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# Linear and Non linear stability for the kinetic plasma *sheath* on a bounded interval

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## Abstract

Plasma sheaths are inhomogeneous equilibrium that form when a plasma is in contact with an absorbing wall. We prove linear and non linear stability of a kinetic sheath equilibrium for a Vlasov-Poisson type system in a bounded interval. Notably, in the linear setting, we obtain exponential decay of the fluctuation provided the rate of injection of particles at equilibrium is smaller than the rate of absorption at the wall. In the non linear setting, we prove a similar result for small enough equilibrium and small localized perturbation of the equilibrium.

*Keywords:* Vlasov-Poisson equations; boundary value problem; plasma sheath; linear stability; non linear stability; first order delayed integral equation; non linear Poisson equation; exit geometric conditions

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## Introduction

In this paper, we study the dynamical behavior of solutions to a kinetic model of plasma-wall interaction. When a plasma made of positive ions is in contact with a partially absorbing wall, the relative difference in mobility between ions and electrons yields the formation of a boundary layer near the wall which is characterized by the formation of a positive space charge. This charge creates an electric field which attracts the ions and repels the electrons at a rate which balances the loss of charges at the wall. It results in a permanent regime. This phenomenon is known as the plasma sheath [9, 27]. The study of the plasma sheath is of significant importance in the design of laboratory plasma devices such as Tokamaks [22, 20, 28]. In this work, we consider a bounded and uni-dimensional plasma made of only one kind of ions in which is immersed an infinite metallic wall. The time scale of interest in our study is that of ions so that electrons are assumed to be at a thermodynamical equilibrium. We shall model the mesoscopic behavior of the ions using a kinetic approach where positions and velocities of the particles in the phase space  $Q = (0, 1) \times \mathbb{R}$  are denoted  $x$  and  $v$ . We then denote by  $f(t, \cdot, \cdot) \geq 0$  the density of ions in the phase space  $Q$  at time  $t \geq 0$  and by  $\phi(t, \cdot)$  the electrostatic potential at time  $t$ . We are interested in long time stability properties of  $(f, \phi)$  solution to the

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Vlasov-Poisson equations,

$$\partial_t f + v \partial_x f - \partial_x \phi(t, x) \partial_v f = 0, \quad t \in (0, +\infty), (x, v) \in Q, \quad (1)$$

$$-\lambda^2 \partial_{xx} \phi(t, x) = \int_{\mathbb{R}} f(t, x, v) dv - n_e(\phi(t, x)), \quad t \geq 0, \quad x \in (0, 1), \quad (2)$$

where  $\lambda > 0$  is a normalized Debye length. Here,

$$n_e : \mathbb{R} \rightarrow \mathbb{R}_*^+ \text{ is } \mathcal{C}^1 \text{ and } n_e' > 0. \quad (3)$$

A typical example of such a function in the context of plasma physics is the so called Boltzman density [9] which is given by  $n_e(\psi) = n_0 e^\psi$  where  $n_0 > 0$  is a normalized density of electrons. To close the system (1)-(2), we have to define boundary conditions and to prescribe the initial data. We thus define the set of incoming particles, the set of outgoing particles and the singular set by

$$\Sigma^{\text{inc}} := \{(0, v) : v > 0\} \cup \{(1, v) : v < 0\}, \quad \Sigma^{\text{out}} := \{(0, v) : v < 0\} \cup \{(1, v) : v > 0\}, \quad (4)$$

$$\Sigma^0 := \{(0, 0)\} \cup \{(1, 0)\}. \quad (5)$$

We therefore prescribe  $f(t, \cdot, \cdot)$  on  $\Sigma^{\text{inc}}$  as follows

$$\forall t > 0, \quad f(t, 0, v) = \mu(v) \text{ if } v > 0 \quad f(t, 1, v) = 0 \quad \text{if } v < 0, \quad (6)$$

where  $\mu(v) > 0$  denotes a stationary density of incoming particles that comes from the plasma core while at the wall ( $x = 1$ ) particles are absorbed. As for the electric potential, we impose the following Dirichlet boundary conditions

$$\phi(t, 0) = 0, \quad \phi(t, 1) = \phi_b, \quad (7)$$

where the reference of potential is chosen to be at  $x = 0$  (the plasma core) and  $\phi_b \in \mathbb{R}$  denotes the voltage at the metallic wall. The system (1)-(7) is eventually completed with the initial condition:

$$f(0, x, v) = f_0(x, v) \quad (8)$$

where  $f_0$  is the initial density of ions.

Up to our knowledge, the wellposedness theory of Vlasov-Poisson equations in bounded domain has not been investigated in full details. Stationary solutions have been obtained in [24, 25, 11, 2, 3]. As for the time dependent problem, existence of weak solutions has been obtained by Ben Abdallah in [5], while uniqueness of the mild solution in the one dimensional case was proven by Bostan in [7] for decreasing electric fields. In the case of the specular reflection boundary condition, global wellposedness of classical solutions has been obtained by H-J Hwang and J.Velazquez in [18] using the assumption that the electric field points outward at the boundary. Recently, Cesbron and Iacobelli in [8] proved a similar result for the Vlasov-Poisson equations with massless electrons. In the case of the half-space, Y. Guo in [14] proved for the Vlasov-Poisson equations, existence and uniqueness of weak solutions in the class of essentially bounded and of bounded variations functions.

As far as the stability of stationary solutions is concerned. The Vlasov Poisson equations have a long history which dates back to the seminal paper by Landau [19]. Indeed, in the absence of spatial boundaries, the stability of homogenous (in space) equilibria (either in the linear or in the non linear setting) has been studied by many authors. A non exhaustive list is [10, 21, 12, 17]. It is known that for a homogenous plasma whose equilibrium density of particles verifies the so called Penrose condition, a slight perturbation of the equilibrium density creates an electric field which tends to zero in the long time asymptotic. This phenomenon is known as the Landau Damping. As far as we know, in the presence of spatial boundaries and with inhomogeneous equilibria, there is no analogue to the Landau Damping. Nevertheless, spectral stability or Lyapunov type stability using the Energy Casimir method have been carried in [23, 26, 6, 1, 13, 15]. We mention that in the use of the Energy Casimir method, very often a structural assumption on the equilibrium density is assumed. Namely, it has to be a decreasing function of the microscopic energy. As for the spectral analysis, a symmetric structure is needed to study the spectrum of the appropriate operator. Unfortunately, in the case of plasma sheath equilibria, we have not been able to use these tools in a satisfactory way, notably because the so called Bohm condition (13) prevents the non trivial equilibrium from being a decreasing function of the microscopic energy.

The approach used in this work is different and is much more in the spirit of the work of Glass, HanKwan and Moussa [16] to study the stability of equilibria for the Vlasov-Navier-Stokes system. Our approach fully exploits the structure of the transport operator for the Vlasov equation, and the fact that the non trivial sheath equilibrium yields an electric field which is uniformly positive. At equilibrium, all the trajectories leave the phase space in some uniform time. This yields a kind of first order delayed Grönwall inequality satisfied by the  $L^1$  norm of the perturbation for times larger than the time of exit of all the trajectories. Our linear stability Theorem (2.1) then roughly says that if the rate of injection of particles at equilibrium in the plasma core is smaller than the rate of absorption of particles at the wall then any perturbation is prone to decay exponentially fast. Our non linear stability Theorem (3.1) is a perturbative variant of the linear stability result. We prove that if the equilibrium density is small enough then a small localized perturbation of the equilibrium yields that all the characteristics associated to the non linear Vlasov-Poisson equations still leave the phase space in some uniform time. We obtain as in the linear analysis an exponential decay of the perturbation for times larger than the time of exit of all the perturbed trajectories. The difficult part in the non linear analysis consists in propagating the local in time stability estimates. This is done by a continuation argument: using the delayed Grönwall inequality we show that if the initial fluctuation is small enough then the perturbation on the electric field stays bounded by a constant, say  $C$ , up to a critical time. We then establish that this bound on the electric field propagates after the critical time provided the semi-norm in  $W^{1,1}$  of the equilibrium density is chosen small enough. So far, we mention that in the non linear analysis, only stability estimates are given, existence and uniqueness of the global mild solution for the non linear system is not addressed directly in this paper and is left for a future work.

This work is organized as follows. We study the stationary solutions in Section (1). Then, we study the linearized equations in Section (2). We notably prove the linear stability result given in Theorem (2.1). The proof relies on linear elliptic estimates for the linearized Poisson equation that are given in Section (2.2), a careful study of the stationary phase portrait that is done in Section (2.3) and a delayed Grönwall inequality satisfied by the  $L^1$  norm of the

perturbation for large times that is studied in Section (2.6). We then study the non linear stability in Section (3) using exactly the same strategy. We establish wellposedness and give non linear elliptic estimates for the non linear Poisson equation in Section (3.2). Then we study, in Section (3.3), the stability of the stationary phase portrait with respect to small perturbations on the equilibrium electric field. We prove local stability estimates in Section (3.4) and conclude the proof of the non linear stability result by a continuation argument in Section (3.5).

## 1. The *sheath* equilibrium for (VP)

In this section, we recall the construction of the non trivial *sheath* equilibrium obtained in [2]. We provide additional estimates that were not in [2]. Let us begin by introducing the concept of solutions we consider. Due to the boundary conditions, solutions to transport equations in bounded domain are rarely classical. We shall generically consider weak solutions for the stationary Vlasov equation in the following sense.

**Definition 1.1.** Let  $\mu \in L^1_{\text{loc}}(\mathbb{R}^+)$ ,  $\phi_b \in \mathbb{R}$ . Let  $(f, \phi) \in L^1_{\text{loc}}(Q) \times W^{2,\infty}(0, 1)$ . We say that  $(f, \phi)$  is a stationary solution for the Vlasov-Poisson system (1)-(7) if

- a)  $f$  is a weak solution of the Vlasov equation: for all  $\psi \in \mathcal{C}_c^1(\overline{Q})$  such that  $\psi|_{\Sigma^{\text{out}}} = 0$

$$\int_Q f(x, v) \Psi(x, v) dx dv = \int_0^{+\infty} \mu(v) v \psi(0, v) dv, \quad (9)$$

where  $\Psi(x, v) = v \partial_x \psi(x, v) - \partial_x \phi(x) \partial_v \psi(x, v)$ .

- b)  $f \in \mathcal{C}^0([0, 1]; L^1(\mathbb{R}))$ ,  $\phi \in \mathcal{C}^2[0, 1]$ , it verifies the boundary conditions  $\phi(0) = 0$ ,  $\phi(1) = \phi_b$  and it is a classical solution of the Poisson equation:

$$\forall x \in [0, 1], \quad -\lambda^2 \partial_{xx} \phi(x) = \int_{\mathbb{R}} f(x, v) dv - n_e(\phi(x)). \quad (10)$$

In our setting, the terminology *sheath* refers to a non trivial equilibrium for (VP) such that

- the plasma is neutral at  $x = 0$ , that is  $\int_{\mathbb{R}} f^\infty(0, v) dv = n_e(0)$ .
- the electric potential is monotone decreasing, concave, and it varies strongly in a neighborhood, of size  $\lambda$ , of  $x = 1$ .

This is what we obtain in the following.

**Theorem 1.2** (The *sheath* equilibrium). *Let  $\phi_b < 0$ . Assume (3) and that the function*

$$s \in \mathbb{R}^- \longmapsto \exp(-s) n_e(s) \text{ is concave.} \quad (11)$$

*Let  $\mu \in L^1(\mathbb{R}^+)$  be such that*

$$\int_0^{+\infty} \mu(v) dv = n_e(0), \quad (12)$$

and

$$\int_0^{+\infty} \frac{\mu(v)}{v^2} dv < n'_e(0). \quad (13)$$

Then, for any  $\lambda > 0$ , there exists a unique weak stationary solution  $(f^\infty, \phi^\infty) \in \mathcal{C}^0([0, 1]; L^1(Q)) \times \mathcal{C}^2[0, 1]$  to **(VP)** in the class of decreasing and concave potentials. Moreover the solution satisfies

a)

$$f^\infty(x, v) = \mathbf{1}_{D^+}(x, v) \mu\left(\sqrt{v^2 + 2\phi^\infty(x)}\right), \quad D^+ = \left\{(x, v) \in Q : v > \sqrt{-2\phi^\infty(x)}\right\}. \quad (14)$$

b) There are constants  $0 < \alpha < \beta$  which depend only  $\mu, n_e$  and  $\phi_b$  but not on  $\lambda$  such that the following estimates hold

$$\int_0^1 \frac{\lambda^2}{2} \left| \partial_x \phi^\infty(x) \right|^2 + \frac{\alpha}{2} |\phi^\infty(x)|^2 dx \underset{\lambda \rightarrow 0^+}{=} \mathcal{O}(\lambda), \quad (15)$$

$$\forall x \in [0, 1], \quad |\phi_b| \frac{\sinh(\frac{\sqrt{\beta}x}{\lambda})}{\sinh(\frac{\sqrt{\beta}}{\lambda})} \leq |\phi^\infty(x)| \leq |\phi_b| \frac{\sinh(\frac{\sqrt{\alpha}x}{\lambda})}{\sinh(\frac{\sqrt{\alpha}}{\lambda})}. \quad (16)$$

$$\partial_x \phi^\infty(0) < 0, \quad |\partial_x \phi^\infty(0)| \leq \frac{\frac{\sqrt{\alpha}}{\lambda}}{\sinh(\frac{\sqrt{\alpha}}{\lambda})}. \quad (17)$$

c)  $\phi^\infty$  converges to zero locally uniformly on  $[0, 1)$  as  $\lambda \rightarrow 0^+$ .

d) The plasma is quasi-neutral

$$\forall p \in [1, +\infty), \quad \lim_{\lambda \rightarrow 0^+} \left\| \int_{\mathbb{R}} f^\infty(\cdot, v) dv - n_e(\phi^\infty) \right\|_{L^p(0,1)} = 0. \quad (18)$$

*Proof.* The existence and uniqueness of a pair  $(f^\infty, \phi^\infty) \in \mathcal{C}^0([0, 1]; L^1(\mathbb{R})) \times \mathcal{C}^2[0, 1]$  with  $f$  given by (14) and  $\phi^\infty$  non increasing and concave follows verbatim the lines of [2]. We thus only prove the claim b), c) and d) which are new. The starting point in the analysis consists in noticing that  $\phi^\infty$  minimizes the functional given by

$$\forall \psi \in \mathcal{C}, \quad J(\psi) = \int_0^1 \frac{\lambda^2}{2} |\psi'(x)|^2 + Q(\psi(x)) dx \quad (19)$$

where  $\mathcal{C} = \{u \in H^1[0, 1] : \phi_b \leq u \leq 0 \text{ in } [0, 1], u(0) = 0, u(1) = \phi_b\}$ . Here,  $Q \in \mathcal{C}^2[\phi_b, 0]$  is a potential that verifies

$$\forall s \in [\phi_b, 0], \quad Q'(s) = - \int_0^{+\infty} \frac{\mu(w)w}{\sqrt{w^2 - 2s}} dw + n_e(s), \quad (20)$$

$$Q(0) = Q'(0) = 0, \quad Q''(0) > 0, \quad (21)$$

$$\phi_b \leq s < 0 \iff Q'(s) < 0. \quad (22)$$

With such a properties of the potential  $Q$ , the function  $s \in [\phi_b, 0[ \mapsto \frac{Q'(s)}{s}$  is continuous and positive, and since  $Q'(0) = 0$  it extends by continuity at  $s = 0$  with  $\lim_{s \rightarrow 0^-} \frac{Q'(s)}{s} = Q''(0) > 0$ . We then set  $\alpha := \inf_{s \in [\phi_b, 0]} \frac{Q'(s)}{s}$  and  $\beta := \sup_{s \in [\phi_b, 0]} \frac{Q'(s)}{s}$  which are well-defined and positive numbers with  $0 < \alpha \leq \beta$ . We then get by definition of the numbers  $\alpha$  and  $\beta$  that

$$\forall s \in [\phi_b, 0], \quad \beta s \leq Q'(s) \leq \alpha s, \quad (23)$$

$$\forall s \in [\phi_b, 0], \quad \frac{\alpha s^2}{2} \leq Q(s) \leq \frac{\beta s^2}{2}, \quad (24)$$

where the second inequality follows from the first inequality by integration. We are now ready to prove claims b),c) and d).

*Proof of b).* Let us start with (15). Using the inequality (24) and the fact  $\phi^\infty$  minimizes  $J$  on  $\mathcal{C}$  we have

$$\int_0^1 \frac{\lambda^2}{2} \left| \partial_x \phi^\infty(x) \right|^2 + \frac{\alpha}{2} |\phi^\infty(x)|^2 dx \leq J(\phi^\infty) \leq J(\psi) \quad (25)$$

where  $\psi \in \mathcal{C}$  is the solution to the linear elliptic equation

$$-\lambda^2 \partial_{xx} \psi + \beta \psi = 0 \text{ a.e in } (0, 1), \quad (26)$$

with the boundary conditions  $\psi(0) = 0$  and  $\psi(1) = \phi_b$ . The solution is given explicitly by

$$\forall x \in [0, 1], \quad \psi(x) = \phi_b \frac{\sinh\left(\frac{\sqrt{\beta}x}{\lambda}\right)}{\sinh\left(\frac{\sqrt{\beta}}{\lambda}\right)}. \quad (27)$$

Taking the  $L^2$  inner product of the linear elliptic equation (26) with  $\frac{\psi}{2}$  and using an integration by parts, we obtain  $J(\psi) = \int_0^1 -\frac{\beta |\psi(x)|^2}{2} + Q(\psi(x)) dx + \frac{\lambda^2}{2} \partial_x \psi(1) \phi_b \leq \frac{\lambda^2}{2} \partial_x \psi(1) \phi_b = \frac{\lambda}{2} \phi_b^2 \sqrt{\beta} \frac{1}{\tanh\left(\frac{\sqrt{\beta}}{\lambda}\right)}$ . For  $\lambda > 0$  small enough, we have  $1 \leq \frac{1}{\tanh\left(\frac{\sqrt{\beta}}{\lambda}\right)} \leq \frac{3}{2}$ . Therefore since  $\phi_b$

is independent of  $\lambda$  we obtain  $\frac{\lambda^2}{2} \partial_x \psi(1) \phi_b \underset{\lambda \rightarrow 0^+}{=} \mathcal{O}(\lambda)$  as expected. We now prove (16). Consider the function  $e = \phi^\infty - \psi$  where  $\psi$  is the solution to (26). Then the difference verifies  $e \in H^2(0, 1) \cap H_0^1(0, 1)$  and

$$-\lambda^2 \partial_{xx} e + \beta e \leq 0 \text{ a.e in } (0, 1).$$

A maximum principle yields  $e \leq 0$  everywhere in  $[0, 1]$  and therefore  $\phi^\infty \leq \psi$  which is the first inequality. Arguing similarly, one gets  $\phi^\infty(x) \geq \phi_b \frac{\sinh\left(\frac{\sqrt{\alpha}x}{\lambda}\right)}{\sinh\left(\frac{\sqrt{\alpha}}{\lambda}\right)}$  for all  $x \in [0, 1]$ . Hence

$$\forall x \in [0, 1], \quad \phi_b \frac{\sinh\left(\frac{\sqrt{\alpha}x}{\lambda}\right)}{\sinh\left(\frac{\sqrt{\alpha}}{\lambda}\right)} \leq \phi^\infty(x) \leq \phi_b \frac{\sinh\left(\frac{\sqrt{\beta}x}{\lambda}\right)}{\sinh\left(\frac{\sqrt{\beta}}{\lambda}\right)}.$$

Then for  $x \in (0, 1]$ ,

$$\phi_b \frac{\sinh\left(\frac{\sqrt{\alpha}x}{\lambda}\right)}{x \sinh\left(\frac{\sqrt{\alpha}}{\lambda}\right)} \leq \frac{\phi^\infty(x)}{x} \leq \phi_b \frac{\sinh\left(\frac{\sqrt{\beta}x}{\lambda}\right)}{x \sinh\left(\frac{\sqrt{\beta}}{\lambda}\right)}.$$

Since  $\phi^\infty(0) = 0$  and  $\phi^\infty$  is differentiable at  $x = 0$ , taking the limit as  $x \rightarrow 0^+$  yields

$$\phi_b \frac{\sqrt{\alpha}}{\lambda \sinh(\frac{\sqrt{\alpha}}{\lambda})} \leq \partial_x \phi^\infty(0) \leq \phi_b \frac{\sqrt{\beta}}{\lambda \sinh(\frac{\sqrt{\beta}}{\lambda})}.$$

It shows in particular that for  $\lambda > 0$  and  $\phi_b < 0$ ,  $\partial_x \phi^\infty(0) < 0$  and that  $|\partial_x \phi^\infty(0)| \leq |\phi_b| \frac{\sqrt{\alpha}}{\lambda \sinh(\frac{\sqrt{\alpha}}{\lambda})}$ .

*Proof of c).* Thanks to (16), for every  $0 < r < 1$ ,

$$\sup_{x \in [0, r]} |\phi^\infty(x)| \leq |\phi_b| \frac{\sinh(\frac{\sqrt{\alpha} r}{\lambda})}{\sinh(\frac{\sqrt{\alpha}}{\lambda})} \rightarrow 0 \text{ as } \lambda \rightarrow 0^+.$$

*Proof of d).* Note that we have the simple bound for all  $p \in [1, +\infty)$  and  $\psi \in [\phi_b, 0]$ ,

$$\left| \int_{\mathbb{R}^+} \frac{\mu(v)v}{\sqrt{v^2 - 2\psi}} dv - n_e(\psi) \right|^p \leq 2^{p-1} \left( \left| \int_{\mathbb{R}^+} \mu(v) dv \right|^p + \left( \sup_{\psi \in [\phi_b, 0]} |n_e(\psi)| \right)^p \right).$$

Also observe thanks to c) that for  $x \in [0, 1]$ ,  $\phi^\infty(x) \rightarrow 0$  as  $\lambda \rightarrow 0^+$ . Then, since  $Q'$  is continuous on  $[\phi_b, 0]$ , we have  $Q'(\phi^\infty(x)) \rightarrow Q'(0) = 0$  as  $\lambda \rightarrow 0^+$ . An application of the Lebesgue dominated convergence theorem yields the expected result.  $\square$

**Remark 1.3.** In [2] the solution of the Vlasov equation is constructed by the method of the characteristics. It is in fact a mild solution. It is of no importance in the sequel.

In the stability analysis, we will further assume that the equilibrium incoming density has the regularity,

$$\mu \in \mathcal{C}^1(\mathbb{R}^+) \cap W^{1,1}(\mathbb{R}^+). \quad (28)$$

In particular, the inequality (13) yields the necessary condition

$$\mu(0) = \mu'(0) = 0. \quad (29)$$

Consequently, the equilibrium density has the regularity  $f^\infty \in W^{1,1}(Q)$  and its partial weak derivative is given by

$$\partial_v f^\infty(x, v) = \mathbf{1}_{D^+}(x, v) \frac{v \mu'(\sqrt{v^2 + 2\phi^\infty(x)})}{\sqrt{v^2 + 2\phi^\infty(x)}}. \quad (30)$$

From now on, it is implicitly assumed that  $n_e$  satisfies (3), (11) and that  $\mu$  has the regularity (28) and verifies the two conditions (12) and (13).

