > \frac{n}{2k}$, we let $\varphi \in H^2_{k,0}(\Omega)$ such that $P\varphi = f$ in the weak sense. Then $\varphi \in C^0(\overline{\Omega}\setminus\{0\})$ and*

$$\varphi(x) = \int_\Omega G(x,\cdot)f\,dy \text{ for all } x \in \Omega\setminus\{0\}.$$

Moreover, such a function G is unique. It is the Green's function for $P-V$. In addition, G is symmetric and for all $x \in \Omega \setminus \{0\}$,

$$G(x,\cdot) \in H^p_{2k,loc}(\Omega \setminus \{0,x\}) \cap H^2_{k,0,loc}(\Omega \setminus \{x\}) \cap C^{2k-1}(\overline{\Omega} \setminus \{0,x\})$$

for all $1 < p < \infty$ and

$$\left\{ \begin{array}{cc} (P-V)G(x,\cdot) = 0 & \text{in } \Omega \setminus \{0,x\} \\ \partial^i_\nu G(x,\cdot)_{|\partial\Omega} = 0 & \text{for } i = 0,...,k-1. \end{array} \right\}$$

In addition, we get the following pointwise control:

Theorem 5.2. *Let Ω be a smooth domain of $\mathbb{R}^n$ such that $0 \in \Omega$ is an interior point. Fix $k \in \mathbb{N}$ such that $2 \le 2k < n$, $L > 0$ and $\mu > 0$. We consider an operator $P = \Delta^k + h$, where $h \in L^\infty(\Omega)$, $\|h\|_{L^\infty} \le L$ and $\int_\Omega uPu\,dx \ge L^{-1}\|u\|^2_{H^2_k}$ for all $u \in H^2_{k,0}(\Omega)$. We let $V \in L^1(\Omega)$ such that $P-V$ is coercive and*

$$|x|^{2k}|V(x)| \le \mu \text{ for all } x \in \Omega.$$

We let G be the Green's function of $P-V$ as in Theorem 5.1.
Then for any $\gamma \in (0, n-2k)$, there exists $\mu_\gamma > 0$ such that for $\mu < \mu_\gamma$, for any $\omega \subset\subset \Omega$, for any $x \in \omega \setminus \{0\}$, $y \in \Omega \setminus \{0\}$ such that $x \neq y$, we have that

- $$|G(x,y)| \le C(\Omega,\gamma,L,\mu,k,\omega)\left(\frac{\max\{|x|,|y|\}}{\min\{|x|,|y|\}}\right)^\gamma |x-y|^{2k-n}$$
- *If $|x| < |y|$ and $l \le 2k-1$, we have that*
$$|\nabla^l_y G(x,y)| \le C(\Omega,\gamma,L,\mu,k,l,\omega)\left(\frac{\max\{|x|,|y|\}}{\min\{|x|,|y|\}}\right)^\gamma |x-y|^{2k-n-l}$$
- *If $|y| < |x|$ and $l \le 2k-1$, we have that*

$$|\nabla^l_y G(x,y)| \le C(\Omega,\gamma,L,\mu,k,l,\omega)\left(\frac{\max\{|x|,|y|\}}{\min\{|x|,|y|\}}\right)^{\gamma+l} |x-y|^{2k-n-l} \tag{31}$$

where $C(\Omega,\gamma,L,\mu,k,l,\omega)$ depends only on Ω, γ, L, μ, k,l and ω.

5.1. **Construction of the Green's function. Preliminary notations:** In addition to the Sobolev inequality (4), we will make a regular use of the Hardy inequality on $\mathbb{R}^n$ (see Theorem 3.3 in Mitidieri [16]): there exists $C_H(n,k) > 0$ such that

$$\int_{\mathbb{R}^n} \frac{\varphi^2}{|X|^{2k}}\,dX \le C_H(n,k)\int_{\mathbb{R}^n} (\Delta^{\frac{k}{2}}\varphi)^2\,dX \text{ for all } \varphi \in D^2_k(\mathbb{R}^n). \tag{32}$$

For $\mu > 0$, we define

$$\mathcal{P}_\mu := \left\{ \begin{array}{c} V \in L^1(\Omega) \text{ such that} \\ |V(x)| \le \mu|x|^{-2k} \text{ for all } x \in \Omega \setminus \{0\} \end{array} \right\}.$$

In the sequel, we consider an operator $P = \Delta^k + h$, where $h \in L^\infty(\Omega)$ is such that

$$\|h\|_{L^\infty} \le L \text{ and } \int_\Omega uPu\,dx \ge L^{-1}\|u\|^2_{H^2_k} \text{ for all } u \in H^2_{k,0}(\Omega). \tag{33}$$

Step 0: Approximation of the potential. We claim that there exists $\mu_0 = \mu_0(k,L)$ such for all $V_0 \in \mathcal{P}_\mu$ with $0 < \mu < \mu_0$, then

$$\int_\Omega (Pu - V_0u)u\,dx \ge \frac{1}{2L}\|u\|^2_{H^2_k} \text{ for all } u \in H^2_{k,0}(\Omega) \tag{34}$$

and there exists a family $(V_\epsilon)_{\epsilon>0} \in L^\infty(\Omega)$ such that:

$$\left\{ \begin{array}{ccc} \lim_{\epsilon\to 0} V_\epsilon(x) = V_0(x) & & \text{for a.e. } x \in \Omega\setminus\{0\} \\ V_\epsilon \in \mathcal{P}_\mu & & \text{for all } \epsilon>0 \\ P-V_\epsilon & & \text{is uniformly coercive for all } \epsilon>0 \end{array} \right\} \tag{35}$$

in the sense that

$$\int_M (Pu - V_\epsilon u)u \geq \frac{1}{2L}\|u\|^2_{H^2_k} \text{ for all } u \in H^2_{k,0}(\Omega) \text{ and } \epsilon>0 \tag{36}$$

We prove the claim. The coercivity of P and the Hardy inequality (32) yield

$$\int_\Omega u(P-V_0)u\,dx \geq \frac{1}{L}\|u\|^2_{H^2_k} - \mu\int_\Omega \frac{u^2}{|x|^{2k}}\,dx \geq \left(\frac{1}{L} - \mu C_H(n,k)\right)\|u\|^2_{H^2_k}$$

for all $u \in H^2_{k,0}(\Omega)$. For $\eta \in C^\infty(\mathbb{R})$ such that $\eta(t)=0$ for $t\leq 1$ and $\eta(t)=1$ for $t\geq 2$, define $V_\epsilon(x) := \eta(|x|/\epsilon)V_0(x)$ for all $\epsilon>0$ and a.e. $x\in\Omega$. As one checks, the claim holds with $0<\mu_0<(2C_H(n,k)L)^{-1}$. This proves the claim.

For any $\epsilon>0$, we let G_ϵ be the Green's function for the operator $P-V_\epsilon$. Since $V_\epsilon \in L^\infty(\Omega)$, the existence of G_ϵ follows from Theorem 6.1 of the Appendix 6.1.

Step 1: Integral bounds. We choose $f\in C^0_c(\Omega)$ and we fix $\epsilon>0$. Since $P-V_\epsilon$ is coercive, it follows from variational methods that there exists a unique function $\varphi_\epsilon \in H^2_{k,0}(\Omega)$ such that

$$\left\{ \begin{array}{ccc} (P-V_\epsilon)\varphi_\epsilon = f & & \text{in } \Omega \\ \partial^i_\nu \varphi_{\epsilon|\partial\Omega} = 0 & & \text{for } i=0,...,k-1 \end{array} \right\} \text{ in the weak sense.}$$

It follows from Theorem 7.3 and Sobolev's embedding theorem that $\varphi_\epsilon \in C^{2k-1}(\overline{\Omega})$ and $\varphi_\epsilon \in H^p_{2k}(\Omega)$ for all $p>1$. The coercivity hypothesis (36) yields

$$\frac{1}{2L}\|\varphi_\epsilon\|^2_{H^2_k} \leq \int_\Omega (P\varphi_\epsilon - V_\epsilon\varphi_\epsilon)\varphi_\epsilon\,dx = \int_\Omega f\varphi_\epsilon\,dx \leq \|f\|_{\frac{2n}{n+2k}}\|\varphi_\epsilon\|_{\frac{2n}{n-2k}}$$

With inequality (4), we get that

$$K(n,k)^{-1}\|\varphi_\epsilon\|_{2^\star} \leq \|\varphi_\epsilon\|_{H^2_k} \leq 2LK(n,k)\|f\|_{\frac{2n}{n+2k}} \tag{37}$$

for all $f\in C^0_c(\Omega)$. We fix $p>1$ such that

$$\frac{n}{2k} < p < \frac{n}{2k-1} \text{ and } \theta_p := 2k - \frac{n}{p} \in (0,1).$$

We fix $\delta \in (0, d(0,\partial\Omega)/4)$. Since $V_\epsilon \in \mathcal{P}_\mu$ for all $\epsilon>0$ and P satisfies (33), it follows from regularity theory, see Theorem 7.1 of Appendix 7) and Sobolev's embedding theorem that

$$\begin{aligned} \|\varphi_\epsilon\|_{C^{0,\theta_p}(\overline{\Omega}\setminus B_\delta(0))} &\leq C(p,\delta,k)\|\varphi_\epsilon\|_{H^p_{2k}(\Omega\setminus B_\delta(0))} \\ &\leq C(p,\delta,k,L,\mu_0)\left(\|f\|_{L^p(\Omega\setminus B_{\delta/2}(0))} + \|\varphi_\epsilon\|_{L^{2^\star}(\Omega\setminus B_{\delta/2}(0))}\right). \end{aligned}$$

With (37) and noting that $\frac{n}{2k} > \frac{2n}{n+2k}$, we get that

$$\|\varphi_\epsilon\|_{C^{0,\theta_p}(\overline{\Omega}\setminus B_\delta(0))} \leq C(p,\delta,k,L,\mu_0)\|f\|_{L^p(\Omega)}.$$

