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GLOBAL WELL-POSEDNESS FOR THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION WITH PERIODIC BOUNDARY CONDITION

HAJER BAHOURI AND GALINA PERELMAN

ABSTRACT. We prove global well-posedness for the derivative nonlinear Schrödinger equation on the torus in the Sobolev space $H^1(\mathbb{T})$ provided that the mass of initial data is strictly less than 8π .

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Keywords: Derivative nonlinear Schrödinger equation, global well-posedness, integrable systems, profile decompositions.

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1. INTRODUCTION

The goal of this paper is to address global well-posedness for the derivative nonlinear Schrödinger equation (DNLS) on the torus:

$$(1.1) \quad iu_t + u_{xx} = -i\partial_x(|u|^2u), \quad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z},$$

with Cauchy data

$$(1.2) \quad u|_{t=0} = u_0 \in H^1(\mathbb{T}).$$

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Note that the equation is L^2 critical as it is invariant under the scaling

$$(1.3) \quad u(t, x) \longrightarrow u_\mu(t, x) = \frac{1}{\sqrt{\mu}} u\left(\frac{t}{\mu^2}, \frac{x}{\mu}\right), \quad \mu > 0.$$

The DNLS equation first appears in the literature in [26, 27] as a model for the propagation of circular polarized nonlinear Alfvén waves in magnetized plasmas and in the study of the one-dimensional compressible magneto-hydrodynamic equation in the presence of the Hall effect.

In [20], Kaup and Newell showed that the DNLS equation is completely integrable, and since that time DNLS has received enduring attention. As a completely integrable equation, DNLS possesses an infinite family of polynomial conservation laws, including the conservation of the mass, the momentum and the energy:

$$(1.4) \quad M(u) = \int |u|^2 dx,$$

$$(1.5) \quad P(u) = \operatorname{Im} \int \bar{u} u_x dx + \frac{1}{2} \int |u|^4 dx,$$

$$(1.6) \quad E(u) = \int \left(|u_x|^2 - \frac{3}{2} \operatorname{Im}(|u|^2 u \bar{u}_x) + \frac{1}{2} |u|^6 \right) dx.$$

To avoid any confusion, all along this paper we will denote by $M_{\mathbb{R}}, P_{\mathbb{R}}, E_{\mathbb{R}}$ and $M_{\mu\mathbb{T}}, P_{\mu\mathbb{T}}, E_{\mu\mathbb{T}}$ the mass, the momentum and the energy on the real line and on the torus $\mathbb{R}/\mu\mathbb{Z}$ respectively.

Local well-posedness for the DNLS equation in H^s for $s \geq \frac{1}{2}$ has been known for some time both on the real line [33] and on the torus [16]. This range is optimal if one requires the solutions to be locally uniformly continuous with respect to initial data, see for instance [5, 28, 34]. The gap between the $s = \frac{1}{2}$ threshold and the scaling critical regularity can be (almost) closed by leaving the H^s -scale and working in more general Fourier-Lebesgue spaces [6, 10, 11].

Concerning global well-posedness on the real line, Wu [35] and Guo-Wu [12] showed that the H^1 -norm of solutions is controlled by their momentum and energy provided that the mass is strictly less than 4π . More precisely, they proved the following inequality (see Lemma 2.2 in [12])

$$(1.7) \quad \|u\|_{\dot{H}^1(\mathbb{R})}^2 \lesssim \frac{P_{\mathbb{R}}^2(u) + |E_{\mathbb{R}}(u)|}{\left(1 - \frac{1}{2\sqrt{\pi}} \|u\|_{L^2(\mathbb{R})}\right)^2}, \quad \forall u \in H^1(\mathbb{R}), \quad \|u\|_{L^2(\mathbb{R})}^2 < 4\pi,$$

which gives immediately global well-posedness of DNLS in $H^1(\mathbb{R})$ under the restriction $M_{\mathbb{R}}(u) < 4\pi$. This global well-posedness result was subsequently extended to $H^{\frac{1}{2}}(\mathbb{R})$ and $H^{\frac{1}{2}}(\mathbb{T})$ by Guo-Wu [12] and Oh-Mosincat [29], Mosincat [28] respectively, and more recently to H^s , for $1/6 \leq s < 1/2$ both on the real line and the torus by Killip-Ntekoume-Visan in [21].

The 4π restriction is related to the lack of coercivity of the DNLS conservation laws that can be readily detected by considering the algebraic solitons:

$$(1.8) \quad u_c(t, x) = 2\sqrt{c} e^{-i\frac{c^2}{4}t + i\frac{c}{2}x} \frac{cx - c^2t + i}{(cx - c^2t - i)^2}, \quad c > 0,$$

for which $M_{\mathbb{R}}(u_c) = 4\pi$, while all other polynomial conservation laws vanish.

In the case of the real line, the first evidence that global existence persists beyond the 4π barrier was obtained by means of the inverse scattering techniques [25, 17, 18, 19, 30, 31]. The most definite result is due to Jenkins, Liu, Perry and Sulem who have proved in [17]

that the Cauchy problem for the DNLS equation is globally well-posed for any initial data u_0 in $H^{2,2}(\mathbb{R}) = \{f \in H^2(\mathbb{R}) : x^2 f \in L^2(\mathbb{R})\}$.

In Sobolev spaces $H^s(\mathbb{R})$, the 4π barrier was overcome only very recently. In [2], the authors proved that the DNLS equation is globally well-posed in $H^{\frac{1}{2}}(\mathbb{R})$ regardless of the size of the mass. Afterwards, Harrop-Griffiths, Killip, Ntekoume and Visan [13, 14, 21] established global well-posedness of the DNLS equation in the critical space $L^2(\mathbb{R})$.

The periodic case is more challenging. To the best of our knowledge, large data global well-posedness for DNLS on the torus is not known even for C^∞ data. In the present paper, we improve the 4π bound to 8π in the H^1 -setting. Our main result is the following:

Theorem 1. *The Cauchy problem (1.1)-(1.2) is globally well-posed in $H^1(\mathbb{T})$ under the restriction $M_{\mathbb{T}}(u) < 8\pi$. Furthermore, for any $R > 0$ and $0 < \delta < 8\pi$, there exists $C(R, \delta) > 0$ so that*

$$(1.9) \quad \sup_{t \in \mathbb{R}} \|u(t)\|_{H^1(\mathbb{T})} \leq C(R, \delta),$$

for any $H^1(\mathbb{T})$ -solution to (1.1) with $M_{\mathbb{T}}(u) \leq 8\pi - \delta$ and $\|u(0)\|_{\dot{H}^1(\mathbb{T})} \leq R$.

The structure of the paper is as follows. In Section 2, we record some preliminary results related to the integrability of the DNLS equation, that will be needed in the proof of Theorem 1. In Section 3 we prove Theorem 1. As in [2], we proceed by contradiction, combining the profile decomposition techniques with the integrability structure of the equation. There are also two appendices: in the first one, we recall some properties of the regularized determinants required for the results of Section 2 and the second one is devoted to the proof of some technical results.

Throughout this article, we shall use the following convention for the Fourier transform on the real line

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx.$$

We will use the notation $\|\cdot\|_2$ for the Hilbert-Schmidt norm, and $\|\cdot\|$ for the operator norm on L^2 either on the real line or on a torus, while the standard hermitian norm in \mathbb{C}^2 and the induced operator norm in the space of complex 2×2 matrices will be denoted by $|\cdot|$. Finally, we mention that the letter C will be used to denote universal constants which may vary from line to line. If we need the implied constant to depend on parameters, we shall indicate this by subscripts. We also use the notation $A \lesssim B$ to denote bounds of the form $A \leq CB$, and $A \lesssim_\alpha B$ for $A \leq C_\alpha B$, where C_α depends only on α . For simplicity, we shall still denote by (u_n) any subsequence of (u_n) .

2. KAUP-NEWELL SPECTRAL PROBLEM

2.1. Lax pair formulation of the DNLS equation. Kaup and Newell showed in [20] that the DNLS equation can be obtained as a compatibility condition of the following linear system

$$(2.1) \quad \begin{aligned} \partial_x \psi &= U(\lambda) \psi, \\ \partial_t \psi &= \Upsilon(\lambda) \psi, \end{aligned}$$

with

$$\begin{aligned} U(\lambda) &= -i\sigma_3(\lambda^2 + i\lambda U), \quad U = \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix}, \\ \Upsilon(\lambda) &= -i(2\lambda^4 - \lambda^2 |u|^2)\sigma_3 + \begin{pmatrix} 0 & 2\lambda^3 u - \lambda |u|^2 u + i\lambda u_x \\ -2\lambda^3 \bar{u} + \lambda |u|^2 \bar{u} + i\lambda \bar{u}_x & 0 \end{pmatrix}, \end{aligned}$$

where $\lambda \in \mathbb{C}$, ψ is a \mathbb{C}^2 -valued function of (t, x, λ) , and σ_3 is the Pauli matrix given by $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In other words, u satisfies the DNLS equation if and only if

$$\frac{\partial \mathcal{U}}{\partial t} - \frac{\partial \Upsilon}{\partial x} + [\mathcal{U}, \Upsilon] = 0.$$

The first equation of (2.1), viewed as a spectral problem, will play a central role in our analysis. We will write it in the form

$$(2.2) \quad L_u(\lambda)\psi = 0,$$

where $L_u(\lambda) = \mathcal{L}_0 - \lambda^2 - i\lambda U$, with $\mathcal{L}_0 = i\sigma_3 \partial_x$. In the next three subsections, we collect the properties of the Kaup-Newell spectral problem (2.2) that will be used in the proof of Theorem 1, we refer to [2, 8, 13, 19, 20, 24, 25, 30, 31] and references therein as well as to Appendices for the proofs and further details.

2.2. Kaup-Newell spectral problem in the periodic case. Given $u \in L^2_{loc}(\mathbb{R})$, we denote $E_u(x, \lambda)$ the canonical fundamental solution of (2.2):

$$(2.3) \quad \begin{cases} L_u(\lambda)E_u(\lambda) & = 0 \\ E_u(0, \lambda) & = \text{Id}. \end{cases}$$

For each $x \in \mathbb{R}$, the fundamental matrix $E_u(x, \lambda)$ is an analytic function of (λ, u, \bar{u}) with the following properties:

$$(2.4) \quad \det E_u(x, \lambda) = 1,$$

$$(2.5) \quad E_{u_\mu}(x, \lambda) = E_u(\mu^{-1}x, \sqrt{\mu}\lambda),$$

$$(2.6) \quad \sigma_3 E_u(x, -\lambda) \sigma_3 = E_u(x, \lambda), \quad \sigma_1 \overline{E_u(x, -\bar{\lambda})} \sigma_1 = E_u(x, \lambda),$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In the case where u is periodic with period $T > 0$: $u(x+T) = u(x)$, we denote by $M_u(\lambda)$ the corresponding monodromy matrix:

$$(2.7) \quad M_u(\lambda) = \begin{pmatrix} M_{11}(\lambda, u) & M_{12}(\lambda, u) \\ M_{21}(\lambda, u) & M_{22}(\lambda, u) \end{pmatrix} = E_u(T, \lambda),$$

which satisfies $\det M_u(\lambda) = 1$, and by $\Delta_u(\lambda)$ the trace of $M_u(\lambda)$:

$$(2.8) \quad \Delta_u(\lambda) = M_{11}(\lambda, u) + M_{22}(\lambda, u).$$

It follows from (2.1) that $\Delta_u(\lambda)$ is time independent if u is a T -periodic solution of the DNLS equation.

The values of λ for which $\Delta_u(\lambda) = \pm 2$ correspond to the eigenvalues of the spectral problem (2.2) with periodic/antiperiodic boundary conditions on $[0, T]$: $\psi(T) = \pm\psi(0)$. They can be also viewed as eigenvalues of $L_u(\lambda)$ considered on $[0, 2T]$ with periodic boundary conditions. We will refer to them as periodic eigenvalues of $L_u(\lambda)$.

We also need to introduce the functions

$$(2.9) \quad A_u^D(\lambda) = \frac{i}{2}(M_{11}(\lambda, u) + M_{12}(\lambda, u) - M_{21}(\lambda, u) - M_{22}(\lambda, u)),$$

$$(2.10) \quad A_u^N(\lambda) = \frac{i}{2}(M_{11}(\lambda, u) - M_{12}(\lambda, u) + M_{21}(\lambda, u) - M_{22}(\lambda, u)).$$

Zeros of the function A_u^D (resp. A_u^N) are the eigenvalues of (2.2) considered on $[0, T]$ with the Dirichlet (resp. Neumann) boundary condition $(\psi_1 - \psi_2)(T) = (\psi_1 - \psi_2)(0) = 0$ (resp. $(\psi_1 + \psi_2)(T) = (\psi_1 + \psi_2)(0) = 0$), where we denote $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$.

Obviously, in the case of $u = 0$, the fundamental solution of (2.2) is given by

$$(2.11) \quad E_0(x, \lambda) = \exp(-ix\lambda^2 \sigma_3) = \begin{pmatrix} e^{-i\lambda^2 x} & 0 \\ 0 & e^{i\lambda^2 x} \end{pmatrix}$$

and so

$$(2.12) \quad M_0(\lambda) = e^{-iT\lambda^2 \sigma_3}, \quad \Delta_0(\lambda) = 2 \cos(T\lambda^2),$$

and

$$(2.13) \quad A_0^D(\lambda) = A_0^N(\lambda) = \sin(T\lambda^2).$$

Thanks to the symmetry relations (2.6), one has

$$(2.14) \quad \sigma_3 M_u(-\lambda) \sigma_3 = M_u(\lambda), \quad \overline{\sigma_1 M_u(-\bar{\lambda}) \sigma_1} = M_u(\lambda), \quad M_{u_\mu}(\lambda) = M_u(\sqrt{\mu}\lambda),$$

and therefore,

$$(2.15) \quad \Delta_u(-\lambda) = \Delta_u(\lambda), \quad \overline{\Delta_u(\bar{\lambda})} = \Delta_u(\lambda), \quad \Delta_{u_\mu}(\lambda) = \Delta_u(\sqrt{\mu}\lambda)$$

$$(2.16) \quad \overline{A_u^D(-\bar{\lambda})} = A_u^D(\lambda), \quad A_u^D(-\lambda) = A_u^N(\lambda).$$

Note that if $A_u^D(\lambda) = 0$ or $A_u^N(\lambda) = 0$, then since $\det M_u(\lambda) = 1$, we have

$$\Delta_u^2(\lambda) - 4 = (M_{12}(\lambda, u) + M_{21}(\lambda, u))^2,$$

which together with (2.14) implies that

$$(2.17) \quad |\Delta_u(\lambda)| \geq 2 \quad \text{if } A_u^D(\lambda) = 0 \text{ or } A_u^N(\lambda) = 0 \text{ with } \lambda \in i\mathbb{R}.$$

To study the behavior of the matrix $E_u(x, \lambda)$ for $|\lambda|$ large, it is more convenient to transform (2.2) into a Zakharov-Shabat system following [20, 30]. Setting

$$(2.18) \quad \tilde{\psi}(x) = \exp\left(\frac{i\sigma_3}{2} \int_0^x dy |u(y)|^2\right) \begin{pmatrix} 1 & 0 \\ -\bar{u}(x) & 2i\lambda \end{pmatrix} \psi(x),$$

one can easily check that ψ is a solution of (2.2) if and only if $\tilde{\psi}$ solves

$$(2.19) \quad (i\sigma_3 \partial_x - \mathcal{Q} - \zeta) \tilde{\psi} = 0,$$

where $\zeta = \lambda^2$ and $\mathcal{Q} = \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix}$ with

$$\begin{aligned} q_1(x) &= \frac{1}{2} u(x) \exp\left(i \int_0^x dy |u(y)|^2\right) \\ q_2(x) &= (i\bar{u}_x + \frac{1}{2}\bar{u}|u|^2)(x) \exp\left(-i \int_0^x dy |u(y)|^2\right). \end{aligned}$$

The representation (2.18) allows us to prove the following proposition, see Appendix B for the proof.

Proposition 2.1. *Let $\hat{E}_u(x, \lambda) = E_u(x, \lambda) - e^{-i\lambda^2 x \sigma_3 - \frac{i\sigma_3}{2} \int_0^x dy |u(y)|^2}$,*

$$\hat{E}_u(x, \lambda) = \begin{pmatrix} \hat{E}_{11}(x, \lambda; u) & \hat{E}_{12}(x, \lambda; u) \\ \hat{E}_{21}(x, \lambda; u) & \hat{E}_{22}(x, \lambda; u) \end{pmatrix}.$$

Then for any $T > 0$ and $R > 0$, there exists $C_{T,R} > 0$ such that

$$(2.20) \quad \begin{aligned} |\hat{E}_{11}(x, \lambda; u)| + |\hat{E}_{22}(x, \lambda; u)| &\leq C_{T,R} \frac{e^{|\operatorname{Im} \lambda^2| x}}{\langle \lambda \rangle^2}, \\ |\hat{E}_{12}(x, \lambda; u)| + |\hat{E}_{21}(x, \lambda; u)| &\leq C_{T,R} \frac{e^{|\operatorname{Im} \lambda^2| x}}{\langle \lambda \rangle}, \end{aligned}$$

for all $(x, \lambda) \in [0, T] \times \mathbb{C}$ and all $u \in H_{loc}^1(\mathbb{R})$ with $\|u\|_{H^1([0, T])} \leq R$.

It follows from (2.20) that as $|\lambda| \rightarrow \infty$,

$$(2.21) \quad \begin{aligned} \Delta_u(\lambda) &= 2 \cos \left(\lambda^2 T + \frac{\|u\|_{L^2([0,T])}^2}{2} \right) + O_T \left(\frac{e^{|\operatorname{Im} \lambda^2| T}}{|\lambda|^2} \right), \\ A_u^D(\lambda), A_u^N(\lambda) &= \sin \left(\lambda^2 T + \frac{\|u\|_{L^2([0,T])}^2}{2} \right) + O_T \left(\frac{e^{|\operatorname{Im} \lambda^2| T}}{|\lambda|} \right), \end{aligned}$$

uniformly with respect to u in bounded sets of $H^1([0, T])$.

These asymptotics lead to the following localization of large periodic, Dirichlet and Neumann eigenvalues of the Kaup-Newell spectral problem, that we formulate in the case of $T = 1$.

Proposition 2.2. *For any $R \geq 0$ there exist an integer $N = N(R) \gg 1$ and a positive constant $C = C(R)$ so that, for all $u \in H^1(\mathbb{T})$ with $\|u\|_{H^1(\mathbb{T})} \leq R$ and all $n \in \mathbb{Z}$ with $|n| \geq N$, the disk*

$$D_n = \left\{ \lambda \in \mathbb{C} : |\lambda - \lambda_n^0(u)| < \frac{C}{|n|} \right\}, \quad \lambda_n^0(u) = \sqrt{\pi n - \frac{1}{2} \|u\|_{L^2(\mathbb{T})}^2} \in \mathbb{R}_+ \cup i\mathbb{R}_+$$

contains exactly two periodic eigenvalues (counted with their multiplicities), one simple Dirichlet eigenvalue and one simple Neumann eigenvalue of $L_u(\lambda)$. If $n < 0$, the corresponding Dirichlet and Neumann eigenvalues are purely imaginary. The functions $\Delta_u^2 - 4$, A_u^D and A_u^N have no zeros in the region

$$\left\{ \lambda \in \mathbb{C} : |\lambda^2 + \frac{1}{2} \|u\|_{L^2(\mathbb{T})}^2| \geq \pi N - \frac{\pi}{4}, \quad |\lambda \pm \lambda_n^0(u)| \geq \frac{C}{|n|}, \quad \forall |n| \geq N \right\}.$$

Proposition 2.2 has the following consequence that will play an important role in the proof of Theorem 1.

Lemma 2.1. *Let $R \geq 0$ and $u \in C([0, 1], H^1(\mathbb{T}))$ with $\|u(0)\|_{H^1(\mathbb{T})} \leq R$ such that, for all $\tau \in [0, 1]$, $\Delta_{u(\tau)} = \Delta_{u(0)}$. Then for any $0 < \gamma \leq \frac{\pi}{4}$, there exists an integer $N_0 = N_0(R, \gamma) \gg 1$ such that, for all $n \leq -N_0$, the interval¹*

$$I_n = [\lambda_n^+, \lambda_n^-] \subset i\mathbb{R}_+, \quad \lambda_n^\pm = \sqrt{n\pi - \frac{1}{2} \|u\|_{L^2}^2} \pm \gamma,$$

contains at least one Dirichlet eigenvalue and one Neumann eigenvalue of $L_{u(\tau)}(\lambda)$.

Proof. Asymptotics (2.21) ensure that for all $0 < \gamma \leq \frac{\pi}{4}$, there exist $\delta_1 > 0$, and $N_1 = N_1(R, \gamma) \in \mathbb{N}^*$ such that

$$|\Delta_{u(0)}(\lambda_n^\pm)| \leq 2 - \delta_1, \quad \forall n \leq -N_1.$$

Since we suppose that $\Delta_{u(\tau)}$ is independent of τ , this inequality remains valid for all $\tau \in [0, 1]$ which together with (2.17) implies that

$$(2.22) \quad A_{u(\tau)}^D(\lambda_n^\pm) \neq 0 \text{ and } A_{u(\tau)}^N(\lambda_n^\pm) \neq 0, \quad \forall n \leq -N_1, \tau \in [0, 1].$$

Taking into account the symmetry properties of $A_u^D(\lambda)$, $A_u^N(\lambda)$ with respect to λ and their continuous dependence on u , one deduces from (2.22) that, for all $n \geq N_1$, the parity of the number of zeros of $A_{u(\tau)}^D$, $A_{u(\tau)}^N$ in the interval I_n , counted with their multiplicity, is independent of τ , which thanks to Proposition 2.2 gives the result. \square

We end this section by the following estimate for the resolvent of $L_u(\lambda)$, the proof of which is given in Appendix B.