## 2. Linear stability

We firstly investigate the linear stability of the equilibrium  $(f^\infty, \phi^\infty)$  given by the Theorem (1.2). For small enough  $\varepsilon > 0$  we want to investigate the dynamic of the first order perturbation of the equilibrium, namely we write

$$\begin{cases} f(t, x, v) &= f^\infty(x, v) + \varepsilon h(t, x, v) + o(\varepsilon), \\ \phi(t, x) &= \phi^\infty(x) + \varepsilon U(t, x) + o(\varepsilon), \end{cases} \quad (31)$$



where  $(h, U)$  denotes the first order fluctuation and  $o(\varepsilon)$  denotes formal higher order fluctuations. Formally, the first order fluctuation verifies the linearized Vlasov-Poisson equations

$$\partial_t h + v \partial_x h - \partial_x \phi^\infty \partial_v h = \partial_x U(t, x) \partial_v f^\infty, \quad t > 0, \quad (x, v) \in Q, \quad (32)$$

$$-\lambda^2 \partial_{xx} U(t) + n'_e(\phi^\infty) U(t) = \int_{\mathbb{R}} h(t, \cdot, v) dv, \quad t \geq 0, \quad x \in (0, 1). \quad (33)$$

We prescribe  $h(t, \cdot, \cdot)$  on  $\Sigma^{\text{inc}}$  as follows

$$\forall t > 0, \quad h(t, 0, v) = 0 \text{ if } v > 0, \quad h(t, 1, v) = 0 \text{ if } v < 0. \quad (34)$$

$$(35)$$

It means that there is no fluctuation on the incoming density of particles at the boundary. For the fluctuating potential, we also neglect the fluctuation at the boundary so that it is assumed to verify the homogeneous Dirichlet boundary conditions

$$U(t, 0) = 0, \quad U(t, 1) = 0. \quad (36)$$

The system is supplemented with the initial condition

$$h(0, x, v) = h_0(x, v) \quad (37)$$

where  $h_0$  represents the initial fluctuation of density in the phase space  $Q$ . We shall denote **(LVP)** the set of linearized equations (32)-(37). The wellposedness study will be performed in the following Banach spaces. For  $T > 0$  and  $\gamma > 0$ , we define the spaces

$$X_{T,\gamma} := L^\infty\left([0, T]; L^1(Q)\right), \quad Y_{T,\gamma} := L^\infty\left([0, T]; W^{2,1}(0, 1)\right), \quad (38)$$

endowed with the norms defined for any couple of functions  $(f, g) \in X_{T,\gamma} \times Y_{T,\gamma}$  by

$$\|f\|_{X_{T,\gamma}} := \text{ess sup}_{0 \leq t \leq T} e^{-\gamma t} \|f(t)\|_{L^1(Q)}, \quad \|\phi\|_{Y_{T,\gamma}} := \text{ess sup}_{0 \leq t \leq T} e^{-\gamma t} \|\phi(t)\|_{W^{2,1}(0,1)}. \quad (39)$$

Before going deeper in the analysis, let us recall that for a generic measurable function  $f : Q \rightarrow \mathbb{R}$  we call support of  $f$  and denote  $\text{supp } f$  the complement in  $Q$  of the largest open set  $O \subset Q$  on which  $f$  is almost everywhere equal to zero. In the sequel we will use the abbreviation a.e in place of almost everywhere. For  $r \geq 0$ , we define the following subset of  $Q$

$$D_r^+ := \left\{ (x, v) \in Q : v > \sqrt{-2\phi^\infty(x) + r^2} \right\} \quad (40)$$

and define the associated (exit) time

$$T_r := \begin{cases} \frac{\sqrt{-2\phi_b}}{|\partial_x \phi^\infty(0)|} & \text{if } r = 0, \\ \frac{1}{r} & \text{if } r > 0. \end{cases} \quad (41)$$

Note that for  $r = 0$  we have  $D_r^+ = D^+$  where  $D^+$  is given in (14). The main result of this section is the following.

**Theorem 2.1** (Linear stability). *Let  $\lambda > 0$ ,  $\phi_b < 0$  and  $r \geq 0$ . Consider the equilibrium  $(f^\infty, \phi^\infty)$  given by Theorem (1.2) where  $\mu$  is such that  $\text{supp } \mu \subset (r + \infty)$ . Assume moreover*

$$\frac{2\|\partial_v f^\infty\|_{L^1(D_r^+)} T_r}{\lambda^2} < 1. \quad (42)$$

*For all  $h_0 \in L^1(Q)$ , the linearized system (**LVP**) admits a unique global mild-strong solution  $(h, U)$  in the sense of Definition (2.3). In addition, we have*

$$(h, U) \in \mathcal{C}\left([0, +\infty); L^1(Q)\right) \times \mathcal{C}\left([0, +\infty); W^{2,1}(0, 1)\right) \quad (43)$$

*and the solution satisfies*

- a) Decay of the  $L^1$  norm in  $Q \setminus D^+$ . *There is a family of non increasing Borel sets  $(\mathcal{O}_t)_{t \geq 0} \subset Q \setminus D^+$  which become empty in finite time such that for every  $t \geq 0$*

$$\|h(t)\|_{L^1(Q \setminus D^+)} = \|h_0\|_{L^1(\mathcal{O}_t)}. \quad (44)$$

- b) Exponential decay of the  $L^1$  norm in  $D_r^+$ . *If  $\text{supp } h_0 \subset D_r^+$  then for every  $t \geq 0$*

$$\text{supp } h(t) \subset D_r^+ \quad (45)$$

*and there are constants  $\kappa > 0$  and  $C \geq 0$  such that for every  $t \geq 0$*

$$\|h(t)\|_{L^1(D_r^+)} \leq C \exp(-\kappa t), \quad (46)$$

$$\|\partial_x U(t)\|_{L^\infty(0,1)} \leq \frac{2C}{\lambda^2} \exp(-\kappa t). \quad (47)$$

Several comments are in order about this result.

**Remark 2.2.** • The wellposedness of the linearized equations will follow from the classical Banach-Picard fixed point theorem. In the proof, a key ingredient is the use of an elliptic estimate for the linearized Poisson equation which enables to obtain a closed estimate on the growth of the  $L^1$  norm of  $h$ .

- As for the stability result, we heavily rely on the method of the characteristics and a detailed study of the associated flow. The proof of stability in  $Q \setminus D^+$  uses the fact that the equilibrium is supported on  $D^+$ . As a matter of fact, the source term vanishes in  $Q \setminus D^+$  for the linearized Vlasov equation (32) and thus the  $L^1$  norm in  $Q \setminus D^+$  is expected to decay thanks to the absorbing boundary conditions.
- The proof of the exponential decay in  $D_r^+$  for  $r \geq 0$  heavily relies on the fact the stationary electric field  $-\partial_x \phi^\infty$  is uniformly positive on  $[0, 1]$  and thus satisfies the exit geometric conditions in time  $T_r$  (2.4)-(2.5). The stability condition (42) essentially conveys the idea that the source term in the linearized Vlasov equation must produce particles at a rate

which is less than the rate of absorption of the particles at the boundary. We mention that in the case when  $r > 0$  the smallness condition (42) is equivalent to

$$\frac{2\|\mu'\|_{L^1(r,+\infty)}}{\lambda^2 r} < 1$$

which is trivially satisfied if  $\mu' \in L^1(\mathbb{R}^+)$  is either small enough or  $r$  is large enough. In the case  $r = 0$ , the time given in (41) is a lower bound for the time of exit of the characteristic which started at time  $t = 0$  from the point of coordinate  $(0, 0)$ . It is the largest time of exit of a particle in the domain  $D^+$ . This time diverges as  $\lambda \rightarrow 0^+$  exponentially fast. Note that the bound (17) on  $\partial_x \phi^\infty(0)$  has an implicit dependence on  $\mu$ . Thus, the stability condition (42) in the case  $r = 0$  is more intricate to treat.

### 2.1. The stationary characteristics.

For the analysis, it is convenient to consider an extension of  $\phi^\infty$  to  $\mathbb{R}$  in such a way its extension still denoted  $\phi^\infty$  verifies

$$\phi^\infty \in W^{2,\infty}(\mathbb{R}), \quad \|\partial_x \phi^\infty\|_{L^\infty(\mathbb{R})} \leq \|\partial_x \phi^\infty\|_{L^\infty(0,1)}. \quad (48)$$

As we work in dimension  $d = 1$ , such an extension is easily constructed by an affine extrapolation of  $\phi$  to  $\mathbb{R} \setminus [0, 1]$ . For  $(t, x, v) \in \mathbb{R} \times \mathbb{R}^2$ , we define the characteristics which passes through the point  $(x, v)$  at time  $t$  as the unique solution of the ordinary differential system

$$\begin{cases} \frac{d}{ds} X_\infty(s; t, x, v) = V_\infty(s; t, x, v), \\ \frac{d}{ds} V_\infty(s; t, x, v) = -\partial_x \phi^\infty(X_\infty(s; t, x, v)), \\ X_\infty(t; t, x, v) = x, V_\infty(t; t, x, v) = v. \end{cases} \quad (49)$$

Note that  $\phi^\infty \in W^{2,\infty}(\mathbb{R})$ , thus  $\partial_x \phi^\infty$  is Lipschitz continuous. Therefore the ordinary differential system (49) has a unique solution with the regularity

$$s \longmapsto \left( X_\infty(s; t, x, v), V_\infty(s; t, x, v) \right) \in \mathcal{C}^1(\mathbb{R}) \cap W_{\text{loc}}^{2,\infty}(\mathbb{R}).$$

As for the spatial regularity, for fixed  $(s, t) \in \mathbb{R}$ , the flow map

$$\mathcal{F}_{s,t}^\infty : (x, v) \in \mathbb{R}^2 \mapsto \left( X_\infty(s; t, x, v), V_\infty(s; t, x, v) \right) \quad (50)$$

is a measure preserving diffeomorphism. For  $(t, x, v) \in \mathbb{R} \times Q$ , we may often be interested in the restriction of the the solution to the interval  $(t_\infty^{\text{inc}}(t, x, v), t_\infty^{\text{out}}(t, x, v))$  where

$$t_\infty^{\text{inc}}(t, x, v) = \inf \left\{ s \leq t : X_\infty(s'; t, x, v) \in (0, 1) \forall s' \in (s, t) \right\}, \quad (51)$$

$$t_\infty^{\text{out}}(t, x, v) = \sup \left\{ s \geq t : X_\infty(s'; t, x, v) \in (0, 1) \forall s' \in (t, s) \right\}. \quad (52)$$

It is the largest open interval on which the characteristic starting at time  $t$  from a point  $(x, v) \in Q$  stays in  $Q$ . (51) (respectively (52)) is called the incoming (respectively the outgoing)

time. The life time of the characteristic which started at time  $t$  from the point  $(x, v)$  is the difference  $t_\infty^{\text{out}}(t, x, v) - t_\infty^{\text{inc}}(t, x, v)$ . Note that the definitions (51)-(52) do not depend on the way  $\phi$  is extended on  $\mathbb{R} \setminus [0, 1]$ . The differential system (49) being autonomous, one readily justifies by a uniqueness argument of the solution that

$$t_\infty^{\text{inc}}(t, x, v) = t + t_\infty^{\text{inc}}(0, x, v), \quad t_\infty^{\text{out}}(t, x, v) = t + t_\infty^{\text{out}}(0, x, v). \quad (53)$$

Moreover, the characteristics verify the conservation of the energy

$$\forall s \in \mathbb{R}, \quad \frac{d}{ds} \left( \frac{1}{2} V_\infty^2 \left( s; t, x, v \right) + \phi^\infty \left( X_\infty \left( s; t, x, v \right) \right) \right) = 0. \quad (54)$$

We shall study more in details the characteristics in Section 2.3.

To solve the Vlasov equation (32), we will use its mild-formulation which consists in integrating backward in time the equation along the characteristics. In doing so, there is two cases : either the characteristics has crossed  $\Sigma^{\text{inc}}$  at some positive time or it has crossed the time axis  $\{t = 0\}$ . We may denote and define when  $t_\infty^{\text{inc}}(t, x, v) > -\infty$ ,

$$X_\infty^{\text{inc}}(t, x, v) = X_\infty \left( t_\infty^{\text{inc}}(t, x, v); t, x, v \right) \in \{0, 1\}, \quad V_\infty^{\text{inc}}(t, x, v) = V_\infty \left( t_\infty^{\text{inc}}(t, x, v); t, x, v \right). \quad (55)$$

We are now ready to define the concept of solutions we consider for **(LVP)**.

**Definition 2.3** (Global mild-strong solution). Let  $\lambda > 0$  and  $h_0 \in L^1(Q)$ . We say that a couple  $(h, U)$  is a global mild-strong solution to **(LVP)** if there exists  $\gamma > 0$  such that for every  $T > 0$

- a)  $(h, U) \in X_{T, \gamma} \times Y_{T, \gamma}$  where  $X_{T, \gamma}$  and  $Y_{T, \gamma}$  are defined in (38).
- b)  $h$  is a mild solution of the Vlasov equation, in the sense that it satisfies for a.e  $(t, x, v) \in [0, T] \times Q$ ,

$$\begin{aligned} h(t, x, v) &= \mathbf{1}_{t_\infty^{\text{inc}}(t, x, v) \leq 0} h_0 \left( X_\infty \left( 0; t, x, v \right), V_\infty \left( 0; t, x, v \right) \right) \\ &+ \mathcal{G}(x, v) \int_0^t \mathbf{1}_{t_\infty^{\text{inc}}(t, x, v) < s} \partial_x U \left( s, X_\infty \left( s; t, x, v \right) \right) V_\infty \left( s; t, x, v \right) ds \end{aligned} \quad (56)$$

where the function  $\mathcal{G}$  is given by

$$\mathcal{G}(x, v) = \mathbf{1}_{D^+}(x, v) \frac{\mu' \left( \sqrt{v^2 + 2\phi^\infty(x)} \right)}{\sqrt{v^2 + 2\phi^\infty(x)}}. \quad (57)$$

- c) For a.e  $0 \leq t \leq T$ ,  $U(t)$  is a strong solution to

$$-\lambda^2 \partial_{xx} U(t) + n'_e(\phi^\infty) U(t) = \int_{\mathbb{R}} h(t, \cdot, v) dv \quad (58)$$

and satisfies the homogeneous Dirichlet boundary conditions  $U(t)(0) = 0$  and  $U(t)(1) = 0$ .

The fact that the function  $\mathcal{G}$  is outside the integral term is due to the conservation of the energy (54) and the fact that the domain  $D^+$  is left invariant by the stationary flow as we prove in Lemma (2.10). For the stability analysis, we need to introduce the concept of exit geometric conditions. The aim is to define precisely the rough picture of the fact some characteristics leaves the phase space in finite time. These definitions are not new since they are borrowed from [16]. The first exit condition is meant to estimate the time spent by a trajectory in  $Q$  starting at a positive time. It will be used to estimate the contribution of the source term in the Vlasov equation.

**Definition 2.4** (Internal Exit Geometric Condition). Let  $K \subset Q$  and  $J$  a subinterval of  $\mathbb{R}_+$ . We say that the equilibrium electric field  $-\partial_x \phi^\infty$  verifies the first exit geometric condition in time  $T$  with respect to  $K$  on  $J$  if

$$\sup_{(s,x,v) \in J \times K} (t_\infty^{\text{out}}(s, x, v) - t) \leq T, \quad (59)$$

$$\forall (s, x, v) \in J \times K, \quad (X_\infty^{\text{out}}, V_\infty^{\text{out}})(s, x, v) \in \Sigma^{\text{out}}. \quad (60)$$

The second exit geometric condition aims at estimating the time spent by a trajectory in  $Q$  starting at time zero. This one will be used to estimate how long the initial condition contributes to the solution of the Vlasov equation.

**Definition 2.5** (Initial Exit Geometric Condition). Let  $K \subset Q$ . We say that the equilibrium electric field  $-\partial_x \phi^\infty$  verifies the initial exit geometric condition in time  $T$  with respect to  $K$  if

$$\sup_{(x,v) \in K} t_\infty^{\text{out}}(0, x, v) \leq T, \quad (61)$$

$$\forall (x, v) \in K, \quad (X_\infty^{\text{out}}, V_\infty^{\text{out}})(0, x, v) \in \Sigma^{\text{out}}. \quad (62)$$

## 2.2. The linear elliptic estimates.

Let  $\rho \in L^1(0, 1)$ . In this section we provide  $W^{1,\infty}$  estimates for the solution of the linear elliptic problem

$$\begin{cases} -\lambda^2 \partial_{xx} V + n'_e(\phi^\infty) V = \rho \text{ a.e in } (0, 1), \\ V(0) = 0, \quad V(1) = 0. \end{cases} \quad (63)$$

We have the following.

**Proposition 2.6** ( $W^{1,\infty}$ -estimates). *Let  $\rho \in L^1(0, 1)$ . Then the problem (63) admits a unique strong solution  $V \in W^{2,1}(0, 1)$ . In addition, the solution satisfies the estimates*

$$\|\partial_{xx} V\|_{L^1(0,1)} \leq \frac{2\|\rho\|_{L^1(0,1)}}{\lambda^2}, \quad (64)$$

$$\|\partial_x V\|_{L^\infty(0,1)} \leq \frac{2\|\rho\|_{L^1(0,1)}}{\lambda^2}, \quad (65)$$

$$\|(n'_e \circ \phi^\infty) V\|_{L^1(0,1)} \leq \|\rho\|_{L^1(0,1)}. \quad (66)$$

*Proof.* In one dimension, the existence and uniqueness in  $W^{2,1}(0, 1)$  of a strong solution to (63) follows directly from the Lax-Milgram theorem and the elliptic regularity theory. In doing so,

may be the the point to be careful with is to establish the continuity in  $H_0^1(0, 1)$  of the linear form  $\varphi \in H_0^1(0, 1) \mapsto \int_0^1 \varphi(x)\rho(x)dx$ . One has using a Hölder inequality that

$$\forall \varphi \in H_0^1(0, 1), \quad \left| \int_0^1 \varphi(x)\rho(x)dx \right| \leq \|\varphi\|_{L^\infty(0,1)} \|\rho\|_{L^1(0,1)} \leq \|\varphi\|_{H_0^1(0,1)} \|\rho\|_{L^1(0,1)},$$

where we used the continuous imbedding of  $H_0^1(0, 1)$  in  $L^\infty(0, 1)$  to obtain the last inequality. We now focus on the estimates. *First estimate.* We obtain directly from the equation (63) that

$$\|\partial_{xx}V\|_{L^1(0,1)} \leq \frac{1}{\lambda^2} \|\rho - (n'_e \circ \phi^\infty)V\|_{L^1} \leq \frac{\|\rho\|_{L^1} + \|(n'_e \circ \phi^\infty)V\|_{L^1}}{\lambda^2}.$$

So it remains to estimate the  $L^1$ -norm of  $(n'_e \circ \phi^\infty)V$ . Let us proceed by regularization of the sign function. Let  $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{C}^1(\mathbb{R})$  a sequence of functions such that for all  $n \in \mathbb{N}$  :

$$\begin{aligned} \varphi_n(0) &= 0, \\ \forall u \in \mathbb{R}, \varphi'_n(u) &> 0, \quad |\varphi_n(u)| \leq 1 \\ \forall u \neq 0, \varphi_n(u) &\longrightarrow \text{sgn}(u) \text{ as } n \rightarrow +\infty. \end{aligned}$$

For each  $n \in \mathbb{N}$ , the function  $\varphi_n \circ V$  belongs to  $W^{1,1}(0, 1)$  and it verifies  $\varphi_n \circ V(0) = \varphi_n \circ V(1) = 0$ . We then multiply the equation (63) by  $\varphi_n \circ V$  and integrate by parts to obtain

$$\int_0^1 \lambda^2 |\partial_x V|^2 \varphi'_n \circ V dx + \int_0^1 (n'_e \circ \phi^\infty)V \varphi_n \circ V dx = \int_0^1 \rho \varphi_n \circ V dx.$$

Since  $\varphi'_n$  is positive on  $\mathbb{R}$  the first term is non negative and therefore

$$\int_0^1 (n'_e \circ \phi^\infty)V \varphi_n \circ V dx \leq \int_0^1 \rho \varphi_n \circ V dx \leq \|\rho\|_{L^1(0,1)}$$

where the last inequality is obtained using both a Hölder inequality and the fact that  $|\varphi_n| \leq 1$  on  $\mathbb{R}$ . Using the Lebesgue dominated convergence theorem, one has besides

$$\int_0^1 (n'_e \circ \phi^\infty)V \varphi_n \circ V dx \longrightarrow \int_0^1 n'_e \circ \phi^\infty |V| dx.$$

So, we get by passing to the limit in the previous inequality that

$$\int_0^1 n'_e \circ \phi^\infty |V| dx \leq \|\rho\|_{L^1(0,1)}$$

which yields (66) since  $n'_e$  is positive and thus (64).

*Second estimate.* Since  $V \in W^{2,1}(0, 1)$  then it belongs to  $\mathcal{C}^1[0, 1]$ . Besides,  $\int_0^1 \partial_x V(x) dx = 0$  because  $V(0) = V(1) = 0$ . By the mean value theorem there exists  $x_0 \in (0, 1)$  such that  $\partial_x V(x_0) = 0$ . Then, for  $x \in [0, 1]$  one has  $\partial_x V(x) = \int_{x_0}^x \partial_{xx} V(t) dt$ . Using a triangular inequality and the elliptic equation (63), we get for all  $x \in [0, 1]$ ,

$$|\partial_x V(x)| \leq \|\partial_{xx} V\|_{L^1(0,1)} \leq \frac{2\|\rho\|_{L^1(0,1)}}{\lambda^2}.$$

□

### 2.3. Study of the stationary characteristics.

Consider the following partition of the phase-space

$$Q = D^+ \cup D^\pm \cup D^- \cup S, \quad (67)$$

where  $D^+$  is given by (14),

$$S = \left\{ (x, v) \in Q : v^2 + 2\phi^\infty(x) = 0 \right\}, \quad (68)$$

$$D^\pm = \left\{ (x, v) \in Q : |v| < \sqrt{-2\phi^\infty(x)} \right\}, \quad (69)$$

$$D^- = \left\{ (x, v) \in Q : v < -\sqrt{-2\phi^\infty(x)} \right\}. \quad (70)$$

We recall here that  $\phi^\infty$  is decreasing on  $[0, 1]$  with  $\phi^\infty(0) = 0$ , hence and the above sets are well defined. These sets corresponds to sub or super level sets of the microscopic energy  $(x, v) \mapsto \frac{v^2}{2} + \phi^\infty(x)$  which is a conserved quantity along the flow (54). In this section we are interested in studying in details the characteristics, notably the invariance of the above sets and providing quasi-explicit formulas for the times of exit of the characteristics. A sketch of the phase portrait is given in Figure (1). As a consequence of this analysis we shall obtain.

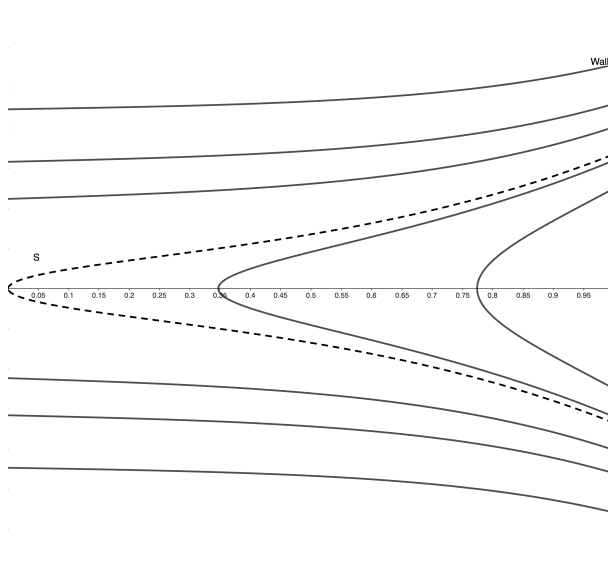


Figure 1: Sketch of the stationary phase portrait. The dashed line is the curve of equation  $\frac{v^2}{2} + \phi(x) = 0$ .

**Proposition 2.7** (Exit Geometric Conditions). *Let  $r \geq 0$ . The equilibrium electric field given by Theorem (1.2) verifies the internal exit geometric condition in time  $T_r$  with respect to  $D_r^+$  on  $\mathbb{R}^+$  and the initial exit geometric condition in time  $T_r$  with respect to  $D_r^+$ .*

We begin with following first result which in particular states that the stationary characteristics are leaving the phase space  $Q$  in finite time.

