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Global well-posedness of the 2D nonlinear Schrödinger equation with multiplicative spatial white noise on the full space

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Abstract

We consider the nonlinear Schrödinger equation with multiplicative spatial white noise and an arbitrary polynomial nonlinearity on the two-dimensional full space domain. We prove global well-posedness by using a gauge-transform introduced by Hairer and Labbé (Electron Commun Probab 20(43):11, 2015) and constructing the solution as a limit of solutions to a family of approximating equations. This paper extends a previous result by Debussche and Martin (Nonlinearity 32(4):1147–1174, 2019) with a sub-quadratic nonlinearity.

Keywords Nonlinear Schrödinger equation · Multiplicative white noise · Global well-posedness

Mathematics Subject Classification 35Q55 · 60H15

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1 Introduction

1.1 NLS with multiplicative space white noise

We study the following Cauchy problem on \mathbb{R}^2 for the stochastic defocusing nonlinear Schrödinger equation (NLS)

$$i\partial_t u = \Delta u + u\xi - \lambda u|u|^p, \quad u(0) = u_0, \quad (1.1)$$

where $p > 0$, $\lambda > 0$ are parameters and $\xi \in \mathcal{S}'(\mathbb{R}^2)$ stands for white noise in space. The unknown u is a complex valued process and u_0 is a randomized initial datum (more precise assumptions are given below).

One can view the NLS (1.1) as a stochastic version of the deterministic nonlinear Schrödinger equation with a power type nonlinearity, which has been studied extensively in recent decades. See [6, 8, 9, 25] and the references therein. On the other hand, if we ignore the nonlinearity, (1.1) can be viewed as the dispersive Anderson model, which is the dispersive counterpart of the well-studied parabolic Anderson model (see, for example, [20, 21]), i.e. with $i\partial_t u$ replaced by $\partial_t u$.

The NLS (1.1) was first considered by the first author and Weber in [13] on a periodic domain $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$. To deal with the ill-defined nature of the term $u\xi$, they used the gauge transform $v = e^Y u$ where Y solves $\Delta Y = \xi$. This gauge transform, which resembles the so-called Doss-Sussmann transformation in [14, 38], was first introduced by Hairer and Labbé in the context of the parabolic Anderson model on \mathbb{R}^2 in [20] (the definition for Y is slightly different on \mathbb{R}^2). The equation for v now formally reads as

$$i\partial_t v = \Delta v - 2\nabla v \cdot \nabla Y + v\nabla Y^2 - \lambda e^{-pY} v|v|^p, \quad v(0) = e^Y u_0,$$

which is easier to solve since the most singular term is canceled. Still, the term ∇Y is merely a distribution, so that ∇Y^2 is replaced by a Wick product $:\nabla Y^2:$ in [13, 20]. See the next subsection for more detailed explanations.

In [13], the first author and Weber showed global well-posedness of the following gauge-transformed NLS on \mathbb{T}^2 with a (sub-)cubic nonlinearity (i.e. with $p \leq 2$):

$$i\partial_t v = \Delta v - 2\nabla v \cdot \nabla Y + v:\nabla Y^2: - \lambda e^{-pY} v|v|^p, \quad v(0) = e^Y u_0. \quad (1.2)$$

The main strategy is to consider a mollified noise ξ_ε and a smoothed process $Y_\varepsilon = \Delta^{-1}\xi_\varepsilon$, and then construct the solution v as a limit of v_ε in probability, which is the solution of the following smoothed equation:

$$i\partial_t v_\varepsilon = \Delta v_\varepsilon - 2\nabla v_\varepsilon \cdot \nabla Y_\varepsilon + v_\varepsilon:\nabla Y_\varepsilon^2: - \lambda e^{-pY_\varepsilon} v_\varepsilon|v_\varepsilon|^p, \quad v_\varepsilon(0) = e^{Y_\varepsilon} u_0.$$

The key ingredient for the convergence of v_ε is a suitable H^2 a-priori bound for v_ε with a logarithmic loss in ε .

Later on, the third author and the fourth author [41] improved the result in [13] by extending global well-posedness of the gauge-transformed NLS (1.2) on \mathbb{T}^2 to the

larger range $p \leq 3$. Specifically, they introduced modified energies that allow them to control the H^2 a-priori bound of v_ε for a larger range of p . In a subsequent paper [42], the third author and the fourth author improved their global well-posedness result by covering all $p > 0$. In particular, they exploited the time averaging effect for dispersive equations and established Strichartz estimates to obtain the H^2 a-priori bound of v_ε for the whole range $p > 0$. Moreover, in [41, 42], the authors improved [13] by proving almost sure convergence of v_ε to v instead of convergence in probability.

We now turn our attention to the NLS (1.1) on the \mathbb{R}^2 setting, which is the main concern in this paper. In this situation, the additional difficulty comes from the logarithmic growth of the white noise. In [12], the first author and Martin showed global well-posedness of a gauge-transformed NLS similar to (1.2) (see (1.7) below) for $0 < p < 1$. To conquer the issue of the unboundedness of the noise, they used weighted Sobolev and Besov spaces and obtained a weighted H^2 a-priori bound for v_ε with a logarithmic loss in ε . This approach of using the weighted Besov spaces in the study of stochastic PDEs was also used in [20, 21, 30, 34]. In the case of the NLS (1.1), such an approach requires more assumptions on the regularity of the initial datum than those on the \mathbb{T}^2 setting.

In this paper, we extend the result in [12] by including all $p > 0$ cases using an intricate combination of the methods mentioned above: the weighted Sobolev and Besov spaces estimates, the modified energies as in [41], and the dispersive effect as in [42]. In addition, we are able to prove a stronger convergence result of v_ε to v (almost sure convergence) than that in [12] (convergence in probability). This is obtained thanks to the estimates on the white noise on \mathbb{R}^2 and Wick products which are of independent interest. See the next subsection for a more detailed set-up and the main result of this paper.

1.2 Set-up and the main result

Let us recall that, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a real valued white noise on \mathbb{R}^2 is a random variable $\xi : \Omega \rightarrow \mathcal{D}'(\mathbb{R}^2)$ such that for each $f \in \mathcal{D}(\mathbb{R}^2)$, (ξ, f) is a real valued centered Gaussian random variable such that $\mathbb{E}((\xi, f)^2) = \|f\|_{L^2}^2$. Along the rest of the paper, (\cdot, \cdot) will denote the duality $\mathcal{D}(\mathbb{R}^2), \mathcal{D}'(\mathbb{R}^2)$ and hence in the special case of classical functions, this is the usual L^2 scalar product. Classically, $f \mapsto (\xi, f)$ can be extended uniquely from $\mathcal{D}(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$, and (ξ, f) is a real valued centered Gaussian random variable such that $\mathbb{E}((\xi, f)^2) = \|f\|_{L^2}^2$ for all $f \in L^2(\mathbb{R}^2)$.

We proceed as in [20] and use a truncated Green's function $G \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ that satisfies $\text{supp } G \subseteq B(0, 1)$ (the unit ball around 0) and $G(x) = -\frac{1}{2\pi} \log |x|$ for $|x|$ small enough, so that $Y := G * \xi$ solves

$$\Delta Y = \xi + \varphi * \xi \quad (1.3)$$

for some $\varphi \in C_c^\infty(\mathbb{R}^2)$. Hence, we introduce the new variable

$$v = e^Y u \quad (1.4)$$

which converts (1.1) into the following gauge-transformed NLS for v :

$$i \partial_t v = \Delta v + v(\nabla Y^2 - \varphi * \xi) - 2 \nabla v \cdot \nabla Y - \lambda v |v|^p e^{-pY}, \quad v(0) = v_0 := e^Y u_0.$$

The term ∇Y^2 is ill-defined as a square of a distribution, but can be replaced by a meaningful object $:\nabla Y^2:$, which is essentially the Wick product of ∇Y with itself. As in [20], we introduce the random variable

$$:\nabla Y^2: = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \nabla G(\cdot - z_1) \nabla G(\cdot - z_2) \xi(dz_1) \xi(dz_2) \tag{1.5}$$

with ξ denoting the Gaussian stochastic measure on \mathbb{R}^2 induced by the white noise ξ (see [24, page 95-99]). The relation (1.5) should be read in the *distributional* sense, i.e. for $\phi \in \mathcal{S}(\mathbb{R}^2)$ we have

$$:\nabla Y^2:(\phi) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \phi(x) \nabla G(x - z_1) \nabla G(x - z_2) dx \right) \xi(dz_1) \xi(dz_2),$$

so that $:\nabla Y^2:$ is only defined (almost surely) as a distribution. Recall that for $f_1, f_2 \in L^2(\mathbb{R}^2)$ we have the following identity for $X_1 := \int_{\mathbb{R}^2} f_1(z_1) \xi(dz_1), X_2 := \int_{\mathbb{R}^2} f_2(z_2) \xi(dz_2)$ (see [24, Theorem 7.26])

$$:X_1 \cdot X_2: = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f_1(z_1) f_2(z_2) \xi(dz_1) \xi(dz_2), \tag{1.6}$$

where $:X_1 \cdot X_2:$ denotes the Wick product between the Gaussian random variables X_1, X_2 . Note that the above integral is a multiple Wiener-Ito integral (see [33]). From this perspective, the definition $:\nabla Y^2:$ can be read as a Wick product of the distribution ∇Y with itself. For an introduction to Wick calculus let us refer to [22, 24, 33]. Thus, we shall focus on the following equation

$$i \partial_t v = \Delta v + v : \widetilde{\nabla Y^2} : - 2 \nabla v \cdot \nabla Y - \lambda v |v|^p e^{-pY}, \quad v(0) = v_0, \tag{1.7}$$

where

$$:\widetilde{\nabla Y^2}: = :\nabla Y^2: - \varphi * \xi. \tag{1.8}$$

In order to construct a solution to (1.7) we consider an approximation v_ε which solves

$$i \partial_t v_\varepsilon = \Delta v_\varepsilon + v_\varepsilon : \widetilde{\nabla Y_\varepsilon^2} : - 2 \nabla v_\varepsilon \cdot \nabla Y_\varepsilon - \lambda v_\varepsilon |v_\varepsilon|^p e^{-pY_\varepsilon}, \quad v_\varepsilon(0) = v_0. \tag{1.9}$$

Here, Y_ε is defined as

$$Y_\varepsilon = \rho_\varepsilon * Y = \rho_\varepsilon * G * \xi = G * \xi_\varepsilon,$$

where $\rho_\varepsilon(x) = \varepsilon^{-2}\rho(\varepsilon^{-1}x)$, $\rho \in C_c^\infty(B(0, 1))$, $\rho \geq 0$, $\int_{\mathbb{R}^2} \rho = 1$, and $\xi_\varepsilon = \rho_\varepsilon * \xi$ is a mollification of the considered noise. Also,

$$\widetilde{:\nabla Y_\varepsilon^2:} = :\nabla Y_\varepsilon^2: - \varphi * \xi_\varepsilon, \tag{1.10}$$

where the Wick product $:\nabla Y_\varepsilon^2:$ is defined as follows:

$$:\nabla Y_\varepsilon^2:(x) = \nabla Y_\varepsilon^2(x) - c_\varepsilon, \quad c_\varepsilon = \mathbb{E} \left(|\nabla Y_\varepsilon|^2 \right) = \|\rho_\varepsilon * \nabla G\|_{L^2}^2. \tag{1.11}$$

We show in this paper that Y_ε converges to Y , ∇Y_ε converges to ∇Y , and $:\nabla Y_\varepsilon^2:$ converges to $:\nabla Y^2:$ almost surely in certain function spaces (see Proposition 3.8).

In the sequel we shall use weighted Sobolev spaces, where the details are in Sect. 2. For the moment, we can consider the following equivalent norm on the space $H_\delta^s(\mathbb{R}^2)$:

$$\| \langle x \rangle^\delta \varphi \|_{H^s(\mathbb{R}^2)}, \quad s \geq 0, \quad \delta \in \mathbb{R},$$

which reduces the weighted Sobolev spaces to the usual unweighted Sobolev spaces once the weight is pulled inside.

We now state the following result regarding global well-posedness of the equation (1.9) for v_ε .

Theorem 1.1 *There exists a full measure event $\Sigma \subset \Omega$ such that for every $\omega \in \Sigma$ and every $\varepsilon \in (0, \frac{1}{2})$, the following property holds. For any $p \geq 1$, $\delta_0 > 0$, $s \in (1, 2)$, and $\delta > 0$, there exists $\delta_1 > 0$ such that the Cauchy problem (1.9) with $v_0 \in H_{\delta_0}^2(\mathbb{R}^2)$ admits one unique global solution*

$$v_\varepsilon(t, x) \in L_{\text{loc}}^\infty((0, \infty); H_{-\delta}^2(\mathbb{R}^2)) \cap \mathcal{C}([0, \infty); H_{\delta_1}^s(\mathbb{R}^2)).$$

Moreover, for every $T > 0$ and $\delta > 0$ there exists constants $C, C(\omega) > 0$ independent of ε such that the following bound holds:

$$\|v_\varepsilon(t, x)\|_{L^\infty((0, T); H_{-\delta}^2(\mathbb{R}^2))} \leq C(\omega) |\ln \varepsilon|^C.$$

The existence and uniqueness of solution to (1.9) is not obvious since the coefficients involved in the linear propagator associated with (1.9) are smooth but unbounded. At the best of our knowledge, because of the lack of decay at the spatial infinity of the derivatives of ξ_ε the classical Strichartz estimates are not available in this framework, and so we cannot apply a classical contraction argument. In fact, along the paper we shall establish some weighted Strichartz estimates that will allow us to perform a compactness argument as in [12] and to deduce the existence of solutions to (1.9) as the limit of solutions $v_{\varepsilon, n}$ to the following further regularized equation at $\varepsilon > 0$ fixed:

$$\begin{aligned} \iota \partial_t v_{\varepsilon, n} &= \Delta v_{\varepsilon, n} + v_{\varepsilon, n} \theta_n \widetilde{:\nabla Y_\varepsilon^2:} - 2 \nabla v_{\varepsilon, n} \cdot \nabla (\theta_n Y_\varepsilon) - \lambda v_{\varepsilon, n} |v_{\varepsilon, n}|^p e^{-p\theta_n Y_\varepsilon}, \\ v_{\varepsilon, n}(0) &= v_0, \end{aligned} \tag{1.12}$$

where $\theta_n(x) = \theta(\frac{x}{n})$ and $\theta \in C_0^\infty(\mathbb{R}^2)$, $\theta \geq 0$, $\theta(x) = 1$ when $|x| \leq 1$. To see that (1.12) is globally well-posed, we let $u_{\varepsilon,n} = e^{-\theta_n Y_\varepsilon} v_{\varepsilon,n}$ and consider the following equation for $u_{\varepsilon,n}$:

$$\begin{aligned} i \partial_t u_{\varepsilon,n} &= \Delta u_{\varepsilon,n} + u_{\varepsilon,n} (\theta_n \cdot \widetilde{\nabla Y_\varepsilon^2} - \nabla(\theta_n Y_\varepsilon)^2 + \Delta(\theta_n Y_\varepsilon)) - \lambda u_{\varepsilon,n} |u_{\varepsilon,n}|^p, \\ u_{\varepsilon,n}(0) &= e^{-\theta_n Y_\varepsilon} v_0. \end{aligned} \tag{1.13}$$

Note that $\theta_n Y_\varepsilon$ is a Schwartz function, so that $e^{-\theta_n Y_\varepsilon} v_0 \in H^2(\mathbb{R}^2)$ given $v_0 \in H^2(\mathbb{R}^2) \subset H_{\delta_0}^2(\mathbb{R}^2)$. Since the equation (1.13) contains only bounded and smooth terms, by classical results as in [9, 16, 25], there exists a unique solution $u_{\varepsilon,n}$ to (1.13) in $\mathcal{C}([0, \infty); H^2(\mathbb{R}^2))$. This shows that there exists a unique solution $v_{\varepsilon,n}$ to (1.12) in $\mathcal{C}([0, \infty); H^2(\mathbb{R}^2))$.

Once we establish a weighted H^2 a-priori bound (independent of n) for $v_{\varepsilon,n}$, we can use a similar argument in [12] to prove Theorem 1.1. See Sect. 8 for a more detailed explanation.

Next, we describe the behavior of the solutions $v_\varepsilon(t, x)$ in the limit $\varepsilon \rightarrow 0$. It is remarkable that this result can be also interpreted in terms of the convergence, up to a phase shift, of $u_\varepsilon(t, x)$ solutions to the smoothed version of (1.1)

$$i \partial_t u_\varepsilon = \Delta u_\varepsilon + u_\varepsilon \xi_\varepsilon - \lambda u_\varepsilon |u_\varepsilon|^p, \quad u(0) = u_0. \tag{1.14}$$

It is worth mentioning that it is not obvious to prove directly the existence and uniqueness of solutions to the smoothed equation above. Nevertheless, by direct computation, one can show that

$$u_\varepsilon = e^{-i c_\varepsilon t} e^{-Y_\varepsilon} v_\varepsilon \tag{1.15}$$

solves (1.14) provided that v_ε is given by Theorem 1.1.

We are now ready to state the main result of this paper.

Theorem 1.2 *There exists a full measure event $\Sigma \subset \Omega$ such that for every $\omega \in \Sigma$, the following property holds. Let $\delta_0 > 0$ and $v_0 \in H_{\delta_0}^2$. For any $p \geq 1$, $s \in (1, 2)$, and $\delta > 0$, there exists $v \in \mathcal{C}([0, \infty); H_{\delta_1}^s(\mathbb{R}^2))$ for some $\delta_1 > 0$, such that the following convergence holds:*

$$\|v_\varepsilon(t, x) - v(t, x)\|_{\mathcal{C}([0,T]; H_{\delta_1}^s(\mathbb{R}^2))} \xrightarrow{\varepsilon \rightarrow 0} 0, \tag{1.16}$$

where v_ε is given by Theorem 1.1 with $v_\varepsilon(0) = v_0$. In particular, $u_\varepsilon = e^{-i c_\varepsilon t} e^{-Y_\varepsilon} v_\varepsilon$ solves (1.14) and

$$\|e^{i c_\varepsilon t} e^{Y_\varepsilon} u_\varepsilon(t, x) - v(t, x)\|_{\mathcal{C}([0,T]; H_{\delta_1}^s(\mathbb{R}^2))} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

where $c_\varepsilon \sim |\ln \varepsilon|$ is the constant in (1.11). Moreover, v is the unique global solution to (1.7) in $\mathcal{C}([0, \infty); H_{\delta_1}^s(\mathbb{R}^2))$.

Theorem 1.2 is an extension of previous result proved in [12] in the case $0 < p < 1$. In particular, we cover the relevant case of cubic nonlinearity. Notice also that our convergence occurs almost surely, which is stronger than the result in [12] where the convergence is in probability.

We conclude this subsection by stating several remarks.

Remark 1.3 Throughout the whole paper, we restrict our attention to the case $\lambda > 0$ of the NLS (1.1), which refers to the defocusing case. For the focusing case (i.e. $\lambda < 0$) of (1.1), Theorem 1.2 also holds for $0 < p < 2$. For $\lambda < 0$ and $p \geq 2$, we need to impose a smallness assumption on the initial data $\|v_0\|_{H_{\delta_0}^1}$ in order to obtain Theorem 1.2. Indeed, the only place that requires a different proof for this case is Proposition 5.1. See Remark 5.2 for details.

Remark 1.4 For the NLS (1.1) in higher dimensions, it is not clear whether the approach in this paper based on a gauge-transform works in showing global well-posedness. The main challenge lies in the fact that the spatial white noise is too rough for our approach when $d \geq 3$. Another challenge is the weaker smoothing properties of the Schrödinger equation in higher dimensions. One can compare the situation with the parabolic setting in [21], where the authors used the theory of regularity structures due to Hairer [19].

Remark 1.5 The authors in [18, 31, 43] introduced another approach to the study of the NLS (1.1) with $p \leq 2$. Their method is based on the realization of the (formal) Anderson Hamiltonian $H = \Delta + \xi$ as a self-adjoint operator on the L^2 space. Specifically, [18] considered the equation on the torus, [31] considered a compact manifold, and [43] considered the full space. In their settings, the initial data u_0 needs to belong to the domain of H . One can compare the initial condition in [18, 43] and that in this paper and in [12, 13, 41, 42], where the initial data is chosen to have a specific structure $e^{-Y}v_0$ with v_0 belonging to a weighted H^2 space. For more discussions on the Anderson Hamiltonian, see [1, 3, 10, 29, 32].

1.3 Notations

We denote by $\mathcal{P}(a, b, \dots)$ a polynomial function depending on the a, b, \dots . The polynomial can change from line to line along the estimates. For any positive number $a > 0$, we use a^+ to denote $a + \eta$ for $\eta > 0$ arbitrarily small. We denote by N dyadic numbers larger than or equal to $\frac{1}{2}$ and by Δ_N the corresponding Littelwood-Paley partition. All the functional spaces that we shall use are based on \mathbb{R}^2 . We denote by $C > 0$ a deterministic constant that can change from line to line and $C(\omega) > 0$ a stochastic constant which is finite almost surely. We denote by (\cdot, \cdot) the L^2 scalar product as well as the duality in $\mathcal{D}(\mathbb{R}^2), \mathcal{D}'(\mathbb{R}^2)$.

1.4 Organization of the paper

This paper is organized as follows. In Sect. 2, we recall the definitions of some useful functional spaces and their properties. In Sect. 3, we discuss properties of the process Y

and its related stochastic objects. In Sect. 4 and Sect. 5, we establish some useful linear and nonlinear estimates for a generalized equation. In Sect. 6, we recall definitions of modified energies and provide estimates for them. In Sect. 7, we prove a key H^2 a-priori bound. Lastly, in Sect. 8, we prove the main result of this paper: Theorem 1.2.

2 Functional spaces and preliminary facts

2.1 Functional spaces

Given $p \in [1, \infty]$, $\mu \in \mathbb{R}$ we introduce respectively the weighted Lebesgue and Sobolev spaces as follows:

$$\|f\|_{L_\mu^p} = \left(\int_{\mathbb{R}^2} \langle x \rangle^{\mu p} |f|^p dx \right)^{\frac{1}{p}}, \quad \langle x \rangle = \sqrt{1 + |x|^2}$$

and

$$\|f\|_{W_\mu^{1,p}} = \|f\|_{L_\mu^p} + \|\nabla f\|_{L_\mu^p}$$

with the usual interpretation if $p = \infty$. If $\mu = 0$ we simply write $L^p = L_0^p$ and $W^{1,p} = W_0^{1,p}$.

Along the paper we shall make extensively use of the Littlewood-Paley multipliers Δ_N , namely

$$\text{Id} = \sum_{N\text{-dyadic}} \Delta_N \tag{2.1}$$

where

$$\Delta_N f = N^2 \int_{\mathbb{R}^2} K(N(x - y)) f(y) dy, \quad N \geq 1$$

with $\hat{K} \in \mathcal{S}(\mathbb{R}^2)$ such that $\text{supp } \hat{K}(\xi) \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$ and

$$\Delta_{\frac{1}{2}} f = \int_{\mathbb{R}^2} L(x - y) f(y) dy$$

with $\hat{L} \in \mathcal{S}(\mathbb{R}^2)$ such that $\text{supp } \hat{L}(\xi) \subset \{|\xi| < 1\}$. We also denote $\Delta_M = 0$ if $M < \frac{1}{2}$.

We can then introduce the weighted inhomogeneous Besov spaces $\mathcal{B}_{p,q,\mu}^\alpha$ as follows:

$$\|f\|_{\mathcal{B}_{p,q,\mu}^\alpha} = \left(\sum_{N\text{-dyadic}} N^{\alpha q} \|\Delta_N f\|_{L_\mu^p}^q \right)^{\frac{1}{q}} \tag{2.2}$$

for every $\alpha, \mu \in \mathbb{R}, p, q \in [1, \infty]$. Notice that for $\mu = 0$ the space $\mathcal{B}_{p,q,0}^\alpha$ reduces to the usual Besov space $\mathcal{B}_{p,q}^\alpha$. A convenient property of the spaces $\mathcal{B}_{p,q,\mu}^\alpha$ is that the weight can be “pulled in”, namely we have the equivalent norms:

$$c \|f \langle x \rangle^\mu\|_{\mathcal{B}_{p,q}^\alpha} < \|f\|_{\mathcal{B}_{p,q,\mu}^\alpha} \leq C \|f \langle x \rangle^\mu\|_{\mathcal{B}_{p,q}^\alpha}, \tag{2.3}$$

for suitable $c, C > 0$ that depend on $\alpha, \mu \in \mathbb{R}$ and $p, q \in [1, \infty]$ (see [40, Theorem 6.5]). Relation (2.3) can be used to translate results from the unweighted spaces to their weighted analogues.