Since $\varphi_\epsilon \in H^p_{2k}(\Omega)$ for all $p>1$, for any $x\in\Omega\setminus\{0\}$, Green's representation formula (see Theorem 6.1) yields

$$\varphi_\epsilon(x) = \int_\Omega G_\epsilon(x,y)f(y)\,dy \text{ for all } x\in\Omega\setminus\{0\},$$

and then when $|x| > \delta$, we get

$$\left| \int_\Omega G_\epsilon(x,y) f(y)\, dy \right| \leq C(p,\delta,k,L,\mu) \|f\|_{L^p(\Omega)}$$

for all $f \in C^0_c(\Omega)$ and $p \in \left(\frac{n}{2k}, \frac{n}{2k-1} \right)$. Via duality, we then deduce that

$$\|G_\epsilon(x,\cdot)\|_{L^q(\Omega)} \leq C(q,\delta,k,L,\mu_0) \text{ for all } q \in \left(1, \frac{n}{n-2k}\right) \text{ and } |x| > \delta. \tag{38}$$

We now fix $x \in \Omega$ such that $|x| > \delta$. We take $f \in C^0_c(\Omega)$ such that $f \equiv 0$ in $B_{\delta/2}(x)$, so that $(P - V_\epsilon)\varphi_\epsilon = 0$ in $B_{\delta/2}(x)$. Since $V_\epsilon \in \mathcal{P}_\mu$ for all $\epsilon > 0$ and P satisfies (33), it follows from regularity theory (Theorem 7.1 of Appendix 7) and Sobolev's embedding theorem that for any $p > \frac{n}{2k}$,

$$\begin{aligned} \|\varphi_\epsilon\|_{C^{0,\theta_p}(\Omega \cap B_{\delta/4}(x))} &\leq C(p,\delta,k) \|\varphi_\epsilon\|_{H^p_{2k}(\Omega \cap B_{\delta/4}(x))} \\ &\leq C(p,\delta,k,L,\mu_0) \|\varphi_\epsilon\|_{L^{2^\star}(\Omega \cap B_{\delta/2}(x))}. \end{aligned}$$

With (37), we get that

$$\|\varphi_\epsilon\|_{C^{0,\theta_p}(\Omega \cap B_{\delta/4}(x))} \leq C(p,\delta,k,L,\mu_0) \|f\|_{L^{\frac{2n}{n+2k}}(\Omega)}.$$

Since $\varphi_\epsilon \in H^p_{2k}(\Omega)$ for all $p > 1$ and $\varphi_\epsilon \in C^{2k-1}(\overline{\Omega}) \cap H^2_{k,0}(\Omega)$, Green's representation formula (see Theorem 6.1) yields

$$\varphi_\epsilon(x) = \int_\Omega G_\epsilon(x,y) f(y)\, dy,$$

and then

$$\left| \int_\Omega G_\epsilon(x,y) f(y)\, dy \right| \leq C(p,\delta,k,L,\mu) \|f\|_{L^{\frac{2n}{n+2k}}(\Omega)}$$

for all $f \in C^0_c(\Omega)$ vanishing in $B_{\delta/2}(x)$. Via duality, we then deduce that

$$\|G_\epsilon(x,\cdot)\|_{L^{2^\star}(\Omega \setminus B_{\delta/2}(x))} \leq C(\delta,k,L,\mu_0) \text{ when } |x| > \delta. \tag{39}$$

Step 2: passing to the limit $\epsilon \to 0$ and Green's function for $P - V_0$. We fix $\delta > 0$ and $x \in \Omega$ such that $|x| > \delta$. For all $\epsilon > 0$, we have that

$$\left\{ \begin{array}{cc} PG_\epsilon(x,\cdot) - V_\epsilon G_\epsilon(x,\cdot) = 0 & \text{in } \Omega \setminus \{x\} \\ \partial^i_\nu G_\epsilon(x,\cdot)_{|\partial\Omega} = 0 & \text{for } i = 0, ..., k-1 \end{array} \right\} \tag{40}$$

Since $V_\epsilon \in \mathcal{P}_\mu$ for all $\epsilon > 0$, we have that $|V_\epsilon(y)| \leq C(\mu,\delta)$ for all $y \in \Omega \setminus B_{\delta/2}(0)$ and $\epsilon > 0$. Since P satisfies (33) and $G_\epsilon(x,\cdot) \in H^p_{2k,loc}(\Omega \setminus \{0,x\}) \cap H^2_{k,0,loc}(\Omega \setminus \{x\})$, it follows from the control (38) and standard regularity theory (see Theorem 7.1) that given $\nu \in (0,1)$, we have that for any $r > 0$,

$$\|G_\epsilon(x,\cdot)\|_{C^{2k-1,\nu}(\Omega - (B_r(0) \cup B_r(x)))} \leq C(\delta,k,\mu,L,r,\nu,\mu_0) \text{ for all } |x| > \delta. \tag{41}$$

It then follows from Ascoli's theorem that, up to extraction of a subfamily, there exists $G_0(x,\cdot) \in C^{2k-1}(\overline{\Omega} - \{x,0\})$ such that

$$\lim_{\epsilon \to 0} G_\epsilon(x,\cdot) = G_0(x,\cdot) \text{ in } C^{2k-1}_{loc}(\overline{\Omega} - \{x,0\}). \tag{42}$$

By Theorem 7.1 again, we also get that

$$\lim_{\epsilon \to 0} G_\epsilon(x,\cdot) = G_0(x,\cdot) \text{ in } H^p_{2k,loc}(\overline{\Omega} - \{x,0\}) \text{ for all } p > 1. \tag{43}$$

Moreover, passing to the limit in (38), we get that

$$\|G_0(x,\cdot)\|_{L^q(\Omega)} \leq C(q,\delta,k,L,\mu_0) \text{ for all } q \in \left(1, \frac{n}{n-2k}\right) \text{ and } |x| > \delta, \tag{44}$$

and then $G_0(x,\cdot) \in L^q(\Omega)$ for all $q \in \left(1, \frac{n}{n-2k}\right)$ and $x \neq 0$. Similarly, using (39), we get that

$$\|G_0(x,\cdot)\|_{L^{2^\star}(\Omega\setminus B_{\delta/2}(x))} \leq C(\delta,k,L,\mu_0) \text{ when } |x| > \delta. \tag{45}$$

So that $G_0(x,\cdot) \in L^{2^\star}_{loc}(\Omega \setminus \{x\})$.

Step 3: Representation formula. We fix $f \in L^{\frac{2n}{n+2k}}(\Omega) \cap L^p_{loc}(\Omega\setminus\{0\})$, $p > \frac{n}{2k} > 1$. Via the coercivity of $P - V_\epsilon$ and $P - V_0$, it follows from variational methods (see also Theorem 7.3) that there exists $\varphi_\epsilon \in H^2_{k,0}(\Omega) \cap H^{\frac{2n}{n+2k}}_{2k}(\Omega)$ and $\varphi_0 \in H^2_{k,0}(\Omega)$ such that

$$\left\{ \begin{array}{cc} (P-V_\epsilon)\varphi_\epsilon = f & \text{in } \Omega \\ \partial^i_\nu \varphi_{\epsilon|\partial\Omega} = 0 & \text{for } i = 0,...,k-1 \end{array} \right\} \text{ and } \left\{ \begin{array}{cc} (P-V_0)\varphi_0 = f & \text{in } \Omega \\ \partial^i_\nu \varphi_{0|\partial\Omega} = 0 & \text{for } i = 0,...,k-1 \end{array} \right\} \tag{46}$$

As one checks,

$$\lim_{\epsilon\to 0} \varphi_\epsilon = \varphi_0 \text{ in } H^2_{k,0}(\Omega) \text{ and } \lim_{\epsilon\to 0} \varphi_\epsilon = \varphi_0 \text{ in } C^0_{loc}(\overline{\Omega}\setminus\{0\}) \tag{47}$$

We now write Green's formula for φ_ϵ to get

$$\varphi_\epsilon(x) = \int_\Omega G_\epsilon(x,\cdot) f\, dy \text{ for } x \neq 0 \text{ and for all } \epsilon > 0.$$

With (38), (39), (42), (44), (45) and (47), we pass to the limit to get

$$\varphi_0(x) = \int_\Omega G_0(x,\cdot) f\, dy.$$

This yields the existence of a Green's function for $P - V_0$ in Theorem 5.1. Concerning uniqueness, let us consider another Green's function as in Theorem 5.1, say $\bar{G}_0$, and, given $x \in \Omega \setminus \{0\}$, let us define $H_x := G_0(x,\cdot) - \bar{G}_0(x,\cdot)$. We then get that $H_x \in L^q(\Omega)$ for all $1 \leq q < \frac{n}{n-2k}$ and $\int_\Omega H_x f\, dy = 0$ for all $f \in C^0_c(\Omega)$. By density, this identity is also valid for all $f \in L^{q'}(\Omega)$ where $\frac{1}{q} + \frac{1}{q'} = 1$. By duality, this yields $H_x \equiv 0$, and then $\bar{G}_0 = G_0$, which proves uniqueness. This ends the proof of Theorem 5.1.