**Lemma 2.8** (Finiteness of the exit times). *Let  $(t, x, v) \in \mathbb{R} \times Q$ . We have the following*

a)  $t_\infty^{\text{inc}}(t, x, v) > -\infty$  and  $t_\infty^{\text{out}}(t, x, v) < +\infty$ .

b)  $\text{sgn}\left(X_\infty^{\text{inc}}(t, x, v) - \frac{1}{2}\right)V_\infty^{\text{inc}}(t, x, v) \leq 0$ .

*Proof. Proof of a).* We only do the proof for  $t_\infty^{\text{inc}}(t, x, v)$  since the proof for  $t_\infty^{\text{out}}(t, x, v)$  is similar. Observe first that by construction, the stationary solution verifies  $\partial_x \phi^\infty < 0$  in  $[0, 1]$  and  $-\partial_x \phi^\infty$  is non decreasing on  $[0, 1]$ . We now argue by contradiction and assume that  $t_\infty^{\text{inc}}(t, x, v) = -\infty$ . Then one has for all  $s \in (-\infty, t]$ ,  $0 < X_\infty(s; t, x, v) < 1$ . Using the monotony of  $\partial_x \phi^\infty$  we deduce from (49) that for all  $s \in (-\infty, t]$ ,  $-\partial_x \phi^\infty(0) \leq \frac{d}{ds} V_\infty(s; t, x, v) \leq -\partial_x \phi^\infty(1)$ . By integration of the previous inequality for  $s \in [u, t]$  with  $u < t$  we get

$$\partial_x \phi^\infty(1)(t - u) + v \leq V(u; t, x, v) \leq \partial_x \phi^\infty(0)(t - u) + v.$$

Integrating again the previous inequality on  $[s, t]$  for  $s < t$ , we get

$$-\partial_x \phi^\infty(0) \frac{(t - s)^2}{2} - v(t - s) + x \leq X_\infty(s; t, x, v) \leq -\partial_x \phi^\infty(1) \frac{(t - s)^2}{2} - v(t - s) + x.$$

Note that  $-\partial_x \phi^\infty > 0$  on  $[0, 1]$  therefore  $\lim_{s \rightarrow -\infty} -\partial_x \phi^\infty(0) \frac{(t - s)^2}{2} - v(t - s) + x = +\infty$ . We then deduce by comparison that for  $|s|$  large enough we have  $X_\infty(s; t, x, v) \geq 1$  which yields the contradiction.

*Proof of b).* By virtue of the previous point, the incoming position and velocity are well-defined. Then b) amounts to prove that if  $X_\infty^{\text{inc}}(t, x, v) = 0$  then  $V_\infty^{\text{inc}}(t, x, v) \geq 0$  and if  $X_\infty^{\text{inc}}(t, x, v) = 1$  then  $V_\infty^{\text{inc}}(t, x, v) \leq 0$ . We only treat the case when  $X_\infty^{\text{inc}}(t, x, v) = 0$  since the reasoning is similar for the other case. So suppose that  $X_\infty^{\text{inc}}(t, x, v) = 0$ . We want to prove that  $V_\infty^{\text{inc}}(t, x, v) \geq 0$ . So assume for the sake of the contradiction that  $V_\infty^{\text{inc}}(t, x, v) < 0$ . Then, since the solution to (49) is continuous in time, for  $0 < \eta < 1$  there exists  $\delta > 0$  such that for all  $s \in (t_\infty^{\text{inc}}(t, x, v) - \delta, t_\infty^{\text{inc}}(t, x, v) + \delta)$

$$V_\infty(s; t, x, v) < \frac{V_\infty^{\text{inc}}(t, x, v)}{2} < 0, \quad -\eta < X_\infty(s; t, x, v) < \eta.$$

Since  $\frac{dX_\infty}{ds}(s; t, x, v) = V_\infty(s; t, x, v)$ ,  $X_\infty(\cdot; t, x, v)$  is decreasing on the interval  $(t_\infty^{\text{inc}}(t, x, v) - \delta, t_\infty^{\text{inc}}(t, x, v)]$ . Therefore,

$$t_\infty^{\text{inc}}(t, x, v) - \delta < s < t_\infty^{\text{inc}}(t, x, v) \implies 0 < X_\infty(s; t, x, v) < \eta.$$

It eventually contradicts the minimality of the incoming time (51). □

We now study the regularity of the incoming time and the associated exit point.

**Lemma 2.9** (Regularity and characterization of the singular set). *Let  $t \in \mathbb{R}$ .*

a) *The three functions  $(x, v) \in Q \mapsto t_\infty^{\text{inc}}(t, x, v)$ ,  $(x, v) \in Q \mapsto X_\infty^{\text{inc}}(t, x, v)$ ,  $(x, v) \in Q \mapsto V_\infty^{\text{inc}}(t, x, v)$  are continuous at every point  $(x, v) \in Q$  such that  $(X_\infty^{\text{inc}}(t, x, v), V_\infty^{\text{inc}}(t, x, v)) \notin \Sigma^0$ .*



b) One has for all  $(x, v) \in Q$ ,

$$\left( (X_\infty^{\text{inc}}(t, x, v), V_\infty^{\text{inc}}(t, x, v)) \in \Sigma^0 \iff X_\infty^{\text{inc}}(t, x, v) = 0 \text{ and } (x, v) \in S \right).$$

*Proof. Proof of a).* A proof of continuity can be found in [16] Lemma 3.3. We nevertheless propose one here for the sake selfcontainedness of this document with slightly more details. We fix  $t \in \mathbb{R}$ . Observe firstly that we can decompose the incoming position and velocity as functions in the form

$$X_\infty^{\text{inc}}(t, \cdot, \cdot) = (X_{\infty, t} \circ S_t)(\cdot, \cdot) \quad V_\infty^{\text{inc}}(t, \cdot, \cdot) = (V_{\infty, t} \circ S_t)(\cdot, \cdot) \quad (71)$$

with

$$\begin{aligned} X_{\infty, t} : (s, x, v) \in \mathbb{R}^3 &\longmapsto X_\infty(s; t, x, v), & V_{\infty, t} : (s, x, v) \in \mathbb{R}^3 &\longmapsto V_\infty(s; t, x, v), \\ S_t : (x, v) \in \mathbb{R}^2 &\longmapsto (t_\infty^{\text{inc}}(t, x, v), x, v). \end{aligned}$$

Let us recall briefly why  $(X_{\infty, t}, V_{\infty, t})$  is continuous. For  $(x, v), (x', v') \in \mathbb{R}^2$ , let us set  $Y_\infty(\cdot; t) = X_\infty(\cdot; t, x, v) - X_\infty(\cdot; t, x', v')$ ,  $W_\infty(\cdot; t) = V_\infty(\cdot; t, x, v) - V_\infty(\cdot; t, x', v')$ . Then integrating the differential system (49) and using a triangle inequality, one gets for  $s \leq t$

$$|Y_\infty(s, t)| + |W_\infty(s, t)| \leq |x - x'| + |v - v'| + C \int_s^t |Y_\infty(\tau, t)| + |W_\infty(\tau, t)| d\tau$$

with  $C = \max(1, \|\partial_{xx}\phi^\infty\|_{L^\infty(\mathbb{R})})$ . A similar inequality holds if  $s > t$  so that a Gronwall Lemma then yields for  $s \in \mathbb{R}$

$$|Y_\infty(s, t)| + |W_\infty(s, t)| \leq (|x - x'| + |v - v'|) e^{C|t-s|}. \quad (72)$$

This estimate associated with the Cauchy-Lipschitz theorem shows that for all  $t \in \mathbb{R}$ , the map  $(X_{\infty, t}, V_{\infty, t})$  is continuous. Then, thanks to the decomposition (71), to establish the expected continuity, it is now sufficient to prove the continuity of the function  $(x, v) \in Q \longmapsto t_\infty^{\text{inc}}(t, x, v)$  at every point  $(x, v) \in Q$  such that  $(X_\infty^{\text{inc}}(t, x, v), V_\infty^{\text{inc}}(t, x, v)) \notin \Sigma^0$ .

We fix  $(x, v) \in Q$  be such that  $(X_\infty^{\text{inc}}(t, x, v), V_\infty^{\text{inc}}(t, x, v)) \notin \Sigma^0$  and suppose without loss of generality that  $V_\infty^{\text{inc}}(t, x, v) > 0$ , and as a consequence of Lemma 2.8 b), that  $X_\infty^{\text{inc}}(t, x, v) = 0$ . Then we have  $t_\infty^{\text{inc}}(t, x, v) > -\infty$  and by definition of the incoming time we also have  $X_\infty(s; t, x, v) \in (0, 1)$  for all  $t_\infty^{\text{inc}}(t, x, v) < s < t$ . The proof is in three steps.

*Step 1: Localization in time.* We are going to prove the following claim: *there exists  $\varepsilon_0 > 0$  such that for any  $0 < \varepsilon < \frac{\varepsilon_0}{2}$ , we have  $X_\infty(s; t, x, v) < 0$  for  $s \in (t_\infty^{\text{inc}}(t, x, v) - \varepsilon_0, t_\infty^{\text{inc}}(t, x, v) - \varepsilon)$ .* Indeed, since  $X_\infty(\cdot; t, x, v)$  is  $\mathcal{C}^1$  a Taylor expansion at the point  $t_\infty^{\text{inc}}(t, x, v)$  yields for  $s < t_\infty^{\text{inc}}(t, x, v)$ ,  $X_\infty(s; t, x, v) = (s - t_\infty^{\text{inc}}(t, x, v)) (V_\infty^{\text{inc}}(t, x, v) + o(1)(s))$  where  $o(1) : \mathbb{R} \rightarrow \mathbb{R}$  is function such that  $o(1)(s) \rightarrow 0$  as  $s \rightarrow t_\infty^{\text{inc}}(t, x, v)$ . Since  $V_\infty^{\text{inc}}(t, x, v) > 0$ , it therefore exists  $\varepsilon_0 > 0$  such that we have  $V_\infty^{\text{inc}}(t, x, v) + o(1) > 0$  for  $t_\infty^{\text{inc}}(t, x, v) - \varepsilon_0 < s < t_\infty^{\text{inc}}(t, x, v)$  and it yields the claim.

*Step 2: Localization in space.* Let  $0 < \varepsilon < \min\{\frac{\varepsilon_0}{2}; t - t_\infty^{\text{inc}}(t, x, v)\}$ . Using the Gronwall estimate (72) at  $s = t_\infty^{\text{inc}}(t, x, v) - 2\varepsilon$  and the fact that by virtue of the previous claim

$X_\infty(t_\infty^{\text{inc}}(t, x, v) - 2\varepsilon; t, x, v) < 0$  we have that there exists  $\delta \equiv \delta_{t,x,v,\varepsilon} > 0$  such that for all  $(x', v') \in Q$ ,

$$|x - x'| + |v - v'| < \delta \implies X_\infty(t_\infty^{\text{inc}}(t, x, v) - 2\varepsilon; t, x', v') < 0.$$

It implies that for  $(x', v')$  in a ball centered at  $(x, v)$  and of radius  $\delta$ , we have  $t_\infty^{\text{inc}}(t, x', v') > -\infty$  and that the associated incoming time is such that  $t_\infty^{\text{inc}}(t, x, v) - 2\varepsilon < t_\infty^{\text{inc}}(t, x', v')$ . We are now going to prove that  $t_\infty^{\text{inc}}(t, x', v') < t_\infty^{\text{inc}}(t, x, v) + 2\varepsilon$ . Observe that by definition of  $t_\infty^{\text{inc}}(t, x, v)$  and since  $\varepsilon$  is small enough we have  $X_\infty(t_\infty^{\text{inc}}(t, x, v) + 2\varepsilon; t, x, v) \in (0, 1)$ . Now using again the Gronwall estimate at (72)  $s = t_\infty^{\text{inc}}(t, x, v) + 2\varepsilon$ , there exists  $\eta \equiv \eta_{t,x,v,\varepsilon} > 0$  such that for all  $(x', v') \in Q$ ,

$$|x - x'| + |v - v'| < \eta \implies X_\infty(t_\infty^{\text{inc}}(t, x, v) + 2\varepsilon; t, x', v') \in (0, 1).$$

It implies that for  $(x', v')$  in a ball centered at  $(x, v)$  of radius  $\eta$  we have  $t_\infty^{\text{inc}}(t, x', v') < t_\infty^{\text{inc}}(t, x, v) + 2\varepsilon$ .

*Conclusion.* We have proven that for  $0 < \varepsilon < \min\{\frac{\varepsilon_0}{2}; t - t_\infty^{\text{inc}}(t, x, v)\}$  there exists  $r = \min\{\delta_{t,x,v,\varepsilon}, \eta_{t,x,v,\varepsilon}\} > 0$  such that for all  $(x', v') \in Q$

$$|x - x'| + |v - v'| < r \implies |t_\infty^{\text{inc}}(t, x', v') - t_\infty^{\text{inc}}(t, x, v)| < 2\varepsilon.$$

It shows that the map  $(x, v) \in Q \mapsto t_\infty^{\text{inc}}(t, x, v)$  is continuous at any point  $(x, v) \in Q$  such that  $(X_\infty^{\text{inc}}(t, x, v), V_\infty^{\text{inc}}(t, x, v)) \notin \Sigma^0$ .

*Proof of b).* We show the necessary condition. Fix  $t \in \mathbb{R}$  and  $(x, v) \in Q$  such that  $(X_\infty^{\text{inc}}(t, x, v), V_\infty^{\text{inc}}(t, x, v)) \in \Sigma^0$ . Then by conservation of the energy (54) we have

$$\phi^\infty(X_\infty^{\text{inc}}(t, x, v)) + \frac{V_\infty^{\text{inc}}(t, x, v)^2}{2} = \phi^\infty(x) + \frac{v^2}{2}.$$

Note that the case  $(X_\infty^{\text{inc}}(t, x, v), V_\infty^{\text{inc}}(t, x, v)) = (0, 0)$  readily yields  $(x, v) \in S$ , while the other case  $(X_\infty^{\text{inc}}(t, x, v), V_\infty^{\text{inc}}(t, x, v)) = (1, 0)$  yields  $\phi^\infty(1) - \phi^\infty(x) = \frac{v^2}{2}$ . Since  $\phi^\infty$  is decreasing on  $[0, 1]$  one has  $\frac{v^2}{2} = \phi^\infty(1) - \phi^\infty(x) < 0$ . It yields a contradiction. The sufficient condition is trivial by conservation of energy.  $\square$

We now study the regions that remain invariant by (50) and give the associated formula for the incoming and outgoing times. Note that by (53) we have

$$\forall (t, x, v) \in \mathbb{R} \times Q, \quad t_\infty^{\text{inc}}(t, x, v) = t + t_\infty^{\text{inc}}(0, x, v), \quad t_\infty^{\text{out}}(t, x, v) = t + t_\infty^{\text{out}}(0, x, v),$$

which encodes the fact that for an autonomous differential equation the dynamic is invariant by translation in time. So it is sufficient to study the characteristics which started at time  $t = 0$ . We prove the following.

**Lemma 2.10** (Invariant regions). *Let  $r \geq 0$ . The sets  $D_r^+$ ,  $D^-$ ,  $D^\pm$  and  $S$  are left invariant by the flow (50). More precisely, for any set  $E \in \{D_r^+, D^-, D^\pm, S\}$  we have*

$$(x, v) \in E \implies \forall s \in (t_\infty^{\text{inc}}(0, x, v), t_\infty^{\text{out}}(0, x, v)), \quad (X_\infty, V_\infty)(s; 0, x, v) \in E. \quad (73)$$

*Proof. Invariance in  $D_r^+$ .* Let  $(x, v) \in D_r^+$ . By conservation of the energy (54) one has for all  $s \in (t_\infty^{\text{inc}}(0, x, v), t_\infty^{\text{out}}(0, x, v))$ ,  $V_\infty^2(s; 0, x, v) = v^2 + 2\phi^\infty(x) - 2\phi^\infty(X_\infty(s; 0, x, v))$  and since  $v^2 + 2\phi^\infty(x) > r^2$  we obtain

$$|V_\infty(s; 0, x, v)| = \sqrt{v^2 + 2(\phi^\infty(x) - \phi^\infty(X_\infty(s; 0, x, v)))} > \sqrt{r^2 - 2\phi^\infty(X_\infty(s; 0, x, v))}.$$

Let us now prove that  $V_\infty(s; 0, x, v)$  is positive. Since  $\phi^\infty$  is decreasing and  $\mathcal{C}^1$ , by (49) we deduce that  $s \mapsto V_\infty(s; 0, x, v)$  is increasing. Therefore if  $s \geq t_\infty^{\text{inc}}(0, x, v)$  then  $V_\infty(s; 0, x, v) \geq V_\infty^{\text{inc}}(0, x, v)$ . We now claim that  $V_\infty^{\text{inc}}(0, x, v) \geq 0$ . For the sake of the contradiction, let us assume the opposite. Since  $V_\infty(t; 0, x, v) = v$  and  $v > 0$ , and the function  $V_\infty(\cdot; t, x, v)$  is continuous on its interval of definition, there exists  $t^0 \in (t_\infty^{\text{inc}}(t, x, v), t)$  such that  $V_\infty(t^0; 0, x, v) = 0$ . Using the conservation of the energy, observe that  $\phi^\infty(X_\infty(t^0; 0, x, v)) = \frac{v^2}{2} + \phi^\infty(x)$  and the right hand side is positive since  $(x, v) \in D_r^+$ . It contradicts the negativity of  $\phi^\infty$ . Hence  $V_\infty^{\text{inc}}(0, x, v) \geq 0$  and thus  $|V_\infty(s; 0, x, v)| = V_\infty(s; 0, x, v) > \sqrt{-2\phi^\infty(X_\infty(s; 0, x, v)) + r^2}$ . It proves the claim for the set  $D_r^+$ .

*Invariance in  $D^-$ .* Consider the symmetry  $\mathcal{R} : (x, v) \in \mathbb{R}^2 \mapsto (x, -v)$  and observe that  $\mathcal{R}(D^+) = D^-$ . Arguing the uniqueness of the characteristics, one has the identity

$$\mathcal{R} \circ \mathcal{F}_{-s, -t}^\infty \circ \mathcal{R} = \mathcal{F}_{s, t}^\infty \quad (74)$$

where  $\mathcal{F}_{s, t}^\infty$  is the flow defined in (50). Let  $(x, v) \in D^-$ . One has  $\mathcal{R}(x, v) \in D^+$  and  $\mathcal{F}_{-s, -t}^\infty(\mathcal{R}(x, v)) \in D^+$  for  $s \in (t_\infty^{\text{inc}}(0, x, v), t_\infty^{\text{out}}(0, x, v))$  by symmetry and invariance in  $D^+$ . It implies  $\mathcal{R} \circ \mathcal{F}_{-s, -t}^\infty \circ \mathcal{R}(x, v) \in \mathcal{R}(D^+) = D^-$ . Using the identity (74) it yields  $\mathcal{F}_{s, t}^\infty(x, v) \in D^-$  which is the expected claim.

*Invariance in  $D^\pm$ .* Let  $(x, v) \in D^\pm$ . By conservation of the energy one has for all  $s \in (t_\infty^{\text{inc}}(0, x, v), t_\infty^{\text{out}}(0, x, v))$ ,  $\frac{V_\infty^2(s; 0, x, v)}{2} + \phi^\infty(X_\infty(s; 0, x, v)) = \frac{v^2}{2} + \phi^\infty(x) < 0$  which is the expected inequality. Eventually the invariance in  $S$  is trivial. The proof is achieved.  $\square$

We now provide quasi-explicit formula for the incoming and the outgoing time defined in (51), (52).

**Lemma 2.11** (Explicit formulas for the incoming and the outgoing time). *One has the following formulas for the incoming and the outgoing times in  $Q \setminus S$ ,*

$$t_\infty^{\text{inc}}(0, x, v) = \begin{cases} -\int_0^x \frac{du}{\sqrt{v^2 + 2(\phi^\infty(x) - \phi^\infty(u))}} & (x, v) \in D^+ \\ -\int_{x_0}^x \frac{du}{\sqrt{v^2 + 2(\phi^\infty(x) - \phi^\infty(u))}} - \int_{x_0}^1 \frac{du}{\sqrt{2(\phi^\infty(x_0) - \phi^\infty(u))}}, & (x, v) \in D^\pm, v > 0, \\ -\int_x^1 \frac{du}{\sqrt{v^2 + 2(\phi^\infty(x) - \phi^\infty(u))}}, & (x, v) \in D^\pm, v \leq 0, \\ -\int_x^1 \frac{du}{\sqrt{v^2 + 2(\phi^\infty(x) - \phi^\infty(u))}} & (x, v) \in D^-, \end{cases} \quad (75)$$

$$t_\infty^{\text{out}}(0, x, v) = \begin{cases} \int_x^1 \frac{du}{\sqrt{v^2 + 2(\phi^\infty(x) - \phi^\infty(u))}} & (x, v) \in D^+, \\ \int_{x_0}^x \frac{du}{\sqrt{v^2 + 2(\phi^\infty(x) - \phi^\infty(u))}} + \int_{x_0}^1 \frac{du}{\sqrt{2(\phi^\infty(x_0) - \phi^\infty(u))}}, & (x, v) \in D^\pm, v < 0, \\ \int_x^1 \frac{du}{\sqrt{v^2 + 2(\phi^\infty(x) - \phi^\infty(u))}}, & (x, v) \in D^\pm, v \geq 0 \\ \int_0^x \frac{du}{\sqrt{v^2 + 2(\phi^\infty(x) - \phi^\infty(u))}} & (x, v) \in D^-, \end{cases} \quad (76)$$

where  $x_0 = (\phi^\infty)^{-1}\left(\frac{v^2}{2} + \phi^\infty(x)\right)$ .

*Proof.* *Formula in  $D^+ \cup D^-$ .* Let  $(x, v) \in D^+$ . Set  $I(x, v) := (t_\infty^{\text{inc}}(0, x, v), t_\infty^{\text{out}}(0, x, v))$ . Observe that by the Lemma 2.10, for all  $s \in I(x, v)$ ,  $V_\infty(s; 0, x, v) > 0$ . Then, the function  $s \in \overline{I(x, v)} \mapsto X_\infty(s; 0, x, v)$  is monotone increasing and  $\mathcal{C}^1$  on  $\overline{I(x, v)}$ . Therefore  $X_\infty(\cdot; 0, x, v)$  is onto and into from  $\overline{I(x, v)}$  to  $[0, 1]$ . By the bijection theorem, it has a unique inverse denoted  $\tau : [0, 1] \rightarrow \overline{I(x, v)}$  which is continuous and verifies

$$\forall u \in (0, 1), X_\infty(\tau(u); 0, x, v) = u. \quad (77)$$

Since  $X_\infty(\cdot; 0, x, v)$  is  $\mathcal{C}^1$  we deduce from the global inverse mapping theorem that the function  $\tau$  is  $\mathcal{C}^1(0, 1)$ . One therefore differentiates (77) to get the Cauchy problem

$$\begin{cases} \forall u \in (0, 1), \frac{d}{du}\tau(u) = \frac{1}{V_\infty(\tau(u); 0, x, v)}, \\ \tau(x) = 0. \end{cases}$$

Invoking the conservation of the energy and the fact that  $V_\infty(\cdot; 0, x, v)$  is positive, it yields  $V_\infty(\tau(u); 0, x, v) = \sqrt{v^2 + 2(\phi^\infty(x) - \phi^\infty(u))}$ . By definition,  $\tau(0) = t_\infty^{\text{inc}}(0, x, v)$  and  $\tau(1) = t_\infty^{\text{out}}(0, x, v)$ . Since  $s \in \mathbb{R} \mapsto V_\infty(s; 0, x, v)$  is continuous and  $\tau \in \mathcal{C}^0[0, 1]$ , the function  $\tau$  is also  $\mathcal{C}^1[0, 1]$ . Integrating the Cauchy problem both on  $(0, x)$  and on  $(x, 1)$  yields the expected formulas for  $t_\infty^{\text{inc}}(0, x, v)$  and  $t_\infty^{\text{out}}(0, x, v)$ . A similar reasoning yields the formulas in  $D^-$ .