In the case $(p, q) = (2, 2)$, the weighted Besov spaces are generalizations of weighted Sobolev spaces:

$$\mathcal{B}_{2,2,\mu}^\alpha = H_\mu^\alpha, \quad \text{where } \|f\|_{H_\mu^\alpha} = \|\mathcal{F}^{-1} \langle \cdot \rangle^\alpha \mathcal{F} f\|_{L_\mu^2}. \tag{2.4}$$

In the sequel we shall make extensively use of the following obvious continuous embedding

$$H_{\mu_1}^{s_1} \subset H_{\mu_2}^{s_2}, \quad s_1 \geq s_2, \quad \mu_1 \geq \mu_2. \tag{2.5}$$

The embedding (2.5) is compact when $s_1 > s_2$ and $\mu_1 > \mu_2$ (see [15, Section 4.2.3]). We shall also use the notation

$$\mathcal{B}_{\infty,\infty,\mu}^\alpha = C_\mu^\alpha. \tag{2.6}$$

In the special case $\alpha \in \mathbb{R}_+ \setminus \{0, 1, 2, \dots\}$, the space C_μ^α is in turn equivalent to the classical weighted Hölder-Zygmund space with the following norm

$$\begin{aligned} \|f\|_{C_\mu^\alpha} &= \sum_{|k| \leq [\alpha]} \sup_{x \in \mathbb{R}^2} \langle x \rangle^\mu |\partial^k f(x)| \\ &+ \sum_{|k| = [\alpha]} \sup_{\substack{x, y \in \mathbb{R}^2 \\ 0 < |x-y| \leq 1}} \langle x \rangle^\mu \frac{|\partial^k f(x) - \partial^k f(y)|}{|x - y|^{\alpha - [\alpha]}}. \end{aligned} \tag{2.7}$$

Setting $\mu = 0$ and restricting (2.7) to $x, y \in D$ for some domain $D \subset \mathbb{R}^2$ gives rise to the local Hölder space $C^\alpha(D)$. For the other values of α (in particular for $\alpha < 0$) we take (2.6) as a definition of C_μ^α .

2.2 Some properties of the Littlewood-Paley localization

We now gather well-known properties of the weighted Besov spaces that will be used along the paper. We first give an elementary, but useful, property of the Littlewood-Paley decomposition in weighted spaces.

Lemma 2.1 *Let $\gamma, \delta \geq 0$ and $\gamma_0 > 0$ be given. Then, there exists $C > 0$ such that*

$$\sum_{N\text{-dyadic}} N^\gamma \|\Delta_N f\|_{L^2_\delta} \leq C \|f\|_{H^{\gamma+\gamma_0}}. \tag{2.8}$$

Proof We have by the Cauchy-Schwarz inequality

$$\sum_{N\text{-dyadic}} N^\gamma \|\Delta_N f\|_{L^2_\delta} \leq C \left(\sum_{N\text{-dyadic}} N^{2(\gamma+\gamma_0)} \|\Delta_N f\|_{L^2_\delta}^2 \right)^{\frac{1}{2}} \tag{2.9}$$

and hence we conclude by recalling (2.2) and (2.4). □

We shall need the following commutator estimates.

Lemma 2.2 *For every $\delta \in (0, 1)$ and $p \in [1, \infty]$, there exists $C > 0$ such that for every N dyadic, we have*

$$\|[\Delta_N, \langle x \rangle^\delta]f\|_{L^p} \leq CN^{-1} \|f\|_{L^p}, \tag{2.10}$$

$$\|[\nabla, [\Delta_N, \langle x \rangle^\delta]]f\|_{L^p} \leq CN^{-1} \|f\|_{L^p}. \tag{2.11}$$

Proof It is easy to check that

$$[\Delta_N, \langle x \rangle^\delta]f = N^2 \int_{\mathbb{R}^2} (\langle y \rangle^\delta - \langle x \rangle^\delta) K(N(x - y)) f(y) dy$$

and hence we easily conclude (2.10) by the Schur test since

$$\begin{aligned} & \sup_x \left(N^2 \int_{\mathbb{R}^2} |\langle y \rangle^\delta - \langle x \rangle^\delta| |K(N(x - y))| dy \right) \\ &= \sup_y \left(N^2 \int_{\mathbb{R}^2} |\langle y \rangle^\delta - \langle x \rangle^\delta| |K(N(x - y))| dx \right) \\ &\leq CN^{-1} \| |x| K(x) \|_{L^1} \end{aligned}$$

where we used that $|\langle y \rangle^\delta - \langle x \rangle^\delta| \leq C|x - y|$ for $\delta < 1$.

Concerning (2.11) we denote by $\tilde{K}_N(x, y) = N^2(\langle y \rangle^\delta - \langle x \rangle^\delta)K(N(x - y))$ the kernel of the operator $[\Delta_N, \langle x \rangle^\delta]$. Hence, we have

$$[\nabla, [\Delta_N, \langle x \rangle^\delta]]f = \int_{\mathbb{R}^2} (\nabla_x \tilde{K}_N(x, y) + \nabla_y \tilde{K}_N(x, y)) f(y) dy$$

and we conclude as above via the Schuur test since

$$\nabla_x \tilde{K}_N(x, y) + \nabla_y \tilde{K}_N(x, y) = N^2[-\nabla_x \langle x \rangle^\delta + \nabla_y \langle y \rangle^\delta]K(N(x - y))$$

and $|\nabla_x \langle x \rangle^\delta + \nabla_y \langle y \rangle^\delta| \leq C|x - y|$. □

Next, we show a useful property of the Littlewood-Paley multipliers Δ_N in weighted Sobolev spaces.

Lemma 2.3 *For every $s \in [0, 2]$ and $\delta \in (0, 1)$, there exists $C > 0$ such that for every N dyadic, we have*

$$\|\Delta_N f\|_{H_\delta^s} \leq CN^s \left(\|\Delta_{\frac{N}{2}} f\|_{L_\delta^2} + \|\Delta_N f\|_{L_\delta^2} + \|\Delta_{2N} f\|_{L_\delta^2} \right). \tag{2.12}$$

Proof We split the proof in several cases.

Case 1: $s \in [0, 1)$.

By combining (2.3) and (2.10) we get:

$$\begin{aligned} \|\Delta_N f\|_{H_\delta^s}^2 &\leq C \sum_{M\text{-dyadic}} M^{2s} \|\Delta_M(\langle x \rangle^\delta \Delta_N f)\|_{L^2}^2 \\ &\leq C \sum_{M\text{-dyadic}} M^{2s} \|\langle x \rangle^\delta \Delta_M(\Delta_N f)\|_{L^2}^2 \\ &\quad + C \sum_{M\text{-dyadic}} M^{2s-2} \|\Delta_N f\|_{L^2}^2 \\ &\leq C \sum_{M=\frac{N}{2}, N, 2N} M^{2s} \|\langle x \rangle^\delta \Delta_M f\|_{L^2}^2 + C \|\Delta_N f\|_{L^2}^2, \end{aligned} \tag{2.13}$$

where we used that $\sum_{M\text{-dyadic}} M^{2s-2} < \infty$ for $s \in [0, 1)$. Hence, we get

$$\begin{aligned} \|\Delta_N f\|_{H_\delta^s}^2 &\leq CN^{2s} \left(\|\Delta_{\frac{N}{2}} f\|_{L_\delta^2}^2 + \|\Delta_N f\|_{L_\delta^2}^2 + \|\Delta_{2N} f\|_{L_\delta^2}^2 \right) \\ &\quad + CN^{-2s} \|\Delta_N f\|_{H_\delta^s}^2 \end{aligned} \tag{2.14}$$

From (2.14), we can deduce (2.12) provided that $N > \bar{N}$, with \bar{N} chosen in such a way that the last term on the r.h.s. can be absorbed on the l.h.s. On the other hand, we have finitely dyadic numbers $1 \leq N \leq \bar{N}$ and hence the corresponding estimate (2.12) for those values of N holds, provided that we choose the multiplicative constant large enough on the r.h.s.

Case 2: $s \in [1, 2)$.

We start by proving that for $s \in [1, 2]$, there exists $C > 0$ such that for every N dyadic:

$$\|\Delta_N f\|_{H_\delta^s} \leq C \left(\|\Delta_N f\|_{H_\delta^{s-1}} + \|\Delta_N \nabla f\|_{H_\delta^{s-1}} \right). \tag{2.15}$$

In fact, by (2.3) we have

$$\begin{aligned}
 \|\Delta_N f\|_{H^s_\delta} &\leq C \langle x \rangle^\delta \|\Delta_N f\|_{H^s} \\
 &\leq C \left(\|\langle x \rangle^\delta \Delta_N f\|_{L^2} + \|\nabla(\langle x \rangle^\delta \Delta_N f)\|_{H^{s-1}} \right) \\
 &\leq C \left(\|\langle x \rangle^\delta \Delta_N f\|_{L^2} + \|\langle x \rangle^\delta (\Delta_N \nabla f)\|_{H^{s-1}} \right. \\
 &\quad \left. + \|[\nabla, \langle x \rangle^\delta] \Delta_N f\|_{H^{s-1}} \right) \\
 &\leq C \left(\|\langle x \rangle^\delta \Delta_N f\|_{L^2} + \|\langle x \rangle^\delta (\Delta_N \nabla f)\|_{H^{s-1}} + \|\Delta_N f\|_{H^{s-1}} \right).
 \end{aligned}
 \tag{2.16}$$

Here, we used the elementary commutator bound

$$\|[\nabla, \langle x \rangle^\delta] f\|_{H^\sigma} \leq C \|f\|_{H^\sigma}, \quad \sigma \in [0, 1],
 \tag{2.17}$$

which follows from interpolating the $L^2 \rightarrow L^2$ bound and the $H^1 \rightarrow H^1$ bound of the commutator $[\nabla, \langle x \rangle^\delta]$.

Next, we show (2.12) for $s \in [1, 2)$. Notice that we have $s - 1 \in [0, 1)$ and hence we can combine (2.15) with the estimate proved in the first case in order to obtain

$$\begin{aligned}
 \|\Delta_N f\|_{H^s_\delta} &\leq CN^{s-1} \left(\|\Delta_{\frac{N}{2}} f\|_{L^2_\delta} + \|\Delta_N f\|_{L^2_\delta} + \|\Delta_{2N} f\|_{L^2_\delta} \right) \\
 &\quad + CN^{s-1} \left(\|\Delta_{\frac{N}{2}} \nabla f\|_{L^2_\delta} + \|\Delta_N \nabla f\|_{L^2_\delta} + \|\Delta_{2N} \nabla f\|_{L^2_\delta} \right).
 \end{aligned}
 \tag{2.18}$$

Then, we get for a generic dyadic M

$$\|\Delta_M \nabla f\|_{L^2_\delta} \leq C \left(\|\nabla(\langle x \rangle^\delta \Delta_M f)\|_{L^2} + \|[\langle x \rangle^\delta, \nabla] \Delta_M f\|_{L^2} \right)$$

and by noticing that $(\Delta_{\frac{M}{2}} + \Delta_M + \Delta_{2M}) \circ \Delta_M = \Delta_M$ and using (2.17), we get

$$\begin{aligned}
 \dots &\leq C \sum_{M'=\frac{M}{2}, M, 2M} \left(\|\nabla \Delta_{M'}(\langle x \rangle^\delta \Delta_M f)\|_{L^2} \right. \\
 &\quad \left. + \|\nabla([\Delta_{M'}, \langle x \rangle^\delta] \Delta_M f)\|_{L^2} + \|\Delta_M f\|_{L^2} \right) \\
 &\leq C \sum_{M'=\frac{M}{2}, M, 2M} \left(M \|\Delta_M f\|_{L^2_\delta} + \|[\Delta_{M'}, \langle x \rangle^\delta] \Delta_M \nabla f\|_{L^2} \right. \\
 &\quad \left. + \|[\nabla, [\Delta_{M'}, \langle x \rangle^\delta]] \Delta_M f\|_{L^2} \right).
 \end{aligned}$$

By using (2.10) and (2.11), we can summarize the estimates above as follows

$$\|\Delta_M \nabla f\|_{L^2_\delta} \leq CM \|\Delta_M f\|_{L^2_\delta}.
 \tag{2.19}$$

We get the desired conclusion by combining (2.18) and (2.19).

Case 3: $s = 2$.

We use (2.15) for $s = 2$ and the fact that (2.12) has been proved for $s = 1$ to obtain

$$\begin{aligned} \|\Delta_N f\|_{H_\delta^2} &\leq C\left(\|\Delta_N f\|_{H_\delta^1} + \|\Delta_N \nabla f\|_{H_\delta^1}\right) \\ &\leq CN\left(\|\Delta_{\frac{N}{2}} f\|_{L_\delta^2} + \|\Delta_N f\|_{L_\delta^2} + \|\Delta_{2N} f\|_{L_\delta^2}\right) \\ &\quad + CN\left(\|\Delta_{\frac{N}{2}} \nabla f\|_{L_\delta^2} + \|\Delta_N \nabla f\|_{L_\delta^2} + \|\Delta_{2N} \nabla f\|_{L_\delta^2}\right). \end{aligned}$$

We conclude by using (2.19). □

We shall also use the following estimates.

Lemma 2.4 *For every $s \in (0, 1)$, $s_0 \in \mathbb{R}$, and $\delta > 0$, there exists $C > 0$ such that for every N dyadic, we have*

$$\|\Delta_N f\|_{H_\delta^s} \leq CN^{s_0} \sum_{\frac{N}{4} \leq M \leq 4N} \|\Delta_M f\|_{H_\delta^{s-s_0}}, \tag{2.20}$$

$$\|\Delta_N f\|_{H_\delta^{1+s}} \leq CN^{s_0} \sum_{\frac{N}{4} \leq M \leq 4N} \|\Delta_M f\|_{H_\delta^{1+s-s_0}}. \tag{2.21}$$

Proof We first prove (2.20) and notice that by (2.13)

$$\|\Delta_N f\|_{H_\delta^s}^2 \leq C \sum_{M=\frac{N}{2}, N, 2N} M^{2s} \|\langle x \rangle^\delta \Delta_M f\|_{L^2}^2 + C \|\Delta_N f\|_{L^2}^2$$

that can be continued as follows

$$\begin{aligned} \dots &\leq CN^{2s_0} \sum_{M=\frac{N}{2}, N, 2N} M^{2s-2s_0} \|\langle x \rangle^\delta \Delta_M f\|_{L^2}^2 + C \|\Delta_N f\|_{L^2}^2 \\ &\leq CN^{2s_0} \sum_{M'=\frac{M}{2}, M, 2M} \sum_{M=\frac{N}{2}, N, 2N} M^{2s-2s_0} \|\Delta_M(\langle x \rangle^\delta \Delta_{M'} f)\|_{L^2}^2 \\ &\quad + CN^{2s_0} \sum_{M'=\frac{M}{2}, M, 2M} \sum_{M=\frac{N}{2}, N, 2N} M^{2s-2s_0-2} \|\Delta_{M'} f\|_{L^2}^2 + C \|\Delta_N f\|_{L^2}^2, \end{aligned}$$

where we used the estimate (2.10) and the fact that $(\Delta_{\frac{M}{2}} + \Delta_M + \Delta_{2M}) \circ \Delta_M = \Delta_M$. The proof follows since exactly as in the proof of (2.12), the term $\|\Delta_N f\|_{L^2}$ on the r.h.s. can be absorbed by $\|\Delta_N f\|_{H_\delta^s}^2$.

Next, we focus on (2.21) and we recall that by (2.15)

$$\|\Delta_N f\|_{H_\delta^{1+s}} \leq C\left(\|\Delta_N f\|_{H_\delta^s} + \|\Delta_N \nabla f\|_{H_\delta^s}\right). \tag{2.22}$$

Arguing as above, we get

$$\|\Delta_N \psi\|_{H_\delta^s}^2 \leq CN^{2s_0} \sum_{\frac{N}{4} \leq M \leq 4N} \|\Delta_M \psi\|_{H_\delta^{s-s_0}}^2 + C\|\Delta_N \psi\|_{L^2}^2$$

and by choosing $\psi = f$ and $\psi = \nabla f$, we get from (2.22)

$$\begin{aligned} \|\Delta_N f\|_{H_\delta^{1+s}}^2 &\leq CN^{2s_0} \sum_{\frac{N}{4} \leq M \leq 4N} \|\Delta_M f\|_{H_\delta^{s-s_0}}^2 + C\|\Delta_N f\|_{L^2}^2 \\ &\quad + CN^{2s_0} \sum_{\frac{N}{4} \leq M \leq 4N} \|\Delta_M \nabla f\|_{H_\delta^{s-s_0}}^2 + C\|\Delta_N \nabla f\|_{L^2}^2. \end{aligned}$$

As in the proof of (2.12), we can absorb $\|\Delta_N f\|_{L^2}^2$ and $\|\Delta_N \nabla f\|_{L^2}^2$ on the l.h.s. and hence we obtain

$$\begin{aligned} \|\Delta_N f\|_{H_\delta^{1+s}}^2 &\leq CN^{2s_0} \sum_{\frac{N}{4} \leq M \leq 4N} \|\Delta_M f\|_{H_\delta^{s-s_0}}^2 \\ &\quad + CN^{2s_0} \sum_{\frac{N}{4} \leq M \leq 4N} \|\Delta_M \nabla f\|_{H_\delta^{s-s_0}}^2 \\ &\leq CN^{2s_0} \sum_{\frac{N}{4} \leq M \leq 4N} \|\Delta_M f\|_{H_\delta^{s-s_0}}^2 + CN^{2s} \sum_{\frac{N}{4} \leq M \leq 4N} \|\Delta_M \nabla f\|_{L_\delta^2}^2 \\ &\leq CN^{2s_0} \sum_{\frac{N}{4} \leq M \leq 4N} \|\Delta_M f\|_{H_\delta^{s-s_0}}^2 \\ &\quad + CN^{2s_0} \sum_{\frac{N}{4} \leq M \leq 4N} M^{2+2s-2s_0} \|\Delta_M f\|_{L_\delta^2}^2, \end{aligned}$$

where we used (2.19). The conclusion follows. □

We close this subsection with the following result.

Lemma 2.5 *For every $\beta, \gamma \in \mathbb{R}, \delta \geq 0$, and $\varphi \in C_c^\infty(\mathbb{R}^2)$, there exists $C > 0$ such that*

$$\|\varphi * f\|_{C_{-\delta}^\beta} \leq C\|f\|_{C_{-\delta}^\gamma}. \tag{2.23}$$

Proof In [2, Lemma 2.2] this estimate is proved for $\delta = 0$. For $\delta > 0$ we proceed as follows. Since $(\Delta_{\frac{N}{2}} + \Delta_N + \Delta_{2N}) \circ \Delta_N = \Delta_N$, we get

$$N^\beta \|\Delta_N(\varphi * f)\|_{L_{-\delta}^\infty} = N^\beta \sum_{M=\frac{N}{2}, N, 2N} \|\Delta_M \varphi * \Delta_N f\|_{L_{-\delta}^\infty}$$

$$\begin{aligned} &\leq N^\beta \|\Delta_N f\|_{L^\infty_{-\delta}} \sum_{M=\frac{N}{2}, N, 2N} \|\Delta_M \varphi\|_{L^1_\delta} \\ &\leq N^\gamma \|\Delta_N f\|_{L^\infty_{-\delta}} \sum_{M=\frac{N}{2}, N, 2N} M^{\beta-\gamma} \|\Delta_M \varphi\|_{L^\infty_{2+\delta}}, \end{aligned}$$

where we used $\langle x \rangle^{-1} \leq 2\langle y \rangle \langle x - y \rangle^{-1}$ and the Hölder inequality. We conclude since $\varphi \in C^{\beta-\gamma}_{2+2\delta}$. □

2.3 Some estimates on the approximation of the identity ρ_ε

We shall need the following estimate proved in [4] and in a different case, but with a similar proof, in [5, Lemma 8].

Lemma 2.6 *For every $\delta > 0$, there exists $C > 0$ such that:*

$$\|\rho_\varepsilon\|_{\mathcal{B}^0_{1,2,\delta}} \leq C\sqrt{|\ln \varepsilon|}, \quad \forall \varepsilon \in \left(0, \frac{1}{2}\right). \tag{2.24}$$

Proof We give the proof for the sake of completeness. First we notice that

$$\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)(\varepsilon x) = N^2 \int_{\mathbb{R}^2} K(\varepsilon N(x - y)) \langle \varepsilon y \rangle^\delta \rho(y) dy$$

and hence it is easy to deduce

$$\begin{aligned} \varepsilon^{-2} \|\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)\|_{L^1} &\leq CN^2 \|K(\varepsilon Nx)\|_{L^1} \|\langle \varepsilon x \rangle^\delta \rho\|_{L^1} \\ &= C\varepsilon^{-2} \|K\|_{L^1} \|\langle \varepsilon x \rangle^\delta \rho\|_{L^1} \\ &\leq C\varepsilon^{-2} \|K\|_{L^1} \|\rho\|_{L^1_\delta}. \end{aligned} \tag{2.25}$$

We fix N_0 as the unique dyadic such that $N_0 \leq \frac{1}{\varepsilon} < 2N_0$ and from (2.25) we get

$$\sum_{\frac{1}{2} \leq N \leq 8N_0} \|\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)\|_{L^1}^2 \leq C|\ln \varepsilon|. \tag{2.26}$$

Next, we focus on the case $N > 8N_0$. First notice the identity

$$\varepsilon^2 \langle x \rangle^\delta \rho_\varepsilon = \sum_{M\text{-dyadic}} \Delta_M(\langle \varepsilon x \rangle^\delta \rho)\left(\frac{x}{\varepsilon}\right), \tag{2.27}$$

along with the following inclusion for the support of the Fourier transform

$$\text{supp } \mathcal{F}\left[\Delta_M(\langle \varepsilon x \rangle^\delta \rho)\left(\frac{x}{\varepsilon}\right)\right] \subset \left\{\frac{M}{2\varepsilon} \leq |\eta| \leq \frac{2M}{\varepsilon}\right\} \subset \left\{\frac{MN_0}{2} \leq |\eta| \leq 4MN_0\right\}.$$

As a consequence, we get

$$\Delta_N \left[\Delta_M(\langle \varepsilon x \rangle^\delta \rho) \left(\frac{x}{\varepsilon} \right) \right] = 0, \quad \forall N \text{ s.t. } \frac{N}{2} > 4MN_0 \text{ or } 2N < \frac{MN_0}{2}.$$

Hence, by (2.27) we get

$$\varepsilon^2 \Delta_N(\langle x \rangle^\delta \rho_\varepsilon) = \sum_{\frac{N}{8N_0} \leq M \leq \frac{4N}{N_0}} \Delta_N \left[\Delta_M(\langle \varepsilon x \rangle^\delta \rho) \left(\frac{x}{\varepsilon} \right) \right], \quad \forall N \geq 8N_0, \quad (2.28)$$

which in turn implies (since there are exactly six dyadic numbers in the interval $[\frac{N}{8N_0}, \frac{4N}{N_0}]$)

$$\begin{aligned} \|\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)\|_{L^1}^2 &\leq \left(\sum_{\frac{N}{8N_0} \leq M \leq \frac{4N}{N_0}} \varepsilon^{-2} \left\| \Delta_M(\langle \varepsilon x \rangle^\delta \rho) \left(\frac{x}{\varepsilon} \right) \right\|_{L^1} \right)^2 \\ &\leq 6 \sum_{\frac{N}{8N_0} \leq M \leq \frac{4N}{N_0}} \|\Delta_M(\langle \varepsilon x \rangle^\delta \rho)\|_{L^1}^2, \quad \forall N \geq 8N_0 \end{aligned}$$

and then

$$\sum_{N > 8N_0} \|\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)\|_{L^1}^2 \leq 36 \sum_{M \text{-dyadic}} \|\Delta_M(\langle \varepsilon x \rangle^\delta \rho)\|_{L^1}^2 \leq C \|\langle \varepsilon x \rangle^\delta \rho\|_{\mathcal{B}_{1,2}^0}^2. \quad (2.29)$$

By combining (2.26), (2.29), and the obvious bound $\sup_{\varepsilon \in (0, \frac{1}{2})} \|\langle \varepsilon x \rangle^\delta \rho\|_{\mathcal{B}_{1,2}^0} < \infty$, we conclude our estimate. □

The following estimate will also be useful.

Lemma 2.7 *For every $\beta \in (0, 1)$, $\zeta > 0$ with $\beta + \zeta \in (0, 1)$, and $\delta \geq 0$, there exists $C > 0$ such that*

$$\|\rho_\varepsilon * f\|_{C_{-\delta}^{\beta+1}} \leq C \varepsilon^{-\beta-\zeta} \|f\|_{C_{-\delta}^{1-\zeta}}, \quad \forall \varepsilon \in (0, 1). \quad (2.30)$$

Proof Arguing as in the proof of (2.23), we get

$$N^{1+\beta} \|\Delta_N(\rho_\varepsilon * f)\|_{L_{-\delta}^\infty} \leq N^{1-\zeta} \|\Delta_N f\|_{L_{-\delta}^\infty} \sum_{M=\frac{N}{2}, N, 2N} M^{\beta+\zeta} \|\Delta_M \rho_\varepsilon\|_{L_\delta^1}$$

and hence we conclude provided that we show

$$(\varepsilon N)^{\beta+\zeta} \|\Delta_N \rho_\varepsilon\|_{L_\delta^1} \leq C, \quad \forall \varepsilon \in (0, 1), \quad \forall N \text{ dyadic}. \quad (2.31)$$

By using (2.10), and since we are assuming $\beta + \zeta \in (0, 1)$, it is sufficient to prove

$$(\varepsilon N)^{\beta+\zeta} \|\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)\|_{L^1} \leq C, \quad \forall \varepsilon \in (0, 1), \quad \forall N \text{ dyadic}. \quad (2.32)$$

In order to do that we introduce the unique dyadic N_0 such that $N_0 \leq \frac{1}{\varepsilon} < 2N_0$. Notice that in case $\frac{N}{N_0} \leq 8$ the estimate above is trivial. On the other hand from (2.28), we get

$$\|\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)\|_{L^1} \leq \sum_{\frac{N}{8N_0} \leq M \leq \frac{4N}{N_0}} \|\Delta_N(\langle \varepsilon x \rangle^\delta \rho)\|_{L^1}, \quad \forall N > 8N_0,$$

and hence

$$\left(\frac{N}{N_0}\right)^{\beta+\zeta} \|\Delta_N(\langle x \rangle^\delta \rho_\varepsilon)\|_{L^1} \leq C \sup_{\varepsilon \in (0,1)} \|\langle \varepsilon x \rangle^\delta \rho\|_{\mathcal{B}_{1,\infty}^{\beta+\zeta}}, \quad \forall N > 8N_0.$$

We conclude since $\sup_{\varepsilon \in (0,1)} \|\langle \varepsilon x \rangle^\delta \rho\|_{\mathcal{B}_{1,\infty}^{\beta+\zeta}} < \infty$. □

We close the subsection with the following result.