Step 4: First pointwise control. As above, we fix $\delta > 0$ and we take $x \in \Omega$ such that $|x| > \delta$. It follows from (40), (41), (42) and regularity theory (see Theorem 7.1) that for all $l \in \{0,...,2k-1\}$, we have that

$$|\nabla^l_y G_\epsilon(x,y)| \leq C(\Omega,\delta,k,\mu_0,L) \text{ for } \{|x-y| \geq \delta,\ |x| \geq \delta,\ |y| \geq \delta\}. \tag{48}$$

and

$$|\nabla^l_y G_0(x,y)| \leq C(\Omega,\delta,k,\mu_0,L) \text{ for } \{|x-y| \geq \delta,\ |x| \geq \delta,\ |y| \geq \delta\}. \tag{49}$$

We fix $\gamma \in (0, n-2k)$. Since $G_\epsilon(x,\cdot)$ satisfies (40) in the weak sense and $G_\epsilon(x,\cdot) \in H^2_k(B_{\delta/2}(0))$, it follows from Lemma 6.1 that for all $p > 1$, there exists $\mu = \mu(\gamma,L,\delta) > 0$, there exists $C = C(\Omega,\gamma,p,L,\delta) > 0$ such that

$$|y|^\gamma |G_\epsilon(x,y)| \leq C\|G_\epsilon(x,\cdot)\|_{L^p(B_{\delta/2}(0))} \text{ for all } y \in B_{\delta/3}(0) - \{0\}$$

when $|x| \geq \delta$. It then follows from (38) that

$$|y|^\gamma |G_\epsilon(x,y)| \leq C(\Omega,\delta,k,L,\gamma) \text{ for all } y \in B_{\delta/2}(0) - \{0\} \text{ and } |x| \geq \delta. \quad (50)$$

With Lemma 6.1, for all $0 \leq l \leq 2k-1$, there exists $C(\delta,k,L,\gamma,l) > 0$ such that

$$|y|^{\gamma+l} |\nabla_y^l G_\epsilon(x,y)| \leq C(\Omega,\delta,k,L,\gamma,l) \text{ for all } y \in B_{\delta/2}(0) - \{0\} \text{ and } |x| \geq \delta. \quad (51)$$

These inequalities are valid for $\epsilon > 0$, and then for $\epsilon = 0$. In order to get the full estimates of Theorem 5.2, we now perform infinitesimal versions of these estimates.

5.2. **Asymptotics for the Green's function close to the singularity.** We prove an infinitesimal version of (48) and (50) for x, y close to the singularity 0.

Theorem 5.3. *Let Ω be a smooth domain of $\mathbb{R}^n$ such that $0 \in \Omega$ is an interior point. Fix $k \in \mathbb{N}$ such that $2 \leq 2k < n$, $L > 0$ and $\mu > 0$. Fix an operator P that satisfies (33), $V \in \mathcal{P}_\mu$ and a family (V_ϵ) as in (35). For $\mu > 0$ sufficiently small, let G_ϵ be the Green's function for $P - V_\epsilon$, $\epsilon \geq 0$. Let us fix $\mathcal{U}, \mathcal{V}$ two open subsets of $\mathbb{R}^n$ such that*

$$\mathcal{U} \subset\subset \mathbb{R}^n - \{0\}, \ \mathcal{V} \subset\subset \mathbb{R}^n \text{ and } \overline{\mathcal{U}} \cap \overline{\mathcal{V}} = \emptyset.$$

We let $\alpha_0 := \alpha_0(\mathcal{U},\mathcal{V}) > 0$ be such that $|\alpha X| < d(0,\partial\Omega)/2$ for all $0 < \alpha < \alpha_0$ and $X \in \mathcal{U} \cup \mathcal{V}$. We fix $\gamma \in (0, n-2k)$. Then there exists $\mu = \mu(\gamma) > 0$, there exists $C(\mathcal{U},\mathcal{V},\mu,k,L) > 0$ such that

$$\left| |X|^\gamma \alpha^{n-2k+l} \nabla_y^l G_\epsilon(\alpha X, \alpha Y) \right| \leq C(\mathcal{U},\mathcal{V},k,L) \quad (52)$$

for all $X \in \mathcal{V} - \{0\}$, $Y \in \mathcal{U}$, $l = 0, ..., 2k-1$, $\alpha \in (0,\alpha_0)$ and $\epsilon \geq 0$.

Proof of Theorem 5.3. We first set $\mathcal{U}', \mathcal{V}'$ two open subsets of $\mathbb{R}^n$ such that

$$\mathcal{U} \subset\subset \mathcal{U}' \subset\subset \mathbb{R}^n - \{0\}, \ \mathcal{V} \subset\subset \mathcal{V}' \subset\subset \mathbb{R}^n \text{ and } \overline{\mathcal{U}'} \cap \overline{\mathcal{V}} = \emptyset.$$

We fix $f \in C_c^\infty(\mathcal{U}')$ and for any $0 < \alpha < \alpha_0$, we set

$$f_\alpha(x) := \frac{1}{\alpha^{\frac{n+2k}{2}}} f\left(\frac{x}{\alpha}\right) \text{ for all } x \in \Omega.$$

As one checks, $f_\alpha \in C_c^\infty(\alpha\mathcal{U}')$ and $\alpha\mathcal{U}' \subset\subset \Omega \setminus \{0\}$. It follows from Theorem 7.3 that there exists $\varphi_{\alpha,\epsilon} \in H_{2k}^q(\Omega) \cap H_{k,0}^q(\Omega)$ for all $q > 1$ be such that

$$\left\{ \begin{array}{cc} P\varphi_{\alpha,\epsilon} - V_\epsilon \varphi_{\alpha,\epsilon} = f_\alpha & \text{in } \Omega \\ \partial_\nu^i \varphi_{\epsilon|\partial\Omega} = 0 & \text{for } i = 0, ..., k-1 \end{array} \right\} \text{ in the weak sense.} \quad (53)$$

It follows from Sobolev's embedding theorem that $\varphi_{\alpha,\epsilon} \in C^{2k-1}(\overline{\Omega})$. We define

$$\tilde{\varphi}_{\alpha,\epsilon}(X) := \alpha^{\frac{n-2k}{2}} \varphi_{\alpha,\epsilon}(\alpha X) \text{ for all } X \in \mathbb{R}^n - \{0\}, \ |\alpha X| < d(0,\partial\Omega). \quad (54)$$

A change of variable yields

$$\|f_\alpha\|_{\frac{2n}{n+2k}}^{\frac{2n}{n+2k}} = \int_\Omega |f_\alpha(x)|^{\frac{2n}{n+2k}} dx = \int_{\alpha\mathcal{U}'} |f_\alpha(x)|^{\frac{2n}{n+2k}} dx = \int_{\mathcal{U}'} |f(X)|^{\frac{2n}{n+2k}} dX.$$

Therefore

$$\|f_\alpha\|_{L^{\frac{2n}{n+2k}}(\Omega)} = \|f\|_{L^{\frac{2n}{n+2k}}(\mathcal{U}')}. \quad (55)$$

With (36), (53) and the Sobolev inequality (4), we get

$$\begin{aligned} \frac{1}{2L} \|\varphi_{\alpha,\epsilon}\|_{H_{k,0}^2(\Omega)}^2 &\leq \int_\Omega \varphi_{\alpha,\epsilon}(P - V_\epsilon)\varphi_{\alpha,\epsilon} \, dx = \int_\Omega f_\alpha \varphi_{\alpha,\epsilon} \, dx \\ &\leq \|f_\alpha\|_{L^{\frac{2n}{n+2k}}(\Omega)} \|\varphi_{\alpha,\epsilon}\|_{L^{\frac{2n}{n-2k}}(\Omega)} \leq \sqrt{K(n,k)} \|f_\alpha\|_{L^{\frac{2n}{n+2k}}(\Omega)} \|\varphi_{\alpha,\epsilon}\|_{H_{k,0}^2(\Omega)}. \end{aligned}$$

Therefore, using again the Sobolev inequality (4) and (55), we get that

$$\|\varphi_{\alpha,\epsilon}\|_{L^{\frac{2n}{n-2k}}(\Omega)} \leq C(n,k,L)\|f\|_{L^{\frac{2n}{n+2k}}(\mathcal{U}')} \tag{56}$$

Equation (53) rewrites

$$\Delta^k\tilde{\varphi}_{\alpha,\epsilon} + \alpha^{2k}h(\alpha\cdot)\tilde{\varphi}_{\alpha,\epsilon} - \alpha^{2k}V_\epsilon(\alpha X)\tilde{\varphi}_{\alpha,\epsilon} = f \tag{57}$$

weakly locally in $\mathbb{R}^n$. Since V_ϵ satisfies (35), we have that

$$\left|\alpha^{2k}V_\epsilon(\alpha X)\right| \leq \mu|X|^{-2k} \text{ for all } X \in \mathcal{V}' - \{0\}$$

Since $f(X) = 0$ for all $X \in \mathcal{V}'$ and $\tilde{\varphi}_{\alpha,\epsilon} \in H^q_{2k,loc}(\mathcal{V}')$, it follows from the regularity Lemma 6.1 that there exists $\mu = \mu(\gamma) > 0$ such that for any $\delta > 0$ such that $B_\delta(0) \subset\subset \mathcal{V}'$, there exists $C(L,\delta,\gamma,\mathcal{V}') > 0$ such that

$$|X|^\gamma\,|\tilde{\varphi}_{\alpha,\epsilon}(X)| \leq C(L,\delta,\gamma,U,U')\|\tilde{\varphi}_{\alpha,\epsilon}\|_{L^{2^\star}(\mathcal{V}')} \text{ for all } X \in B_\delta(0) - \{0\}$$

Since the coefficients are uniformly bounded outside 0, classical elliptic regularity yields

$$|\tilde{\varphi}_{\alpha,\epsilon}(X)| \leq C(L,\delta,\gamma,\mathcal{V},\mathcal{V}')\|\tilde{\varphi}_{\alpha,\epsilon}\|_{L^{2^\star}(\mathcal{V}')} \text{ for all } X \in \mathcal{V} - B_\delta(0)$$

These two inequalities yield the existence of $C(L,\delta,\gamma,\mathcal{V},\mathcal{V}')$ such that

$$|X|^\gamma\,|\tilde{\varphi}_{\alpha,\epsilon}(X)| \leq C\|\tilde{\varphi}_{\alpha,\epsilon}\|_{L^{2^\star}(\mathcal{V}')} \text{ for all } X \in \mathcal{V} - \{0\} \tag{58}$$