¹Since u depends continuously on τ , the conservation of Δ_u implies the conservation of the masse of u .

Proposition 2.3. *There exists a positive constant C such that, for any periodic function $u \in L^2$ with period T and all $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, we have*

$$(2.23) \quad \|L_u^{-1}(\lambda)\| \leq C \exp\left(C \frac{|\lambda|^2}{|\operatorname{Im} \lambda|^2} \frac{e^{T|\operatorname{Im} \lambda^2|}}{|\sin(\frac{T\lambda^2}{2})|^2} \|u\|_{L^2([0,T])}^2\right) \left| \frac{\cos(T\lambda^2) - 1}{(\Delta_u(\lambda) - 2)\operatorname{Im} \lambda^2} \right|,$$

provided that $\Delta_u(\lambda) \neq 2$. Here the operator $L_u(\lambda)$ is considered with the periodic boundary conditions on $[0, T]$.

2.3. Kaup-Newell spectral problem on the real line. In order to establish Theorem 1, we also need to recall a few results about the KN system on the real line. Given $u \in \mathcal{S}(\mathbb{R})$, for any $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda^2 \geq 0$ there are unique solutions² $\psi_1^-(x, \lambda; u)$ and $\psi_2^+(x, \lambda; u)$ to (2.2), the so-called Jost solutions, with the following behavior at $\pm\infty$:

$$\begin{aligned} \psi_1^-(x, \lambda; u) &= e^{-i\lambda^2 x} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1) \right], \quad \text{as } x \rightarrow -\infty, \\ \psi_2^+(x, \lambda; u) &= e^{i\lambda^2 x} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1) \right], \quad \text{as } x \rightarrow +\infty. \end{aligned}$$

The solutions ψ_1^-, ψ_2^+ are holomorphic functions of λ on $\Omega_+ = \{\lambda \in \mathbb{C} : \operatorname{Im} \lambda^2 > 0\}$, C^∞ up to the boundary.

Similarly, for $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda^2 \leq 0$, there are unique solutions $\psi_1^+(x, \lambda; u)$, $\psi_2^-(x, \lambda; u)$ to (2.2) such that

$$\begin{aligned} \psi_1^+(x, \lambda; u) &= e^{-i\lambda^2 x} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1) \right], \quad \text{as } x \rightarrow +\infty, \\ \psi_2^-(x, \lambda; u) &= e^{i\lambda^2 x} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1) \right], \quad \text{as } x \rightarrow -\infty. \end{aligned}$$

One can easily check that the Jost solutions satisfy the following symmetry relations

$$(2.24) \quad \psi_1^-(x, \lambda; u) = \sigma_3 \psi_1^-(x, -\lambda; u), \quad \psi_2^+(x, \lambda; u) = -\sigma_3 \psi_2^+(x, -\lambda; u),$$

$$(2.25) \quad \psi_1^-(x, \lambda; u) = -\sigma_1 \sigma_3 \overline{\psi_2^-(x, \bar{\lambda}; u)}, \quad \psi_2^+(x, \lambda; u) = \sigma_1 \sigma_3 \overline{\psi_1^+(x, \bar{\lambda}; u)}.$$

We denote by $a_u(\lambda)$ the Wronskian of the Jost solutions ψ_1^-, ψ_2^+ defined above:

$$(2.26) \quad a_u(\lambda) = \det(\psi_1^-(x, \lambda; u), \psi_2^+(x, \lambda; u)).$$

Using the second equation in (2.1), it can be shown that $a_u(\lambda)$ is time-independent if u is a solution of the DNLS equation.

Clearly, a_u is a holomorphic even function of $\lambda \in \Omega_+$, C^∞ up to the boundary, with $a_u(0) = 1$, and thanks to the symmetries (2.25), it verifies

$$(2.27) \quad |a_u(\lambda)|^2 \leq 1, \quad \forall \lambda \in \mathbb{R},$$

$$(2.28) \quad |a_u(\lambda)|^2 \geq 1, \quad \forall \lambda \in i\mathbb{R}.$$

According to (2.26), the zeros of a_u in Ω_+ are the eigenvalues of the spectral problem (2.2) (considered in $L^2(\mathbb{R})$). Let us recall the following result from [2].

Lemma 2.2. *For any $R \geq 0$ there exists a positive constant C_R such that $a_u(\lambda) \neq 0$, for all $\lambda \in \Omega_+$ with $\operatorname{Re} \lambda^2 \leq -C_R$, provided that $\|u\|_{H^{\frac{1}{2}}(\mathbb{R})} \leq R$.*

Since a_u is an even function of λ , it is convenient to introduce $\tilde{a}_u(\zeta) = a_u(\sqrt{\zeta})$. The function $\tilde{a}_u(\zeta)$ is holomorphic in the open upper half plane \mathbb{C}_+ , C^∞ up to the boundary and verifies, in view of (2.27)-(2.28),

$$(2.29) \quad |\tilde{a}_u(\zeta)| \geq 1 \text{ for } \zeta < 0, \quad |\tilde{a}_u(\zeta)| \leq 1 \text{ for } \zeta > 0 \quad \text{and} \quad \tilde{a}_u(0) = 1.$$

²If $\operatorname{Im} \lambda^2 > 0$, the solutions ψ_1^-, ψ_2^+ remain well defined for $u \in L^2(\mathbb{R})$.

From the equivalence between the systems (2.2) and (2.19), one deduces that

$$(2.30) \quad \lim_{|\zeta| \rightarrow \infty, \zeta \in \overline{\mathbb{C}}_+} \tilde{a}_u(\lambda) = e^{-\frac{i}{2} \|u\|_{L^2(\mathbb{R})}^2}.$$

Furthermore, one has the following asymptotic expansion as $|\zeta| \rightarrow +\infty$, $\text{Im } \zeta \geq 0$:

$$(2.31) \quad \ln \tilde{a}_u(\zeta) = \sum_{k \geq 0} E_k(u) \zeta^{-k}.$$

The coefficients³ $E_k(u)$ are polynomial with respect to u, \bar{u} and their derivatives, and homogeneous with respect to the scaling $u(x) \rightarrow u_\mu(x) = \frac{1}{\sqrt{\mu}} u(\frac{x}{\mu})$, $\mu > 0$, since

$$(2.32) \quad \tilde{a}_{u_\mu}(\zeta) = \tilde{a}_u(\mu\zeta).$$

As $\tilde{a}_u(\zeta)$ is time-independent, the quantities $E_k(u)$ are conservation laws. The first three of them coincide, up to constants, with the mass, momentum and energy:

$$E_0(u) = -\frac{i}{2} \|u\|_{L^2(\mathbb{R})}^2, \quad E_1(u) = \frac{i}{4} P_{\mathbb{R}}(u), \quad E_2(u) = -\frac{i}{8} E_{\mathbb{R}}(u).$$

Furthermore, one can show that $|\tilde{a}_u(\zeta)|^2 \in 1 + \mathcal{S}(\mathbb{R})$.

The analyticity of \tilde{a}_u in \mathbb{C}_+ allows to express the functionals E_k in terms of the zeros of \tilde{a}_u in \mathbb{C}_+ and of its trace on \mathbb{R} . In the simplest case where \tilde{a}_u does not vanish⁴ on \mathbb{R}_+ , denoting by ζ_1, \dots, ζ_N the zeros of \tilde{a}_u in \mathbb{C}_+ counted with their multiplicity, we have

$$(2.33) \quad \tilde{a}_u(\zeta) = e^{-\frac{i}{2} \|u\|_{L^2(\mathbb{R})}^2} \prod_{j=1}^N \left(\frac{\zeta - \zeta_j}{\zeta - \bar{\zeta}_j} \right) \exp \left(\frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\xi - \zeta} \ln |\tilde{a}_u(\xi)|^2 \right), \quad \forall \zeta \in \mathbb{C}_+,$$

which implies that

$$(2.34) \quad E_k(u) = -\frac{2i}{k} \sum_{j=1}^N \text{Im } \zeta_j^k + \frac{i}{2\pi} \int_{-\infty}^{\infty} d\xi \xi^{k-1} \ln |\tilde{a}_u(\xi)|^2, \quad \forall k \in \mathbb{N}^*.$$

The set of Schwartz functions u such that \tilde{a}_u does not vanish on \mathbb{R}_+ , is dense in $\mathcal{S}(\mathbb{R})$ [4, 19, 24, 31]. We will denote this set by $\tilde{\mathcal{S}}_{reg}(\mathbb{R})$.

The continuity⁵ of \tilde{a}_u with respect to u , together with (2.29) and (2.30) ensure that for all $u \in \mathcal{S}(\mathbb{R})$,

$$(2.35) \quad \int_{-\infty}^0 \frac{\tilde{a}'_u(s)}{\tilde{a}_u(s)} ds = \frac{i}{2} \|u\|_{L^2(\mathbb{R})}^2.$$

As a straightforward consequence of this formula and of the analyticity of \tilde{a}_u in \mathbb{C}_+ , one obtains the following result which was crucial for the analysis in [2].

Lemma 2.3. *i) Let $u \in \mathcal{S}(\mathbb{R})$, $\theta \in]0, \pi[$ such that $\tilde{a}_u(\zeta) \neq 0$, for all ζ in \mathbb{C}_+ with $\arg \zeta = \theta$ and let $n(u; \theta)$ be the number of zeros of $\tilde{a}_u(\zeta)$ in the angle $\{\zeta \in \mathbb{C}_+ : \theta < \arg \zeta < \pi\}$ counted with their multiplicity. Then*

$$(2.36) \quad n(u; \theta) = \frac{1}{2i\pi} \int_0^{+\infty} \frac{\tilde{a}'_u(s)}{\tilde{a}_u(s)} ds + \frac{1}{4\pi} \|u\|_{L^2(\mathbb{R})}^2,$$

³We choose the branch of the logarithm so that $\lim_{|\zeta| \rightarrow \infty, \zeta \in \overline{\mathbb{C}}_+} \ln \tilde{a}_u(\zeta) = -\frac{i}{2} \|u\|_{L^2(\mathbb{R})}^2$.

⁴It follows from (2.29)-(2.30) that in that case, \tilde{a}_u has only a finite number of zeros in \mathbb{C}_+ .

⁵The map $H^{1,1}(\mathbb{R}) \ni u \rightarrow \tilde{a}_u \in L^\infty(\mathbb{R})$ is Lipschitz continuous on bounded sets of $H^{1,1}(\mathbb{R}) = \{u \in H^1(\mathbb{R}) : xu \in L^2(\mathbb{R})\}$, see [31].

where, all along this paper, $\int_0^{+\infty} e^{i\theta} ds$ denotes the integral along the path $\gamma = \{z = \rho e^{i\theta}, \rho \in \mathbb{R}_+\}$.

ii) Let $u \in \widetilde{\mathcal{S}}_{reg}(\mathbb{R})$ and let ζ_1, \dots, ζ_N be the zeros of \tilde{a}_u in \mathbb{C}_+ counted with their multiplicity. Then

$$(2.37) \quad M_{\mathbb{R}}(u) = 4 \sum_{j=1}^N \arg(\zeta_j) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \ln |\tilde{a}_u(\xi)|^2,$$

with, all along this paper, $\arg(\zeta) \in [0, 2\pi[$.

Remark 2.1. Observe that due to (2.29) and (2.37), we have:

$$(2.38) \quad n(u; \theta) \leq \frac{\|u\|_{L^2(\mathbb{R})}^2}{4\theta},$$

for all $u \in \widetilde{\mathcal{S}}_{reg}(\mathbb{R})$ and all $\theta \in]0, \pi[$.

2.3.1. *Regularized determinant realization of a_u .* A convenient way to study the function a_u is to use its representation as a regularized perturbation determinant⁶:

$$(2.39) \quad a_u(\lambda) = \det_2(\mathbb{I} - T_u(\lambda)), \quad \forall \lambda \in \Omega_+, \quad u \in L^2(\mathbb{R}),$$

where

$$(2.40) \quad T_u(\lambda) = i\lambda(\mathcal{L}_0 - \lambda^2)^{-1}U, \quad U = \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix}.$$

The operator $T_u(\lambda)$, considered on $L^2(\mathbb{R}, \mathbb{C}^2)$, is a Hilbert-Schmidt operator with

$$(2.41) \quad \|T_u(\lambda)\|_2^2 = \frac{|\lambda|^2}{\text{Im}(\lambda^2)} \|u\|_{L^2(\mathbb{R})}^2.$$

The representation (2.39) together with the corresponding properties of the regularized determinants (see Appendix A) leads immediately to the following bound.

Proposition 2.4. *There exists a positive constant C such that the following estimate holds*

$$(2.42) \quad |a_{u_1}(\lambda) - a_{u_2}(\lambda)| \leq C e^{C \frac{|\lambda|^2}{\text{Im}(\lambda^2)} (\|u_1\|_{L^2(\mathbb{R})}^2 + \|u_2\|_{L^2(\mathbb{R})}^2)} \frac{|\lambda|}{\sqrt{\text{Im}(\lambda^2)}} \|u_1 - u_2\|_{L^2(\mathbb{R})},$$

$$\forall \lambda \in \Omega_+, \quad u_1, u_2 \in L^2(\mathbb{R}).$$

Combining this proposition with (2.29), (2.30), Remark 2.1 and Lemma 2.3 i), and taking into account the density of $\widetilde{\mathcal{S}}_{reg}(\mathbb{R})$ in $L^2(\mathbb{R})$, we obtain:

Corollary 2.1. *Let u be a function in $L^2(\mathbb{R})$.*

i) For all $0 < \delta < \frac{\pi}{2}$, there holds

$$(2.43) \quad \lim_{\zeta \rightarrow 0, \zeta \in \Gamma_\delta} \tilde{a}_u(\zeta) = 1, \quad \lim_{|\zeta| \rightarrow \infty, \zeta \in \Gamma_\delta} \tilde{a}_u(\zeta) = e^{-\frac{i}{2} \|u\|_{L^2(\mathbb{R})}^2},$$

where we denote $\Gamma_\delta = \{\zeta \in \mathbb{C}_+ : \delta < \arg \zeta < \pi - \delta\}$.

ii) For all $\theta \in]0, \pi[$,

$$(2.44) \quad n(u; \theta) \leq \frac{\|u\|_{L^2(\mathbb{R})}^2}{4\theta},$$

where, as before, $n(u; \theta)$ denotes the number of zeros of $\tilde{a}_u(\zeta)$ in the angle $\{\zeta \in \mathbb{C}_+ : \theta < \arg \zeta < \pi\}$ counted with their multiplicity.

⁶See Appendix A for the definition of the regularized determinants \det_n and their basic properties.

iii) Let $\theta \in]0, \pi[$ such that $\tilde{a}_u(\zeta) \neq 0$, for all ζ belonging to the ray $e^{i\theta}\mathbb{R}_+$. Then,

$$(2.45) \quad \frac{1}{2i\pi} \int_0^{+\infty} ds \frac{\tilde{a}'_u(s)}{\tilde{a}_u(s)} + \frac{1}{4\pi} \|u\|_{L^2(\mathbb{R})}^2 \in \mathbb{N}.$$

Remark 2.2. If $u \in H^1(\mathbb{R})$, then one has the following lemma, the proof of which is sketched in Appendix B:

Lemma 2.4. Let $u \in H^1(\mathbb{R})$. Then,

$$(2.46) \quad \ln \tilde{a}_u(\zeta) = \sum_{k=0}^2 E_k(u) \zeta^{-k} + o(|\zeta|^{-2}), \quad \text{as } |\zeta| \rightarrow \infty, \quad \zeta \in \Gamma_\delta.$$

For our proof of Theorem 1, we will also need the following lemma which can be viewed as a slightly modified version of Lemma 4.3 in [14].

Lemma 2.5. Let $M \geq 0$, $0 < \theta_1 < \theta_2 < \pi$, and $0 < t \leq \frac{1}{2}$. Then there exists a positive constant $C = C(\theta_1, \theta_2, t, M)$ so that

$$(2.47) \quad \left| \frac{1}{\tilde{a}_u(\zeta)} \right| \leq C,$$

for all $\zeta \in \mathbb{C}_+$, with $(1-t)\theta_1 + t\theta_2 \leq \arg \zeta \leq t\theta_1 + (1-t)\theta_2$ and all $u \in L^2(\mathbb{R})$ with $M_{\mathbb{R}}(u) \leq M$, and such that \tilde{a}_u has no zeros in the angle $\theta_1 \leq \arg \zeta \leq \theta_2$.

Proof. In order to establish the lemma, we proceed by approximation: let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $\tilde{S}_{reg}(\mathbb{R})$ that converges to u in $L^2(\mathbb{R})$. Denoting by ζ_j^n , $j = 1, \dots, N_n$ the zeros of $\tilde{a}_{u_n}(\zeta)$ in \mathbb{C}_+ counted with their multiplicity, we have

$$(2.48) \quad \frac{1}{\tilde{a}_{u_n}(\zeta)} = e^{\frac{i}{2}\|u\|_{L^2(\mathbb{R})}^2} \prod_{j=1}^{N_n} \left(\frac{\zeta - \bar{\zeta}_j^n}{\zeta - \zeta_j^n} \right) \exp \left(\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\xi - \zeta} \ln |\tilde{a}_{u_n}(\xi)|^2 \right), \quad \forall \zeta \in \mathbb{C}_+ \setminus \{\zeta_1^n, \dots, \zeta_{N_n}^n\}.$$

Clearly, for all $0 < \delta < \frac{\pi}{2}$,

$$\sup_{\xi \in \mathbb{R}, \zeta \in \Gamma_\delta} \left| \frac{\xi}{\xi - \zeta} \right| \leq C_\delta,$$

and therefore

$$\sup_{\zeta \in \Gamma_\delta} \left| \int_{-\infty}^{\infty} \frac{d\xi}{\xi - \zeta} \ln |\tilde{a}_{u_n}(\xi)|^2 \right| \leq -C_\delta \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \ln |\tilde{a}_{u_n}(\xi)|^2.$$

Furthermore, the stability estimate (2.42) together with Corollary 2.1 ensure that, for all n sufficiently large, \tilde{a}_{u_n} does not vanish in the angle $\theta_1 \leq \arg \zeta \leq \theta_2$, and as a consequence

$$\prod_{j=1}^{N_n} \left| \frac{\zeta - \bar{\zeta}_j^n}{\zeta - \zeta_j^n} \right| \leq \exp \left(C_{\theta_1, \theta_2, t} \sum_{j=1}^{N_n} \arg \zeta_j^n \right),$$

for all n sufficiently large, and all ζ with $(1-t)\theta_1 + t\theta_2 \leq \arg \zeta \leq t\theta_1 + (1-t)\theta_2$.

Combining the two last inequalities with (2.37) and (2.48), and passing to the limit $n \rightarrow \infty$, we obtain (2.47). \square

We conclude this subsection by the following H^1 -version of Lemma 2.6 in [2].

Lemma 2.6. Let $u \in H^1(\mathbb{R})$ be such that the corresponding function \tilde{a}_u has no zeros in \mathbb{C}_+ . Then $E_{\mathbb{R}}(u) \geq 0$. Furthermore, $E_{\mathbb{R}}(u) = 0$ if and only if

$$(2.49) \quad \tilde{a}_u(\zeta) = 1, \quad \forall \zeta \in \mathbb{C}_+.$$

In that case, one also has $\|u\|_{L^2(\mathbb{R})}^2 \in 4\pi\mathbb{N}$ and $P_{\mathbb{R}}(u) = 0$.

Proof. We proceed as in the proof of Lemma 2.5. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $\tilde{\mathcal{S}}_{reg}(\mathbb{R})$ that converges to u in $H^1(\mathbb{R})$. Let ζ_j^n , $j = 1, \dots, N_n$, be the zeros of $\tilde{a}_{u_n}(\zeta)$ in \mathbb{C}_+ , counted with their multiplicity. Thanks to (2.34), we have

$$(2.50) \quad E_{\mathbb{R}}(u_n) = 16 \underbrace{\sum_{\substack{1 \leq j \leq N_n \\ \operatorname{Re} \zeta_j^n < 0}} \operatorname{Re} \zeta_j^n \operatorname{Im} \zeta_j^n}_{\geq 0} + 16 \underbrace{\sum_{\substack{1 \leq j \leq N_n \\ \operatorname{Re} \zeta_j^n > 0}} \operatorname{Re} \zeta_j^n \operatorname{Im} \zeta_j^n}_{\geq 0} - \underbrace{\frac{4}{\pi} \int_{-\infty}^{\infty} d\xi \xi \ln |\tilde{a}_{u_n}(\xi)|^2}_{\geq 0}.$$

Combining Corollary 2.1 with the fact that \tilde{a}_u does not vanish on \mathbb{C}_+ , and taking into account the stability estimate (2.42), we infer that

$$(2.51) \quad \sup_{j=1, \dots, N_n} \frac{\operatorname{Im} \zeta_j^n}{|\operatorname{Re} \zeta_j^n|} \xrightarrow{n \rightarrow \infty} 0.$$

Therefore, invoking Lemma 2.2 together with the bound

$$(2.52) \quad \#\{\zeta_i^n, \operatorname{Re}(\zeta_j^n) < 0\} \lesssim \|u^{(n)}\|_{L^2(\mathbb{R})}^2 \lesssim \|u\|_{L^2(\mathbb{R})}^2,$$

we obtain

$$(2.53) \quad \sum_{\substack{1 \leq j \leq N_n \\ \operatorname{Re} \zeta_j^n < 0}} \operatorname{Re} \zeta_j^n \operatorname{Im} \zeta_j^n \xrightarrow{n \rightarrow \infty} 0,$$

which, after passing to the limit in (2.50), gives $E_{\mathbb{R}}(u) \geq 0$.