Lemma 2.8 *For every $\alpha \in \mathbb{R}$, $\eta \in (0, 1)$, and $\delta \geq 0$, there exists $C > 0$ such that*

$$\|\rho_\varepsilon * f - f\|_{C_{-\delta}^\alpha} \leq C\varepsilon^\eta \|f\|_{C_{-\delta}^{\alpha+\eta}}, \quad \forall \varepsilon \in (0, 1). \tag{2.33}$$

Proof For $\varepsilon N \geq 1$, we have the bound

$$\begin{aligned} N^\alpha \|\Delta_N(\rho_\varepsilon * f) - \Delta_N f\|_{L_{-\delta}^\infty} &\leq N^\alpha \|\rho_\varepsilon * \Delta_N f\|_{L_{-\delta}^\infty} + N^\alpha \|\Delta_N f\|_{L_{-\delta}^\infty} \\ &\leq N^\alpha \|\rho_\varepsilon\|_{L_{-\delta}^1} \|\Delta_N f\|_{L_{-\delta}^\infty} + N^\alpha \|\Delta_N f\|_{L_{-\delta}^\infty} \\ &\leq CN^{\alpha+\eta} \varepsilon^\eta \|\Delta_N f\|_{L_{-\delta}^\infty}, \end{aligned} \tag{2.34}$$

where we used $\langle x \rangle^{-1} \leq 2\langle y \rangle \langle x - y \rangle^{-1}$. To deal with the case $\varepsilon N < 1$, we notice that

$$\langle x \rangle^{-\delta} \left(\Delta_N(\rho_\varepsilon * f) - \Delta_N f \right) = \int_{\mathbb{R}^2} \tilde{K}_{\varepsilon,N}(x, y) \langle y \rangle^{-\delta} f(y) dy$$

where

$$\begin{aligned} \tilde{K}_{\varepsilon,N}(x, y) &= N^2 \frac{\langle y \rangle^\delta}{\langle x \rangle^\delta} \left(\int_{\mathbb{R}^2} K(Nx - Nz) \rho_\varepsilon(z - y) dz - K(Nx - Ny) \right) \\ &= N^2 \frac{\langle y \rangle^\delta}{\langle x \rangle^\delta} \int_{\mathbb{R}^2} \left(K(Nx - Nz) - K(Nx - Ny) \right) \rho_\varepsilon(z - y) dz. \end{aligned}$$

Next, for fixed x , we compute

$$\begin{aligned}
 & \int_{\mathbb{R}^2} |\tilde{K}_{\varepsilon,N}(x, y)| dy \\
 & \leq \frac{N^2}{\langle x \rangle^\delta} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |K(Nx - Nz) - K(Nx - Ny)| \rho_\varepsilon(z - y) \langle y \rangle^\delta dy dz \\
 & = \frac{(N\varepsilon)^2}{\langle x \rangle^\delta} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |K(Nx - N\varepsilon z) - K(Nx - N\varepsilon y)| \rho(z - y) \langle \varepsilon y \rangle^\delta dy dz \\
 & \leq \frac{(N\varepsilon)^3}{\langle x \rangle^\delta} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\sup_{[Nx - N\varepsilon z, Nx - N\varepsilon y]} |\nabla K| \right) |z - y| \rho(z - y) \langle \varepsilon y \rangle^\delta dy dz
 \end{aligned} \tag{2.35}$$

and notice that for every $\bar{x} \in \mathbb{R}^2$ and $\lambda \in (0, 1)$, we get

$$\begin{aligned}
 & \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\sup_{[\bar{x} - \lambda z, \bar{x} - \lambda y]} |\nabla K| \right) |z - y| \rho(z - y) \langle \varepsilon y \rangle^\delta dy dz \\
 & \leq C \int_{\mathbb{R}^2} \int_{|z - y| < 1} \left(\sup_{B(\bar{x} - \lambda z, \lambda)} |\nabla K| \right) \langle \varepsilon z \rangle^\delta dz dy \\
 & \leq C \int_{\mathbb{R}^2} \left(\sup_{B(\bar{x} - \lambda z, 1)} |\nabla K| \right) \langle \varepsilon z \rangle^\delta dz.
 \end{aligned}$$

Here, we denoted by $[a, b]$ the segment between a and b , $B(x, r)$ the ball in \mathbb{R}^2 centered in x of radius r and we used the inclusion $[\bar{x} - \lambda z, \bar{x} - \lambda y] \subset B(\bar{x} - \lambda z, 1)$ since $|z - y| < 1$ and $\lambda \in (0, 1)$. In particular, by introducing the function

$$G(w) = \sup_{B(w, 1)} |\nabla K|, \tag{2.36}$$

we get from the estimate above

$$\begin{aligned}
 & \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\sup_{[\bar{x} - \lambda z, \bar{x} - \lambda y]} |\nabla K| \right) |z - y| \rho(z - y) \langle \varepsilon z \rangle^\delta dy dz \\
 & \leq \int_{\mathbb{R}^2} G(\bar{x} - \lambda z) \langle \varepsilon z \rangle^\delta dz \\
 & = \lambda^{-2} \int_{\mathbb{R}^2} G(u) \langle \varepsilon \lambda^{-1} (\bar{x} - u) \rangle^\delta du.
 \end{aligned} \tag{2.37}$$

By combining (2.35) with (2.37), where we choose $\lambda = \varepsilon N$ and $\bar{x} = Nx$, we conclude

$$\sup_{x, N} \int_{\mathbb{R}^2} |\tilde{K}_{\varepsilon,N}(x, y)| dy = O(\varepsilon N), \tag{2.38}$$

where we used

$$\sup_{\bar{x}, N} \int_{\mathbb{R}^2} G(u) \langle N^{-1}(\bar{x} - u) \rangle^\delta du < \infty.$$

Similarly, one can show

$$\sup_{y, N} \int_{\mathbb{R}^2} |\tilde{K}_{\varepsilon, N}(x, y)| dx = O(\varepsilon N). \tag{2.39}$$

In fact for y fixed we get:

$$\begin{aligned} & \int_{\mathbb{R}^2} |\tilde{K}_{\varepsilon, N}(x, y)| dx \\ &= N^2 \langle y \rangle^\delta \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |K(Nx - N\varepsilon z - Ny) - K(Nx - Ny)| \rho(z) \langle x \rangle^{-\delta} dx dz \\ &\leq C\varepsilon N^3 \langle y \rangle^\delta \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\sup_{[Nx - N\varepsilon z - Ny, Nx - Ny]} |\nabla K| \right) |z| \rho(z) \langle x \rangle^{-\delta} dx dz \\ &\leq C\varepsilon N^3 \langle y \rangle^\delta \int_{\mathbb{R}^2} \int_{|z| < 1} G(N(x - y)) \langle x \rangle^{-\delta} dx dz, \end{aligned}$$

where the function $G(w)$ is defined in (2.36) and we used the inclusion

$$[Nx - N\varepsilon z - Ny, Nx - Ny] \subset B((Nx - Ny), 1)$$

for $|z| < 1$ and $\varepsilon N < 1$. By a change of variable, we conclude

$$\int_{\mathbb{R}^2} |\tilde{K}_{\varepsilon, N}(x, y)| dx \leq C\varepsilon N \langle y \rangle^\delta \int_{\mathbb{R}^2} G(x) \langle \frac{x}{N} + y \rangle^{-\delta} dx \leq C\varepsilon N \int_{\mathbb{R}^2} G(x) \langle x \rangle^\delta dx.$$

Summarizing by (2.38) and (2.39) we can apply the Schuur Lemma and we get

$$\| \Delta_N(\rho_\varepsilon * f) - \Delta_N f \|_{L^\infty_{-\delta}} \leq C\varepsilon N \| \Delta_N f \|_{L^\infty_{-\delta}}, \quad \varepsilon N < 1,$$

which in turn implies

$$\| \Delta_N(\rho_\varepsilon * f) - \Delta_N f \|_{L^\infty_{-\delta}} \leq C\varepsilon^\eta N^\eta \| \Delta_N f \|_{L^\infty_{-\delta}}, \quad \varepsilon N < 1.$$

We conclude by combining this estimate with (2.34). □

2.4 Embeddings and product rules

We first record the useful interpolation inequality in weighted Besov spaces (see [36, Theorem 3.8])

Lemma 2.9 *Let $p_0, q_0, p_1, q_1, p, q \in [1, \infty], \alpha_0, \alpha_1, \mu_0, \mu_1, \alpha, \mu \in \mathbb{R}$ be such that*

$$\begin{aligned} \frac{1}{p} &= \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1}, \quad \frac{1}{q} = \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1}, \\ \alpha &= (1 - \Theta)\alpha_0 + \Theta\alpha_1, \quad \mu = (1 - \Theta)\mu_0 + \Theta\mu_1 \end{aligned}$$

for a suitable $\Theta \in [0, 1]$. Then there exists $C > 0$ such that

$$\|f\|_{\mathcal{B}_{p,q,\mu}^\alpha} \leq C \|f\|_{\mathcal{B}_{p_0,q_0,\mu_0}^{1-\Theta}} \|f\|_{\mathcal{B}_{p_1,q_1,\mu_1}^\Theta}. \tag{2.40}$$

Next, we state a Sobolev embedding for weighted Sobolev spaces. The proof follows from the corresponding unweighted version along with (2.3).

Lemma 2.10 *Let $p \in [2, \infty), \alpha \geq 1 - \frac{2}{p}$, and $\mu, \nu \in \mathbb{R}$ such that $\nu \leq \mu$. Then, we have the continuous embedding*

$$H_\mu^\alpha \subset L_\nu^p. \tag{2.41}$$

Moreover for every $\alpha > 1$ we have

$$H_\mu^\alpha \subset L_\nu^\infty. \tag{2.42}$$

As a consequence, we can prove the following estimate.

Lemma 2.11 *For every $q \in [2, \infty)$ and $\delta \in (0, 1]$, there exists $C > 0$ such that*

$$\|f\|_{W_\delta^{1,q}} \leq C \|f\|_{H_{-\delta}^2}^{1-\frac{1}{q}} \|f\|_{L_{(2q-1)\delta}^2}^{\frac{1}{q}}. \tag{2.43}$$

Proof We have the following chain of estimates

$$\begin{aligned} \|f\|_{W_\delta^{1,q}} &\leq C \|f\|_{H_\delta^{2-\frac{2}{q}}} \\ &\leq C \|f\|_{H_{-\delta}^{2-\frac{2}{q}}} \|f\|_{H_{(q-1)\delta}^1}^{\frac{2}{q}} \\ &\leq C \|f\|_{H_{-\delta}^2}^{1-\frac{2}{q}} \|f\|_{H_{-\delta}^2}^{\frac{1}{q}} \|f\|_{L_{(2q-1)\delta}^2}^{\frac{1}{q}} \end{aligned}$$

where the first inequality is a consequence of (2.17) and (2.41). Moreover we have also used twice special cases of (2.40). □

We shall also need the following product estimate (see [2] or [35]).

Lemma 2.12 *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 > 0, \alpha = \min(\alpha_1, \alpha_2, \alpha_1 + \alpha_2), \mu_1, \mu_2 \in \mathbb{R}, \mu = \mu_1 + \mu_2, p_1, p_2 \in [1, \infty]$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, for any $\kappa > 0$, we have*

$$\|f_1 \cdot f_2\|_{\mathcal{B}_{p,p,\mu}^{\alpha-\kappa}} \leq C \|f_1\|_{\mathcal{B}_{p_1,p_1,\mu_1}^{\alpha_1}} \|f_2\|_{\mathcal{B}_{p_2,p_2,\mu_2}^{\alpha_2}}. \tag{2.44}$$

Also, we have

$$\|f_1 \cdot f_2\|_{C_\mu^\alpha} \leq C \|f_1\|_{C_{\mu_1}^{\alpha_1}} \|f_2\|_{C_{\mu_2}^{\alpha_2}}. \tag{2.45}$$

The following duality estimate will also be useful (see [39, Theorem 2.11.2]).

Lemma 2.13 *Let $\alpha, \mu \in \mathbb{R}$ and $p, q \in [1, \infty)$. Then, we have*

$$\left| \int_{\mathbb{R}^2} f_1 \cdot f_2 dx \right| \leq C \|f_1\|_{B_{p,q,\mu}^\alpha} \|f_2\|_{B_{p',q',-\mu}^{-\alpha}}, \tag{2.46}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

3 Stochastics bounds

The following results are improvements of some results from [20] and [12], where similar estimates were given in terms of moments. In this paper, these estimates hold almost surely.

3.1 Estimates in classical spaces

We first show some estimates in classical spaces.

Proposition 3.1 *Given $\delta \in (0, 1)$, $r \in (2, \infty)$ such that $\delta \cdot r > 2$, and $\varepsilon \in (0, \frac{1}{2})$, we have*

$$\|\nabla Y_\varepsilon\|_{L_{-\delta}^r}^2 + \|\nabla Y_\varepsilon^2\|_{L_{-\delta}^r} \leq C(\omega) |\ln \varepsilon|, \quad \text{a.s. } \omega. \tag{3.1}$$

Moreover, for $\alpha \in (0, 1)$, $\delta > 0$, $\beta \in \mathbb{R}$, $\varepsilon \in (0, 1)$, and $\varphi \in C_c^\infty(\mathbb{R}^2)$, there exists $\kappa \in (0, 1)$ such that

$$\|Y_\varepsilon - Y\|_{C_{-\delta}^\alpha} + \|\varphi * \xi_\varepsilon - \varphi * \xi\|_{C_{-\delta}^\beta} \leq C(\omega) \varepsilon^\kappa, \quad \text{a.s. } \omega. \tag{3.2}$$

To prove Proposition 3.1, we shall use the following result (see [23, Proposition 3.1] and [44, Proposition 2.3]).

Lemma 3.2 *Let $(X, \|\cdot\|_X)$ be a separable Banach space and (η_n) be a sequence of X -valued random variables. Assume that there exists a sequence σ_n of real numbers such that:*

$$\mathbb{E}\left((\eta_n, f)^2\right) \leq \sigma_n^2 \|f\|_{X'}^2, \quad \forall f \in X'.$$

Then

$$\mathbb{E}\left(\sup_n \|\eta_n\|_X\right) \leq \sup_n \mathbb{E}(\|\eta_n\|_X) + 3\rho(\sigma_n),$$

where

$$\rho(\sigma_n) = \inf \left\{ \delta \mid \sum_n \sigma_n^2 \delta^{-2} \exp(-2^{-1}(\delta \sigma_n^{-1})^2) \leq 1 \right\}.$$

Remark 3.3 As recalled in [44], if $\sigma_n \leq \alpha^n$ then $\rho(\sigma_n) \sim \sqrt{\ln(1 - \alpha)^{-1}}$.

We also need the following result, which is an immediate consequence of [12, Lemma 2.5].

Lemma 3.4 For any $\alpha \in (0, 1)$ and $\delta > 0$, we have

$$\|Y\|_{C_{-\delta}^\alpha} + \|\xi\|_{C_{-\delta}^{\alpha-2}} \leq C(\omega), \quad a.s. \omega. \tag{3.3}$$

Proof of Prop. 3.1 We split the proof in several steps. First we show the following property:

$$\mathbb{E} \left(\|\nabla Y\|_{B_{r, \infty, -\delta}^0} \right) < \infty, \quad \delta \cdot r > 2, \quad r \in (2, \infty), \tag{3.4}$$

whose proof is a generalization of [44, Theorem 3.4].

For every fixed $x \in \mathbb{R}^2$, we have that $\Delta_N \xi(x)$ is a Gaussian random random variable and hence

$$\begin{aligned} \mathbb{E} \left(\|\Delta_N \xi\|_{L_{r, \delta}^r}^r \right) &= \int_{\mathbb{R}^2} \mathbb{E}(|\Delta_N \xi(x)|^r) \langle x \rangle^{-\delta r} dx \\ &\leq C \int_{\mathbb{R}^2} \left(\mathbb{E}|\Delta_N \xi(x)|^2 \right)^{r/2} \langle x \rangle^{-\delta r} dx. \end{aligned}$$

Moreover, we have

$$\mathbb{E}(|\Delta_N \xi(x)|^2) = N^4 \mathbb{E} \left[\left(\xi, K(N(x - \cdot)) \right)^2 \right] = N^4 \|K(N(x - \cdot))\|_{L^2}^2 = N^2 \|K\|_{L^2}^2$$

and hence

$$\mathbb{E} \left(\|\Delta_N \xi\|_{L_{r, \delta}^r} \right) \leq CN \tag{3.5}$$

provided that $\delta \cdot r > 2$. Also, for $f \in L_{\delta}^{r'}$, where $\frac{1}{r} + \frac{1}{r'} = 1$, we get

$$\begin{aligned} \mathbb{E} \left((\Delta_N \xi, f)^2 \right) &= \mathbb{E} \left((\xi, \Delta_N f)^2 \right) \\ &= N^4 \|K(N \cdot) * f\|_{L^2}^2 \\ &\leq N^4 \|K(N \cdot)\|_{L^{\frac{2r}{r+2}}}^2 \|f\|_{L^{r'}}^2 \\ &\leq CN^{\frac{2(r-2)}{r}} \|f\|_{L_{\delta}^{r'}}^2. \end{aligned} \tag{3.6}$$

By using the fact that $L_{\delta}^{r'} = (L_{-\delta}^r)'$ and (3.5), we can apply Lemma 3.2, where we choose: $X = L_{-\delta}^r, \eta_N = N^{-1} \Delta_N \xi, \sigma_N = N^{-\frac{1}{r}}$ (see (3.6)). Hence, we obtain

$$\mathbb{E}(\|\xi\|_{B_{r,\infty,-\delta}^{-1}}) = \mathbb{E}\left(\sup_N N^{-1} \|\Delta_N \xi\|_{L_{-\delta}^r}\right) \leq C + C\rho(N^{-\frac{2}{r}}) \leq C. \tag{3.7}$$

where we have used Remark 3.3 at the last step. The estimate (3.4) clearly follows from (3.7).

Next, we claim that

$$\|\rho_{\varepsilon} * \nabla Y\|_{B_{r,2,-\delta}^0} \leq C \|\rho_{\varepsilon}\|_{B_{1,2,\delta}^0} \|\nabla Y\|_{B_{r,\infty,-\delta}^0}. \tag{3.8}$$

Once this estimate is established, by combining (3.4) with (2.24), we get

$$\|\nabla Y_{\varepsilon}\|_{L_{-\delta}^r}^2 \leq C(\omega) |\ln \varepsilon|$$

and in turn also

$$\|:\nabla Y_{\varepsilon}^2:\|_{L_{-\delta}^r}^2 \leq C(\omega) |\ln \varepsilon|$$

since $\nabla Y_{\varepsilon}^2 := \nabla Y_{\varepsilon}^2 - c_{\varepsilon}$ with $c_{\varepsilon} \sim |\ln \varepsilon|$. Hence, (3.1) follows from (3.8), whose proof without weight is in [28, Theorem 2.2]. Here, by using $(\Delta_{\frac{N}{2}} + \Delta_N + \Delta_{2N}) \circ \Delta_N = \Delta_N$, we get

$$\begin{aligned} \|\rho_{\varepsilon} * \nabla Y\|_{B_{r,2,-\delta}^0}^2 &= \sum_{N\text{-dyadic}} \|\Delta_N(\rho_{\varepsilon} * \nabla Y)\|_{L_{-\delta}^r}^2 \\ &= \sum_{N\text{-dyadic}} \sum_{M=\frac{N}{2}, N, 2N} \|\Delta_M \rho_{\varepsilon} * \Delta_N(\nabla Y)\|_{L_{-\delta}^r}^2 \\ &\leq C \sum_{N\text{-dyadic}} \sum_{M=\frac{N}{2}, N, 2N} \|\Delta_M \rho_{\varepsilon}\|_{L_{\delta}^1}^2 \|\Delta_N \nabla Y\|_{L_{-\delta}^r}^2 \\ &\leq C \|\rho_{\varepsilon}\|_{B_{1,2,\delta}^0}^2 \|\nabla Y\|_{B_{r,\infty,-\delta}^0}^2, \end{aligned}$$

where we have used the inequality $\langle x \rangle^{-1} \leq 2\langle y \rangle \langle x - y \rangle^{-1}$. Thus, (3.8) follows.

Then, we consider (3.2). The bound

$$\|\varphi * \xi_{\varepsilon} - \varphi * \xi\|_{C_{-\delta}^{\beta}} \leq C(\omega) \varepsilon^{\kappa}, \quad \text{a.s. } \omega$$

follows by combining (2.33) with (2.23) and (3.3). In order to conclude the proof of (3.2), we notice that again by (2.33) and (3.3) we get

$$\|Y_{\varepsilon} - Y\|_{C_{-\delta}^{\alpha}} \leq C(\omega) \varepsilon^{\kappa}, \quad \text{a.s. } \omega,$$

as desired. □

We also establish the following uniform bound and convergence result for e^{aY_ε} for any $a \in \mathbb{R}$.

Proposition 3.5 *For $\alpha \in (0, 1)$, $\delta > 0$, and $a \in \mathbb{R}$, we have*

$$\sup_{\varepsilon \in (0,1)} \|e^{aY_\varepsilon}\|_{C^\alpha_{-\delta}} \leq C(\omega), \quad \text{a.s. } \omega. \tag{3.9}$$

Moreover, there exists $\kappa \in (0, 1)$ such that

$$\|e^{aY_\varepsilon} - e^{aY}\|_{L^\infty_{-\delta}} \leq C(\omega)\varepsilon^\kappa, \quad \text{a.s. } \omega. \tag{3.10}$$

To prove Proposition 3.5, we shall use the following lemma (see [12, Corollary 2.6]).

Lemma 3.6 *For any $\alpha \in (0, 1)$, $a \in \mathbb{R}$, and $\delta > 0$, we have*

$$\|e^{aY}\|_{C^\alpha_{-\delta}} \leq C(\omega), \quad \text{a.s. } \omega. \tag{3.11}$$

We also need the following result (see [12, Lemma 2.3] and [1, Lemma 5.3]). Below we shall use the functions $\chi_k \in C_c^\infty(\mathbb{R}^2)$ with $k \in \mathbb{N}$, $\text{supp } \chi_k \subseteq [-k - 1, k + 1]^2$, and $\chi_k = 1$ on $[-k, k]^2$.

Lemma 3.7 *For $\alpha < 1$, there exist $\lambda, \lambda' > 0$ such that*

$$\sup_{k \in \mathbb{N}} \frac{\mathbb{E}\left(\exp(\lambda \|\chi_k \xi\|_{C^{\alpha-2}}^2)\right)}{k^{\lambda'}} \leq C. \tag{3.12}$$

Proof of Prop. 3.5 We first establish the uniform bound (3.9), whose proof is similar to that of Corollary 2.6 in [12]. For every $k \in \mathbb{N}$, we easily get

$$\begin{aligned} \|Y_\varepsilon\|_{C^\alpha([-k,k]^2)} &= \|\rho_\varepsilon * Y\|_{C^\alpha([-k,k]^2)} \\ &\leq \|Y\|_{C^\alpha([-k-1,k+1]^2)} \\ &\leq C \|G * \chi_{k+2\xi}\|_{C^\alpha} \\ &\leq C \|\chi_{k+2\xi}\|_{C^{\alpha-2}} \end{aligned} \tag{3.13}$$

uniformly in $\varepsilon \in (0, 1)$, where in the second inequality we used $Y = G * \xi$ and the fact that $\text{supp } G \subseteq B(0, 1)$, and in the last inequality we used a Schauder estimate (see [37]). We also note that

$$\|e^{aY_\varepsilon}\|_{C^\alpha_{-\delta}} \leq C \sup_{k \in \mathbb{N}} \frac{\|e^{aY_\varepsilon}\|_{C^\alpha([-k,k]^2)}}{k^\delta} \leq C \sup_{k \in \mathbb{N}} \frac{\exp(C|a|\|Y_\varepsilon\|_{C^\alpha([-k,k]^2)})}{k^\delta}. \tag{3.14}$$

Thus, for $p > 1$ big enough, by (3.14), (3.13), and (3.12), we get

$$\mathbb{E}\left(\sup_{\varepsilon \in (0,1)} \|e^{aY_\varepsilon}\|_{C^\alpha_{-\delta}}^p\right) \leq C \mathbb{E}\left(\left|\sup_{\varepsilon \in (0,1)} \sup_{k \in \mathbb{N}} \frac{\exp(C|a|\|Y_\varepsilon\|_{C^\alpha([-k,k]^2)})}{k^\delta}\right|^p\right)$$

$$\begin{aligned}
 &\leq C \mathbb{E} \left(\left| \sup_{k \in \mathbb{N}} \frac{\exp(C|a| \|\chi_{k+2\xi}\|_{C^{\alpha-2}})}{k^\delta} \right|^p \right) \\
 &\leq C \sum_{k=1}^\infty \frac{\mathbb{E} \left(\exp(pC|a| \|\chi_k \xi\|_{C^{\alpha-2}}) \right)}{k^{\delta p}} \\
 &\leq C \sum_{k=1}^\infty \frac{\mathbb{E} \left(\exp(\lambda \|\chi_k \xi\|_{C^{\alpha-2}}^2) \right)}{k^{2+\lambda'}} \\
 &\leq C,
 \end{aligned}$$

where in the last step we used $\exp(pC|a|x) \leq C \exp(\lambda x^2)$ and we picked p big enough such that $\delta p \geq 2 + \lambda'$. The bound (3.9) then follows.