*Formula in  $D^\pm$ .* Let  $(x, v) \in D^\pm$ . Set  $I(x, v) := (t_\infty^{\text{inc}}(0, x, v), t_\infty^{\text{out}}(0, x, v))$ . There is two cases. If  $v > 0$  then by a standard continuity and monotony argument there exists a unique  $t^0(x, v) \in (t_\infty^{\text{inc}}(0, x, v), 0)$  such that  $V_\infty(t^0(x, v); 0, x, v) = 0$ . Then set  $x_0 = X_\infty(t^0(x, v); 0, x, v)$ . Using the conservation of the energy and the fact that  $\phi^\infty$  is a bijection from  $[0, 1]$  to  $[\phi_b, 0]$  one has  $x_0 = (\phi^\infty)^{-1}\left(\frac{v^2}{2} + \phi^\infty(x)\right)$ . Following backward in time the characteristic which started at time  $t = 0$  from the point of coordinate  $(x, v)$ , one decomposes the incoming time as follows,

$$t_\infty^{\text{inc}}(0; x, v) = t^0(x, v) + t_\infty^{\text{inc}}(t^0(x, v); x_0, 0).$$

Observe that for  $t > t^0(x, v)$ ,  $V_\infty(t; 0, x, v) > 0$  and for  $t < t^0(x, v)$ ,  $V_\infty(t; 0, x, v) < 0$ . One may again use the global inversion theorem on each branch of the characteristic (for  $t > t^0(x, v)$  and for  $t < t^0(x, v)$ ) to obtain

$$t^0(x, v) = - \int_{x_0}^x \frac{du}{\sqrt{v^2 + 2(\phi^\infty(x) - \phi^\infty(u))}} \text{ and } t_\infty^{\text{inc}}(t^0(x, v); x_0, 0) = - \int_{x_0}^1 \frac{du}{\sqrt{v^2 + 2(\phi^\infty(x) - \phi^\infty(u))}}.$$

where the singularity at  $u = x_0$  is integrable because one has by concavity and monotony of  $\phi^\infty$ , for all  $u \in [0, 1]$ ,  $\phi^\infty(u) \leq \phi^\infty(x_0) + (u - x_0)\partial_x\phi^\infty(x_0)$  with  $\partial_x\phi^\infty(x_0) < 0$ . Combining both expressions, one obtains the expected formula for  $t_\infty^{\text{inc}}(0, x, v)$ . To get the formula for  $t_\infty^{\text{out}}(0, x, v)$ , it suffices to follow the same characteristic but forward in time. Eventually, the case  $v \leq 0$ , is treated analogously to the case when  $(x, v) \in D^-$ .  $\square$

As a corollary of the above explicit formulas, we obtain the following bounds which depend on the equilibrium potential  $\phi^\infty$ .

**Corollary 2.12** (Bound on the incoming and outgoing times). *We have the following bounds*

a) For every  $r > 0$  we have the bounds in  $D_r^+$ ,

$$-\frac{1}{r} \leq \inf_{(x,v) \in D_r^+} t_\infty^{\text{inc}}(0, x, v), \quad \sup_{(x,v) \in D_r^+} t_\infty^{\text{out}}(0, x, v) \leq \frac{1}{r}. \quad (78)$$

b) More generally, we for all  $(x, v) \in Q \setminus S$

$$|t_\infty^{\text{inc}}(0, x, v)| \leq t^-(x, v), \quad |t_\infty^{\text{out}}(0, x, v)| \leq t^+(x, v), \quad (79)$$

where

$$t^-(x, v) = \begin{cases} \frac{\sqrt{-2\phi^\infty(x)}}{|\partial_x \phi^\infty(0)|} & (x, v) \in D^+, \\ \frac{\sqrt{2(\phi^\infty(x_0) - \phi^\infty(x))}}{|\partial_x \phi^\infty(x_0)|} + \frac{\sqrt{2(\phi^\infty(x_0) - \phi^\infty(1))}}{|\partial_x \phi^\infty(x_0)|} & (x, v) \in D^\pm, v > 0, \\ \frac{\sqrt{2(\phi^\infty(x) - \phi^\infty(1))}}{|\partial_x \phi^\infty(x)|} & (x, v) \in D^\pm, v \leq 0, \\ \frac{\sqrt{-2\phi^\infty(1) - \sqrt{2\phi^\infty(x)}}}{|\partial_x \phi^\infty(0)|} & (x, v) \in D^-, \end{cases} \quad (80)$$

$$t^+(x, v) = \begin{cases} \frac{\sqrt{-2\phi^\infty(1) - \sqrt{2\phi^\infty(x)}}}{|\partial_x \phi^\infty(0)|} & (x, v) \in D^+, \\ \frac{\sqrt{2(\phi^\infty(x_0) - \phi^\infty(x))}}{|\partial_x \phi^\infty(x_0)|} + \frac{\sqrt{2(\phi^\infty(x_0) - \phi^\infty(1))}}{|\partial_x \phi^\infty(x_0)|} & (x, v) \in D^\pm, v < 0, \\ \frac{\sqrt{2(\phi^\infty(x) - \phi^\infty(1))}}{|\partial_x \phi^\infty(x)|} & (x, v) \in D^\pm, v \geq 0, \\ \frac{\sqrt{-2\phi^\infty(x)}}{|\partial_x \phi^\infty(0)|} & (x, v) \in D^-. \end{cases} \quad (81)$$

with  $x_0 = (\phi^\infty)^{-1}(\frac{v^2}{2} + \phi^\infty(x))$  are bounded functions. Moreover we have the uniform bounds in  $D^+$

$$\sup_{(x,v) \in D^+} |t^-(x, v)| \leq \frac{\sqrt{-2\phi_b}}{|\partial_x \phi^\infty(0)|}, \quad \sup_{(x,v) \in D^+} |t^+(x, v)| \leq \frac{\sqrt{-2\phi_b}}{|\partial_x \phi^\infty(0)|} \quad (82)$$

where we recall that  $\phi_b < 0$  is independent of  $\lambda > 0$  and  $\partial_x \phi^\infty(0) < 0$  thanks to (16).

*Proof. Proof of a)* Let  $r > 0$  and  $(x, v) \in D_r^+$  where  $D_r^+$  is defined in (40). Then we have  $v^2 + 2\phi^\infty(x) > r^2$ . Since  $\phi$  is non positive we obtain

$$t_\infty^{\text{inc}}(0, x, v) = - \int_0^x \frac{du}{\sqrt{v^2 + 2\phi^\infty(x) - 2\phi^\infty(u)}} > -\frac{x}{r} > -\frac{1}{r}.$$

Taking the infimum on  $D_r^+$  yields  $\inf_{(x,v) \in D_r^+} t_\infty^{\text{inc}}(0, x, v) \geq -\frac{1}{r}$ . The other bound is obtained in the same manner. *Proof of b)* We only do the proof for the incoming time since the proof is similar for the outgoing time. We have three cases: if  $(x, v) \in D^+ \cup D^-$ , then  $v^2 + 2\phi^\infty(x) > 0$ . Therefore one has,

$$|t_\infty^{\text{inc}}(0, x, v)| \leq \begin{cases} \int_0^x \frac{du}{\sqrt{-2\phi^\infty(u)}} & (x, v) \in D^+, \\ \int_x^1 \frac{du}{\sqrt{-2\phi^\infty(u)}} & (x, v) \in D^-. \end{cases}$$

Using the change of variable  $u \mapsto \varphi = -\phi^\infty(u)$  and the fact that  $\partial_x \phi^\infty$  is decreasing, we obtain the expected estimate. If  $(x, v) \in D^\pm$  with  $v > 0$  then  $2\phi^\infty(x_0) = v^2 + 2\phi^\infty(x)$ . So,

$$|t_\infty^{\text{inc}}(0, x, v)| \leq \int_{x_0}^x \frac{du}{\sqrt{2(\phi^\infty(x_0) - \phi^\infty(u))}} + \int_{x_0}^1 \frac{du}{\sqrt{2(\phi^\infty(x_0) - \phi^\infty(u))}}.$$

Using again the change of variable  $u \mapsto \varphi = -\phi^\infty(u)$  and the fact that  $\phi^\infty$  is decreasing and that  $\partial_x \phi^\infty$  is also decreasing with  $\partial_x \phi^\infty < 0$  in  $[0, 1]$ , we get the expected estimate. Eventually, if  $(x, v) \in D^\pm$  with  $v \leq 0$ , we use the fact that  $v^2 + 2(\phi^\infty(x) - \phi^\infty(u)) \geq 2(\phi^\infty(x) - \phi^\infty(u))$  for all  $u \in [x, 1]$ . So using again the change of variable  $u \mapsto -\phi^\infty(u)$  and the fact that  $\partial_x \phi^\infty$  is decreasing we get the desired estimate.  $\square$

Using the fact that for any time  $t \in \mathbb{R}$  and  $(x, v) \in Q$ ,  $t_\infty^{\text{inc}}(t, x, v) = t + t_\infty^{\text{inc}}(0, x, v)$ ,  $t_\infty^{\text{out}}(t, x, v) = t + t_\infty^{\text{out}}(0, x, v)$  we obtain that Proposition (2.7) is a consequence of the above corollary.

#### 2.4. Measurability and change of variables in the integrals.

We introduce different subsets of the phase-space defined for  $t \geq 0$  by

$$\mathcal{A}_t := \{(x, v) \in Q \setminus S : t_\infty^{\text{inc}}(t, x, v) < 0\}, \quad (83)$$

$$\mathcal{B}_t := \{(x, v) \in Q \setminus S : t_\infty^{\text{inc}}(t, x, v) = 0\}. \quad (84)$$

Points in  $\mathcal{A}_t$  are on characteristics that do not leave  $Q$  on the interval  $[0, t]$ . Points in  $\mathcal{B}_t$  are on characteristics which reach  $\Sigma^-$  at time zero. We do not consider points which are in  $S$  because the incoming and outgoing times (51) and (52) are not continuous in  $S$ . Since  $S$  is a set of measure zero, it does not pose any difficulty to define the solution  $h$  to (32) outside  $S$  since we are concerned with a mild solution and therefore a function which is defined a.e.

**Lemma 2.13** (Measurability and change of variable). *Let  $t \geq 0$ . We have*

a) *The sets  $\mathcal{A}_t, \mathcal{B}_t$ , are Borel sets and the set  $\mathcal{B}_t$  is of Lebesgue measure zero.*

b)  $\mathcal{F}_{0,t}^\infty(\mathcal{A}_t) = \{(x', v') \in Q \setminus S : t_\infty^{\text{out}}(0, x', v') > t\}$ .

c) *For any measurable function  $f : Q \rightarrow \mathbb{R}^+$ , we have*

$$\int_{\mathcal{A}_t} f(\mathcal{F}_{0,t}^\infty(x, v)) dx dv = \int_{\mathcal{F}_{0,t}^\infty(\mathcal{A}_t)} f(x', v') dx' dv' \leq \int_Q f(x', v') dx' dv'.$$

*Proof.* Let  $t \geq 0$ .

*Proof of a).* By virtue of the Lemma 2.9, the function  $(x, v) \in Q \setminus S \mapsto t_\infty^{\text{inc}}(t, x, v)$  is continuous. Therefore the sets  $\mathcal{A}_t$  and  $\mathcal{B}_t$  are Borel sets since they are respectively open and closed. Since  $\mathcal{F}_{0,t}^\infty$  is measure preserving the measure of  $\mathcal{B}_t$  is equal to the measure of  $\mathcal{F}_{0,t}^\infty(\mathcal{B}_t)$ . By definition of  $\mathcal{B}_t$  we have  $\mathcal{F}_{0,t}^\infty(\mathcal{B}_t) \subset \Sigma^{\text{inc}}$  where  $\Sigma^{\text{inc}}$  is a set of measure zero.

*Proof of b).* We begin with the embedding  $\{\mathcal{F}_{0,t}^\infty(x, v) : (x, v) \in \mathcal{A}_t\} \subset \{(x', v') \in Q \setminus S : t_\infty^{\text{out}}(0, x', v') > t\}$ . Let  $(x, v) \in \mathcal{A}_t$  and set  $(x', v') = \mathcal{F}_{0,t}^\infty(x, v)$ . For  $s \in [0, t]$  we have,

$$\mathcal{F}_{s,0}^\infty(x', v') = \mathcal{F}_{s,0}^\infty \circ \mathcal{F}_{0,t}^\infty(x, v) = \mathcal{F}_{s,t}^\infty(x, v).$$

By definition of  $\mathcal{A}_t$ , we have  $t_\infty^{\text{inc}}(t, x, v) < 0$ . So,  $\mathcal{F}_{s,t}^\infty(x, v) \in Q$  for all  $s \in [0, t]$ . By energy conservation (54) we have  $\mathcal{F}_{s,t}^\infty(x, v) \notin S$  for all  $s \in [0, t]$  since  $(x, v) \notin S$ . Thus, for  $s \in [0, t]$  we have  $\mathcal{F}_{s,0}^\infty(x', v') \in Q \setminus S$  and consequently  $t_\infty^{\text{out}}(0, x', v') > t$ . We now show the reverse embedding. Let  $(x', v') \in Q \setminus S$  such that  $t_\infty^{\text{out}}(0, x', v') > t$ . Therefore  $\mathcal{F}_{s,0}^\infty(x', v') \in Q$  for all  $s \in [0, t]$ . By conservation of energy (54) we have  $\mathcal{F}_{s,0}^\infty(x', v') \notin S$  for all  $s \in [0, t]$  since  $(x', v') \notin S$ . Then set  $(x, v) = \mathcal{F}_{t,0}^\infty(x', v')$ . Since, for all  $s \in [0, t]$  we have  $\mathcal{F}_{s,t}^\infty(x, v) = \mathcal{F}_{s,0}^\infty(x', v') \in Q \setminus S$  we deduce that  $t_\infty^{\text{inc}}(t, x, v) < 0$ . Therefore  $(x, v) \in \mathcal{A}_t$  and  $(x', v') = \mathcal{F}_{0,t}^\infty(x, v)$ . It shows the claim.

*Proof of c).* It is just a consequence of the measure preserving change of variables  $(x', v') \mapsto (X_\infty, V_\infty)(0; t, x, v)$  in the integral.  $\square$

### 2.5. The proof of wellposedness for (LVP).

Consider the assumptions of Theorem 2.1. Let  $T > 0$  and fix  $\gamma > \frac{2\|\partial_v f^\infty\|_{L^1(Q)}}{\lambda^2}$  and note that  $\gamma$  is chosen independently of  $T$ . Consider  $\mathcal{S}$  the operator defined for  $h \in X_{T,\gamma}$  by

$$\begin{aligned} \mathcal{S}(h)(t, x, v) &= \mathbf{1}_{\mathcal{A}_t \cup \mathcal{B}_t}(x, v) h_0\left(X_\infty\left(0; t, x, v\right), V_\infty\left(0; t, x, v\right)\right) \\ &+ \mathcal{G}(x, v) \int_0^t \mathbf{1}_{t_\infty^{\text{inc}}(t, x, v) < s} \partial_x U(h)\left(s, X_\infty\left(s; t, x, v\right)\right) V_\infty\left(s; t, x, v\right) ds \end{aligned} \quad (85)$$

for a.e  $(t, x, v) \in [0, T] \times Q$ , where for a.e  $0 \leq s \leq T$ ,  $U(h)(s, \cdot)$  is the solution to the linearized Poisson equation (58) with the source term  $\rho_h(s, \cdot) = \int_{\mathbb{R}} h(s, \cdot, v) dv$  and the sets  $\mathcal{A}_t$  and  $\mathcal{B}_t$  are given in (83)-(84). Remark that the indicatrix functions are a.e equal to those appearing in the mild formulation (56) so that both formulations are in fact equivalent. Set for a.e  $(t, x, v) \in [0, T] \times Q$ ,

$$\begin{aligned} I_1(t, x, v) &= \mathbf{1}_{\mathcal{A}_t \cup \mathcal{B}_t}(x, v) h_0\left(X_\infty\left(0; t, x, v\right), V_\infty\left(0; t, x, v\right)\right), \\ I_2(t, x, v) &= \mathcal{G}(x, v) \int_0^t \mathbf{1}_{t_\infty^{\text{inc}}(t, x, v) < s} \partial_x U(h)\left(s, X_\infty\left(s; t, x, v\right)\right) V_\infty\left(s; t, x, v\right) ds. \end{aligned}$$

*Step 1: Stability estimate in  $X_{T,\gamma}$ .* We will show separately that  $I_1$  and  $I_2$  belong to  $X_{T,\gamma}$ . According to Lemma 2.13 a), the sets  $\mathcal{A}_t, \mathcal{B}_t$  are Borel sets for every  $t \in [0, T]$ . Therefore,  $I_1(t, \cdot, \cdot), I_2(t, \cdot, \cdot)$  are measurable functions. Using Lemma 2.13 c) we have for a.e  $t \in [0, T]$ ,

$$\int_Q |I_1(t, x, v)| dx dv = \int_{\mathcal{A}_t} \left| h_0\left(\mathcal{F}_{0,t}^\infty(x, v)\right) \right| dx dv \leq \int_Q |h_0(x', v')| dx' dv',$$

where we have used the fact that  $\mathcal{B}_t$  is a set of measure zero. Therefore  $I_1 \in X_{T,\gamma}$ . We now treat  $I_2$ . Using the fact that  $\mathcal{G}$  is supported in  $D^+$ , we have for a.e  $t \in [0, T]$ ,

$$\int_Q |I_2(t, x, v)| dx dv = \int_{D^+} |\mathcal{G}(x, v)| \left| \int_0^t \mathbf{1}_{t_\infty^{\text{inc}}(t, x, v) < s} \partial_x U(h)(s; X_\infty(s; t, x, v)) V_\infty(s; t, x, v) ds \right| dx dv.$$

Using a triangular inequality and the Fubini-Tonelli theorem we have,

$$\int_Q |I_2(t, x, v)| \, dx dv \leq \int_0^t \int_{D^+} \mathbf{1}_{t_{\text{inc}}^\infty(t, x, v) < s} |\mathcal{G}(x, v)| |\partial_x U(h)(s; X_\infty(s; t, x, v))| |V_\infty(s; t, x, v)| \, dx dv ds.$$

For  $s \in (0, t)$ , we use the measure preserving change of variable  $(x, v) \in D^+ \mapsto (x', v') = \mathcal{F}_{s,t}^\infty(x, v)$  we therefore obtain

$$\int_Q |I_2(t, x, v)| \, dx dv \leq \int_0^t \int_{\mathcal{F}_{s,t}^\infty(D^+)} \mathbf{1}_{t_{\text{inc}}^\infty(t, \mathcal{F}_{t,s}^\infty(x', v')) < s} |\mathcal{G}(\mathcal{F}_{t,s}^\infty(x', v'))| |\partial_x U(h)(s; x')| |v'| \, dx' dv' ds.$$

By the linear elliptic estimates of Proposition (2.6), we have that for a.e  $s \in (0, t)$ ,  $\partial_x U(h)(s, \cdot) \in L^\infty(0, 1)$  and therefore,

$$\int_Q |I_2(t, x, v)| \, dx dv \leq \int_0^t \|\partial_x U(h)(s)\|_{L^\infty(0,1)} \int_{\mathcal{F}_{s,t}^\infty(D^+)} \mathbf{1}_{t_{\text{inc}}^\infty(t, \mathcal{F}_{t,s}^\infty(x', v')) < s} |\mathcal{G}(\mathcal{F}_{t,s}^\infty(x', v'))| |v'| \, dx' dv' ds.$$

By Lemma (2.10),  $D^+$  is invariant by the flow. Therefore on the set defined for  $s \in (0, t)$  by

$$\mathcal{D}_{s,t} = \left\{ (x', v') \in \mathcal{F}_{s,t}^\infty(D^+) : t_{\text{inc}}^\infty(t, \mathcal{F}_{t,s}^\infty(x', v')) < s \right\} \subset D^+$$

we have  $\mathcal{G}(\mathcal{F}_{t,s}^\infty(x', v')) = \mathcal{G}(x', v')$  because  $\mathcal{G}$  depends only on the microscopic energy which is a conserved quantity along the flow (54) and  $v' > 0$ . We thus get,

$$\int_Q |I_2(t, x, v)| \, dx dv \leq \int_0^t \|\partial_x U(h)(s)\|_{L^\infty(0,1)} \int_{\mathcal{D}_{s,t}} |\mathcal{G}(x', v')| v' \, dx' dv' ds.$$

Since  $\mathcal{D}_{s,t} \subset D^+$ , by monotony of the integral we obtain the crude bound

$$\int_Q |I_2(t, x, v)| \, dx dv \leq \int_0^t \|\partial_x U(h)(s)\|_{L^\infty(0,1)} \int_{D^+} |\mathcal{G}(x', v')| v' \, dx' dv' ds.$$

Remark that  $\|\partial_v f^\infty\|_{L^1(Q)} = \int_{D^+} |\mathcal{G}(x', v')| v' \, dx' dv'$ , we eventually get

$$\int_Q |I_2(t, x, v)| \, dx dv \leq \|\partial_v f^\infty\|_{L^1(Q)} \int_0^t \|\partial_x U(h)(s)\|_{L^\infty(0,1)} \, ds.$$

Combining this upper bound with the linear elliptic estimates (65) and the fact that  $\|\rho_h(s)\|_{L^1(0,1)} \leq \|h(s)\|_{L^1(Q)}$ , we thus glean

$$\int_Q |I_2(t, x, v)| \, dx dv \leq \frac{2\|\partial_v f^\infty\|_{L^1(Q)}}{\lambda^2} \int_0^t \|h(s)\|_{L^1(Q)} \, ds. \quad (86)$$

Now observe that for  $0 \leq t \leq T$ ,

$$\int_0^t \|h(s)\|_{L^1(Q)} \, ds \leq \|h\|_{X_{T,\gamma}} \frac{e^{\gamma t} - 1}{\gamma},$$



So that we obtain that for a.e  $0 \leq t \leq T$ ,

$$e^{-\gamma t} \int_Q |I_2(t, x, v)| \, dx dv \leq \frac{2 \|\partial_v f^\infty\|_{L^1(Q)}}{\lambda^{2\gamma}} \|f\|_{X_{T,\gamma}} \quad (87)$$

This last estimate shows that  $I_2 \in X_{T,\gamma}$ .

*Step 2: Contraction.* Since  $\gamma > \frac{2 \|\partial_v f^\infty\|_{L^1(Q)}}{\lambda^2}$  and  $\mathcal{S}$  is an affine map, the estimate (87) shows that for  $(f, g) \in X_{T,\gamma} \times X_{T,\gamma}$ , we have  $\|\mathcal{S}(f) - \mathcal{S}(g)\|_{X_{T,\gamma}} \leq a \|f - g\|_{X_{T,\gamma}}$  with  $a = \frac{2 \|\partial_v f^\infty\|_{L^1(Q)}}{\lambda^{2\gamma}} < 1$ . Since  $\gamma$  is independent of  $T$ , where  $T > 0$  was chosen arbitrarily, the Banach-Picard theorem applies in  $X_{T,\gamma}$  for any  $T > 0$ . Therefore, for any  $T > 0$ , there exists a unique  $h \in X_{T,\gamma}$  such that  $\mathcal{S}(h) = h$  in  $X_{T,\gamma}$ . It shows that  $h$  is a mild solution of the Vlasov equation (56). By definition of  $h$ , we have that  $s \mapsto \rho_h(s, \cdot) = \int_{\mathbb{R}} h(s, \cdot, v) \, dv \in L^\infty([0, T]; L^1(0, 1))$ . Then, we know from Proposition (2.6) that there is a unique  $U(h) \in Y_{T,\gamma}$  which solves the linearized Poisson equation (58) with the source term  $\rho_h$ .

