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► **To cite this version:**

Hamza Ennaji, Jalal M. Fadili, Hedy Attouch. Stochastic monotone inclusion with closed loop distributions. 2024. hal-04652333v1

**HAL Id: hal-04652333**

**<https://hal.science/hal-04652333v1>**

Preprint submitted on 18 Jul 2024 (v1), last revised 13 Sep 2024 (v2)

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# STOCHASTIC MONOTONE INCLUSION WITH CLOSED LOOP DISTRIBUTIONS

HAMZA ENNAJI\*, JALAL FADILI<sup>c,‡</sup>, AND HEDY ATTOUCH<sup>◊</sup>

*Dedicated to the memory of Hedy Attouch, outstanding mathematician and beloved collaborator.*

ABSTRACT. In this paper, we study in a Hilbertian setting, first and second-order monotone inclusions related to stochastic optimization problems with decision dependent distributions. The studied dynamics are formulated as monotone inclusions governed by Lipschitz perturbations of maximally monotone operators where the concept of equilibrium plays a central role. We discuss the relationship between the  $W_1$ -Wasserstein Lipschitz behavior of the distribution and the so-called coarse Ricci curvature. As an application, we consider the monotone inclusions associated with stochastic optimisation problems involving the sum of a smooth function with Lipschitz gradient, a proximable function and a composite term.

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## 1. INTRODUCTION

Recently, many problems in machine learning and risk management come in form of stochastic optimisation problems. Such problems aim to learn a decision rule from a data

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2020 *Mathematics Subject Classification.* Primary 34G25, 37N40, 46N10, 47H05, 49M30, 60J20.

*Key words and phrases.* Monotone inclusions; Dynamical systems; Gradient flows; Inertial dynamics; Ollivier-Ricci curvature.

sample. This can be formulated in terms of optimization problems of the form

$$\min_x \mathbb{E}_{\xi \sim m} f(x, \xi) + g(x), \quad (1)$$

where  $m$  is a probability measure,  $f(x, \xi)$  is a loss function of the decision  $x$  at data point  $\xi$  and  $g$  is a convex regularizer. In this work, we are interested in (1) in the case where the distribution  $m$  depends itself on the decision  $x$ , i.e., problems of the form

$$\min_x \mathbb{E}_{\xi \sim m_x} f(x, \xi) + g(x), \quad (2)$$

In this case, one tries to learn a decision rule from a decision-dependent data distribution  $m_x$ . Problems of the form (2) were addressed in the framework of performative prediction proposed in [43, 47] and discussed with further algorithmic aspects in [35]. A typical example concerns prediction of loan default risks, that is the chance that a borrower won't be able to repay their loan. More precisely, banks take into account several parameters, including the default risk, to decide whether to accept a consumer's loan application and, if so, what interest rate will apply. It is clear that a high default risk implies a high interest rate, but a high interest rate increases the consumer's default risk. Thus, the predictive performance of the bank's model is not calibrated with respect to future results obtained by acting on the model. Another example concerns navigation apps, such as Google Maps (see, e.g., [33, 41]), which suggest routes with low travel time to users. This influences users' decisions to pick such routes and consequently, increases traffic on these routes, impacting travel times. Further applications and illustrations can be found in [47, Appendix A].

In general, problems of the form (2) are difficult to solve. However, a natural approach consists in performing a repeated minimization procedure, i.e., throughout iterations, one solves

$$x_{t+1} = \arg \min_x \mathbb{E}_{\xi \sim m_{x_t}} f(x, \xi) + g(x), \quad (3)$$

and then updates the distribution  $m_{x_{t+1}}$ . Under suitable assumptions that will be specified later, the sequence  $(x_t)_t$  generated by the repeated minimization procedure (3) admits a fixed point  $\bar{x}$ . Such a point turns out to be an equilibrium with respect to the distribution  $m_{(\cdot)}$  in the following sense:

$$\bar{x} = \arg \min_x \mathbb{E}_{\xi \sim m_{\bar{x}}} f(x, \xi) + g(x), \quad (4)$$

that is,  $\bar{x}$  solves (2) for the induced distribution  $m_{\bar{x}}$ . So instead of solving (2), we look at an equilibrium point in the sense of (4). Notice that in terms of operators, (4) can be written (formally, for instance) as a monotone inclusion:

$$A(\bar{x}) + G(\bar{x}) \ni 0 \quad (5)$$

with  $A(x) = \partial g(x)$  and  $G(x) = \mathbb{E}_{\xi \sim m_x} \nabla_x f(x, \xi)$ . That is  $\bar{x}$  is a zero of the sum of two monotone operators. One strategy to solve problems of the form (5) is to consider some continuous and discrete dynamical systems whose trajectories may converge, under suitable assumptions, to an element in  $(A + B)^{-1}(0)$ , the zero set of the sum  $A + B$ . For instance, when  $B \equiv 0$  and  $A = \nabla h$  where  $h$  is a given convex lower semicontinuous function on  $\mathbb{R}^n$ , it is well known since the works of Brézis, Baillon and Bucker [21, 29], that each trajectory of the gradient flow

$$\dot{x}(t) = -\nabla h(x) \text{ with } x(0) = x_0 \in \mathbb{R}^n, \quad (6)$$

converges to a minimizer of  $h$ , and thus a zero of  $\nabla h$ , provided  $\arg \min h \neq \emptyset$ .

Designing algorithms and dynamical systems with rapid convergence properties to solve monotone inclusions is at the core of many fields in modern optimization, partial differential equation, game theory, etc. The literature is extensive, and to name only a few, the reader

is referred to [1, 2, 7, 9, 17, 25, 38, 40, 52] and the references therein.

In this work, we address monotone inclusions of the form

$$\dot{x}(t) + A(x(t)) + \mathbb{E}_{\xi \sim \mathbf{m}_{x(t)}}(B(x(t), \xi)) \ni 0, \quad (7)$$

where  $A$  is a maximally monotone operator, and  $B$  is a single valued mapping. The particularity of such a dynamic is, of course, the presence of the random operator  $\mathbb{E}_{\xi \sim \mathbf{m}_{(\cdot)}}(B(\cdot, \xi))$  where the random variable  $\xi$  has a trajectory-dependent distribution  $\mathbf{m}_{x(t)}$ . Thus, it is not straightforward how to address (7) within the classical framework (see, e.g., [22, 27]). Yet, a clever reformulation of (7), based on the notion of equilibrium, as a monotone inclusion governed by a Lipschitz perturbation of a maximally monotone operators will allow us to tackle this issue. Then we investigate inertial dynamics related to (7). Indeed, since the work of Polyak [48], who considered a system of the form

$$\ddot{x}(t) + \gamma \dot{x}(t) + \nabla f(x(t)) = 0, \quad (\text{HBF})$$

where  $\gamma > 0$  is called the viscous damping coefficient, the introduction of inertial dynamics to accelerate optimization methods has gained a lot of attention and led to many developments (see, e.g., [3, 10, 15, 18, 19, 52] and the references therein).

In this paper, we then consider second-order dynamics of the form

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla f_{\mathbf{m}_{\bar{x}}}(x(t)) + \omega \nabla^2 f_{\mathbf{m}_{\bar{x}}}(x(t))\dot{x}(t) + \mathbf{e}_{\bar{x}}(x(t)) = 0, \quad (8)$$

where  $\omega$  is the so-called Hessian-driven damping coefficient. When  $\gamma(t) \equiv \gamma$ ,  $f_{\mathbf{m}_{\bar{x}}} = f$  (i.e., without a stochastic structure) and  $\mathbf{e}_{\bar{x}} = 0$ , systems of the form (8) were first studied in [3]. Later, this system was combined with an asymptotic vanishing damping  $\gamma(t) = \frac{\alpha}{t}$ , for  $\alpha > 0$  in [19]. Several recent studies has been devoted to this topic (see, e.g., [9, 25, 38, 40, 51]).

**1.1. Statement of the problem.** Throughout,  $\mathcal{H}$  is a real Hilbert space and  $\Xi$  is a Polish metric space, i.e., separable and completely metrizable. Given a maximal monotone operator  $A : \mathcal{H} \rightarrow 2^{\mathcal{H}}$  such that  $\text{int}(\text{dom}(A)) \neq \emptyset$ , a single-valued mapping  $B : \overline{\text{dom}(A)} \times \Xi \rightarrow \mathcal{H}$ , and consider the closed-loop inclusion

$$\begin{cases} \dot{x}(t) + A(x(t)) + \mathbb{E}_{\xi \sim \mathbf{m}_{x(t)}}(B(x(t), \xi)) \ni 0, \text{ a.e } t > t_0 > 0 \\ x(t_0) = x_0 \in \overline{\text{dom}(A)}, \end{cases} \quad (\text{SMI})$$

where  $\mathbf{m}_x$  is a family of probability distributions on  $\Xi$  indexed by  $x \in \mathcal{H}$ , such that  $\mathbf{m}_x(C)$  is a measurable function on  $\mathcal{H}$  for each fixed  $C \in \mathcal{B}$ , where  $\mathcal{B}$  is the Borel  $\sigma$ -algebra of  $\Xi$ .

We call the above dynamic (SMI), which stands for Stochastic Monotone Inclusion.

- **Example 1.1.** (1) Typically, (SMI) covers the case of stochastic optimization problems with a state-dependent distribution studied in [35, 47] by taking  $A = \partial g$  and  $B = \nabla f$  for  $g \in \Gamma_0(\mathcal{H})$  and  $f(\cdot, \xi) \in C_{\beta}^{1,1}(\mathcal{H}) \cap \Gamma_0(\mathcal{H})$ .
- (2) Taking  $A = N_C$  the normal cone of convex set  $C$  of admissible decisions, we recover the framework of variational inequalities addressed recently in [32].

**Notation.** To simplify the presentation, we set, for any measure  $\mathbf{m} \in \mathcal{P}(\Xi)$

$$B_{\mathbf{m}}(x) = \mathbb{E}_{\xi \sim \mathbf{m}}(B(x, \xi)) \text{ and } F_{\mathbf{m}}(x) = A(x) + B_{\mathbf{m}}(x) \quad (9)$$

where  $\mathbb{E}_{\xi}(\cdot)$  is the expectation with respect to the random variable  $\xi$ .

Using this notation, the system (SMI) can be simply rewritten as

$$\begin{cases} \dot{x}(t) + F_{\mathbf{m}_{x(t)}}(x(t)) \ni 0, \text{ a.e } t > t_0, \\ x(t_0) = x_0 \in \overline{\text{dom}(A)}. \end{cases} \quad (\text{SMI})$$

In the sequel, we need one of the following assumptions.

**Assumption 1.** We suppose that  $\xi \mapsto B(x, \xi)$  is measurable for any  $x \in \mathcal{H}$  and that  $x \mapsto B(x, \xi)$  is  $\beta$ -Lipschitz continuous for all  $\xi$ .

The following assumption is essential for the convergence analysis. It describes the sensitivity of the distribution to shifts in the index (here trajectory). It is widely used in the literature (see, e.g., [35, 43, 47, 54]). We will discuss how it closely relates to the so-called coarse Ricci curvature in Section 5.

**Assumption 2** (Lipschitz distributions). There exists  $\tau > 0$  such that

$$W_1(\mathfrak{m}_x, \mathfrak{m}_y) \leq \tau \|x - y\|, \text{ for all } x, y \in \mathcal{H}.$$

The last two assumptions are standard monotonicity assumptions essential to the well-posedness of the dynamics and for the existence and uniqueness of equilibria.

**Assumption 3** (Strong monotonicity).  $F_{\mathfrak{m}_x}$  is  $\mu$ -strongly monotone for all  $x \in \mathcal{H}$ , for  $\mu > 0$ .

Rather than using Assumption 3, one can use uniform monotonicity in the sense of Definition 3 below.

**Assumption 4** (Uniform monotonicity). For all  $x \in \mathcal{H}$ ,  $F_{\mathfrak{m}_x}$  is uniformly monotone with a modulus  $\phi$  such that

$$\phi(t) > \beta\tau t, \quad \forall t > 0.$$

Finally, define the following parameter  $\rho := \frac{\beta\tau}{\mu}$ . As we will see, (see also [35, 47]), the parameter regime  $\rho < 1$  will play a crucial role in the analysis of the convergence of the trajectories.

**1.2. Organization of the paper.** The paper is organized as follows. In Section 3 we address first-order monotone inclusions with closed-loop distributions. We prove the existence of equilibria as well as the well-posedness of the dynamics and convergence properties of the trajectories. In Section 4 we study asymptotic convergence properties of the trajectories of second-order dynamics with closed-loop distributions via Hessian damping. This allows us in particular to cover problems of the form (2). Section 5 contains a discussion concerning the Lipschitz behavior of the family  $(\mathfrak{m}_x)_x$  with respect to  $W_1$ -distance and some consequences in the framework of Markov chains on metric random walk spaces. In Section 6 we discuss the inertial primal-dual algorithm as an application of our results. Finally, Section 7 contains some conclusions and discusses some future works.

## 2. NOTATION AND PRELIMINARIES

In this section we fix some notation and present some notions and results that will be used. Throughout this paper,  $\mathcal{H}$  is a real Hilbert space,  $\langle \cdot, \cdot \rangle$  is the scalar product on  $\mathcal{H}$  and  $\|\cdot\|$  is the induced norm.