Arguing as in the proof of (55), we have that

$$\|\tilde{\varphi}_{\alpha,\epsilon}\|_{L^{\frac{2n}{n-2k}}(\mathcal{V}')} \leq \|\varphi_{\alpha,\epsilon}\|_{L^{\frac{2n}{n-2k}}(\Omega)}. \tag{59}$$

Putting together (54), (56), (58) and (59) we get that

$$|X|^\gamma\left|\alpha^{\frac{n-2k}{2}}\varphi_{\alpha,\epsilon}(\alpha X)\right| \leq C(L,\delta,\mu,\gamma,\mathcal{V},\mathcal{V}')\|f\|_{L^{\frac{2n}{n+2k}}(\mathcal{U}')} \tag{60}$$

for all $X \in \mathcal{V} - \{0\}$. For $\alpha > 0$, we define

$$\tilde{G}_{\alpha,\epsilon}(X,Y) := \alpha^{n-2k}G_\epsilon(\alpha X,\alpha Y) \text{ for } (X,Y) \in \mathcal{V}' \times \mathcal{U}',\ X \neq 0 \tag{61}$$

It follows from Green's representation formula for G_ϵ, $\epsilon > 0$, and (53) that

$$\varphi_{\alpha,\epsilon}(\alpha X) = \int_\Omega G_\epsilon(\alpha X, y)\, f_\alpha(y)\, dy$$

for all $X \in \mathcal{V} - \{0\}$. With a change of variable, we then get that

$$\alpha^{\frac{n-2k}{2}}\varphi_{\alpha,\epsilon}(\alpha X) = \int_{\mathcal{U}'}\tilde{G}_{\alpha,\epsilon}(X,Y)f(Y)\,dY \tag{62}$$

for all $X \in \mathcal{V} - \{0\}$. Putting together (60) and (62), we get that

$$\left||X|^\gamma\int_{\mathcal{U}'}\tilde{G}_{\alpha,\epsilon}(X,Y)f(Y)\,dY\right| \leq C(L,\delta,\mu,\gamma,\mathcal{V},\mathcal{V}',\omega')\|f\|_{L^{\frac{2n}{n+2k}}(\mathcal{U}')}$$

for all $f \in C^\infty_c(\mathcal{U}')$ and $X \in \mathcal{V} - \{0\}$. It then follows from duality arguments that

$$\||X|^\gamma\tilde{G}_{\alpha,\epsilon}(X,\cdot)\|_{L^{2^\star}(\mathcal{U}')} \leq C(L,\delta,\mu,\gamma,\mathcal{V},\mathcal{V}',\mathcal{U}') \text{ for } X \in \mathcal{V} - \{0\} \tag{63}$$

Since $G_\epsilon(x,\cdot)$ is a solution to $(P - V_\epsilon)G_\epsilon(x,\cdot) = 0$ in $\Omega - \{0,x\}$, as in (57), we get that

$$\begin{aligned}&\Delta^k\tilde{G}_{\alpha,\epsilon}(X,\cdot) + \alpha^{2k}h(\alpha\cdot)\tilde{G}_{\alpha,\epsilon}(X,\cdot)\\ &-\alpha^{2k}V_\epsilon(\alpha\cdot)\tilde{G}_{\alpha,\epsilon}(X,\cdot) = 0 \text{ in } \mathcal{U}' \subset\subset \mathbb{R}^n - \{0,X\}\end{aligned}$$

Since $\mathcal{U}' \subset\subset \mathbb{R}^n - \{0, X\}$, there exists $c_{\mathcal{U}'} > 0$ such that $|Y| \geq c_{\mathcal{U}'}$ for all $Y \in \mathcal{U}'$. Since V_ϵ satisfies (35), we have that

$$\left|\alpha^{2k} V_\epsilon(\alpha Y)\right| \leq \mu c_{\omega'}^{-2k} \text{ for all } Y \in \mathcal{U}'.$$

It then follows from elliptic regularity theory (see Theorem 7.1) that

$$|X|^\gamma |\tilde{G}_{\alpha,\epsilon}(X,Y)| \leq C(k, L, \mu, \mathcal{U}', \mathcal{V}') \||X|^\gamma \tilde{G}_{\alpha,\epsilon}(X,\cdot)\|_{L^{2^\star}(\mathcal{U}')}$$

for all $Y \in \mathcal{U} \subset\subset \mathcal{U}'$ and $X \in \mathcal{V} - \{0\}$. The conclusion (52) of Theorem 5.3 then follows from this inequality, (63), the definition (61) of $\tilde{G}_{\alpha,\epsilon}$, the limit (42) and elliptic regularity for the derivatives along y.

5.3. **Asymptotics for the Green's function far from the singularity.** We prove an infinitesimal version of (48) and (50) for x, y far from the singularity.

Theorem 5.4. *We fix $p \in \Omega \setminus \{0\}$ and $\mathcal{U}, \mathcal{V}$ two open subsets of $\mathbb{R}^n$ such that*

$$\mathcal{U} \subset\subset \mathbb{R}^n\,,\ \mathcal{V} \subset\subset \mathbb{R}^n \text{ and } \overline{\mathcal{U}} \cap \overline{\mathcal{V}} = \emptyset.$$

We let $\alpha_0 > 0$ be such that

$$|\alpha X| < \frac{1}{2} \min\{d(0, \partial\Omega), |p|, d(p, \partial\Omega)\} \text{ for all } 0 < \alpha < \alpha_0 \text{ and } X \in \mathcal{V} \cup \mathcal{U}. \tag{64}$$

Then for all $\gamma \in (0, n-2k)$, there exists $\mu = \mu(\gamma) > 0$ and $C(\mathcal{V}, \mathcal{U}, L, \alpha_0, \gamma, \mu) > 0$ such that

$$\left|\alpha^{n-2k+l} \nabla_y^l G_\epsilon(p + \alpha X, p + \alpha Y)\right| \leq C(U, \omega, L, \alpha_0, \gamma, \mu) \tag{65}$$

for all $X \in \mathcal{V}$ and $Y \in \mathcal{U}$, $l = 0, ..., 2k-1$, $\alpha \in (0, \alpha_0)$ and $\epsilon \geq 0$ small enough.

Proof of Theorem 5.4. We first set $\mathcal{U}', \mathcal{V}'$ two open subsets of $\mathbb{R}^n$ such that

$$\mathcal{U} \subset\subset \mathcal{U}' \subset\subset \mathbb{R}^n\,,\ \mathcal{V} \subset\subset \mathcal{V}' \subset\subset \mathbb{R}^n \text{ and } \overline{\mathcal{U}'} \cap \overline{\mathcal{V}'} = \emptyset$$

and (64) still holds for $X \in \mathcal{V}' \cup \mathcal{U}'$. We fix $f \in C_c^\infty(\mathcal{U}')$ and for any $0 < \alpha < \alpha_0$, we set

$$f_\alpha(x) := \frac{1}{\alpha^{\frac{n+2k}{2}}} f\left(\frac{x-p}{\alpha}\right) \text{ for all } x \in \Omega.$$

As one checks, $f_\alpha \in C_c^\infty(p + \alpha\mathcal{U}')$ and $p + \alpha\mathcal{U}' \subset\subset \Omega \setminus \{0\}$. It follows from Theorem 7.3 that there exists $\varphi_{\alpha,\epsilon} \in H_{2k}^q(\Omega) \cap H_{k,0}^q(\Omega)$ for all $q > 1$ such that

$$\left\{ \begin{array}{cc} P\varphi_{\alpha,\epsilon} - V_\epsilon \varphi_{\alpha,\epsilon} = f_\alpha & \text{in } \Omega \\ \partial_\nu^i \varphi_{\alpha,\epsilon|\partial\Omega} = 0 & \text{for } i = 0, ..., k-1. \end{array} \right\} \tag{66}$$

It follows from Sobolev's embedding theorem that $\varphi_{\alpha,\epsilon} \in C^{2k-1}(\overline{\Omega})$. We define

$$\tilde{\varphi}_{\alpha,\epsilon}(X) := \alpha^{\frac{n-2k}{2}} \varphi_{\alpha,\epsilon}(p + \alpha X) \text{ for all } X \in \mathbb{R}^n,\ |\alpha X| < d(p, \partial\Omega). \tag{67}$$

As in (55) and (56), we get

$$\|f_\alpha\|_{L^{\frac{2n}{n+2k}}(\Omega)} = \|f\|_{L^{\frac{2n}{n+2k}}(\mathcal{U}')} \text{ and } \|\varphi_{\alpha,\epsilon}\|_{L^{\frac{2n}{n-2k}}(\Omega)} \leq C(n,k,L)\|f\|_{L^{\frac{2n}{n+2k}}(\mathcal{U}')}. \tag{68}$$

Equation (66) rewrites

$$\Delta^k \tilde{\varphi}_{\alpha,\epsilon} + \alpha^{2k} h(p + \alpha\cdot)\tilde{\varphi}_{\alpha,\epsilon} - \alpha^{2k} V_\epsilon(p + \alpha X)\tilde{\varphi}_{\alpha,\epsilon} = f \tag{69}$$

weakly in $\mathbb{R}^n$. Since V_ϵ satisfies (35), we have that

$$\left|\alpha^{2k} V_\epsilon(p + \alpha X)\right| \leq \mu \alpha^{2k} |p + \alpha X|^{-2k} \text{ for all } X \in \mathcal{V}'$$

With (64), we get that

$$\left|\alpha^{2k}V_\epsilon(p+\alpha X)\right| \le \mu\left(\frac{2|p|}{\alpha}\right)^{-2k} \le C(\mu_0) \text{ for all } X \in \mathcal{V}'$$

Since $f(X) = 0$ for all $X \in \mathcal{V}'$, it follows from standard regularity theory (see Theorem 7.1) that there exists $C(k, L, \mathcal{V}, \mathcal{V}', \mathcal{U}, \mathcal{U}', \alpha_0) > 0$ such that

$$|\tilde\varphi_{\alpha,\epsilon}(X)| \le C\|\tilde\varphi_{\alpha,\epsilon}\|_{L^{2^\star}(\mathcal{V}')} \text{ for all } X \in \mathcal{V} \tag{70}$$

Arguing as in the proof of (55), we have that

$$\|\tilde\varphi_{\alpha,\epsilon}\|_{L^{\frac{2n}{n-2k}}(\mathcal{V}')} \le \|\varphi_{\alpha,\epsilon}\|_{L^{\frac{2n}{n-2k}}(\Omega)}. \tag{71}$$

Putting together (67), (70), (71) and (68) we get that

$$\left|\alpha^{\frac{n-2k}{2}}\varphi_{\alpha,\epsilon}(p+\alpha X)\right| \le C(k, L, \mathcal{V}, \mathcal{V}', \mu)\|f\|_{L^{\frac{2n}{n+2k}}(\mathcal{U}')} \text{ for all } X \in \mathcal{V}. \tag{72}$$

We now just follow verbatim the proof of Theorem 5.3 above to get the conclusion (65) of Theorem 5.4. We leave the details to the reader.