In the case of $E_{\mathbb{R}}(u) = 0$, we deduce from (2.50) and (2.53) that

$$(2.54) \quad \sum_{\substack{1 \leq j \leq N_n \\ \operatorname{Re} \zeta_j^n > 0}} \operatorname{Re} \zeta_j^n \operatorname{Im} \zeta_j^n \xrightarrow{n \rightarrow \infty} 0,$$

and

$$(2.55) \quad \int_{-\infty}^{\infty} d\xi \xi \ln |\tilde{a}_{u_n}(\xi)|^2 \xrightarrow{n \rightarrow \infty} 0.$$

By (2.37) and (2.51), we also have

$$(2.56) \quad - \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \ln |\tilde{a}_{u_n}(\xi)|^2 \lesssim \|u_n\|_{L^2(\mathbb{R})}^2 \lesssim 1,$$

and

$$(2.57) \quad \sum_{1 \leq j \leq N_n} \frac{\operatorname{Im} \zeta_j^n}{|\operatorname{Re} \zeta_j^n|} \lesssim \|u_n\|_{L^2(\mathbb{R})}^2 \lesssim 1,$$

for all n sufficiently large. Combining (2.53), (2.54), (2.55) and (2.56), we obtain

$$(2.58) \quad \int_{-\infty}^{\infty} d\xi |\ln |\tilde{a}_{u_n}(\xi)|^2| \xrightarrow{n \rightarrow \infty} 0,$$

and

$$(2.59) \quad \sum_{1 \leq j \leq N_n} \operatorname{Im} \zeta_j^n \xrightarrow{n \rightarrow \infty} 0,$$

which in light of (2.33) implies that $\tilde{a}_{u_n}(\zeta) \xrightarrow{n \rightarrow \infty} e^{-\frac{i}{2}\|u\|_{L^2(\mathbb{R})}^2}$, for all $\zeta \in \mathbb{C}_+$, and therefore $\tilde{a}_u \equiv e^{-\frac{i}{2}\|u\|_{L^2(\mathbb{R})}^2}$. In view of Corollary 2.1 this means that $\|u\|_{L^2(\mathbb{R})}^2 \in 4\pi\mathbb{N}$ and $\tilde{a}_u \equiv 1$. Conversely if $\tilde{a}_u \equiv 1$, then by (2.46) $E_{\mathbb{R}}(u) = P_{\mathbb{R}}(u) = 0$. \square

Remark 2.3. *The set of potential verifying the condition $\tilde{a}_u \equiv 1$ is not trivial: it contains the algebraic solitons, see for instance [2, 14]. Furthermore, they are the only potentials with $\tilde{a}_u \equiv 1$ and $\|u\|_{L^2(\mathbb{R})}^2 = 4\pi$:*

Lemma 2.7. *Let $u \in H^1(\mathbb{R})$ such that $\tilde{a}_u \equiv 1$ and $\|u\|_{L^2(\mathbb{R})}^2 = 4\pi$. Then*

$$u(x) = 2\sqrt{c}e^{i\varphi+icx/2} \frac{c(x-x_0) + i}{(c(x-x_0) - i)^2},$$

for some $\varphi, x_0 \in \mathbb{R}$ and $c > 0$.

This result is an immediate consequence of the following sharp Gagliardo-Nirenberg inequality due to Agueh [1]:

$$(2.60) \quad \forall f \in L^4(\mathbb{R}) \cap \dot{H}^1(\mathbb{R}), \quad \|f\|_{L^6(\mathbb{R})} \leq C_{GN} \|f\|_{L^4(\mathbb{R})}^{\frac{8}{9}} \|f\|_{\dot{H}^1(\mathbb{R})}^{\frac{1}{9}}, \quad C_{GN} = 3^{\frac{1}{6}} (2\pi)^{-\frac{1}{9}},$$

with equality if and only if there exist $z \in \mathbb{C}$, $c > 0$, and $x_0 \in \mathbb{R}$ such that $f(x) = \frac{z}{\sqrt{c^2(x-x_0)^2+1}}$. For the sake of completeness, we give the proof of Lemma 2.7 in Appendix B.

2.4. Bäcklund transformation. In this paragraph, we define the Bäcklund transform for the Kaup-Newell spectral problem following closely [30] (see also [15] and the references therein).

Given $u \in \mathcal{S}(\mathbb{R})$, $\lambda \in \mathbb{C}_{++} = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0, \operatorname{Im} \lambda > 0\}$ and $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ a non zero smooth solution of the KN spectral problem $L_u(\lambda)\psi = 0$, one defines the Bäcklund transformation $\mathcal{B}_\lambda(\psi)$ by⁷

$$(2.61) \quad \mathcal{B}_\lambda(\psi)u = G_\lambda(\psi) \left[-G_\lambda(\psi)u + \mathcal{S}_\lambda(\psi) \right],$$

where

$$(2.62) \quad G_\lambda(\psi) = \frac{d_{\bar{\lambda}}(\psi)}{d_\lambda(\psi)}, \quad \mathcal{S}_\lambda(\psi) = 2i(\lambda^2 - \bar{\lambda}^2) \frac{\psi_1 \bar{\psi}_2}{d_\lambda(\psi)},$$

with $d_\lambda(\psi) = \lambda|\psi_1|^2 + \bar{\lambda}|\psi_2|^2$. Since ψ depends implicitly of u , the transformation (2.61) is nonlinear with respect to the potential u . One can easily check that $\mathcal{B}_\lambda(\psi)u \in \mathcal{S}(\mathbb{R})$. Observe also that

$$(2.63) \quad |G_\lambda(\psi)| = 1 \quad \text{and} \quad |\mathcal{S}_\lambda(\psi)| \leq 4 \operatorname{Im} \lambda.$$

Moreover, by straightforward computations, one can check that (see [2] for the proof)

$$(2.64) \quad \left| \frac{d}{dx} G_\lambda(\psi)(x) \right| \leq 8(\operatorname{Im} \lambda)^2 + 4 \operatorname{Im}(\lambda) |u(x)|.$$

Setting

$$(2.65) \quad \psi^{(1)} = \begin{pmatrix} \psi_1^{(1)} \\ \psi_2^{(1)} \end{pmatrix}, \quad \psi_1^{(1)} = \frac{\bar{\psi}_2}{d_\lambda(\psi)}, \quad \psi_2^{(1)} = \frac{\bar{\psi}_1}{d_{\bar{\lambda}}(\psi)},$$

one gets a solution of the KN system associated with the potential $u^{(1)} = \mathcal{B}_\lambda(\psi)u$: $L_{u^{(1)}}(\lambda)\psi^{(1)} = 0$. The Bäcklund transformation $\mathcal{B}_\lambda(\psi^{(1)})$ is a left inverse of $\mathcal{B}_\lambda(\psi)$:

$$(2.66) \quad u = \mathcal{B}_\lambda(\psi^{(1)})u^{(1)}.$$

The key property of the Bäcklund transformation (2.61) is that it allows to add or to remove eigenvalues of the Kaup-Newell spectral problem. In particular, assume that $\tilde{a}_u(\zeta)$ has a simple zero $\zeta_1 \in \mathbb{C}_+$ and let $\psi \in L^2(\mathbb{R}, \mathbb{C}^2) \setminus \{0\}$ be the corresponding eigenfunction: $L_u(\lambda_1)\psi = 0$, $\lambda_1 = \sqrt{\zeta_1} \in \mathbb{C}_{++}$. Then the function $\tilde{a}_{u^{(1)}}$ associated to the potential $u^{(1)} = \mathcal{B}_{\lambda_1}(\psi)u$ is given by (see for instance [15, 30])

$$(2.67) \quad \tilde{a}_{u^{(1)}}(\zeta) = \tilde{a}_u(\zeta) \frac{\zeta_1(\zeta - \bar{\zeta}_1)}{\bar{\zeta}_1(\zeta - \zeta_1)}.$$

⁷Note that, for any constant $C \neq 0$, $\mathcal{B}_\lambda(\psi) = \mathcal{B}_\lambda(C\psi)$.

Thus, $\tilde{a}_{u^{(1)}}$ does not vanish at ζ_1 . Note also that thanks to (2.31) and (2.35), we have

$$(2.68) \quad M_{\mathbb{R}}(u^{(1)}) = M_{\mathbb{R}}(u) - 4 \arg \zeta_1,$$

and

$$(2.69) \quad E_k(u^{(1)}) = E_k(u) + \frac{2i}{k} \operatorname{Im} \zeta_1^k, \quad \forall k \geq 1.$$

The Jost solutions of the KN system associated with the new potential $u^{(1)} = \mathcal{B}_{\lambda_1}(\psi)u$ can be obtained from the old Jost solutions by means of the following transformation matrix:

$$(2.70) \quad A(x, \lambda; \psi, \lambda_1) = \frac{\lambda_1}{\bar{\lambda}_1} \frac{1}{\lambda^2 - \lambda_1^2} \begin{pmatrix} \lambda^2 G_{\lambda_1}(\psi) - |\lambda_1|^2 & \frac{-\lambda}{2i} \mathcal{S}_{\lambda_1}(\psi) \\ \frac{\lambda}{2i} \mathcal{S}_{\lambda_1}(\psi) & -\lambda^2 \bar{G}_{\lambda_1}(\psi) + |\lambda_1|^2 \end{pmatrix}.$$

More precisely, one has the following result (see Lemma 4 in [30]).

Lemma 2.8. *Let $\lambda_1 \in \mathbb{C}_{++}$ be an eigenvalue of the Kaup-Newell spectral problem (2.2) and $\psi \in L^2(\mathbb{R}, \mathbb{C}^2) \setminus \{0\}$ a corresponding eigenfunction. Then the Jost solutions associated with the new potential $u^{(1)} = \mathcal{B}_{\lambda_1}(\psi)u$ are given by*

$$\begin{aligned} \psi^-(x, \lambda; u^{(1)}) &= A(x, \lambda; \psi, \lambda_1) \psi^-(x, \lambda; u) \\ \psi^+(x, \lambda; u^{(1)}) &= -A(x, \lambda; \psi, \lambda_1) \psi^+(x, \lambda; u). \end{aligned}$$

Remark 2.4. *It follows from (2.67) that under the assumptions of Lemma 2.8, one has*

$$\det A(x, \lambda; \psi, \lambda_1) = -\frac{\lambda_1^2(\lambda^2 - \bar{\lambda}_1^2)}{\bar{\lambda}_1^2(\lambda^2 - \lambda_1^2)}.$$

3. PROOF OF THE MAIN THEOREM

3.1. Strategy of proof. Thanks to the local well-posedness result of Herr [16], to prove Theorem 1 it is enough to show that, for any $R > 0$ and any $0 < \delta < 8\pi$, there exists $C = C(R, \delta) > 0$ such that

$$(3.1) \quad \sup_{0 \leq t \leq T} \|u(t)\|_{H^1(\mathbb{T})} \leq C,$$

for any solution $u \in C([0, T], H^1(\mathbb{T}))$ of the DNLS equation with $u(0) \in C^\infty(\mathbb{T})$, $\|u(0)\|_{H^1(\mathbb{T})} \leq R$ and $\|u\|_{L^2(\mathbb{T})}^2 \leq 8\pi - \delta$. In order to establish the bound (3.1), we proceed by contradiction,

assuming that there exists a sequence of initial data $(u_0^{(n)})$ in $C^\infty(\mathbb{T})$, bounded in $H^1(\mathbb{T})$ with $\sup_{n \in \mathbb{N}} \|u_0^{(n)}\|_{L^2(\mathbb{T})}^2 < 8\pi$, and a sequence of times $(t_n) \subset \mathbb{R}_+$ such that, denoting u_n

the solution of the DNLS equation with initial data $u_0^{(n)}$ we have: $u_n \in C([0, t_n], C^\infty(\mathbb{T}))$ and $\|u_n(t_n)\|_{H^1(\mathbb{T})} \rightarrow +\infty$, as n goes to infinity. After eventually passing to a subsequence, we can assume that $\|u_0^{(n)}\|_{L^2(\mathbb{T})} \rightarrow m$, with⁸ $4\pi \leq m < 8\pi$. Setting

$$(3.2) \quad U_n^{(0)}(y) = \frac{1}{\sqrt{\mu_n}} u_n(t_n, \frac{y}{\mu_n}) \quad \text{with} \quad \mu_n = \|u_n(t_n)\|_{H^1(\mathbb{T})},$$

we obtain a sequence of C^∞ functions satisfying

$$(3.3) \quad U_n^{(0)}(y + \mu_n) = U_n^{(0)}(y), \quad \|U_n^{(0)}\|_{L^2(\mu_n \mathbb{T})} = \|u_0^{(n)}\|_{L^2(\mathbb{T})} \quad \text{and} \quad \|\partial_y U_n^{(0)}\|_{L^2(\mu_n \mathbb{T})} = 1, \quad \forall n \in \mathbb{N}.$$

⁸We assume that $m \geq 4\pi$ since the case of $m < 4\pi$ is precluded by the global well-posedness results of Oh-Mosincat [29] and Killip-Ntekoume-Visan [21]. The bound $m \geq 4\pi$ can be also recovered from Theorem 3.

We next truncate $U_n^{(0)}$ as follows. We fix a sequence $(N_n)_{n \in \mathbb{N}} \subset \mathbb{N}^*$ such that

$$(3.4) \quad N_n \xrightarrow{n \rightarrow \infty} \infty \quad \text{and} \quad \frac{\mu_n}{N_n} \xrightarrow{n \rightarrow \infty} \infty,$$

and for each n , split $[0, \mu_n]$ into N_n intervals of the same length. Then we choose $k \in \{0, \dots, N_n - 1\}$ such that

$$(3.5) \quad \|U_n^{(0)}\|_{H^1([\frac{k\mu_n}{N_n}, \frac{(k+1)\mu_n}{N_n}])}^2 \leq \frac{\|u_0^{(n)}\|_{L^2(\mathbb{T})}^2 + 1}{N_n},$$

and set

$$(3.6) \quad V_n^{(1)}(y) = \chi\left(\frac{y}{R_n}\right) U_n^{(1)}(y) \chi\left(\frac{\mu_n - y}{R_n}\right),$$

where $U_n^{(1)}(y) = U_n^{(0)}(y + (k + \frac{1}{2})R_n)$, $R_n = \frac{\mu_n}{N_n}$ and χ is a C^∞ function valued in the interval $[0, 1]$ and such that

$$(3.7) \quad \chi \equiv 1 \text{ on } \mathbb{R}_+ \quad \text{and} \quad \chi \equiv 0 \text{ on } (-\infty, -\frac{1}{2}].$$

The constructed function $V_n^{(1)}$ is supported in $[-\frac{R_n}{2}, \mu_n + \frac{R_n}{2}]$, equal to $U_n^{(1)}$ on $[0, \mu_n]$, and according to (3.5) satisfies

$$(3.8) \quad \|V_n^{(1)}\|_{H^1((-\infty, \frac{R_n}{2}] \cup [\mu_n - \frac{R_n}{2}, +\infty))}^2 \xrightarrow{n \rightarrow \infty} 0.$$

Therefore the sequence $(V_n^{(1)})_{n \in \mathbb{N}}$ is bounded in $H^1(\mathbb{R})$ and

$$\lim_{n \rightarrow \infty} \|V_n^{(1)}\|_{L^2(\mathbb{R})}^2 = m, \quad \lim_{n \rightarrow \infty} \|V_n^{(1)}\|_{\dot{H}^1(\mathbb{R})}^2 = 1.$$

The first step towards proving Theorem 1 consists in analyzing the profile decomposition of the sequence $(V_n^{(1)})_{n \in \mathbb{N}}$ with respect to the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$, $p > 2$, and showing that after eventually passing to a subsequence⁹, we have

$$V_n^{(1)}(y) = V(y - y_n^{(1)}) + r_n^{(1)}(y),$$

where the profile V coincides, up to the symmetries, with the algebraic soliton (1.8), the core $y_n^{(1)}$ satisfies $y_n^{(1)} \xrightarrow{n \rightarrow \infty} +\infty$, $\mu_n - y_n^{(1)} \xrightarrow{n \rightarrow \infty} +\infty$ and $\|\partial_y r_n^{(1)}\|_{L^2(\mathbb{R})} \xrightarrow{n \rightarrow \infty} 0$. Compared to the profile decomposition of [2], the situation is much simpler here since we are working in the H^1 -setting which will allow us to obtain this result essentially as a consequence of the conservation of energy.

We next refine the above construction introducing a family of H^1 functions V_n^K depending on an additional parameter $K \in \mathbb{N}$, $M^2 \leq K \leq M^{-2}\mu_n$, for some large fixed constant M , so that the following properties hold: for all n and K , the function V_n^K is supported in the interval $[-M, \mu_n + M]$, on $[0, \mu_n]$ it coincides up to a translation with $U_n^{(0)}$, and it is of order $\frac{1}{\sqrt{\mu_n}}$ in a M -vicinity of 0 and μ_n and of order $\frac{1}{\sqrt{K}}$ in a $\frac{\mu_n}{K}$ vicinity of these points (see further (3.26)). The decomposition proved for the sequence $(V_n^{(1)})$ ensures that there exists $(y_n(K)) \subset [0, \mu_n]$ so that

$$V_n^K(y) = V(y - y_n(K)) + r_n^{(K)}(y),$$

with $\max_K \|r_n^{(K)}\|_{\dot{H}^1(\mathbb{R})} \xrightarrow{n \rightarrow \infty} 0$.

To arrive to a contradiction, we perform a refined study of the monodromy matrix associated to $V_n^K|_{[0, \mu_n]}$ for some suitable choice of $K = K_n$. First, following [2], we show that for all n sufficiently large and all K , the function $\tilde{a}_{V_n^K}$ has a unique zero $\zeta_{0,n}^{(K)}$ in the angle $\theta_0 \leq \arg \zeta < \pi$ provided that θ_0 is chosen sufficiently close to π , and that this

⁹See Theorem 2 for a precise statement.

zero is simple and satisfies $\max_K \operatorname{Im} \zeta_{0,n}^{(K)} \xrightarrow{n \rightarrow \infty} 0$, and $-c_1 \leq \operatorname{Re} \zeta_{0,n}^{(K)} \leq -c_2 < 0$, for some constants c_1, c_2 . We next remove this zero by applying to V_n^K the corresponding Bäcklund transformation. The resulting potential $\tilde{r}_n^{(K)}$ has mass strictly less than 4π , and energy and momentum of order $\operatorname{Im} \zeta_{0,n}^{(K)} + \frac{1}{\mu_n}$, which in view of (1.7) ensures that

$$\|\tilde{r}_n^{(K)}\|_{\dot{H}^1(\mathbb{R})}^2 \lesssim \operatorname{Im} \zeta_{0,n}^{(K)} + \frac{1}{\mu_n} \xrightarrow{n \rightarrow \infty} 0.$$

Accordingly to this bound, we distinguish two regimes:

- $\mu_n \min_K \operatorname{Im} \zeta_{0,n}^{(K)} \xrightarrow{n \rightarrow \infty} \infty$,
- up to a subsequence, $\mu_n \min_K \operatorname{Im} \zeta_{0,n}^{(K)} \xrightarrow{n \rightarrow \infty} \beta \in \mathbb{R}_+$.

Then, we fix K_n in such a way that $K_n \xrightarrow{n \rightarrow \infty} \infty$, $\frac{K_n}{\mu_n \min_K \operatorname{Im} \zeta_{0,n}^{(K)}} \xrightarrow{n \rightarrow \infty} 0$ in the first case,

and $\operatorname{Im} \zeta_{0,n}^{(K_n)} = \min_K \operatorname{Im} \zeta_{0,n}^{(K)}$ in the second case.

The smallness of $\tilde{r}_n = \tilde{r}_n^{(K_n)}$ in $\dot{H}^1(\mathbb{R})$, allows to approximate the Jost solutions $\psi_{\tilde{r}_n}^\pm$ by a suitable WKB ansatz. Thanks to Lemma 2.8, this leads to an explicit approximation of the monodromy matrix of $V_n^{K_n}|_{[0, \mu_n]}$. In the case of $\mu_n \operatorname{Im} \zeta_{0,n}^{(K_n)} \xrightarrow{n \rightarrow \infty} \infty$, we use the conservation of the Floquet discriminant to show that the obtained approximation is in contradiction with the asymptotics given by (2.21). In the second case, we get a contradiction by invoking Lemma 2.1.

3.2. Profile decomposition. The goal of this section is to prove the following profile decomposition for the sequence $(V_n^{(1)})_{n \in \mathbb{N}}$ defined by (3.6).

Theorem 2. *Under the notations of Section 3.1, there exist $\gamma \in \mathbb{R}$, $c > 0$, and a sequence $(y_n^{(1)})_{n \in \mathbb{N}}$ of $[0, \mu_n]$, such that, up to a subsequence,*

$$(3.9) \quad V_n^{(1)}(y) = V(y - y_n^{(1)}) + r_n^{(1)}(y) \quad \text{with} \quad V(y) = 2\sqrt{c}e^{i\gamma + icy/2} \frac{cy + i}{(cy - i)^2},$$

where $\|\partial_y r_n^{(1)}\|_{L^2(\mathbb{R})} \xrightarrow{n \rightarrow \infty} 0$, and where in addition,

$$(3.10) \quad y_n^{(1)} \xrightarrow{n \rightarrow \infty} \infty \quad \text{and} \quad \mu_n - y_n^{(1)} \xrightarrow{n \rightarrow \infty} +\infty.$$

In fact we will prove a more general result given by the following theorem.