**2.1. Convex analysis.** We denote by  $\Gamma_0(\mathcal{H})$  the class of proper, l.s.c and convex function on  $\mathcal{H}$  with values in  $\mathbb{R} \cup \{\infty\}$ . Given  $g \in \Gamma_0(\mathcal{H})$ , its domain is defined by  $\text{dom}(g) = \{x \in \mathcal{H} : g(x) < \infty\}$ . We say that  $g$  is  $\alpha$ -strongly convex, for  $\alpha > 0$ , if  $g - \frac{\alpha}{2}\|\cdot\|^2$  is convex.

The subdifferential of  $g$  is defined as

$$\partial g : x \in \mathcal{H} \mapsto \{v \in \mathcal{H} : g(y) \geq g(x) + \langle v, y - x \rangle\}.$$

We recall the following Fermat's optimality condition for  $g \in \Gamma_0(\mathcal{H})$ ,

$$0 \in \partial g(x^*) \Leftrightarrow x^* \in \arg \min g(\mathcal{H}).$$

**Definition 1** (Differentiability). Let  $g : \mathcal{H} \rightarrow \mathbb{R} \cup \{\infty\}$  and  $x \in \text{int}(\text{dom}(g))$ . We say that  $g$  is differentiable at  $x$  if there exists  $v \in \mathcal{H}$  such that

$$\lim_{h \rightarrow 0} \frac{g(x+h) - g(x) - \langle v, h \rangle}{\|h\|} = 0.$$

The unique vector  $v$  satisfying this condition is the gradient of  $g$  at  $x$  and is denote by  $\nabla g(x)$ .

This being said, we recall the following

**Definition 2** ( $L$ -smoothness). Let  $L \geq 0$  and  $g : \mathcal{H} \rightarrow \mathbb{R} \cup \{\infty\}$ . We say that  $g$  is  $L$ -smooth over  $D \subset \mathcal{H}$  if it is differentiable over  $D$  and

$$\|\nabla g(x) - \nabla g(y)\| \leq L\|x - y\| \text{ for any } x, y \in D.$$

We denote by  $C_L^{1,1}(D)$  the class of  $L$ -smooth functions over  $D$ .

Given a function  $g : \mathcal{H} \rightarrow \mathbb{R} \cup \{\infty\}$ , its proximal mapping is defined through

$$\text{prox}_f(x) = \arg \min_y \left\{ g(y) + \frac{1}{2}\|y - x\|^2 \right\} \text{ for any } x \in \mathcal{H}.$$

When  $g = \iota_K$  the indicator function of a nonempty closed convex set  $K \subset \mathcal{H}$ , then  $\text{prox}_f = \text{pr}_K$ . For further details and notion, we refer the reader to [23].

**2.2. Operator theory.** We write  $A : \mathcal{H} \rightrightarrows \mathcal{H}$  to denote that  $A$  is a set-valued operator on  $\mathcal{H}$ , its domain is defined as  $\text{dom}(A) = \{x \in \mathcal{H} : A(x) \neq \emptyset\}$ , its graph as  $\text{gra}(A) = \{[x, u] \in \mathcal{H} \times \mathcal{H} : u \in Ax\}$  and its zeros set as:  $\text{zer}(A) = \{x \in \mathcal{H} : 0 \in A(x)\} := A^{-1}(0)$ . A selection of  $A$  is an operator  $T : \text{dom } A \rightarrow \mathcal{H}$  such that,  $Tx \in Ax$  for any  $x \in \text{dom } A$ . We write  $A : \mathcal{H} \rightarrow \mathcal{H}$  to indicate that  $A$  is single-valued. In the following, we gather some main properties that are essential for the rest of the paper.

**Definition 3.** • We say that  $A$  is  $\beta$ -Lipschitz continuous if it is single-valued over  $\text{dom } A$  and

$$\|Ax - Ay\| \leq \beta\|x - y\| \quad \forall x, y \in \text{dom } A. \quad (10)$$

• We say that  $A : \mathcal{H} \rightrightarrows \mathcal{H}$  is monotone if

$$\langle x - y, u - v \rangle \geq 0 \quad \forall [x, u], [y, v] \in \text{gra } A. \quad (11)$$

• We say that  $A$  is maximal monotone if there exists no monotone operator  $B$ , i.e., satisfying (11), such that  $\text{gra } A \subset \text{gra } B$ .

• We say that  $A$  is uniformly monotone with modulus  $\phi : [0, \infty) \rightarrow [0, \infty)$  if  $\phi$  is increasing,  $\phi(0) = 0$ ,  $\lim_{t \rightarrow \infty} \phi(t) = \infty$  and

$$\langle x - y, u - v \rangle \geq \|x - y\|\phi(\|x - y\|) \quad \forall [x, u], [y, v] \in \text{gra } A. \quad (12)$$

• We say that  $A$  is  $\mu$ -strongly monotone, with  $\mu > 0$ , if

$$\langle x - y, u - v \rangle \geq \mu\|x - y\|^2 \quad \forall [x, u], [y, v] \in \text{gra } A. \quad (13)$$

**Remark 2.1.** • Note that if  $A$  is  $\mu$ -strongly monotone is equivalent to saying that  $A - \mu\text{Id}$  is monotone.

• The definition of uniform monotonicity is slightly different from the one in [23, Definition 22.1].

• If  $A$  is  $\mu$ -strongly monotone, then it is uniformly monotone with modulus  $\phi(t) = \mu t$ .

► **Example 2.1.** • The typical example of a maximal monotone operator is the subdifferential  $\partial g$  of a function  $g \in \Gamma_0(\mathcal{H})$ . We usually refer to such an operator as a subpotential maximal monotone operator.

**2.3. Monotone inclusions.** Let  $A$  be an operator on  $\mathcal{H}$  such that  $\text{int}(\text{dom } A) \neq \emptyset$  and single-valued mapping  $D : [0, \infty) \times \overline{\text{dom}(A)} \rightarrow \mathcal{H}$  and consider the following differential inclusion

$$\begin{cases} \dot{x}(t) + A(x(t)) + D(t, x(t)) \ni 0, & t \in [0, T] \\ x(0) = x_0 \in \text{dom}(A). \end{cases} \quad (14)$$

**Definition 4.** A strong solution to (14) is a function  $x \in W^{1,1}((0, T]; \mathcal{H}) \cap C([0, T]; \mathcal{H})$  such that

$$-\dot{x}(t) \in A(x(t)) + D(t, x(t)), \quad \text{a.e. } t \in (0, T), x(0) = x_0.$$

Here,  $W^{1,1}((0, T]; \mathcal{H}) = \{x \in L^1(0, T; \mathcal{H}); \dot{x} \in L^1(\epsilon, T; \mathcal{H}), \forall \epsilon \in (0, T)\}$ .

A trajectory  $x : [0, \infty) \rightarrow \mathcal{H}$  is a strong global solution of (14) if it is a strong solution on  $[0, T]$  for any  $T > 0$ .

For further details, we refer the reader to the classical monograph [27] or [22].

**2.4. Transportation distance.** For  $m_1, m_2 \in \mathcal{P}(\Xi)$ , the  $W_1$ -Wasserstein distance is defined by

$$W_1(m_1, m_2) = \sup_{u \in \text{Lip}_1} \mathbb{E}_{\xi \sim m_1} u(\xi) - \mathbb{E}_{\zeta \sim m_2} u(\zeta), \quad (15)$$

where  $\text{Lip}$  is the set of Lipschitz continuous function on  $\Xi$ .

### 3. FIRST-ORDER MONOTONE INCLUSIONS

In this section we perform the analysis of the first-order monotone inclusion (SMI). More precisely, we discuss the existence and uniqueness of solutions as well as the convergence of trajectories. Recall that the dynamic (SMI) is governed by the operator  $F_{m_x} = A + B_{m_x}$ . We first prove the existence of an equilibrium point  $\bar{x}$  which will allow us to reformulate (SMI) in a suitable form.

#### 3.1. Existence and uniqueness of equilibria.

**Definition 5.** (Equilibrium point) We say that  $\bar{x} \in \mathcal{H}$  is at equilibrium with respect to the distribution  $m(\cdot)$  if

$$0 \in F_{m_{\bar{x}}}(\bar{x}). \quad (16)$$

In case  $A = \partial g$  and  $B = \nabla f$  where  $g \in \Gamma_0(\mathcal{H})$  and  $f(\cdot, \xi) \in C_{\beta}^{1,1}(\mathcal{H}) \cap \Gamma_0(\mathcal{H})$ , this definition is to be compared to the one introduced in [47] (see also [35]). Indeed, (16) reduces to:

$$\bar{x} \in \arg \min_x \mathbb{E}_{\xi \sim m_{\bar{x}}} f(x, \xi) + g(x). \quad (17)$$

Solutions of (17) are exactly the fixed points of the repeated minimization procedure, that is, starting from some  $x_0$ , we generate the following sequence for  $t \geq 0$

$$x_{t+1} = S(x_t) := \arg \min_x \mathbb{E}_{\xi \sim m_{x_t}} f(x, \xi) + g(x). \quad (18)$$

In [47, Theorem 3.5] it is shown that if  $f$  is  $C^1$  in both variables,  $\xi \mapsto \nabla f(x, \xi)$  is  $\beta$ -Lipschitz and  $\mathbb{E}_{\xi \sim m_x} f(\cdot, \xi)$  is  $\mu$ -strongly convex for all  $x \in \mathcal{H}$  with  $\rho < 1$ , then, under Assumption 2, the iterates of (18) converge to a unique stable point. Their proof is essentially based on a fixed point argument. In [47, Propostion 4.1], they show the existence of equilibrium points under weaker assumptions on the loss  $f$ . Specifically, they demonstrate that if  $f$  is convex and jointly continuous, then equilibrium points exists provided  $\text{dom}(g)$  is compact. However, in this case the equilibrium is not necessarily unique. In the following lemma, we show the existence of equilibrium in the sense of (16).

**Theorem 1** (Existence and uniqueness of equilibrium point). *Under [Assumption 4](#), the map*

$$S : x \in \mathcal{H} \mapsto \text{zer}(F_{m_x}) = \{u \in \mathcal{H} : 0 \in F_{m_x}(u)\}$$

*is a contraction. In particular, the equilibrium is unique. If moreover, [Assumption 3](#) holds instead, i.e.,  $F_{m_x}$  is  $\mu$ -strongly monotone for  $\mu > 0$ , the mapping  $S$  is  $\rho$ -Lipschitz.*

*Proof.* First, we see that  $S$  is well defined. Indeed, for any  $x \in \mathcal{H}$ ,  $\text{zer}(F_{m_x})$  is nonempty, and is in fact a singleton due to [Assumption 4](#). To see this, we argue as in [[23](#), Proposition 22.11]. Fix  $[y_0, u_0] \in \text{gra } F_{m_x}$ . We have for any  $[y, u] \in \text{gra } F_{m_x}$ :

$$\begin{aligned} \|y - y_0\| \|u\| &\geq \langle y - y_0, u \rangle = \langle y - y_0, u - u_0 \rangle + \langle y - y_0, u_0 \rangle \\ &\geq \|y - y_0\| \phi(\|y - y_0\|) - \|y - y_0\| \|u_0\|. \end{aligned} \quad (19)$$

Since  $\lim_{t \rightarrow \infty} \phi(t) = \infty$  we infer that  $\inf_{u \in \text{gra } F_{m_x}(y)} \|u\| \rightarrow \infty$  as  $\|y\| \rightarrow \infty$  and thus  $F_{m_x}$  is surjective (see [[23](#), Corollary 21.25]). Moreover, in view of strict monotonicity of  $F_{m_x}$ ,  $\text{zer}(F_{m_x})$  is a singleton (see, e.g., [[23](#), Proposition 23.35]).

Now, pick  $x, y \in \mathcal{H}$ . We have that  $0 \in F_{m_x}(S(x))$  and  $0 \in F_{m_y}(S(y))$ . In particular,  $-B_{m_y}(S(y)) \in A(S(y))$ , which gives that  $B_{m_x}(S(y)) - B_{m_y}(S(y)) \in F_{m_x}(S(y))$ . We get, thanks to [Assumption 4](#)

$$\|S(x) - S(y)\| \phi(\|S(x) - S(y)\|) \leq \langle u - v, S(x) - S(y) \rangle, \text{ for any } (u, v) \in F_{m_x}(S(y)) \times F_{m_x}(S(x)).$$

Then, taking  $u = B_{m_x}(S(y)) - B_{m_y}(S(y))$  and  $v = 0$ , we get, using Cauchy-Schwarz inequality

$$\phi(\|S(x) - S(y)\|) \leq \|B_{m_x}(S(y)) - B_{m_y}(S(y))\|,$$

We get, using [Corollary 1](#) below

$$\phi(\|S(x) - S(y)\|) \leq \beta\tau \|x - y\|. \quad (20)$$

Since  $\phi$  is strictly increasing, the last inequality gives thanks to [Assumption 4](#)

$$\|S(x) - S(y)\| \leq \phi^{-1}(\beta\tau \|x - y\|) < \|x - y\|,$$

and by Banach's fixed-point theorem (see [Theorem 6](#)),  $S$  has a unique fixed point  $\bar{x}$ .

Now if  $F_{m_x}$  is  $\mu$ -strongly monotone, it is in particular uniformly monotone with modulus  $\phi(t) = \mu t$ . Equation-([20](#)) gives

$$\|S(x) - S(y)\| \leq \rho \|x - y\|,$$

with  $\frac{\beta\tau}{\mu} := \rho$ . Again, we conclude using [Assumption 4](#) and [Theorem 6](#). ■

*Remark 3.1.* In the strongly monotone case, [Assumption 4](#) incorporates the parameter regime  $\rho < 1$  which appears in particular in [[47](#), Theorem 3.5].