We now consider (3.10). By (3.2), (3.9), and (3.11), we get

$$\|e^{aY_\varepsilon} - e^{aY}\|_{L^\infty_{-\delta}} \leq C \|Y_\varepsilon - Y\|_{L^\infty_{-\frac{\delta}{2}}} \left(\|e^{aY_\varepsilon}\|_{L^\infty_{-\frac{\delta}{2}}} + \|e^{aY}\|_{L^\infty_{-\frac{\delta}{2}}} \right) \leq C(\omega)\varepsilon^\kappa,$$

which is the desired estimate. □

3.2 Estimates in spaces at negative regularity

The main point of this subsection is the following result where the convergence in negative regularity occurs almost surely in $\omega \in \Omega$.

Proposition 3.8 *For $\alpha \in (0, 1)$, $\delta > 0$, and $\varepsilon \in (0, \frac{1}{2})$, there exists $\kappa \in (0, 1)$ such that*

$$\|\nabla Y_\varepsilon - \nabla Y\|_{C^{\alpha-1}_{-\delta}} + \|\widetilde{:\nabla Y_\varepsilon^2:} - \widetilde{:\nabla Y^2:}\|_{C^{\alpha-1}_{-\delta}} \leq C(\omega)\varepsilon^\kappa, \quad \text{a.s. } \omega. \tag{3.15}$$

Moreover, we have

$$\|\widetilde{:\nabla Y_\varepsilon^2:} - \widetilde{:\nabla Y^2:}\|_{C^{\alpha-1}_{-\delta}} \leq C(\omega)\varepsilon^\kappa, \quad \text{a.s. } \omega, \tag{3.16}$$

where $\widetilde{:\nabla Y_\varepsilon^2:}$ and $\widetilde{:\nabla Y^2:}$ are defined in (1.10) and (1.8), respectively.

To prove Proposition 3.8, we need the following result which follows from [20].

Lemma 3.9 *For any $\alpha \in (0, 1)$ and $\delta > 0$, we have the bound*

$$\|\nabla Y\|_{C^{\alpha-1}_{-\delta}} + \|\widetilde{:\nabla Y^2:}\|_{C^{\alpha-1}_{-\delta}} \leq C(\omega), \quad \text{a.s. } \omega. \tag{3.17}$$

Proof of Proposition 3.8 The estimate $\|\nabla Y_\varepsilon - \nabla Y\|_{C^{\alpha-1}_{-\delta}} \leq C(\omega)\varepsilon^\kappa$ follows by combining (2.33) with $\|\nabla Y\|_{C^{\alpha-1}_{-\delta}} \leq C(\omega)$ (see (3.17)). Also, the estimate (3.16) follows immediately from (3.15) and (3.2)

Hence, we focus on the proof of $\|:\nabla Y_\varepsilon^2:-:\nabla Y^2:\|_{C_{-\delta}^{\alpha-1}} \leq C(\omega)\varepsilon^\kappa$. The argument is a little more complicated since Wick products cannot be estimated pathwise. It is shown in [20] that there exists $\kappa_0 > 0$ such that for all $k \geq 1$ we have:

$$\mathbb{E} \left(\|:\nabla Y_\varepsilon^2:-:\nabla Y^2:\|_{C_{-\delta}^{\alpha-1}}^k \right) \leq C\varepsilon^{k\kappa_0}. \tag{3.18}$$

Recall (1.11) and notice that by elementary considerations we have:

$$\begin{aligned} |c_\varepsilon - c_\eta| &\leq \|(\rho_\varepsilon - \rho_\eta) * \nabla G\|_{L^2} \left(\|\rho_\varepsilon * \nabla G\|_{L^2} + \|\rho_\eta * \nabla G\|_{L^2} \right) \\ &\leq C |\ln \varepsilon|^{\frac{1}{2}} \|\rho_\varepsilon - \rho_\eta\|_{L^q}, \quad \forall q \in (2, \infty), \quad 0 < \varepsilon < \eta < \frac{1}{2}, \end{aligned} \tag{3.19}$$

where we used $\nabla G \in L^p$ for any $p \in (1, 2)$, along with Young convolution inequality and the classical bound $\|\rho_\varepsilon * \nabla G\|_{L^2}^2 = c_\varepsilon \sim |\ln \varepsilon|$ when $\varepsilon \rightarrow 0$. By combining the following inequality (whose proof is elementary)

$$\|\rho_\varepsilon - \rho_\eta\|_{L^q} \leq C\varepsilon^{-3+2/q} |\varepsilon - \eta|, \quad 0 < \varepsilon < \eta < \frac{1}{2}, \quad q \geq 1, \tag{3.20}$$

with (3.19), we get

$$|c_\varepsilon - c_\eta| \leq C\varepsilon^{-\gamma} |\varepsilon - \eta|, \quad 0 < \varepsilon < \eta < \frac{1}{2}, \quad \gamma > 1. \tag{3.21}$$

Moreover, by (2.45) and the estimate $\|\nabla G * f\|_{C_\chi^\gamma} \leq C\|f\|_{C_\chi^{\gamma-1}}$ ($\gamma, \chi \in \mathbb{R}$) from [20], we obtain that for $1 > \beta > 1 - \alpha$ and $p > \frac{2}{2-\alpha}$,

$$\begin{aligned} &\|\nabla Y_\varepsilon^2 - \nabla Y_\eta^2\|_{C_{-\delta}^{\alpha-1}} \\ &\leq C\|\nabla Y_\varepsilon - \nabla Y_\eta\|_{C_{-\frac{\delta}{2}}^{\alpha-1}} \left(\|\nabla Y_\varepsilon\|_{C_{-\frac{\delta}{2}}^\beta} + \|\nabla Y_\eta\|_{C_{-\frac{\delta}{2}}^\beta} \right) \\ &\leq C\|\xi_\varepsilon - \xi_\eta\|_{C_{-\frac{\delta}{2}}^{\alpha-2}} \left(\|Y_\varepsilon\|_{C_{-\frac{\delta}{2}}^{\beta+1}} + \|Y_\eta\|_{C_{-\frac{\delta}{2}}^{\beta+1}} \right) \\ &\leq C\|\xi_\varepsilon - \xi_\eta\|_{L_{-\frac{\delta}{2}}^p} \left(\|Y_\varepsilon\|_{C_{-\frac{\delta}{2}}^{\beta+1}} + \|Y_\eta\|_{C_{-\frac{\delta}{2}}^{\beta+1}} \right), \end{aligned} \tag{3.22}$$

where we used at the last step $L_{-\frac{\delta}{2}}^p \subset C_{-\frac{\delta}{2}}^{\alpha-2}$ for $p > \frac{2}{2-\alpha}$. In fact, this embedding comes from the following computation (recall (2.3)):

$$\sup_N \left(N^{\alpha-2} \|\Delta_N(\langle x \rangle^{-\frac{\delta}{2}} f)\|_{L^\infty} \right) \leq C \sup_N N^{\alpha-2} \|\Delta_N(\langle x \rangle^{-\frac{\delta}{2}} f)\|_{W^{2-\alpha,p}} \leq C\|f\|_{L_{-\frac{\delta}{2}}^p},$$

where we used the classical Sobolev embedding $W^{2-\alpha,p} \subset L^\infty$ for $(2 - \alpha) \cdot p > 2$.

Now, using the Gaussianity of ξ and Minkowski’s inequality, we have the following estimates for $k \geq p$:

$$\begin{aligned} \mathbb{E} \left(\|\xi_\varepsilon - \xi_\eta\|_{L^p_{-\frac{\delta}{2}}}^k \right) &\leq \left(\int_{\mathbb{R}^2} \langle x \rangle^{-\frac{p\delta}{2}} \mathbb{E} \left(|\xi_\varepsilon(x) - \xi_\eta(x)|^k \right)^{\frac{p}{k}} dx \right)^{\frac{k}{p}} \\ &\leq C \left(\int_{\mathbb{R}^2} \langle x \rangle^{-\frac{p\delta}{2}} \mathbb{E} \left(|\xi_\varepsilon(x) - \xi_\eta(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{k}{p}} \tag{3.23} \\ &= C \|\rho_\varepsilon - \rho_\eta\|_{L^2}^k \\ &\leq C \varepsilon^{-2k} |\varepsilon - \eta|^k, \end{aligned}$$

where $C > 0$ depends on k and we have used (3.20) at the last step.

Then, by combining (3.22), (3.23), and (2.30), using Cauchy-Schwarz and [12, Lemma 2.7], we have that for any k large enough,

$$\mathbb{E} \left(\|\nabla Y_\varepsilon^2 - \nabla Y_\eta^2\|_{C^{\alpha-1}_{-\delta}}^k \right) \leq C \varepsilon^{-(2+\beta+\zeta)k} |\varepsilon - \eta|^k. \tag{3.24}$$

By combining (3.24) with (3.21) and recalling (1.11), we get

$$\mathbb{E} \left(\|\nabla Y_\varepsilon^2: - : \nabla Y_\eta^2: \|_{C^{\alpha-1}_{-\delta}}^k \right) \leq C \varepsilon^{-(2+\beta+\zeta)k} |\varepsilon - \eta|^k. \tag{3.25}$$

On the other hand, by (3.18), we have

$$\mathbb{E} \left(\|\nabla Y_\varepsilon^2: - : \nabla Y_\eta^2: \|_{C^{\alpha-1}((x)-\delta)}^k \right) \leq C \eta^{k\kappa_0}, \quad 0 < \varepsilon < \eta < \frac{1}{2}. \tag{3.26}$$

Let us now consider several cases.

Case 1: $2\varepsilon < \eta$. We have necessarily $\eta < 2|\varepsilon - \eta|$ and by (3.26),

$$\mathbb{E} \left(\|\nabla Y_\varepsilon^2: - : \nabla Y_\eta^2: \|_{C^{\alpha-1}((x)-\delta)}^k \right) \leq C |\varepsilon - \eta|^{k\kappa_0} \leq C |\varepsilon - \eta|^{\frac{k\kappa_0}{2+\kappa_0+\beta+\zeta}}.$$

Case 2: $\varepsilon < \eta \leq 2\varepsilon$ and $\varepsilon < |\varepsilon - \eta|^{\frac{1}{\kappa_0+2+\beta+\zeta}}$. Then, again by (3.26) we get

$$\mathbb{E} \left(\|\nabla Y_\varepsilon^2: - : \nabla Y_\eta^2: \|_{C^{\alpha-1}((x)-\delta)}^k \right) \leq C \varepsilon^{k\kappa_0} \leq C |\varepsilon - \eta|^{\frac{k\kappa_0}{2+\kappa_0+\beta+\zeta}}.$$

Case 3: $\varepsilon < \eta \leq 2\varepsilon$ and $\varepsilon \geq |\varepsilon - \eta|^{\frac{1}{\kappa_0+2+\beta+\zeta}}$. In this case, we use (3.25) to get

$$\mathbb{E} \left(\|\nabla Y_\varepsilon^2: - : \nabla Y_\eta^2: \|_{C^{\alpha-1}((x)-\delta)}^k \right) \leq C \varepsilon^{-(2+\beta+\zeta)k} |\varepsilon - \eta|^k \leq C |\varepsilon - \eta|^{\frac{k\kappa_0}{2+\kappa_0+\beta+\zeta}}.$$

Summarizing, we get

$$\mathbb{E} \left(\|\nabla Y_\varepsilon^2 - \nabla Y_\eta^2\|_{C_{-\delta}^{\alpha-1}}^k \right) \leq C|\varepsilon - \eta|^{\frac{k\kappa_0}{2+\kappa_0+\beta+\zeta}}, \quad \forall \varepsilon, \eta \in \left(0, \frac{1}{2}\right).$$

It remains to choose k large enough so that $\frac{k\kappa_0}{2+\kappa_0+\beta+\zeta} > 1$ and we may invoke Kolmogorov continuity criterion (see [11, Theorem 3.3]) to deduce that $\varepsilon \mapsto \nabla Y_\varepsilon^2$: from $[0, 1]$ to $C_{-\delta}^{\alpha-1}$ is almost surely Hölder continuous of exponent arbitrarily less than $\frac{\kappa_0}{2+\kappa_0} - \frac{1}{k}$ on $[0, 1]$. The proof is complete. \square

4 Linear estimates

We introduce the propagator $S_{A,V}(t)$ associated with

$$i \partial_t w = \Delta w - 2\nabla A \cdot \nabla w + Vw. \tag{4.1}$$

We also denote for shortness

$$H_{A,V} = \Delta - 2\nabla A \cdot \nabla + V. \tag{4.2}$$

In the sequel we shall assume that A and V satisfy:

$$\forall \delta > 0, \exists C > 0 \text{ s.t. } \|\langle x \rangle^{-\delta} e^{\pm A}\|_{L^\infty} \leq C; \tag{4.3}$$

$$\forall \delta > 0, \exists C > 0 \text{ s.t. } \left| \int_{\mathbb{R}^2} V|\varphi|^2 e^{-2A} dx \right| \leq C\|\varphi\|_{H_\delta^{\frac{1}{2}}}^2, \quad \forall \varphi \in H_\delta^{\frac{1}{2}}. \tag{4.4}$$

Under the assumption (4.3), we have that by (2.3) and classical elliptic regularity,

$$\forall \delta > 0, \exists c, C > 0 \text{ s.t. } c\|u\|_{H_\delta^2} \leq \|\Delta u e^{-A}\|_{L^2} + \|u e^{-A}\|_{L^2} \leq C\|u\|_{H_\delta^2}. \tag{4.5}$$

Here, we emphasize that the constants c and C in (4.5) depend only on the constant C in (4.3). It is easy to check that any solution to (4.1) satisfies the following conservation laws:

$$\frac{d}{dt} \int_{\mathbb{R}^2} |w(t)|^2 e^{-2A} dx = 0; \tag{4.6}$$

$$\frac{d}{dt} \int_{\mathbb{R}^2} \left(|\nabla w(t)|^2 e^{-2A} - V|w(t)|^2 e^{-2A} \right) dx = 0. \tag{4.7}$$

Next we associate to any couple (A, V) the following quantity for any given $\delta > 0$, $r \in (2, \infty)$:

$$\begin{aligned}
 |(A, V)|_{\delta,r} &= \|\langle x \rangle^{-\delta} \nabla A e^{-A}\|_{L^r} + \|\langle x \rangle^{-2\delta} \nabla A e^{-2A} V\|_{L^{\frac{r}{2}}} \\
 &\quad + \|\langle x \rangle^{-\delta} V e^{-A}\|_{L^r} + \|\langle x \rangle^{-\delta} e^{-(p+2)A}\|_{L^\infty} \\
 &\quad + \|\langle x \rangle^{-\delta} V e^{-(p+2)A}\|_{L^r} + \|\langle x \rangle^{-\delta} \nabla A e^{-(p+2)A}\|_{L^r} \tag{4.8} \\
 &\quad + \|\langle x \rangle^{-\delta} e^{-(p+1)A}\|_{L^\infty} + \|\langle x \rangle^{-\delta} \nabla A e^{-(p+1)A}\|_{L^r} \\
 &\quad + \|\langle x \rangle^{-\delta} e^{-pA}\|_{L^\infty} + \|\langle x \rangle^{-\delta} \nabla A e^{-pA}\|_{L^r}.
 \end{aligned}$$

4.1 Linear energy estimates

We first prove some L^2 weighted estimates for the propagator $S_{A,V}(t)$ associated with (4.1).

Proposition 4.1 *Assume A, V satisfy (4.3) and (4.4).*

(i) *For every $\delta > 0$, we have*

$$\|S_{A,V}(t)\varphi\|_{L^\infty((0,\infty);L^2_{-\delta})} \leq C\|\varphi\|_{L^2_{\delta}}. \tag{4.9}$$

(ii) *For every $T > 0$ and $0 < \delta < \delta^+$ satisfying $\frac{\delta}{2} + 2\delta^+ < 1$, we have*

$$\|S_{A,V}(t)\varphi\|_{L^\infty((0,T);L^2_{\delta})} \leq C\|\varphi\|_{H^1_{\delta^+}}. \tag{4.10}$$

(iii) *For every $T > 0, s \in (0, 1)$, and $0 < \delta < \delta^+$ satisfying $\delta + 9\delta^+ < 4s$, we have*

$$\|S_{A,V}(t)\varphi\|_{L^\infty((0,T);L^2_{\delta})} \leq C\|\varphi\|_{H^{\frac{\delta^+}{s}}_{\delta^+}}. \tag{4.11}$$

(iv) *For every $T > 0, r \in (2, \infty)$, and $0 < \delta < \delta^+$ satisfying $\frac{\delta}{2} + 2\delta^+ < \frac{r-2}{3r+2}$, we have*

$$\|S_{A,V}(t)\varphi\|_{L^\infty((0,T);H^2_{-\delta})} \leq \mathcal{P}\left(|(A, V)|_{\delta,r}\right)\|\varphi\|_{H^2_{(\frac{3r+2}{r-2})\delta^+}}, \tag{4.12}$$

where we recall that $\mathcal{P}(a)$ is a polynomial function depending on a .

Proof We denote for simplicity $w(t) = S_{A,V}(t)\varphi$. The conservation of mass (4.6) along with (4.3) imply (4.9). In order to prove (4.10) we rely on (4.6) and (4.7). After integrating (4.6) and (4.7) in time, by recalling (4.4) with δ replaced by $\frac{\delta}{4}$, we get the following bound:

$$\begin{aligned}
 &\|e^{-A}w(t)\|_{L^2}^2 + \|e^{-A}\nabla w(t)\|_{L^2}^2 \\
 &\leq \|e^{-A}w(0)\|_{L^2}^2 + \|e^{-A}\nabla w(0)\|_{L^2}^2 + C\|w(0)\|_{H^{\frac{1}{2}}_{\frac{\delta}{4}}}^2 + C\|w(t)\|_{H^{\frac{1}{2}}_{\frac{\delta}{4}}}^2
 \end{aligned}$$

$$\begin{aligned} &\leq C \|w(0)\|_{H^1_{\frac{\delta}{2}}}^2 + C \|w(t)\|_{L^2_{\delta}} \|w(t)\|_{H^1_{-\frac{\delta}{2}}} \\ &\leq C \|w(0)\|_{H^1_{\frac{\delta}{2}}}^2 + \frac{C}{\eta} \|w(t)\|_{L^2_{\delta}}^2 + \eta \|w(t)\|_{H^1_{-\frac{\delta}{2}}}^2, \end{aligned}$$

where we used interpolation, (4.3), and Cauchy’s inequality with $\eta > 0$ small. By using again (4.3) and taking $\eta > 0$ to be sufficiently small, we get the bound

$$\|w(t)\|_{H^1_{-\frac{\delta}{2}}} \leq C \|w(0)\|_{H^1_{\delta^+}} + C \|w(t)\|_{L^2_{\delta}}. \tag{4.13}$$

Next, by fixing $\delta_1 \in (\delta, \delta^+)$ and following the proof of Lemma 3.1 in [12], we get

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^2} \langle x \rangle^{2\delta_1} |w(t)|^2 e^{-2A} \\ &= 2\text{Re} \int_{\mathbb{R}^2} \langle x \rangle^{2\delta_1} \partial_t w(t) \bar{w}(t) e^{-2A} dx \\ &= 2\text{Im} \int_{\mathbb{R}^2} \langle x \rangle^{2\delta_1} (\Delta w(t) - 2\nabla w(t) \cdot \nabla A) \bar{w}(t) e^{-2A} dx \tag{4.14} \\ &= -2\text{Im} \int_{\mathbb{R}^2} \nabla \langle x \rangle^{2\delta_1} \cdot \nabla w(t) \bar{w}(t) e^{-2A} dx \\ &\leq C \int_{\mathbb{R}^2} \langle x \rangle^{2\delta_1-1} |\nabla w(t)| |w(t)| e^{-2A} dx. \end{aligned}$$

After integration in time, by using (4.3) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} &\|w(t)e^{-A}\|_{L^2_{\delta_1}}^2 \\ &\leq \|w(0)e^{-A}\|_{L^2_{\delta_1}}^2 + C \int_0^t \|\nabla w(\tau)\|_{L^2_{2\delta_1-1}} \|w(\tau)\|_{L^2_{\delta}} d\tau \tag{4.15} \\ &\leq \|w(0)e^{-A}\|_{L^2_{\delta_1}}^2 + C \int_0^t \|\nabla w(\tau)\|_{L^2_{-\frac{\delta}{2}}}^2 d\tau + C \int_0^t \|w(\tau)\|_{L^2_{\delta}}^2 d\tau, \end{aligned}$$

where we used the condition $\frac{\delta}{2} + 2\delta^+ < 1$ in order to guarantee $\langle x \rangle^{2\delta_1-1} \leq \langle x \rangle^{-\frac{\delta}{2}}$. By inserting in (4.15) the estimate (4.13) and by using again (4.3), we deduce

$$\|w(t)\|_{L^2_{\delta}}^2 \leq C(1 + T)\|w(0)\|_{H^1_{\delta^+}}^2 + C \int_0^t \|w(\tau)\|_{L^2_{\delta}}^2 d\tau$$

and we conclude by the Gronwall inequality.

Next, we focus on (4.11) and we fix $\eta, \mu > 0$ such that

$$s\eta - (1 - s)\mu = \delta, \quad \eta = \frac{\delta}{s} + \frac{(\delta^+ - \delta)}{2s}. \tag{4.16}$$

Then, for every $t \in [0, T]$ we use interpolation to obtain

$$\begin{aligned} \|S_{A,V}(t)\varphi\|_{L^2_\delta} &\leq \sum_{N\text{-dyadic}} \|S_{A,V}(t)\Delta_N\varphi\|_{L^2_\delta} \\ &\leq C \sum_{N\text{-dyadic}} \|S_{A,V}(t)\Delta_N\varphi\|_{L^2_\eta}^s \|S_{A,V}(t)\Delta_N\varphi\|_{L^{2-\mu}}^{1-s} \\ &\leq C \sum_{N\text{-dyadic}} \|S_{A,V}(t)\Delta_N\varphi\|_{L^2_\eta}^s \|\Delta_N\varphi\|_{L^{2-\frac{\delta}{s}}}^{1-s}, \end{aligned}$$

where we have used (4.9) at the last step. By (4.10) (that we can use thanks to the conditions $\delta + 9\delta^+ < 4s$ and $\delta^+ > \delta$), we continue the estimate above as follows

$$\begin{aligned} \dots &\leq C \sum_{N\text{-dyadic}} \|\Delta_N\varphi\|_{H^1_{\frac{\delta^+}{s}}}^s \|\Delta_N\varphi\|_{L^{2-\frac{\delta}{s}}}^{1-s} \\ &\leq C \sum_{N\text{-dyadic}} N^s \|\Delta_N\varphi\|_{L^{2-\frac{\delta^+}{s}}}^s \|\Delta_N\varphi\|_{L^{2-\frac{\delta}{s}}}^{1-s} \leq C \|\varphi\|_{H^{s^+}_{\frac{\delta^+}{s}}}, \end{aligned}$$

where we used (2.8) and (2.12).

For (4.12), we postpone the proof to Sect. 6, since the proof requires a special case of modified energies. Nevertheless, the tools needed to prove (4.12) are elementary (such as Hölder’s inequality) and do not rely on any estimates in Subsect. 4.2 and Sect. 5. □

4.2 Linear Strichartz estimates

In this subsection, we shall need the following norm associated with A, V :

$$\|(A, V)\|_{\delta,k} = \|V\|_{L^k_{-\delta}} + \|\nabla A\|_{L^k_{-\delta}}, \quad k \in [1, \infty), \quad \delta > 0. \tag{4.17}$$

We start with some useful lemmas.