Theorem 3. *Let $(u_0^{(n)})$ be a bounded sequence in $H^1(\mathbb{T})$ and let $m = \limsup_{n \rightarrow \infty} \|u_0^{(n)}\|_{L^2(\mathbb{T})}^2$.*

Denote by $u_n \in C([0, T_n[, H^1(\mathbb{T}))$ the solution of the DNLS equation with initial data $u_0^{(n)}$ and assume that there exists a sequence $(t_n) \subset \mathbb{R}_+$ with $t_n < T_n$ such that $\lim_{n \rightarrow \infty} \|u_n(t_n)\|_{H^1(\mathbb{T})} =$

$+\infty$. Let $(V_n^{(1)})$ be the corresponding $H^1(\mathbb{R})$ -sequence constructed as in Section 3.1. Then there exist an integer $1 \leq L_0 \leq \frac{m}{4\pi}$, a family of profiles $(\varphi_\ell)_{1 \leq \ell \leq L_0}$ in $H^1(\mathbb{R}) \setminus \{0\}$ with $\tilde{a}_{\varphi_\ell} \equiv 1$, for all $\ell \in \{1, \dots, L_0\}$, and a family of orthogonal cores¹⁰ $(\underline{y}^{(\ell)})_{1 \leq \ell \leq L_0}$, in the sense that for all $\ell \neq \ell'$, we have $|\underline{y}_n^{(\ell)} - \underline{y}_n^{(\ell')}| \xrightarrow{n \rightarrow \infty} \infty$, such that, up to a subsequence,

$$V_n^{(1)}(y) = \sum_{\ell=1}^{L_0} \varphi_\ell(y - \underline{y}_n^{(\ell)}) + r_n^{(1)}(y),$$

where $\|\partial_y r_n^{(1)}\|_{L^2(\mathbb{R})} \xrightarrow{n \rightarrow \infty} 0$, and where in addition, for all $1 \leq \ell \leq L_0$,

$$(3.11) \quad \underline{y}_n^{(\ell)} \xrightarrow{n \rightarrow \infty} \infty \quad \text{and} \quad \mu_n - \underline{y}_n^{(\ell)} \xrightarrow{n \rightarrow \infty} +\infty.$$

¹⁰Following the vocabulary of P. Gérard in [7], we designate by a core $\underline{y}^{(\ell)}$ any real sequence $(y_n^{(\ell)})_{n \in \mathbb{N}}$.

We start the proof of Theorem 3 with the following proposition which can be easily derived from the work of P. Gérard [7].

Proposition 3.1. *There exist a sequence of profiles $(\varphi_\ell)_{\ell \geq 1}$ in $H^1(\mathbb{R})$ which are not all zero and a sequence of orthogonal cores $(y^{(\ell)})_{\ell \geq 1}$ such that, up to a subsequence extraction, we have, for all $L \geq 1$,*

$$(3.12) \quad V_n^{(1)}(y) = \sum_{\ell=1}^L \varphi_\ell(y - y_n^{(\ell)}) + r_n^L(y),$$

where

$$(3.13) \quad r_n^L(\cdot + y_n^{(\ell)}) \xrightarrow{n \rightarrow \infty} 0 \text{ in } H^1(\mathbb{R}), \quad \forall \ell = 1, \dots, L,$$

and

$$(3.14) \quad \limsup_{n \rightarrow \infty} \|r_n^L\|_{L^p(\mathbb{R})} \xrightarrow{L \rightarrow \infty} 0,$$

for all $2 < p < \infty$.

In addition we have

$$(3.15) \quad y_n^{(\ell)} \xrightarrow{n \rightarrow \infty} +\infty \quad \text{and} \quad \mu_n - y_n^{(\ell)} \xrightarrow{n \rightarrow \infty} +\infty, \quad \forall \ell \geq 1.$$

Proof. The sequence $(V_n^{(1)})$ is bounded in $H^1(\mathbb{R})$, and for all $2 < p$, $\limsup_{n \rightarrow \infty} \|V_n^{(1)}\|_{L^p(\mathbb{R})} > 0$.

Indeed, $\|V_n^{(1)}\|_{L^p(\mathbb{R})} \xrightarrow{n \rightarrow \infty} 0$ would imply that

$$E_{\mu_n \mathbb{T}}(U_n) = \|U_n^{(0)}\|_{\dot{H}^1(\mu_n \mathbb{T})}^2 + o(1) = 1 + o(1), \quad n \rightarrow \infty,$$

which is impossible since by the conservation of energy and the boundedness of $(u_0^{(n)})$ in $H^1(\mathbb{T})$, we have

$$(3.16) \quad E_{\mu_n \mathbb{T}}(U_n) = \mu_n^{-2} E_{\mathbb{T}}(u_0^{(n)}) \xrightarrow{n \rightarrow \infty} 0.$$

Therefore applying [7], we obtain the existence of a sequence of profiles $(\varphi_\ell)_{\ell \geq 1}$ in $H^1(\mathbb{R})$ which are not all zero and a sequence of orthogonal cores $(y^{(\ell)})_{\ell \geq 1}$ such that, up to a subsequence extraction, the properties (3.12)-(3.14) are satisfied. Finally, the property (3.15) is a direct consequence of (3.8). \square

Remark 3.1. *Note that thanks to (3.12)-(3.14) we have, for all $L \geq 1$,*

$$(3.17) \quad \|V_n^{(1)}\|_{L^2(\mathbb{R})}^2 = \sum_{\ell=1}^L \|\varphi_\ell\|_{L^2(\mathbb{R})}^2 + \|r_n^L\|_{L^2(\mathbb{R})}^2 + o(1),$$

$$(3.18) \quad \|\partial_y V_n^{(1)}\|_{L^2(\mathbb{R})}^2 = \sum_{\ell=1}^L \|\partial_y \varphi_\ell\|_{L^2(\mathbb{R})}^2 + \|\partial_y r_n^L\|_{L^2(\mathbb{R})}^2 + o(1),$$

$$(3.19) \quad E_{\mathbb{R}}(V_n^{(1)}) = \sum_{\ell=1}^L E_{\mathbb{R}}(\varphi_\ell) + E_{\mathbb{R}}(r_n^L) + o(1),$$

as $n \rightarrow \infty$. In particular,

$$\sum_{\ell=1}^{\infty} \|\varphi_\ell\|_{L^2(\mathbb{R})}^2 \leq m \quad \text{and} \quad \sum_{\ell=1}^{\infty} \|\varphi_\ell\|_{\dot{H}^1(\mathbb{R})}^2 \leq 1.$$

To conclude the proof of Theorem 3, it remains to show that $\limsup_{n \rightarrow \infty} \|\partial_y r_n^L\|_{L^2(\mathbb{R})} \xrightarrow{L \rightarrow \infty} 0$ and that, for all ℓ , $\tilde{a}_{\varphi_\ell} \equiv 1$, which, in view of Remarks 2.3 and 3.1, will also imply that the number of non-zero profiles in (3.12) is bounded by $\frac{m}{4\pi}$. These properties will be deduced by combining (3.19) with Lemma 2.6. In order to apply this lemma, we have to check first that the functions \tilde{a}_{φ_ℓ} do not vanish on \mathbb{C}_+ .

Lemma 3.1. *For each profile φ_ℓ involved in the decomposition (3.12), the spectral coefficient \tilde{a}_{φ_ℓ} does not vanish on \mathbb{C}_+ .*

Proof. We proceed by contradiction, assuming that there exist $\ell_0 \geq 1$, $\lambda_0 \in \mathbb{C}_{++}$ and $\psi_0 \in H^1(\mathbb{R})$ such that $\|\psi_0\|_{L^2(\mathbb{R})} = 1$ and

$$(3.20) \quad L_{\varphi_{\ell_0}}(\lambda_0)\psi_0 = 0.$$

Let Θ be a \mathbb{C}^∞ function supported in $[-1, 1]$ and such that $\Theta \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. Set $\Theta_n(y) = \Theta\left(\frac{y-y_n^{(\ell_0)}}{A_n}\right)$ and $\psi_n = \Theta_n(y)\psi_0(y - y_n^{(\ell_0)})$, with $A_n = \min(y_n^{(\ell_0)}, \mu_n - y_n^{(\ell_0)})$. Then, we have

$$(3.21) \quad L_{V_n^{(1)}}(\lambda_0)\psi_n = \mathcal{R}_n,$$

where

$$\mathcal{R}_n(y) = \Theta_n \mathcal{R}_n^1(y) + i(\partial_y \Theta_n) \sigma_3 \psi_0(y - y_n^{(\ell_0)}),$$

with

$$\mathcal{R}_n^1(y) = -i\lambda_0 \begin{pmatrix} 0 & \sum_{\substack{\ell \neq \ell_0 \\ 1 \leq \ell \leq L}} \varphi_\ell(y - y_n^{(\ell)}) + r_n^L(y) \\ \sum_{\substack{\ell \neq \ell_0 \\ 1 \leq \ell \leq L}} \varphi_\ell(y - y_n^{(\ell)}) + r_n^L(y) & 0 \end{pmatrix} \psi_0(y - y_n^{(\ell_0)}).$$

Combining the orthogonality condition between the cores with (3.14) and (3.15), one can easily check that

$$\|\mathcal{R}_n\|_{L^2(\mathbb{R})} \xrightarrow{n \rightarrow +\infty} 0.$$

Given that $V_n^{(1)}$ coincides with $U_n^{(1)}$ on $[0, \mu_n]$ and ψ_n satisfies the periodic boundary conditions on $[0, \mu_n]$, we can apply Proposition 2.3 to deduce, in view of (2.21),

$$\|\psi_n\|_{L^2([0, \mu_n])} \leq C(\lambda_0, \|u_0^{(n)}\|_{L^2(\mathbb{T})}) \left| \frac{\cos(\mu_n \lambda_0^2) - 1}{\Delta_{u_0}(\mu_n^{\frac{1}{2}} \lambda_0) - 2} \right| \|\mathcal{R}_n\|_{L^2(\mathbb{R})} \xrightarrow{n \rightarrow +\infty} 0,$$

which leads to a contradiction since by construction, $\|\psi_n\|_{L^2([0, \mu_n])} \xrightarrow{n \rightarrow +\infty} \|\psi_0\|_{L^2(\mathbb{R})} = 1$. \square

We are now in position to finish the proof of Theorem 3. Due to (3.8) and (3.16), we have

$$E_{\mathbb{R}}(V_n^{(1)}) \xrightarrow{n \rightarrow +\infty} 0,$$

which together with (3.14) and (3.19) implies that

$$(3.22) \quad \limsup_{n \rightarrow \infty} \left| \sum_{\ell=1}^L E_{\mathbb{R}}(\varphi_\ell) + \|r_n^L\|_{\dot{H}^1(\mathbb{R})}^2 \right| \xrightarrow{L \rightarrow \infty} 0.$$

Since by Lemmas 2.6 and 3.1, $E_{\mathbb{R}}(\varphi_\ell) \geq 0$ for all $\ell \geq 1$, we obtain that

$$\limsup_{n \rightarrow \infty} \|r_n^L\|_{\dot{H}^1(\mathbb{R})}^2 \xrightarrow{L \rightarrow \infty} 0,$$

$$E_{\mathbb{R}}(\varphi_\ell) = 0, \quad \forall \ell \geq 1,$$

and thus $\tilde{a}_{\varphi_\ell} \equiv 1$, for all $\ell \geq 1$. Accordingly to Remark 3.1, this ensures that the number of non zero profiles in the decomposition (3.12) is finite and bounded by $\frac{m}{4\pi}$. Denoting this number by L_0 and setting $r_n = r_n^{L_0}$, we get

$$\lim_{n \rightarrow \infty} \|r_n\|_{\dot{H}^1(\mathbb{R})} = 0,$$

which concludes the proof of Theorem 3.

3.3. Preliminary results. To proceed further, we need to introduce a refined version of the construction of Subsection 3.1. We fix once forever a large integer $M \gg 1$. For each¹¹ $K \in \left\{M^2, \dots, \left\lfloor \frac{\mu_n}{M^2} \right\rfloor\right\}$, we split the interval $[0, \mu_n]$ into K subintervals of the same length and choose $k_0 = k_0(K) \in \{0, \dots, K-1\}$ such that

$$(3.23) \quad \|U_n^{(1)}\|_{H^1\left(\left[\frac{k_0\mu_n}{K}, \frac{(k_0+1)\mu_n}{K}\right]\right)}^2 \leq \frac{\|u_0^{(n)}\|_{L^2(\mathbb{T})}^2 + 1}{K}.$$

Next we take the interval $I_n^0(K) = \left[\frac{(k_0+\frac{1}{4})\mu_n}{K}, \frac{(k_0+\frac{3}{4})\mu_n}{K}\right]$ and divide it into $K_1 = \left\lfloor \frac{\mu_n}{8MK} \right\rfloor$ subintervals of the same size and choose again $k_1 \in \{0, \dots, K_1-1\}$ such that

$$(3.24) \quad \|U_n^{(1)}\|_{H^1(I_n^1(K))}^2 \leq \frac{\|U_n^{(1)}\|_{H^1(I_n^0(K))}^2}{K_1} \lesssim \frac{\|u_0^{(n)}\|_{L^2(\mathbb{T})}^2 + 1}{\mu_n},$$

where $I_n^1(K) = \left[\frac{\mu_n}{K} \left(k_0 + \frac{1}{4} + \frac{k_1}{2K_1}\right), \frac{\mu_n}{K} \left(k_0 + \frac{1}{4} + \frac{k_1+1}{2K_1}\right)\right]$.

Finally, we set

$$(3.25) \quad U_n^K(y) = U_n^{(1)}(y + \tilde{y}_n(K)), \quad V_n^K(y) = \chi\left(\frac{y}{M}\right)U_n^K(y)\chi\left(\frac{\mu_n - y}{M}\right),$$

where $\tilde{y}_n(K)$ is the center of the interval $I_n^1(K)$ and χ is the function defined by (3.7). The function V_n^K is supported in $[-\frac{M}{2}, \mu_n + \frac{M}{2}]$, coincides with U_n^K on $[0, \mu_n]$ and satisfies

$$(3.26) \quad \|V_n^K\|_{H^1((-\infty, M] \cup [\mu_n - M, +\infty))}^2 \lesssim \frac{\|u_0^{(n)}\|_{L^2(\mathbb{T})}^2 + 1}{\mu_n},$$

$$\|V_n^K\|_{H^1((-\infty, \frac{\mu_n}{4K}] \cup [\mu_n - \frac{\mu_n}{4K}, +\infty))}^2 \lesssim \frac{\|u_0^{(n)}\|_{L^2(\mathbb{T})}^2 + 1}{K},$$

for all n sufficiently large and all $K \in \left\{M^2, \dots, \left\lfloor \frac{\mu_n}{M^2} \right\rfloor\right\}$. As a consequence, we get

$$(3.27) \quad |M_{\mathbb{R}}(V_n^K) - M_{\mathbb{T}}(u_0^{(n)})| \lesssim \frac{1}{\mu_n},$$

$$(3.28) \quad |P_{\mathbb{R}}(V_n^K)| \lesssim \frac{1}{\mu_n},$$

$$(3.29) \quad |E_{\mathbb{R}}(V_n^K)| \lesssim \frac{1}{\mu_n}.$$

Furthermore, from (3.9) and (3.24), we deduce that

$$(3.30) \quad \min_{M^2 \leq K \leq \frac{\mu_n}{M^2}} |\tilde{y}_n(K) - y_n^{(1)}| \xrightarrow{n \rightarrow \infty} \infty.$$

According to (3.9), we write

$$(3.31) \quad V_n^K(y) = V(y - y_n(K)) + r_n^{(K)}(y),$$

where $y_n(K) \in (0, \mu_n)$ is given by

$$y_n(K) = \begin{cases} y_n^{(1)} - \tilde{y}_n(K) & \text{if } y_n^{(1)} \geq \tilde{y}_n(K) \\ y_n^{(1)} + \mu_n - \tilde{y}_n(K) & \text{if } y_n^{(1)} < \tilde{y}_n(K). \end{cases}$$

Thanks to (3.24), (3.30) and Theorem 2, we have

$$(3.32) \quad \max_{M^2 \leq K \leq \frac{\mu_n}{M^2}} \|r_n^{(K)}\|_{\dot{H}^1(\mathbb{R})} \xrightarrow{n \rightarrow \infty} 0,$$

¹¹We assume that n is sufficiently large so that $\left\lfloor \frac{\mu_n}{M^2} \right\rfloor \geq M^2$.

which in particular, implies that

$$(3.33) \quad \max_{M^2 \leq K \leq \frac{\mu_n}{M^2}} \left| \|V_n^K\|_{L^2(\mathbb{R})}^2 - \|\mathfrak{r}_n^{(K)}\|_{L^2(\mathbb{R})}^2 - 4\pi \right| \xrightarrow{n \rightarrow +\infty} 0.$$

Taking into account (3.27) we obtain that

$$(3.34) \quad \max_{M^2 \leq K \leq \frac{\mu_n}{M^2}} |M_{\mathbb{R}}(\mathfrak{r}_n^{(K)}) - m + 4\pi| \xrightarrow{n \rightarrow \infty} 0.$$

Consider the spectral coefficients $\tilde{a}_{V_n^K}(\zeta)$. The following result inspired by [2] will play a crucial role in our arguments.

Lemma 3.2. *Let $\frac{m}{8} < \theta_0 < \pi$. Then, for all n sufficiently large and all $K \in \{M^2, \dots, [\frac{\mu_n}{M^2}]\}$, the function $\tilde{a}_{V_n^{(K)}}$ has a unique zero $\zeta_{0,n}^{(K)}$ in the angle $\{\zeta \in \mathbb{C} : \theta_0 \leq \arg \zeta < \pi\}$. This zero is simple and satisfies*

$$(3.35) \quad \max_K \operatorname{Im} \zeta_{0,n}^{(K)} \xrightarrow{n \rightarrow \infty} 0.$$

Furthermore, there exist $c_1 > c_2 > 0$ such that

$$(3.36) \quad -c_1 \leq \operatorname{Re} \zeta_{0,n}^{(K)} \leq -c_2,$$

for all n sufficiently large and all $K \in \{M^2, \dots, [\frac{\mu_n}{M^2}]\}$.

Proof. We shall establish Lemma 3.2 following the arguments used in the proof of Lemma 3.4 in [2]. For that purpose, we need the following proposition which is a particular case of Proposition 3.3 in [2].

Proposition 3.2. *Let $V \in H^1(\mathbb{R})$ such that $\tilde{a}_V \equiv 1$. Then, for all $0 < \delta < \frac{\pi}{2}$ and all $m > 0$, we have,*

$$(3.37) \quad \tilde{a}_{V+r}(\zeta) - \tilde{a}_r(\zeta) \rightarrow 0,$$

as $\|r\|_{L^4} \rightarrow 0$, $r \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ with $\|r\|_{L^2} \leq m$, uniformly with respect to $\zeta \in \Gamma_\delta$.

To go further, let us recall that we have:

$$V_n^K(y) = V(y - y_n(K)) + \mathfrak{r}_n^{(K)}(y),$$

where $V(y) = 2\sqrt{c}e^{i\gamma+icy/2} \frac{cy+i}{(cy-i)^2}$, for some $\gamma \in \mathbb{R}$ and $c > 0$, and where $\mathfrak{r}_n^{(K)}$ satisfies (3.32) and (3.34). Since $\theta_0 > \frac{m}{8} > \frac{m}{4} - \pi$, it follows from (2.44) and (3.34) that, for all n large enough and all $K \in \{M^2, \dots, [\frac{\mu_n}{M^2}]\}$, the function $\tilde{a}_{\mathfrak{r}_n^{(K)}}$ does not vanish in the angle $\{\zeta \in \mathbb{C} : \theta_0 \leq \arg \zeta < \pi\}$, which in view of Lemma 2.5, ensures that for any $\theta \in [\theta_0, \pi)$ there exists a positive constant C_θ such that

$$(3.38) \quad \frac{1}{C_\theta} \leq \left| \frac{1}{\tilde{a}_{\mathfrak{r}_n^{(K)}}(\zeta)} \right| \leq C_\theta,$$

for all n sufficiently large, all $K \in \{M^2, \dots, [\frac{\mu_n}{M^2}]\}$ and all $\zeta \in e^{i\theta}\mathbb{R}_+$.