*Remark 3.2.* Since  $\text{zer}(F_{m_{\bar{x}}}) = \{\bar{x}\}$ , we clearly see that  $\bar{x} \in \text{zer}(F_{m_{\bar{x}}} + \mathbf{e}_{\bar{x}})$ . Indeed, taking into account ([23](#)), we have that  $\mathbf{e}_{\bar{x}}(\bar{x}) = 0$  so that  $0 \in (F_{m_{\bar{x}}} + \mathbf{e}_{\bar{x}})(\bar{x})$ .

*Notation 1.* Before stating the main result of this section, let us fix the following notation. Assume that [Assumption 4](#) holds, and denote by

$$\varphi(t) = \phi(t) - \beta\tau t, \quad (21)$$

where  $\phi$  is the modulus of uniform monotonicity of  $F_{m_{\bar{x}}}$ .

We prove the following



**Lemma 1.** *Let  $a > 0$  and define*

$$\theta(z) := \int_z^a \frac{ds}{\varphi(s)}. \quad (22)$$

*Then  $\theta$  is nonincreasing and  $\lim_{z \rightarrow 0^+} \theta(z) = \infty$ .*

*Proof.* Indeed, we have  $\dot{\theta}(z) = -1/\varphi(z) < 0$  since  $\phi$  satisfies [Assumption 4](#). Moreover, since  $\phi(t) \geq \varphi(t)$ , and  $\lim_{z \rightarrow 0^+} \int_z^a \frac{ds}{\phi(s)} = \infty$ , the result follows.  $\blacksquare$

*Remark 3.3.* In the literature of ordinary differential equations, the above lemma is related to the fact that  $\varphi$  is somehow an *Osgood modulus of continuity* (see, e.g., [20, Definitions 2.108 and 3.1]). If  $\varphi(s) = s$ , which corresponds to Lipschitz regularity, and  $a = 1$  then  $\theta(z) = \log_+(z) = \max\{0, \log(1/z)\}$ . If  $a = 1/e$  and  $\varphi(s) = s \log(1/s)$ , which corresponds to log-Lipschitz regularity, then  $\theta(z) = \log \log_+(z)$ . More generally,  $\varphi(s) = s (\log(1/s))^r$  for  $r \leq 1$  are admissible choices.

**3.2. Well-posedness and convergence.** Now let us define the following gap

$$\mathbf{e}_{\bar{x}}(x) = B_{\mathbf{m}_x}(x) - B_{\mathbf{m}_{\bar{x}}}(x). \quad (23)$$

Using the notation in (9), we may rewrite (SMI) in the following form

$$\begin{cases} \dot{x}(t) + F_{\mathbf{m}_{\bar{x}}}(x(t)) + \mathbf{e}_{\bar{x}}(x(t)) \ni 0, & \text{a.e } t > t_0 \\ x(t_0) = x_0, \end{cases} \quad (\text{p-SMI})$$

One advantage of this formulation is that the mapping  $x \mapsto \mathbf{e}_{\bar{x}}(x)$  exhibits Lipschitz behaviour, and the operator  $F_{\mathbf{m}}$  depends only on  $\mathbf{m}_{\bar{x}}$  instead of  $\mathbf{m}_{x(t)}$ . This allows us to treat (p-SMI) within the framework of evolution equations governed by Lipschitz perturbations of maximal monotone operators (cf. [27, Chapter III]). This is behind the notation (p-SMI), which stands for perturbed stochastic monotone inclusion. This being said, our aim is to use [27, Proposition 3.13] and show the existence of a unique strong solution (cf. [Definition 4](#)) to (p-SMI) (see also [27, Definition 3.1]). To this end we begin with the following lemmas.

**Lemma 2.** *The operator  $F_{\mathbf{m}_{\bar{x}}}$  is maximally monotone.*

*Proof.* Since  $A$  is maximal monotone, and  $B(\cdot, \xi)$  is  $\beta$ -Lipschitz continuous for all  $\xi$ , the result is a direct consequence of [27, Lemma 2.4].  $\blacksquare$

For completeness of the presentation, we also recall the following elementary results.

**Lemma 3.** *Under [Assumption 1](#), we have, for any  $\mathbf{m}, \nu \in \mathcal{P}(\Xi)$*

$$\sup_{x \in \mathcal{H}} \|B_{\mathbf{m}}(x) - B_{\nu}(x)\| \leq \beta \mathbb{W}_1(\mathbf{m}, \nu).$$

*Proof.* By [Assumption 1](#),  $B$  is  $\beta$ -Lipschitz continuous. We then have from the definition of the  $\mathbb{W}_1$ -Wasserstein distance (15)

$$B_{\mathbf{m}}(x) - B_{\nu}(x) = \mathbb{E}_{\xi \sim \mathbf{m}} B(x, \xi) - \mathbb{E}_{\zeta \sim \nu} B(x, \zeta) \leq \beta \mathbb{W}_1(\mathbf{m}, \nu).$$

Taking the supremum over  $x \in \mathcal{H}$  yields the result.  $\blacksquare$

Combining [Lemma 3](#) and [Assumption 2](#) we get the following.

**Corollary 1.** *Under [Assumption 1](#) and [Assumption 2](#), for all  $y, z \in \mathcal{H}$*

$$\sup_{x \in \mathcal{H}} \|B_{\mathbf{m}_y}(x) - B_{\mathbf{m}_z}(x)\| \leq \beta \tau \|y - z\|.$$

**Lemma 4.** *Under [Assumption 1](#), we have for any  $x, y \in \mathcal{H}$  and  $\mathbf{m} \in \mathcal{P}(\Xi)$*

$$\|B_{\mathbf{m}}(y) - B_{\mathbf{m}}(x)\| \leq \beta \|x - y\|.$$

*Proof.* Let  $\mathfrak{m} \in \mathcal{P}(\Xi)$  and  $x, y \in \mathcal{H}$ . Since  $B(\cdot, \xi)$  is  $\beta$ -Lipschitz continuous, we have

$$B_{\mathfrak{m}}(x) - B_{\mathfrak{m}}(y) = \mathbb{E}_{\xi \sim \mathfrak{m}} (B(x, \xi) - B(y, \xi)) \leq \beta \|x - y\|.$$

The result follows immediately.  $\blacksquare$

The key ingredient to use [27, Proposition 3.13] is the Lipschitz continuity of the perturbation  $\mathbf{e}_{\bar{x}}(\cdot)$ . This is the content of the following statement.

**Lemma 5.** *Suppose that Assumption 1 and Assumption 2 hold, then  $\mathbf{e}_{\bar{x}}(\cdot)$  is  $\beta(2+\tau)$ -Lipschitz continuous.*

*Proof.* Let  $x, z \in \overline{\text{dom}(F_{\mathfrak{m}_{\bar{x}}})}$ . We then have

$$\begin{aligned} \mathbf{e}_{\bar{x}}(x) - \mathbf{e}_{\bar{x}}(z) &= (B_{\mathfrak{m}_x}(x) - B_{\mathfrak{m}_{\bar{x}}}(x)) - (B_{\mathfrak{m}_z}(z) - B_{\mathfrak{m}_{\bar{x}}}(z)) \\ &= (B_{\mathfrak{m}_x}(x) - B_{\mathfrak{m}_x}(z)) + (B_{\mathfrak{m}_z}(z) - B_{\mathfrak{m}_{\bar{x}}}(x)) + (B_{\mathfrak{m}_x}(z) - B_{\mathfrak{m}_z}(z)) \\ &\leq \beta \|x - z\| + \beta \|x - z\| + \beta \tau \|x - z\| \\ &= \beta(2 + \tau) \|x - z\|, \end{aligned} \quad (24)$$

where we have used Corollary 1 and Lemma 4 in the inequality.  $\blacksquare$

**Proposition 1.** *Assume Assumption 1 and Assumption 2 hold. Then, given  $x_0 \in \overline{\text{dom}(F_{\mathfrak{m}_{\bar{x}}})}$ , the dynamic (p-SMI), and hence (SMI), admits a unique solution  $x \in W^{1,1}(t_0, T; \mathcal{H})$ .*

*Proof.* Thanks to Lemma 2 and Lemma 5  $F_{\mathfrak{m}_{\bar{x}}}$  is maximal monotone and  $\mathbf{e}_{\bar{x}}$  is Lipschitz continuous. Since  $t \mapsto \mathbf{e}_{\bar{x}}(x)$  is trivially in  $L^\infty(t_0, T; \mathcal{H})$ , we deduce, thanks to [27, Proposition 3.13] the existence of a unique solution  $x \in W^{1,1}(t_0, T; \mathcal{H})$  to (SMI).  $\blacksquare$

*Remark 3.4.* Let us point that evolution problems of the form (p-SMI) were also addressed in [13] with a possibly multivalued operator  $\mathbf{e}_{\bar{x}}$  and without requiring that  $x \mapsto \mathbf{e}_{\bar{x}}$  is Lipschitz continuous.

**Theorem 2.** *Let  $x$  be the solution of (SMI), and assume that Assumption 4 holds. We then have for all  $t \geq t_0$*

$$\|x(t) - \bar{x}\| \leq \theta^{-1} (t - \hat{t}), \quad (25)$$

where  $\theta$  is defined in Lemma 1 and  $\hat{t} = 2t_0 - \theta (\|x_0 - \bar{x}\|)$ .

Moreover, if Assumption 3 holds instead, we have

$$\|x(t) - \bar{x}\| \leq C e^{-2\mu(1-\rho)t} \text{ for all } t \geq t_0, \quad (26)$$

with  $C = \|x_0 - \bar{x}\| e^{2\mu(1-\rho)t_0}$  and  $\rho := \frac{\beta\tau}{\mu}$ .

*Proof.* Let  $y \in \mathcal{H}$ , we observe that

$$\frac{1}{2} \frac{d}{dt} \|x(t) - y\|^2 = \langle \dot{x}(t), x(t) - y \rangle.$$

We then have, for any selection  $u(t)$  of  $F_{\mathfrak{m}_{\bar{x}}}(x(t))$

$$\frac{1}{2} \frac{d}{dt} \|x(t) - y\|^2 = \langle \dot{x}(t), x(t) - y \rangle = -\langle u(t), x(t) - y \rangle - \langle \mathbf{e}_{\bar{x}}(x(t)), x(t) - y \rangle. \quad (27)$$

Taking  $y \in \text{zer}(F_{\mathfrak{m}_{\bar{x}}}) = \{\bar{x}\}$ , we get by Assumption 3

$$\langle u(t) - 0, x(t) - \bar{x} \rangle \geq \|x(t) - \bar{x}\| \phi(\|x(t) - \bar{x}\|).$$

Then, thanks to [Lemma 3](#) and [Assumption 2](#), equation (27) gives

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|x(t) - \bar{x}\|^2 &= -\langle u(t) - 0, x(t) - \bar{x} \rangle - \langle \mathbf{e}_{\bar{x}}(x(t)), x(t) - \bar{x} \rangle \\
&\leq -\|x(t) - \bar{x}\| \phi(\|x(t) - \bar{x}\|) + \|\mathbf{e}_{\bar{x}}(x(t))\| \|x(t) - \bar{x}\| \\
&\leq -\|x(t) - \bar{x}\| \phi(\|x(t) - \bar{x}\|) + \beta\tau \|x(t) - \bar{x}\|^2 \\
&= -\|x(t) - \bar{x}\| \varphi(\|x(t) - \bar{x}\|),
\end{aligned} \tag{28}$$

where  $\varphi(t) = \phi(t) - \beta\tau t$  is as introduced in (21). Hence, dividing by  $h(t) := \|x(t) - \bar{x}\| \neq 0$  in both sides of (28), we get

$$\dot{h}(t) \leq -2\varphi(h(t)). \tag{29}$$

We infer from (28) and (29) that

$$\frac{d}{dt} \theta(h(t)) = \frac{-1}{\varphi(h(t))} \dot{h}(t) \geq \left( \frac{1}{\varphi(h(t))} \right) (2\varphi(h(t))) = 2. \tag{30}$$

Integrating (30) between  $t_0$  and  $t > 0$ , we get

$$\theta(h(t)) - \theta(h(t_0)) \geq 2(t - t_0).$$

Using [Lemma 1](#), we deduce that

$$h(t) \leq \theta^{-1}(2t - \hat{t}), \tag{31}$$

where  $\hat{t} = 2t_0 - \theta(h(t_0))$ . This proves (25), as desired.

To prove the second claim, we observe that if  $F_{\mathbf{m}_{\bar{x}}}$  is  $\mu$ -strongly monotone,  $\phi(t) = \mu t$  so that  $\varphi(t) = (\mu - \beta\tau)t$ . Assuming that  $\mu \neq \beta\tau$ , we get  $\theta(s) = \frac{-\log(s)}{\mu - \beta\tau}$ . So that  $\theta^{-1}(s) = e^{-(\mu - \beta\tau)s}$ . Plugging this in (31) we obtain

$$h(t) \leq \|x_0 - \bar{x}\| e^{-2(\mu - \beta\tau)(t - t_0)}, \tag{32}$$

as desired. ■

*Remark 3.5.* Observe that one can obtain (26) from (28) using Gronwall's [Lemma 9](#). Indeed, taking  $\phi(t) = \beta\mu t$ , we infer from (28)

$$\frac{1}{2} \frac{d}{dt} \|x(t) - \bar{x}\|^2 \leq -(\mu - \beta\tau) \|x(t) - \bar{x}\|^2.$$

As a consequence of [Assumption 2](#) and [Theorem 2](#), we obtain a rate of convergence, as  $t \rightarrow \infty$  of the measure  $\mathbf{m}_{x(t)}$  to  $\mathbf{m}_{\bar{x}}$  in  $\mathbb{W}_1$ .