*Step 3: Justification of the regularity in time.* Let us prove that  $(h, U) \in \mathcal{C}([0, +\infty), L^1(Q)) \times \mathcal{C}([0, +\infty); W^{2,1}(0, 1))$ . We firstly show that

$$t \in \mathbb{R}^+ \mapsto \int_Q \mathbf{1}_{t_{\text{inc}}(t,x,v) \leq 0} \left| h_0 \left( X_\infty(0; t, x, v), V_\infty(0; t, x, v) \right) \right| \, dx dv$$

is continuous. Let  $t \in \mathbb{R}^+$ . By a direct computation we have,

$$\begin{aligned} & \int_Q \mathbf{1}_{t_{\text{inc}}(t,x,v) \leq 0} \left| h_0 \left( X_\infty(0; t, x, v), V_\infty(0; t, x, v) \right) \right| \, dx dv \\ &= \int_{Q \setminus S} \mathbf{1}_{t_{\text{inc}}(t,x,v) < 0} \left| h_0 \left( X_\infty(0; t, x, v), V_\infty(0; t, x, v) \right) \right| \, dx dv = \int_{\mathcal{A}_t} \left| h_0 \left( X_\infty(0; t, x, v), V_\infty(0; t, x, v) \right) \right| \, dx dv \\ &= \int_{\mathcal{F}_{0,t}^\infty(\mathcal{A}_t)} |h_0(x', v')| \, dx' dv' = \int_{Q \setminus S} \mathbf{1}_{t_{\text{out}}(0,x',v') > t} |h_0(x', v')| \, dx' dv' \end{aligned}$$

where in the last equality we used Lemma (2.13) b). Let  $(\delta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$  a sequence such that  $\delta_n \rightarrow 0$  as  $n \rightarrow +\infty$ . Fix  $(x', v') \in (Q \setminus S)$  such that  $t_{\text{out}}(0, x', v') \neq t$ . Therefore,

$$\mathbf{1}_{t_{\text{out}}(0,x',v') > t + \delta_n} \rightarrow \mathbf{1}_{t_{\text{out}}(0,x',v') > t} \text{ as } n \rightarrow +\infty.$$

Since the set  $\{(x', v') \in Q \setminus S : t_{\text{out}}(0, x', v') = t\}$  is of measure zero. It shows that the previous convergence holds a.e in  $Q$ . Observe that  $h_0$  is integrable in  $Q$ , therefore, by the Lebesgue dominated convergence theorem, we deduce that

$$\int_{Q \setminus S} \mathbf{1}_{t_{\text{out}}(0,x',v') > t + \delta_n} |h_0(x', v')| \, dx' dv' \rightarrow \int_{Q \setminus S} \mathbf{1}_{t_{\text{out}}(0,x',v') > t} |h_0(x', v')| \, dx' dv'$$

and thus the conclusion follows.

Let now  $T > 0$  and assume to be given some function  $g \in \mathcal{C}([0, T]; L^1(Q))$ . We shall justify that  $\mathcal{S}(g) \in \mathcal{C}([0, T]; L^1(Q))$  where  $\mathcal{S}$  is defined in (85). Note that we have proven few lines above that the first term in (85) belongs to  $\mathcal{C}([0, T]; L^1(Q))$ . We focus on the second map

$$t \in [0, T] \longmapsto \int_Q |\mathcal{G}(x, v)| \left| \int_0^t \mathbf{1}_{t_{\infty}^{\text{inc}}(t, x, v) < s} \partial_x U(g)(s, X_{\infty}(s; t, x, v)) V_{\infty}(s; t, x, v) ds \right| dx dv.$$

Since  $g \in \mathcal{C}([0, T]; L^1(Q))$  we have in particular that for every  $s \in [0, T]$ ,  $\rho_g(s, \cdot) := \int_{\mathbb{R}} g(s, \cdot, v) dv$  defines a function in  $L^1(0, 1)$  and moreover the map  $s \in [0, T] \longmapsto \rho_g(s, \cdot) \in L^1(0, 1)$  is continuous. From the elliptic regularity theory and notably thanks to the estimates of Proposition (2.6), we deduce that  $U(g) \in \mathcal{C}([0, T]; W^{2,1}(0, 1))$ . Therefore for every  $(x, v) \in Q$  and  $t \in \mathbb{R}$  we have

$$s \in [0, T] \longmapsto \partial_x U(g)(s, X_{\infty}(s; t, x, v)) V_{\infty}(s; t, x, v) \in \mathcal{C}([0, T]),$$

and for every  $(x, v) \in Q$  we have

$$(s, t) \in [0, T]^2 \longmapsto \partial_x U(g)(s, X_{\infty}(s; t, x, v)) V_{\infty}(s; t, x, v) \in \mathcal{C}([0, T]^2).$$

We now denote for ease  $j(s; t, x, v) = \partial_x U(g)(s, X_{\infty}(s; t, x, v)) V_{\infty}(s; t, x, v)$ . Fix  $t \in (0, T)$  and  $(x, v) \in Q \setminus S$  so that the incoming time  $t_{\infty}^{\text{inc}}(t, x, v)$  is well defined. Let  $(\delta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$  a sequence such that  $\delta_n \rightarrow 0$  as  $n \rightarrow +\infty$ . We shall prove that

$$\int_0^{t+\delta_n} \mathbf{1}_{t_{\infty}^{\text{inc}}(t+\delta_n, x, v) < s} j(s; t+\delta_n, x, v) ds \longrightarrow \int_0^t \mathbf{1}_{t_{\infty}^{\text{inc}}(t, x, v) < s} j(s; t, x, v) ds.$$

For all  $n \in \mathbb{N}$  we decompose,

$$\begin{aligned} & \int_0^{t+\delta_n} \mathbf{1}_{t_{\infty}^{\text{inc}}(t+\delta_n, x, v) < s} j(s; t+\delta_n, x, v) ds - \int_0^t \mathbf{1}_{t_{\infty}^{\text{inc}}(t, x, v) < s} j(s; t, x, v) ds \\ &= \int_0^{+\infty} \mathbf{1}_{t_{\infty}^{\text{inc}}(t+\delta_n, x, v) < s < t+\delta_n} (j(s; t+\delta_n, x, v) - j(s; t, x, v)) ds \\ &+ \int_0^{+\infty} j(s; t, x, v) (\mathbf{1}_{t_{\infty}^{\text{inc}}(t+\delta_n, x, v) < s < t+\delta_n} - \mathbf{1}_{t_{\infty}^{\text{inc}}(t, x, v) < s < t}) ds. \end{aligned}$$

For the first integral we have for  $n \in \mathbb{N}$  large enough,

$$\left| \int_0^{+\infty} \mathbf{1}_{t_{\infty}^{\text{inc}}(t+\delta_n, x, v) < s < t+\delta_n} (j(s; t+\delta_n, x, v) - j(s; t, x, v)) ds \right| \leq \int_0^T |j(s; t+\delta_n, x, v) - j(s; t, x, v)| ds.$$

By continuity, we have for all  $s \in [0, T]$ ,  $j(s; t+\delta_n, x, v) \rightarrow j(s; t, x, v)$  as  $n \rightarrow +\infty$ . Moreover we have for  $n \in \mathbb{N}$  large enough

$$|j(s; t+\delta_n, x, v)| \leq \sup_{(s, t) \in [0, T]^2} |j(s; t, x, v)|.$$

Since we integrate on the compact set  $[0, T]$ , we deduce by the Lebesgue dominated convergence theorem that  $\int_0^T |j(s; t+\delta_n, x, v) - j(s; t, x, v)| ds \rightarrow 0$  and thus

$$\left| \int_0^{+\infty} \mathbf{1}_{t_{\infty}^{\text{inc}}(t+\delta_n, x, v) < s < t+\delta_n} (j(s; t+\delta_n, x, v) - j(s; t, x, v)) ds \right| \longrightarrow 0.$$

For the second integral we have for  $n \in \mathbb{N}$  large enough,

$$\left| \int_0^{+\infty} j(s; t, x, v) \left( \mathbf{1}_{t_\infty^{\text{inc}}(t+\delta_n; x, v) < s < t+\delta_n} - \mathbf{1}_{t_\infty^{\text{inc}}(t; x, v) < s < t} \right) ds \right| \quad (88)$$

$$\leq \sup_{s \in [0, T]} |j(s; t, x, v)| \int_0^T \left| \mathbf{1}_{t_\infty^{\text{inc}}(t+\delta_n; x, v) < s < t+\delta_n} - \mathbf{1}_{t_\infty^{\text{inc}}(t; x, v) < s < t} \right| ds. \quad (89)$$

For a fixed  $s \in [0, T]$  with  $s \neq t$  and  $s \neq t_\infty^{\text{inc}}(t, x, v)$  we have  $\left| \mathbf{1}_{t_\infty^{\text{inc}}(t+\delta_n; x, v) < s < t+\delta_n} - \mathbf{1}_{t_\infty^{\text{inc}}(t; x, v) < s < t} \right| \rightarrow 0$  because  $t_\infty^{\text{inc}}(t+\delta_n, x, v) \rightarrow t_\infty^{\text{inc}}(t, x, v)$ . Since we integrate on the compact set  $[0, T]$ , we obtain

$$\int_0^T \left| \mathbf{1}_{t_\infty^{\text{inc}}(t+\delta_n; x, v) < s < t+\delta_n} - \mathbf{1}_{t_\infty^{\text{inc}}(t; x, v) < s < t} \right| ds \rightarrow 0.$$

Thus, we have proven for all  $t \in (0, T)$  and any sequence  $(\delta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$  which tends to zero that

$$\begin{aligned} & \int_0^{t+\delta_n} \mathbf{1}_{t_\infty^{\text{inc}}(t+\delta_n, x, v) < s} \partial_x U(g)(s, X_\infty(s; t+\delta_n, x, v)) V_\infty(s; t+\delta_n, x, v) ds \\ & \rightarrow \int_0^t \mathbf{1}_{t_\infty^{\text{inc}}(t, x, v) < s} \partial_x U(g)(s, X_\infty(s; t, x, v)) V_\infty(s; t, x, v) ds. \end{aligned}$$

In particular, the absolute value of the integral at the left hand side converges to the absolute value of the integral at right hand side. The same conclusion holds if  $t \in \{0, T\}$  by considering any sequence  $(\delta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$  which tends to zero, by either greater values in the case  $t = 0$ , or lower values in the case  $t = T$ . Eventually, observe, since  $\text{supp } \mathcal{G} \subset D^+$ , that we have a simple bound for  $(x, v) \in D^+$ ,

$$\begin{aligned} |\mathcal{G}(x, v)| & \left| \int_0^t \mathbf{1}_{t_\infty^{\text{inc}}(t, x, v) < s} \partial_x U(g)(s, X_\infty(s; t, x, v)) V_\infty(s; t, x, v) ds \right| \leq \\ & |\mathcal{G}(x, v)| v \int_0^T \|\partial_x U(g)(s)\|_{L^\infty(0,1)} ds \end{aligned}$$

because  $0 < V_\infty(s; t, x, v) \leq v$  for  $s \in [t_\infty^{\text{inc}}(t, x, v), t]$ . Finally, since  $(x, v) \mapsto |\mathcal{G}(x, v)| v \in L^1(Q)$  we conclude by the Lebesgue dominated convergence theorem. Therefore, the map

$$t \in [0, T] \mapsto \int_Q |\mathcal{G}(x, v)| \left| \int_0^t \mathbf{1}_{t_\infty^{\text{inc}}(t, x, v) < s} \partial_x U(g)(s, X_\infty(s; t, x, v)) V_\infty(s; t, x, v) ds \right| dx dv$$

is continuous on  $[0, T]$ . We thus have proven that for  $g \in \mathcal{C}([0, T]; L^1(Q))$  and  $h_0 \in L^1(Q)$  we have  $\mathcal{S}(g) \in \mathcal{C}([0, T]; L^1(Q))$ . Since the mild mild solution  $(h, U)$  to **(LVP)** is obtained by the Banach-Picard theorem, we have that it verifies  $h \in \mathcal{C}([0, T]; L^1(Q))$  and therefore from the elliptic estimates (2.6) that  $U \in \mathcal{C}([0, T]; W^{2,1}(0, 1))$ . Since the result holds for arbitrary  $T > 0$ , one obtains the conclusion.

## 2.6. A delayed Grönwall lemma.

A consequence of the exit geometric conditions (2.7) will be that  $t \mapsto \|h(t)\|_{L^1(Q)}$  verifies for  $t \geq T_r$  a first order delayed integral equation. We show in the following that such an integral equation has exponential decaying solutions provided that the rate of increase of the solution is small enough. We have.

**Lemma 2.14** (A delayed Grönwall lemma). *Let  $T > 0$  and  $\alpha > 0$  such that  $\alpha T < 1$ . Let  $y \in \mathcal{C}(\mathbb{R}^+)$  such that for  $t \geq T$*

$$y(t) \leq \alpha \int_{t-T}^t y(s) ds. \quad (90)$$

*Then there exists  $\kappa > 0$  such that for  $t \geq 0$*

$$y(t) \leq C_{T,\kappa} \exp(-\kappa t) \quad (91)$$

where  $C_{T,\kappa} = \sup_{t \in [0, T]} |y(t)| \exp(\kappa t)$ .

*Proof.* Let us first remark that if  $y$  is negative on  $[0, T]$  then  $y$  is also negative on  $[T, +\infty)$ . Indeed, if it is not the case then let  $t^* := \sup\{t > T : y(s) < 0 \forall s \in [0, t]\}$ . By continuity of  $y$  this point exists and  $y(t^*) = 0$ . Then applying (90) for  $t = t^*$  we have

$$y(t^*) \leq \alpha \int_{t^*-T}^{t^*} y(s) ds < 0 \quad (92)$$

which yields a contradiction. We now refine this result by proving that if  $y$  is non positive on  $[0, T]$  then it is also non positive on  $[T, +\infty)$ . Indeed, consider the auxiliary sequence of functions defined for all  $n \in \mathbb{N}$  by  $y_n : t \in \mathbb{R}^+ \mapsto y(t) - \frac{1}{n+1}$ . By virtue of the previous result, for each  $n \in \mathbb{N}$  we have that  $y_n$  is negative on  $[0, T]$  and therefore all  $t \geq 0$ ,  $y(t) < \frac{1}{n+1}$ . Passing to the limit as  $n \rightarrow +\infty$  yields  $y(t) \leq 0$ . This shows that  $y$  is non positive on  $[0, +\infty)$ .

Consequently, we deduce that if  $z \in \mathcal{C}^1(\mathbb{R}^+)$  verifies the case of equality in (90) and is such that  $y|_{[0, T]} \leq z|_{[0, T]}$  then for all  $t \geq 0$ ,  $y(t) \leq z(t)$ . We then look for a function of the form  $t \in \mathbb{R}^+ \mapsto \exp(-\kappa t)$  where  $\kappa > 0$  which verifies the equality in (90). It yields the equation for  $\kappa$

$$-\kappa = \alpha(1 - \exp(\kappa T)). \quad (93)$$

The function  $a : \kappa \in \mathbb{R}^+ \mapsto \alpha(1 - \exp(\kappa T)) + \kappa$  is continuous with  $a(0) = 0$  and  $\lim_{\kappa \rightarrow +\infty} a(\kappa) = -\infty$ . It therefore suffices that  $a'(0) > 0$  to obtain the existence of  $\kappa > 0$  such that  $a(\kappa) = 0$ . Note that  $a'(0) > 0$  is equivalent to  $\alpha T < 1$ . Thus, we conclude that if  $\alpha T < 1$  then there is  $\kappa > 0$  which solves (93) and such that the function  $z : t \in \mathbb{R}^+ \mapsto C_T \exp(-\kappa t)$  verifies  $z|_{[0, T]} \geq y|_{[0, T]}$  (by definition of  $C_T$ ) and  $z(t) = \alpha \int_{t-T}^t z(s) ds$  for all  $t \geq T$ . We then conclude that for all  $t \geq 0$ ,  $y(t) \leq z(t)$  which shows the claim.  $\square$

### 2.7. The proof of stability for (LVP).

We are now ready to prove the linear stability theorem. We begin with the *Decay of the  $L^1$  norm in  $Q \setminus D^+$* . For every  $t \geq 0$  and a.e  $(x, v) \in Q$  we have,

$$\begin{aligned} h(t, x, v) &= \mathbf{1}_{\mathcal{A}_t \cup \mathcal{B}_t}(x, v) h_0 \left( X_\infty(0; t, x, v), V_\infty(0; t, x, v) \right) \\ &+ \mathcal{G}(x, v) \int_0^t \mathbf{1}_{t_\infty^{\text{inc}}(t, x, v) < s} \partial_x U(h) \left( s, X_\infty(s; t, x, v) \right) V_\infty(s; t, x, v) ds \end{aligned}$$

where the set  $\mathcal{A}_t, \mathcal{B}_t$  are given in (83)-(84). Since  $\text{supp } \mathcal{G} \subset D^+$ , we obtain for a.e  $(x, v) \in Q \setminus D^+$  that

$$h(t, x, v) = \mathbf{1}_{\mathcal{A}_t \cup \mathcal{B}_t}(x, v) h_0 \left( X_\infty(0; t, x, v), V_\infty(0; t, x, v) \right).$$

Then

$$\|h(t)\|_{L^1(Q \setminus D^+)} = \int_{(Q \setminus (D^+ \cap S)) \cap \mathcal{A}_t} \left| h_0 \left( \mathcal{F}_{0,t}^\infty(x, v) \right) \right| dx dv$$

where we have used again the fact  $\mathcal{B}_t$  is of measure zero. Using the measure preserving change of variable  $(x, v) \mapsto (x' = X_\infty(0; t, x, v), v' = V_\infty(0; t, x, v))$  we get

$$\|h(t)\|_{L^1(Q \setminus D^+)} = \int_{\mathcal{F}_{0,t}^\infty \left( (Q \setminus (D^+ \cap S)) \cap \mathcal{A}_t \right)} |h_0(x', v')| dx' dv'.$$

Set

$$\mathcal{O}_t := \mathcal{F}_{0,t}^\infty \left( (Q \setminus (D^+ \cap S)) \cap \mathcal{A}_t \right).$$

Following Lemma 2.13 b) we deduce that,

$$\mathcal{O}_t = \left\{ (x, v) \in Q \setminus (D^+ \cap S) : t^{\text{out}}(0, x, v) > t \right\}. \quad (94)$$

Since the outgoing time is a continuous function in  $Q \setminus S$ , the set  $\mathcal{O}_t$  is a Borel set. For  $t = 0$ , we have obviously  $\mathcal{O}_0 = Q \setminus (D^+ \cap S)$ . If  $t_1$  and  $t_2$  are two positive numbers such that  $t_1 < t_2$  then  $\mathcal{O}_{t_2} \subset \mathcal{O}_{t_1}$ . Therefore, the family  $(\mathcal{O}_t)_{t \in \mathbb{R}^+}$  is non increasing. According to Corollary (2.12), the outgoing time is bounded in  $Q \setminus (D^+ \cap S)$  and therefore if  $t \geq \sup_{(x,v) \in Q \setminus (D^+ \cap S)} t_\infty^{\text{out}}(0, x, v)$  then

$\mathcal{O}_t = \emptyset$ . It yields the claim.

We now prove the *Exponential decay in  $D_r^+$* . Let  $r \geq 0$ . Assume that  $\text{supp } h_0 \subset D_r^+$ . Since  $\text{supp } \mu \subset (r, +\infty)$ , we have that  $\partial_v f^\infty$  given in (30) is such that  $\text{supp } f^\infty \subset D_r^+$ . Thanks to the representation formula (56) and the invariance of  $D_r^+$  by the stationary flow, we infer for every  $t \geq 0$  that  $\text{supp } h(t) \subset D_r^+$ . We are now going to prove that  $\|h(t)\|_{L^1(D_r^+)}$  verifies a delayed Gronwall type inequality of the form (90). We study separately the contribution of the initial data and the source term.

*Study of the initial data.* Let  $t > 0$  and  $(x, v) \in Q$  such that  $t_\infty^{\text{inc}}(t, x, v) \leq 0$ . We have,

$$(X_\infty, V_\infty)(0; t, x, v) \in D_r^+ \implies \exists (x', v') \in D_r^+, t_\infty^{\text{out}}(0, x', v') > t, (x, v) = (X_\infty, V_\infty)(t; 0, x', v').$$

According to the Proposition (2.7), the equilibrium electric field  $-\partial_x \phi^\infty$  satisfies the initial exit geometric condition in time  $T_r$  with respect to  $D_r^+$ . Therefore,  $t_\infty^{\text{out}}(0, x', v') \leq T_r$  for all  $(x', v') \in D_r^+$ . We then deduce, since  $\text{supp } h_0 \subset D_r^+$ , that

$$t \geq T_r \implies \mathbf{1}_{t_\infty^{\text{inc}}(t, x, v) \leq 0} h_0 \left( X_\infty(0; t, x, v), V_\infty(0; t, x, v) \right) = 0.$$

Hence, for  $t \geq T_r$  and a.e  $(x, v) \in D_r^+$ ,

$$h(t, x, v) = \mathcal{G}(x, v) \int_0^t \mathbf{1}_{t_\infty^{\text{inc}}(t, x, v) < s} \partial_x U(h) \left( s, X_\infty(s; t, x, v) \right) V_\infty(s; t, x, v) ds.$$

*Study of the source term.* Let  $t \geq T_r$ . We now estimate the  $L^1$  norm of  $h$  on  $D_r^+$  by a direct computation. Using a triangular inequality and an  $L^\infty$  bound on  $\partial_x U$  we have,

$$\begin{aligned} \int_{D_r^+} |h(t, x, v)| dx dv &\leq \int_{D_r^+} |\mathcal{G}(x, v)| \left| \int_0^t \mathbf{1}_{t_{\infty}^{\text{inc}}(t, x, v) < s} \|\partial_x U(h)(s)\|_{L^\infty(0,1)} |V_\infty(s; t, x, v)| ds dx dv \right. \\ &= \int_0^t \|\partial_x U(h)(s)\|_{L^\infty(0,1)} \int_{D_r^+} |\mathcal{G}(x, v)| \mathbf{1}_{t_{\infty}^{\text{inc}}(t, x, v) < s} |V_\infty(s; t, x, v)| dx dv ds \end{aligned}$$

where we have used Fubini's theorem for the last equality. Define for  $s \in (0, t)$

$$\mathcal{D}_{s,t}^r = \{(x, v) \in D_r^+ : t_{\infty}^{\text{inc}}(t, x, v) < s\}. \quad (95)$$

Using the measure preserving change of variable  $(x, v) \in \mathcal{D}_{s,t}^r \mapsto (x', v') = \mathcal{F}_{s,t}^\infty(x, v)$ , we then obtain

$$\int_{D_r^+} |\mathcal{G}(x, v)| \mathbf{1}_{t_{\infty}^{\text{inc}}(t, x, v) < s} |V_\infty(s; t, x, v)| dx dv = \int_{\mathcal{F}_{s,t}^\infty(\mathcal{D}_{s,t}^r)} |\mathcal{G}(x', v')| v' dx' dv',$$

where, arguing as in the proof Lemma (2.13) b), we have

$$\mathcal{F}_{s,t}^\infty(\mathcal{D}_{s,t}^r) = \{(x', v') \in D_r^+ : t_{\infty}^{\text{out}}(s, x', v') > t\}, \quad (96)$$

and where we have used the fact that the function  $(x, v) \in D^+ \mapsto \mathcal{G}(x, v)$  depends only on the microscopic energy (which is a conserved quantity (54)) and the fact that if  $(x', v') \in \mathcal{F}_{s,t}^\infty(\mathcal{D}_{s,t}^r)$  then  $v' > 0$ . Thus,

$$\int_{D_r^+} |\mathcal{G}(x, v)| \mathbf{1}_{s > t_{\infty}^{\text{inc}}(t, x, v)} |V_\infty(s; t, x, v)| dx dv = \int_{D_r^+} \mathbf{1}_{t_{\infty}^{\text{out}}(s, x', v') > t} |\mathcal{G}(x', v')| v' dx' dv'.$$

Gathering this term with the other, we obtain

$$\begin{aligned} \int_{D_r^+} |h(t, x, v)| dx dv &\leq \int_0^t \|\partial_x U(h)(s)\|_{L^\infty(0,1)} \int_{D_r^+} \mathbf{1}_{t_{\infty}^{\text{out}}(s, x', v') > t} |\mathcal{G}(x', v')| v' dx' dv' ds \\ &= \int_{D_r^+} |\mathcal{G}(x', v')| v' \int_0^t \mathbf{1}_{t_{\infty}^{\text{out}}(s, x', v') > t} \|\partial_x U(h)(s)\|_{L^\infty(0,1)} ds dx' dv' \\ &\leq \int_{D_r^+} |\mathcal{G}(x', v')| v' dx' dv' \times \sup_{(x', v') \in D_r^+} \int_0^t \mathbf{1}_{t_{\infty}^{\text{out}}(s, x', v') > t} \|\partial_x U(h)(s)\|_{L^\infty(0,1)} ds. \end{aligned}$$

According to the Proposition (2.7), the equilibrium electric field  $-\partial_x \phi^\infty$  satisfies the internal exit geometric condition in time  $T_r$  with respect to  $D_r^+$  on  $\mathbb{R}^+$ . So,

$$\sup_{(s, x, v) \in \mathbb{R}^+ \times D_r^+} (t_{\infty}^{\text{out}}(s, x, v) - s) \leq T_r.$$

Therefore for every  $s \in (0, t]$  and  $(x', v') \in D_r^+$ , we have

$$t_{\infty}^{\text{out}}(s, x', v') > t \implies s > t - T_r.$$

Therefore,

$$\sup_{(x',v') \in D^+} \int_0^t \mathbf{1}_{t_{\infty}^{\text{out}}(s,x',v') \geq t} \|\partial_x U(h)(s)\|_{L^\infty(0,1)} ds \leq \int_{t-T_r}^t \|\partial_x U(h)(s)\|_{L^\infty(0,1)} ds.$$

Using the elliptic estimate (65) and recalling that  $\text{supp } h(t) \subset D_r^+$  we eventually obtain a closed estimate for  $\|h(t)\|_{L^1(D_r^+)}$  which writes

$$\forall t \geq T_r, \quad \|h(t)\|_{L^1(D_r^+)} \leq \frac{2\|\partial_v f^\infty\|_{L^1(D_r^+)}}{\lambda^2} \int_{t-T_r}^t \|h(s)\|_{L^1(D_r^+)} ds. \quad (97)$$

where  $\|\partial_v f^\infty\|_{L^1(D_r^+)} = \int_{D_r^+} |\mathcal{G}(x',v')| v' dx' dv' = \int_r^{+\infty} |\mu'(v)| dv$ . By virtue of Lemma (2.14) we thus get the expected exponential decay provided that  $\frac{2\|\partial_v f^\infty\|_{L^1(D_r^+)} T_r}{\lambda^2} < 1$ . We thus infer (46), and from the elliptic estimates (65) we also have (47). It concludes the study of the linearized Vlasov-Poisson equations.