5.4. **Proof of Theorem 5.2.** We let Ω, k, μ, L, P, V as in the statement of Theorem 5.2. With $\mu > 0$ small enough, we let G_0 be the Green's function of $P - V$ as in Theorem 5.1. Given $\gamma \in (0, n-2k)$, we let $\mu_\gamma > 0$ as in (50) and Theorems 5.3 and 5.4 hold when $0 < \mu < \mu_\gamma$. We prove here the first estimate of Theorem 5.2 by contradiction. We fix $\omega \subset\subset \Omega$ and we assume that there is a family of operators $(P_i)_{i\in\mathbb{N}}$ such that P_i satisfies (33) for all i, a family of potentials $(V_i)_{i\in\mathbb{N}} \in \mathcal{P}_{\mu_\gamma}$, sequences $(x_i), (y_i) \in \Omega \setminus \{0\}$ such that $x_i \neq y_i$ and $x_i \in \omega$ for all $i \in \mathbb{N}$ and

$$\lim_{i\to+\infty} \frac{|x_i - y_i|^{n-2k}|G_i(x_i, y_i)|}{\left(\frac{\max\{|x_i|,|y_i|\}}{\min\{|x_i|,|y_i|\}}\right)^\gamma} = +\infty, \tag{73}$$

where G_i denotes the Green's function of $P_i - V_i$ for all $i \in \mathbb{N}$. We distinguish 5 cases:

Case 1: $|x_i - y_i| = o(|x_i|)$ as $i \to +\infty$. It then follows from the triangle inequality that $|x_i - y_i| = o(|y_i|)$ and $|x_i| = (1 + o(1))|y_i|$. Therefore

$$\left(\frac{\max\{|x_i|,|y_i|\}}{\min\{|x_i|,|y_i|\}}\right)^\gamma = 1 + o(1)$$

and then (73) yields

$$\lim_{i\to+\infty} |x_i - y_i|^{n-2k}|G_i(x_i, y_i)| = +\infty \tag{74}$$

We let $Y_i \in \mathbb{R}^n$ be such that $y_i := x_i + |x_i - y_i|Y_i$. In particular, $|Y_i| = 1$, so, up to a subsequence, there exists $Y_\infty \in \mathbb{R}^n$ such that $\lim_{n\to+\infty} Y_i = Y_\infty$ with $|Y_\infty| = 1$ Note that since $x_i \in \omega$, there exists $\epsilon_0 > 0$ such that $d(x_i, \partial\Omega) \ge \epsilon_0$ for all i. We apply Theorem 5.4 with $p := x_i$, $\alpha := |x_i - y_i|$, $\mathcal{V} = B_{1/3}(0)$, $\mathcal{U} = B_{1/3}(Y_\infty)$: for $i \in \mathbb{N}$ large enough, taking $X = 0$ and $Y = Y_i$ in (65), we get that

$$|x_i - y_i|^{n-2k}|G_i(x_i, y_i)| = |x_i - y_i|^{n-2k}|G_i(x_i + |x_i - y_i|\cdot 0, x_i + |x_i - y_i|\cdot Y_i)| \le C(L, \gamma, \mu)$$

which contradicts (74). This ends Case 1.

The case $|x_i - y_i| = o(|y_i|)$ as $i \to +\infty$ is equivalent to Case 1.

Case 2: $|x_i| = o(|x_i - y_i|)$ and $|x_i - y_i| \not\to 0$ as $i \to +\infty$. Therefore (73) rewrites

$$\lim_{i\to+\infty} |x_i|^\gamma |G_i(x_i, y_i)| = +\infty \tag{75}$$

This is a contradiction with (50) when $\epsilon = 0$. This ends Case 2 by using the symmetry of G.

Case 3: $|x_i| = o(|x_i - y_i|)$ and $|x_i - y_i| \to 0$ as $i \to +\infty$. Then $|x_i| = o(|y_i|)$ and $|x_i - y_i| = (1 + o(1))|y_i|$. In particular, $x_i, y_i \to 0$ as $i \to +\infty$. Therefore (73) rewrites

$$\lim_{i\to+\infty} |x_i - y_i|^{n-2k-\gamma} |x_i|^\gamma |G_i(x_i, y_i)| = +\infty \tag{76}$$

We let $X_i, Y_i \in \mathbb{R}^n$ be such that $x_i := |x_i - y_i|X_i$ and $y_i := |x_i - y_i|Y_i$. In particular, $\lim_{i\to+\infty} |X_i| = 0$ and $|Y_i| = 1 + o(1)$. So, up to a subsequence, there exists $Y_\infty \in \mathbb{R}^n$ such that $\lim_{n\to+\infty} Y_i = Y_\infty$ with $|Y_\infty| = 1$. We apply Theorem 5.3 with $\alpha := |x_i - y_i|$, $\mathcal{V} = B_{1/3}(0)$, $\mathcal{U} = B_{1/3}(Y_\infty)$: for $i \in \mathbb{N}$ large enough, taking $X = X_i \neq 0$ and $Y = Y_i$ in (52), we get that

$$|X_i|^\gamma |x_i - y_i|^{n-2k} |G_i(|x_i - y_i|X_i, |x_i - y_i|Y_i)| \leq C(\mu, k, L),$$

and, coming back to the definitions of X_i and Y_i, we get a contradiction with (76). This ends Case 3.

Case 4: $|y_i| = o(|x_i - y_i|)$ as $i \to +\infty$. Since the Green's function is symmetric, this is similar to Case 2 and 3.

Case 5: $|x_i| \asymp |y_i| \asymp |x_i - y_i|$. Then (73) rewrites

$$\lim_{i\to+\infty} |x_i - y_i|^{n-2k} |G_i(x_i, y_i)| = +\infty \tag{77}$$

Case 5.1: $|x_i - y_i| \not\to 0$ as $i \to +\infty$. Then it follows from (48) that $|G_i(x_i, y_i)| \leq C(\mu, k, L)$ for all i, which contradicts (77).

Case 5.2: $|x_i - y_i| \to 0$ as $i \to +\infty$. We let $X_i, Y_i \in \mathbb{R}^n$ be such that $x_i := |x_i - y_i|X_i$ and $y_i := |x_i - y_i|Y_i$. In particular, there exists $c > 0$ such that $c^{-1} < |X_i|, |Y_i| < c$ and $|X_i - Y_i| \geq c^{-1}$ for all i. So, up to a subsequence, there exists $X_\infty, Y_\infty \in \mathbb{R}^n$ such that $\lim_{n\to+\infty} X_i = X_\infty \neq 0$ and $\lim_{n\to+\infty} Y_i = Y_\infty \neq 0$ and $X_\infty \neq Y_\infty$. We apply Theorem 5.3 with $\alpha := \alpha_i = |x_i - y_i|$, $\mathcal{V} = B_{r_0}(X_\infty)$, $\mathcal{U} = B_{r_0}(Y_\infty)$ for some $r_0 > 0$ small enough. So for $i \in \mathbb{N}$ large enough, taking $X = X_i \neq 0$ and $Y = Y_i$ in (52), we get that

$$|X_i|^\gamma \alpha_i^{n-2k} |G_i(\alpha_i X_i, \alpha_i Y_i)| \leq C(U, \omega, L, \gamma, \mu)$$

and, coming back to the definitions of X_i and Y_i, we get that a contradiction with (77). This ends Case 5.

Therefore, in all 5 cases, we have obtained a contradiction with (73). This proves the first estimate of Theorem 5.2. The proof of the estimates on the derivative uses the same method by contradiction, with a few more cases to study using regularity theory (Theorem 7.1). We leave the details to the reader.

6. The regularity Lemma

For any domain $D \subset \mathbb{R}^n$, $k \in \mathbb{N}$ such that $2 \leq 2k < n$ and $L > 0$, we say that an operator P is of type $O_{k,L}(D)$ if $P := \Delta^k + h$, where $h \in L^\infty(D)$ and $\|h\|_\infty \leq L$.