**Corollary 2.** *Let  $x : [t_0, \infty) \rightarrow \mathbb{R}$  be the solution of (SMI), and assume that [Assumption 4](#) then for all  $t \geq t_0$*

$$\mathbb{W}_1(\mathbf{m}_{x(t)}, \mathbf{m}_{\bar{x}}) \leq \tau \theta^{-1}(t - \hat{t}),$$

where  $\theta$  is defined in [Lemma 1](#) and  $\hat{t} = 2t_0 - \theta(\|x_0 - \bar{x}\|)$ . If [Assumption 3](#) holds instead, we have

$$\mathbb{W}_1(\mathbf{m}_{x(t)}, \mathbf{m}_{\bar{x}}) \leq C e^{-2\mu(1-\rho)t},$$

with  $C = \|x_0 - \bar{x}\| e^{2\mu(1-\rho)t_0}$ .

## 4. SECOND-ORDER MONOTONE INCLUSIONS VIA HESSIAN DAMPING

This section is devoted to the analysis of the following second-order monotone inclusion

$$\begin{cases} \ddot{x}(t) + \gamma(t)\dot{x}(t) + F_{m_{x(t)}}(x(t)) \ni 0, & \text{a.e } t \in [t_0, T] \\ x(t_0) = x_0 \in \overline{\text{dom}(F_{m_{x(t)}})}, \end{cases} \quad (\text{ISMI})$$

where  $\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a continuous function, usually referred to as the viscous damping coefficient. We refer to this dynamic as an Inertial Stochastic Monotone Inclusion, (**ISMI**) for short. In a smooth deterministic setting, i.e.,  $F_{m_x} = \nabla f$  where  $f$  is a smooth, strongly convex function, systems of the form (**ISMI**) were first studied by Polyak in [48] with a fixed viscous damping coefficient. Later on, this kind of systems were studied by [15] and then [52] where the authors establish the link between the continuous dynamics with  $\gamma(t) = \frac{3}{t}$  and the Nesterov's methods [44]. Further results and extensions followed (see, e.g., [5, 11, 16, 39] and the references therein).

Now, coming back to our setting, we reformulate (**ISMI**) using the notations of Section 3, as follows

$$\begin{cases} \ddot{x}(t) + \gamma(t)\dot{x}(t) + F_{m_{\bar{x}}}(x(t)) + \mathbf{e}_{\bar{x}}(x(t)) \ni 0, & \text{a.e } t \in [t_0, T] \\ x(t_0) = x_0 \in \overline{\text{dom}(F_{m_{\bar{x}}})}. \end{cases} \quad (\text{p-ISMI})$$

One of the drawbacks of the perturber stochastic inertial monotone inclusion (**p-ISMI**) is that it cannot be written as a dynamical system governed by Lipschitz perturbations of monotone operators. This being said, we shall consider (**p-ISMI**) with the presence of the so called Hessian damping. To avoid technicalities, we restrict ourselves to the smooth case, i.e.,  $A = \nabla g$  for  $g \in \Gamma_0(\mathcal{H}) \cap C^1(\mathcal{H})$  so that

$$F_{m_{\bar{x}}} = \nabla G_{m_{\bar{x}}} \text{ with } G_{m_{\bar{x}}} = g + \mathbb{E}_{\xi \sim m_{\bar{x}}} f(x, \xi). \quad (33)$$

In this case the perturbation defined in (23) is given by

$$\mathbf{e}_{\bar{x}} = \mathbb{E}_{\xi \sim m_x} \nabla f(x, \xi) - \mathbb{E}_{\xi \sim m_{\bar{x}}} \nabla f(x, \xi), \quad (34)$$

and it measure the gradient deviation. This being said, (**p-ISMI**) reduces to an inertial gradient system

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla G_{m_{\bar{x}}}(x(t)) + \mathbf{e}_{\bar{x}}(x(t)) = 0, \quad (35)$$

which, when  $\gamma(t) \equiv \gamma$ , is essentially a Polyak's heavy ball (HBF) system [48] with the presence of the perturbation  $\mathbf{e}_{\bar{x}}$ . The Hessian-driven damped system we shall consider reads

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla G_{m_{\bar{x}}}(x(t)) + \omega \nabla^2 G_{m_{\bar{x}}}(x(t))\dot{x}(t) + \mathbf{e}_{\bar{x}}(x(t)) = 0, \quad (\text{ISEHD}_{m, \gamma})$$

where  $\omega : [0, \infty) \rightarrow \mathbb{R}^+$  is a continuous function usually referred to as the Hessian-driven damping coefficient, and which will be taken to be constant, i.e.,  $\omega(t) \equiv \omega > 0$ . We call the above dynamic an Inertial System with Explicit Hessian Damping, (**ISEHD** <sub>$m, \gamma, \omega$</sub> ) for short. The subscript  $m$  is here to emphasize the stochastic structure of the problem. Here, the Hessian damping is said to be explicit since since, when  $G_{m_{\bar{x}}}$  is of class  $C^2$ , we have

$$\frac{d}{dt} (\nabla G_{m_{\bar{x}}}(x(t))) = \nabla^2 G_{m_{\bar{x}}}(x(t))\dot{x}(t).$$

One of the main advantages of considering dynamics with Hessian-driven damping is the to attenuate the oscillation that might occur with (HBF)-like dynamics or more generally with inertial dynamics with viscous damping as was observed in [3]. Variants and generalizations were studied by multiple authors (see, e.g., [9, 18, 19]). The study of the effect of perturbed Hessian-driven damping, i.e., problems of the form (**ISEHD** <sub>$m, \gamma, \omega$</sub> ) was carried in the recent work [14].

In particular, this will allow us to proceed as in [Section 3](#) to prove the well-posedness of the dynamics. As for convergence properties, this setting will allow us to exploit some techniques from [\[14\]](#).

#### 4.1. Well-posedness.

##### 4.1.1. Equivalent formulation.

**Proposition 2.** *Suppose that  $\gamma(t) \geq 0, \omega > 0$ . For any initial conditions  $(x_0, v_0) \in \mathcal{H} \times \mathcal{H}$ , the dynamics  $(\text{ISEHD}_{\mathbf{m}, \gamma, \omega})$  admits an equivalent formulation of the form*

$$\begin{cases} \dot{x}(t) + \omega \left( \nabla G_{\mathbf{m}_{\bar{x}}} x(t) + \mathbf{e}_{\bar{x}}(x(t)) \right) - \left( \frac{1}{\omega} - \gamma(t) \right) x(t) + \frac{1}{\omega} y(t) & = 0 \\ \dot{y}(t) - \left( \frac{1}{\omega} - \gamma(t) - \dot{\gamma}(t)\omega \right) x(t) + \frac{1}{\omega} y(t) & = 0, \end{cases} \quad (36)$$

with initial conditions  $x(t_0) = x_0, y(t_0) = -\omega(v_0 + \omega \nabla G_{\mathbf{m}_{\bar{x}}}(x_0)) + (1 - \omega\gamma(t_0))x_0 - \omega^2 \mathbf{e}_{\bar{x}}(x_0)$ .

*Proof.* Let  $(x, y)$  be a solution of [\(36\)](#). By differentiation of the first equation in [\(36\)](#), we get

$$\ddot{x}(t) + \omega \nabla^2 G_{\mathbf{m}_{\bar{x}}}(x(t)) \dot{x}(t) + \dot{\gamma}(t)x(t) - \left( \frac{1}{\omega} - \gamma(t) \right) \dot{x}(t) + \frac{1}{\omega} \dot{y}(t) = 0.$$

Replacing  $\dot{y}$  by its expression from the second equation in [\(36\)](#), we obtain:

$$\begin{aligned} & \ddot{x}(t) + \omega \nabla^2 G_{\mathbf{m}_{\bar{x}}}(x(t)) \dot{x}(t) + \dot{\gamma}(t)x(t) - \left( \frac{1}{\omega} - \gamma(t) \right) \dot{x}(t) \\ & + \frac{1}{\omega} \left( \left( \frac{1}{\omega} - \gamma(t) - \dot{\gamma}(t)\omega \right) x(t) - \frac{1}{\omega} y(t) \right) = 0, \end{aligned} \quad (37)$$

using again the first equation in [\(36\)](#) to eliminate  $y(t)$ , we get:

$$\begin{aligned} & \ddot{x}(t) + \omega \nabla^2 G_{\mathbf{m}_{\bar{x}}}(x(t)) \dot{x}(t) + \dot{\gamma}(t)x(t) - \left( \frac{1}{\omega} - \gamma(t) \right) \dot{x}(t) + \frac{1}{\omega} \left( \left( \frac{1}{\omega} - \gamma(t) \right) \right. \\ & \left. - \dot{\gamma}(t)\omega \right) x(t) + \dot{x}(t) + \omega \left( \nabla G_{\mathbf{m}_{\bar{x}}} x(t) + \mathbf{e}_{\bar{x}}(x(t)) \right) - \left( \frac{1}{\omega} - \gamma(t) \right) x(t) = 0, \end{aligned} \quad (38)$$

and after simplifications, we recover  $(\text{ISEHD}_{\mathbf{m}, \gamma, \omega})$ . Conversely, let  $x$  be a trajectory solution to  $(\text{ISEHD}_{\mathbf{m}, \gamma, \omega})$  with initial conditions  $(x_0, v_0) \in \mathcal{H} \times \mathcal{H}$  and define

$$y(t) = -\omega \left( \dot{x}(t) + \omega \left( \nabla G_{\mathbf{m}_{\bar{x}}} x(t) + \mathbf{e}_{\bar{x}}(x(t)) \right) - \left( \frac{1}{\omega} - \gamma(t) \right) x(t) \right).$$

By differentiating the previous formula and using  $(\text{ISEHD}_{\mathbf{m}, \gamma, \omega})$ , we recover the second equation of [\(36\)](#), as desired.  $\blacksquare$

4.1.2. *Well-posedness.* Thanks to [Proposition 2](#), one can also consider [\(36\)](#) in the case where  $G_{\mathbf{m}_{\bar{x}}} \in \Gamma_0(\mathcal{H})$ . In fact, we have the following

**Definition 6.** Suppose that  $\gamma(t) \geq 0, \omega > 0$ . For any initial conditions  $(x_0, v_0) \in \overline{\text{dom}(G_{\mathbf{m}_{\bar{x}}})} \times \mathcal{H}$ , the dynamics  $(\text{ISEHD}_{\mathbf{m}, \gamma, \omega})$  admits an equivalent formulation of the form

$$\begin{cases} \dot{x}(t) + \omega \left( \nabla G_{\mathbf{m}_{\bar{x}}} x(t) + \mathbf{e}_{\bar{x}}(x(t)) \right) - \left( \frac{1}{\omega} - \gamma(t) \right) x(t) + \frac{1}{\omega} y(t) & \ni 0 \\ \dot{y}(t) - \left( \frac{1}{\omega} - \gamma(t) - \dot{\gamma}(t)\omega \right) x(t) + \frac{1}{\omega} y(t) & = 0, \end{cases} \quad (39)$$

with initial conditions  $x(t_0) = x_0, y(t_0) = v_0$ .

One of the main advantages of (39) is that it can be easily recast as a monotone inclusion governed by Lipschitz perturbation of a maximal monotone operator on the product space  $\mathcal{H} \times \mathcal{H}$ . Indeed, setting  $Z(t) = (x(t), y(t))$ ,  $\mathcal{A}(x, y) = (\omega \partial G_{m_{\bar{x}}}(x), 0)$  and

$$\mathcal{E}(t, x, y) = \left( \omega \mathbf{e}_{\bar{x}}(x(t)) - \left( \frac{1}{\omega} - \gamma(t) \right) x(t) + \frac{1}{\omega} y(t), - \left( \frac{1}{\omega} - \gamma(t) - \dot{\gamma}(t) \omega \right) x(t) + \frac{1}{\omega} y(t) \right),$$

we immediately see that (39) can be written as

$$\dot{Z}(t) + \mathcal{A}(Z(t)) + \mathcal{E}(t, Z(t)) \ni 0_{\mathcal{H} \times \mathcal{H}}, \quad Z(0) = (x_0, v_0). \quad (40)$$

Then (39) can be written as monotone inclusion in the product space  $\mathcal{H} \times \mathcal{H}$

$$\begin{cases} \dot{Z}(t) + \mathcal{A}(Z(t)) + \mathcal{E}(t, Z(t)) \ni 0_{\mathcal{H} \times \mathcal{H}}, & \text{a.e } t \in [t_0, T] \\ Z(0) = (x_0, v_0) \in \mathcal{H} \times \mathcal{H}, \end{cases} \quad (\text{MIS})$$

which fits in the framework of Lipschitz perturbations of maximal monotone operators as in Section 3. Notice that this formulation is different from the classical Hamiltonian one.