Lemma 4.2 *For every $s \in (0, 1)$, there exists $C > 0$ such that*

$$\|(H_{A,V} - \Delta)u\|_{L^2} \leq C \|(A, V)\|_{\delta, \frac{2}{s}} \|u\|_{H^{s^+}_\delta}, \quad \forall \delta > 0. \tag{4.18}$$

Proof We have for every $s \in (0, 1)$ and $\delta > 0$,

$$\|\nabla u \cdot \nabla A\|_{L^2} \leq \|\nabla A\|_{L^{\frac{2}{s}}_{-\delta}} \|\nabla u\|_{L^{\frac{2}{1-s}}_\delta} \leq C \|\nabla A\|_{L^{\frac{2}{s}}_{-\delta}} \|u\|_{H^{1+s}_\delta},$$

where we have used the embedding $H^s_\delta \subset L^{\frac{2}{1-s}}_\delta$ (see (2.41)) along with (2.17) and (2.3). By a similar argument,

$$\|Vu\|_{L^2} \leq C \|V\|_{L_{-\delta}^{\frac{2}{\delta}}} \|u\|_{H_{\delta}^s}.$$

The proof is complete. □

Lemma 4.3 *Let $T > 0$ be fixed and A, V satisfy (4.3) and (4.4). There exists $\bar{\delta} > 0$ such that for $\delta \in (0, \bar{\delta})$ and for every $r \in [4, \infty)$, we have¹*

$$\|S_{A,V}(t)\varphi\|_{L^\infty((0,T);H_{\frac{\delta}{2}(1-3\delta)}^{1+\delta})} \leq \mathcal{P}\left(|(A, V)|_{\delta,r}\right) \|\varphi\|_{H_{\frac{\sqrt{\delta}(1-\delta)}{4\sqrt{\delta}}+1+\delta+}}. \tag{4.19}$$

Proof By (4.11) (where we choose $s = \sqrt{\delta}$), there exists $\bar{\delta} > 0$ such that for $0 < \delta < \bar{\delta}$, we have

$$\|S_{A,V}(t)\varphi\|_{L^\infty((0,T);L_{2\delta}^2)} \leq C \|\varphi\|_{H_{\frac{\sqrt{2\delta}}{4\sqrt{\delta}}}}. \tag{4.20}$$

From (4.12), under the extra assumption $r \geq 4$, we get

$$\|S_{A,V}(t)\varphi\|_{L^\infty((0,T);H_{-\delta}^2)} \leq \mathcal{P}\left(|(A, V)|_{\delta,r}\right) \|\varphi\|_{H_{\frac{\delta}{8}}^2}. \tag{4.21}$$

Next, we notice that by special case of (2.40) and by (4.20), (4.21) we get:

$$\begin{aligned} & \|S_{A,V}(t)\varphi\|_{L^\infty((0,T);H_{\frac{\delta}{2}(1-3\delta)}^{1+\delta})} \\ & \leq \sum_{N\text{-dyadic}} \|S_{A,V}(t)\Delta_N\varphi\|_{L^\infty((0,T);H_{\frac{\delta}{2}(1-3\delta)}^{1+\delta})} \\ & \leq \sum_{N\text{-dyadic}} \|S_{A,V}(t)\Delta_N\varphi\|_{L^\infty((0,T);L_{2\delta}^2)}^{\frac{1-\delta}{2}} \|S_{A,V}(t)\Delta_N\varphi\|_{L^\infty((0,T);H_{-\delta}^2)}^{\frac{1+\delta}{2}} \\ & \leq \mathcal{P}\left(|(A, V)|_{\delta,r}\right) \sum_{N\text{-dyadic}} \|\Delta_N\varphi\|_{H_{\frac{\sqrt{2\delta}}{4\sqrt{\delta}}}}^{\frac{1-\delta}{2}} \|\Delta_N\varphi\|_{H_{\frac{\delta}{8}}^2}^{\frac{1+\delta}{2}} \\ & \leq \mathcal{P}\left(|(A, V)|_{\delta,r}\right) \sum_{N\text{-dyadic}} N^{\frac{\sqrt{\delta}(1-\delta)}{\sqrt{2}}+1+\delta} \|\Delta_N\varphi\|_{L_{\frac{4\sqrt{\delta}}{2}}^2}. \end{aligned}$$

The conclusion follows by (2.8). □

¹ Notice that the estimate (4.19) is very weak since the norm on the l.h.s is much weaker than the one on the r.h.s in both derivatives and weights. In fact, we have

$$1 + \delta \ll \frac{\sqrt{\delta}(1-\delta)}{\sqrt{2}} + 1 + \delta^+ \quad \text{and} \quad \frac{\delta}{2}(1-3\delta) \ll 4\sqrt{\delta}$$

for $\delta \ll 1$. Nevertheless, this weak estimate will be useful in the sequel (in particular it implies (4.24) below).

Lemma 4.4 *Let $T > 0$ be fixed and A, V satisfy (4.3) and (4.4). There exists $\bar{\delta} > 0$ such that for every $\delta \in (0, \bar{\delta})$ and $r \in [4, \infty)$, we have*

$$\|S_{A,V}(t)\varphi\|_{L^\infty((0,T);H^{\frac{3}{2}})} \leq \mathcal{P}\left(|(A, V)|_{\delta,r}\right)\|\varphi\|_{H^{\frac{3}{2}+\frac{\sqrt{\delta}^+}{4}}}. \tag{4.22}$$

Proof By combining a special case of (2.40), (4.12) and (4.11) (here we assume $\delta > 0$ small enough in order to guarantee $\delta + 9\delta^+ < 4\sqrt{\delta}$, and hence we can apply (4.11) with $s = \sqrt{\delta}$), we get

$$\begin{aligned} & \|S_{A,V}(t)\varphi\|_{H^{\frac{3}{2}}} \\ & \leq \sum_{N\text{-dyadic}} \|S_{A,V}(t)\Delta_N\varphi\|_{L^\infty((0,T);H^{\frac{3}{2}})} \\ & \leq C \sum_{N\text{-dyadic}} \|S_{A,V}(t)\Delta_N\varphi\|_{L^\infty((0,T);H^2_{-\frac{\delta}{3}})}^{\frac{3}{4}} \|S_{A,V}(t)\Delta_N\varphi\|_{L^\infty((0,T);L^2_{\delta})}^{\frac{1}{4}} \\ & \leq \mathcal{P}\left(|(A, V)|_{\delta,r}\right) \sum_{N\text{-dyadic}} \|\Delta_N\varphi\|_{H^2_{\frac{\delta}{3}}}^{\frac{3}{4}} \|\Delta_N\varphi\|_{H^{\sqrt{\delta}^+}}^{\frac{1}{4}} \\ & \leq \mathcal{P}\left(|(A, V)|_{\delta,r}\right) \sum_{N\text{-dyadic}} N^{\frac{3}{2}+\frac{\sqrt{\delta}^+}{4}} \|\Delta_N\varphi\|_{L^2_{\sqrt{\delta}^+}}. \end{aligned}$$

The conclusion follows by (2.8). □

We are now ready to prove the Strichartz estimates associated with $S_{A,V}(t)$.

Proposition 4.5 *Let $T > 0$ be fixed and A, V satisfy (4.3) and (4.4). For every $\delta, s > 0$, there exist $\tilde{\delta}, \delta_1, s_1 > 0$ such that $\frac{\delta_1}{s_1} > 1$ and for every $r \in [4, \infty)$,*

$$\|S_{A,V}(t)\varphi\|_{L^l((0,T);L^q)} \leq \mathcal{P}\left(|(A, V)|_{\tilde{\delta},r}\right)\mathcal{P}\left(\|(A, V)\|_{\delta_1, \frac{2}{s_1}}\right)\|\varphi\|_{H^{\frac{1}{l}+s}}, \tag{4.23}$$

where $\frac{1}{l} + \frac{1}{q} = \frac{1}{2}$ and $l > 2$.

Proof Notice that by (4.19), for every $\delta_2, s_2 > 0$, there exist $\tilde{\delta}, \delta_1, s_1 > 0$ such that for every $r \in [4, \infty)$,

$$\|S_{A,V}(t)\varphi\|_{L^\infty((0,T);H^{1+s_1}_{\delta_1})} \leq \mathcal{P}\left(|(A, V)|_{\tilde{\delta},r}\right)\|\varphi\|_{H^{\frac{1}{\delta_2}+\frac{s_2}{2}}}. \tag{4.24}$$

Moreover, by (2.5) we can also assume $\frac{\delta_1}{s_1}$ to be arbitrarily large and in particular larger than 1. We also have the following bound for any given $s_2, \delta_2 > 0$ as a consequence of (4.11):

$$\|S_{A,V}(t)\varphi\|_{L^\infty((0,T);L^2)} \leq C\|\varphi\|_{H^{\frac{s_2}{\delta_2}}} . \tag{4.25}$$

Next, following [7, 27] we split the interval $[0, T]$ in an essentially disjoint union of intervals of size N^{-1} as

$$[0, T] = \bigcup_j I_j \tag{4.26}$$

and we aim to estimate $\|S_{A,V}(t)\Delta_N\varphi\|_{L^1(I_j;L^q)}$. Suppose that $I_j = [a, b]$. Then, for $t \in [a, b]$ we can write:

$$\begin{aligned} S_{A,V}(t)\Delta_N\varphi &= e^{i(t-a)\Delta} S_{A,V}(a)\Delta_N\varphi \\ &+ i \int_a^t e^{i(t-\tau)\Delta} (H_{A,V} - \Delta) S_{A,V}(\tau)\Delta_N\varphi d\tau. \end{aligned} \tag{4.27}$$

We now estimate each term in the r.h.s. of (4.27). Using the Strichartz estimates on \mathbb{R}^2 (see [9, 17, 26]), and (4.25), we get

$$\begin{aligned} &\|e^{i(t-a)\Delta} S_{A,V}(a)\Delta_N\varphi\|_{L^1(I_j;L^q)} \\ &\leq C\|S_{A,V}(a)\Delta_N\varphi\|_{L^2} \\ &\leq C\|\Delta_N\varphi\|_{H^{\frac{s_2}{\delta_2}}} \\ &\leq CN^{-\frac{1}{l}-\frac{s_2}{2}} \sum_{\frac{N}{4} \leq M \leq 4N} \|\Delta_M\varphi\|_{H^{\frac{1}{l}+s_2}} \\ &\leq CN^{-\frac{1}{l}-\frac{s_2}{2}} \|\varphi\|_{H^{\frac{1}{\delta_2}+s_2}} \end{aligned}$$

where we used (2.20) at the second last step. Now we estimate the second term in the r.h.s. of (4.27). Using Minkowski’s inequality, the Strichartz estimates on \mathbb{R}^2 , (4.18), and (4.24) (with s_2 small), we get

$$\begin{aligned} &\left\| \int_a^t e^{i(t-\tau)\Delta} (H_{A,V} - \Delta) S_{A,V}(\tau)\Delta_N\varphi d\tau \right\|_{L^1(I_j;L^q)} \\ &\leq C \int_{I_j} \|(H_{A,V} - \Delta) S_{A,V}(\tau)\Delta_N\varphi\|_{L^2} d\tau \\ &\leq C\|(A, V)\|_{\delta_1, \frac{2}{s_1}} \mathcal{P}\left(|(A, V)|_{\tilde{\delta}, r}\right) N^{-1} \|\Delta_N\varphi\|_{H^{\frac{1}{\delta_2}+s_2}} \\ &\leq C\|(A, V)\|_{\delta_1, \frac{2}{s_1}} \mathcal{P}\left(|(A, V)|_{\tilde{\delta}, r}\right) N^{-1} N^{1-\frac{1}{l}-\frac{s_2}{2}} \|\varphi\|_{H^{\frac{1}{\delta_2}+s_2}}, \end{aligned}$$

where we used (2.21). Summarizing, we get

$$\|S_{A,V}(t)\Delta_N\varphi\|_{L^1(I_j;L^q)}$$

$$\leq C\mathcal{P}\left(|(A, V)|_{\tilde{\delta},r}\right)\left(1 + \|(A, V)\|_{\delta_1, \frac{2}{s_1}}\right)N^{-\frac{1}{l}-\frac{s_2}{2}}\|\varphi\|_{H_{\delta_2}^{\frac{1}{l}+s_2}}.$$

Using that the number of I_j in (4.26) is $O(TN)$, taking the l 'th power of the previous bound, and summing over j , we get the estimate

$$\begin{aligned} &\|S_{A,V}(t)\Delta_N\varphi\|_{L^l((0,T);L^q)} \\ &\leq CT^{\frac{1}{l}}\mathcal{P}\left(|(A, V)|_{\tilde{\delta},r}\right)\left(1 + \|(A, V)\|_{\delta_1, \frac{2}{s_1}}\right)N^{-\frac{s_2}{2}}\|\varphi\|_{H_{\delta_2}^{\frac{1}{l}+s_2}} \end{aligned}$$

and hence we conclude by summation over N . □

In the sequel, we shall need the following Strichartz estimate.

Proposition 4.6 *Assume (4.3) and (4.4) and let $T > 0$ be fixed. For every $s, \delta > 0$ small enough, there exist $s_1, \delta_1, \tilde{\delta} > 0$ such that $\frac{\delta_1}{s_1} > 1$ and for every $r \in [4, \infty)$, we have*

$$\begin{aligned} &\|S_{A,V}(t)\varphi\|_{L^4((0,T);W^{\frac{3}{4}-s,4})} \\ &\leq \mathcal{P}\left(|(A, V)|_{\frac{\delta^2}{4},r}\right)\mathcal{P}\left(|(A, V)|_{\tilde{\delta},r}\right)\mathcal{P}\left(\|(A, V)\|_{\delta_1, \frac{2}{s_1}}\right)\|\varphi\|_{H_{\delta}^1} \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} &\left\| \int_0^t S_{A,V}(t-\tau)f(\tau)d\tau \right\|_{L^4((0,T);W^{\frac{3}{4}-s,4})} \\ &\leq \mathcal{P}\left(|(A, V)|_{\frac{\delta^2}{4},r}\right)\mathcal{P}\left(|(A, V)|_{\tilde{\delta},r}\right)\mathcal{P}\left(\|(A, V)\|_{\delta_1, \frac{2}{s_1}}\right)\|f\|_{L^1((0,T);H_{\delta}^1)}. \end{aligned} \tag{4.29}$$

Proof Notice that it is not restrictive to assume δ, s small enough (we will exploit this fact later). Notice also that (4.29) follows by combining (4.28) with Minkowski's inequality. Thus, we focus on the proof of (4.28).

For every $s \in (0, \frac{1}{2})$, there exists $q \in (2, \infty)$ such that the following Gagliardo-Nirenberg inequality holds:

$$\|u\|_{W^{\frac{3}{4}-s,4}} \leq C\|u\|_{L^q}^{\frac{1}{2}}\|u\|_{H^{\frac{3}{2}}}^{\frac{1}{2}},$$

and hence by integration in time and the Hölder inequality in time we get

$$\begin{aligned} \|S_{A,V}(t)\varphi\|_{L^4((0,T);W^{\frac{3}{4}-s,4})}^4 &\leq C\|S_{A,V}(t)\varphi\|_{L^2((0,T);L^q)}^2\|S_{A,V}(t)\varphi\|_{L^\infty((0,T);H^{\frac{3}{2}})}^2 \\ &\leq C\mathcal{P}\left(|(A, V)|_{\frac{\delta^2}{4},r}\right)\|S_{A,V}(t)\varphi\|_{L^1((0,T);L^q)}^2\|\varphi\|_{H^{\frac{3}{2}+(\frac{q}{8})^+}}^2, \end{aligned}$$

where $\frac{1}{l} + \frac{1}{q} = 1, l > 2$ are Strichartz admissible and we used (4.22) (where $r \in [4, \infty)$ is arbitrary and we have replaced δ by $\frac{\delta^2}{4}$). By using the Strichartz estimates (4.23),

we can continue the estimate above as follows

$$\dots \leq \mathcal{P}\left(|(A, V)|_{\frac{\delta^2}{4}, r}\right) \mathcal{P}\left(|(A, V)|_{\tilde{\delta}, r}\right) \mathcal{P}\left(\|(A, V)\|_{\delta_1, \frac{2}{s_1}}\right) \|\varphi\|_{H_\delta^{\frac{1}{l}+s}}^2 \|\varphi\|_{H_{\left(\frac{\delta}{2}\right)^+}^{\frac{3}{2}+\left(\frac{\delta}{8}\right)^+}}^2,$$

where $\tilde{\delta}, \delta_1, s_1 > 0$ depend on s, δ . Notice that for initial datum $\varphi = \Delta_N \varphi$ which is localized at dyadic frequency N , we get from the previous bound that

$$\begin{aligned} & \|S_{A, V}(t) \Delta_N \varphi\|_{L^4((0, T); W^{\frac{3}{4}-s, 4})} \\ & \leq \mathcal{P}\left(|(A, V)|_{\frac{\delta^2}{4}, r}\right) \mathcal{P}\left(|(A, V)|_{\tilde{\delta}, r}\right) \mathcal{P}\left(\|(A, V)\|_{\delta_1, \frac{2}{s_1}}\right) \|\Delta_N \varphi\|_{H_\delta^{\frac{1}{2l}+\frac{s}{2}+\frac{3}{4}+\left(\frac{\delta}{16}\right)^+}}. \end{aligned}$$

We conclude (4.28) by summing over N and using (2.8), once we notice that $\frac{1}{2l} + \frac{3}{4} < 1$ for $l > 2$ and $s, \delta > 0$ are small enough. □

5 Nonlinear estimates

Along this section, we focus on solutions to the following nonlinear problem

$$i \partial_t v = H_{A, V} v - \lambda e^{-pA} |v|^p, \quad \lambda > 0 \tag{5.1}$$

where $H_{A, V}$ is defined in (4.2).

5.1 Nonlinear energy estimates

We have the following conserved quantities for any solution to (5.1):

$$\frac{d}{dt} \int_{\mathbb{R}^2} |v(t)|^2 e^{-2A} dx = 0; \tag{5.2}$$

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} \left(\frac{1}{2} |\nabla v|^2 e^{-2A} - \frac{1}{2} |v(t)|^2 V e^{-2A} \right. \\ & \left. + \frac{\lambda}{p+2} |v(t)|^{p+2} e^{-(p+2)A} \right) dx = 0. \end{aligned} \tag{5.3}$$

Proposition 5.1 *Assume (4.3) and (4.4) and let $T > 0$ be fixed. Then, for every $\delta \in (0, \frac{1}{9})$, there exists $C > 0$ such that for every solution v to (5.1), we have*

$$\|v\|_{L^\infty((0, T); L_\delta^2)} \leq C \left(1 + \|v(0)\|_{H_{4\delta}^{\frac{p+2}{2}}} \right); \tag{5.4}$$

$$\|v\|_{L^\infty((0, T); H_{-\delta}^1)} \leq C \left(1 + \|v(0)\|_{H_{8\delta}^{\frac{p+2}{2}}} \right). \tag{5.5}$$

Moreover, for every $q \in [2, \infty)$ and $\delta \in \left(0, \frac{1}{36(2q-1)}\right)$, we have

$$\|v\|_{L^\infty((0,T);W_\delta^{1,q})} \leq C \|v\|_{L^\infty((0,T);H_{-\delta}^2)}^{1-\frac{1}{q}} \mathcal{P}\left(\|v(0)\|_{H_{4(2q-1)\delta}^1}\right). \tag{5.6}$$

Proof By using conservations (5.2) and (5.3) and recalling that $\lambda > 0$, we get

$$\begin{aligned} & \int_{\mathbb{R}^2} \frac{1}{2} (|\nabla v(t)|^2 + |v(t)|^2) e^{-2A} dx - \int_{\mathbb{R}^2} \frac{1}{2} |v(t)|^2 V e^{-2A} dx \\ & \leq \int_{\mathbb{R}^2} \frac{1}{2} (|\nabla v(0)|^2 + |v(0)|^2) e^{-2A} dx - \int_{\mathbb{R}^2} \frac{1}{2} |v(0)|^2 V e^{-2A} dx \\ & \quad + \int_{\mathbb{R}^2} \frac{\lambda}{p+2} |v(0)|^{p+2} e^{-(p+2)A} dx \end{aligned}$$

and by (4.4) we get

$$\begin{aligned} & \int_{\mathbb{R}^2} \frac{1}{2} (|\nabla v(t)|^2 + |v(t)|^2) e^{-2A} dx \\ & \leq C \|v(t)\|_{H_\chi^{\frac{1}{2}}}^2 + \int_{\mathbb{R}^2} \frac{1}{2} (|\nabla v(0)|^2 + |v(0)|^2) e^{-2A} dx \\ & \quad + \int_{\mathbb{R}^2} \frac{\lambda}{p+2} |v(0)|^{p+2} e^{-(p+2)A} dx. \end{aligned} \tag{5.7}$$

In turn by (4.3) and the Sobolev embedding $H_\delta^1 \subset L_\delta^{p+2}$, we get

$$\|v(t)\|_{H_{-\delta}^1}^2 \leq C \left(1 + \|v(0)\|_{H_\delta^1}^{p+2}\right) + C \|v(t)\|_{H_{-\delta}^1} \|v(t)\|_{L_{2\delta}^2}, \tag{5.8}$$

where we used a special case of (2.40) and we chose $\chi = \frac{\delta}{2}$. Hence, by elementary considerations, we get

$$\|v(t)\|_{H_{-\delta}^1}^2 \leq C \left(1 + \|v(0)\|_{H_\delta^1}^{p+2}\right) + C \|v(t)\|_{L_{2\delta}^2}^2. \tag{5.9}$$

Next, by following the computation in [12, Lemma 3.1] (see also (4.14)), we get

$$\frac{d}{dt} \int_{\mathbb{R}^2} |\langle x \rangle^\delta v(t)|^2 e^{-2A} dx \leq C \int_{\mathbb{R}^2} \langle x \rangle^{2\delta-1} |\nabla v(t)| |v(t)| e^{-2A} dx,$$

which, by integration in time, (4.3), and the Cauchy-Schwarz inequality, implies that

$$\|v(t) e^{-A}\|_{L_\delta^2}^2 \leq \|e^{-A} v(0)\|_{L_\delta^2}^2 + C \int_0^t \|\nabla v(\tau)\|_{L_{2\delta-1}^2} \|v(\tau)\|_{L_{\frac{\delta}{2}}^2} d\tau,$$

so that by using again (4.3),

$$\|v(t)\|_{L^2_{\frac{\delta}{2}}}^2 \leq C\|v(0)\|_{L^2_{2\delta}}^2 + C \int_0^t \|\nabla v(\tau)\|_{L^2_{-\frac{\delta}{4}}}^2 d\tau + C \int_0^t \|v(\tau)\|_{L^2_{\frac{\delta}{2}}}^2 d\tau$$

where we assumed $\delta \in (0, \frac{1}{9})$ in order to guarantee $\langle x \rangle^{2\delta-1} \leq \langle x \rangle^{-\frac{\delta}{4}}$. By combining this estimate with (5.9), where we replace δ by $\frac{\delta}{4}$, we get

$$\|v(t)\|_{L^2_{\frac{\delta}{2}}}^2 \leq C\|v(0)\|_{L^2_{2\delta}}^2 + Ct \left(1 + \|v(0)\|_{H^1_{\frac{\delta}{4}}}^{p+2}\right) + C \int_0^t \|v(\tau)\|_{L^2_{\frac{\delta}{2}}}^2 d\tau.$$

We deduce (5.4) by the Gronwall inequality. The estimate (5.5) follows by combining (5.4) and (5.9). Concerning (5.6) we can combine (2.43) with (5.4). □

Remark 5.2 We consider (5.1) with $\lambda < 0$. If $0 < p < 2$, (5.5) and (5.4) follow from Proposition 3.2 and Corollary 3.3 in [12], respectively. If $p \geq 2$, we further assume that $\|v(0)\|_{H^1_{\delta_0}} \ll 1$ for some $\delta_0 > 0$. In this case, (5.7) needs to be replaced by

$$\begin{aligned} & \int_{\mathbb{R}^2} \frac{1}{2} (|\nabla v(t)|^2 + |v(t)|^2) e^{-2A} dx \\ & \leq C\|v(t)\|_{H^{\frac{1}{2}}_{\frac{\delta}{2}}}^2 + \int_{\mathbb{R}^2} \frac{1}{2} (|\nabla v(0)|^2 + |v(0)|^2) e^{-2A} dx \\ & \quad + \int_{\mathbb{R}^2} \frac{\lambda}{p+2} |v(0)|^{p+2} e^{-(p+2)A} dx - \int_{\mathbb{R}^2} \frac{\lambda}{p+2} |v(t)|^{p+2} e^{-(p+2)A} dx. \end{aligned}$$

Thus, instead of (5.8), we obtain

$$\begin{aligned} \|v(t)\|_{H^1_{-\delta}}^2 & \leq C\|v(0)\|_{H^1_{\delta}}^2 + C\|v(t)\|_{H^1_{-\delta}} \|v(t)\|_{L^2_{2p\delta}} + \|v(t)\|_{L^{\frac{p+2}{\frac{3p\delta}{p+2}}}}^{p+2} \\ & \leq C\|v(0)\|_{H^1_{\delta}}^2 + \frac{1}{2}\|v(t)\|_{H^1_{-\delta}}^2 + C\|v(t)\|_{L^2_{2p\delta}}^2 + \|v(t)\|_{H^{\frac{\frac{p}{3p\delta}}{\frac{p+2}}{p+2}}}^{p+2} \\ & \leq C\|v(0)\|_{H^1_{\delta}}^2 + \frac{1}{2}\|v(t)\|_{H^1_{-\delta}}^2 + C\|v(t)\|_{L^2_{2p\delta}}^2 + \|v(t)\|_{H^1_{-\delta}}^p \|v(t)\|_{L^2_{2p\delta}}^2 \end{aligned}$$

where we take $\delta > 0$ to be sufficiently small and we used (4.3), the Sobolev embedding $H^{\frac{p}{\chi}}_{\chi} \subset L^{\frac{p+2}{\chi}}$ for any $\chi \in \mathbb{R}$, and (2.40). Assume that $\|v(t)\|_{H^1_{-\delta}} \leq 1$, so that we have the bound

$$\|v(t)\|_{H^1_{-\delta}}^2 \leq C\|v(0)\|_{H^1_{\delta}}^2 + C\|v(t)\|_{L^2_{2p\delta}}^2. \tag{5.10}$$

Then, by arguing as in the proof of Proposition 5.1 and using (5.10) instead of (5.9), we get

$$\|v(t)\|_{L^2_{2p\delta}}^2 \leq C(1+t)\|v(0)\|_{H^1_{4p\delta}}^2 + C \int_0^t \|v(\tau)\|_{L^2_{2p\delta}}^2 d\tau.$$

By applying the Gronwall inequality and taking $\|v(0)\|_{H^1_{4p\delta}}$ to be sufficiently small, we obtain $\|v(t)\|_{L^2_{2p\delta}}^2 \leq \frac{1}{100C}$, so that (5.10) gives $\|v(t)\|_{H^1_{-\delta}} \leq \frac{1}{2}$. Thus, by a standard continuity argument, we get the following two uniform bounds for $\delta > 0$ sufficiently small:

$$\begin{aligned} \|v\|_{L^\infty((0,T);L^2_\delta)} &\leq 1, \\ \|v\|_{L^\infty((0,T);H^1_{-\delta})} &\leq 1. \end{aligned}$$

The next proposition will also be useful.