### 3. Non linear stability

We now investigate the non linear stability of the Vlasov-Poisson system (1)-(7) written in a perturbative form. Using the same notation, we denote the fluctuation

$$h(t, x, v) = f(t, x, v) - f^\infty(x, v), \quad U(t, x) = \phi(t, x) - \phi^\infty(x). \quad (98)$$

It satisfies the perturbative form of **(VP)** that we denote in short **(PVP)**

$$\text{(PVP)} : \begin{cases} \partial_t h + v \partial_x h - \partial_x \phi \partial_v h = \partial_x U \partial_v f^\infty, & (t, x, v) \in (0, +\infty) \times Q, \\ -\lambda^2 \partial_{xx} U + n_e(\phi^\infty + U) - n_e(\phi^\infty) = \int_{\mathbb{R}} h(\cdot, v) dv, & (t, x) \in [0, +\infty) \times (0, 1), \\ \forall t > 0, \quad h(t, 0, v > 0) = 0, \text{ if } v > 0 \quad h(t, 1, v < 0) = 0, \quad \text{if } v < 0, \\ U(t, 0) = 0, \quad U(t, 1) = 0, \quad t \geq 0, \\ h(t = 0, x, v) = h_0(x, v), \quad (x, v) \in Q. \end{cases}$$

We shall prove a non linear version of Theorem (2.1) b) for **(PVP)** under smallness condition on the equilibrium and on the initial perturbation. Though it is not the main purpose of this work to prove it, we need  $U \in \mathcal{C}(\mathbb{R}^+; W^{2,\infty}(0, 1))$  to define properly the characteristics. Following the work of BenAbdallah [5] we restrict the set of admissible initial conditions. We define

$$\mathcal{A} = \left\{ h \in (L^1 \cap L^\infty)(Q) : \int_Q h(x, v) |v|^2 dx dv < +\infty, |v|^2 h(x, v) \in L^\infty(Q) \right\}. \quad (99)$$

Our main result is the following.

**Theorem 3.1** (Non linear stability). *Let  $\lambda > 0$ ,  $\phi_b < 0$  and  $r^* \equiv r^*(|\phi_b|)$  be defined in (156). Let  $r > r^*$  and consider the equilibrium  $(f^\infty, \phi^\infty)$  given by Theorem (1.2) with  $\mu$  such that  $\text{supp } \mu \subset (r + \infty)$ . There are constants  $\varepsilon_{\infty,r}, \varepsilon_{0,r} > 0$  such that for  $f^\infty$  satisfying*

$$\|\partial_v f^\infty\|_{L^1(D_r^+)} < \varepsilon_{\infty,r} \quad (100)$$

and  $h_0 \in \mathcal{A}$  satisfying

$$\text{supp } h_0 \subset D_r^+, \quad \|h_0\|_{L^1(Q)} < \varepsilon_{0,r},$$

any global mild-strong solution  $(h, U) \in \mathcal{C}(\mathbb{R}^+, L^1(Q)) \times \mathcal{C}(\mathbb{R}^+, W^{2,\infty}(0,1))$  to **(PVP)** in the sense of Definition (3.3) associated with the initial data  $h_0$  and the equilibrium  $(f^\infty, \phi^\infty)$  satisfies

a) Propagation of the support. For every  $t \geq 0$

$$\text{supp } h(t) \subset D_{\frac{r}{2}}^+. \quad (101)$$

b) Exponential Decay. There are constants  $\kappa > 0$  and  $C \geq 0$  such that for every  $t \geq 0$

$$\|h(t)\|_{L^1\left(D_{\frac{r}{2}}^+\right)} \leq C \exp(-\kappa t), \quad (102)$$

$$\|\partial_x U(t)\|_{L^\infty(0,1)} \leq \frac{2C}{\lambda^2} \exp(-\kappa t). \quad (103)$$

Several comments are in order about this result

**Remark 3.2.** • The constants  $\varepsilon_{\infty,r}, \varepsilon_{0,r}$  can be made explicit in the proof.

• The smallness condition (100) is equivalent to

$$\|\mu'\|_{L^1(r,+\infty)} < \varepsilon_{\infty,r}.$$

It depends only on the equilibrium incoming density. In the proof  $\varepsilon_{\infty,r}$  is chosen in such a way that it also implies the natural condition (regarding the linear analysis)

$$\frac{4\|\partial_v f^\infty\|_{L^1(D_r^+)} T_r}{\lambda^2} < 1.$$

It is needed for the first order delayed integral equation (90) to have exponentially decaying solutions. Note that the left hand side in this inequality is by two times larger than the left hand side in (42). This is because we prove that the exit geometric conditions are rather satisfied in time  $2T_r$  instead of  $T_r$ .

• We propagate the support of the initial data and the equilibrium up to the domain  $D_{\frac{r}{2}}^+ \supset D_r^+$ . The supports were exactly preserved in the linear case thanks to the energy conservation (54). In the non linear case, we prove a stability result of the stationary phase portrait in Lemma (3.8). This result is a key ingredient in the proof, both for the stability of the phase-portrait and for the propagation of the exit geometric conditions for the perturbed characteristics.

To prove this theorem, we shall again rely on the concept of mild-strong solutions to **(PVP)**, we therefore need to define the characteristics associated with a generic time dependent potential.



### 3.1. The generic characteristics.

Let be  $\phi \in \mathcal{C}(\mathbb{R}^+; W^{2,\infty}(0,1))$  a generic potential. For every  $s \in \mathbb{R}^+$ , we consider and extension of  $\phi(s, \cdot)$  to  $\mathbb{R}$  still denoted  $\phi(s, \cdot)$  such that:

$$\forall s \in \mathbb{R}^+, \quad \phi(s, \cdot) \in W^{2,\infty}(\mathbb{R}), \quad \|\partial_x \phi(s)\|_{L^\infty(\mathbb{R})} \leq \|\partial_x \phi(s)\|_{L^\infty(0,1)}. \quad (104)$$

Such an extension is easily constructed in one dimension by extrapolation to  $\mathbb{R} \setminus [0,1]$  of the boundary value of  $\phi(s, \cdot)$ .

For  $(t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^2$  we define the characteristics which passes through  $(x, v)$  at time  $t$  as the solution to the system of differential equations:

$$\begin{cases} \frac{d}{ds} X_\phi(s; t, x, v) = V_\phi(s; t, x, v), \\ \frac{d}{ds} V_\phi(s; t, x, v) = -\partial_x \phi\left(s, X_\phi(s; t, x, v)\right). \\ X_\phi(t; t, x, v) = x, V_\phi(t; t, x, v) = v. \end{cases} \quad (105)$$

Invoking the Cauchy-Lipschitz theorem, there exists a unique solution

$$s \in \mathbb{R}^+ \mapsto (X_\phi(s; t, x, v), V_\phi(s; t, x, v)) \in \mathcal{C}^1(\mathbb{R}^+) \cap W_{\text{loc}}^{2,\infty}(\mathbb{R}^+).$$

When  $(t, x, v) \in \mathbb{R}^+ \times Q$ , we define the incoming time in  $Q$  and the outgoing times of  $Q$  by

$$t_\phi^{\text{inc}}(t, x, v) = \inf\{s \in (-\infty, t) \cap \mathbb{R}^+ : X_\phi(s'; t, x, v) \in (0, 1) \forall s' \in (s, t)\}, \quad (106)$$

$$t_\phi^{\text{out}}(t, x, v) = \sup\{s \in (t, +\infty) : X_\phi(s'; t, x, v) \in (0, 1) \forall s' \in (t, s)\}. \quad (107)$$

The interval  $(t_\phi^{\text{inc}}(t, x, v), t_\phi^{\text{out}}(t, x, v))$  is largest open interval contained in  $\mathbb{R}^+$  on which the characteristic which started at time  $t$  from the point  $(x, v)$  is in  $Q$ . By definition,

$$t_\phi^{\text{inc}}(t, x, v) > 0 \implies X_\phi^{\text{inc}}(t, x, v) \in \{0, 1\}, \quad (108)$$

$$t_\phi^{\text{out}}(t, x, v) < +\infty \implies X_\phi^{\text{out}}(t, x, v) \in \{0, 1\}. \quad (109)$$

We also define for notational convenience,

$$X_\phi^{\text{inc}}(t, x, v) = X_\phi\left(t_\phi^{\text{inc}}(t, x, v); t, x, v\right), \quad V_\phi^{\text{inc}}(t, x, v) = V_\phi\left(t_\phi^{\text{inc}}(t, x, v); t, x, v\right), \quad (110)$$

and when  $t_\phi^{\text{out}}(t, x, v) < +\infty$ ,

$$X_\phi^{\text{out}}(t, x, v) = X_\phi\left(t_\phi^{\text{out}}(t, x, v); t, x, v\right), \quad V_\phi^{\text{out}}(t, x, v) = V_\phi\left(t_\phi^{\text{out}}(t, x, v); t, x, v\right). \quad (111)$$

We also denote for  $(s, t) \in \mathbb{R}^+ \times \mathbb{R}^+$  the associated flow by

$$\mathcal{F}_{s,t}^\phi : (x, v) \in \mathbb{R}^2 \mapsto \left(X_\phi(s; t, x, v), V_\phi(s; t, x, v)\right). \quad (112)$$

It is a measure preserving diffeomorphism.

We now define the notion of solutions we consider for **(PVP)**.

**Definition 3.3** (Global mild-strong solution to **(PVP)**). Let  $\lambda > 0$ ,  $h_0 \in \mathcal{A}$ . We say that  $(h, U)$  is a global mild-strong solution to **(PVP)** if

a)  $(h, U) \in \mathcal{C}(\mathbb{R}^+; L^1(Q)) \times \mathcal{C}(\mathbb{R}^+; W^{2,\infty}(0, 1))$ .

b)  $h$  is a mild solution of the non linear Vlasov equation, in the sense that it satisfies for every  $t \geq 0$  and a.e  $(x, v) \in Q$ ,

$$\begin{aligned} h(t, x, v) &= \mathbf{1}_{t_\phi^{\text{inc}}(t,x,v)=0} h_0\left(X_\phi\left(0; t, x, v\right), V_\phi\left(0; t, x, v\right)\right) \\ &+ \int_0^t \mathbf{1}_{t_\phi^{\text{inc}}(t,x,v)<s} \partial_x U\left(s, X_\phi\left(s; t, x, v\right)\right) \partial_v f^\infty\left(X_\phi\left(s; t, x, v\right), V_\phi\left(s; t, x, v\right)\right) ds, \end{aligned} \quad (113)$$

where  $\phi = \phi^\infty + U$  where  $\phi^\infty$  is the stationary solution given by theorem (1.2) and  $(X_\phi, V_\phi)$  are the characteristics defined in (105).

c) For every  $t \geq 0$ ,  $U(t)$  is a strong solution to the non linear Poisson equation

$$-\lambda^2 \partial_{xx} U(t) + n_e(\phi^\infty + U(t)) - n_e(\phi^\infty) = \int_{\mathbb{R}} h(t, \cdot, v) dv, \quad (114)$$

and  $U(t)(0) = U(t)(1) = 0$ .

As for the linear analysis, we define two types of exit geometric conditions which depends on a generic time dependent potential.

**Definition 3.4.** Let  $\phi \in \mathcal{C}(\mathbb{R}^+; W^{2,\infty}(0, 1))$ . Let  $K \subset Q$  and  $J$  a subinterval of  $\mathbb{R}^+$ . We say that the electric field  $-\partial_x \phi$  satisfies the internal exit geometric condition in time  $T > 0$  with respect to  $K$  on  $J$  if

$$\sup_{(t,x,v) \in J \times K} (t_\phi^{\text{out}}(t, x, v) - t) \leq T, \quad (115)$$

$$\forall (t, x, v) \in J \times K, \quad (X_\phi^{\text{out}}, V_\phi^{\text{out}})(t, x, v) \in \Sigma^+ \quad (116)$$

**Definition 3.5.** Let  $\phi \in \mathcal{C}(\mathbb{R}^+; W^{2,\infty}(0, 1))$ . Let  $K \subset Q$ . We say that the electric field  $-\partial_x \phi$  satisfies the initial exit geometric condition in time  $T > 0$  with respect to  $K$  if

$$\sup_{(x,v) \in K} t_\phi^{\text{out}}(0, x, v) \leq T, \quad (117)$$

$$\forall (x, v) \in K, \quad (X_\phi^{\text{out}}, V_\phi^{\text{out}})(0, x, v) \in \Sigma^+. \quad (118)$$

### 3.2. The non linear elliptic estimates.

In this section we prove the wellposedness and give elliptic estimates for the non linear Poisson equation

$$\begin{cases} -\lambda^2 \partial_{xx} W + n_e(\phi^\infty + W) - n_e(\phi^\infty) = \rho, & \text{a.e in } (0, 1), \\ W(0) = 0, \quad W(1) = 0, \end{cases} \quad (119)$$

where  $\rho \in L^1(0, 1)$  is a given source term. We have the following.

**Proposition 3.6.** *Let  $\rho \in L^1(0, 1)$ . Then the non linear Poisson problem (119) admits a unique strong solution  $W \in W^{2,1}(0, 1)$ . In addition, the solution satisfies the estimates*

$$\lambda^2 \|\partial_x W\|_{L^2(0,1)} \leq \|\rho\|_{L^1(0,1)}, \quad (120)$$

$$\|n_e(\phi^\infty + W) - n_e(\phi^\infty)\|_{L^1(0,1)} \leq \|\rho\|_{L^1(0,1)}, \quad (121)$$

$$\lambda^2 \|\partial_{xx} W\|_{L^1(0,1)} \leq 2\|\rho\|_{L^1(0,1)}, \quad (122)$$

$$\lambda^2 \|\partial_x W\|_{L^\infty(0,1)} \leq 2\|\rho\|_{L^1(0,1)}. \quad (123)$$

If  $\rho \in L^\infty(0, 1)$  then  $W \in W^{2,\infty}(0, 1)$  and we have the estimates

$$\|n_e(\phi^\infty + W) - n_e(\phi^\infty)\|_{L^1(0,1)} \leq \|\rho\|_{L^\infty(0,1)}, \quad (124)$$

$$\lambda^2 \|\partial_x W\|_{L^\infty(0,1)} \leq 2\|\rho\|_{L^\infty(0,1)}, \quad (125)$$

$$\|n_e(\phi^\infty + W) - n_e(\phi^\infty)\|_{L^\infty(0,1)} \leq \frac{2M}{\lambda^2} \|\rho\|_{L^\infty(0,1)}, \quad (126)$$

$$\lambda^2 \|\partial_{xx} W\|_{L^\infty(0,1)} \leq \left( \frac{2M}{\lambda^2} + 1 \right) \|\rho\|_{L^\infty(0,1)}. \quad (127)$$

where  $M$  is a positive constant that depends only on  $\lambda^2$ ,  $\phi_\infty$  and  $\rho$ .

*Proof. Step 1: Existence and uniqueness.* The proof follows from a classical variational argument. Let  $E : H_0^1(0, 1) \rightarrow \mathbb{R}$  be the functional given for all  $\psi \in H_0^1(0, 1)$  by

$$E(\psi) = \int_0^1 \frac{\lambda^2}{2} |\partial_x \psi(x)|^2 + N_e(\phi^\infty(x) + \psi(x)) - (n_e(\phi^\infty(x)) - \rho(x))\psi(x) dx.$$

where the function  $N_e$  is the function given for all  $s \in \mathbb{R}$  by  $N_e(s) = \int_0^s n_e(s') ds'$  and where we recall that  $H_0^1(0, 1)$  is equipped with the usual norm  $\psi \in H_0^1(0, 1) \mapsto \|\psi\|_{H_0^1(0,1)} = \left( \int_0^1 |\partial_x \psi(x)|^2 dx \right)^{\frac{1}{2}}$ . The functional  $E$  is well defined and of class  $\mathcal{C}^1$  on  $H_0^1$  because the space  $H_0^1$  is continuously imbedded in  $L^\infty(0, 1)$  and  $n_e \in \mathcal{C}^1(\mathbb{R})$ . In addition note that since the function  $n_e'$  is positive on  $\mathbb{R}$ , the function  $N_e$  is strictly convex and thus the functional  $\psi \in H_0^1(0, 1) \mapsto \int_0^1 N_e(\phi^\infty(x) + \psi(x)) - (n_e(\phi^\infty(x)) + \rho(x))\psi(x) dx$  is strictly convex. Therefore  $E$  is a strictly convex functional as the sum of two strictly convex functionals. We then deduce that  $W$  is a weak solution to (119) if and only if  $W$  minimizes  $E$  on  $H_0^1(0, 1)$ . Let us therefore prove the existence of the minimizer. Using a Hölder inequality we have for all  $\psi \in H_0^1(0, 1)$ ,

$$E(\psi) \geq \frac{\lambda^2}{2} \|\psi\|_{H_0^1(0,1)}^2 + \int_0^1 N_e(\phi^\infty(x) + \psi(x)) dx - \|n_e \circ \phi^\infty - \rho\|_{L^1(0,1)} \|\psi\|_{L^\infty(0,1)}.$$

Since  $N_e$  is a positive function and using the fact that for all  $\psi \in H_0^1(0, 1)$  we have  $\|\psi\|_{L^\infty(0,1)} \leq \|\psi\|_{H_0^1(0,1)}$  we get

$$E(\psi) \geq \frac{\lambda^2}{2} \|\psi\|_{H_0^1(0,1)}^2 - \|n_e \circ \phi^\infty + \rho\|_{L^1(0,1)} \|\psi\|_{H_0^1(0,1)}.$$

Let  $0 < \varepsilon < \lambda$ . Using an  $\varepsilon$  dependent Young inequality for the second term we obtain

$$E(\psi) \geq \frac{\lambda^2 - \varepsilon^2}{2} \|\psi\|_{H_0^1(0,1)}^2 - \frac{\|n_e \circ \phi^\infty - \rho\|_{L^1(0,1)}^2}{2\varepsilon^2}. \quad (128)$$

It is a coercivity estimate for the functional  $E$ . By Mazur's Lemma, since  $E$  is convex and continuous in  $H_0^1(0,1)$ , it is lower semi-continuous for the weak topology of  $H_0^1(0,1)$ . Consequently, the coercivity of the functional  $E$  with its lower semi continuity for the weak topology of  $H_0^1(0,1)$  yields the existence of a minimizer in  $H_0^1(0,1)$ . Since  $E$  is strictly convex the minimizer is unique in  $H_0^1(0,1)$ . Let therefore  $W = \underset{\psi \in H_0^1(0,1)}{\operatorname{argmin}} E(\psi)$ . Since  $E$  is  $\mathcal{C}^1$  on  $H_0^1(0,1)$ , the

Fréchet differential of  $E$  evaluated at  $W$  vanishes, it yields for all  $\psi \in H_0^1(0,1)$ ,

$$\int_0^1 \lambda^2 \partial_x W \partial_x \psi + (n_e(\phi^\infty + W) - n_e(\phi^\infty)) \psi \, dx = \int_0^1 \rho \psi \, dx. \quad (129)$$

Since  $n_e(\phi^\infty + W) - n_e(\phi^\infty) - \rho \in L^1(0,1)$ , it is a consequence of the weak formulation (129) that  $W \in W^{2,1}(0,1)$  and that  $W$  is a strong solution to (119). Arguing similarly, if  $\rho \in L^\infty(0,1)$  then  $W \in W^{2,\infty}(0,1)$ .