Before stating the main result, let us recall that we endow the product space with the scalar product  $\langle (u, v), (u^*, v^*) \rangle_{\mathcal{H} \times \mathcal{H}} = \langle u, u^* \rangle + \langle v, v^* \rangle$ , and the induced norm  $\|(u, v)\|_{\mathcal{H} \times \mathcal{H}} = \sqrt{\|u\|^2 + \|v\|^2}$ . We have the following auxiliary results

**Lemma 6.** *Under Assumption 3, the operator  $\mathcal{E}$  is Lipschitz continuous.*

*Proof.* Let  $u, v, u^*, v^* \in \mathcal{H}$  and set  $p = (\frac{1}{\omega} - \gamma(t))$ ,  $q = (\frac{1}{\omega} - \gamma(t) - \dot{\gamma}(t)\omega)$ . We have, for  $t \in [t_0, T]$

$$\begin{aligned} \|\mathcal{E}(t, u, v) - \mathcal{E}(t, u^*, v^*)\|_{\mathcal{H} \times \mathcal{H}} &= \left\| \left( \omega(\mathbf{e}_{\bar{x}}(u) - \mathbf{e}_{\bar{x}}(u^*)) + p(u^* - u) + \frac{1}{\omega}(v - v^*), \frac{1}{\omega}(v - v^*) + q(u^* - u) \right) \right\|_{\mathcal{H} \times \mathcal{H}} \\ &= \sqrt{\|(\omega(\mathbf{e}_{\bar{x}}(u) - \mathbf{e}_{\bar{x}}(u^*)) + p(u^* - u) + \frac{1}{\omega}(v - v^*))\|^2 + \|\frac{1}{\omega}(v - v^*) + q(u^* - u)\|^2} \\ &= \sqrt{2\omega^2 \|\mathbf{e}_{\bar{x}}(u) - \mathbf{e}_{\bar{x}}(u^*)\|^2 + (4p^2 + 2p^2)\|u - u^*\|^2 + \frac{6}{\omega^2}\|v - v^*\|^2} \\ &= \sqrt{(2\omega^2\beta^2(2 + \tau)^2 + 4p^2 + 2p^2)\|u - u^*\|^2 + \frac{6}{\omega^2}\|v - v^*\|^2} \\ &\leq \left( \sqrt{2}\omega\beta(2 + \tau) + 2|p| + \sqrt{2}|q| + \frac{\sqrt{6}}{\omega} \right) \|(u, v) - (u^*, v^*)\|_{\mathcal{H} \times \mathcal{H}} \\ &= K(\omega, \beta, \tau, \gamma) \|(u, v) - (u^*, v^*)\|_{\mathcal{H} \times \mathcal{H}} \end{aligned}$$

where we have used Young's inequality and Lemma 5 for the Lipschitz continuity of the operator  $\mathbf{e}_{\bar{x}}$ .  $\blacksquare$

**Proposition 3.** *Assume that Assumption 3 and Assumption 4 hold. Then, for any initial data  $x_0 \in \overline{\text{dom}(G_{m_{\bar{x}}})}$  and  $v_0 \in \mathcal{H}$ , there exists a unique global solution  $x \in C^2([t_0, \infty[, \mathcal{H})$  to (MIS) such that  $x(t_0) = x_0$  and  $\dot{x}(t_0) = v_0$ . Moreover, the solution  $Z = (x, y)$  satisfies the following properties*

- (i)  $y$  is continuously differentiable on  $[t_0, +\infty[$ , and  $\dot{y}(t) - (\frac{1}{\omega} - \gamma(t) - \omega\dot{\gamma}(t))x(t) + \frac{1}{\omega}y(t) = 0$ , for all  $t \geq t_0$ ;
- (ii)  $x$  is absolutely continuous on  $[t_0, T]$  and  $\dot{x} \in L^2(t_0, T; \mathcal{H})$  for all  $T > t_0$ ;
- (iii)  $x(t) \in \text{dom}(\partial G_{m_{\bar{x}}})$  for all  $t > t_0$ ;
- (iv)  $x$  is Lipschitz continuous on any compact subinterval of  $]t_0, +\infty[$ ;
- (v) the function  $t \in [t_0, +\infty[ \mapsto G_{m_{\bar{x}}}(x(t))$  is absolutely continuous on  $[t_0, T]$  for all  $T > t_0$ ;

- (vi) there exists a function  $\xi : [t_0, +\infty[ \rightarrow \mathcal{H}$  such that
- (a)  $\xi(t) \in \partial G_{\bar{m}_x}(x(t))$  for all  $t > t_0$ ;
  - (b)  $\dot{x}(t) + \omega \xi(t) - \left(\frac{1}{\omega} - \gamma(t)\right)x(t) + \frac{1}{\omega}y(t) = 0$  for almost every  $t > t_0$ ;
  - (c)  $\xi \in L^2(t_0, T; \mathcal{H})$  for all  $T > t_0$ ;
  - (d)  $\frac{d}{dt}\phi(x(t)) = \langle \xi(t), \dot{x}(t) \rangle$  for almost every  $t > t_0$ .

*Proof.* Thanks to [Assumption 4](#) and Moreau's theorem (see, e.g., [23, Theorem 20.25])  $\mathcal{A}$  is maximal monotone and  $\mathcal{E}$  is Lipschitz continuous thanks to [Lemma 6](#). We conclude again the existence of a strong global solution  $Z = (x, y) : [t_0, T] \rightarrow \mathcal{H} \times \mathcal{H}$  of (MIS) via [27, Proposition 3.12]. The verification of items (i)-(vi) can be done by following the main arguments of [19, Theorem 4.4]  $\blacksquare$

As a consequence, we have the following

**Corollary 3.** *Suppose that  $G_{\bar{m}_x} : \mathcal{H} \rightarrow \mathbb{R}$  is a convex  $C^2$  function. For any  $t_0 > 0$ , and any Cauchy data  $(x_0, v_0) \in \mathcal{H} \times \mathcal{H}$ , there exists a unique classical global solution  $x : [t_0, +\infty[ \rightarrow \mathcal{H}$  to*

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla G_{\bar{m}_x}(x(t)) + \omega \nabla^2 G_{\bar{m}_x}(x(t))\dot{x}(t) + \mathbf{e}_{\bar{x}}(x(t)) = 0, \quad (\text{ISEHD}_{\mathbf{m}, \gamma, \omega})$$

with  $x(t_0) = x_0, \dot{x}(t_0) = v_0$ .

*Proof.* We have thanks to (36) that

$$\dot{x}(t) + \omega \left( \nabla G_{\bar{m}_x} x(t) + \mathbf{e}_{\bar{x}}(x(t)) \right) - \left( \frac{1}{\omega} - \gamma(t) \right) x(t) + \frac{1}{\omega} y(t) = 0,$$

so thanks to [Proposition 3 Item \(i\)-Item \(ii\)](#), we have  $\dot{x} \in C^1([t_0, \infty))$ , and thus  $x \in C^2([t_0, \infty))$   $\blacksquare$

**4.2. Convergence properties.** Now let us examine the convergence properties of (ISEHD $_{\mathbf{m}, \gamma, \omega}$ ). Recall that we restrict ourselves here to the smooth case, that is, we assume that  $F_{\bar{m}_x} = \nabla G_{\bar{m}_x}$  satisfies [Assumption 3](#), i.e.,  $G_{\bar{m}_x}$  is  $\mu$ -strongly monotone for  $\mu > 0$ . Moreover, we tune the viscous damping function to the modulus of strong convexity  $\mu$  of  $G_{\bar{m}_x}$ , by taking  $\gamma(t) \equiv 2\sqrt{\mu}$ . From now on, we focus on the following system

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla G_{\bar{m}_x}(x(t)) + \omega \nabla^2 G_{\bar{m}_x}(x(t))\dot{x}(t) + \mathbf{e}_{\bar{x}}(x(t)) = 0, \quad (\text{ISEHD}_{\mathbf{m}, 2\sqrt{\mu}})$$

To perform Lyapunov analysis, let us define the following function  $\mathcal{V} : [t_0, \infty[ \rightarrow \mathbb{R}^+$  by

$$\mathcal{V}(t) := G_{\bar{m}_x}(x(t)) - G_{\bar{m}_x}^* + \frac{1}{2}\|v(t)\|^2, \quad \text{where } v(t) := \sqrt{\mu}(x(t) - \bar{x}) + \dot{x}(t) + \omega \nabla G_{\bar{m}_x}(x(t)).$$

Following the main ideas of [9, 14] we prove the following result.

**Theorem 3.** *Assume that  $G_{\bar{m}_x}$  satisfies [Assumption 3](#) and let  $x : [t_0, \infty[ \rightarrow \mathbb{R}$  be the solution of (ISEHD $_{\mathbf{m}, 2\sqrt{\mu}}$ ). Suppose that  $0 \leq \omega \leq \frac{1}{2\sqrt{\mu}}$  and that  $\rho$  and the damping coefficient  $\omega$  satisfy*

$$8\rho^2 + \omega < 1. \quad (41)$$

We then have:

- for all  $t \geq t_0$

$$\mathcal{V}(t) \leq C e^{-\frac{\sqrt{\mu}}{4}t}, \quad \text{where } C = \mathcal{V}(t_0) e^{\frac{\sqrt{\mu}}{4}t_0}.$$

In particular

$$\frac{\mu}{2}\|x(t) - \bar{x}\|^2 \leq G_{\bar{m}_x}(x(t)) - G_{\bar{m}_x}^* \leq \mathcal{V}(t_0) e^{-\frac{\sqrt{\mu}}{4}(t-t_0)}.$$

- There exists  $C_1 > 0$  such that,

$$e^{\sqrt{\mu}t} \int_{t_0}^t e^{\sqrt{\mu}s} \|\nabla G_{\mathbf{m}_{\bar{x}}}(x(s))\|^2 ds \leq C_1 e^{-\frac{\sqrt{\mu}}{4}t}, \quad \forall t \geq t_0.$$

*Proof.* We have

$$\begin{aligned} \dot{\mathcal{V}}(t) &= \langle \nabla G_{\mathbf{m}_{\bar{x}}}(x(t)), \dot{x}(t) \rangle + \langle v(t), \sqrt{\mu}\dot{x}(t) + \ddot{x}(t) + \omega \nabla G_{\mathbf{m}_{\bar{x}}}(x(t))\dot{x}(t) \rangle \\ &= \langle \nabla G_{\mathbf{m}_{\bar{x}}}(x(t)), \dot{x}(t) \rangle + \langle v(t), -\sqrt{\mu}\dot{x}(t) - \nabla G_{\mathbf{m}_{\bar{x}}}(x(t)) - \mathbf{e}_{\bar{x}}(x(t)) \rangle. \end{aligned} \quad (42)$$

We get from (ISEHD<sub>m,2 $\sqrt{\mu}$</sub> ) after some simplifications

$$\begin{aligned} \dot{\mathcal{V}}(t) + \mu \langle \dot{x}(t), x(t) - \bar{x} \rangle + \sqrt{\mu} \|\dot{x}(t)\|^2 + \sqrt{\mu} \langle \nabla G_{\mathbf{m}_{\bar{x}}}(x(t)), x(t) - \bar{x} \rangle \\ + \omega \sqrt{\mu} \langle \nabla G_{\mathbf{m}_{\bar{x}}}(x(t)), \dot{x}(t) \rangle + \omega \|\nabla G_{\mathbf{m}_{\bar{x}}}(x(t))\|^2 = -\langle v(t), \mathbf{e}_{\bar{x}}(x(t)) \rangle. \end{aligned} \quad (43)$$

Using  $\mu$ -strong convexity of  $G_{\mathbf{m}_{\bar{x}}}$ , we have

$$\langle \nabla G_{\mathbf{m}_{\bar{x}}}(x(t)), x(t) - \bar{x} \rangle \geq G_{\mathbf{m}_{\bar{x}}}(x(t)) - G_{\mathbf{m}_{\bar{x}}}^* + \frac{\mu}{2} \|x(t) - \bar{x}\|^2, \quad (44)$$

and using this in (76), we get

$$\dot{\mathcal{V}}(t) + \sqrt{\mu} \Theta(t) \leq \|v(t)\| \|\mathbf{e}_{\bar{x}}(x(t))\|, \quad (45)$$

where

$$\Theta(t) := G_{\mathbf{m}_{\bar{x}}}(x(t)) - G_{\mathbf{m}_{\bar{x}}}^* + \frac{\mu}{2} \|x(t) - \bar{x}\|^2 + \sqrt{\mu} \langle \dot{x}(t), x(t) - \bar{x} \rangle + \|\dot{x}(t)\|^2 + \omega \langle \nabla G_{\mathbf{m}_{\bar{x}}}(x(t)), \dot{x}(t) \rangle + \frac{\omega}{\sqrt{\mu}} \|\nabla G_{\mathbf{m}_{\bar{x}}}(x(t))\|^2.$$

Using the definition of  $\mathcal{V}(t)$ , we may rewrite  $\Theta(t)$  as

$$\Theta(t) = \mathcal{V}(t) + \frac{1}{2} \|\dot{x}(t)\|^2 - \omega \sqrt{\mu} \langle \nabla G_{\mathbf{m}_{\bar{x}}}(x(t)), x(t) - \bar{x} \rangle + \left( \frac{\omega}{\sqrt{\mu}} - \frac{\omega^2}{2} \right) \|\nabla G_{\mathbf{m}_{\bar{x}}}(x(t))\|^2.$$

Consequently, (45) becomes

$$\dot{\mathcal{V}}(t) + \sqrt{\mu} \mathcal{V}(t) + \frac{\sqrt{\mu}}{2} \|\dot{x}(t)\|^2 + \left( \left( \frac{\omega}{\sqrt{\mu}} - \frac{\omega^2}{2} \right) \|\nabla G_{\mathbf{m}_{\bar{x}}}(x(t))\|^2 - \omega \sqrt{\mu} \langle \nabla G_{\mathbf{m}_{\bar{x}}}(x(t)), x(t) - \bar{x} \rangle \right) \leq \|v(t)\| \|\mathbf{e}_{\bar{x}}(x(t))\|,$$