Lemma 6.1. *Let $k \in \mathbb{N}$ be such that $2 \leq 2k < n$ and $\delta, L > 0$. Fix $p > 1$ and $\delta_1, \delta_2 > 0$ such that $0 < \delta_1 < \delta_2$. We consider a differential operator $P \in O_{k,L}(B_{\delta_2}(0))$ where $B_{\delta_2}(0) \subset \mathbb{R}^n$. Then for all $0 < \gamma < n - 2k$, there exists $\mu = \mu(\gamma, p, L, \delta_1, \delta_2) > 0$ and $C_0 = C_0(\gamma, p, L, \delta_1, \delta_2) > 0$ such that for any $V \in L^1(B_{\delta_2}(0))$ such that*

$$|V(x)| \leq \mu |x|^{-2k} \text{ for all } x \in B_{\delta_2}(0),$$

then for any $\varphi \in H^2_k(B_{\delta_2}(0)) \cap H^s_{2k,loc}(B_{\delta_2}(0) - \{0\})$ (for some $s > 1$) such that

$$P\varphi - V \cdot \varphi = 0 \text{ weakly in } H^2_k(B_{\delta_2}(0)),$$

then we have that

$$|x|^\gamma |\varphi(x)| \leq C_0 \cdot \|\varphi\|_{L^p(B_{\delta_2}(0))} \text{ for all } x \in B_{\delta_1}(0) - \{0\}. \tag{78}$$

and

$$\|\varphi\|_{H^2_k(B_{\delta_1}(0))} \leq C_0 \cdot \|\varphi\|_{L^p(B_{\delta_2}(0))}.$$

Moreover, for any $0 < l < 2k$, there exists $C_l = C_l(\gamma, p, L, \delta_1, \delta_2) > 0$ such that

$$|x|^{\gamma+l} |\nabla^l \varphi(x)| \leq C_l \cdot \|\varphi\|_{L^p(B_{\delta_2}(0))} \text{ for all } x \in B_{\delta_1}(0) - \{0\} \tag{79}$$

For the reader's convenience, we set $\delta := \delta_1$ and we assume that $\delta_2 = 3\delta_1 = 3\delta$. The general case follows the same proof by changing 2δ, 2.9δ, etc, into various radii $\delta', \delta'', ...$ such that $\delta_1 < \delta' < \delta'' < \delta_2$, etc. We split the proof of the Lemma in two steps.

Step 1: Proof of (78) when $V \equiv 0$ around 0. We prove (78) by contradiction under the assumption that V vanishes around 0. We assume that there exists $\gamma \in (0, n - 2k)$, $p > 1$, $L > 0$, $\delta > 0$ such that for all $\mu > 0$, there exists a differential operator $P_\mu = \Delta^k + h_\mu$ and a potential $V_\mu \in L^1(B_{3\delta}(0))$ such that there exists $\varphi_\mu \in H^2_k(B_{3\delta}(0)) \cap H^s_{2k,loc}(B_{3\delta}(0) - \{0\})$ (for some $s > 1$) such that

$$\left\{ \begin{array}{l} (P_\mu - V_\mu)\psi_\mu = 0 \text{ weakly in } H^2_k(B_{3\delta}(0)) \cap H^s_{2k,loc}(B_{3\delta}(0) - \{0\}) \\ \|\psi_\mu\|_{L^p(B_{3\delta}(0)} = 1 \\ |V_\mu(x)| \leq \mu |x|^{-2k} \text{ for all } x \in B_{3\delta}(0) - \{0\} \\ V_\mu \equiv 0 \text{ around } 0 \\ \sup_{x \in \overline{B_\delta(0)}} |x|^\gamma |\psi_\mu(x)| > \frac{1}{\mu} \to +\infty \text{ as } \mu \to 0 \end{array} \right\} \tag{80}$$

With our assumption that V_μ vanishes around 0, we get that $V_\mu \in L^\infty(B_{3\delta}(0))$. Then, by regularity theory (see Theorem 7.1), we get that $\psi_\mu \in C^0(B_{2\delta}(0))$. Therefore, there exists $x_\mu \in \overline{B_\delta(0)}$ such that

$$|x_\mu|^\gamma |\psi_\mu(x_\mu)| = \sup_{x \in \overline{B_\delta(0)}} |x|^\gamma |\psi_\mu(x)| > \frac{1}{\mu} \to +\infty \tag{81}$$

as $\mu \to 0$.

Step 1.1: We claim that $\lim_{\mu \to 0} x_\mu = 0$.

We prove the claim. For any $r > 0$, we have that $|V_\mu(x)| \leq \mu r^{-2k}$ for all $x \in B_{3\delta}(0) \setminus B_r(0)$. So, with regularity theory (see Theorem 7.1), we get that for all $q > 1$, then $\|\psi_\mu\|_{H^q_{2k}(B_{2\delta}(0) \setminus B_{2r}(0))} = C(r, q, L, p,)\|\psi_\mu\|_{L^p(B_{3\delta}(0)} \leq C(r, q, L, p)$.

Taking $q > \frac{n}{2k}$, we get that $|\psi_\mu(x)| \leq C(r,q,L,p)$ for all $x \in B_{2\delta}(0) \setminus B_{2r}(0)$. With (81), this forces $\lim_{\mu\to 0} x_\mu = 0$. The claim is proved.

Step 1.2: Convergence after rescaling. We set $r_\mu := |x_\mu| > 0$ and we define

$$\tilde{\psi}_\mu(X) := \frac{\psi_\mu(r_\mu X)}{\psi_\mu(x_\mu)} \text{ for } X \in \mathbb{R}^n - \{0\} \text{ such that } |X| < \frac{\delta}{r_\mu}.$$

We define $X_\mu \in \mathbb{R}^n$ such that $x_\mu = r_\mu X_\mu$. In particular $|X_\mu| = 1$. With the definition of x_μ, for any $X \in \mathbb{R}^n$ such that $0 < |X| < \frac{\delta}{r_\mu}$, we have that

$$r_\mu^\gamma |X|^\gamma |\tilde{\psi}_\mu(X)| \quad = \quad \frac{|r_\mu X|^\gamma |\psi_\mu(r_\mu X)|}{|\psi_\mu(x_\mu)|} \leq \frac{|x_\mu|^\gamma |\psi_\mu(x_\mu)|}{|\psi_\mu(x_\mu)|} = r_\mu^\gamma.$$

Therefore, we get that

$$|X|^\gamma |\tilde{\psi}_\mu(X)| \leq 1 \text{ for all } X \in \mathbb{R}^n \text{ such that } 0 < |X| < \frac{\delta}{r_\mu} \text{ and } \tilde{\psi}_\mu(X_\mu) = 1. \quad (82)$$

The equation satisfied by $\tilde{\psi}_\mu$ in (80) rewrites

$$\Delta^k \tilde{\psi}_\mu + r_\mu^{2k} h_\mu(r_\mu \cdot)\tilde{\psi}_\mu - r_\mu^{2k} V_\mu(r_\mu X)\tilde{\psi}_\mu = 0 \quad (83)$$

weakly in $B_{3\delta/r_\mu}(0) - \{0\}$. Note that

$$|r_\mu^{2k} V_\mu(r_\mu X)| \leq \mu |X|^{-2k} \text{ for all } \mu > 0 \text{ and } 0 < |X| < \frac{3\delta}{r_\mu}. \quad (84)$$

With the bound (82) and the bounds of the coefficient h_μ, it follows from regularity theory (see Theorem 7.1) that for any $R > 0$ and any $0 < \nu < 1$, there exists $C(R) > 0$ such that $\|\tilde{\psi}_\mu\|_{C^{2k-1,\nu}(B_R(0) - B_{R^{-1}}(0))} \leq C(R,\nu)$ for all $\mu > 0$. Ascoli's theorem yields the existence of $\tilde{\psi} \in C^{2k-1}(\mathbb{R}^n - \{0\})$ such that $\tilde{\psi}_\mu \to \tilde{\psi}$ in $C^{2k-1}_{loc}(\mathbb{R}^n - \{0\})$ as $\mu \to 0$. Passing to the limit $\mu \to 0$ in (83), we get that $\Delta^k \tilde{\psi} = 0$ weakly in $\mathbb{R}^n - \{0\}$ and regularity yields $\tilde{\psi} \in C^{2k}(\mathbb{R}^n - \{0\})$. We define $X_0 := \lim_{\mu\to 0} X_\mu$, so that $|X_0| = 1$. Finally, passing to the limit in (82) yields

$$\left\{ \begin{array}{l} \tilde{\psi} \in C^{2k}(\mathbb{R}^n - \{0\}) \\ \Delta^k \tilde{\psi} = 0 \text{ in } \mathbb{R}^n - \{0\} \\ \tilde{\psi}(X_0) = 1 \text{ with } |X_0| = 1 \\ |\tilde{\psi}(X)| \leq |X|^{-\gamma} \text{ for all } X \in \mathbb{R}^n - \{0\}. \end{array} \right\} \quad (85)$$

By standard elliptic theory (see Theorems 7.1 and 7.2), for any $l = 1, ..., 2k$, there exists $C_l > 0$ such that

$$|\nabla^l \tilde{\psi}(X)| \leq C_l |X|^{-\gamma - l} \text{ for all } X \in \mathbb{R}^n - \{0\}. \quad (86)$$

Step 1.3: Contradiction via Green's formula. Let us consider the Poisson kernel of Δ^k at X_0, namely

$$\Gamma_{X_0}(X) := C_{n,k} |X - X_0|^{2k-n} \text{ for all } X \in \mathbb{R}^n - \{X_0\},$$

where

$$C_{n,k} := \frac{1}{(n-2)\omega_{n-1} \Pi_{i=1}^{k-1} (n - 2k + 2(i-1))(2k - 2i)}. \quad (87)$$

Let us choose $R > 3$ and $0 < \epsilon < 1/2$ and define the domain

$$\Omega_{R,\epsilon} := B_R(0) \setminus \left(B_{R^{-1}}(0) \cup B_\epsilon(X_0) \right).$$