*Step 2: Estimates.* We prove the first estimate (120). Since  $W^{2,1}(0,1) \subset H^1(0,1)$ , we take  $\psi = W$  in the weak formulation (129). Then we obtain since  $n_e$  is increasing that  $\int_0^1 (n_e(\phi^\infty + W) - n_e(\phi^\infty)) W \, dx \geq 0$  and therefore

$$\int_0^1 \lambda^2 |\partial_x W|^2 \, dx \leq \int_0^1 \rho W \, dx \leq \|\rho\|_{L^1(0,1)} \|W\|_{L^\infty(0,1)} \leq \|\rho\|_{L^1(0,1)} \|\partial_x W\|_{L^2(0,1)}$$

where we have used the Hölder inequality and the fact  $\|W\|_{L^\infty(0,1)} \leq \|\partial_x W\|_{L^2(0,1)}$ . It yields the expected estimate. We now prove the second estimate which follows the same lines as in Proposition (2.6). So we repeat the same argument. Let  $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{C}^1(\mathbb{R})$  a sequence of functions such that for all  $n \in \mathbb{N}$ :

$$\begin{aligned} \varphi_n(0) &= 0, \\ \forall u \in \mathbb{R}, \varphi_n'(u) &> 0, |\varphi_n(u)| \leq 1 \\ \forall u \neq 0, \varphi_n(u) &\longrightarrow \operatorname{sgn}(u) \text{ as } n \rightarrow +\infty. \end{aligned}$$

Since  $W \in W^{2,1}(0,1)$ , for each  $n \in \mathbb{N}$ , the function  $\varphi_n \circ W$  belongs to  $H^1(0,1)$  and it verifies  $\varphi_n \circ W(0) = \varphi_n \circ W(1) = 0$ . We then use  $\varphi_n \circ W$  as a test function in (129). We thus get

$$\int_0^1 \lambda^2 |\partial_x W|^2 \varphi_n'(W) \, dx + \int_0^1 (n_e(\phi^\infty + W) - n_e(\phi^\infty)) \varphi_n \circ W \, dx = \int_0^1 \rho \varphi_n \circ W \, dx.$$

Note that the first integral is non negative therefore we get

$$\int_0^1 (n_e(\phi^\infty + W) - n_e(\phi^\infty)) \varphi_n \circ W \, dx \leq \int_0^1 \rho \varphi_n \circ W \, dx \leq \|\rho\|_{L^1(0,1)}$$

where the last inequality was obtained using the Hölder inequality combined with the  $L^\infty$  bound  $\|\varphi_n \circ W\|_{L^\infty} \leq 1$ . Using the Lebesgue Dominated convergence theorem we have, because  $n_e$  is increasing,

$$\int_0^1 (n_e(\phi^\infty + W) - n_e(\phi^\infty))\varphi_n \circ W dx \xrightarrow{n \rightarrow +\infty} \int_0^1 |(n_e(\phi^\infty + W) - n_e(\phi^\infty))| dx.$$

It yields the second estimate. The third estimate is obtained using the strong form of the equation (119) together with the estimate (121). As for the last estimate, since  $\partial_x W \in W^{1,1}(0,1) \subset C^0[0,1]$  and  $\int_0^1 \partial_x W(x') dx' = 0$  because of the Dirichlet boundary condition, we have that there exists  $x_0 \in [0,1]$  such that for all  $x \in [0,1]$ ,  $\partial_x W(x) = \int_{x_0}^x \partial_{xx} W(x') dx'$ . Then  $|\partial_x W(x)| \leq \|\partial_{xx} W\|_{L^1(0,1)}$  and using the estimate (122) we conclude.

We now prove the estimates in the case when  $\rho \in L^\infty(0,1)$ . Note that (124) is obtained exactly as previously at the difference that we use rather the following bound for every  $n \in \mathbb{N}$ ,

$$\int_0^1 \rho \varphi_n \circ W dx \leq \|\rho\|_{L^\infty(0,1)} \int_0^1 |\varphi_n \circ W| dx \leq \|\rho\|_{L^\infty(0,1)}$$

since  $|\varphi_n| \leq 1$ . From (124) and the strong form of the equation (119) we infer  $\lambda^2 \|\partial_{xx} W\|_{L^1(0,1)} \leq 2\|\rho\|_{L^\infty(0,1)}$  where we have used the continuous imbedding  $L^\infty(0,1) \hookrightarrow L^1(0,1)$ . Then, we deduce because  $\int_0^1 \partial W(x) dx = 0$  that  $\|\partial_x W\|_{L^\infty(0,1)} \leq \|\partial_{xx} W\|_{L^1(0,1)} \leq \frac{2}{\lambda^2} \|\rho\|_{L^\infty(0,1)}$  which is (125). We now prove (126). Since  $W(0) = 0$  we have for all  $x \in [0,1]$ ,  $|W(x)| \leq |\int_0^x \partial_x W(x') dx'| \leq \|\partial_x W\|_{L^\infty(0,1)} \leq \frac{2}{\lambda^2} \|\rho\|_{L^\infty(0,1)}$ . Therefore by the mean value theorem we have

$$|n_e(\phi^\infty(x) + W(x)) - n_e(\phi^\infty(x))| \leq M|W(x)| \leq \frac{2M}{\lambda^2} \|\rho\|_{L^\infty(0,1)},$$

where  $M = \sup_{s \in [-a,a]} n'_e(s)$  where  $a = \|\phi^\infty\|_{L^\infty(0,1)} + \frac{2}{\lambda^2} \|\rho\|_{L^\infty(0,1)}$ . It proves (126). Lastly, the estimate (127) is obtained from the strong form of the equation (119) using (126).  $\square$

### 3.3. Study of the characteristics.

In this section, we study the properties of the generic characteristics. In particular, we are interested in stability properties of the stationary characteristics with respect to small perturbation the equilibrium electric field. As for the linear case, we state a regularity result of the incoming time of a characteristic with respect to its starting point. For fixed  $t > 0$  and  $0 < s < t$  we define the sets

$$\mathcal{D}_t := \{(x, v) \in Q : t_\phi^{\text{inc}}(t, x, v) = 0\}, \quad (130)$$

$$\mathcal{E}_s := \{(x, v) \in Q : t_\phi^{\text{inc}}(t, x, v) < s\}. \quad (131)$$

The set  $\mathcal{D}_t$  is the set of points in  $Q$  which are on characteristics that stay in  $Q$  on interval  $(0, t]$ . The set  $\mathcal{E}_s$  is the set of points in  $Q$  which are on characteristics that reach possibly  $\Sigma^{\text{inc}}$  at a time smaller than  $s$ . As for the linear case, we may justify the measurability of these sets. We have the following.

**Lemma 3.7** (Regularity of the incoming time and measurability). *Let  $\phi \in \mathcal{C}(\mathbb{R}^+; W^{2,\infty}(0,1))$  and fix  $t > 0$ . We have*

a) The three functions  $(x, v) \in Q \mapsto t_\phi^{\text{inc}}(t, x, v)$ ,  $(x, v) \in Q \mapsto X_\phi^{\text{inc}}(t, x, v)$ ,  $(x, v) \in Q \mapsto V_\phi^{\text{inc}}(t, x, v)$  are continuous at every point  $(x, v) \in Q$  such that  $t_\phi^{\text{inc}}(t, x, v) > 0$  and  $(X_\phi^{\text{inc}}(t, x, v), V_\phi^{\text{inc}}(t, x, v)) \notin \Sigma^0$ .

b) The sets  $\mathcal{D}_t$  and  $\mathcal{E}_s$  for every  $s \in (0, t)$  given in (130), (131) are Borel sets.

*Proof.* The proof of a) is in every point similar to that of the point a) of Lemma (2.9). For b), we have  $\mathcal{D}_t := Q \setminus \{(x, v) \in Q : t_\phi^{\text{inc}}(t, x, v) > 0\}$ . Therefore, by stability of the Borel sigma algebra, it is sufficient to prove that  $\{(x, v) \in Q : t_\phi^{\text{inc}}(t, x, v) > 0\}$  is a Borel. One has a decomposition

$$\begin{aligned} \{(x, v) \in Q : t_\phi^{\text{inc}}(t, x, v) > 0\} &= F_t \cup G_t, \\ F_t &= \{(x, v) \in Q : t_\phi^{\text{inc}}(t, x, v) > 0, (X_\phi^{\text{inc}}(t, x, v), V_\phi^{\text{inc}}(t, x, v)) \in \Sigma^0\}, \\ G_t &= \{(x, v) \in Q : t_\phi^{\text{inc}}(t, x, v) > 0, (X_\phi^{\text{inc}}(t, x, v), V_\phi^{\text{inc}}(t, x, v)) \notin \Sigma^0\}. \end{aligned}$$

One may adapt the proof of Proposition 2.3 in [4] to prove that  $F_t$  is a Borel set of measure zero.  $G_t$  is an open set as a consequence of the continuity of  $t_\phi^{\text{inc}}(t, \cdot, \cdot)$  outside the points that reach transversally the boundary. The same argument of continuity yields the claim for the set  $\mathcal{E}_s$ .  $\square$

We now come to a key lemma in the analysis. Unlike the linear case, the microscopic energy  $(x, v) \in Q \mapsto \frac{v^2}{2} + \phi^\infty(x)$  is not conserved along the generic characteristics (105). Since we shall ask the supports of the initial data  $h_0$  and the equilibrium  $f^\infty$  to be imbedded in  $D_r^+$ . It is natural to track the evolution of the set  $D_r^+$  by the flow (112). On that occasion, we recall that the equilibrium electric field  $-\partial_x \phi^\infty$  verifies the exit geometric conditions (2.4), (2.5) at time  $T_r = \frac{1}{r}$  with respect to  $D_r^+$ . We would like that the perturbed electric field  $-\partial_x \phi$  does so, at least, at a larger time than  $T_r$ . In that regard, we set for ease in the reading

$$\tilde{T}_r = 2T_r = \frac{2}{r} \text{ if } r > 0. \quad (132)$$

We will show that under an appropriate smallness assumption,  $-\partial_x \phi$  verifies the exit geometric conditions (3.4)-(3.5) in time  $\tilde{T}_r$ . We have.

**Lemma 3.8** (Stability of the the microscopic energy and exit geometric conditions). *Fix  $r > 0$ . Let  $J$  a subinterval of  $\mathbb{R}^+$ . For any  $U \in \mathcal{C}(\mathbb{R}^+; W^{2,\infty}(0, 1))$  such that*

$$\forall s \in J, \quad \int_s^{s+\tilde{T}_r} \frac{\|\partial_x U(\tau)\|_{L^\infty(0,1)}^2}{2} d\tau + \tilde{T}_r \|\phi^\infty\|_{L^\infty(0,1)} < r^2 \left( \exp(-\tilde{T}_r) - \frac{1}{8} \right). \quad (133)$$

we have

a) for all  $s \in J$  and  $(x, v) \in D_r^+$ , the characteristic (105) associated with the potential  $\phi = U + \phi^\infty$  which started at time  $s$  from  $(x, v)$  verifies

$$\forall t \in [s, s + \tilde{T}_r], \quad V_\phi(t; s, x, v) > \frac{r}{2}. \quad (134)$$

b) In addition,  $\partial_x \phi$  verifies the internal exit geometric condition in time  $\tilde{T}_r$  with respect to  $D_r^+$  on  $J$  and we have

$$\forall t \in [s, t_\phi^{\text{out}}(s, x, v)), \quad (X_\phi, V_\phi)(t; s, x, v) \in D_r^+. \quad (135)$$

Furthermore, if  $0 \in J$  then  $\partial_x \phi$  satisfies also the initial exit geometric condition in time  $\tilde{T}_r$  with respect to  $D_r^+$ .

*Proof.* Fix  $r > 0$ ,  $J$  a subinterval of  $\mathbb{R}^+$  and consider  $U \in \mathcal{C}(\mathbb{R}^+; W^{2,\infty}(0, 1))$  which verifies (133). *Proof of a).* Let  $(s, x, v) \in J \times D_r^+$ . We have for all  $t \in \mathbb{R}^+$

$$\frac{d}{ds} X_\phi(t; s, x, v) = V_\phi(t; s, x, v), \quad (136)$$

$$\frac{d}{ds} V_\phi(t; s, x, v) = -\partial_x U(t; X_\phi(t; s, x, v)) - \partial_x \phi^\infty(X_\phi(t; s, x, v)). \quad (137)$$

We multiply (137) by  $V_\phi(t; s, x, v)$  to get,

$$\frac{d}{dt} \left( \frac{V_\phi^2(t; s, x, v)}{2} + \phi^\infty(X_\phi(t; s, x, v)) \right) = -\partial_x U(t; X_\phi(t; s, x, v)) V_\phi(t; s, x, v). \quad (138)$$

For  $t \geq s$ , we get after an integration on  $[s, t]$ ,

$$\frac{V_\phi^2(t; s, x, v)}{2} + \phi^\infty(X_\phi(t; s, x, v)) = \frac{v^2}{2} + \phi^\infty(x) + \int_s^t \partial_x U(t'; X_\phi(t'; s, x, v)) V_\phi(t'; s, x, v) dt'. \quad (139)$$

Using the Young inequality, for  $ab \geq -\frac{a^2+b^2}{2}$  for all real numbers  $a$  and  $b$ , we obtain

$$\frac{V_\phi^2(t; s, x, v)}{2} + \phi^\infty(X_\phi(t; s, x, v)) \geq \frac{v^2}{2} + \phi^\infty(x) - \int_s^t \frac{V_\phi^2(t'; s, x, v)}{2} + \phi^\infty(X_\phi(t'; s, x, v)) dt' \quad (140)$$

$$- B(t, s) \quad (141)$$

where  $B(t, s) = \int_s^t \frac{1}{2} \|\partial_x U(t')\|_{L^\infty(0,1)}^2 dt' + (t-s) \|\phi^\infty\|_{L^\infty(0,1)}$ . Since  $(x, v) \in D_r^+$  we have  $\frac{v^2}{2} + \phi^\infty(x) > r^2$ . Setting  $I(t) = \frac{V_\phi^2(t; s, x, v)}{2} + \phi^\infty(X_\phi(t; s, x, v))$  we thus get the integral inequality

$$\forall t \geq s, \quad I(t) > r^2 - \int_s^t I(t') dt' - B(t, s). \quad (142)$$

Since the function  $t \mapsto B(t, s)$  is increasing, using a Grönwall Lemma, we get

$$\forall t \geq s, \quad I(t) > r^2 \exp(s-t) - B(t, s). \quad (143)$$

Remark that the function  $t \in [s, s + \tilde{T}_r] \mapsto r^2 \exp(s - t) - B(t, s)$  is decreasing, therefore we obtain

$$\forall t \in [s, s + T_r], \quad I(t) > r^2 \exp(s - t) - B(t, s) \geq r^2 \exp(-\tilde{T}_r) - B(s + \tilde{T}_r, s) \quad (144)$$

By assumption (133), we have  $r^2 \exp(-\tilde{T}_r) - B(s + \tilde{T}_r, s) > \frac{r^2}{8}$ . It yields

$$\forall t \in [s, s + \tilde{T}_r], \quad \frac{V_\phi^2(t; s, x, v)}{2} + \phi^\infty(X_\phi(t; s, x, v)) > \frac{r^2}{8}. \quad (145)$$

Observe that  $\phi^\infty$  is non positive on  $[0, 1]$ . Therefore, since  $V_\phi(t; t, x, v) = v > 0$  we deduce thanks to (145) and a standard continuity argument that  $V_\phi(t; s, x, v) > 0$  for all  $t \in [s, s + \tilde{T}_r]$ . Thus for all  $t \in [s, s + \tilde{T}_r]$  we have  $V_\phi(t; s, x, v) > \frac{r}{2}$ .

*Proof of b).* From the previous point we have for all  $t \in [s, s + \tilde{T}_r]$ ,  $V_\phi(t; s, x, v) > \frac{r}{2}$ . By integration on  $[s, s + \tilde{T}_r]$  we obtain  $X_\phi(s + \tilde{T}_r; s, x, v) > x + \frac{\tilde{T}_r \times r}{2} = x + 1 > 1$ . By definition of the outgoing time we infer  $t_\phi^{\text{out}}(s, x, v) < s + \tilde{T}_r$ . Since it holds for every  $s \in J$  and  $(x, v) \in D_r^+$ , taking the supremum on  $J \times D_r^+$  yields internal exit geometric condition. Eventually, remark that if  $0 \in J$  then the previous arguments applied at  $s = 0$  show that the initial exit geometric condition with respect to  $D_r^+$  is also satisfied. Finally, remark that since  $s < t_\phi^{\text{out}}(s, x, v) < s + \tilde{T}_r$  we thus have for  $t \in [s, t_\phi^{\text{out}}(s, x, v))$ ,  $(X_\phi, V_\phi)(t; s, x, v) \in D_{\frac{r}{2}}^+$ .  $\square$

To conclude this study of the generic characteristics let us prove the following elementary lemma.

**Lemma 3.9.** *Let  $\phi \in \mathcal{C}(\mathbb{R}^+; W^{2,\infty}(0, 1))$ . Let  $P \subset Q$  and fix  $(t, x, v) \in \mathbb{R}_*^+ \times Q$ . Assume there exists  $t_\phi^{\text{inc}}(t, x, v) < s < t$  such that  $(X_\phi, V_\phi)(s; t, x, v) \in P$ . Then there exists  $(x', v') \in P$  such  $(x, v) = (X_\phi, V_\phi)(t; s, x', v')$  and  $t_\phi^{\text{out}}(s, x', v') > t$ .*

*Proof.* Let  $P \subset Q$  and fix  $(t, x, v) \in \mathbb{R}_*^+ \times Q$ . Assume there exists  $s \in (t_\phi^{\text{inc}}(t, x, v), t)$  such that  $(X_\phi, V_\phi)(s; t, x, v) \in P$ . Set  $(x', v') = (X_\phi, V_\phi)(s; t, x, v) \in P$ . Then,  $(X_\phi, V_\phi)(t; s, x', v') = (x, v)$ . Since  $t_\phi^{\text{inc}}(t, x, v) < s < t$ , we have for all  $s' \in [s, t]$ ,  $(X_\phi, V_\phi)(s'; s, x', v') = (X_\phi, V_\phi)(s'; t, x, v) \in Q$ . Therefore,  $t_\phi^{\text{out}}(s, x', v') > t$ .  $\square$

To study the contribution of the initial data and of the source term in the solution of the Vlasov equation (113) we shall make a great use of this lemma in its opposite version: let  $P \subset Q$  and fix  $(t, x, v) \in \mathbb{R}_*^+ \times Q$ . If there is some  $0 < s_0 < t$  such that for all  $s > s_0$  and all  $(x', v') \in P$ , we have  $t_\phi^{\text{out}}(s, x', v') \leq t$  then for all  $s$  such that  $t_\phi^{\text{inc}}(t, x, v) < s < t$  we have  $(X_\phi, V_\phi)(s; t, x, v) \notin P$ .

### 3.4. Local stability estimates for (PVP).

From now, we consider  $(h, U) \in \mathcal{C}(\mathbb{R}^+; L^1(Q)) \times \mathcal{C}(\mathbb{R}^+; W^{2,\infty}(0, 1))$  a global mild-strong solution to (PVP) associated with an initial data  $h_0 \in \mathcal{A}$  and an equilibrium  $(f^\infty, \phi^\infty)$  given by Theorem (1.2). We recall that we have for every  $t \geq 0$  and a.e  $(x, v) \in Q$

$$\begin{aligned} h(t, x, v) &= \mathbf{1}_{t_\phi^{\text{inc}}(t, x, v) = 0} h_0(X_\phi(0; t, x, v), V_\phi(0; t, x, v)) \\ &+ \int_0^t \mathbf{1}_{t_\phi^{\text{inc}}(t, x, v) < s} \partial_x U(s, X_\phi(s; t, x, v)) \partial_v f^\infty(X_\phi(s; t, x, v), V_\phi(s; t, x, v)) ds. \end{aligned} \quad (146)$$



Thanks to this formula, we deduce the following naive estimate

$$\forall t \geq 0, \quad \|h(t)\|_{L^1(Q)} \leq \|h_0\|_{L^1(Q)} + \|\partial_v f^\infty\|_{L^1(Q)} \int_0^t \|\partial_x U(s')\|_{L^\infty(0,1)} ds'. \quad (147)$$

Thanks to the elliptic estimate (123), we deduce for every  $t \geq 0$

$$\|h(t)\|_{L^1(Q)} \leq \|h_0\|_{L^1(Q)} \exp\left(\frac{2\|\partial_v f^\infty\|_{L^1(Q)}t}{\lambda^2}\right), \quad (148)$$

$$\|\partial_x U(t)\|_{L^\infty(0,1)} \leq \frac{2}{\lambda^2} \|h_0\|_{L^1(Q)} \exp\left(\frac{2\|\partial_v f^\infty\|_{L^1(Q)}t}{\lambda^2}\right). \quad (149)$$

We now fix  $r > 0$ . Our goal now is to refine the analysis by showing that under an appropriate smallness assumption on  $h_0$  we have for  $t^* > \tilde{T}_r$ ,

$$\forall t \in [0, t^*], \quad \text{supp } h(t) \subset D_{\frac{r}{2}}^+,$$

$$\forall t \in [\tilde{T}_r, t^*], \quad h(t, x, v) = \int_{t-\tilde{T}_r}^t \mathbf{1}_{t_\phi^{\text{inc}}(t,x,v) < s} \partial_x U\left(s, X_\phi\left(s; t, x, v\right)\right) \partial_v f^\infty\left(X_\phi\left(s; t, x, v\right), V_\phi\left(s; t, x, v\right)\right) ds,$$

for a.e  $(x, v) \in D_{\frac{r}{2}}^+$ . In that regards, the strategy is as follows:

- We prove thanks to the bounds (149) that  $\partial_x U$  verifies the smallness condition (133) on  $J = [0, t^*]$  provided  $\|h_0\|_{L^1(Q)}$  is small enough and  $r > 0$  is chosen large enough. Then our key Lemma (3.8) applies.
- We then use the fact that the perturbed electric field  $-\partial_x \phi$  satisfies the initial exit geometric condition at time  $\tilde{T}_r$ . So for  $t \geq \tilde{T}_r$ , in (146) the contribution of the initial condition vanishes because either the condition  $t_\phi^{\text{inc}}(t, x, v) = 0$  is not met or the characteristics at time zero is outside the support of the initial data.
- Lastly, we use the fact that the perturbed electric field  $-\partial_x \phi$  satisfies the internal exit geometric condition at time  $\tilde{T}_r$  to prove that in (146) the piece of integral on  $[0, t - \tilde{T}_r]$  vanishes because in (146) the characteristic starting at time  $t$  from  $(x, v) \in D_r^+$  has spent a priori a time longer than  $\tilde{T}_r$  in  $D_r^+$  so it cannot be in the support of  $\partial_v f^\infty$  for  $s \in [0, t - \tilde{T}_r]$ .

To reach our goal, we somehow proceed in a reversed way compared to the strategy sketched above. We begin to give a sufficient condition on  $\partial_x U$  to get the expected formula for  $h$  as  $t \geq \tilde{T}_r$ . We will show at the end of this section that this sufficient condition is fulfilled provided  $\|h_0\|_{L^1(D_r^+)}$  is small enough and  $r > 0$  is large enough.

**Lemma 3.10.** *Let  $r > 0$ . Suppose*

$$\text{supp } h_0, \quad \text{supp } f^\infty \subset D_r^+ \quad (150)$$

*and in addition, that  $U \in \mathcal{C}(\mathbb{R}^+; W^{2,\infty}(0,1))$  verifies for some  $t^* > T_r$*

$$\forall \hat{s} \in [\tilde{T}_r; t^* + \tilde{T}_r], \quad \int_{\hat{s}-\tilde{T}_r}^{\hat{s}} \frac{\|\partial_x U(\tau)\|_{L^\infty(0,1)}^2}{2} d\tau + \tilde{T}_r \|\phi^\infty\|_{L^\infty(0,1)} < r^2 \left( \exp(-\tilde{T}_r) - \frac{1}{8} \right). \quad (151)$$

Then we have

$$\forall t \in [0, t^*], \quad \text{supp } h(t) \subset D_{\frac{r}{2}}^+, \quad (152)$$

$$\forall t \in [\tilde{T}_r, t^*], \quad \|h(t)\|_{L^1(D_{\frac{r}{2}}^+)} \leq \|\partial_v f^\infty\|_{L^1(D_r^+)} \int_{t-\tilde{T}_r}^t \|\partial_x U(s)\|_{L^\infty(0,1)} ds, \quad (153)$$

$$\forall t \in [\tilde{T}_r, t^*], \quad \|h(t)\|_{L^1(D_{\frac{r}{2}}^+)} \leq \frac{2\|\partial_v f^\infty\|_{L^1(D_r^+)}}{\lambda^2} \int_{t-\tilde{T}_r}^t \|h(s)\|_{L^1(D_{\frac{r}{2}}^+)} ds. \quad (154)$$

*Proof.* We begin to prove (152). For  $t \in [0, t^*]$  and a.e.  $(x, v) \in Q$  we have,

$$\begin{aligned} h(t, x, v) &= \mathbf{1}_{t_\phi^{\text{inc}}(t, x, v)=0} h_0\left(X_\phi(0; t, x, v), V_\phi(0; t, x, v)\right) \\ &+ \int_0^t \mathbf{1}_{t_\phi^{\text{inc}}(t, x, v) < s} \partial_x U\left(s, X_\phi(s; t, x, v)\right) \partial_v f^\infty\left(X_\phi(s; t, x, v), V_\phi(s; t, x, v)\right) ds. \end{aligned}$$

We denote for ease  $h_1(t, x, v) = \mathbf{1}_{t_\phi^{\text{inc}}(t, x, v)=0} h_0\left(X_\phi(0; t, x, v), V_\phi(0; t, x, v)\right)$  and  $h_2(t, x, v) = \int_0^t \mathbf{1}_{t_\phi^{\text{inc}}(t, x, v) < s} \partial_x U\left(s, X_\phi(s; t, x, v)\right) \partial_v f^\infty\left(X_\phi(s; t, x, v), V_\phi(s; t, x, v)\right) ds$ . Let us study separately each term.