Invoking Proposition 3.2, we deduce from (3.32), (3.34) and (3.38) that, for all $\theta \in [\theta_0, \pi)$,

$$(3.39) \quad \max_{M^2 \leq K \leq \frac{\mu_n}{M^2}} \sup_{\substack{\zeta \in \mathbb{C}_+ \\ \arg \zeta = \theta}} \left| 1 - \frac{\tilde{a}_{V_n^K}(\zeta)}{\tilde{a}_{\mathfrak{r}_n^{(K)}}(\zeta)} \right| \xrightarrow{n \rightarrow +\infty} 0.$$

In particular, for n large enough, $\tilde{a}_{V_n^K}$ does not vanish on the ray $e^{i\theta}\mathbb{R}_+$. Applying Lemma 2.3 i) and Corollary 2.1 iii), we get:

$$n(V_n^K; \theta) \geq \frac{1}{2i\pi} \int_0^{+\infty} e^{i\theta} \left(\frac{\tilde{a}'_{V_n^K}(s)}{\tilde{a}_{V_n^K}(s)} - \frac{\tilde{a}'_{r_n^{(K)}}(s)}{\tilde{a}_{r_n^{(K)}}(s)} \right) ds + \frac{1}{4\pi} \left(\|V_n^K\|_{L^2(\mathbb{R})}^2 - \|r_n^{(K)}\|_{L^2(\mathbb{R})}^2 \right).$$

By virtue of (3.39), we have

$$\max_{M^2 \leq K \leq \frac{\mu_n}{M^2}} \left| \int_0^{+\infty} e^{i\theta} \left(\frac{\tilde{a}'_{V_n^K}(s)}{\tilde{a}_{V_n^K}(s)} - \frac{\tilde{a}'_{r_n^{(K)}}(s)}{\tilde{a}_{r_n^{(K)}}(s)} \right) ds \right| \xrightarrow{n \rightarrow +\infty} 0,$$

which together with (3.33) ensures that for all $\theta \in [\theta_0, \pi)$ there exists $N(\theta)$ such that

$$n(V_n^K; \theta) \geq 1, \quad \forall n \geq N(\theta), \quad K \in \left\{ M^2, \dots, \left\lfloor \frac{\mu_n}{M^2} \right\rfloor \right\}.$$

Given that $\|V_n^K\|_{L^2(\mathbb{R})}^2 < 8\theta_0$, we have necessarily $n(V_n^K; \theta) = 1$, which means that for all n large enough and $K \in \left\{ M^2, \dots, \left\lfloor \frac{\mu_n}{M^2} \right\rfloor \right\}$, the function $\tilde{a}_{V_n^{(K)}}$ has a unique zero $\zeta_{0,n}^{(K)}$ in the angle $\{\zeta \in \mathbb{C} : \theta_0 \leq \arg \zeta < \pi\}$, and furthermore this zero is simple and satisfies:

$$(3.40) \quad \min_K \arg \zeta_{0,n}^{(K)} \xrightarrow{n \rightarrow +\infty} \pi.$$

To complete the proof of the lemma, it remains to prove the bounds (3.36). The family V_n^K being bounded in $H^1(\mathbb{R})$, Lemma 2.2 readily ensures that there exists $c_1 > 0$ such that

$$(3.41) \quad -c_1 \leq \operatorname{Re} \zeta_{0,n}^{(K)}.$$

In order to establish the upper bound, we remove the zero $\zeta_{0,n}^{(K)}$ by applying to V_n^K the corresponding Bäcklund transformation. More precisely, invoking (2.61), we define, for all n sufficiently large and $K \in \left\{ M^2, \dots, \left\lfloor \frac{\mu_n}{M^2} \right\rfloor \right\}$,

$$(3.42) \quad \tilde{r}_n^{(K)} = \mathcal{B}_{\lambda_{0,n}^{(K)}}(\psi_n^{(K)})V_n^K,$$

with $\lambda_{0,n}^{(K)} = \sqrt{\zeta_{0,n}^{(K)}} \in \mathbb{C}_{++}$, $\psi_n^{(K)} \in L^2(\mathbb{R}, \mathbb{C}^2) \setminus \{0\}$ such that $L_{V_n^K}(\lambda_{0,n}^{(K)})\psi_n^{(K)} = 0$. It then follows from (2.68) that

$$(3.43) \quad \|\tilde{r}_n^{(K)}\|_{L^2(\mathbb{R})}^2 = \|V_n^K\|_{L^2(\mathbb{R})}^2 - 4\arg(\zeta_{0,n}^{(K)}),$$

which in view of (3.27) and (3.40) implies that

$$(3.44) \quad \max_K \left| \|\tilde{r}_n^{(K)}\|_{L^2(\mathbb{R})}^2 - m + 4\pi \right| \xrightarrow{n \rightarrow +\infty} 0.$$

Recall that, according to (2.61),

$$(3.45) \quad \mathcal{B}_{\lambda_{0,n}^{(K)}}(\psi_n^{(K)})V_n^K = G_{\lambda_{0,n}^{(K)}}(\psi_n^{(K)}) \left[-G_{\lambda_{0,n}^{(K)}}(\psi_n^{(K)})V_n^K + \mathcal{S}_{\lambda_{0,n}^{(K)}}(\psi_n^{(K)}) \right],$$

where, by virtue of (2.63),

$$(3.46) \quad |G_{\lambda_{0,n}^{(K)}}(\psi_n^{(K)})| = 1 \quad \text{and} \quad |\mathcal{S}_{\lambda_{0,n}^{(K)}}(\psi_n^{(K)})| \leq 4 \operatorname{Im} \lambda_{0,n}^{(K)}.$$

Furthermore, combining (3.43) with (3.45), we infer that there exists a positive constant C such that, for all n large enough and $K \in \left\{ M^2, \dots, \left\lfloor \frac{\mu_n}{M^2} \right\rfloor \right\}$, there holds

$$(3.47) \quad \|\mathcal{S}_{\lambda_{0,n}^{(K)}}(\psi_n^{(K)})\|_{L^2(\mathbb{R})} \leq C.$$

Taking into account (3.31), we deduce from (3.45) and (3.46) that

$$\|\tilde{r}_n^{(K)}\|_{L^2(\mathbb{R})}^2 = \|V(\cdot - y_n(K)) + \mathcal{R}_n^{(K)}\|_{L^2(\mathbb{R})}^2,$$

with $\mathcal{R}_n^{(K)} = r_n^{(K)} - \overline{G_{\lambda_{0,n}^{(K)}}(\psi_n^{(K)})} \mathcal{S}_{\lambda_{0,n}^{(K)}}(\psi_n^{(K)})$. Therefore, for all $2 \leq p < \infty$ we have

$$\|\tilde{r}_n^{(K)}\|_{L^2(\mathbb{R})}^2 \geq 4\pi + \|\mathcal{R}_n^{(K)}\|_{L^2(\mathbb{R})}^2 - C_p \|\mathcal{R}_n^{(K)}\|_{L^p(\mathbb{R})},$$

which implies that

$$\|\mathcal{R}_n^{(K)}\|_{L^p(\mathbb{R})} \geq C_p(4\pi - \|\tilde{r}_n^{(K)}\|_{L^2(\mathbb{R})}^2).$$

This leads to the bound

$$\|S_{\lambda_{0,n}^{(K)}}(\psi_n^{(K)})\|_{L^p(\mathbb{R})} \geq C_p(4\pi - \|\tilde{r}_n^{(K)}\|_{L^2(\mathbb{R})}^2) - \|r_n^{(K)}\|_{L^p(\mathbb{R})}, \quad \forall p \in [2, \infty).$$

Combining this inequality with (3.44), (3.46), (3.47) and (3.31), we obtain that

$$\operatorname{Im} \lambda_{0,n}^{(K)} \gtrsim 1,$$

for all n large enough and $K \in \{M^2, \dots, \lfloor \frac{\mu_n}{M^2} \rfloor\}$. Since $\frac{\pi}{2} < \theta_0 < \pi$, this gives the desired upper bound for $\operatorname{Re} \zeta_{0,n}^{(K)}$. \square

To go further, we shall distinguish two cases depending on whether

$$\mu_n \min_K \operatorname{Im} \zeta_{0,n}^{(K)} \xrightarrow{n \rightarrow \infty} \infty,$$

or up to a subsequence,

$$\mu_n \min_K \operatorname{Im} \zeta_{0,n}^{(K)} \xrightarrow{n \rightarrow \infty} \beta \in \mathbb{R}_+.$$

In the first case, we fix $K = K_n$ in such a way that

$$(3.48) \quad K_n \xrightarrow{n \rightarrow \infty} \infty \quad \text{and} \quad \frac{K_n}{\mu_n \min_K \operatorname{Im} \zeta_{0,n}^{(K)}} \xrightarrow{n \rightarrow \infty} 0.$$

In the second case, we take $K = K_n$ where the minimum is achieved:

$$(3.49) \quad \operatorname{Im} \zeta_{0,n}^{(K_n)} = \min_K \operatorname{Im} \zeta_{0,n}^{(K)}.$$

In both cases, we denote $\zeta_{0,n}^{(K_n)} = \zeta_{0,n}$, $V_n^{(K_n)} = V_n$, $U_n^{(K_n)} = U_n$, $y_n(K_n) = y_n$ and $\tilde{r}_n^{K_n} = \tilde{r}_n$, where $\tilde{r}_n^{K_n}$ is defined by (3.42). Explicitly,

$$(3.50) \quad \tilde{r}_n = \mathcal{B}_{\lambda_{0,n}}(\psi_n) V_n,$$

with $\lambda_{0,n} = \sqrt{\zeta_{0,n}} \in \mathbb{C}_{++}$ and ψ_n in $L^2(\mathbb{R}) \setminus \{0\}$ solving $L_{V_n}(\lambda_{0,n})\psi_n = 0$. It follows from (3.44) that

$$(3.51) \quad \|\tilde{r}_n\|_{L^2(\mathbb{R})}^2 = m - 4\pi + o(1), \quad n \rightarrow \infty.$$

Furthermore we have:

Lemma 3.3. *For all n sufficiently large, \tilde{r}_n is a C^∞ function supported in $[-\frac{M}{2}, \mu_n + \frac{M}{2}]$ and one has:*

$$(3.52) \quad \|\tilde{r}_n\|_{H^1(\mathbb{R})}^2 \lesssim \operatorname{Im} \zeta_{0,n} + \frac{1}{\mu_n} \xrightarrow{n \rightarrow \infty} 0.$$

Proof. The fact that \tilde{r}_n is supported in $[-\frac{M}{2}, \mu_n + \frac{M}{2}]$ is a direct consequence of the definition of \tilde{r}_n . To prove (3.52), we apply (2.69), which gives

$$(3.53) \quad P_{\mathbb{R}}(\tilde{r}_n) = P_{\mathbb{R}}(V_n) + 8 \operatorname{Im} \zeta_{0,n}, \quad E_{\mathbb{R}}(\tilde{r}_n) = E_{\mathbb{R}}(V_n) - 8 \operatorname{Im} \zeta_{0,n}^2.$$

Thanks to (3.26), we have

$$(3.54) \quad \begin{aligned} |P_{\mathbb{R}}(V_n)| &\lesssim \frac{1}{\mu_n} |P_{\mathbb{T}}(u_0^{(n)})| + \frac{1}{\mu_n} \lesssim \frac{1}{\mu_n}, \\ |E_{\mathbb{R}}(V_n)| &\lesssim \frac{1}{\mu_n^2} |E_{\mathbb{T}}(u_0^{(n)})| + \frac{1}{\mu_n} \lesssim \frac{1}{\mu_n}. \end{aligned}$$

Combining (3.53), (3.54), and taking into account Lemma 3.2, we deduce that

$$(3.55) \quad |P_{\mathbb{R}}(\tilde{r}_n)| + |E_{\mathbb{R}}(\tilde{r}_n)| \lesssim \text{Im } \zeta_{0,n} + \frac{1}{\mu_n},$$

which by virtue of (1.7) ends the proof of (3.52). \square

To study the behavior of the monodromy matrix $M_{U_n}(\lambda)$ as $n \rightarrow \infty$, we apply Lemma 2.8 to express $M_{U_n}(\lambda)$ in terms of the Jost solutions $\psi^-(x, \lambda; \tilde{r}_n)$ and $\psi^+(x, \lambda; \tilde{r}_n)$ of the Kaup-Newell system associated with \tilde{r}_n :

$$(3.56) \quad i\sigma_3 \partial_x \psi - \lambda^2 \psi - i\lambda \begin{pmatrix} 0 & \tilde{r}_n \\ \tilde{r}_n & 0 \end{pmatrix} \psi = 0.$$

Lemma 3.4. *For any $\lambda \in \bar{\Omega}_+$, with $a_{V_n}(\lambda) \neq 0$, there holds*

$$(3.57) \quad M_{U_n}(\lambda) = A_n^{-1}(\mu_n, \lambda)(\psi^- \psi^+)(\mu_n, \lambda; \tilde{r}_n)(\psi^- \psi^+)^{-1}(0, \lambda; \tilde{r}_n)A_n(0, \lambda),$$

where $A_n(x, \lambda) = A(x; \psi_n, \lambda, \lambda_{0,n})$ with ψ_n in $L^2(\mathbb{R}) \setminus \{0\}$ satisfying $L_{V_n}(\lambda_{0,n})\psi_n = 0$.

Proof. Since, for all $\lambda \in \bar{\Omega}_+$ such that $a_{V_n}(\lambda) \neq 0$, we have

$$M_{U_n}(\lambda) = (\psi^- \psi^+)(\mu_n, \lambda; V_n)(\psi^- \psi^+)^{-1}(0, \lambda; V_n),$$

the result follows from Lemma 2.8 which asserts that

$$(\psi^- \psi^+)(x, \lambda; V_n) = A_n^{-1}(x, \lambda)(\psi^- \psi^+)(x, \lambda; \tilde{r}_n)\sigma_3.$$

\square

The asymptotics of $A_n(0, \lambda)$ and $A_n(\mu_n, \lambda)$ for large n are described by the following lemma.

Lemma 3.5. *Let $\hat{A}_n(x, \lambda) = \frac{\bar{\lambda}_{0,n}}{\lambda_{0,n}}(\lambda^2 - \lambda_{0,n}^2)A_n(x, \lambda)$ and*

$$D_n(\lambda) = \begin{pmatrix} \frac{\bar{\lambda}_{0,n}}{\lambda_{0,n}}(\lambda^2 - \lambda_{0,n}^2) & 0 \\ 0 & \frac{\lambda_{0,n}}{\bar{\lambda}_{0,n}}(\bar{\lambda}_{0,n}^2 - \lambda^2) \end{pmatrix}.$$

Then,

$$\left| \hat{A}_n(0, \lambda) - D_n(\lambda) \right| + \left| \hat{A}_n(\mu_n, \lambda) - \overline{D_n(\bar{\lambda})} \right| \lesssim \frac{\text{Im } \zeta_{0,n}}{\sqrt{\mu_n}},$$

uniformly with respect to λ in bounded subsets of \mathbb{C} .

Proof. Given that V_n is supported in $[-M, \mu_n + M]$, the bounds (3.26) ensure the following estimates for the Jost solutions $\psi^-(x, \lambda; V_n)$ and $\psi^+(x, \lambda; V_n)$:

$$(3.58) \quad \begin{aligned} \left| e^{i\lambda^2 x} \psi^-(x, \lambda; V_n) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right| &\lesssim \frac{1}{\sqrt{\mu_n}}, \quad \forall x \in [-M, M], \\ \left| e^{-i\lambda^2 x} \psi^+(x, \lambda; V_n) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| &\lesssim \frac{1}{\sqrt{\mu_n}}, \quad \forall x \in [\mu_n - M, \mu_n + M], \end{aligned}$$

uniformly with respect to λ in bounded sets of $\bar{\Omega}_+$. Since by hypothesis ψ_n is a non zero L^2 solution of $L_{V_n}(\lambda_{0,n})\psi = 0$, and therefore $\psi_n = C_n^\pm \psi^\pm(\lambda_{0,n}; V_n)$ for some constants C_n^- and C_n^+ , invoking (2.62) together with Lemma 3.2, we deduce from (3.58) that

$$\left| G_{\lambda_{0,n}}(\psi_n)(0) - \frac{\bar{\lambda}_{0,n}}{\lambda_{0,n}} \right| \lesssim \frac{\text{Im } \zeta_{0,n}}{\mu_n}, \quad \left| G_{\lambda_{0,n}}(\psi_n)(\mu_n) - \frac{\lambda_{0,n}}{\bar{\lambda}_{0,n}} \right| \lesssim \frac{\text{Im } \zeta_{0,n}}{\mu_n},$$

and

$$|\mathcal{S}_{\lambda_{0,n}}(\psi_n)(0)| \lesssim \frac{\text{Im } \zeta_{0,n}}{\sqrt{\mu_n}}, \quad |\mathcal{S}_{\lambda_{0,n}}(\psi_n)(\mu_n)| \lesssim \frac{\text{Im } \zeta_{0,n}}{\sqrt{\mu_n}}.$$

This completes the proof of the lemma according to (2.70). \square

Next, we investigate the Jost solutions $\psi^-(x, \lambda; \tilde{r}_n)$, $\psi^+(x, \lambda; \tilde{r}_n)$ of the system (3.56), taking advantage of the smallness of $\partial_x \tilde{r}_n$. To this end, we set

$$(3.59) \quad \psi(x) = B_n(x, \lambda) \hat{\psi}(x)$$

with

$$(3.60) \quad B_n(x, \lambda) = \frac{1}{\sqrt{1 + |\tilde{r}_n(x)|^2/4\lambda^2}} \begin{pmatrix} 1 & \frac{-i\tilde{r}_n(x)}{2\lambda} \\ \frac{-i\tilde{r}_n(x)}{2\lambda} & 1 \end{pmatrix} e^{-i\sigma_3 \Phi_n(x, \lambda)},$$

$$\Phi_n(x, \lambda) = \int_{-M}^x \frac{2\lambda^2 |\tilde{r}_n(y)|^2 + \text{Im}(\tilde{r}_n(y) \partial_y \tilde{r}_n(y))}{4\lambda^2 + |\tilde{r}_n(y)|^2} dy.$$

Then the system (3.56) takes the following form

$$(3.61) \quad (i\sigma_3 \partial_x - \lambda^2 - Q_n(\lambda)) \hat{\psi} = 0,$$

where

$$(3.62) \quad Q_n(\lambda) = \begin{pmatrix} 0 & q_n(\lambda) \\ -q_n(\bar{\lambda}) & 0 \end{pmatrix}, \quad q_n(x, \lambda) = -\frac{\lambda e^{2i\Phi_n(x, \lambda)} (2\partial_x \tilde{r}_n(x) + i|\tilde{r}_n(x)|^2 \tilde{r}_n(x))}{4\lambda^2 + |\tilde{r}_n(x)|^2}.$$

Writing

$$(3.63) \quad \psi^-(x, \lambda; \tilde{r}_n) = e^{-i\lambda^2 x} B_n(x, \lambda) \hat{\eta}_n^-(x, \lambda), \quad \psi^+(x, \lambda; \tilde{r}_n) = e^{-i\Phi_n(\mu_n + M, \lambda) + i\lambda^2 x} B_n(x, \lambda) \hat{\eta}_n^+(x, \lambda),$$

where $\hat{\eta}_n^\mp(x, \lambda) = \begin{pmatrix} \hat{\eta}_{n,1}^\mp(x, \lambda) \\ \hat{\eta}_{n,2}^\mp(x, \lambda) \end{pmatrix}$, and taking advantage of (3.51), (3.52), (3.61) and (3.62), we obtain the following bounds.

Lemma 3.6. *For all $R \geq 1$, there exists a positive constant $C = C(R)$ such that, for all n sufficiently large (depending on R) and all $\lambda \in \Omega_+$ with $\frac{1}{R} \leq |\lambda| \leq R$, one has*

$$(3.64) \quad \|\hat{\eta}_{n,1}^-(\lambda) - 1\|_{L^\infty(\mathbb{R})} + \|\hat{\eta}_{n,2}^+(\lambda) - 1\|_{L^\infty(\mathbb{R})} \leq C e^{C \frac{\text{Im } \zeta_{n,0} + \mu_n^{-1}}{\text{Im } \lambda^2}} \frac{\text{Im } \zeta_{n,0} + \mu_n^{-1}}{\text{Im } \lambda^2},$$

$$(3.65) \quad \|\hat{\eta}_{n,2}^-(\lambda)\|_{L^\infty(\mathbb{R})} + \|\hat{\eta}_{n,1}^+(\lambda)\|_{L^\infty(\mathbb{R})} \leq C e^{C \frac{\text{Im } \zeta_{n,0} + \mu_n^{-1}}{\text{Im } \lambda^2}} \sqrt{\frac{\text{Im } \zeta_{n,0} + \mu_n^{-1}}{\text{Im } \lambda^2}}.$$

Furthermore

$$(3.66) \quad \|\hat{\eta}_n^\pm(\lambda) - \hat{\eta}_n^{\pm,0}(\lambda)\|_{L^\infty(\mathbb{R})} \leq C e^{C \frac{\text{Im } \zeta_{n,0} + \mu_n^{-1}}{\text{Im } \lambda^2}} \sqrt{\frac{\text{Im } \zeta_{n,0} + \mu_n^{-1}}{\text{Im } \lambda^2}} (|\lambda - \lambda_{0,n}| + (\text{Im } \zeta_{n,0})^{\frac{1}{2}} + \mu_n^{-\frac{1}{2}}),$$

where $e^{-i\lambda^2 x} \hat{\eta}_n^-(x, \lambda)$ and $e^{i\lambda^2 x} \hat{\eta}_n^+(x, \lambda)$ are the Jost solutions of the system

$$(3.67) \quad (i\sigma_3 \partial_x - \lambda^2 - Q_n^0) \psi = 0, \quad Q_n^0 = \begin{pmatrix} 0 & q_n^0 \\ q_n^0 & 0 \end{pmatrix},$$

$$q_n^0(x) = \frac{i}{4 \text{Im } \lambda_{0,n}} e^{i \int_{-M}^x |\tilde{r}_n(y)|^2 dy} (2\partial_x \tilde{r}_n(x) + i|\tilde{r}_n(x)|^2 \tilde{r}_n(x)).$$

Proof. Since in view of (3.51) and Lemma 3.3, for all n sufficiently large and $\lambda \in \Omega_+$ with $\frac{1}{R} \leq |\lambda| \leq R$, we have

$$(3.68) \quad \|q_n(\lambda)\|_{L^2(\mathbb{R})} \lesssim_R \|\tilde{r}_n\|_{\dot{H}^1(\mathbb{R})} \lesssim_R (\operatorname{Im} \zeta_{0,n} + \mu_n^{-1})^{\frac{1}{2}}$$

and

$$(3.69) \quad \begin{aligned} \|q_n(\lambda) - q_n^0\|_{L^2(\mathbb{R})} + \|\overline{q_n(\lambda)} + \overline{q_n^0}\|_{L^2(\mathbb{R})} &\lesssim_R |\lambda - i \operatorname{Im} \lambda_{0,n}| \|\tilde{r}_n\|_{\dot{H}^1(\mathbb{R})} + \|\tilde{r}_n\|_{\dot{H}^1(\mathbb{R})}^2 \\ &\lesssim_R |\lambda - \lambda_{0,n}| (\operatorname{Im} \zeta_{0,n} + \mu_n^{-1})^{\frac{1}{2}} + \operatorname{Im} \zeta_{0,n} + \mu_n^{-1}, \end{aligned}$$

the bounds (3.64)-(3.66) readily follow from Lemma B.1. \square

Remark 3.2. The spectral problem (3.67) is self-adjoint, which leads to the following low bound for $\hat{\eta}_n^{\mp,0} = \begin{pmatrix} \hat{\eta}_{n,1}^{\mp,0} \\ \hat{\eta}_{n,2}^{\mp,0} \end{pmatrix}$:

$$(3.70) \quad |\hat{\eta}_{n,1}^-,0(x, \lambda)|^2 - |\hat{\eta}_{n,2}^-,0(x, \lambda)|^2 \geq 1, \quad |\hat{\eta}_{n,2}^+,0(x, \lambda)|^2 - |\hat{\eta}_{n,1}^+,0(x, \lambda)|^2 \geq 1, \quad \forall (x, \lambda) \in \mathbb{R} \times \overline{\Omega}_+.$$

Indeed, $\hat{\eta}_n^-,0$ solves

$$(3.71) \quad \begin{cases} i\partial_x \eta_{n,1}^-,0 - q_n^0 \eta_{n,2}^-,0 = 0 \\ i\partial_x \eta_{n,2}^-,0 + 2\lambda^2 \eta_{n,2}^-,0 + \overline{q_n^0} \eta_{n,1}^-,0 = 0 \\ \eta_{n,1}^-,0|_{x \leq -M} = 1, \quad \eta_{n,2}^-,0|_{x \leq -M} = 0. \end{cases}$$

As a consequence,

$$\partial_x (|\eta_{n,1}^-,0|^2 - |\eta_{n,2}^-,0|^2) = 4 \operatorname{Im} \lambda^2 |\eta_{n,2}^-,0|^2 \geq 0, \quad \forall (x, \lambda) \in \mathbb{R} \times \overline{\Omega}_+,$$

and $|\eta_{n,1}^-,0(x, \lambda)|^2 - |\eta_{n,2}^-,0(x, \lambda)|^2 = 1$ for $x \leq -M$, which gives the first inequality in (3.70). The second one follows in a similar way.