Proposition 5.3 *Assume (4.3) and (4.4). Let $0 < \delta < \min\{\frac{1}{\delta}, \frac{\bar{\eta}}{9(6-4\bar{\eta})}\}$ and $\bar{\eta} \in (0, 1)$. For every $t > 0$ and for every solution v to (5.1), we have the bound*

$$\|v(t)\|_{H^{\frac{2}{\delta}-\bar{\eta}}} \leq C\mathcal{P}\left(\|v(0)\|_{H^1_{\frac{4\delta(6-4\bar{\eta})}{\bar{\eta}}}}\right) \|v(t)\|_{H^2_{-\delta(\frac{1-2\bar{\eta}}{\bar{\eta}})}}^{\bar{\eta}}. \tag{5.11}$$

In particular, for every $\eta_0 \in (0, \frac{1}{3})$, $\delta_0 > 0$, there exists $\bar{\delta} > 0$ such that

$$\|v(t)\|_{H^{\frac{2}{\delta}-\eta_0}} \leq C\mathcal{P}\left(\|v(0)\|_{H^1_{\delta_0}}\right) \|v(t)\|_{H^2_{-\bar{\delta}}}^{\eta_0}, \quad \forall \delta \in (0, \bar{\delta}). \tag{5.12}$$

Proof By special case of (2.40), we get

$$\|f\|_{H^{1+s}_\delta} \leq C\|f\|_{H^{1+s}_{2\delta}}^{\frac{1-s}{1+s}} \|f\|_{H^2_{-\delta(\frac{1-3s}{2s})}}^{\frac{2s}{1+s}}, \quad \delta > 0, \quad s \in (0, 1)$$

and also

$$\|f\|_{H^{1-s}_{2\delta}} \leq C\|f\|_{H^{1-s}_{-\delta}}^{\frac{1-s}{1+s}} \|f\|_{L^2_{\delta(\frac{3-s}{5})}}^s, \quad \delta > 0, \quad s \in (0, 1).$$

Summarizing, we obtain

$$\|f\|_{H^{1+s}_\delta} \leq C\|f\|_{H^{1-s}_{-\delta}}^{\frac{(1-s)^2}{1+s}} \|f\|_{L^2_{\delta(\frac{3-s}{5})}}^{\frac{s(1-s)}{1+s}} \|f\|_{H^2_{-\delta(\frac{1-3s}{2s})}}^{\frac{2s}{1+s}}, \quad \delta > 0, \quad s \in (0, 1)$$

Next, we select $s \in (0, 1)$ such that $\bar{\eta} = \frac{2s}{s+1}$, namely $s = \frac{\bar{\eta}}{2-\bar{\eta}}$. We recall that (5.4) and (5.5) are available for solutions v to (5.1) and so we get

$$\|v(t)\|_{H_{\delta}^{\frac{2}{2-\bar{\eta}}}} \leq C\mathcal{P}\left(\|v(0)\|_{H_{8\delta}^1}\right)\mathcal{P}\left(\|v(0)\|_{H_{4\delta(\frac{6-4\bar{\eta}}{\bar{\eta}})}^1}\right)\|v(t)\|_{H_{-\delta(\frac{1-2\bar{\eta}}{\bar{\eta}})}^2}^{\bar{\eta}}.$$

The conclusion follows since $8\delta \leq 4\delta(\frac{6-4\bar{\eta}}{\bar{\eta}})$ for every $\bar{\eta} \in (0, 1)$. The bound (5.12) is an easy consequence of (5.11). □

5.2 Nonlinear Strichartz estimates

Along this subsection, v denotes any solution to (5.1) and $\delta_0 > 0$ is a fixed number such that $v(0) \in H_{\delta_0}^2$. We aim at proving local in time Strichartz space-time bounds for the solution v . We shall use the quantities introduced respectively in (4.8) and (4.17).

Proposition 5.4 *Assume (4.3) and (4.4). Let $T > 0$ and $\bar{\eta} \in (0, 1)$ be fixed. Then, for every $s > 0$ and $0 < \delta < \min\{\frac{1}{18}, \frac{\bar{\eta}}{18(6-4\bar{\eta})}\}$, there exist $s_1, \delta_1, \bar{\delta} > 0$ such that $\frac{\delta_1}{s_1} > 1$ and for every $r \in [4, \infty)$, we have the following bound:*

$$\begin{aligned} & \|v\|_{L^4((0,T);W^{\frac{3}{4}-s,4})} \\ & \leq \mathcal{P}\left(|(A, V)|_{\delta,r}\right)\mathcal{P}\left(|(A, V)|_{\bar{\delta},r}\right)\mathcal{P}\left(|(A, V)|_{\frac{\delta^2}{4},r}\right)\mathcal{P}\left(\|(A, V)\|_{\delta_1, \frac{2}{s_1}}^2\right) \quad (5.13) \\ & \quad \times \left[\|v(0)\|_{H_{\delta}^1} + \mathcal{P}\left(\|v(0)\|_{H_{8\delta(\frac{6-4\bar{\eta}}{\bar{\eta}})}^1}\right)\|v\|_{L^\infty((0,T);H_{-\frac{2\delta}{2\delta(\frac{6-4\bar{\eta}}{\bar{\eta}})}^2}^{(p+1)\bar{\eta}})}\right]. \end{aligned}$$

In particular, for every given $\eta_0 \in (0, \frac{2}{5})$ and $s > 0$, there exists $\bar{\delta} > 0$ such that for every $\delta \in (0, \bar{\delta})$, there are $s_1, \delta_1, \bar{\delta} > 0$ with $\frac{\delta_1}{s_1} > 1$ and for every $r \in [4, \infty)$,

$$\begin{aligned} & \|v\|_{L^4((0,T);W^{\frac{3}{4}-s,4})} \\ & \leq \mathcal{P}\left(|(A, V)|_{\delta,r}\right)\mathcal{P}\left(|(A, V)|_{\bar{\delta},r}\right)\mathcal{P}\left(|(A, V)|_{\frac{\delta^2}{4},r}\right)\mathcal{P}\left(\|(A, V)\|_{\delta_1, \frac{2}{s_1}}^2\right) \quad (5.14) \\ & \quad \times \mathcal{P}\left(\|v(0)\|_{H_{\delta_0}^1}\right)\|v\|_{L^\infty((0,T);H_{-\delta}^{(p+1)\eta_0}}. \end{aligned}$$

Proof We get by (4.28) and (4.29) that

$$\begin{aligned} \|v\|_{L^4((0,T);W^{\frac{3}{4}-s,4})} & \leq \mathcal{P}\left(|(A, V)|_{\bar{\delta},r}\right)\mathcal{P}\left(|(A, V)|_{\frac{\delta^2}{4},r}\right)\mathcal{P}\left(\|(A, V)\|_{\delta_1, \frac{2}{s_1}}^2\right) \\ & \quad \times \left(\|v(0)\|_{H_{\delta}^1} + |\lambda| \int_0^T \|e^{-pA}v(\tau)|v(\tau)|^p\|_{H_{\delta}^1}d\tau\right). \end{aligned}$$

On the other hand, we have

$$\|e^{-pA}v|v|^p\|_{H_{\delta}^1}$$

$$\begin{aligned}
 &\leq C \|e^{-\rho A} v|v|^p\|_{L^2_\delta} + C \|\nabla A e^{-\rho A} v|v|^p\|_{L^2_\delta} + C \|e^{-\rho A} \nabla v|v|^p\|_{L^2_\delta} \\
 &\leq C \|\langle x \rangle^{-\delta} e^{-\rho A} \|v\|_{H^{2\delta}_1} \|v\|_{L^\infty}^p + C \|\langle x \rangle^{-\delta} \nabla A e^{-\rho A} \|v\|_{L^r} \|v\|_{L^{2\delta}^{\frac{2r}{r-2}}}^p \|v\|_{L^\infty}^p \\
 &\leq C |(A, V)|_{\delta,r} \|v\|_{H^{2\delta}^{\frac{2}{2-\eta}}}^{p+1},
 \end{aligned}$$

where $v = v(t)$ for some fixed $t \in [0, T]$ and we used the Sobolev embedding $H^{2\delta}_{\frac{2}{2-\eta}} \subset L^q_\delta$ for every $q \in [2, \infty]$. By using (5.11), we get

$$\|e^{-\rho A} v|v|^p\|_{L^\infty((0,T);H^1_\delta)} \leq C \mathcal{P} \left(\|v(0)\|_{H^1_{\frac{8\delta(6-4\bar{\eta})}{\eta}}} \right) \|v\|_{L^\infty((0,T);H^2_{-2\delta(\frac{1-2\bar{\eta}}{\eta})})}^{(p+1)\bar{\eta}}.$$

The bound (5.14) easily follows from (5.13). □

As a consequence, we can show the following estimates.

Proposition 5.5 *Assume (4.3) and (4.4). Let $T > 0$, $\eta_0 \in (0, \frac{2}{3})$, and $s \in (0, \frac{1}{8})$ be given. Then, there exists $\bar{\delta} > 0$ such that for every $\delta \in (0, \bar{\delta})$, there exist $s_1, \delta_1, \bar{\delta} > 0$ with $\frac{\delta_1}{s_1} > 1$ and for every $r \in [4, \infty)$,*

$$\begin{aligned}
 &\|v\|_{L^2((0,T);W^{1,4})} \\
 &\leq \mathcal{P} \left(|(A, V)|_{\delta,r} \right) \mathcal{P} \left(|(A, V)|_{\bar{\delta},r} \right) \mathcal{P} \left(|(A, V)|_{\frac{\delta_1}{4},r} \right) \mathcal{P} \left(\|(A, V)\|_{\delta_1, \frac{2}{s_1}} \right) \\
 &\quad \times \mathcal{P} \left(\|v(0)\|_{H^1_{\delta_0}} \right) \|v\|_{L^\infty((0,T);H^2_{-\delta})}^{\frac{2(1-2s)(p+1)\eta_0}{3}} \|v\|_{L^\infty((0,T);H^2_{-\frac{s\delta_0}{4(4-s)}})}^{\frac{(1+4s)(2-s)}{6}}.
 \end{aligned} \tag{5.15}$$

In particular, by choosing $s = \frac{1}{16}$ and $0 < \delta < \min \left\{ \bar{\delta}, \frac{\delta_0}{64(4-\frac{1}{16})} \right\}$, we get

$$\begin{aligned}
 &\|v\|_{L^2((0,T);W^{1,4})} \\
 &\leq \mathcal{P} \left(|(A, V)|_{\delta,r} \right) \mathcal{P} \left(|(A, V)|_{\bar{\delta},r} \right) \mathcal{P} \left(|(A, V)|_{\frac{\delta_1}{4},r} \right) \mathcal{P} \left(\|(A, V)\|_{\delta_1, \frac{2}{s_1}} \right) \\
 &\quad \times \mathcal{P} \left(\|v(0)\|_{H^1_{\delta_0}} \right) \|v\|_{L^\infty((0,T);H^2_{-\delta})}^{\frac{7}{12}(p+1)\eta_0 + \frac{155}{384}}.
 \end{aligned} \tag{5.16}$$

Proof We have the following interpolation bound at time fixed:

$$\|v(t)\|_{W^{1,4}} \leq C \|v(t)\|_{W^{\frac{3}{4}-s,4}}^{\frac{2(1-2s)}{3}} \|v(t)\|_{H^{2-s}}^{\frac{1+4s}{3}}, \quad s \in \left(0, \frac{1}{2}\right)$$

and hence by integration in time and using Hölder in time, we get

$$\begin{aligned}
 &\|v(t)\|_{L^2((0,T);W^{1,4})} \\
 &\leq C \|v(t)\|_{L^4((0,T);W^{\frac{3}{4}-s,4})}^{\frac{2(1-2s)}{3}} \|v(t)\|_{L^\infty((0,T);H^{2-s})}^{\frac{1+4s}{3}}, \quad s \in \left(0, \frac{1}{8}\right).
 \end{aligned} \tag{5.17}$$

Next, notice that we have

$$\|v(t)\|_{H^{2-s}_{-\frac{s\delta_0}{4(4-s)}}} \leq C \|v(t)\|_{L^2_{-\frac{\delta_0}{4}}}^{\frac{s}{2}} \|v(t)\|_{H^2_{-\frac{s\delta_0}{4(4-s)}}}^{\frac{2-s}{2}}. \tag{5.18}$$

We conclude (5.15) by combining (5.4) with (5.14), (5.17), and (5.18). The estimate (5.16) follows by (5.15) once we notice that the condition $0 < \delta < \frac{\delta_0}{64(4-\frac{1}{16})}$ implies that we have the embedding $H^2_{-\delta} \subset H^2_{-\frac{s\delta_0}{4(4-s)}}$ for $s = \frac{1}{16}$. Then one can absorb the term $\|v\|_{L^\infty((0,T); H^2_{-\frac{s\delta_0}{4(4-s)}})}$ in the factor $\|v\|_{L^\infty((0,T); H^2_{-\delta})}$ on the r.h.s. in (5.15). \square

6 Modified energies

Along this section we denote by v a solution to the following equation with time-independent A and V :

$$i\partial_t v = \Delta v - 2\nabla A \cdot \nabla v + Vv - \lambda e^{-pA} v|v|^p, \quad \lambda \geq 0 \tag{6.1}$$

and $\delta_0 > 0$ will denote a fixed given number such that $v(0) \in H^2_{\delta_0}$. The following result has already been used in the linear case ($\lambda = 0$).

Proposition 6.1 *Let v be solution to (6.1), then we have the following identity:*

$$\frac{d}{dt} \mathcal{E}_{A,V}(v(t)) = -\lambda \mathcal{H}_{A,V}(v(t)), \tag{6.2}$$

where

$$\mathcal{E}_{A,V}(v(t)) = \int_{\mathbb{R}^2} |\Delta v(t)|^2 e^{-2A} dx + \mathcal{F}_{A,V}(v(t)) - \lambda \mathcal{G}_{A,V}(v(t))$$

and the energies $\mathcal{F}_{A,V}, \mathcal{G}_{A,V}, \mathcal{H}_{A,V}$ are defined as follows:

$$\begin{aligned} \mathcal{F}_{A,V}(v(t)) &:= -4\text{Re} \int_{\mathbb{R}^2} \Delta v(t) \nabla A \cdot \nabla \bar{v}(t) e^{-2A} dx \\ &\quad - 4 \int_{\mathbb{R}^2} (\nabla A \cdot \nabla v)^2 e^{-2A} dx + 2\text{Re} \int_{\mathbb{R}^2} v(t) V \nabla \bar{v}(t) \cdot \nabla (e^{-2A}) dx \\ &\quad + 2\text{Re} \int_{\mathbb{R}^2} \Delta v(t) \bar{v}(t) V e^{-2A} dx + \int_{\mathbb{R}^2} |v(t)|^2 V^2 e^{-2A} dx, \\ \mathcal{G}_{A,V}(v(t)) &:= - \int_{\mathbb{R}^2} |\nabla v(t)|^2 |v(t)|^p e^{-(p+2)A} dx \\ &\quad - 2\text{Re} \int_{\mathbb{R}^2} v(t) \nabla (|v(t)|^p) \cdot \nabla \bar{v}(t) e^{-(p+2)A} dx \end{aligned}$$

$$\begin{aligned}
 & + \frac{p}{4} \int_{\mathbb{R}^2} |\nabla(|v(t)|^2)|^2 |v(t)|^{p-2} e^{-(p+2)A} dx \\
 & + \frac{2}{p+2} \int_{\mathbb{R}^2} |v(t)|^{p+2} V e^{-(p+2)A} dx \\
 & + 2p \operatorname{Re} \int_{\mathbb{R}^2} v(t) |v(t)|^p \nabla A \cdot \nabla \bar{v}(t) e^{-(p+2)A} dx, \\
 \mathcal{H}_{A,v}(v(t)) := & - \int_{\mathbb{R}^2} |\nabla v(t)|^2 \partial_t (|v(t)|^p) e^{-(p+2)A} dx \\
 & - 2 \operatorname{Re} \int_{\mathbb{R}^2} \partial_t v(t) \nabla (|v(t)|^p) \cdot \nabla \bar{v}(t) e^{-(p+2)A} dx \\
 & - \frac{p}{4} \int_{\mathbb{R}^2} |\nabla(|v(t)|^2)|^2 \partial_t (|v(t)|^{p-2}) e^{-(p+2)A} dx \\
 & + 2p \operatorname{Re} \int_{\mathbb{R}^2} \partial_t (v(t) |v(t)|^p) \nabla A \cdot \nabla \bar{v}(t) e^{-(p+2)A} dx.
 \end{aligned}$$

Proof In the sequel, we denote by (\cdot, \cdot) the usual scalar product in L^2 . We start with the following computation:

$$\begin{aligned}
 \frac{d}{dt} (\Delta v, \Delta v e^{-2A}) & = 2 \operatorname{Re} (\partial_t \Delta v, \Delta v e^{-2A}) \\
 & = 2 \operatorname{Re} (\partial_t \Delta v, \iota \partial_t v e^{-2A}) + 4 \operatorname{Re} (\partial_t \Delta v, \nabla A \cdot \nabla v e^{-2A}) \\
 & \quad - 2 \operatorname{Re} (\partial_t \Delta v, v V e^{-2A}) + 2 \lambda \operatorname{Re} (\partial_t \Delta v, v |v|^p e^{-(p+2)A}) \\
 & = I + II + III + IV.
 \end{aligned} \tag{6.3}$$

Notice that

$$\begin{aligned}
 I & = -2 \operatorname{Re} (\partial_t \nabla v, \iota \partial_t \nabla v e^{-2A}) - 2 \operatorname{Re} (\partial_t \nabla v, \iota \partial_t v \nabla (e^{-2A})) \\
 & = -2 \operatorname{Im} (\partial_t \nabla v, \partial_t v \nabla (e^{-2A})).
 \end{aligned} \tag{6.4}$$

Moreover we have

$$\begin{aligned}
 II & = 4 \operatorname{Re} (\partial_t \Delta v, \nabla A \cdot \nabla v e^{-2A}) \\
 & = 2 \operatorname{Re} (\Delta v, \partial_t \nabla v \cdot \nabla (e^{-2A})) + 4 \frac{d}{dt} \operatorname{Re} (\Delta v, \nabla A \cdot \nabla v e^{-2A})
 \end{aligned}$$

and using again the equation solved by v , we obtain

$$\begin{aligned}
 II & = 4 \frac{d}{dt} \operatorname{Re} (\Delta v, \nabla A \cdot \nabla v e^{-2A}) - 2 \operatorname{Im} (\partial_t v, \partial_t \nabla v \cdot \nabla (e^{-2A})) \\
 & \quad + 4 \operatorname{Re} (\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla (e^{-2A})) - 2 \operatorname{Re} (V v, \partial_t \nabla v \cdot \nabla (e^{-2A})) \\
 & \quad + 2 \lambda \operatorname{Re} (e^{-pA} v |v|^p, \partial_t \nabla v \cdot \nabla (e^{-2A})) \\
 & = 4 \frac{d}{dt} \operatorname{Re} (\Delta v, \nabla A \cdot \nabla v e^{-2A}) + 2 \operatorname{Im} (\partial_t \nabla v, \partial_t v \nabla (e^{-2A}))
 \end{aligned}$$

$$\begin{aligned}
& + 4\operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla(e^{-2A})) - 2\operatorname{Re}(Vv, \partial_t \nabla v \cdot \nabla(e^{-2A})) \\
& + 2\lambda \operatorname{Re}(e^{-pA} v |v|^p, \partial_t \nabla v \cdot \nabla(e^{-2A})).
\end{aligned}$$

Hence by (6.4) we get

$$\begin{aligned}
II &= 4 \frac{d}{dt} \operatorname{Re}(\Delta v, \nabla A \cdot \nabla v e^{-2A}) - I + 4\operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla(e^{-2A})) \\
& \quad - 2\operatorname{Re}(Vv, \partial_t \nabla v \cdot \nabla(e^{-2A})) + 2\lambda \operatorname{Re}(e^{-pA} v |v|^p, \partial_t \nabla v \cdot \nabla(e^{-2A})) \\
&= 4 \frac{d}{dt} \operatorname{Re}(\Delta v, \nabla A \cdot \nabla v e^{-2A}) - I + 4\operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla(e^{-2A})) \\
& \quad - 2 \frac{d}{dt} \operatorname{Re}(Vv, \nabla v \cdot \nabla(e^{-2A})) + 2\operatorname{Re}(V \partial_t v, \nabla v \cdot \nabla(e^{-2A})) \\
& \quad + 2\lambda \operatorname{Re}(e^{-pA} v |v|^p, \partial_t \nabla v \cdot \nabla(e^{-2A})).
\end{aligned}$$

Namely, we have

$$\begin{aligned}
I + II &= 2\operatorname{Re}(V \partial_t v, \nabla v \cdot \nabla(e^{-2A})) + 4 \frac{d}{dt} \operatorname{Re}(\Delta v, \nabla A \cdot \nabla v e^{-2A}) \\
& \quad + 4\operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla(e^{-2A})) - 2 \frac{d}{dt} \operatorname{Re}(Vv, \nabla v \cdot \nabla(e^{-2A})) \\
& \quad + 2\lambda \operatorname{Re}(e^{-pA} v |v|^p, \partial_t \nabla v \cdot \nabla(e^{-2A})).
\end{aligned}$$

On the other hand, we can compute the last term in the above identity

$$\begin{aligned}
& 2\lambda \operatorname{Re}(e^{-pA} v |v|^p, \partial_t \nabla v \cdot \nabla(e^{-2A})) \\
&= -2\lambda \operatorname{Re}(\nabla(e^{-pA} v |v|^p), \partial_t \nabla v e^{-2A}) - 2\lambda \operatorname{Re}(e^{-pA} v |v|^p, \partial_t \Delta v e^{-2A}) \\
&= -2\lambda \operatorname{Re}(\nabla(e^{-pA} v |v|^p), \partial_t \nabla v e^{-2A}) - IV \\
&= -2\lambda \operatorname{Re}(e^{-pA} \nabla v |v|^p, \partial_t \nabla v e^{-2A}) - 2\lambda \operatorname{Re}(e^{-pA} v \nabla(|v|^p), \partial_t \nabla v e^{-2A}) \\
& \quad - 2\lambda \operatorname{Re}(\nabla(e^{-pA} v |v|^p), \partial_t \nabla v e^{-2A}) - IV \\
&= -\lambda(\partial_t(|\nabla v|^2)|v|^p, e^{-(p+2)A}) - 2\lambda \frac{d}{dt} \operatorname{Re}(v \nabla(|v|^p), \nabla v e^{-(p+2)A}) \\
& \quad + 2\lambda \operatorname{Re}(\partial_t v \nabla(|v|^p), \nabla v e^{-(p+2)A}) + 2\lambda \operatorname{Re}(v \nabla \partial_t(|v|^p), \nabla v e^{-(p+2)A}) \\
& \quad - 2\lambda \operatorname{Re}(\nabla(e^{-pA} v |v|^p), \partial_t \nabla v e^{-2A}) - IV \\
&= -\lambda \frac{d}{dt} (|\nabla v|^2 |v|^p, e^{-(p+2)A}) + \lambda (|\nabla v|^2 \partial_t(|v|^p), e^{-(p+2)A}) \\
& \quad - 2\lambda \frac{d}{dt} \operatorname{Re}(v \nabla(|v|^p), \nabla v e^{-(p+2)A}) + 2\lambda \operatorname{Re}(\partial_t v \nabla(|v|^p), \nabla v e^{-(p+2)A}) \\
& \quad + \frac{\lambda p}{2} (\partial_t(\nabla(|v|^2)|v|^{p-2}), \nabla(|v|^2) e^{-(p+2)A}) \\
& \quad - 2\lambda \operatorname{Re}(\nabla(e^{-pA} v |v|^p), \partial_t \nabla v e^{-2A}) - IV \\
&= -\lambda \frac{d}{dt} (|\nabla v|^2 |v|^p, e^{-(p+2)A}) + \lambda (|\nabla v|^2 \partial_t(|v|^p), e^{-(p+2)A})
\end{aligned}$$

$$\begin{aligned}
 & -2\lambda \frac{d}{dt} \operatorname{Re}(v \nabla(|v|^p), \nabla v e^{-(p+2)A}) + 2\lambda \operatorname{Re}(\partial_t v \nabla(|v|^p), \nabla v e^{-(p+2)A}) \\
 & + \frac{\lambda p}{4} (\partial_t (|\nabla(|v|^2)|^2) |v|^{p-2}, e^{-(p+2)A}) + \frac{\lambda p}{2} (\nabla(|v|^2) \partial_t (|v|^{p-2}), \nabla(|v|^2) e^{-(p+2)A}) \\
 & - 2\lambda \operatorname{Re}(\nabla(e^{-pA}) v |v|^p, \partial_t \nabla v e^{-2A}) - IV \\
 = & -\lambda \frac{d}{dt} (|\nabla v|^2 |v|^p, e^{-(p+2)A}) + \lambda (|\nabla v|^2 \partial_t (|v|^p), e^{-(p+2)A}) \\
 & - 2\lambda \frac{d}{dt} \operatorname{Re}(v \nabla(|v|^p), \nabla v e^{-(p+2)A}) + 2\lambda \operatorname{Re}(\partial_t v \nabla(|v|^p), \nabla v e^{-(p+2)A}) \\
 & + \frac{\lambda p}{4} \frac{d}{dt} (|\nabla(|v|^2)|^2 |v|^{p-2}, e^{-(p+2)A}) + \frac{\lambda p}{4} (|\nabla(|v|^2)|^2 \partial_t (|v|^{p-2}), e^{-(p+2)A}) \\
 & - 2\lambda \operatorname{Re}(\nabla(e^{-pA}) v |v|^p, \partial_t \nabla v e^{-2A}) - IV.
 \end{aligned}$$