Using strong convexity we obtain again

$$\mathcal{V}(t) = \frac{1}{2} \mathcal{V}(t) + \frac{1}{2} \mathcal{V}(t) \geq \frac{1}{2} \mathcal{V}(t) + \frac{\mu}{4} \|x(t) - \bar{x}\|^2,$$

and observing that  $\frac{\omega}{2\sqrt{\mu}} \leq \frac{\omega}{\sqrt{\mu}} - \frac{\omega^2}{2}$  for  $0 \leq \omega \leq \frac{1}{\sqrt{\mu}}$ . We end up with

$$\begin{aligned} \dot{\mathcal{V}}(t) + \frac{\sqrt{\mu}}{2} \mathcal{V}(t) + \frac{\sqrt{\mu}}{2} \|\dot{x}(t)\|^2 \\ + \left( \frac{\mu}{4} \|x(t) - \bar{x}\|^2 + \frac{\omega}{2\sqrt{\mu}} \|\nabla G_{\mathbf{m}_{\bar{x}}}(x(t))\|^2 - \omega \sqrt{\mu} \|\nabla G_{\mathbf{m}_{\bar{x}}}(x(t))\| \|x(t) - \bar{x}\| \right) \leq \|v(t)\| \|\mathbf{e}_{\bar{x}}(x(t))\|. \end{aligned} \quad (46)$$

Now let us treat the right hand side of this inequality. Since  $\mathcal{V}(t) \geq \frac{1}{2} \|v(t)\|^2$ , we have, using Young's inequality

$$\|v(t)\| \|\mathbf{e}_{\bar{x}}(x(t))\| \leq \frac{\sqrt{\mu}}{8} \|v(t)\|^2 + \frac{2}{\sqrt{\mu}} \|\mathbf{e}_{\bar{x}}(x(t))\|^2 \leq \frac{\sqrt{\mu}}{4} \mathcal{V}(t) + \frac{2\beta^2\tau^2}{\sqrt{\mu}} \|x(t) - \bar{x}\|^2. \quad (47)$$

Using Lemma 5, we get after rearranging the terms

$$\dot{\mathcal{V}}(t) + \frac{\sqrt{\mu}}{4} \mathcal{V}(t) + \frac{\sqrt{\mu}}{2} \|\dot{x}(t)\|^2 + \sqrt{\mu} \Psi(t) \leq 0, \quad (48)$$



where

$$\Psi(t) = \left( \frac{\mu}{4} - \frac{2\beta^2\tau^2}{\mu} \right) \|x(t) - \bar{x}\|^2 + \frac{\omega}{2\sqrt{\mu}} \|\nabla G_{\mathbf{m}_{\bar{x}}}(x(t))\|^2 - \omega\sqrt{\mu} \|\nabla G_{\mathbf{m}_{\bar{x}}}(x(t))\| \|x(t) - \bar{x}\|.$$

Setting

$$\mathbf{a} = \frac{\mu}{4} - \frac{2\beta^2\tau^2}{\mu}, \mathbf{b} = \frac{\omega}{2\sqrt{\mu}}, \mathbf{c} = -\frac{\omega\sqrt{\mu}}{2}$$

and  $X = \|x(t) - \bar{x}\|$  and  $Y = \|\nabla f_{\mathbf{m}_{\bar{x}}}(x(t))\|$ , we see that  $\Psi$  can be written as a quadratic form  $\mathcal{Q} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$  with  $\mathcal{Q}(X, Y) = \mathbf{a}\|X\|^2 + 2\mathbf{c}\langle X, Y \rangle + \mathbf{d}\|Y\|^2$ . By assumption  $\mathbf{a}, \mathbf{b} \geq 0$ , and the discriminant  $\mathbf{c}^2 - \mathbf{a}\mathbf{b}$  of  $\mathcal{Q}$  is nonpositive. Indeed, since  $0 \leq \omega \leq \frac{1}{2\sqrt{\mu}}$

$$\mathbf{c}^2 - \mathbf{a}\mathbf{b} = \frac{\omega^2\mu}{4} - \frac{\omega}{2\sqrt{\mu}} \left( \frac{\mu}{4} - \frac{2\beta^2\tau^2}{\mu} \right) \leq \frac{\omega^2\sqrt{\mu}}{8} - \frac{\omega\sqrt{\mu}}{2} \left( \frac{1}{4} - 2\rho^2 \right) = \frac{\omega\sqrt{\mu}}{8} (\omega - 1 + 8\rho^2) \leq 0.$$

Hence  $\Psi(t) \geq 0$  and

$$\dot{\mathcal{V}}(t) + \frac{\sqrt{\mu}}{4}\mathcal{V}(t) + \frac{\sqrt{\mu}}{2}\|\dot{x}(t)\|^2 \leq 0,$$

which gives after integration

$$\mathcal{V}(t) \leq \mathcal{V}(t_0)e^{-\frac{\sqrt{\mu}}{4}(t-t_0)}. \quad (49)$$

Therefore,  $\lim_{t \rightarrow \infty} \mathcal{V}(t) = 0$  and in particular

$$\lim_{t \rightarrow \infty} G_{\mathbf{m}_{\bar{x}}}(x(t)) - G_{\mathbf{m}_{\bar{x}}}^* = 0 \text{ and } \lim_{t \rightarrow \infty} \|v(t)\| = 0. \quad (50)$$

This implies, using strong convexity of  $G_{\mathbf{m}_{\bar{x}}}$

$$\lim_{t \rightarrow \infty} \|x(t) - \bar{x}\| = 0,$$

which gives that

$$\lim_{t \rightarrow \infty} \|\nabla G_{\mathbf{m}_{\bar{x}}}(x(t))\| = 0.$$

We deduce from (50) that  $\lim_{t \rightarrow \infty} \|\dot{x}(t)\| = 0$ .

Coming back to (49), we have, by definition of  $\mathcal{V}$ , that

$$G_{\mathbf{m}_{\bar{x}}}(x(t)) - G_{\mathbf{m}_{\bar{x}}}^* \leq \mathcal{V}(t_0)e^{-\frac{\sqrt{\mu}}{4}(t-t_0)} \text{ and } \|v(t)\|^2 \leq 2\mathcal{V}(t_0)e^{-\frac{\sqrt{\mu}}{4}(t-t_0)}. \quad (51)$$

Developing in (51) we have

$$\begin{aligned} \mu\|x(t) - \bar{x}\|^2 + \|\dot{x}(t)\|^2 + \omega^2\|\nabla G_{\mathbf{m}_{\bar{x}}}(x(t))\|^2 + 2\omega\sqrt{\mu}\langle \nabla G_{\mathbf{m}_{\bar{x}}}(x(t)), x(t) - \bar{x} \rangle \\ + 2\omega\langle \nabla G_{\mathbf{m}_{\bar{x}}}(x(t)), \dot{x}(t) \rangle + 2\sqrt{\mu}\langle \dot{x}(t), x(t) - \bar{x} \rangle \leq Ce^{-\frac{\sqrt{\mu}}{4}t}, \end{aligned} \quad (52)$$

where  $C = 2\mathcal{V}(t_0)e^{\frac{\sqrt{\mu}}{4}t_0}$ .

Since  $\langle \nabla G_{\mathbf{m}_{\bar{x}}}(x(t)), x(t) - \bar{x} \rangle \geq G_{\mathbf{m}_{\bar{x}}}(x(t)) - G_{\mathbf{m}_{\bar{x}}}^*$ , we deduce from (52) that

$$\dot{U}(t) + \sqrt{\mu}U(t) + \omega^2\|\nabla G_{\mathbf{m}_{\bar{x}}}(x(t))\|^2 \leq Ce^{-\frac{\sqrt{\mu}}{4}t}, \quad (53)$$

where  $U(t) := \sqrt{\mu}\|x(t) - \bar{x}\|^2 + 2\omega(G_{\mathbf{m}_{\bar{x}}}(x(t)) - G_{\mathbf{m}_{\bar{x}}}^*)$ . Integrating (53), we obtain after elementary computation

$$e^{\sqrt{\mu}t} \int_{t_0}^t e^{\sqrt{\mu}s} \|\nabla G_{\mathbf{m}_{\bar{x}}}(x(s))\|^2 ds \leq C_1 e^{-\frac{\sqrt{\mu}}{4}t},$$

as desired. ■

*Remark 4.1.* Notice that contrary to [17], no assumption on the integrability of  $\mathbf{e}_{\bar{x}}$  is made in [Theorem 3](#). In fact, thanks to [Lemma 4](#), the norm of the error term  $\mathbf{e}_{\bar{x}}$  can be absorbed in the righthandside of (45).

*Remark 4.2.* Notice that in case  $\omega = 0$ , i.e., when the inertial dynamic is considered only with the viscous damping coefficient, the condition (41) reduces to  $\rho < \frac{\sqrt{2}}{4}$ . This contrasts with the convergence condition for first-order dynamics (cf. Theorem 2), where convergence is ensured in parameter regime  $\rho < 1$ . One possible explanation for this difference, is the potential occurrence of oscillations, which may necessitate a stricter compatibility condition between the parameters  $\tau, \beta, \mu$ .

We end this section with a similar result to Corollary 2, which is a direct consequence of Assumption 2 and Theorem 3-3.

**Corollary 4.** *Let  $x : [t_0, \infty) \rightarrow \mathbb{R}$  be the solution of (ISEHD $_{m, 2\sqrt{\mu}}$ ), then  $\forall t \geq t_0$*

$$W_1(m_{x(t)}, m_{\bar{x}}) \leq C e^{-\frac{\sqrt{\mu}}{8}t},$$

with  $C = \tau \sqrt{\frac{2}{\mu} \mathcal{V}(t_0)} e^{\frac{\sqrt{\mu}}{8}t_0}$ .

## 5. ON COARSE RICCI CURVATURE

In this section we discuss some dynamical and geometrical properties of the family  $(m_x)_x$ , particularly the notion of Ollivier-Ricci curvature and how it is tightly related to Assumption 2. In fact, the family of probabilities  $m = (m_x)_{x \in \mathcal{H}}$  and its Lipschitz behavior with respect to the  $W_1$ -Wasserstein distance, reveals that a natural setting to address monotone inclusions of the form (SMI), and thus stochastic optimization problems with decision-dependent distributions is the framework of *metric random walk spaces* (see, e.g., [42, 45]). All definitions of this section can be found in [36, 42].

**5.1. Metric random walk spaces.** Before going further, let us recall the following definitions to introduce a couple of probabilistic notions.

**Definition 7** (Random walks [45]). Given a Polish space  $(X, d)$ . A family of probabilities  $m = (m_x)_{x \in X}$  is a random walk on  $X$  if  $m_x \in \mathcal{P}(\Xi)$  for each  $x \in X$  and

- $m_x$  depends measurably on  $x \in X$ ,
- Each  $m_x$  has finite first-order moment, i.e., for some  $x^o \in X$ ,  $\mathbb{E}_{y \sim m_x} d(y, x^o) < \infty$ .

Then  $(X, d)$  equipped with a random walk first-order  $m$  is a metric random walk space (m.r.w.s for short), and we denote it by  $[X, d, m]$ .

Let us recall the notion of invariant and ergodic measures.

**Definition 8** (Invariance). Let  $\nu$  be a  $\sigma$ -finite measure on  $X$  and  $m$  a random walk on  $(X, \mathcal{B})$ . We say that  $\nu$  is invariant with respect to  $m$  if  $\nu \star m = \nu$ , where  $\nu \star m$  is the convolution of  $\nu$  by the random walk  $m$  and is defined by

$$\nu \star m(A) = \int_X m_x(A) d\nu(x) \text{ for all } A \in \mathcal{B}.$$

As pointed out in [45], each measure  $m_x$  can be seen as a replacement of a sphere around  $x$ . While in a probabilistic framework one think about a Markov chain whose transition kernel from  $x$  to  $y$  in  $n$  steps is defined by

$$dm_x^{*n}(y) = \int_{z \in X} dm_x^{*(n-1)}(z) dm_z(y), \quad (54)$$

with  $m_x^1 = m_x$  and  $m_x^0 = \delta_x$ .

In the sequel, we assume that  $(\mathcal{H}, d)$  is a separable real Hilbert space, and thus a Polish space, where  $d(x, y) = \langle x - y, x - y \rangle^{1/2}$ .

**5.2. Feller Property.** Let us recall the following definition.

**Definition 9.** We say that  $\mathfrak{m} := (\mathfrak{m}_x)_x$  has the weak-Feller property if and only if for every sequence  $x_n \rightarrow x^o \in \mathcal{H}$  we have  $\mathfrak{m}_{x_n} \rightarrow \mathfrak{m}_{x^o}$ , i.e.,  $\int f d\mathfrak{m}_{x_n} \rightarrow \int f d\mathfrak{m}_{x^o}$  for any  $f \in C_b(\mathcal{H})$ .

It turns that [Assumption 2](#) implies directly that the family  $\mathfrak{m}$  is weak-Feller.

**Proposition 4.** Under [Assumption 2](#),  $\mathfrak{m}$  has the weak-Feller property. Moreover, for each  $x \in \mathcal{H}$ ,  $\mathfrak{m}_x$  has finite first-order moment.