3.4. Arriving to a contradiction. To achieve the proof of Theorem 1, we shall investigate separately the regimes (3.48) and (3.49).

3.4.1. First regime. We start by regime (3.48):

$$(3.72) \quad \mu_n \operatorname{Im} \zeta_{0,n} \xrightarrow{n \rightarrow \infty} \infty, \quad K_n \xrightarrow{n \rightarrow \infty} \infty, \quad \frac{\mu_n \operatorname{Im} \zeta_{0,n}}{K_n} \xrightarrow{n \rightarrow \infty} \infty.$$

Consider $\Delta_{U_n}(\lambda)$ with $\lambda = \sqrt{\operatorname{Re} \zeta_{n,0} + p \operatorname{Im} \zeta_{0,n}} \in \mathbb{C}_{++}$, $p \in \mathbb{C}_+$. Since $\Delta_{U_n}(\lambda) = \Delta_{u_0^{(n)}}(\sqrt{\mu_n} \lambda)$, applying (2.21) we obtain

$$(3.73) \quad e^{i\mu_n \lambda^2} \Delta_{U_n}(\lambda) = e^{-\frac{i}{2}m} + o(1), \quad n \rightarrow \infty,$$

uniformly with respect to p in compact subsets of \mathbb{C}_+ . To arrive to a contradiction, we shall compare this approximation of $e^{i\mu_n \lambda^2} \Delta_{U_n}(\lambda)$ with the one that can be deduced from Lemmas 3.5 and 3.6. Under the assumption (3.72), Lemma 3.6 together with Lemma 3.2 ensure that, for all n sufficiently large and $\lambda \in \mathcal{D} = \{\lambda \in \mathbb{C}_{++} : |\lambda^2 - \zeta_{0,n}| \leq (\operatorname{Im} \zeta_{0,n})^{\frac{1}{2}}, \operatorname{Im}(\lambda^2) \geq \frac{1}{2} \operatorname{Im} \zeta_{0,n}\}$, we have

$$(3.74) \quad \|\hat{\eta}_n^\pm(\lambda)\|_{L^\infty(\mathbb{R})} \lesssim 1 \quad \text{and} \quad \|\hat{\eta}_n^\pm(\lambda) - \hat{\eta}_n^{\pm,0}(\lambda)\|_{L^\infty(\mathbb{R})} \lesssim (\operatorname{Im} \zeta_{n,0})^{\frac{1}{2}}.$$

Building on lemmas 3.4, 3.5 and bounds (3.74), we get the following key result.

Proposition 3.3. *There exists a constant C so that, for all n sufficiently large and all $\lambda \in \mathcal{D}$, there holds*

$$(3.75) \quad \left| e^{i\mu_n \lambda^2} \Delta_{U_n}(\lambda) - \Sigma_n(\lambda) \right| \leq C \left(1 + \frac{\operatorname{Im} \zeta_{0,n}}{|\lambda^2 - \zeta_{0,n}|} \right) \left(e^{-\mu_n \operatorname{Im} \zeta_{0,n}} + (\operatorname{Im} \zeta_{0,n})^{\frac{1}{4}} \right),$$

where

$$\Sigma_n(\lambda) = e^{-\frac{i}{2}\|u_n\|_{L^2(\mathbb{T})}^2} \frac{\lambda^2 - \zeta_{0,n}}{\lambda^2 - \bar{\zeta}_{0,n}} \hat{\eta}_{n,1}^{-,0}(\mu_n, \lambda) - e^{\frac{i}{2}\|u_n\|_{L^2(\mathbb{T})}^2} \frac{\lambda^2 - \bar{\zeta}_{0,n}}{\lambda^2 - \zeta_{0,n}} \frac{\hat{\eta}_{n,2}^{-,0}(\mu_n, \lambda) \hat{\eta}_{n,1}^{+,0}(0, \lambda)}{\hat{\eta}_{n,2}^{+,0}(0, \lambda)}.$$

Proof. In view of (3.63) and Lemma 3.4, we have

$$\Delta_{U_n}(\lambda) = \text{Tr}[B_n^{-1}(0, \lambda)A_n(0, \lambda)A_n^{-1}(\mu_n, \lambda)B_n(\mu_n, \lambda)(\hat{\eta}_n^- \hat{\eta}_n^+)(\mu_n, \lambda)e^{-i\mu_n\lambda^2\sigma_3}(\hat{\eta}_n^- \hat{\eta}_n^+)^{-1}(0, \lambda)].$$

It follows from Lemmas 3.3 and 3.5 that, for all n sufficiently large and $\lambda \in \mathcal{D}$,

$$(3.76) \quad \left| B_n^{-1}(0, \lambda)A_n(0, \lambda)A_n^{-1}(\mu_n, \lambda)B_n(\mu_n, \lambda) - \Lambda_n(\lambda^2) \right| \lesssim \left(1 + \frac{\text{Im } \zeta_{0,n}}{|\lambda^2 - \zeta_{0,n}|} \right) (\text{Im } \zeta_{0,n})^{\frac{1}{4}},$$

with

$$\Lambda_n(\zeta) = e^{-\frac{i}{2}\|V_n\|_{L^2(\mathbb{R})}^2\sigma_3} \begin{pmatrix} \frac{\zeta - \zeta_{0,n}}{\zeta - \bar{\zeta}_{0,n}} & 0 \\ 0 & \frac{\zeta - \bar{\zeta}_{0,n}}{\zeta - \zeta_{0,n}} \end{pmatrix}.$$

We next address the expression $(\hat{\eta}_n^- \hat{\eta}_n^+)(\mu_n, \lambda)e^{-i\mu_n\lambda^2\sigma_3}(\hat{\eta}_n^- \hat{\eta}_n^+)^{-1}(0, \lambda)$. By (3.68), we have

$$\left| \hat{\eta}_n^-(0, \lambda) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right| + \left| \hat{\eta}_n^+(\mu_n, \lambda) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| \lesssim (\text{Im } \zeta_{0,n})^{\frac{1}{2}},$$

which together with the bounds (3.74) and (3.70) implies that

$$(3.77) \quad (\hat{\eta}_n^- \hat{\eta}_n^+)(\mu_n, \lambda) \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\mu_n\lambda^2} \end{pmatrix} (\hat{\eta}_n^- \hat{\eta}_n^+)^{-1}(0, \lambda) = \begin{pmatrix} \hat{\eta}_{n,1}^{-,0}(\mu_n, \lambda) & -\frac{\hat{\eta}_{n,1}^{+,0}(0, \lambda)}{\hat{\eta}_{n,2}^{+,0}(0, \lambda)} \hat{\eta}_{n,1}^{-,0}(\mu_n, \lambda) \\ \hat{\eta}_{n,2}^{-,0}(\mu_n, \lambda) & -\frac{\hat{\eta}_{n,1}^{+,0}(0, \lambda)}{\hat{\eta}_{n,2}^{+,0}(0, \lambda)} \hat{\eta}_{n,2}^{-,0}(\mu_n, \lambda) \end{pmatrix} + O\left(e^{-\mu_n \text{Im } \zeta_{0,n}} + (\text{Im } \zeta_{n,0})^{\frac{1}{2}}\right), \text{ as } n \rightarrow \infty,$$

uniformly with respect to $\lambda \in \mathcal{D}$. Combining (3.76) with (3.77) and taking into account (3.27), we obtain (3.75). \square

For $p \in \mathbb{C}_+$, define

$$(3.78) \quad \varphi_n^-(p) = \hat{\eta}_{n,2}^{-,0}(\mu_n, \sqrt{\text{Re } \zeta_{0,n} + p \text{Im } \zeta_{0,n}}),$$

$$(3.79) \quad \varphi_n^+(p) = \hat{\eta}_{n,1}^{+,0}(0, \sqrt{\text{Re } \zeta_{0,n} + p \text{Im } \zeta_{0,n}}).$$

The functions φ_n^\pm are analytic in \mathbb{C}_+ and in view of Lemma 3.6 are uniformly bounded on compact subsets of \mathbb{C}_+ . Therefore, there exist φ^+ , φ^- analytic in \mathbb{C}_+ such that, after eventually passing to a subsequence, we have

$$(3.80) \quad \varphi_n^\pm(p) \xrightarrow{n \rightarrow \infty} \varphi^\pm(p),$$

uniformly with respect to p in compact subsets of \mathbb{C}_+ . We claim that the following property holds.

Proposition 3.4. *With the above notations, we have*

$$(3.81) \quad \varphi^\pm(i) = 0.$$

Admitting for a while Proposition 3.4, let us achieve the proof of Theorem 1 in the case of (3.72). In view of Proposition 3.3 and the bounds (3.74), Proposition 3.4 ensures that

$$\lim_{p \rightarrow i} \lim_{n \rightarrow \infty} e^{i\mu_n(\text{Re } \zeta_{0,n} + p \text{Im } \zeta_{0,n})} \Delta_{U_n}(\sqrt{\text{Re } \zeta_{0,n} + p \text{Im } \zeta_{0,n}}) = 0,$$

which contradicts (3.73).

Proof of Proposition 3.4. To establish the result, we proceed by contradiction, assuming for instance that $\varphi^-(i) \neq 0$. Because of (3.51) and (3.52), we clearly have

$$|\hat{\eta}_n^{-,0}(\mu_n, \lambda_{0,n}) - \hat{\eta}_n^{-,0}(\mu_n + M, \lambda_{0,n})| \lesssim (\text{Im } \zeta_{0,n})^{\frac{1}{2}}.$$

Therefore, the assumption $\varphi^-(i) \neq 0$ together with (3.66) and (3.80) implies that there exists $\varepsilon_0 > 0$ such that, up to a subsequence,

$$(3.82) \quad |\hat{\eta}_{n,2}^{-}(\mu_n + M, \lambda_{0,n})| \geq \varepsilon_0,$$

for all n sufficiently large.

Let $\Psi_n = \begin{pmatrix} \Psi_{n,1} \\ \Psi_{n,2} \end{pmatrix}$ be the solution of the system $L_{\tilde{r}_n}(\lambda_{0,n})\psi = 0$ satisfying

$$(3.83) \quad \Psi_n(x) = e^{-i\lambda_{0,n}^2 x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } x \geq \mu_n + M.$$

Then accordingly to (2.65) and (2.66),

$$(3.84) \quad V_n = \mathcal{B}_{\lambda_{0,n}}(\Psi_n)\tilde{r}_n.$$

To get a contradiction, it will be enough to show that for all n sufficiently large, there exists $a_n \in [\mu_n - \frac{\mu_n}{4K_n}, \mu_n + M]$ such that

$$(3.85) \quad |\Psi_{n,1}(a_n)| = |\Psi_{n,2}(a_n)|.$$

Indeed, in view of (3.84), this would imply that

$$V_n(a_n) = -\tilde{r}_n(a_n) + 2i(\lambda_{0,n} - \bar{\lambda}_{0,n}) \frac{\Psi_{n,1}(a_n)}{\Psi_{n,2}(a_n)} \quad \text{with} \quad \left| \frac{\Psi_{n,1}(a_n)}{\Psi_{n,2}(a_n)} \right| = 1,$$

which together with Lemmas 3.2 and 3.3 gives, for all n sufficiently large,

$$|V_n(a_n)| \geq 4 \text{Im } \lambda_{0,n} - \|\tilde{r}_n\|_{L^\infty(\mathbb{R})} \gtrsim 1,$$

contradicting

$$\|V_n\|_{H^1([\mu_n - \frac{\mu_n}{4K_n}, \mu_n + M])} \lesssim \frac{1}{\sqrt{K_n}} \xrightarrow{n \rightarrow \infty} 0.$$

Thus to complete the proof of the proposition, it suffices to establish (3.85). To this end we decompose¹² Ψ_n in the basis of the Jost solutions $\psi^-(x, \lambda_{0,n}; \tilde{r}_n), \psi^+(x, \lambda_{0,n}; \tilde{r}_n)$:

$$(3.86) \quad \Psi_n(x) = \frac{e^{-i\lambda_{0,n}^2(\mu_n + M)}}{\psi_1^-(\mu_n + M, \lambda_{0,n}; \tilde{r}_n)} \psi^-(x, \lambda_{0,n}; \tilde{r}_n) - e^{-2i\lambda_{0,n}^2(\mu_n + M)} \frac{\psi_2^-(\mu_n + M, \lambda_{0,n}; \tilde{r}_n)}{\psi_1^-(\mu_n + M, \lambda_{0,n}; \tilde{r}_n)} \psi^+(x, \lambda_{0,n}; \tilde{r}_n).$$

Taking advantage of (3.63), one can rewrite (3.86) in the form

$$(3.87) \quad \Psi_n(x) = -e^{i\lambda_{0,n}^2(x-2(\mu_n+M))+i\theta_n} \frac{\hat{\eta}_{n,2}^{-}(\mu_n + M, \lambda_{0,n})}{\hat{\eta}_{n,1}^{-}(\mu_n + M, \lambda_{0,n})} \Psi_n^\sharp(x)$$

with

$$\Psi_n^\sharp = B_n(x, \lambda_{0,n}) \left[\hat{\eta}_n^+(x, \lambda_{0,n}) - \frac{e^{2i\lambda_{0,n}^2(\mu_n + M - x)}}{\hat{\eta}_{n,2}^{-}(\mu_n + M, \lambda_{0,n})} \hat{\eta}_n^-(x, \lambda_{0,n}) \right], \quad \theta_n = \Phi_n(\mu_n + M, \lambda_{0,n}).$$

The bounds (3.82), (3.74) together with (3.51) and Lemma 3.3 guarantee that, for all n sufficiently large,

$$|\Psi_n^\sharp(x) - e^{-\frac{i}{2}\sigma_3 \int_{-M}^x |\tilde{r}_n(y)|^2 dy} \hat{\eta}_n^{+,0}(x, \lambda_{0,n})| \lesssim (\text{Im } \zeta_{0,n})^{1/4} + e^{-2 \text{Im } \zeta_{0,n}(\mu_n + M - x)}, \quad x \in \mathbb{R}.$$

Taking $x = \mu_n - \frac{\mu_n}{8K_n}$ and invoking (3.70), (3.72) and (3.74), we obtain

$$|\Psi_{n,2}^\sharp(\mu_n - \frac{\mu_n}{8K_n})|^2 - |\Psi_{n,1}^\sharp(\mu_n - \frac{\mu_n}{8K_n})|^2 > 0,$$

¹²Since $\tilde{a}_{\tilde{r}_n}(\zeta_{0,n}) \neq 0$, the Jost solutions $\psi^-(x, \lambda_{0,n}; \tilde{r}_n), \psi^+(x, \lambda_{0,n}; \tilde{r}_n)$ are linearly independent.

and therefore,

$$|\Psi_{n,2}(\mu_n - \frac{\mu_n}{8K_n})|^2 - |\Psi_{n,1}(\mu_n - \frac{\mu_n}{8K_n})|^2 > 0.$$

Since in view of (3.83), $|\Psi_{n,2}(\mu_n + M)|^2 - |\Psi_{n,1}(\mu_n + M)|^2 = -|\Psi_{n,1}(\mu_n + M)|^2 < 0$, this implies that there exists a_n in $[\mu_n - \frac{\mu_n}{8K_n}, \mu_n + M]$ such that

$$|\Psi_{n,2}(a_n)|^2 = |\Psi_{n,1}(a_n)|^2,$$

which achieves the proof of the proposition. \square

3.4.2. *Second regime.* We now investigate the second regime where

$$(3.88) \quad \mu_n \operatorname{Im} \zeta_{0,n} \xrightarrow{n \rightarrow +\infty} \beta \geq 0.$$

Then, in view of (3.52), we have

$$(3.89) \quad \|\tilde{r}_n\|_{\dot{H}^1(\mathbb{R})}^2 \lesssim \frac{1}{\mu_n}.$$

We write

$$(3.90) \quad \mu_n \operatorname{Re} \zeta_{0,n} = 2\pi N_n + 2\pi\alpha_n,$$

with $N_n = [\frac{\mu_n \operatorname{Re} \zeta_{0,n}}{2\pi}] \in \mathbb{Z}$ and $\alpha_n = \{\frac{\mu_n \operatorname{Re} \zeta_{0,n}}{2\pi}\} \in [0, 1[$. In view of Lemma 3.2, $N_n \xrightarrow{n \rightarrow +\infty} -\infty$. After passing to a subsequence, we can assume that $\alpha_n \xrightarrow{n \rightarrow +\infty} \alpha^*$.

We first exclude the case of $\beta = 0$. To this end, we consider again the function $\Psi_n = \begin{pmatrix} \Psi_{n,1} \\ \Psi_{n,2} \end{pmatrix}$ used in the proof of Proposition 3.4. Recall that it solves $L_{\tilde{r}_n}(\lambda_{0,n})\Psi_n = 0$, and satisfies, in view of (3.83) and (2.65),

$$(3.91) \quad \Psi_n(x) = e^{-i\lambda_{0,n}^2 x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ if } x \geq \mu_n + M, \text{ and } \Psi_n(x) = c_n e^{i\lambda_{0,n}^2 x} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ if } x \leq -M,$$

for some constant c_n . According to (3.59), we write Ψ_n in the form:

$$\Psi_n(x) = B_n(x, \lambda_{0,n}) \hat{\Psi}_n(x).$$

Then, for $\hat{\Psi}_n = \begin{pmatrix} \hat{\Psi}_{n,1} \\ \hat{\Psi}_{n,2} \end{pmatrix}$ we get

$$(3.92) \quad (i\sigma_3 \partial_x - \lambda_{0,n}^2 - Q_n(\lambda_{0,n})) \hat{\Psi}_n = 0,$$

and

$$(3.93) \quad \hat{\Psi}_n(x) = \begin{cases} c_n e^{i\lambda_{0,n}^2 x} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } x \leq -M \\ e^{-i\lambda_{0,n}^2 x + i\theta_n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } x \geq \mu_n + M. \end{cases}$$

It follows from (3.68) and (3.88) that $\|Q_n(\lambda_{0,n})\|_{L^1(\mathbb{R})} \lesssim 1$, which leads to the bound

$$(3.94) \quad \|\hat{\Psi}_n\|_{L^\infty([-M, \mu_n + M])} \lesssim 1.$$

In view of (3.92), we have

$$(3.95) \quad \partial_x (|\hat{\Psi}_{n,1}|^2 - |\hat{\Psi}_{n,2}|^2) = 2 \operatorname{Im} \zeta_{0,n} (|\hat{\Psi}_{n,1}|^2 + |\hat{\Psi}_{n,2}|^2) + 2 \operatorname{Im} (q_n(\lambda_{0,n}) \hat{\Psi}_{n,2} \overline{\hat{\Psi}_{n,1}} - \overline{q_n(\bar{\lambda}_{0,n})} \overline{\hat{\Psi}_{n,2}} \hat{\Psi}_{n,1}).$$

Combining (3.62) together with (3.94), we infer that

$$\begin{aligned} |\operatorname{Im}(q_n(\lambda_{0,n})\widehat{\Psi}_{n,2}\overline{\widehat{\Psi}_{n,1}} - \overline{q_n(\lambda_{0,n})}\widehat{\Psi}_{n,2}\widehat{\Psi}_{n,1})| &= |\operatorname{Im}((q_n(\lambda_{0,n}) + \overline{q_n(\lambda_{0,n})})\widehat{\Psi}_{n,2}\widehat{\Psi}_{n,1})| \\ &\lesssim \operatorname{Re}(\lambda_{0,n})(|\partial_x \tilde{r}_n| + |\tilde{r}_n|^3) \\ &\lesssim \operatorname{Im}(\zeta_{0,n})(|\partial_x \tilde{r}_n| + \mu_n^{-\frac{3}{4}}), \end{aligned}$$

which, according to (3.95), ensures that

$$(3.96) \quad \partial_x(|\widehat{\Psi}_{n,1}|^2 - |\widehat{\Psi}_{n,2}|^2) \lesssim \operatorname{Im} \zeta_{0,n}(1 + |\partial_x \tilde{r}_n(x)|).$$

Since by (3.93), we have

$$\begin{aligned} |\widehat{\Psi}_{n,1}(\mu_n + M)|^2 - |\widehat{\Psi}_{n,2}(\mu_n + M)|^2 &= |\widehat{\Psi}_{n,1}(\mu_n + M)|^2 \gtrsim 1, \\ |\widehat{\Psi}_{n,1}(-M)|^2 - |\widehat{\Psi}_{n,2}(-M)|^2 &= -|\widehat{\Psi}_{n,2}(\mu_n + M)|^2 < 0, \end{aligned}$$

the bound (3.96) implies that

$$\mu_n \operatorname{Im} \zeta_{0,n} \gtrsim 1.$$

In order to complete the proof of Theorem 1, we start by considering the monodromy matrices $M_{U_n}(\lambda)$ with $\lambda = \sqrt{\operatorname{Re} \zeta_{0,n} + \mu_n^{-1}p} \in \mathbb{C}_+$, $p \in \mathbb{C}$, and proving the following convergence result.