By combining the identities above we get

$$\begin{aligned}
 I + II + IV & = 2\operatorname{Re}(\partial_t v V, \nabla v \cdot \nabla(e^{-2A})) + 2 \frac{d}{dt} \operatorname{Re}(\Delta v, 2\nabla A \cdot \nabla v e^{-2A}) \\
 & + 4\operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla(e^{-2A})) - 2 \frac{d}{dt} \operatorname{Re}(V v, \nabla v \cdot \nabla(e^{-2A})) \\
 & - \lambda \frac{d}{dt} \operatorname{Re}(|\nabla v|^2 |v|^p, e^{-(p+2)A}) + \lambda \operatorname{Re}(|\nabla v|^2 \partial_t (|v|^p), e^{-(p+2)A}) \\
 & - 2\lambda \frac{d}{dt} \operatorname{Re}(v \nabla(|v|^p), \nabla v e^{-(p+2)A}) + 2\lambda \operatorname{Re}(\partial_t v \nabla(|v|^p), \nabla v e^{-(p+2)A}) \\
 & + \frac{\lambda p}{4} \frac{d}{dt} (|\nabla(|v|^2)|^2 |v|^{p-2}, e^{-(p+2)A}) + \frac{\lambda p}{4} (|\nabla(|v|^2)|^2 \partial_t (|v|^{p-2}), e^{-(p+2)A}) \\
 & - 2\lambda \operatorname{Re}(\nabla(e^{-pA}) v |v|^p, \partial_t \nabla v e^{-2A}).
 \end{aligned}$$

Since $III = -2 \frac{d}{dt} \operatorname{Re}(\Delta v, V v e^{-2A}) + 2\operatorname{Re}(\Delta v, \partial_t v V e^{-2A})$, we obtain the following identity:

$$\begin{aligned}
 I + II + III + IV & = 2\operatorname{Re}(V \partial_t v, \nabla v \cdot \nabla(e^{-2A})) + 4 \frac{d}{dt} \operatorname{Re}(\Delta v, \nabla A \cdot \nabla v e^{-2A}) \\
 & + 4\operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla(e^{-2A})) - 2 \frac{d}{dt} \operatorname{Re}(V v, \nabla v \cdot \nabla(e^{-2A})) \\
 & - \lambda \frac{d}{dt} \operatorname{Re}(|\nabla v|^2 |v|^p, e^{-(p+2)A}) + \lambda \operatorname{Re}(|\nabla v|^2 \partial_t (|v|^p), e^{-(p+2)A}) \\
 & - 2\lambda \frac{d}{dt} \operatorname{Re}(v \nabla(|v|^p), \nabla v e^{-(p+2)A}) + 2\lambda \operatorname{Re}(\partial_t v \nabla(|v|^p), \nabla v e^{-(p+2)A}) \\
 & + \frac{\lambda p}{4} \frac{d}{dt} (|\nabla(|v|^2)|^2 |v|^{p-2}, e^{-(p+2)A}) + \frac{\lambda p}{4} (|\nabla(|v|^2)|^2 \partial_t (|v|^{p-2}), e^{-(p+2)A}) \\
 & - 2\lambda \frac{d}{dt} \operatorname{Re}(\nabla(e^{-pA}) v |v|^p, \nabla v e^{-2A}) + 2\lambda \operatorname{Re}(\nabla(e^{-pA}) \partial_t (v |v|^p), \nabla v e^{-2A})
 \end{aligned}$$

$$-2 \frac{d}{dt} \operatorname{Re}(\Delta v, v V e^{-2A}) + 2 \operatorname{Re}(\Delta v, \partial_t v V e^{-2A}).$$

Next, by using the equation solved by v , we compute the first, the third, and the last term on the r.h.s. above as

$$\begin{aligned} & 2 \operatorname{Re}(\partial_t v V, \nabla v \cdot \nabla(e^{-2A})) + 4 \operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla(e^{-2A})) \\ & \quad + 2 \operatorname{Re}(\Delta v, \partial_t v V e^{-2A}) \\ & = 2 \operatorname{Re}(\partial_t v V, \nabla v \cdot \nabla(e^{-2A})) - 8 \operatorname{Re}(\nabla A \cdot \nabla v, \partial_t \nabla v \cdot \nabla A e^{-2A}) \\ & \quad + 4 \operatorname{Re}(\nabla v \cdot \nabla A, \partial_t v V e^{-2A}) - 2 \operatorname{Re}(V v, \partial_t v V e^{-2A}) \\ & \quad + 2 \lambda \operatorname{Re}(e^{-pA} v |v|^p, \partial_t v V e^{-2A}) \\ & = -4 \operatorname{Re}(\partial_t v V, \nabla v \cdot \nabla A e^{-2A}) - 4 \frac{d}{dt} ((\nabla A \cdot \nabla v)^2, e^{-2A}) \\ & \quad + 4 \operatorname{Re}(\nabla v \cdot \nabla A, \partial_t v V e^{-2A}) - 2 \operatorname{Re}(V v, \partial_t v V e^{-2A}) \\ & \quad + 2 \lambda \operatorname{Re}(v |v|^p, \partial_t v V e^{-(p+2)A}) \\ & = -4 \frac{d}{dt} ((\nabla A \cdot \nabla v)^2, e^{-2A}) - \frac{d}{dt} (V |v|^2, V e^{-2A}) \\ & \quad + \frac{2 \lambda}{p+2} \frac{d}{dt} \operatorname{Re}(|v|^{p+2}, V e^{-(p+2)A}). \end{aligned}$$

The proof of Proposition 6.1 easily follows by combining the above two identities with (6.3). \square

Next, we provide some useful estimates on the energies $\mathcal{F}_{A,V}$, $\mathcal{G}_{A,V}$, $\mathcal{H}_{A,V}$. We recall that the quantity $|(A, V)|_{\delta,r}$ has been introduced in (4.8).

Proposition 6.2 *For every $\delta > 0$ and $r \in (2, \infty)$, we have*

$$|\mathcal{F}_{A,V}(w)| \leq \mathcal{P}\left(|(A, V)|_{\delta,r}\right) \left(\|e^{-A} \Delta w\|_{L^2} \|w\|_{W_\delta^{1, \frac{2r}{r-2}}} + \|w\|_{W_\delta^{1, \frac{2r}{r-2}}}^2 \right) \quad (6.5)$$

for any generic time-independent function w . Moreover, there exists $\bar{\delta} > 0$ such that for any given $T > 0$ and for every $\delta \in (0, \bar{\delta})$ and $r \in [4, \infty)$, we have

$$\begin{aligned} \sup_{t \in (0, T)} |\mathcal{F}_{A,V}(v(t))| & \leq \mathcal{P}\left(|(A, V)|_{\delta,r}\right) \mathcal{P}\left(\|v(0)\|_{H_{\delta_0}^1}\right) \\ & \quad \times \left(\|e^{-A} \Delta v\|_{L^\infty((0, T); L^2)} \|v\|_{L^\infty((0, T); H_{\delta}^{\frac{r+2}{2r}})} + \|v\|_{L^\infty((0, T); H_{\delta}^2)}^{\frac{r+2}{r}} \right). \end{aligned} \quad (6.6)$$

Proof By the Hölder inequality we get:

$$\begin{aligned} |\mathcal{F}_{A,V}(w)| & \leq C \|e^{-A} \Delta w\|_{L^2} \|\langle x \rangle^\delta \nabla w(t)\|_{L^{\frac{2r}{r-2}}} \|\langle x \rangle^{-\delta} \nabla A e^{-A}\|_{L^r} \\ & \quad + C \|\langle x \rangle^{-\delta} e^{-A} \nabla A\|_{L^r}^2 \|\langle x \rangle^\delta \nabla w\|_{L^{\frac{2r}{r-2}}}^2 \end{aligned}$$

$$\begin{aligned}
 &+ C \|\langle x \rangle^{-2\delta} \nabla A e^{-2A} V\|_{L^{\frac{r}{2}}} \|\langle x \rangle^\delta \nabla w\|_{L^{\frac{2r}{r-2}}} \|\langle x \rangle^\delta w\|_{L^{\frac{2r}{r-2}}} \\
 &+ C \|e^{-A} \Delta w\|_{L^2} \|\langle x \rangle^\delta w\|_{L^{\frac{2r}{r-2}}} \|\langle x \rangle^{-\delta} V e^{-A}\|_{L^r} \\
 &+ C \|\langle x \rangle^{-\delta} V e^{-A}\|_{L^r}^2 \|\langle x \rangle^\delta w\|_{L^{\frac{2r}{r-2}}}^2,
 \end{aligned}$$

which in turn implies (6.5). On the other hand, by (5.6) for $\delta \in (0, \frac{r-2}{36(3r+2)})$, we have

$$\begin{aligned}
 \|v\|_{L^\infty((0,T); W_\delta^{1, \frac{2r}{r-2}})} &\leq C \|v\|_{L^\infty((0,T); H_{-\delta}^2)} \mathcal{P}\left(\|v(0)\|_{H_{4(\frac{3r+2}{r-2})\delta}^1}\right) \\
 &\leq C \|v\|_{L^\infty((0,T); H_{-\delta}^2)} \mathcal{P}\left(\|v(0)\|_{H_{\delta_0}^1}\right),
 \end{aligned} \tag{6.7}$$

where at the last step we used that for $r > 4$ we have $4(\frac{3r+2}{r-2})\delta \leq 28\delta$ and hence $\|v(0)\|_{H_{4(\frac{3r+2}{r-2})\delta}^1} \leq \|v(0)\|_{H_{\delta_0}^1}$ provided that $\delta > 0$ is small enough. The proof is complete. □

We now have the necessary tools to provide the proof of (4.12) in Proposition 4.1.

Proof of (4.12) For simplicity, we denote $w(t) = S_{A,V}(t)\varphi$. To prove (4.12), we use (6.2) with $\lambda = 0$ to obtain

$$\frac{d}{dt} \int_{\mathbb{R}^2} \left(|\Delta w(t)|^2 e^{-2A} + \mathcal{F}_{A,V}(w(t)) \right) dx = 0,$$

which in turn after integration in time implies

$$\int_{\mathbb{R}^2} |\Delta w(t)|^2 e^{-2A} dx = \int_{\mathbb{R}^2} |\Delta w(0)|^2 e^{-2A} dx - \mathcal{F}_{A,V}(w(t)) + \mathcal{F}_{A,V}(w(0)).$$

By (6.5) (whose proof only involves the Hölder inequality) in conjunction with (4.6), we get that for every $\mu > 0$,

$$\begin{aligned}
 &\int_{\mathbb{R}^2} (|w(t)|^2 + |\Delta w(t)|^2) e^{-2A} dx \\
 &\leq \int_{\mathbb{R}^2} (|w(0)|^2 + |\Delta w(0)|^2) e^{-2A} dx \\
 &\quad + \mathcal{P}\left(|(A, V)|_{\delta,r}\right) \left(\mu \|e^{-A} \Delta w(t)\|_{L^2}^2 + \left(1 + \frac{1}{2\mu}\right) \|w(t)\|_{W_\delta^{1, \frac{2r}{r-2}}}^2 \right) \\
 &\quad + \mathcal{P}\left(|(A, V)|_{\delta,r}\right) \left(\|e^{-A} \Delta w(0)\|_{L^2} \|w(0)\|_{W_\delta^{1, \frac{2r}{r-2}}} + \|w(0)\|_{W_\delta^{1, \frac{2r}{r-2}}}^2 \right).
 \end{aligned}$$

In particular, we can choose $\mu = \frac{1}{2\mathcal{P}(|(A,V)|_{\delta,r})}$ and get

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^2} (|w(t)|^2 + |\Delta w(t)|^2) e^{-2A} dx \\ & \leq \int_{\mathbb{R}^2} (|w(0)|^2 + |\Delta w(0)|^2) e^{-2A} dx + \mathcal{P}(|(A, V)|_{\delta,r}) \|w(t)\|_{W_\delta^1, \frac{2r}{r-2}}^2 \\ & \quad + \mathcal{P}(|(A, V)|_{\delta,r}) \left(\|e^{-A} \Delta w(0)\|_{L^2} \|w(0)\|_{W_\delta^1, \frac{2r}{r-2}} + \|w(0)\|_{W_\delta^1, \frac{2r}{r-2}}^2 \right) \end{aligned} \tag{6.8}$$

Then, notice that by (2.43), where we choose $q = \frac{2r}{r-2}$, we get

$$\begin{aligned} \|w(t)\|_{W_\delta^1, \frac{2r}{r-2}} & \leq C \|w(t)\|_{H_\delta^{\frac{r+2}{2r}}}^{\frac{r+2}{2r}} \|w(t)\|_{L^2_{(\frac{3r+2}{r-2})\delta}}^{\frac{r-2}{2r}} \\ & \leq C \|w(t)\|_{H_\delta^{\frac{r+2}{2r}}}^{\frac{r+2}{2r}} \|w(0)\|_{H^1_{(\frac{3r+2}{r-2})\delta+}}^{\frac{r-2}{2r}}, \end{aligned} \tag{6.9}$$

where we used at the last step (4.10) (recall our assumptions $\frac{\delta}{2} + 2\delta^+ < \frac{r-2}{3r+2}$ and $\delta^+ > \delta$). We conclude by combining (6.8), (6.9) and (4.5). \square

We move on and prove the following estimates for $\mathcal{G}_{A,V}$ and $\mathcal{H}_{A,V}$.

Proposition 6.3 *For any given $T > 0$ and $\eta_0 \in (0, \frac{1}{3})$, there exists $\bar{\delta} > 0$ such that for every $r \in [4, \infty)$ and $\delta \in (0, \bar{\delta})$, we have the bound*

$$\begin{aligned} & \sup_{t \in (0,T)} |\mathcal{G}_{A,V}(v(t))| \\ & \leq |(A, V)|_{\delta,r} \mathcal{P} \left(\|v(0)\|_{H_{\delta_0}^1} \right) \left(1 + \|v\|_{L^\infty((0,T); H_{-\delta}^2)}^{\frac{1}{r} + \frac{1}{2} + \eta_0(p+2)} \right). \end{aligned} \tag{6.10}$$

Proof We have by the Hölder inequality

$$\begin{aligned} |\mathcal{G}_{A,V}(v(t))| & \leq C \|\nabla v(t)\|_{L^{\frac{2r}{r-2}}}^2 \|\langle x \rangle^\delta |v(t)|^p\|_{L^{\frac{r}{2}}} \|\langle x \rangle^{-\delta} e^{-(p+2)A}\|_{L^\infty} \\ & \quad + C \|\langle x \rangle^\delta |v(t)|^{p+2}\|_{L^{\frac{r}{r-1}}} \|\langle x \rangle^{-\delta} V e^{-(p+2)A}\|_{L^r} \\ & \quad + C \|\nabla v(t)\|_{L^{\frac{2r}{r-2}}} \|\langle x \rangle^\delta |v(t)|^{p+1}\|_{L^2} \|\langle x \rangle^{-\delta} \nabla A e^{-(p+2)A}\|_{L^r}. \end{aligned} \tag{6.11}$$

Next, notice that by combining the Sobolev embedding $L_\rho^q \subset H_\rho^{\frac{2}{2-\eta_0}}$ for $q \in [2, \infty]$ with (5.12), we get

$$\|\langle x \rangle^\delta |v(t)|^p\|_{L^{\frac{r}{2}}} = \|v(t)\|_{L^{\frac{rp}{\delta}}}^p \leq \mathcal{P} \left(\|v(0)\|_{H_{\delta_0}^1} \right) \|v(t)\|_{H_{-\delta}^2}^{\eta_0 p}$$

$$\begin{aligned} \|\langle x \rangle^\delta |v(t)|^{p+2}\|_{L^{\frac{r}{r-1}}} &= \|v(t)\|_{L^{\frac{r(p+2)}{\frac{\delta}{p+2}}}}^{p+2} \leq \mathcal{P}\left(\|v(0)\|_{H_{\delta_0}^1}\right) \|v(t)\|_{H_{-\delta}^2}^{\eta_0(p+2)} \\ \|\langle x \rangle^\delta |v(t)|^{p+1}\|_{L^2} &= \|v(t)\|_{L^{\frac{2(p+1)}{\frac{\delta}{p+1}}}}^{p+1} \leq \mathcal{P}\left(\|v(0)\|_{H_{\delta_0}^1}\right) \|v(t)\|_{H_{-\delta}^2}^{\eta_0(p+1)}. \end{aligned}$$

By combining the estimates above with (6.11) and (5.6), where we choose $q = \frac{2r}{r-2}$, we conclude the proof. \square

Proposition 6.4 *For any given $T > 0$ and $\eta_0 \in (0, \frac{1}{3})$, there exists $\bar{\delta} > 0$ such that for every $r \in [4, \infty)$ and $\delta \in (0, \bar{\delta})$, we have the bound*

$$\begin{aligned} \int_0^T |\mathcal{H}_{A,V}(v(\tau))| d\tau &\leq |(A, V)|_{r,\delta} \mathcal{P}\left(\|v(0)\|_{H_{\delta_0}^1}\right) \|\partial_t v e^{-A}\|_{L^\infty((0,T);L^2)} \\ &\times \left(1 + \|\nabla v\|_{L^2((0,T);L^4)}^2\right) \left(1 + \|v\|_{L^\infty((0,T);H_{-\delta}^2)}\right)^{\eta_0 p}. \end{aligned} \tag{6.12}$$

Proof We only focus on the case when $p > 1$. The case $p = 1$ will follow in a similar (and easier) manner. Notice that by the Hölder inequality, one can estimate

$$\begin{aligned} |\mathcal{H}_{A,V}(v(t))| &\leq C \|\nabla v(t)\|_{L^4}^2 \|\partial_t v e^{-A}\|_{L^2} \|\langle x \rangle^\delta |v(t)|^{p-1}\|_{L^\infty} \|\langle x \rangle^{-\delta} e^{-(p+1)A}\|_{L^\infty} \\ &+ C \|\nabla v(t)\|_{L^4} \|\partial_t v e^{-A}\|_{L^2} \|\langle x \rangle^\delta |v(t)|^p\|_{L^{\frac{4r}{r-4}}} \|\langle x \rangle^{-\delta} |\nabla A| e^{-(p+1)A}\|_{L^r}. \end{aligned}$$

By combining (5.12) with the Sobolev embedding $H_\rho^{\frac{2}{2-\eta_0}} \subset L_\rho^q$ for $q \in [2, \infty]$, we get

$$\|\langle x \rangle^\delta |v(t)|^p\|_{L^{\frac{4r}{r-4}}} = \|v(t)\|_{L^{\frac{4rp}{\frac{\delta}{p}}}}^p \leq \mathcal{P}\left(\|v(0)\|_{H_{\delta_0}^1}\right) \|v(t)\|_{H_{-\delta}^2}^{\eta_0 p}$$

and

$$\|\langle x \rangle^\delta |v(t)|^{p-1}\|_{L^\infty} = \|v(t)\|_{L^{\frac{\delta}{p-1}}}^{p-1} \leq \mathcal{P}\left(\|v(0)\|_{H_{\delta_0}^1}\right) \|v(t)\|_{H_{-\delta}^2}^{\eta_0(p-1)}.$$

After integration in time and the Hölder inequality w.r.t. time variable, we obtain (6.12). \square

7 H^2 a-priori bound

We introduce the following family of regularized and localized potentials, for $\varepsilon > 0$ and $n \in \mathbb{N}$:

$$A_{\varepsilon,n} = \theta_n Y_\varepsilon, \quad V_{\varepsilon,n} = \theta_n \widetilde{\nabla Y_\varepsilon^2}.$$

where $\theta_n(x) = \theta(\frac{x}{n})$, $\theta \in C_0^\infty(\mathbb{R}^2)$, $\theta \geq 0$, and $\theta(0) = 1$. We first notice that due to (3.9), if we choose $A = A_{\varepsilon,n}$ and $V = V_{\varepsilon,n}$, then the condition (4.3) holds uniformly w.r.t. n and ε with the constant C replaced by a random constant $C(\omega)$. The same for (4.4) which is satisfied uniformly w.r.t. n, ε with a random constant $C(\omega)$. In order to prove this fact combine (3.15), (3.17), (3.2), (3.3), and [12] (more precisely, see at page 1160 three lines below (33)).

We introduce, following the notation (4.2), the operators

$$H_{\varepsilon,n} = H_{A_{\varepsilon,n}, V_{\varepsilon,n}}$$

as well as the associated nonlinear Cauchy problem

$$i \partial_t v_{\varepsilon,n} = H_{\varepsilon,n} v_{\varepsilon,n} + \lambda e^{-pA_{\varepsilon,n}} v_{\varepsilon,n} |v_{\varepsilon,n}|^p, \quad v_{\varepsilon,n}(0) = v_0, \quad \lambda \geq 0 \tag{7.1}$$

where $v_0 = u_0 e^Y \in H_{\delta_0}^2$ for some fixed $\delta_0 > 0$. The main point in this section is to establish the following a-propri bound for any sufficiently small $\delta > 0$:

$$\sup_{n \in \mathbb{N}} \|v_{\varepsilon,n}\|_{L^\infty((0,T); H_{\delta}^2)} \leq C(\omega) |\ln \varepsilon|^C \mathcal{P}\left(\|v_0\|_{H_{\delta_0}^2}\right), \quad \text{a.s. } \omega \tag{7.2}$$

for any given $T > 0$.