*Proof.* Let  $x^o \in \mathcal{H}$  and  $(x_n)_n$  a sequence of  $\mathcal{H}$  such that  $x_n \rightarrow x^o$  as  $n \rightarrow \infty$ . Then [Assumption 2](#) gives

$$W_1(\mathfrak{m}_{x_n}, \mathfrak{m}_{x^o}) \leq \tau \|x_n - x^o\|,$$

and thus  $\lim_{n \rightarrow \infty} W_1(\mathfrak{m}_{x_n}, \mathfrak{m}_{x^o}) = 0$ . Thanks to [[4](#), Proposition 7.1.5],  $(\mathfrak{m}_{x_n})$  has uniformly integrable  $p$ -moments with  $p \geq 1$  and narrowly converges towards  $\mathfrak{m}_{x^o}$ . In particular  $\mathfrak{m}$  is weak-Feller and each  $\mathfrak{m}_x$  has finite first-order moments.  $\blacksquare$

*Remark 5.1.* We already know that  $\mathfrak{m}_x \in \mathcal{P}(\Xi)$  for each  $x \in \mathcal{H}$  and that  $x \mapsto \mathfrak{m}_x(C)$  is measurable for each  $C \in \mathcal{B}$ . Moreover, thanks to [Proposition 4](#), we have finiteness of first-order moments of each  $\mathfrak{m}_x$ , so that the family  $\mathfrak{m}$  satisfies the requirements of [Definition 7](#). This shows that a natural setting to address dynamics of the form (SMI) is the metric random walk space  $[\mathcal{H}, d, \mathfrak{m}]$ . Many diffusion and variational problems has been studies within this framework, with allows in particular consider nonlocal continuum problems or problems on weighted graphs (see, e.g., [[42](#)] and the references therein).

*Remark 5.2.* Let us point out that if  $\nu$  is an invariant measure with respect to  $\mathfrak{m}$  then it is also an invariant measure with respect to  $\mathfrak{m}^{*n}$  for every  $n \in \mathbb{N}$ , where  $\mathfrak{m}^{*n}$  is the  $n$ -step transition probability function given by ([54](#)). It turns out that weak-Feller property implies that every weak- $*$  limit  $\nu$  of  $(\mathfrak{m}^{*n})_n$  is an invariant measure of  $\mathfrak{m}$  cf. [[36](#), Proposition 7.2.2] (see also [[34](#), Proposition 12.3.4]). However, without assuming at first the existence of an invariant measure with respect to  $\mathfrak{m}$ , the measure  $\nu$  may be trivial. Without further compactness assumptions on the metric space (see, e.g., [[36](#), Theorem 7.2.3]) one needs some Lyapunov like condition to ensure the existence of an invariant measure  $\nu$  of the weak-Feller family  $\mathfrak{m}$  (see, e.g., [[36](#), Theorem 7.2.4] or [[34](#), Theorem 12.3.3]). As we will see in [Corollary 6](#), another way to obtain the existence of invariant measures is having a positive lower bound on the coarse Ricci curvature of  $[\mathcal{H}, d, \mathfrak{m}]$ .

**5.3. Ollivier-Ricci curvature.** Let us discuss here the connexion between [Assumption 2](#) and the so-called coarse or Ollivier-Ricci curvature (ORC for short). The results can be found in [[45](#)] or [[46](#)]. A more recent presentation can be found in [[42](#)].

**Definition 10** (Ollivier-Ricci curvature [[45](#)]). Let  $[\mathcal{H}, d, \mathfrak{m}]$  be a m.r.w.s. Then, for any distinct points  $x, y \in \mathcal{H}$ , the ORC along  $(x, y)$  is defined as:

$$\kappa_{\mathfrak{m}}(x, y) = 1 - \frac{W_1(\mathfrak{m}_x, \mathfrak{m}_y)}{d(x, y)}, \quad (55)$$

The ORC of  $[\mathcal{H}, d, \mathfrak{m}]$  is defined as

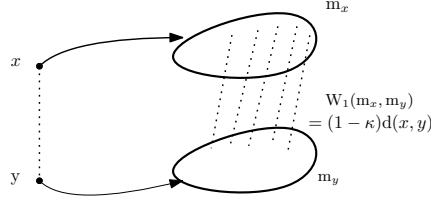
$$\kappa_{\mathfrak{m}} := \inf_{x \neq y} \kappa_{\mathfrak{m}}(x, y). \quad (56)$$

We clearly see from ([55](#)) that  $\kappa_{\mathfrak{m}}(x, y) \leq 1$ . Moreover, rearranging the terms, we have

$$W_1(\mathfrak{m}_x, \mathfrak{m}_y) = (1 - \kappa_{\mathfrak{m}}(x, y)) d(x, y). \quad (57)$$

Consequently, having some lower bound  $\kappa_{\mathfrak{m}}(x, y) \geq c \in \mathbb{R}$  for any  $x, y \in \mathcal{H}$  gives

$$W_1(\mathfrak{m}_x, \mathfrak{m}_y) \leq (1 - c)d(x, y), \quad (58)$$



**Figure 1:** Illustration of the ORC.

which describes a Lipschitz behavior of the random walk  $\mathbf{m}$ . This has to be compared to [Assumption 2](#). Indeed, we see from [Assumption 2](#) that, for  $x \neq y$

$$1 - \tau \leq \kappa_{\mathbf{m}}(x, y) \leq 1, \quad (59)$$

so according to the values of  $\tau$  we have different regimes on the ORC  $\kappa_{\mathbf{m}}$  (cf. [Table 1](#)).

**Table 1:** Relation between the values of  $\tau$  and  $\kappa_{\mathbf{m}}$ .

Values of $\tau$	0	$< 1$	$\leq 1$
Values of $\kappa_{\mathbf{m}}$	1	$(0, 1]$	$[0, 1]$

Notice that [Assumption 2](#) excludes both the cases  $\kappa_{\mathbf{m}} \equiv 1$  and  $\kappa_{\mathbf{m}} < 0$ . Typically,  $\tau = 0$ , would give that  $\kappa_{\mathbf{m}} = 1$  in other words  $\mathbb{W}_1(\mathbf{m}_x, \mathbf{m}_y) = 0$  for any  $x, y \in \mathcal{H}$ , i.e., the distribution  $\mathbf{m}$  is constant.

Moreover, it turns that there is equivalence between the lower bound on  $\kappa_{\mathbf{m}}$  in (59) and the Lipschitz behavior (58). This is directly related to a  $\mathbb{W}_1$ -contraction property cf. [[45](#), Proposition 20].

**Proposition 5.** *Let  $\mathbf{m}$  be a random walk on  $(\mathcal{H}, d)$  and assume that  $\mathbf{m}_x$  has finite moment for all  $x \in \mathcal{H}$ . Then*

$$\kappa_{\mathbf{m}}(x, y) \geq c \in \mathbb{R}, \quad \forall x \neq y \iff \mathbb{W}_1(\nu_1 \star \mathbf{m}, \nu_2 \star \mathbf{m}) \leq (1 - c)\mathbb{W}_1(\nu_1, \nu_2) \quad \forall \nu_1, \nu_2 \in \mathcal{P}_1(\Xi).$$

In view of [Proposition 5](#), taking  $\nu_1 = \delta_x$  and  $\nu_2 = \delta_y$  for  $x \neq y$ , and  $c = 1 - \tau$ , we get

$$\mathbb{W}_1(\mathbf{m}_x, \mathbf{m}_y) = \mathbb{W}_1(\delta_x \star \mathbf{m}, \delta_y \star \mathbf{m}) \leq \tau \mathbb{W}(\delta_x, \delta_y) = \tau d(x, y)$$

which is exactly [Assumption 2](#) since the above inequality is trivial for  $x = y$ .

In the case of positive curvature, this contraction result implies the existence of a unique invariant measure for the random walk  $\mathbf{m}$  when the ORC is positive.

**Corollary 5** ([[42](#)]). *Assume that  $\kappa_{\mathbf{m}}(x, y) \geq c > 0$  for all  $x \neq y$ . Then, the random walk  $\mathbf{m}$  has a unique invariant measure  $\nu \in \mathcal{P}_1(\Xi)$ . Moreover, for any  $\nu \in \mathcal{P}_1(\Xi)$*

- (1)  $\mathbb{W}_1(\nu \star \mathbf{m}^{*n}, \nu) \leq (1 - c)^n \mathbb{W}_1(\nu, \nu), \quad \forall n \in \mathbb{N}$ .
- (2)  $\mathbb{W}_1(\mathbf{m}^{*n}, \nu) \leq \frac{(1-c)^n}{c} \mathbb{W}_1(\delta_x, \nu), \quad \forall n \in \mathbb{N}, \forall x \in \mathcal{H}$ .

In our setting, the parameter regime  $\tau < 1$  would define a positive ORC. Consequently, we have the following

**Corollary 6.** *Assume that  $\mathbf{m}$  satisfies [Assumption 2](#) with  $\tau < 1$ . Then, there exists a unique invariant measure  $\nu \in \mathcal{P}_1(\Xi)$  with respect to  $\mathbf{m}$ .*

## 6. APPLICATION: INERTIAL PRIMAL-DUAL ALGORITHM

This section is devoted to the application of the developed results in [Section 3](#) and [Section 4](#) to the following class of optimization problems

$$\min_x f_{m_x}(x) + g(x) + h(Kx). \quad (60)$$

In what follows, we make the following assumptions.

- Assumption 5.** (a)  $f_{m_x}(x) = \mathbb{E}_{\xi \sim m_x} f(x, \xi)$ , with  $f(\cdot, \xi) \in C_{\beta_f}^{1,1}(\mathcal{H}) \cap \Gamma_0(\mathcal{H})$  and  $f(\cdot, \xi)$  is  $\mu$ -strongly convex for all  $\xi \in \Xi$ , and  $\xi \in \Xi \mapsto f(x, \xi)$  is measurable for any  $x \in \mathcal{H}$ ,  
 (b)  $h \in C_{\beta_h}^{1,1}(\mathcal{H}) \cap \Gamma_0(\mathcal{H})$ ,  $g \in \Gamma_0(\mathcal{H})$  and  $K \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ ,  
 (c)  $0 \in \text{sri}(K \text{ dom}(g) - \text{dom}(r))$ ,

where  $\mu, \beta_f, \beta_h$  are positive constants and  $\text{sri}(K \text{ dom}(g) - \text{dom}(r))$  is the strong relative interior of  $(K \text{ dom}(g) - \text{dom}(r))$  (see, e.g., [23, Definition 6.9]).

[Assumption 5\(a\)](#) is to be compared to [Assumption 1](#). In particular, it ensure that for any measure  $\nu \in \mathcal{P}(\Xi)$ ,  $f_\nu$  is differentiable with  $\nabla f_\nu(x) = \mathbb{E}_{\xi \sim \nu} \nabla f(x, \xi)$ , moreover, strong convexity of  $f(\cdot, \xi)$  for all  $\xi \in \Xi$  implies strong convexity of  $f_{m_x}$  for all  $x \in \mathcal{H}$ . While [Assumption 5\(b\)](#) and [\(c\)](#) are classical for strong duality as discussed below.

Thanks to Fenchel-Rockafellar duality (see, e.g., [23, Chapter 15]) the dual problem of [\(60\)](#) reads

$$\min_y f_{m_x}^* \square g^*(-K^*y) + h^*(y), \quad (61)$$

where  $f_{m_x}^* \square g^*$  is the infimal convolution of  $f_{m_x}^*$  and  $g^*$  given by

$$f_{m_x}^* \square g^*(z) = (f_{m_x} + g)^*(z) = \inf_{y \in \mathcal{H}} f_{m_x}^*(y) + g^*(z - y),$$

and  $K^*$  is the adjoint operator of  $K$ . Notice that the dual problem [\(61\)](#) is difficult to tackle directly since  $f_{m_x}^*$  cannot be computed explicitly due to the dependence of both the loss function  $f$  and the measure  $m_x$  on the state  $x$ .

Problems of the form [\(60\)](#) arise in many fields such as image and signal processing, machine learning and partial differential equations. In the deterministic case, i.e.,  $f$  does not depend on the distribution  $m$  and  $g \equiv 0$ , such problems were studied in [30]. Later on, extensions were addressed in several works (see, e.g., [31, 49, 53]). In [24], the authors studied a fully stochastic variant of [\(60\)](#), i.e.,  $f_{m_x}(x) = \mathbb{E}_{\xi \sim m} f(x, \xi)$ ,  $g(x) = \mathbb{E}_{\xi \sim m} g(x, \xi)$  and  $K(x) = \mathbb{E}_{\xi \sim m} K(x)$  for some suitable functions  $f, g$  and  $K$ . Yet, the distribution  $m$  does not depend on the state  $x$ . To the best of our knowledge composite problems of the form [\(60\)](#) with state-dependent distributions have not been addressed in the literature.