Proposition 3.5. *Let $\mathcal{M}_n(p) = M_{U_n}(\sqrt{\operatorname{Re} \zeta_{0,n} + \mu_n^{-1}p})$. There exists $q \in L^2([0, 1])$ such that, up to a subsequence extraction,*

$$(3.97) \quad \mathcal{M}_n(p) \xrightarrow{n \rightarrow +\infty} \mathcal{M}(p) = \begin{pmatrix} \mathcal{M}_{11}(p) & \mathcal{M}_{12}(p) \\ \mathcal{M}_{21}(p) & \mathcal{M}_{22}(p) \end{pmatrix},$$

uniformly with respect to p in compact subsets of \mathbb{C} , where

$$(3.98) \quad \mathcal{M}(p) = e^{i\theta^* \sigma_3} \begin{pmatrix} \frac{p-i\beta}{p+i\beta} & 0 \\ 0 & -1 \end{pmatrix} \mathcal{M}^*(p) \begin{pmatrix} 1 & 0 \\ 0 & -\frac{p+i\beta}{p-i\beta} \end{pmatrix}, \quad \theta^* = -2\pi\alpha^* - \frac{m}{2},$$

$\mathcal{M}^*(p)$ being the monodromy matrix of the system

$$i\sigma_3 \partial_y \psi - p\psi + \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix} \psi = 0$$

on $[0, 1]$. Explicitly, $\mathcal{M}^*(p) = E(1, p)$ with $E(y, p)$ solving

$$\begin{cases} i\sigma_3 \partial_y E - pE + \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix} E = 0, \\ E(0, p) = \operatorname{Id}. \end{cases}$$

Proof. By Lemma 2.8, we have

$$M_{U_n}(\lambda) = A_n^{-1}(\mu_n, \lambda) E_{\tilde{r}_n}(\mu_n, \lambda) A_n(0, \lambda),$$

where $E_{\tilde{r}_n}(x, \lambda)$ solves

$$\begin{cases} L_{\tilde{r}_n}(\lambda) E_{\tilde{r}_n} = 0 \\ E_{\tilde{r}_n}(0, \lambda) = \operatorname{Id}. \end{cases}$$

According to (3.59), for n sufficiently large uniformly with respect to λ in compact sets of $\mathbb{C} \setminus \{0\}$, we can write $E_{\tilde{r}_n}$ in the form $E_{\tilde{r}_n}(x, \lambda) = B_n(x, \lambda) E_n(x, \lambda) B_n^{-1}(0, \lambda)$, where $E_n(x, \lambda)$ is the fundamental solution of (3.61):

$$(i\sigma_3 \partial_x - \lambda^2 - Q_n(\lambda)) E_n = 0, \quad E_n(0, \lambda) = \operatorname{Id}.$$

Thanks to (3.69), (3.88) and (3.89), we have:

$$\|Q_n(\lambda)\big|_{\lambda=\sqrt{\operatorname{Re}\zeta_0+\mu_n^{-1}p}} - Q_n^0\|_{L^1(\mathbb{R})} \lesssim \frac{1}{\sqrt{\mu_n}}, \quad \|Q_n^0\|_{L^1(\mathbb{R})} \lesssim 1,$$

uniformly with respect to p in bounded sets of \mathbb{C} . Therefore, denoting $E_n^0(x, p)$ the fundamental solution of the system (3.67) with $\lambda^2 = \operatorname{Re}\zeta_0 + \mu_n^{-1}p$, we obtain:

$$\|E_n^0(p)\|_{L^\infty([0, \mu_n])} \lesssim 1, \quad \|E_n(\sqrt{\operatorname{Re}\zeta_0 + \mu_n^{-1}p}) - E_n^0(p)\|_{L^\infty([0, \mu_n])} \lesssim \frac{1}{\sqrt{\mu_n}},$$

uniformly with respect to p in bounded subsets of \mathbb{C} . Combining these bounds with Lemma 3.5 and taking into account (3.60) and (3.89), we deduce that

$$(3.99) \quad \mathcal{M}_n(p) = e^{-i\frac{m}{2}\sigma_3} \begin{pmatrix} \frac{p-i\beta}{p+i\beta} & 0 \\ 0 & -1 \end{pmatrix} E_n^0(\mu_n, p) \begin{pmatrix} 1 & 0 \\ 0 & -\frac{p+i\beta}{p-i\beta} \end{pmatrix} + o(1), \quad \text{as } n \rightarrow \infty,$$

uniformly with respect to p in compact subsets of $\mathbb{C} \setminus \{i\beta, -i\beta\}$.

We next set

$$(3.100) \quad E_n^0(x, p) = e^{-i\operatorname{Re}\zeta_0, n x \sigma_3} \tilde{E}_n^0(\mu_n^{-1}x, p).$$

Then $\tilde{E}_n^0(y, p)$ solves

$$(3.101) \quad i\sigma_3 \partial_y \tilde{E}_n^0 + p \tilde{E}_n^0 - \tilde{Q}_n^0 \tilde{E}_n^0 = 0, \quad \tilde{E}_n^0(0, p) = \operatorname{Id},$$

where

$$\tilde{Q}_n^0 = \begin{pmatrix} 0 & \tilde{q}_n^0 \\ \tilde{q}_n^0 & 0 \end{pmatrix}, \quad \tilde{q}_n^0(y) = \mu_n e^{2i\operatorname{Re}\zeta_0, n \mu_n y} q_n^0(\mu_n y),$$

with q_n^0 given by (3.67). By virtue of (3.89), the sequence (\tilde{q}_n^0) is bounded in $L^2([0, 1])$ and therefore there exists $q \in L^2([0, 1])$ such that, up to a subsequence extraction,

$$(3.102) \quad \tilde{q}_n^0 \xrightarrow{n \rightarrow +\infty} q \text{ in } L^2([0, 1]),$$

which implies that¹³

$$(3.103) \quad \tilde{E}_n^0(y, p) \xrightarrow{n \rightarrow +\infty} E(y, p),$$

uniformly with respect to $y \in [0, 1]$ and p in bounded subsets of \mathbb{C} . In view of (3.99) and (3.100), this allows us to conclude that

$$(3.104) \quad \mathcal{M}_n(p) \xrightarrow{n \rightarrow +\infty} \mathcal{M}(p) = e^{i\theta^* \sigma_3} \begin{pmatrix} \frac{p-i\beta}{p+i\beta} & 0 \\ 0 & -1 \end{pmatrix} E(1, p) \begin{pmatrix} 1 & 0 \\ 0 & -\frac{p+i\beta}{p-i\beta} \end{pmatrix},$$

uniformly with respect to p in compact subsets of $\mathbb{C} \setminus \{i\beta, -i\beta\}$. By analyticity of $M_{U_n}(\lambda)$, the limiting matrix \mathcal{M} is an entire function of p and the convergence in (3.104) is in fact uniform with respect to p in bounded subsets of \mathbb{C} . □

Remark 3.3. *Because of the analyticity of \mathcal{M} , the matrix $\mathcal{M}^* = \begin{pmatrix} \mathcal{M}_{11}^* & \mathcal{M}_{12}^* \\ \mathcal{M}_{21}^* & \mathcal{M}_{22}^* \end{pmatrix}$ has to satisfy:*

$$\mathcal{M}_{11}^*(-i\beta) = \mathcal{M}_{22}^*(i\beta) = 0.$$

Since $\det \mathcal{M}^* = 1$, this implies that $\mathcal{M}_{12}^*(\pm i\beta) \neq 0$ and $\mathcal{M}_{21}^*(\pm i\beta) \neq 0$. Taking into account that $\mathcal{M}_{12}(p) = -e^{i\theta^*} \mathcal{M}_{12}^*(p)$ and $\mathcal{M}_{21}(p) = -e^{-i\theta^*} \mathcal{M}_{21}^*(p)$, we get

$$(3.105) \quad \mathcal{M}_{12}(\pm i\beta) \neq 0 \quad \text{and} \quad \mathcal{M}_{21}(\pm i\beta) \neq 0.$$

As a by product of the above proposition, we obtain the following corollary.

Corollary 3.1. *The matrix $\mathcal{M}(p)$ enjoys the following properties:*

¹³See Appendix B.

- (i) $\operatorname{Tr} \mathcal{M}(p) = 2 \cos(p - \theta^*)$;
(ii) $\mathcal{M}(p) = e^{i(\theta^* - p)\sigma_3} + o(e^{|\operatorname{Im} p|})$, as $|p| \rightarrow \infty$.

Proof. Recalling that $\operatorname{Tr} \mathcal{M}_n(p) = \Delta_{u_n}(\sqrt{\mu_n \operatorname{Re} \zeta_{0,n} + p})$ and applying (2.21), we obtain

$$\operatorname{Tr} \mathcal{M}_n(p) = \cos(\mu_n \operatorname{Re} \zeta_{0,n} + \frac{1}{2} \|u_n\|_{L^2(\mathbb{T})}^2 + p) + o(1), \quad \text{as } n \rightarrow \infty,$$

which after passing to the limit $n \rightarrow \infty$, gives (i). Finally, (ii) follows from (3.98) and the fact that $\mathcal{M}^* = e^{-ip\sigma_3} + o(e^{|\operatorname{Im} p|})$, as $|p| \rightarrow \infty$. \square

Since $\det \mathcal{M} = 1$, item (i) of Corollary 3.1 implies that for $p = p_k$ with $p_k = \pi k + \theta^*$, $k \in \mathbb{Z}$, either

$$(3.106) \quad \mathcal{M}(p_k) = (-1)^k I,$$

or

$$(3.107) \quad |\mathcal{A}^D(p_k)|^2 + |\mathcal{A}^N(p_k)|^2 > 0,$$

where the functions \mathcal{A}^D , \mathcal{A}^N are defined by:

$$\begin{aligned} \mathcal{A}^D(p) &= \frac{i}{2} (\mathcal{M}_{11}(p) + \mathcal{M}_{12}(p) - \mathcal{M}_{22}(p) - \mathcal{M}_{21}(p)), \\ \mathcal{A}^N(p) &= \frac{i}{2} (\mathcal{M}_{11}(p) - \mathcal{M}_{12}(p) + \mathcal{M}_{21}(p) - \mathcal{M}_{22}(p)). \end{aligned}$$

We claim that thanks to Lemma 2.1, one necessarily has (3.106), for all $k \in \mathbb{Z}$.

Indeed, if not then there exists $k_0 \in \mathbb{Z}$ such that, for example¹⁴, $|\mathcal{A}^D(p_{k_0})| = c_0 > 0$. Since \mathcal{M} is continuous, there exists $\eta > 0$ such that, for all p in $[p_{k_0} - \eta, p_{k_0} + \eta]$, we have

$$|\mathcal{A}^D(p)| \geq \frac{c_0}{2}.$$

Since $\mathcal{M}_n(p) \xrightarrow{n \rightarrow \infty} \mathcal{M}(p)$, uniformly with respect to p in compact subsets of \mathbb{C} , this implies that there exists an integer n_0 such that, for all $n \geq n_0$, we have

$$(3.108) \quad |A_{U_n}^D(\lambda)| \geq \frac{c_0}{4},$$

for all λ such that $\lambda^2 = \operatorname{Re} \zeta_{0,n} + p \mu_n^{-1}$ with $p \in [p_{k_0} - \eta, p_{k_0} + \eta]$ and $\operatorname{Im} \lambda > 0$. Recalling that $A_{U_n}^D(\lambda) = A_{\tilde{u}_n}^D(\sqrt{\mu_n} \lambda)$ with $\tilde{u}_n(x) = u_n(t_n, x + x_n)$ for some $x_n \in \mathbb{R}$, we obtain that, for all $n \geq n_0$,

$$(3.109) \quad |A_{\tilde{u}_n}^D(\lambda)| \geq \frac{c_0}{4},$$

for all $\lambda \in \mathbb{C}_+$ such that $\lambda^2 \in [\mu_n \operatorname{Re} \zeta_{0,n} + p_{k_0} - \eta, \mu_n \operatorname{Re} \zeta_{0,n} + p_{k_0} + \eta]$.

Taking into account that

$$\mu_n \operatorname{Re} \zeta_{0,n} + p_{k_0} = \pi(2N_n + k_0) - \frac{1}{2} \|u_n\|_{L^2(\mathbb{T})}^2 + o(1), \quad \text{as } n \rightarrow \infty,$$

we deduce that there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, $A_{\tilde{u}_n}^D(\lambda)$ does not vanish for all $\lambda \in \sqrt{\mathcal{J}_n} \subset i\mathbb{R}_+$ with

$$J_n = \left[\pi(2N_n + k_0) - \frac{1}{2} \|u_n\|_{L^2(\mathbb{T})}^2 - \frac{\eta}{2}, \pi(2N_n + k_0) - \frac{1}{2} \|u_n\|_{L^2(\mathbb{T})}^2 + \frac{\eta}{2} \right],$$

where $N_n \xrightarrow{n \rightarrow \infty} -\infty$. In view of the boundedness of $(u_0^{(n)})$ in $H^1(\mathbb{T})$, this contradicts Lemma 2.1. Thus,

$$(3.110) \quad \mathcal{M}(p_k) = (-1)^k I, \quad \forall k \in \mathbb{Z}.$$

¹⁴The case $|\mathcal{A}^N(p_{k_0})| > 0$ can be treated in the same way.

To complete the proof of the theorem, we will show that (3.110) can not hold. Indeed, the property (3.110) means that

$$(3.111) \quad \mathcal{M}_{12}(p_k) = \mathcal{M}_{21}(p_k) = 0, \forall k \in \mathbb{Z}.$$

This implies that the function

$$\varphi(p) = \frac{\mathcal{M}_{12}(p + \theta^*)}{\sin p}$$

is an entire function of $p \in \mathbb{C}$, which according to Corollary 3.1 (ii), satisfies

$$(3.112) \quad \varphi(p) \xrightarrow{|p| \rightarrow \infty} 0.$$

So $\mathcal{M}_{12} \equiv 0$ which contradicts (3.105). This completes the proof of the theorem.

APPENDIX A. REGULARIZED DETERMINANTS

In this appendix, we recall the basic properties of the regularized determinants $\det_n(\mathbf{I} - A)$ for A in \mathcal{C}_n , the set of bounded operators A on a separable Hilbert space¹⁵ \mathcal{H} such that $|A|^n$ is of trace-class, endowed with the norm $\|A\|_n = [\text{Tr}(|A|^n)]^{\frac{1}{n}}$. For further details, we refer to the monograph of Simon [32] and the references therein.

To introduce the regularized determinants, let us start by defining, for any bounded operator A on \mathcal{H} ,

$$R_n(A) = \mathbf{I} - (\mathbf{I} - A) \exp\left(\sum_{k=1}^{n-1} \frac{A^k}{k}\right).$$

Clearly,

$$(A.1) \quad R_n(A) = A^n h_n(A),$$

where h_n is an entire function on \mathbb{C} . This shows that $R_n(A)$ belongs to \mathcal{C}_1 if A is in \mathcal{C}_n , which justifies the following definition:

Definition A.1. For any operator A in \mathcal{C}_n , $n \geq 2$, we define

$$(A.2) \quad \det_n(\mathbf{I} - A) = \det(\mathbf{I} - R_n(A)) = \det\left((\mathbf{I} - A) \exp\left(\sum_{k=1}^{n-1} \frac{A^k}{k}\right)\right).$$

Note that, for all A in \mathcal{C}_n such that $\|A\| < 1$ (or more generally $\|A^p\| < 1$, for some p), one has:

$$(A.3) \quad \det_n(\mathbf{I} - A) = \exp\left(-\text{Tr} \sum_{k=n}^{\infty} \frac{A^k}{k}\right).$$

In the following proposition, we summarize some useful properties of the regularized determinants:

Proposition A.1. For any integer $n \geq 1$, there exists a positive constant C_n such that the following estimates hold.

(1) For all $A \in \mathcal{C}_n$,

$$(A.4) \quad |\det_n(\mathbf{I} - A)| \leq \exp(C_n \|A\|_n^n),$$

$$(A.5) \quad |\det_n(\mathbf{I} - A) - 1| \leq C_n \|A^n\|_1 \exp(C_n \|A\|_n^n).$$

(2) For all A, B in \mathcal{C}_n ,

$$(A.6) \quad |\det_n(\mathbf{I} - A) - \det_n(\mathbf{I} - B)| \leq \|A - B\|_n \exp(C_n (\|A\|_n + \|B\|_n + 1)^n).$$

¹⁵In our case \mathcal{H} is either $L^2(\mathbb{R}, \mathbb{C}^2)$ or $L^2(\mathbb{T}, \mathbb{C}^2)$.

(3) Let $A \in \mathcal{C}_n$. Then $I - A$ is invertible if and only if $\det_n(I - A) \neq 0$, and furthermore, one has

$$(A.7) \quad \|(I - A)^{-1}\| \leq \frac{C_n}{|\det_n(I - A)|} \exp(C_n \|A\|_n^n).$$

APPENDIX B. PROOF OF TECHNICAL RESULTS

B.1. Some estimates of the solutions of the Zakharov-Shabat spectral problem and proof of Propositions 2.1 and 2.3. In this subsection, we record some estimates for the solutions of the Zakharov-Shabat system:

$$(B.1) \quad (\mathcal{L}_{\mathbf{q}} - \zeta)\psi = 0, \quad \mathcal{L}_{\mathbf{q}} = i\sigma_3\partial_x - Q, \quad Q = \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix},$$

with $\mathbf{q} = (q_1, q_2) \in L^2_{loc}(\mathbb{R}, \mathbb{C}^2)$, that have been used in the proof of Theorem 1, and also prove Propositions 2.1 and 2.3.

B.1.1. Estimates. Let $\mathcal{E}_{\mathbf{q}}(x, \zeta)$ be the fundamental solution of (B.1) satisfying $\mathcal{E}_{\mathbf{q}}(0, \zeta) = \text{Id}$. We denote the columns of $\mathcal{E}_{\mathbf{q}}(x, \zeta)$ by $e_1(x, \zeta; \mathbf{q})$ and $e_2(x, \zeta; \mathbf{q})$. The fundamental solution $\mathcal{E}_{\mathbf{q}}$ can be characterized by the following integral equation

$$(B.2) \quad \mathcal{E}_{\mathbf{q}}(x, \zeta) = e^{-i\zeta x\sigma_3} - i \int_0^x e^{-i\zeta(x-y)\sigma_3} \sigma_3 Q(y) \mathcal{E}_{\mathbf{q}}(y, \zeta) dy,$$

from which one readily deduces (see for example [8, 9]) that for all $x \in \mathbb{R}_+$, $\mathcal{E}_{\mathbf{q}}(x, \zeta)$ is an analytic function of ζ and \mathbf{q} admitting the following estimate

$$(B.3) \quad |\mathcal{E}_{\mathbf{q}}(x, \zeta)| \leq \exp(|\text{Im } \zeta|x + \|\mathbf{q}\|_{L^2([0,x], \mathbb{C}^2)}\sqrt{x}), \quad \forall (x, \zeta) \in \mathbb{R}_+ \times \mathbb{C}.$$

Furthermore as $|\zeta| \rightarrow \infty$, one has

$$(B.4) \quad \mathcal{E}_{\mathbf{q}}(x, \zeta) = e^{-ix\zeta\sigma_3} + o(e^{|\text{Im } \zeta|x}),$$

locally uniformly with respect to $x \in \mathbb{R}$. We also recall the following continuity property of $\mathcal{E}_{\mathbf{q}}$: if $(\mathbf{q}_n)_{n \in \mathbb{N}}$ converges weakly to \mathbf{q} in L^2_{loc} , as $n \rightarrow \infty$, then

$$(B.5) \quad \mathcal{E}_{\mathbf{q}_n}(x, \zeta) \xrightarrow{n \rightarrow \infty} \mathcal{E}_{\mathbf{q}}(x, \zeta),$$

uniformly on bounded subsets of $\mathbb{R} \times \mathbb{C}$.

If $\text{Im } \zeta > 0$, then one has the following bounds.

Lemma B.1. *There exists $C > 0$ such that the following estimates hold*

$$(B.6) \quad \left| e^{i\zeta x} \mathbf{p} e_1(x, \zeta; \mathbf{q}^{(1)}) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right| \leq \exp\left(\frac{\|\mathbf{q}^{(1)}\|_{L^2([0,x])}^2}{4 \text{Im } \zeta}\right) \frac{\|\mathbf{q}^{(1)}\|_{L^2([0,x])}^2}{4 \text{Im } \zeta}, \quad \mathbf{p} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$(B.7) \quad |e^{i\zeta x} (e_1(x, \zeta; \mathbf{q}^{(1)}) - e_1(x, \zeta; \mathbf{q}^{(2)}))| \leq C e^{\frac{C}{\text{Im } \zeta} (\|\mathbf{q}^{(1)}\|_{L^2([0,x])}^2 + \|\mathbf{q}^{(2)}\|_{L^2([0,x])}^2)} \frac{\|\mathbf{q}^{(1)} - \mathbf{q}^{(2)}\|_{L^2([0,x])}}{\sqrt{\text{Im } \zeta}},$$

for all $(x, \zeta) \in \mathbb{R}_+ \times \mathbb{C}_+$ and all $\mathbf{q}^{(1)} = (q_1^{(1)}, q_2^{(1)})$, $\mathbf{q}^{(2)} = (q_1^{(2)}, q_2^{(2)}) \in L^2_{loc}(\mathbb{R}, \mathbb{C}^2)$.