We now focus on proving the bound (7.2). Recall the quantity $|(A, V)|_{\delta,r}$ defined in (4.8). Assume that $\delta > 0$ has been fixed in such a way that (6.6), (6.10) and (6.12) are satisfied with the choice $A = A_{\varepsilon,n}$, $V = V_{\varepsilon,n}$. On the other hand, one can show that once δ is fixed, one can choose $r \geq 4$ large enough in such a way that

$$\sup_{n \in \mathbb{N}} |(A_{\varepsilon,n}, V_{\varepsilon,n})|_{\delta,r} \leq C(\omega) \mathcal{O}(|\ln \varepsilon|^C), \quad \text{a.s. } \omega. \tag{7.3}$$

This bound follows from (4.8), (3.1), and (3.9) (the cut-off $\theta(\frac{x}{n})$ can be easily handled by an elementary argument and the bounds are uniform in $n \in \mathbb{N}$). Moreover, by (6.2) we get

$$\begin{aligned} & \int_{\mathbb{R}^2} |\Delta v_{\varepsilon,n}(t)|^2 e^{-2A_{\varepsilon,n}} dx \\ &= \int_{\mathbb{R}^2} |\Delta v_{\varepsilon,n}(0)|^2 e^{-2A_{\varepsilon,n}} dx - \mathcal{F}_{\varepsilon,n}(v_{\varepsilon,n}(t)) + \mathcal{F}_{\varepsilon,n}(v_{\varepsilon,n}(0)) \\ & \quad - \lambda \mathcal{G}_{\varepsilon,n}(v_{\varepsilon,n}(t)) + \lambda \mathcal{G}_{\varepsilon,n}(v_{\varepsilon,n}(0)) + \int_0^t \mathcal{H}_{\varepsilon,n}(v_{\varepsilon,n}(\tau)) d\tau \end{aligned} \tag{7.4}$$

where

$$\mathcal{F}_{\varepsilon,n} = \mathcal{F}_{A_{\varepsilon,n}, V_{\varepsilon,n}}, \quad \mathcal{G}_{\varepsilon,n} = \mathcal{G}_{A_{\varepsilon,n}, V_{\varepsilon,n}}, \quad \mathcal{H}_{\varepsilon,n} = \mathcal{H}_{A_{\varepsilon,n}, V_{\varepsilon,n}}$$

are the energies defined along Proposition 6.1 with $A = A_{\varepsilon,n}$ and $V = V_{\varepsilon,n}$. Next, we apply (6.6), (6.10) and (6.12) with $A = A_{\varepsilon,n}$ and $V = V_{\varepsilon,n}$. By using (6.6) in

conjunction with (7.3) and by choosing r large enough, we deduce

$$\begin{aligned} \sup_{t \in [0, T]} |\mathcal{F}_{\varepsilon, n}(v_{\varepsilon, n}(t))| &\leq C(\omega) |\ln \varepsilon|^C \mathcal{P} \left(\|v_{\varepsilon, n}(0)\|_{H_{\delta_0}^1} \right) \\ &\times \left(\|e^{-A_{\varepsilon, n}} \Delta v_{\varepsilon, n}\|_{L^\infty((0, T); L^2)} \|v_{\varepsilon, n}\|_{L^\infty((0, T); H_{-\delta}^2)}^{1^-} + \|v_{\varepsilon, n}\|_{L^\infty((0, T); H_{-\delta}^2)}^{2^-} \right) \end{aligned} \tag{7.5}$$

for suitable $1^- \in (0, 1)$ and $2^- \in (1, 2)$. Moreover, we can choose η_0 in such a way that

$$\frac{1}{r} + \frac{1}{2} + \eta_0(p + 2) \in (0, 1); \tag{7.6}$$

$$\frac{7}{6}(p + 1)\eta_0 + \frac{155}{192} + \eta_0 p \in (0, 1); \tag{7.7}$$

$$\eta_0(p + 1) \in (0, 1). \tag{7.8}$$

Hence by (6.10), (7.3), and (7.6), we obtain

$$\begin{aligned} \sup_{t \in [0, T]} |\mathcal{G}_{\varepsilon, n}(v_{\varepsilon, n}(t))| \\ \leq C(\omega) |\ln \varepsilon|^C \mathcal{P} \left(\|v_{\varepsilon, n}(0)\|_{H_{\delta_0}^1} \right) \left(1 + \|v_{\varepsilon, n}(t)\|_{L^\infty((0, T); H_{-\delta}^2)}^{1^-} \right), \end{aligned} \tag{7.9}$$

where $1^- \in (0, 1)$. Notice also that we have (by choosing r even larger than above)

$$\sup_n |(A_{\varepsilon, n}, V_{\varepsilon, n})|_{\frac{\delta^2}{4}, r} \leq C(\omega) |\ln \varepsilon|^C, \quad \sup_n |(A_{\varepsilon, n}, V_{\varepsilon, n})|_{\tilde{\delta}, r} \leq C(\omega) |\ln \varepsilon|^C \text{ a.s. } \omega,$$

where $\tilde{\delta} > 0$ is the one that appears in (5.16). Moreover, by (3.1) in conjunction with the fact that in (5.16) we can assume $\frac{\delta_1}{s_1} > 1$, we also have (see (4.17))

$$\sup_n \|(A_{\varepsilon, n}, V_{\varepsilon, n})\|_{\delta_1, \frac{2}{s_1}} \leq C(\omega) O(|\ln \varepsilon|^C) \text{ a.s. } \omega \tag{7.10}$$

and hence by combining (6.12) with (5.16) and (7.7), we get

$$\begin{aligned} \int_0^T |\mathcal{H}_{\varepsilon, n}(v_{\varepsilon, n}(\tau))| d\tau &\leq C(\omega) |\ln \varepsilon|^C \mathcal{P} \left(\|v_{\varepsilon, n}(0)\|_{H_{\delta_0}^1} \right) \\ &\times \|\partial_t v_{\varepsilon, n} e^{-A_{\varepsilon, n}}\|_{L^\infty((0, T); L^2)} \left(1 + \|v_{\varepsilon, n}\|_{L^\infty((0, T); H_{-\delta}^2)} \right)^{1^-}, \end{aligned} \tag{7.11}$$

where $1^- \in (0, 1)$. Next we notice that by using the equation solved by $v_{\varepsilon, n}$, we get by the Hölder inequality and the Sobolev embedding $H_\delta^{\frac{2}{2-\eta_0}} \subset L_\delta^q$, $q \in [2, \infty]$, in

conjunction with (5.6), (5.12), (7.10), and (7.8),

$$\begin{aligned}
 & \|\partial_t v_{\varepsilon,n}(t)e^{-A_{\varepsilon,n}}\|_{L^2} \\
 & \leq \|\Delta v_{\varepsilon,n}(t)e^{-A_{\varepsilon,n}}\|_{L^2} \\
 & \quad + C(\omega)|\ln \varepsilon|^C \mathcal{P}\left(\|v_{\varepsilon,n}(0)\|_{H_{\delta_0}^1}\right) \|v_{\varepsilon,n}\|_{L^\infty((0,T);H_{-\delta}^2)}^{1^-} \\
 & \quad + C(\omega)|\ln \varepsilon|^C \|v_{\varepsilon,n}(t)|v_{\varepsilon,n}(t)|^p\|_{L_\delta^2} \\
 & \leq \|\Delta v_{\varepsilon,n}(t)e^{-A_{\varepsilon,n}}\|_{L^2} \\
 & \quad + C(\omega)|\ln \varepsilon|^C \mathcal{P}\left(\|v_{\varepsilon,n}(0)\|_{H_{\delta_0}^1}\right) \|v_{\varepsilon,n}\|_{L^\infty((0,T);H_{-\delta}^2)}^{1^-} \\
 & \quad + C(\omega)|\ln \varepsilon|^C \mathcal{P}\left(\|v_{\varepsilon,n}(0)\|_{H_{\delta_0}^1}\right) \|v_{\varepsilon,n}\|_{L^\infty((0,T);H_{-\delta}^2)}^{n_0(p+1)} \\
 & \leq \|\Delta v_{\varepsilon,n}(t)e^{-A_{\varepsilon,n}}\|_{L^2} \\
 & \quad + C(\omega)|\ln \varepsilon|^C \mathcal{P}\left(\|v_{\varepsilon,n}(0)\|_{H_{\delta_0}^1}\right) \|v_{\varepsilon,n}\|_{L^\infty((0,T);H_{-\delta}^2)}^{1^-}
 \end{aligned} \tag{7.12}$$

for a suitable $1^- \in (0, 1)$. Hence, we can gather together (7.4), (7.5), (7.9), (7.11), and (7.12) and we get by elementary manipulations that

$$\begin{aligned}
 & \int_{\mathbb{R}^2} |\Delta v_{\varepsilon,n}(t)|^2 e^{-2A_{\varepsilon,n}} dx \leq C \int_{\mathbb{R}^2} |\Delta v_{\varepsilon,n}(0)|^2 e^{-2A_{\varepsilon,n}} dx \\
 & \quad + C \|v_{\varepsilon,n}\|_{L^\infty((0,T);H_{-\delta}^2)}^{2^-} + C(\omega)|\ln \varepsilon|^C + \mathcal{P}\left(\|v_{\varepsilon,n}(0)\|_{H_{\delta_0}^1}\right),
 \end{aligned}$$

which in turn by (4.5) implies

$$\|v_{\varepsilon,n}\|_{L^\infty((0,T);H_{-\delta}^2)}^2 \leq \mathcal{P}\left(\|v_{\varepsilon,n}(0)\|_{H_{\delta_0}^2}\right) + C \|v_{\varepsilon,n}\|_{L^\infty((0,T);H_{-\delta}^2)}^{2^-} + C(\omega)|\ln \varepsilon|^C.$$

By an elementary continuity argument we get (7.2).

8 Proof of the main result

In this section, we first prove Theorem 1.1, global well-posedness of the mollified equation (1.9). After that, we prove Theorem 1.2, the convergence of the solutions of the mollified problem to a unique solution of (1.7).

8.1 Proof of Theorem 1.1

We first follow the steps from [12, Proposition 2.11] to show the existence of a solution v_ε to (1.9), where $\varepsilon \in (0, \frac{1}{2})$ is fixed. Fix $\delta_0 > 0$, $T > 0$, and let $v_0 \in H_{\delta_0}^2$. By (7.2),

(5.12), and (2.40), we have the following bound for any $\gamma \in (1, 2)$ and for some $\delta > 0$:

$$\sup_{n \in \mathbb{N}} \|v_{\varepsilon, n}\|_{\mathcal{C}([0, T]; H_\delta^\gamma)} \leq C(\omega) |\ln \varepsilon|^C \mathcal{P}\left(\|v_0\|_{H_{\delta_0}^2}\right), \quad \text{a.s. } \omega \tag{8.1}$$

where the bound is uniform in $n \in \mathbb{N}$. Also, by (3.2), (3.15), and (3.9), for any $\alpha \in (0, 1)$, $0 < \delta^- < \delta$, and $\varepsilon \in (0, \frac{1}{2})$ we have the following bounds:

$$\sup_{n \in \mathbb{N}} \left(\|\theta_n Y_\varepsilon\|_{\mathcal{C}^{\alpha}_{-\delta^-}} + \|\theta_n \widetilde{\nabla Y_\varepsilon^2}\|_{\mathcal{C}^{\alpha-1}_{-\delta^-}} + \|e^{-p\theta_n Y_\varepsilon}\|_{L^\infty_{-\delta^-}} \right) \leq C(\omega), \quad \text{a.s. } \omega. \tag{8.2}$$

Using the equation (1.12), (8.1), (8.2), and (2.41), we can easily deduce that $\{\partial_t v_{\varepsilon, n}\}_{n \in \mathbb{N}}$ is bounded (uniformly in $n \in \mathbb{N}$) in $\mathcal{C}([0, T]; H_{\delta'}^{\gamma-2})$ for any $0 < \delta' < \delta$. By the Arzelà-Ascoli theorem along with the compact embedding (2.5), we obtain a convergent subsequence $\{v_{\varepsilon, n_k}\}_{k \in \mathbb{N}}$ in $\mathcal{C}([0, T]; H_{\delta''}^{\gamma_1-2})$ for any $\gamma_1 < \gamma$ and $\delta'' < \delta$, and we denote the limit as v_ε . By (7.2) and (2.40), the convergence also holds in $\mathcal{C}([0, T]; H_{\delta_1}^s)$ for any $s \in (1, 2)$ and some $\delta_1 > 0$. Also, by (7.2), the Banach-Alaoglu theorem, and taking a further subsequence if necessary, we obtain the following bound:

$$\|v_\varepsilon\|_{L^\infty((0, T); H_{-\delta}^2)} \leq C(\omega) |\ln \varepsilon|^C \mathcal{P}\left(\|v_0\|_{H_{\delta_0}^2}\right), \quad \text{a.s. } \omega \tag{8.3}$$

for some $\tilde{\delta} > 0$. Furthermore, v_ε satisfies the equation (1.9).

Next, we show the uniqueness of v_ε in $\mathcal{C}([0, T]; H_{\delta_1}^s)$. Assume that v_ε and w_ε are two solutions to (1.9). Define

$$r_\varepsilon(t) = v_\varepsilon(t) - w_\varepsilon(t), \quad t \in [0, T].$$

Then, r_ε satisfies the equation:

$$i \partial_t r_\varepsilon = \Delta r_\varepsilon + r_\varepsilon \widetilde{\nabla Y_\varepsilon^2} - 2 \nabla r_\varepsilon \nabla Y_\varepsilon - \lambda e^{-pY_\varepsilon} (|v_\varepsilon|^p v_\varepsilon - |w_\varepsilon|^p w_\varepsilon), \quad r_\varepsilon(0) = 0.$$

Using the equation for r_ε , we can deduce that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |r_\varepsilon(t)|^2 e^{-2Y_\varepsilon} dx = \lambda \text{Im} \int_{\mathbb{R}^2} \bar{r}_\varepsilon(t) (|v_\varepsilon(t)|^p v_\varepsilon(t) - |w_\varepsilon(t)|^p w_\varepsilon(t)) e^{-(p+2)Y_\varepsilon} dx.$$

Thus, using the embedding $H_{\delta_1}^s \subset L_{\delta_1}^\infty$ and (3.9), there exists $\delta > 0$ small enough such that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |r_\varepsilon(t)|^2 e^{-2Y_\varepsilon} dx &\leq C \|r_\varepsilon e^{-Y_\varepsilon}\|_{L^2}^2 \left(\|v_\varepsilon(t)\|_{L_{p\delta}^\infty}^p + \|w_\varepsilon(t)\|_{L_{p\delta}^\infty}^p \right) \|e^{-Y_\varepsilon}\|_{L_{-\delta}^\infty}^p \\ &\leq C(\omega) \|r_\varepsilon e^{-Y_\varepsilon}\|_{L^2}^2 \left(\|v_\varepsilon(t)\|_{H_{\delta_1}^s}^p + \|w_\varepsilon(t)\|_{H_{\delta_1}^s}^p \right). \end{aligned}$$

By the Gronwall inequality, we obtain $r_\varepsilon(t) = 0$, so the uniqueness result follows. This implies that the whole sequence $\{v_{\varepsilon, n}\}_{n \in \mathbb{N}}$ converges to v_ε in $\mathcal{C}([0, T]; H_{\delta_1}^s)$.

8.2 Proof of Theorem 1.2

The scheme of the proof is similar to the previous works on NLS equation with white noise potential.

We first note that by taking

$$A = Y_\varepsilon, \quad V = \widetilde{:\nabla Y_\varepsilon^2:},$$

the conditions (4.3) and (4.4) hold almost surely uniformly in $\varepsilon \in (0, \frac{1}{2})$ by using the same reasoning as in the beginning of Sect. 7. In particular, we can use (5.4), (5.5), and (5.12) with v replaced by v_ε .

Fix $\delta_0 > 0$ and assume that $v_0 \in H_{\delta_0}^2$. Let $0 < \varepsilon_2 < \varepsilon_1 < \frac{1}{2}$ and define

$$r(t) = v_{\varepsilon_1}(t) - v_{\varepsilon_2}(t), \quad t \in [0, T].$$

Then, r satisfies the equation:

$$\begin{aligned} i \partial_t r &= \Delta r + r : \widetilde{\nabla Y_{\varepsilon_1}^2} : + v_{\varepsilon_2} (: \widetilde{\nabla Y_{\varepsilon_1}^2} : - : \widetilde{\nabla Y_{\varepsilon_2}^2} :) - 2 \nabla r \nabla Y_{\varepsilon_1} \\ &\quad - 2 \nabla v_{\varepsilon_2} (\nabla Y_{\varepsilon_1} - \nabla Y_{\varepsilon_2}) - \lambda |v_{\varepsilon_1} e^{-Y_{\varepsilon_1}}|^p v_{\varepsilon_1} + \lambda |v_{\varepsilon_2} e^{-Y_{\varepsilon_2}}|^p v_{\varepsilon_2}, \\ r(0) &= 0. \end{aligned}$$

Using the equation for r , we can deduce that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |r(t)|^2 e^{-2Y_{\varepsilon_1}} dx \\ &= \text{Im} \int_{\mathbb{R}^2} \bar{r}(t) v_{\varepsilon_2}(t) (: \widetilde{\nabla Y_{\varepsilon_1}^2} : - : \widetilde{\nabla Y_{\varepsilon_2}^2} :) e^{-2Y_{\varepsilon_1}} dx \\ &\quad - 2 \text{Im} \int_{\mathbb{R}^2} \bar{r}(t) \nabla v_{\varepsilon_2}(t) (\nabla Y_{\varepsilon_1} - \nabla Y_{\varepsilon_2}) e^{-2Y_{\varepsilon_1}} dx \\ &\quad - \lambda \text{Im} \int_{\mathbb{R}^2} \bar{r}(t) \left(|v_{\varepsilon_1}(t) e^{-Y_{\varepsilon_1}}|^p v_{\varepsilon_1}(t) - |v_{\varepsilon_2}(t) e^{-Y_{\varepsilon_2}}|^p v_{\varepsilon_2}(t) \right) e^{-2Y_{\varepsilon_1}} dx \\ &=: I + II + III. \end{aligned} \tag{8.4}$$

Using (2.46), (2.44), (3.16), (3.9), and interpolation with (5.4) and (5.5), we get that for $\alpha \in (0, \frac{1}{100})$, there exists $\delta > 0$ small enough and $\kappa > 0$ such that

$$\begin{aligned} |I| &\leq C \|\bar{r}(t) v_{\varepsilon_2}(t) e^{-2Y_{\varepsilon_1}}\|_{\mathcal{B}_{1,1,\delta}^\alpha} \| : \widetilde{\nabla Y_{\varepsilon_1}^2} : - : \widetilde{\nabla Y_{\varepsilon_2}^2} : \|_{\mathcal{C}_{-\delta}^{-\alpha}} \\ &\leq C(\omega) \varepsilon_1^\kappa \|r(t)\|_{H_\delta^{2\alpha}} \|v_{\varepsilon_2}(t)\|_{H_\delta^{3\alpha}} \|e^{-2Y_{\varepsilon_1}}\|_{\mathcal{C}_{-\delta}^{3\alpha}} \\ &\leq C(\omega) \varepsilon_1^\kappa \mathcal{P} \left(\|v_0\|_{H_{\delta_0}^1} \right). \end{aligned} \tag{8.5}$$

Similarly, using (2.46), (2.44), (3.15), (3.9), (5.12), and (8.3), there exists $\delta > 0$ small enough and $\kappa > 0$ such that

$$\begin{aligned}
 |II| &\leq C \|\bar{r}(t) \nabla v_{\varepsilon_2}(t) e^{-2Y_{\varepsilon_1}}\|_{\mathcal{B}_{1,1,\delta}^\alpha} \|\nabla Y_{\varepsilon_1} - \nabla Y_{\varepsilon_2}\|_{\mathcal{C}_{-\delta}^{-\alpha}} \\
 &\leq C(\omega) \varepsilon_1^\kappa \|r(t)\|_{H_\delta^{2\alpha}} \|\nabla v_{\varepsilon_2}(t)\|_{H_\delta^{3\alpha}} \|e^{-2Y_{\varepsilon_1}}\|_{\mathcal{C}_{-\delta}^{3\alpha}} \\
 &\leq C(\omega) \varepsilon_1^\kappa \mathcal{P}\left(\|v_0\|_{H_{\delta_0}^1}\right) \|v_{\varepsilon_2}(t)\|_{H_{-\delta}^2} \\
 &\leq C(\omega) \varepsilon_1^\kappa |\ln \varepsilon_2|^C \mathcal{P}\left(\|v_0\|_{H_{\delta_0}^2}\right).
 \end{aligned}
 \tag{8.6}$$

Concerning III, using the embedding $H_\delta^{\frac{2}{2-\eta}} \subset L_\delta^\infty$ ($\eta, \tilde{\delta} > 0$ small enough), (3.9), (5.4), (3.10), (5.12), and (8.3), there exists $\delta > 0$ small enough and $\kappa > 0$ such that

$$\begin{aligned}
 |III| &\leq C \|r e^{-Y_{\varepsilon_1}}\|_{L^2}^2 \left(\|v_{\varepsilon_1}(t)\|_{L_{\rho\delta}^\infty}^p + \|v_{\varepsilon_2}(t)\|_{L_{\rho\delta}^\infty}^p \right) \|e^{-Y_{\varepsilon_1}}\|_{L_{-\delta}^\infty}^p \\
 &\quad + \|r\|_{L_\delta^2} \|v_{\varepsilon_2}(t)\|_{L_\delta^{2p+2}}^{p+1} \|e^{-pY_{\varepsilon_1}} - e^{-pY_{\varepsilon_2}}\|_{L_{-\rho\delta}^\infty} \|e^{-Y_{\varepsilon_1}}\|_{L_{-\delta}^\infty} \\
 &\leq C(\omega) \|r e^{-Y_{\varepsilon_1}}\|_{L^2}^2 \left(\|v_{\varepsilon_1}(t)\|_{H_{\rho\delta}^{\frac{2}{2-\eta}}}^p + \|v_{\varepsilon_2}(t)\|_{H_{\rho\delta}^{\frac{2}{2-\eta}}}^p \right) \\
 &\quad + C(\omega) \varepsilon_1^\kappa \|v_{\varepsilon_2}(t)\|_{H_\delta^{\frac{2}{2-\eta}}} \mathcal{P}\left(\|v_0\|_{H_{\delta_0}^1}\right) \\
 &\leq C(\omega) |\ln \varepsilon_2|^\gamma \mathcal{P}\left(\|v_0\|_{H_{\delta_0}^2}\right) \|r e^{-Y_{\varepsilon_1}}\|_{L^2}^2 \\
 &\quad + C(\omega) \varepsilon_1^\kappa |\ln \varepsilon_2|^C \mathcal{P}\left(\|v_0\|_{H_{\delta_0}^2}\right),
 \end{aligned}
 \tag{8.7}$$

for some $\gamma \in (0, 1)$.

Now, letting $\varepsilon_1 = 2^{-k}$ and $\varepsilon_2 = 2^{-(k+1)}$ for $k \in \mathbb{N}$, we combine (8.4), (8.5), (8.6), and (8.7) and apply the Gronwall inequality to obtain

$$\begin{aligned}
 &\sup_{t \in (0, T)} \int_{\mathbb{R}^2} |r(t)|^2 e^{-2Y_{2^{-k}}} dx \\
 &\leq C(\omega) 2^{-k\kappa} (k+1)^C \mathcal{P}\left(\|v_0\|_{H_{\delta_0}^2}\right) e^{C(\omega)T |\ln 2^{-(k+1)}|^\gamma \mathcal{P}\left(\|v_0\|_{H_{\delta_0}^2}\right)} \\
 &\leq C(\omega) 2^{-\frac{k\kappa}{2}} 2^{\tilde{\gamma}(k+1)C(\omega)T \mathcal{P}\left(\|v_0\|_{H_{\delta_0}^2}\right)} \mathcal{P}\left(\|v_0\|_{H_{\delta_0}^2}\right),
 \end{aligned}$$

where we used $e^{|\ln(1+x)|^\gamma} \leq Cx^{\tilde{\gamma}}$ for $\gamma \in (0, 1)$, $\tilde{\gamma} > 0$ arbitrarily small, and $x > 0$ large. Thus, by (3.9), for any $\delta > 0$ we obtain

$$\|v_{2^{-k}} - v_{2^{-(k+1)}}\|_{\mathcal{C}([0, T]; L_{-\delta}^2)}^2 \leq C(\omega) 2^{-\frac{k\kappa}{4}}.$$

Using interpolation along with the bounds (5.12) and (8.3), it is not hard to deduce that for any $s \in (1, 2)$, there exists $\delta_1 > 0$ such that $\{v_{2^{-k}}\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0, T]; H_{\delta_1}^s)$ and converges to some function $v \in \mathcal{C}([0, T]; H_{\delta_1}^s)$. By using similar steps as above, we can also deduce that

$$\sup_{\varepsilon \in (2^{-(k+1)}, 2^{-k})} \|v_\varepsilon - v_{2^{-k}}\|_{\mathcal{C}([0, T]; H_{\delta_1}^s)} \leq C(\omega) 2^{-k\tilde{\kappa}} \mathcal{P}\left(\|v_0\|_{H_{\delta_1}^2}\right),$$

for some $\tilde{\kappa} > 0$, so that the whole sequence $\{v_\varepsilon\}_{\varepsilon \in (0, \frac{1}{2})}$ converges to v in $\mathcal{C}([0, T]; H_{\delta_1}^s)$ as $\varepsilon \rightarrow 0$. This finishes the convergence part of the theorem.

Lastly, we prove the uniqueness of the solution v in $\mathcal{C}([0, T]; H_{\delta_1}^s)$ to the equation (1.7). Assume that v and w are two solutions to (1.7). Define

$$r'(t) = v(t) - w(t), \quad t \in [0, T).$$

Then, r' satisfies the equation:

$$i \partial_t r' = \Delta r' + r' : \widetilde{\nabla Y^2} : - 2 \nabla r' \nabla Y - \lambda e^{-pY} (|v|^p v - |w|^p w), \quad r'(0) = 0.$$

Using the equation for r' , we can deduce that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |r'(t)|^2 e^{-2Y} dx = \lambda \operatorname{Im} \int_{\mathbb{R}^2} \overline{r'(t)} (|v(t)|^p v(t) - |w(t)|^p w(t)) e^{-(p+2)Y} dx.$$

Thus, using the embedding $H_{\delta_1}^s \subset L_{\delta_1}^\infty$ and (3.11), there exists $\delta > 0$ small enough such that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |r'(t)|^2 e^{-2Y} dx &\leq C \|r' e^{-Y}\|_{L^2}^2 \left(\|v(t)\|_{L_{p\delta}^\infty}^p + \|w(t)\|_{L_{p\delta}^\infty}^p \right) \|e^{-Y}\|_{L_{-\delta}^\infty}^p \\ &\leq C(\omega) \|r' e^{-Y}\|_{L^2}^2 \left(\|v(t)\|_{H_{\delta_1}^s}^p + \|w(t)\|_{H_{\delta_1}^s}^p \right). \end{aligned}$$

By the Gronwall inequality, we obtain $r'(t) = 0$, so the uniqueness result follows.

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Declarations

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