Note that all the balls involved here have boundaries that do not intersect. With (25), we get

$$\int_{\Omega_{R,\epsilon}} (\Delta^k \Gamma_{X_0})\tilde{\psi}\, dX = \int_{\Omega_{R,\epsilon}} \Gamma_{X_0}(\Delta^k \tilde{\psi})\, dX + \int_{\partial\Omega_{R,\epsilon}} \sum_{i=0}^{k-1} \mathcal{B}^{(i)}(\Gamma_{X_0}, \tilde{\psi})\, d\sigma \qquad (88)$$

where the $\mathcal{B}^{(i)}$ are as in (26). We have that $\partial\Omega_{R,\epsilon} = \partial B_R(0) \cup \partial B_{R^{-1}}(0) \cup \partial B_\epsilon(X_0)$. Using that Γ_{X_0} is smooth at 0, that $\tilde{\psi}$ is smooth at X_0, using the bounds (86) and the corresponding ones for Γ_{X_0}, for any $i = 0, ..., k-1$, we get that

$$\left| \int_{\partial B_R(0)} \mathcal{B}^{(i)}(\Gamma_{X_0}, \tilde{\psi})\, d\sigma \right| \le CR^{-\gamma}\,, \quad \left| \int_{\partial B_\epsilon(X_0)} \mathcal{B}^{(i)}(\Gamma_{X_0}, \tilde{\psi})\, d\sigma \right| \le C\epsilon^{2i}$$

$$\left| \int_{\partial B_{R^{-1}}(0)} \mathcal{B}^{(i)}(\Gamma_{X_0}, \tilde{\psi})\, d\sigma \right| \le CR^{2-n+\gamma+2i} \le CR^{-(n-2k-\gamma)}$$

and

$$\left| \int_{\partial B_\epsilon(X_0)} \Delta^{k-1}\Gamma_{X_0} \partial_\nu \tilde{\psi}\, d\sigma \right| \le C\epsilon^{n-1}\epsilon^{2k-n-2(k-1)} \le C\epsilon.$$

Therefore, since $0 < \gamma < n-2k$, all the terms involving R go to 0 as $R \to +\infty$, the terms involving ϵ go to 0 when $i \neq 0$. Since $\Delta^k \Gamma_{X_0} = 0$, $\Delta^k \tilde{\psi} = 0$, it follows from (88) and the inequalities above that

$$\int_{\partial B_\epsilon(X_0)} \partial_\nu \Delta^{k-1}\Gamma_{X_0} \tilde{\psi}\, d\sigma = o(1) \text{ as } \epsilon \to 0.$$

With the definition of Γ_{X_0}, we get that

$$-\partial_\nu \Delta^{k-1}\Gamma_{X_0}(X) = \frac{1}{\omega_{n-1}} |X - X_0|^{1-n} \text{ for } X \neq X_0. \qquad (89)$$

So that, with a change of variable, we get that

$$\int_{\partial B_1(0)} \tilde{\psi}(X_0 + \epsilon X)\, d\sigma = o(1) \text{ as } \epsilon \to 0.$$

Passing to the limit, we get that $\tilde{\psi}(X_0) = 0$, which is a contradiction with (85). This proves (78) when V vanishes around 0.

Step 2: The general case. Let $\eta \in C^\infty(\mathbb{R})$ be such that $\eta(t) = 0$ if $t \le 1$, $\eta(t) = 1$ if $t \ge 2$ and $0 \le \eta \le 1$. For any $\epsilon > 0$, define $V_\epsilon(x) := \eta(|x|/\epsilon)V(x)$ for all $x \in B_{\delta_2}(0)$. Up to taking $\delta_2 > 0$ small enough to get coercivity, for any $\epsilon > 0$, there exists $\varphi_\epsilon \in H^2_k(B_{\delta_2}(0)) \cap H^q_{2k}(B_{\delta_2}(0))$ for all $q > 1$ such that

$$\begin{cases} (P - V_\epsilon)\varphi_\epsilon = 0 & \text{in } B_{\delta_2}(0) \\ \partial^i_\nu \varphi_\epsilon = \partial^i_\nu \varphi & \text{on } \partial B_{\delta_2}(0) \text{ for } i = 0, \cdots, k-1 \end{cases} \qquad (90)$$

As one checks, $\lim_{\epsilon\to 0} \varphi_\epsilon = \varphi$ in $H^2_k(B_{\delta_2}(0))$ and $\lim_{\epsilon\to 0} \varphi_\epsilon(x) = \varphi(x)$ for all $x \in \bar{B}_{\delta_1}(0) - \{0\}$. Since V_ϵ vanishes around 0, we apply (85) to φ_ϵ and let $\epsilon \to 0$. We leave the details to the reader. The estimates on the derivatives are consequence of elliptic theory.

6.1. **Green's function for elliptic operators with bounded coefficients.**

Definition 6.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$. Fix $k \in \mathbb{N}$ such that $n > 2k \geq 2$. Let P be an elliptic operator of order $2k$. A* Green's function *for P is a function $(x,y) \mapsto G(x,y) = G_x(y)$ defined for all $x \in \Omega$ and a.e. $y \in \Omega$ such that*

(i) *$G_x \in L^1(\Omega)$ for all $x \in \Omega$,*
(ii) *for all $x \in \Omega$ and all $\varphi \in C^{2k}(\overline{\Omega})$ such that $\partial_\nu^i \varphi_{|\partial\Omega} = 0$ for all $i = 0, .., k-1$, we have that*

$$\int_\Omega G_x P\varphi \, dx = \varphi(x).$$

Theorem 6.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 2$. Fix $k \in \mathbb{N}$ such that $n > 2k \geq 2$ and $L > 0$. Let P be an elliptic operator such that (33) holds. Then there exists a unique Green's function for P. Moreover,*

- *G extends to $\Omega \times \Omega \setminus \{(x,x)/x \in \Omega\}$ and for any $x \in \Omega$, $G_x \in H^2_{k,0,loc}(\Omega - \{x\}) \cap H^p_{2k,loc}(\Omega - \{x\})$ for all $p > 1$ and $G_x \in C^{2k-1}(\overline{\Omega} - \{x\})$*
- *G is symmetric;*
- *For all $x \in \Omega$, we have that*

$$\left\{ \begin{array}{cc} PG_x = 0 & \text{in } \Omega \setminus \{x\} \\ \partial_\nu^i G_{x\,|\partial\Omega} = 0 & \text{for } i = 0, ..., k-1. \end{array} \right\}$$

- *For all $f \in L^p(\Omega)$, $p > \frac{n}{2k}$, and $\varphi \in H^p_{2k}(\Omega) \cap H^p_{k,0}(\Omega)$ such that $P\varphi = f$ weakly, then*

$$\varphi(x) = \int_\Omega G_x P\varphi \, dx \text{ for all } x \in \Omega.$$

- *For all $\varphi \in C^{2k}(\overline{\Omega})$, we have that*

$$\varphi(x) = \int_\Omega G_x P\varphi \, dy - \int_{\partial\Omega} C_P(\varphi, G_x) \, d\sigma \text{ for all } x \in \Omega.$$

where

$$C_P(\varphi, G_x) := - \sum_{2i+1 \leq k-1} \partial_\nu \Delta^i \varphi \Delta^{k-1-i} G_x + \sum_{2i \leq k-1} \Delta^i \varphi \partial_\nu \Delta^{k-1-i} G_x.$$

If $\partial_\nu^i \varphi = 0$ on $\partial\Omega$ for all $i = 0, ..., k-1$, then $C_P(\varphi, G_x) \equiv 0$ on $\partial\Omega$.

- *For all $\omega \subset\subset \Omega$, There exists $C(k, L, \omega) > 0$ such that*

$$|G_x(y)| \leq C(k, L, \omega) \cdot |x - y|^{2k-n} \text{ for all } x \in \omega, \, y \in \Omega, \, x \neq y,$$

- *For all $l = 1, ..., 2k-1$, there exists $C_l(k, L, \omega) > 0$ such that*

$$|\nabla^l G_x(y)| \leq C_l(k, L, \omega) \cdot |x - y|^{2k-n-l} \text{ for all } x \in \omega, \, y \in \Omega, \, x \neq y;$$

The sequel of this subsection is devoted to the proof of Theorem 6.1. We build the Green's function via the classical Neumann series following Robert [21]. Let $\eta \in C^\infty(\mathbb{R})$ be such that $\eta(t) = 1$ if $t \leq 1/4$ and $\eta(t) = 0$ if $t \geq 1/2$. We define

$$\Gamma_x(y) = \Gamma(x,y) := C_{n,k} |x - y|^{2k-n} \text{ for all } x, y \in \Omega, \; x \neq y.$$

where $C_{n,k}$ is defined in (87). Note that $\Gamma_x \in C^\infty(\overline{\Omega} - \{x\})$.

Step 1: As in the proof of Step 1.3, see formula (88), for all $x \in \Omega$, there exists $f_x \in L^1(\Omega)$ such that

$$\left\{ \begin{array}{cc} P\Gamma_x = \delta_x - f_x & \text{weakly in } \Omega \\ |f_x(y)| \leq C(k, L)|x - y|^{2k-n} & \text{for all } x, y \in \Omega, \, x \neq y, \end{array} \right\} \tag{91}$$

Where the equality is to be taken in the distribution sense, that is

$$\int_\Omega \Gamma_x P\varphi\,dx = \varphi(x) - \int_\Omega f_x\varphi\,dx + \int_{\partial\Omega}\sum_{i=0}^{k-1}\mathcal{B}^{(i)}(\varphi,\Gamma_x)\,d\sigma \text{ for all } \varphi\in C^{2k}(\overline{\Omega}),$$

where the $\mathcal{B}^{(i)}$'s are defined in (26) and where $f_x := -(\Delta^k\Gamma_x + h\Gamma_x)$.