Proof. Set $e_1(x, \zeta, \mathbf{q}^{(i)}) = e^{-i\zeta x} f^{(i)}(x, \zeta)$, $f^{(i)} = \begin{pmatrix} f_1^{(i)} \\ f_2^{(i)} \end{pmatrix}$, $i = 1, 2$. Then $f^{(i)}$ satisfies the following Volterra equation

$$(B.8) \quad f_1^{(i)}(x, \zeta) = 1 + \int_0^x k^{(i)}(x, y, \zeta) f_1^{(i)}(y, \zeta) dy,$$

$$(B.9) \quad f_2^{(i)}(x, \zeta) = i \int_0^x e^{2i\zeta(x-y)} q_2^{(i)}(y) f_1^{(i)}(y, \zeta) dy,$$

where

$$k^{(i)}(x, y, \zeta) = q_2^{(i)}(y) \int_y^x e^{2i\zeta(s-y)} q_1^{(i)}(s) ds.$$

Since

$$|k^{(i)}(x, y, \zeta)| \leq |q_2^{(i)}(y)| \int_y^x e^{-2\operatorname{Im}\zeta(s-y)} |q_1^{(i)}(s)| ds, \quad x \geq y,$$

and for all $(x, \zeta) \in \mathbb{R}_+ \times \mathbb{C}_+$,

$$\int_0^x dy |q_2^{(i)}(y)| \int_y^x ds e^{-2\operatorname{Im}\zeta(s-y)} |q_1^{(i)}(s)| \leq \frac{\|q_1^{(i)}\|_{L^2([0,x])} \|q_2^{(i)}\|_{L^2([0,x])}}{2\operatorname{Im}\zeta},$$

Equation (B.8) implies that

$$(B.10) \quad |f_1^{(i)}(x, \zeta)| \leq \exp\left(\frac{\|\mathbf{q}^{(i)}\|_{L^2([0,x])}^2}{4\operatorname{Im}\zeta}\right), \quad i = 1, 2, \quad (x, \zeta) \in \mathbb{R}_+ \times \mathbb{C}_+,$$

$$|f_1^{(i)}(x, \zeta) - 1| \leq \exp\left(\frac{\|\mathbf{q}^{(i)}\|_{L^2([0,x])}^2}{4\operatorname{Im}\zeta}\right) \frac{\|\mathbf{q}^{(i)}\|_{L^2([0,x])}^2}{4\operatorname{Im}\zeta}, \quad i = 1, 2,$$

and

$$(B.11) \quad |f_1^{(1)}(x, \zeta) - f_1^{(2)}(x, \zeta)| \leq e^{\frac{\|\mathbf{q}^{(1)}\|_{L^2([0,x])}^2 + \|\mathbf{q}^{(2)}\|_{L^2([0,x])}^2}{4\operatorname{Im}\zeta}} \frac{(\|\mathbf{q}^{(1)}\|_{L^2([0,x])} + \|\mathbf{q}^{(2)}\|_{L^2([0,x])}) \|\mathbf{q}^{(1)} - \mathbf{q}^{(2)}\|_{L^2([0,x])}}{2\operatorname{Im}\zeta}.$$

Combining (B.10) and (B.11) with (B.9), we get the required bound for $f_2^{(1)} - f_2^{(2)}$. \square

In the case where $q_1 \in H_{loc}^1(\mathbb{R})$, one has the following partial improvement of (B.4).

Lemma B.2. *Let $\mathbf{q} = (q_1, q_2) \in H_{loc}^1(\mathbb{R}) \times L_{loc}^2(\mathbb{R})$. Then for all $T > 0$, there exists $C_T > 0$ so that, for all $(x, \zeta) \in [0, T] \times \mathbb{C}$, one has*

$$(B.12) \quad \left| e_2(x, \zeta; \mathbf{q}) - e^{i\zeta x} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| \leq C_T \exp(|\operatorname{Im}\zeta|x + C_T \|\mathbf{q}\|_{L^2([0,T])}) \frac{\|q_1\|_{H^1([0,T])}}{|\zeta|}.$$

Proof. Set $e_2(x, \zeta; \mathbf{q}) = e^{i\zeta x} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{|\operatorname{Im}\zeta|x} \chi(x, \zeta)$. Then, the function χ satisfies the following integral equation

$$\chi(x, \zeta) = \chi_0(x, \zeta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i \int_0^x e^{(-i\zeta\sigma_3 - |\operatorname{Im}\zeta|)(x-y)} \sigma_3 \mathcal{Q}(y) \chi(y, \zeta) dy,$$

where¹⁶

$$\chi_0(x, \zeta) = -ie^{-(|\operatorname{Im}\zeta| + i\zeta)x} \int_0^x e^{2i\zeta y} q_1(y) dy.$$

Since clearly,

$$|e^{-(|\operatorname{Im}\zeta| \pm i\zeta)(x-y)}| \leq 1, \quad \forall x \geq y, \quad \zeta \in \mathbb{C},$$

and by integration by parts,

$$|\chi_0(x, \zeta)| \leq C_T \frac{\|q_1\|_{H^1([0,T])}}{|\zeta|}, \quad \forall (x, \zeta) \in [0, T] \times \mathbb{C},$$

the result follows by Gronwall's lemma. \square

¹⁶To avoid heaviness, we omit here the dependence of the functions χ and χ_0 on the potential \mathbf{q} .

We are now in position to prove Proposition 2.1. The equivalence between the systems (2.2) and (2.19) together with the bounds (B.2) and (B.3) ensure that, for any $T > 0$, there exists a constant $C_T > 0$ so that

$$\begin{aligned} |\hat{E}_{22}(x, \lambda; u)| &\leq C_T e^{|\operatorname{Im} \lambda^2|x + C_T(\|u\|_{H^1([0,T])} + \|u\|_{H^1([0,T])}^3)} \frac{\|u\|_{H^1([0,T])}}{\langle \lambda \rangle^2}, \\ |\hat{E}_{21}(x, \lambda; u)| &\leq C_T e^{|\operatorname{Im} \lambda^2|x + C_T(\|u\|_{H^1([0,T])} + \|u\|_{H^1([0,T])}^3)} \frac{\|u\|_{H^1([0,T])}}{\langle \lambda \rangle}, \end{aligned}$$

for all $(x, \lambda) \in [0, T] \times \mathbb{C}$ and all $u \in H_{loc}^1(\mathbb{R})$. Since $E_u(\lambda) = \overline{\sigma_1 E_u(x, -\bar{\lambda}) \sigma_1}$, this concludes the proof of Proposition 2.1.

B.1.2. Proof of Proposition 2.3. To establish Proposition 2.3, we will use a periodic analogue of (2.39). Because of the scaling invariance of (2.23), it is enough to consider the case of $T = 1$. For $\mathbf{q} = (q_1, q_2) \in L^2(\mathbb{T}, \mathbb{C}^2)$, we define

$$\mathcal{T}_{\mathbf{q}}(\zeta) = (\mathcal{L}_0 - \zeta)^{-1} Q, \quad Q = \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix}, \quad \zeta \in \mathbb{C} \setminus \{2\pi n, n \in \mathbb{Z}\}.$$

Using the explicit kernel of the free resolvent $(\mathcal{L}_0 - \zeta)^{-1}$:

$$(B.13) \quad (\mathcal{L}_0 - \zeta)^{-1}(x, y) = i \begin{pmatrix} e^{-i\zeta(x-y)} [\mathbf{1}_{\{x < y\}} + \frac{1}{e^{-i\zeta} - 1}] & 0 \\ 0 & e^{i\zeta(x-y)} [\mathbf{1}_{\{x > y\}} + \frac{1}{e^{-i\zeta} - 1}] \end{pmatrix}, \quad (x, y) \in \mathbb{T}^2,$$

and the fact that $(\mathcal{L}_0 - \zeta)^{-1}(x, y) = -\sigma_1 (\mathcal{L}_0 + \zeta)^{-1}(x, y) \sigma_1$, one can easily check that, for all $\zeta \in \mathbb{C} \setminus \{2\pi n, n \in \mathbb{Z}\}$, $\mathcal{T}_{\mathbf{q}}(\zeta)$ is a Hilbert-Schmidt operator on $L^2(\mathbb{T}, \mathbb{C}^2)$ admitting the following bound

$$(B.14) \quad \|\mathcal{T}_{\mathbf{q}}(\zeta)\|_2^2 \lesssim \frac{\|\mathbf{q}\|_{L^2(\mathbb{T})}^2}{|\operatorname{Im} \zeta| |1 - e^{\pm i\zeta}|^2}, \quad \pm \zeta \in \mathbb{C}_+, \quad \mathbf{q} \in L^2(\mathbb{T}, \mathbb{C}^2).$$

We shall deduce the bound (2.23) from the following result:

Lemma B.3. *Let $\mathbf{q} = (q_1, q_2) \in L^2(\mathbb{T}, \mathbb{C}^2)$. Then, for all $\zeta \in \mathbb{C} \setminus \{2\pi n, n \in \mathbb{Z}\}$, we have:*

$$(B.15) \quad \det_2(\mathbf{I} - \mathcal{T}_{\mathbf{q}}(\zeta)) = \frac{1 - \frac{1}{2} \Delta_{\mathbf{q}}(\zeta)}{1 - \cos \zeta},$$

where $\Delta_{\mathbf{q}}(\zeta)$ is the Floquet discriminant of the Zakharov-Shabat system (B.1): $\Delta_{\mathbf{q}}(\zeta) = \operatorname{Tr} \mathcal{M}_{\mathbf{q}}(\zeta)$, $\mathcal{M}_{\mathbf{q}}(\zeta) = \mathcal{E}_{\mathbf{q}}(1, \zeta)$.

Since $L_u^{-1}(\lambda) = (\mathbf{I} - T_u(\lambda))^{-1} (\mathcal{L}_0 - \lambda^2)^{-1}$, with $T_u(\lambda) = i\lambda (\mathcal{L}_0 - \lambda^2)^{-1} U$, $U = \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix}$, combining (A.7) together with (B.14) and (B.15) and taking into account that $\|(\mathcal{L}_0 - \lambda^2)^{-1}\| \leq |\operatorname{Im} \lambda^2|^{-1}$, one readily gets (2.23).

The identity (B.15) is well known in the literature. For completeness, we give a simple proof here.

Proof of Lemma B.3. Denote

$$\mathbf{d}(\zeta) = \frac{1 - \frac{1}{2} \Delta_{\mathbf{q}}(\zeta)}{1 - \cos \zeta}, \quad d(\zeta) = \det_2(\mathbf{I} - \mathcal{T}_{\mathbf{q}}(\zeta)).$$

Since \mathbf{d} and d are holomorphic functions of $\zeta \in \mathbb{C} \setminus \{2\pi n, n \in \mathbb{Z}\}$ and since in view of (B.4), (B.14) and (A.6),

$$\mathbf{d}(\zeta) \rightarrow 1, \quad d(\zeta) \rightarrow 1, \quad \text{as } |\operatorname{Im} \zeta| \rightarrow \infty,$$

it is enough to show that

$$\frac{\mathbf{d}'(\zeta)}{\mathbf{d}(\zeta)} = \frac{d'(\zeta)}{d(\zeta)},$$

for $\text{Im } \zeta$ sufficiently large.

Consider $d(\zeta)$. The bound (B.14) guarantees that for $|\text{Im } \zeta| \geq R$ with $R = R(\|\mathbf{q}\|_{L^2(\mathbb{T})})$ sufficiently large, $\|\mathcal{T}_{\mathbf{q}}(\zeta)\| < 1$ and therefore we may write

$$d(\zeta) = \exp\left(-\sum_{m=2}^{\infty} \frac{1}{m} \text{Tr}(\mathcal{T}_{\mathbf{q}}^m(\zeta))\right),$$

which implies

$$\begin{aligned} \frac{d'(\zeta)}{d(\zeta)} &= -\sum_{m=2}^{\infty} \text{Tr}(\mathcal{T}_{\mathbf{q}}^m(\zeta)(\mathcal{L}_0 - \zeta)^{-1}) \\ &= \text{Tr}\left((\mathcal{L}_0 - \zeta)^{-1} - (\mathcal{L}_{\mathbf{q}} - \zeta)^{-1}\right), \quad \text{Im } \zeta \geq R. \end{aligned}$$

We next compute the trace of $(\mathcal{L}_0 - \zeta)^{-1} - (\mathcal{L}_{\mathbf{q}} - \zeta)^{-1}$ using that

$$\begin{aligned} (\mathcal{L}_0 - \zeta)^{-1}(x, y) - (\mathcal{L}_{\mathbf{q}} - \zeta)^{-1}(x, y) &= -i \left(\mathcal{E}_{\mathbf{q}}(x, \zeta) \mathcal{E}_{\mathbf{q}}^{-1}(y, \zeta) - e^{-i\zeta(x-y)\sigma_3} \right) \mathbf{1}_{x \leq y} \sigma_3 \\ &\quad + \frac{i}{2 - \Delta_{\mathbf{q}}(\zeta)} \mathcal{E}_{\mathbf{q}}(x, \zeta) (\mathbf{I} - \mathcal{M}_{\mathbf{q}}^{-1}(\zeta)) \mathcal{E}_{\mathbf{q}}^{-1}(y, \zeta) \sigma_3 \\ &\quad - \frac{i}{2(1 - \cos \zeta)} (\mathbf{I} - e^{i\zeta\sigma_3}) e^{-i\zeta(x-y)\sigma_3} \sigma_3. \end{aligned}$$

This gives

$$(B.16) \quad \frac{d'(\zeta)}{d(\zeta)} = -\frac{\sin \zeta}{1 - \cos \zeta} - \frac{i}{2 - \Delta_{\mathbf{q}}(\zeta)} \int_0^1 \text{Tr} \left(\mathcal{E}_{\mathbf{q}}(x, \zeta) \mathcal{M}_{\mathbf{q}}^{-1}(\zeta) \mathcal{E}_{\mathbf{q}}^{-1}(x, \zeta) \sigma_3 \right) dx.$$

Observing that

$$\partial_{\zeta} \mathcal{E}_{\mathbf{q}}(x, \zeta) = -i \int_0^x \mathcal{E}_{\mathbf{q}}(x, \zeta) \mathcal{E}_{\mathbf{q}}^{-1}(y, \zeta) \sigma_3 \mathcal{E}_{\mathbf{q}}(y, \zeta) dy,$$

and therefore,

$$\mathcal{M}_{\mathbf{q}}^{-1}(\zeta) \frac{d}{d\zeta} \mathcal{M}_{\mathbf{q}}(\zeta) = -i \int_0^1 \mathcal{E}_{\mathbf{q}}^{-1}(y, \zeta) \sigma_3 \mathcal{E}_{\mathbf{q}}(y, \zeta) dy,$$

we can rewrite (B.16) as

$$\frac{d'(\zeta)}{d(\zeta)} = -\frac{\sin \zeta}{1 - \cos \zeta} - \frac{1}{2 - \Delta_{\mathbf{q}}(\zeta)} \frac{d}{d\zeta} \text{Tr} \mathcal{M}_{\mathbf{q}}^{-1}(\zeta).$$

Since $\det \mathcal{M}_{\mathbf{q}}(\zeta) = 1$ and hence $\text{Tr} \mathcal{M}_{\mathbf{q}}^{-1}(\zeta) = \Delta_{\mathbf{q}}(\zeta)$, we obtain

$$\begin{aligned} \frac{d'(\zeta)}{d(\zeta)} &= -\frac{\sin \zeta}{1 - \cos \zeta} - \frac{\Delta_{\mathbf{q}}'(\zeta)}{2 - \Delta_{\mathbf{q}}(\zeta)} \\ &= \frac{\mathbf{d}'(\zeta)}{\mathbf{d}(\zeta)}. \end{aligned}$$

□

B.2. Proof of Lemma 2.4. Lemma 2.4 can be recover from the analysis performed in [3], we sketch its proof here for the convenience of the reader. Recall that¹⁷

$$(B.17) \quad \ln a_u(\lambda) = - \sum_{k=1}^{\infty} \frac{\text{Tr}(T_u(\lambda)^{2k})}{2k}, \quad \text{if } \|T_u(\lambda)\| < 1.$$

Since clearly

$$(B.18) \quad \|T_u(\lambda)\| \leq \frac{|\lambda| \|u\|_{L^\infty(\mathbb{R})}}{\text{Im}(\lambda^2)} \lesssim \frac{|\lambda| \|u\|_{H^1(\mathbb{R})}}{\text{Im}(\lambda^2)}, \quad \forall \lambda \in \Omega_+,$$

we deduce that (B.17) holds for all $\lambda \in \Omega_+$ satisfying $\lambda^2 \in \Gamma_\delta$, $|\lambda|^2 \geq R_0 = R_0(\|u\|_{H^1(\mathbb{R})}, \delta)$ (where δ is fixed so that $0 < \delta < \frac{\pi}{2}$).

Thanks to (B.17), (B.18) and (2.41), we have

$$(B.19) \quad \ln a_u(\lambda) = - \sum_{k=1}^3 \frac{\text{Tr}(T_u(\lambda)^{2k})}{2k} + o(|\lambda|^{-4}), \quad \text{as } |\lambda| \rightarrow \infty, \lambda \in \Gamma_\delta.$$

Following [3], we next write

$$(B.20) \quad - \frac{\text{Tr}(T_u(\lambda)^{2k})}{2k} = \sum_{j=k-1}^2 \frac{\mu_{j,k}(u)}{\lambda^{2j}} + \tau_k(u, \lambda), \quad k = 1, 2, 3,$$

where $\mu_{j,k}(u)$ are homogeneous polynomials of degree $2k$ in u, \bar{u} , and their derivatives, satisfying

$$(B.21) \quad E_j(u) = \sum_{k=1}^{j+1} \mu_{j,k}(u), \quad j = 0, 1, 2.$$

Since¹⁸

$$(B.22) \quad \text{Tr } T_u^2(\lambda) = 2i\lambda^2 \int_{\mathbb{R}} d\xi \frac{|\hat{u}(\xi)|^2}{\xi + 2\lambda^2},$$

we have

$$\tau_1(u, \lambda) = \frac{i}{8\lambda^4} \int_{\mathbb{R}} \frac{\xi^3 |\hat{u}(\xi)|^2}{\xi + 2\lambda^2} d\xi.$$

Applying Lebesgue dominated convergence theorem, we infer that

$$(B.23) \quad \tau_1(u, \lambda) = o_\delta(|\lambda|^{-4}), \quad \text{as } |\lambda| \rightarrow \infty, \lambda \in \Gamma_\delta.$$

Finally, according to Lemma 2.1 in [3],

$$(B.24) \quad |\tau_2(u, \lambda)| + |\tau_3(u, \lambda)| \leq \frac{C(\|u\|_{H^1}, \delta)}{|\lambda|^6},$$

which completes the proof of the lemma.

B.3. Proof of Lemma 2.7. Let $u \in H^1(\mathbb{R})$ such that $\tilde{a}_u \equiv 1$ and $\|u\|_{L^2(\mathbb{R})}^2 = 4\pi$. Then, it follows from Lemma 2.6 that $E_{\mathbb{R}}(u) = P_{\mathbb{R}}(u) = 0$. Applying the gauge transformation

$$v(x) = e^{\frac{3}{4}i \int_{-\infty}^x |u(s)|^2 ds} u(x),$$

one can easily check that

$$P_{\mathbb{R}}(u) = \text{Im} \int \bar{v} v_x dx - \frac{1}{4} \|v\|_{L^4(\mathbb{R})}^4 = \tilde{P}(v) \quad \text{and} \quad E_{\mathbb{R}}(u) = \|\partial_x v\|_{L^2(\mathbb{R})}^2 - \frac{1}{16} \|v\|_{L^6(\mathbb{R})}^6 = \tilde{E}(v).$$

Then defining, for $\alpha \in \mathbb{R}$, $w_\alpha(x) = e^{-i\alpha x} v(x)$, we deduce that, for all α ,

$$-\tilde{E}(w_\alpha) = -\alpha^2 \|v\|_{L^2(\mathbb{R})}^2 + \frac{\alpha}{2} \|v\|_{L^4(\mathbb{R})}^4.$$

¹⁷In view of (2.41), (B.18) and (B.22), this series expansion is consistent with our definition of $\ln a_u(\lambda)$.

¹⁸see for instance (2.23) in [2]

A direct computation shows that the function $\mathbb{R} \ni \alpha \rightarrow -\alpha^2 \|v\|_{L^2(\mathbb{R})}^2 + \frac{\alpha}{2} \|v\|_{L^4(\mathbb{R})}^4$ reaches its maximum at $\alpha_* = \frac{\|v\|_{L^4(\mathbb{R})}^4}{4\|v\|_{L^2(\mathbb{R})}^2}$:

$$(B.25) \quad -\tilde{E}(w_\alpha) \leq -\tilde{E}(w_{\alpha_*}) = \frac{\|v\|_{L^4(\mathbb{R})}^8}{16\|v\|_{L^2(\mathbb{R})}^2}, \quad \forall \alpha.$$

Applying the sharp Gagliardo-Nirenberg inequality (2.60) to w_{α_*} we obtain that

$$(B.26) \quad 1 \leq \|v\|_{L^2(\mathbb{R})}^2 \Psi \left(\frac{\|v\|_{L^6(\mathbb{R})}^6}{\|v\|_{L^4(\mathbb{R})}^8} \right), \quad \Psi(X) = X \left(1 - \frac{16}{C_{GN}^{18}} X^2 \right),$$

with equality if and only if

$$(B.27) \quad v(x) = z \frac{e^{i\alpha_* x}}{\sqrt{c^2(x-x_0)^2 + 1}}$$

for some $z \in \mathbb{C}$, $c > 0$, and $x_0 \in \mathbb{R}$. Recalling that $C_{GN} = 3^{\frac{1}{6}}(2\pi)^{-\frac{1}{9}}$, one easily gets

$$(B.28) \quad \Psi(X) \leq \Psi(X_*) = \frac{1}{4\pi} \quad \text{with} \quad X_* = \frac{C_{GN}^9}{4\sqrt{3}} = \frac{3}{8\pi}.$$

Since $\|v\|_{L^2(\mathbb{R})}^2 = 4\pi$, it follows from (B.26) and (B.28) that v has the form (B.27) with $|z| = 2\sqrt{c}$, and since the algebraic solitons satisfy $E_{\mathbb{R}}(u) = P_{\mathbb{R}}(u) = 0$, necessarily, $\alpha_* = \frac{c}{2}$.

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