*Study of  $h_1$ .* Note that if  $h_1(t, x, v) \neq 0$  therefore  $t_\phi^{\text{inc}}(t, x, v) = 0$  and  $(X_\phi, V_\phi)(0, t, x, v) \in D_r^+$ . Therefore there is  $(x', v') \in D_r^+$  such that  $(x, v) = (X_\phi, V_\phi)(t; 0, x', v')$  and  $t_\phi^{\text{out}}(0, x', v') > t$ . By assumption,  $\partial_x U$  verifies (151) which means that it verifies (133) on the interval  $J = [0, t^*]$ . Since  $t \leq t^*$  we deduce by application of Lemma (3.8) b) that  $(x, v) = (X_\phi, V_\phi)(t; 0, x', v') \in D_{\frac{r}{2}}^+$  provided  $t < t_\phi^{\text{out}}(0, x', v')$ . This shows that  $(x, v) \in D_{\frac{r}{2}}^+$ .

*Study of  $h_2$ .* Observe that  $h_2(t, x, v) \neq 0$  if there is a least one  $s$  such such that  $t_\phi^{\text{inc}}(t, x, v) < s < t$  and  $(X_\phi, V_\phi)(s; t, x, v) \in D_r^+$ . By Lemma (3.9), there is  $(x', v') \in D_r^+$  such that  $(x, v) = (X_\phi, V_\phi)(t; s, x', v')$  and  $t_\phi^{\text{out}}(s, x', v') > t$ . By assumption,  $\partial_x U$  verifies (133) on the interval  $J = [0, t^*]$ . Since  $t \leq t^*$  we deduce by application of Lemma (3.8) b) that  $(x, v) = (X_\phi, V_\phi)(t; s, x', v') \in D_{\frac{r}{2}}^+$  provided  $t < t_\phi^{\text{out}}(s, x', v')$ . Therefore  $(x, v) \in D_{\frac{r}{2}}^+$ . The first claim is proven.

We now prove the estimates (153) and (154). As previously we study separately each term.

*Study of  $h_1$ .* By the preceding study, we know that if  $h_1(t, x, v) \neq 0$  then there is  $(x', v') \in D_r^+$  such that  $(x, v) = (X_\phi, V_\phi)(t; 0, x', v')$  and  $t_\phi^{\text{out}}(0, x', v') > t$ . Since  $\partial_x U$  verifies (133) on  $[0, t^*]$ , according to Lemma (3.8) a), we have that  $-\partial_x \phi$  satisfies the initial exit geometric condition at time  $\tilde{T}_r$  with respect to  $D_r^+$ . Therefore if  $t \geq \tilde{T}_r$  we have  $t \geq t_\phi^{\text{out}}(0, x', v')$  and thus  $h_1(t, x, v) = 0$ .

*Study of  $h_2$ .* If  $h_2(t, x, v) \neq 0$  then there is at least one  $s$  such that  $t_\phi^{\text{inc}}(t, x, v) < s < t$  and  $(X_\phi, V_\phi)(s; t, x, v) \in D_r^+$ . By Lemma (3.9), there is  $(x', v') \in D_r^+$  such that  $(x, v) = (X_\phi, V_\phi)(t; s, x', v')$  and  $t_\phi^{\text{out}}(s, x', v') > t$ . Since  $\partial_x U$  verifies (151), it means that  $\partial_x U$  verifies (133) on  $J = [0, t^* - \tilde{T}_r]$ . According to Lemma (3.8) a), we deduce that  $\partial_x \phi$  verifies the internal exit geometric condition in time  $\tilde{T}_r$  with respect to  $D_r^+$  on  $[0, t^* - \tilde{T}_r]$ . For  $s \in [t_\phi^{\text{inc}}(t, x, v), t^* - \tilde{T}_r]$  we then have  $t_\phi^{\text{out}}(s, x', v') \leq s + \tilde{T}_r$ . Therefore if  $s \leq t - \tilde{T}_r$  then  $t_\phi^{\text{out}}(s, x', v') \leq t$ . Thus,

$$h_2(t, x, v) = \int_{t-\tilde{T}_r}^t \mathbf{1}_{t_\phi^{\text{inc}}(t, x, v) < s} \partial_x U\left(s, X_\phi(s; t, x, v)\right) \partial_v f^\infty\left(X_\phi(s; t, x, v), V_\phi(s; t, x, v)\right) ds.$$

*Conclusion.* For  $t \in [\tilde{T}_r, t^*]$  we have for a.e.  $(x, v) \in D_{\frac{r}{2}}^+$ ,

$$h(t, x, v) = \int_{t-\tilde{T}_r}^t \mathbf{1}_{t_\phi^{\text{inc}}(t,x,v) < s} \partial_x U\left(s, X_\phi\left(s; t, x, v\right)\right) \partial_v f^\infty\left(X_\phi\left(s; t, x, v\right), V_\phi\left(s; t, x, v\right)\right) ds.$$

We now estimate  $\|h(t)\|_{L^1\left(D_{\frac{r}{2}}^+\right)}$ . Using a triangular inequality and the Fubini-Tonelli theorem we get

$$\|h(t)\|_{L^1\left(D_{\frac{r}{2}}^+\right)} \leq \int_{t-\tilde{T}_r}^t \int_{D_{\frac{r}{2}}^+} \mathbf{1}_{t_\phi^{\text{inc}}(t,x,v) < s} \left| \partial_x U\left(s, X_\phi\left(s; t, x, v\right)\right) \right| \left| \partial_v f^\infty\left(X_\phi\left(s; t, x, v\right), V_\phi\left(s; t, x, v\right)\right) \right| dx dv ds.$$

Using the  $L^\infty$  bound (123), we obtain

$$\|h(t)\|_{L^1\left(D_{\frac{r}{2}}^+\right)} \leq \int_{t-\tilde{T}_r}^t \|\partial_x U(s)\|_{L^\infty(0,1)} \int_{D_{\frac{r}{2}}^+} \mathbf{1}_{t_\phi^{\text{inc}}(t,x,v) < s} \left| \partial_v f^\infty\left(X_\phi\left(s; t, x, v\right), V_\phi\left(s; t, x, v\right)\right) \right| dx dv ds.$$

Using the measure preserving change of variable  $(x, v) \mapsto (x', v') = (X_\phi, V_\phi)(s; t, x, v)$  we obtain (153). Using the elliptic estimate (123) and the fact that  $h$  is supported in  $D_{\frac{r}{2}}^+$  on  $[0, t^*]$  we deduce (154) from (153). □

As expected, we conclude this section by showing that the condition (151) is satisfied provided the norm of  $h_0$  is chosen small enough and  $r$  is large enough. We thus define

$$\forall r > 0, \quad \delta_r := r^2 \left( \exp(-\tilde{T}_r) - \frac{1}{4} \right) - \tilde{T}_r |\phi_b|, \quad (155)$$

where we recall that  $|\phi_b| = \|\phi^\infty\|_{L^\infty(0,1)}$  because  $\phi^\infty$  is monotone decreasing with  $\phi^\infty(0) = 0$  and  $\phi^\infty(1) = \phi_b$ . We then define  $r^* \equiv r^*(|\phi_b|) > 0$  to be the unique positive number such that

$$\forall r > r^*, \quad \delta_r > 0 \quad (156)$$

where  $\delta_r$  is given in (155). Since  $|\phi_b| > 0$ , the continuity and the strict monotony of the function  $r > 0 \mapsto \delta_r$  show that this number exists and is unique.

**Lemma 3.11.** *Let  $r > r^*$  and fix  $t^* > \tilde{T}_r$ . Let  $h_0 \in \mathcal{A}$  such that  $\text{supp } h_0 \subset D_r^+$ , and let  $(f^\infty, \phi^\infty)$  the equilibrium given by Theorem (1.2) be such that  $\text{supp } f^\infty \subset D_r^+$ . There exists  $\varepsilon_{0,r} > 0$  such that if*

$$\|h_0\|_{L^1(D_r^+)} < \varepsilon_{0,r} \quad (157)$$

*then any global mild strong solution  $(h, U) \in \mathcal{C}(\mathbb{R}^+; L^1(Q)) \times \mathcal{C}(\mathbb{R}^+; W^{2,\infty}(0,1))$  to (PVP) associated with an initial data  $h_0$  and equilibrium  $(f^\infty, \phi^\infty)$  as above satisfies (151) and thus the conclusion of Lemma (3.10).*

*Proof.* Fix  $r > r^*$  and  $t^* > \tilde{T}_r$ . Let  $(h, U) \in \mathcal{C}(\mathbb{R}^+; L^1(Q)) \times \mathcal{C}(\mathbb{R}^+; W^{2,\infty}(0, 1))$  be a global mild strong solution to **(PVP)** associated with an initial data  $h_0$  and an equilibrium  $(f^\infty, \phi^\infty)$  such that  $\text{supp } f^\infty \subset D_r^+$ . We obtain at first, thanks to the a priori exponential growth estimate (149), for all  $s \in [0, t^*]$

$$\int_s^{s+\tilde{T}_r} \frac{\|\partial_x U(\tau)\|_{L^\infty(0,1)}^2}{2} d\tau \leq \frac{2}{\lambda^2} \|h_0\|_{L^1(D_r^+)}^2 e^{\frac{4\|\partial_v f^\infty\|_{L^1(D_r^+)}(t^*+\tilde{T}_r)}{\lambda^2}} \tilde{T}_r.$$

A sufficient condition is then

$$\frac{2}{\lambda^2} \|h_0\|_{L^1(D_r^+)}^2 e^{\frac{4\|\partial_v f^\infty\|_{L^1(D_r^+)}(t^*+\tilde{T}_r)}{\lambda^2}} \tilde{T}_r < \delta_r.$$

For  $\|h_0\|_{L^1(D_r^+)}$  small enough it is trivially satisfied. Therefore there is  $\varepsilon_{0,r} > 0$  such that for  $\|h_0\|_{L^1(D_r^+)} < \varepsilon_{0,r}$  we have

$$\forall \hat{s} \in [\tilde{T}_r; t^* + \tilde{T}_r], \quad \int_{\hat{s}-\tilde{T}_r}^{\hat{s}} \frac{\|\partial_x U(\tau)\|_{L^\infty(0,1)}^2}{2} d\tau + \tilde{T}_r \|\phi^\infty\|_{L^\infty(0,1)} < r^2 \left( \exp(-\tilde{T}_r) - \frac{1}{4} \right).$$

Thus (151) is satisfied and Lemma (3.10) applies.  $\square$

### 3.5. The continuation argument and the global exponential decay.

We now come to the final argument which consists in establishing that the estimate (153) holds for  $t^* = +\infty$ . Let  $\lambda > 0$ ,  $\phi_b < 0$  and fix  $r > r^*$  where  $r^*$  is defined in (156). Consider an equilibrium  $(f^\infty, \phi^\infty)$  given by Theorem (1.2) such that  $\text{supp } \mu \subset (r, +\infty)$  so that  $\text{supp } f^\infty \subset D_r^+$ . Consider  $\varepsilon_{0,r}$  given by Lemma (3.11) associated with the time  $2\tilde{T}_r$ . Let therefore  $h_0 \in \mathcal{A}$  be such that

$$\text{supp } h_0 \subset D_r^+, \quad \|h_0\|_{L^1(D_r^+)} < \varepsilon_{0,r}. \quad (158)$$

Consider  $(h, U)$  a global mild solution to **(PVP)** associated with the initial data  $h_0$  and the equilibrium  $(f^\infty, \phi^\infty)$ . Thanks to Lemma (3.11) we have,

$$\forall \hat{s} \in [\tilde{T}_r, 3\tilde{T}_r], \quad \int_{\hat{s}-\tilde{T}_r}^{\hat{s}} \frac{\|\partial_x U(\tau)\|_{L^\infty(0,1)}^2}{2} d\tau < \delta_r \quad (159)$$

where  $\delta_r$  is given by (155). Let us define

$$t^* = \sup \left\{ t > 2\tilde{T}_r : \forall \hat{s} \in [\tilde{T}_r, t + \tilde{T}_r], \int_{\hat{s}-\tilde{T}_r}^{\hat{s}} \frac{\|\partial_x U(\tau)\|_{L^\infty(0,1)}^2}{2} d\tau < \delta_r \right\}. \quad (160)$$

Since (159) holds at  $s = 3\tilde{T}_r$ , a continuity argument shows that  $t^*$  is well-defined (it is either finite or infinite). Our goal is to prove that  $t^* = +\infty$ . So we assume temporarily that  $t^* < +\infty$ . By virtue of Lemma (3.10) we have

$$\forall t \in [0, t^*], \quad \text{supp } h(t) \subset D_{\frac{r}{2}}^+, \quad (161)$$

$$\forall t \in [\tilde{T}_r, t^*], \quad \|h(t)\|_{L^1\left(D_{\frac{r}{2}}^+\right)} \leq \|\partial_v f^\infty\|_{L^1(D_r^+)} \int_{t-\tilde{T}_r}^t \|\partial_x U(s)\|_{L^\infty(0,1)} ds, \quad (162)$$

$$\forall t \in [\tilde{T}_r, t^*], \quad \|h(t)\|_{L^1\left(D_{\frac{r}{2}}^+\right)} \leq \frac{2\|\partial_v f^\infty\|_{L^1(D_r^+)}}{\lambda^2} \int_{t-\tilde{T}_r}^t \|h(s)\|_{L^1\left(D_{\frac{r}{2}}^+\right)} ds. \quad (163)$$

Thanks to (162) and (160), we obtain after a Cauchy-Schwarz inequality, for all  $t \in [\tilde{T}_r, t^*]$

$$\|h(t)\|_{L^1\left(D_{\frac{r}{2}}^+\right)} \leq \|\partial_v f^\infty\|_{L^1(D_r^+)} \sqrt{2\tilde{T}_r \delta_r}. \quad (164)$$

By continuity of the map  $t \in \mathbb{R}^+ \mapsto \|h(t)\|_{L^1(Q)}$  we obtain,

$$\|h(t^*)\|_{L^1\left(D_{\frac{r}{2}}^+\right)} \leq \|\partial_v f^\infty\|_{L^1(D_r^+)} \sqrt{2\tilde{T}_r \delta_r}. \quad (165)$$

We now consider  $h(t^*)$  as an initial data for the Vlasov equation. A Grönwall lemma combined with (165) yields for  $t \geq t^*$

$$\|h(t)\|_{L^1(Q)} \leq \|\partial_v f^\infty\|_{L^1(D_r^+)} \sqrt{2\tilde{T}_r \delta_r} \exp\left(\frac{2\|\partial_v f^\infty\|_{L^1(Q)}(t - t^*)}{\lambda^2}\right). \quad (166)$$

Using the elliptic estimate (123) we obtain for  $t \geq t^*$ ,

$$\|\partial_x U(t)\|_{L^\infty(0,1)} \leq \frac{2}{\lambda^2} \|\partial_v f^\infty\|_{L^1(D_r^+)} \sqrt{2\tilde{T}_r \delta_r} \exp\left(\frac{2\|\partial_v f^\infty\|_{L^1(Q)}(t - t^*)}{\lambda^2}\right). \quad (167)$$

Using (167), we have for  $s \in [t^*, t^* + \tilde{T}_r]$

$$\int_s^{s+\tilde{T}_r} \frac{\|\partial_x U(\tau)\|_{L^\infty(0,1)}^2}{2} d\tau \leq \int_{t^*}^{t^*+2\tilde{T}_r} \frac{\|\partial_x U(\tau)\|_{L^\infty(0,1)}^2}{2} d\tau \leq \quad (168)$$

$$\frac{4}{\lambda^2} \|\partial_v f^\infty\|_{L^1(D_r^+)} \tilde{T}_r \delta_r \left( \exp\left(\frac{8\|\partial_v f^\infty\|_{L^1(D_r^+)} \tilde{T}_r}{\lambda^2}\right) - 1 \right). \quad (169)$$

We therefore see that if

$$\frac{4}{\lambda^2} \|\partial_v f^\infty\|_{L^1(D_r^+)} \tilde{T}_r \left( \exp\left(\frac{12\|\partial_v f^\infty\|_{L^1(D_r^+)} \tilde{T}_r}{\lambda^2}\right) - 1 \right) < 1 \quad (170)$$

then for all  $\hat{s} \in [t^* + \tilde{T}_r, t^* + 2\tilde{T}_r]$ ,

$$\int_{\hat{s}-\tilde{T}_r}^{\hat{s}} \frac{\|\partial_x U(\tau)\|_{L^\infty(0,1)}^2}{2} d\tau < \delta_r \quad (171)$$

which contradicts the maximality of  $t^*$ . Wee that the condition (170) depends only  $\|\partial_v f^\infty\|_{L^1(D_r^+)}$ ,  $\lambda^2$  and  $r$  but not on  $h_0$ . Therefore if  $\|\partial_v f^\infty\|_{L^1(D_r^+)}$  is small enough then (170) is satisfied. We can therefore consider a constant  $\varepsilon_{\infty,r} > 0$  such that if

$$\|\partial_v f^\infty\|_{L^1(D_r^+)} < \varepsilon_{\infty,r} \quad (172)$$

then we have (170) and

$$\frac{\|\partial_v f^\infty\|_{L^1(D_r^+)} \tilde{T}_r}{2\lambda^2} < 1. \quad (173)$$

*Conclusion.* We infer that if  $h_0$  and  $f^\infty$  are such that

$$\|h_0\|_{L^1(D_r^+)} < \varepsilon_{0,r}, \quad \|\partial_v f^\infty\|_{L^1(D_r^+)} < \varepsilon_{\infty,r} \quad (174)$$

then  $t^* = +\infty$ . Thus, any global mild strong solution to **(PVP)** verifies (161)-(163) with  $t^* = +\infty$ . The delayed Grönwall Lemma (2.14) thus applies and we obtain the estimate (102). The estimate (103) is a consequence of the non linear elliptic estimate (123). Theorem (3.1) is proven.

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## References

- [1] Badsı, M., 2017. Linear electron stability for a bi-kinetic sheath model. *Journal of Mathematical Analysis and Applications* 453, 954–872.
- [2] Badsı, M., Campos Pinto, M., Després, B., 2016. A minimization formulation of a bi-kinetic sheath. *Kinetic and related models* 9.
- [3] Badsı, M., Godard-Cadillac, L., 2023. Existence of solutions for a bi-species kinetic model of a cylindrical langmuir probe. *Communication in Mathematical Sciences* 21, 1097–1134.
- [4] Bardos, 1970. Problèmes aux limites pour les équations aux dérivées partielles du premier ordre à coefficients réels; théorèmes d’approximations; application à l’équation de transport. *Annales scientifiques de l’E.N.S* , 185–233.
- [5] Ben Abdallah, N., 1994. Weak solutions of the initial-boundary value problem for the vlasov-poisson system. *M2AS* 17, 451–476.
- [6] Ben-Artzi, J., 2011. Instability of nonsymmetric nonmonotone equilibria of the vlasov-maxwell system. *J. Math. Phys* 52.
- [7] Bostan, M., 2005. Existence and uniqueness of the mild solution for the 1d Vlasov-Poisson initial-boundary value problem. *SIAM J. Math.Anal.* 37, 156–188.
- [8] Cesbron, L., Iacobelli, M., 2023. Global well-posedness of vlasov-poisson-type systems in bounded domains. *Journal of differential equations* , 154–203.
- [9] Chen, F.F., 1984. *Introduction to Plasma Physics and controlled fusion*. Springer.
- [10] Degond, P., 1986. Spectral theory of the linearized vlasov-poisson equation. *Trans. Amer. Math. Soc* , 435–453.

- [11] Degond, P., Jaffard, S., Poupaud, F., Raviart, P.A., 1996. The child-langmuir asymptotics of the vlasov-poisson equation for cylindrically or spherically symmetric diodes. part 1: Statement of the problem and basic estimates. *Math. Meth. Appl. Sci.* 19, 287–312.
- [12] Despres, B., 2014. Symmetrization of vlasov-poisson equations. *SIAM J. Math. Anal.* 46.
- [13] Esenturk, E., Hwang, H.J., 2016. Linear stability of the vlasov-poisson system with boundary conditions. *Non linear analysis : Theory, Methods and Applications* 139, 75–105.
- [14] Guo, Y., 1994. Regularity for the Vlasov equations in a half space. *Univ. Math. J.* .
- [15] Guo, Y., Lin, Z., 2017. The existence of stable bgk waves. *Communications in Mathematical Physics* 352, 1121–1152.
- [16] Han-Kwan, D., Glass, O., Moussa, A., 2018. The vlasov–navier–stokes system in a 2d pipe: Existence and stability of regular equilibria. *Arch. Rational Mech. Anal* 230, 593–639.
- [17] Han-Kwan, D., Rousset, F., Nguyen, T., 2021. Asymptotic stability of equilibria for screened vlasov-poisson systems via pointwise dispersive estimates. *Annals of PDE* 7.
- [18] Hwang, H.J., Velazquez, J.J., 2009. Global existence for the vlasov-poisson system in bounded domains. *Arch. Rat. Mech. Anal.* .
- [19] Landau, L., 1946. On the vibration of the electronic plasma. *J. Phys. USSR* 10, 25.
- [20] Manfredi, G., Devaux, S., 2008. Magnetized plasma-wall transition. consequences for wall sputtering and erosion,. *Institute of Physics Publishing* .
- [21] Mouhot, C., Villani, C., 2011. On landau damping. *Acta mathematica* .
- [22] Munschy, Y., al, 2024. Kinetic plasma-sheath self-organization. *Nuclear Fusion* 64.
- [23] Nguyen, T., Strauss, W., 2013. Stability analysis of collisionless plasmas with specularly reflecting boundary. *SIAM J. Math. Anal.* 45, 777–808.
- [24] Raviart, P., Greengard, C., 1990. A boundary-value problem for the stationary vlasov-poisson equations: the plane diode. *Communications on Pure and Applied Mathematics* .
- [25] Rein, G., 1992. Existence of stationary, collisionless plasmas in bounded domains. *Math. Methods Appl. Sci.* 15, 365–374.
- [26] Rein, G., 1994. Non-linear stability for the vlasov-poisson system. the energy-casimir method. *Mathematical methods in the applied sciences* 17, 1129–1140.
- [27] Stangeby, P., 2000. *The Plasma Boundary of Magnetic Fusion Devices*. *Institute of Physics Publishing*.
- [28] Valsaque, F., Manfredi, G., 2001. Numerical study of plasma wall transition in an oblique magnetic field. *Journal of nuclear materials* .