Step 2: We are now in position to define the Green's function G. We define

$$\left\{\begin{array}{cc} \Gamma_1(x,y) := f_x(y) & \text{for } x,y\in\Omega,\ x\neq y,\\ \Gamma_{i+1}(x,y) := \int_\Omega \Gamma_i(x,z)f_z(y)\,dz & \text{for } x,y\in\Omega,\ x\neq y,\ i\in\mathbb{N}\end{array}\right\}$$

With straightforward computations (Giraud's Lemma [9], as stated in [5] for instance), the definition of Γ and (91), for all $i\in\mathbb{N}$, we have that

$$|\Gamma_i(x,y)| \le C_i(k,L,\Omega)\left\{\begin{array}{cc} |x-y|^{2ki-n} & \text{if } 2ki<n;\\ 1+|\ln|x-y||& \text{if } 2ki=n;\\ 1 & \text{if } 2ki>n.\end{array}\right. \tag{92}$$

for all $x,y\in\Omega$, $x\neq y$. We then get that $\Gamma_i(x,\cdot)\in L^\infty(\Omega)$ for all $x\in\Omega$ and $i>\frac{n}{2k}$. We fix $p>n/k$. For $x\in\Omega$, we take $u_x\in H^2_{2k}(\Omega)\cap C^{2k-1}(\overline{\Omega})$ that will be fixed later, and we define

$$G_x(y) := \Gamma_x(y) + \sum_{i=1}^p\int_\Omega \Gamma_i(x,z)\Gamma(z,y)\,dz + u_x(y) \text{ for a.e } y\in\Omega. \tag{93}$$

We fix $\varphi\in C^{2k}(\overline{\Omega})$. Via Fubini's theorem, using the definition of the Γ_i's and the definition of P, we get that

$$\begin{aligned}\int_\Omega G_x P\varphi\,dy &= \int_\Omega \Gamma_x P\varphi\,dy + \sum_{i=1}^p\int_{\Omega\times\Omega}\Gamma_i(x,z)\Gamma(z,y)P\varphi(y)\,dzdy\\ &+\int_\Omega Pu_x\varphi\,dy + \int_{\partial\Omega}\sum_{i=0}^{k-1}\mathcal{B}^{(i)}(\varphi,u_x)\,d\sigma\\ &= \varphi(x) - \int_\Omega\Gamma_1(x,\cdot)\varphi\,dx + \sum_{i=1}^p\int_\Omega\Gamma_i(x,z)\varphi(z)\,dz\\ &-\sum_{i=1}^p\int_\Omega\left(\int_\Omega\Gamma_i(x,z)f_z(y)\,dz\right)\varphi(y)\,dy\\ &+\int_\Omega Pu_x\varphi\,dx + \int_{\partial\Omega}\sum_{i=0}^{k-1}\mathcal{B}^{(i)}(\varphi,G_x)\,d\sigma\\ &= \varphi(x) + \int_\Omega(Pu_x - \Gamma_{p+1}(x,\cdot))\varphi\,dy + \int_{\partial\Omega}\sum_{i=0}^{k-1}\mathcal{B}^{(i)}(\varphi,G_x)\,d\sigma\end{aligned}$$

Since $\Gamma_{p+1}(x,\cdot)\in L^\infty(\Omega)$, we choose $u_x\in\cap_{q>1}H^q_{2k}(\Omega)\cap H^q_{k,0}(\Omega)$ such that

$$\left\{\begin{array}{cc} Pu_x = \Gamma_{p+1}(x,\cdot) & \text{in } \Omega.\\ \partial^i_\nu u_x = -\partial^i_\nu\left(\Gamma_x + \sum_{i=1}^p\int_\Omega\Gamma_i(x,z)\Gamma(z,\cdot)\,dz\right) & \text{on } \partial\Omega\end{array}\right\}$$

The existence follows from Theorem 7.3. Sobolev's embedding theorem yields $u_x\in C^{2k-1}(\overline{\Omega})$ and Theorem 7.3 yields $C(k,L,p,\omega)>0$ such that

$$|u_x(y)|\le C(k,L,p,\omega) \text{ for all } x\in\omega,\ y\in\Omega. \tag{94}$$

In particular, $G_x \in C^{2k-1}(\overline{\Omega} \setminus \{x\})$ and $\partial_\nu^i G_x = 0$ on $\partial\Omega$ and $i = 0, ..., k-1$. Finally, we get that

$$\int_\Omega G_x P\varphi \, dy = \varphi(x) + \int_{\partial\Omega} \sum_{i=0}^{k-1} \mathcal{B}^{(i)}(\varphi, G_x) \, d\sigma \text{ for all } \varphi \in C^{2k}(\overline{\Omega}). \tag{95}$$

Note that since $\partial_\nu^i G_x = 0$ on $\partial\Omega$ and $i = 0, ..., k-1$, then $\nabla^i G_x = 0$ on $\partial\Omega$ for $i = 0, ..., k-1$ and then we have that

$$\sum_{i=0}^{k-1} \mathcal{B}^{(i)}(\varphi, G_x) = - \sum_{2i+1 \le k-1} \partial_\nu \Delta^i \varphi \Delta^{k-1-i} G_x + \sum_{2i \le k-1} \Delta^i \varphi \partial_\nu \Delta^{k-i-1} G_x$$

The controls (92) and (94), the definition (93) and Giraud's Lemma yield

$$|G_x(y)| \le C(k, L, \omega)|x-y|^{2k-n} \quad \text{for all } x \in \omega, y \in \Omega, \, x \neq y. \tag{96}$$

This proves the existence of a Green's function for P. Moreover, the construction yields $G_x \in H^p_{2k,loc}(\Omega - \{x\}) \cap H^p_{k,0,loc}(\Omega - \{x\})$ for all $p > 1$ and $PG_x = 0$ in $\Omega - \{x\}$. The validity of (95) for $u \in H^p_{2k}(\Omega) \cap H^p_{k,0}(\Omega)$ and $f \in L^p(\Omega)$ such that $Pu = f$ and $p > n/(2k)$ follows by density of $C^\infty_c(\Omega)$ in $L^p(\Omega)$ and the regularity Theorem 7.3. The symmetry of G follows from the self-adjointness of the operator P. The uniqueness goes as the proof of uniqueness of Theorem 5.1. The pointwise control for $|G_x(y)|$ is (96). The control of the gradient of G_x is a consequence of elliptic theory. Since the details of these points are exactly the same as in the case of a second-order operator $\Delta + h$, we refer to the detailed construction [21].

7. Regularity theorems

The following theorems are reformulations of Agmon-Douglis-Nirenberg [1].

Theorem 7.1. *We fix $k \in \mathbb{N}$, $L > 0$ and $\delta > 0$. Let Ω be a smooth domain of $\mathbb{R}^n$, $n > 2k \ge 2$ and $x_0 \in \overline{\Omega} = \Omega \cup \partial\Omega$. Let $P = \Delta^k + h$ be a differential operator such that $h \in L^\infty(\Omega \cap B_\delta(x_0))$ and $\|h\|_\infty \le L$. Let $u \in H^s_{2k}(\Omega \cap B_\delta(x_0))$ be such that $\eta u \in H^s_{k,0}(\Omega)$ for all $\eta \in C^\infty_c(B_\delta(x_0))$ and $f \in L^p(\Omega \cap B_\delta(x_0))$, $p, s \in (1, +\infty)$ be such that $Pu = f$. Then for all $r < \delta$, $u \in H^p_{2k}(\Omega \cap B_r(x_0))$. Moreover, for all $q > 1$, we have that*

$$\|u\|_{H^p_{2k}(\Omega \cap B_r(x_0))} \le C(n, \Omega, k, L, p, q, \delta, r) \left(\|f\|_{L^p(\Omega \cap B_\delta(x_0))} + \|u\|_{L^q(\Omega \cap B_\delta(x_0))} \right)$$

where $C(n, \Omega, k, L, p, q, \delta, r)$ depends only on n, Ω, k, L, p, q, δ and r.

Theorem 7.2. *We fix $k \in \mathbb{N}$ and $L > 0$ and $\delta > 0$. Let Ω be a smooth domain of $\mathbb{R}^n$, $n > 2k \ge 2$ and $x_0 \in \overline{\Omega} = \Omega \cup \partial\Omega$. Let $P = \Delta^k + h$ be a differential operator such that $h \in C^{0,\alpha}(\Omega \cap B_\delta(x_0))$ and $\|h\|_{C^{0,\alpha}} \le L$ for some $\alpha \in (0,1)$. Let $u \in C^{2k,\alpha}(\Omega \cap B_\delta(x_0))$ be such that $\partial_\nu^i u = 0$ on $B_\delta(x_0) \cap \partial\Omega$ for all $i = 0, ..., k-1$ and $f \in C^{0,\alpha}(\Omega \cap B_\delta(x_0))$ be such that $Pu = f$. Then for all $r < \delta$, we have that*

$$\|u\|_{C^{2k,\alpha}(\Omega \cap B_r(x_0))} \le C(n, \Omega, k, L, \alpha, \delta, r) \left(\|f\|_{C^{0,\alpha}(\Omega \cap B_\delta(x_0))} + \|u\|_{C^0(\Omega \cap B_\delta(x_0))} \right)$$

where $C(n, \Omega, k, L, \alpha, \delta, r)$ depends only on n, Ω, k, L, α, δ and r.

Theorem 7.3. *We fix $k \in \mathbb{N}$ and $L > 0$. Let Ω be a smooth domain of $\mathbb{R}^n$, $n > 2k \ge 2$. Let P be a differential operator such that (33) holds and fix $p \in (1, +\infty)$. Then for all $f \in L^p(\Omega)$, there exists $u \in H^p_{2k}(\Omega) \cap H^p_{k,0}(\Omega)$ unique such that $Pu = f$. Moreover,*

$$\|u\|_{H^p_{2k}(\Omega)} \le C(\Omega, k, L, p) \|f\|_{L^p(\Omega)}$$

where $C(\Omega, k, L, p)$ depends only on Ω, k, L and p.

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