**6.1. Formulation as a monotone inclusion.** As seen in [Section 3](#), the appropriate notion of solutions of [\(60\)](#) is that of equilibria. Thus, our aim is to find an equilibrium point  $\bar{x}$ , i.e., a solution of the static problem

$$\bar{x} \in \arg \min_x f_{m_{\bar{x}}}(x) + g(x) + h(Kx). \quad (62)$$

Problems [\(62\)](#) and [\(61\)](#) can be written in an inf-sup form

$$\inf_{x \in \mathcal{H}} \sup_{y \in \mathcal{H}} f_{m_{\bar{x}}}(x) + g(x) + \langle Kx, y \rangle - h^*(y), \quad (63)$$

and  $(\bar{x}, \bar{y})$  is optimal for [\(63\)](#) if and only if the following optimality condition holds

$$\begin{cases} 0 \in K^* \bar{y} + \nabla f_{m_{\bar{x}}}(\bar{x}) + \partial g(\bar{x}) \\ 0 \in -K \bar{x} + \partial h^*(\bar{y}). \end{cases} \quad (64)$$

Notice that the problem (63) is a saddle point problem

$$\inf_{x \in \mathcal{H}} \sup_{y \in \mathcal{H}} \mathcal{L}_{m_{\bar{x}}}(x, y), \quad (65)$$

of the Lagrangian functional:  $\mathcal{L}_{m_{\bar{x}}} : \text{dom}(\partial g) \times \text{dom}(\partial h^*) \rightarrow \mathbb{R}$  defined by

$$\mathcal{L}_{m_{\bar{x}}}(x, y) = f_{m_{\bar{x}}}(x) + g(x) + \langle Kx, y \rangle - h^*(y).$$

For any pair  $(x_0, y_0) \in \text{dom}(\partial g) \times \text{dom}(\partial h^*)$  we define the following restrictions of the Lagrangian functional

$$\mathcal{L}_{m_{\bar{x}}}^{y_0}(x) = \begin{cases} \mathcal{L}_{m_{\bar{x}}}(x, y_0) & \text{if } x \in \text{dom}(\partial g), \\ \infty & \text{otherwise} \end{cases}, \quad \mathcal{L}_{m_{\bar{x}}}^{x_0}(x_0, y) = \begin{cases} \mathcal{L}_{m_{\bar{x}}}(x_0, y) & \text{if } y \in \text{dom}(\partial h^*) \\ -\infty & \text{otherwise} \end{cases}.$$

Then, the optimality system (64) can be written as a monotone inclusion

$$0_{\mathcal{H} \times \mathcal{H}} \in \mathcal{T}_{m_{\bar{x}}}(\bar{x}, \bar{y}), \quad (66)$$

where

$$\mathcal{T}_{m_{\bar{x}}}(x, y) = (\partial \mathcal{L}_{m_{\bar{x}}}^y(x), -\partial \mathcal{L}_{m_{\bar{x}}}^x(y)). \quad (67)$$

Notice that  $\mathcal{T}_{m_{\bar{x}}}$  can be split into the sum of two operators:  $\mathcal{T}_{m_{\bar{x}}} = A + B_{m_{\bar{x}}}$  where

$$A : (x, y) \mapsto (\partial g(x) + K^*(y), \partial h^*(y) - K(x)), \quad B_{m_{\bar{x}}} : (x, y) \mapsto (\nabla f_{m_{\bar{x}}}(x), 0). \quad (68)$$

*Remark 6.1.* Similar problems to (65) were addressed in [54] where the authors studied problems of the form

$$\min_{x \in X} \max_{y \in Y} \mathbb{E}_{\xi \sim m(\bar{x}, \bar{y})} \phi(x, y, \xi), \quad (69)$$

where  $X, Y$  are compact sets and  $\phi$  is a convex-concave function that plays the role of the Lagrangian in our case. Such problems fall into the scope of (65). Indeed, changing the Lagrange functional in (65) to  $\mathcal{L}_{m(\bar{x}, \bar{y})}(x, y) + \delta_X(x) - \delta_Y(y)$ , where  $\delta_X$  and  $\delta_Y$  are the indicator functions of the sets  $X$  and  $Y$  respectively, and considering a distribution  $m$  depending on both the primal and dual variables  $(x, y)$  allows recovering (69).

First, let us recapitulate some facts in the following lemma.

**Lemma 7.** *Under Assumption 5(a)(b)(c), the following properties hold*

- (i) *The operator  $A$  is maximally monotone,*
- (ii) *The operator  $B_{m_{\bar{x}}}(\cdot, \cdot)$  is maximally monotone and  $\beta_f$ -Lipschitz continuous for any  $x \in \mathcal{H}$ ,*
- (iii) *For any  $y, z \in \mathcal{H}$*

$$\sup_{(x', y') \in \mathcal{H} \times \mathcal{H}} \|B_{m_y}(x', y') - B_{m_z}(x', y')\| \leq \beta_f \tau \|y - z\|.$$

- (iv) *The operator  $\mathcal{T}_{m_{\bar{x}}}$  is maximally monotone and  $\tilde{\mu}$ -strongly monotone with  $\tilde{\mu} = \min(\mu, \beta_h^{-1})$ .*

*Proof.* First, we see that  $A$  is the sum of the maximally monotone operator  $(x, y) \mapsto (\partial g(x), \partial h^*(y))$  (see, e.g., [23, Theorem 21.2 and Proposition 20.23]) and a skew-symmetric operator  $(x, y) \mapsto (K^*(y), -K(x))$ , which is maximally monotone (see, e.g., [23, Example 20.35]), this proves (i). The operator  $B_{m_{\bar{x}}}$  is the gradient of the function  $\Phi(z) = f_{m_{\bar{x}}}(z)$  and thus is maximally monotone. Now take  $(x, y), (x', y') \in \mathcal{H}$ . We have

$$\begin{aligned} \|B_{m_x}(x, y) - B_{m_{x'}}(x', y')\|_{\mathcal{H} \times \mathcal{H}} &= \left\| \left( \nabla f_{m_x}(x) - \nabla f_{m_{x'}}(x'), 0 \right) \right\|_{\mathcal{H} \times \mathcal{H}} \\ &= \|\nabla f_{m_x}(x) - \nabla f_{m_{x'}}(x')\| \\ &\leq \beta \|x - x'\| \leq \beta \|(x, y) - (x', y')\|_{\mathcal{H} \times \mathcal{H}}, \end{aligned}$$

this proves (ii). The proof of (iii) can be directly deduced from [Corollary 1](#). Indeed, let us fix  $y, z \in \mathcal{H}$ , we then have, for any  $(x', y') \in \mathcal{H} \times \mathcal{H}$

$$\begin{aligned} \|B_{m_y}(x', y') - B_{m_z}(x', y')\|_{\mathcal{H} \times \mathcal{H}} &= \left\| \left( 0, \nabla f_{m_y}(x') - \nabla f_{m_z}(x') \right) \right\|_{\mathcal{H} \times \mathcal{H}} \\ &= \|\nabla f_{m_y}(x') - \nabla f_{m_z}(x')\| \\ &\leq \beta_f \mathbf{W}(m_y, m_z) \leq \tau \beta_f \|y - z\|, \end{aligned}$$

where the last inequality follows from [Assumption 2](#) and [Assumption 5](#). As for (iv), we see that  $A$  is maximally monotone, and by (iii),  $B_{m_x}$  is Lipschitz continuous, we conclude again using [\[27, Lemma 2.4\]](#). Now take  $(x, y), (x', y') \in \mathcal{H} \times \mathcal{H}$  and  $(u, v) \in \text{gra } \mathcal{T}_{m_x}(x, y) \times \text{gra } \mathcal{T}_{m_x}(x', y')$  with  $u = (u_1, u_2), v = (v_1, v_2)$ . Since by [Assumption 5\(a\)](#)  $\nabla f_{m_x}$  is  $\mu$ -strongly monotone and by [Assumption 5\(b\)](#)  $\partial h^*$  is  $\beta_h^{-1}$ -strongly monotone (see, e.g., [\[23, Theorem 18.15\]](#)), we get

$$\begin{aligned} \langle u - v, (x, y) - (x', y') \rangle &= \langle \nabla f_{m_x}(x) - \nabla f_{m_x}(x'), x - x' \rangle + \langle u_1 - v_1, y - y' \rangle + \langle u_2 - v_2, x - x' \rangle \\ &\quad + \langle \mathbf{K}(x' - x), y - y' \rangle + \langle \mathbf{K}^*(y - y'), x - x' \rangle \\ &\geq \mu \|x - x'\|^2 + \beta_h^{-1} \|y - y'\|^2 \\ &\geq \min(\mu, \beta_h^{-1}) \|z - z'\|_{\mathcal{H} \times \mathcal{H}}^2, \end{aligned} \tag{70}$$

which proves  $\min(\mu, \beta_h^{-1})$ -strong monotonicity of  $\mathcal{T}_{m_x}$ .  $\blacksquare$

**6.2. Existence and uniqueness of equilibrium point.** We are now in a position to prove the existence and uniqueness of an equilibrium to the problem [\(63\)](#)

**Theorem 4** (Existence and uniqueness of equilibrium point). *Under [Assumption 2](#) and [Assumption 5](#), the map*

$$S : (x, y) \in \mathcal{H} \mapsto \text{zer}(\mathcal{T}_{m_x}) = \{(u, v) \in \mathcal{H} : (0, 0)_{\mathcal{H} \times \mathcal{H}} \in \mathcal{T}_{m_x}(u, v)\}$$

is  $\tilde{\rho}$ -Lipschitz with

$$\tilde{\rho} = \frac{\tau \beta_f}{\tilde{\mu}},$$

and  $\tilde{\mu} = \min(\mu, \beta_h^{-1})$ . In particular, if  $\tilde{\rho} < 1$ , the equilibrium  $(\bar{x}, \bar{y})$  is unique.

*Proof.* Following the same lines as in [Theorem 1](#) and using [Lemma 7\(i\)](#), we conclude.  $\blacksquare$

*Remark 6.2.* [Theorem 4](#) is to be compared to [\[54, Theorem 2.6\]](#). Notice that the monotone inclusion [\(66\)](#) defining the equilibrium in [\[54\]](#) becomes a variational inequality due to the presence of constraints on  $x$  and  $y$  ([\[50, Example 12.50\]](#) for details).

Following the main steps of [Sections-3-4](#), let us define the following gap function

$$\mathbf{E}_{\bar{x}}(x, y) = B_{m_x}(x, y) - B_{m_{\bar{x}}}(x, y). \tag{71}$$

Mimicking the proof of [Lemma 5](#) and using [Lemma 7](#), we prove a similar result

**Lemma 8.** *Under [Assumption 2](#) and [Assumption 5](#),  $\mathbf{E}_{\bar{x}}(\cdot)$  is  $\beta(2 + \tau)$ -Lipschitz continuous.*

**6.3. Related first and second-order dynamics.**

6.3.1. *First ordre system.* To simplify the presentation, we set again  $Z(t) = (x(t), y(t))$ . Then, given an initial data  $Z(t_0) = (x(t_0), y(t_0)) \in \text{dom}(\partial g) \times \text{dom}(\partial h^*)$ , we consider the following first-order system associated to the monotone inclusion (66):

$$\dot{Z}(t) + \mathcal{T}_{m_x}(Z(t)) + \mathbf{E}_x(Z(t)) \ni 0_{\mathcal{H} \times \mathcal{H}} \quad (72)$$

Or more explicitly

$$\begin{cases} \dot{x}(t) + K^*y(t) + \nabla f_{m_x}(x(t)) + \partial g(x(t)) + \mathbf{e}_x(x(t)) \ni 0 \\ \dot{y}(t) - Kx(t) + \partial h^*(y(t)) \ni 0 \end{cases} \quad (\text{SPDS})$$

with  $\mathbf{e}_x(x(t)) = \nabla f_{m_x}(x(t)) - \nabla f_{m_x}(x(t))$ . Arguing as in Proposition 1 and Theorem 2 we have the following result.

**Proposition 6.** *Assume that Assumption 2 and Assumption 5 hold. Then, for any initial data  $Z(t_0) = (x(t_0), y(t_0)) \in \text{dom}(\partial g) \times \text{dom}(\partial h^*)$ , (SPDS) admits a unique global solution  $Z : t \in [t_0, \infty) \mapsto (x(t), y(t))$ . Moreover,*

$$\|(x(t), y(t)) - (\bar{x}, \bar{y})\| \leq Ce^{-2\tilde{\mu}(1-\tilde{\rho})t}, \quad \forall t \geq t_0,$$

with  $C = \|(x(t_0), y(t_0)) - (\bar{x}, \bar{y})\|e^{2\tilde{\mu}(1-\tilde{\rho})t_0}$ ,  $\tilde{\rho} = \frac{\tau\beta_t}{\tilde{\mu}}$  and  $\tilde{\mu} = \min(\mu, \beta_h^{-1})$ .

6.3.2. *second-order system.* To simplify the presentation, we assume that the following assumptions hold

- Assumption 5\*.**
- (a)  $f_{m_x}(x) = \mathbb{E}_{\xi \sim m_x} f(x, \xi)$ , with  $f(\cdot, \xi) \in C^{1,1}(\mathcal{H}) \cap \Gamma_0(\mathcal{H})$  and  $f(\cdot, \xi)$  is  $\mu$ -strongly convex for all  $\xi \in \Xi$ , and  $\xi \in \Xi \mapsto f(x, \xi)$  is measurable for any  $x \in \mathcal{H}$ ,
  - (b)  $h \in C^{1,1}(\mathcal{H}) \cap \Gamma_0(\mathcal{H})$  is a strictly convex function,  $g \in C^1(\mathcal{H})$  and  $K \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ ,
  - (c)  $0 \in \text{sri}(K \text{dom}(g) - \text{dom}(r))$ .

The main difference between Assumption 5 and Assumption 5\* is strict convexity of  $h$  which ensures differentiability of  $h^*$  (see, e.g., [23, Corollary 18.12] or [37, Theorem 4.1.1]). In this case we have

$$\mathcal{T}_{m_x}(x, y) = (\nabla_x \mathcal{L}_{m_x}(x, y), -\nabla_y \mathcal{L}_{m_x}(x, y)). \quad (73)$$

By analogy with (ISEHD $_{m,2\sqrt{\tilde{\mu}}}$ ), we propose the following inertial system associated to (66)

$$\begin{cases} \ddot{x}(t) + 2\sqrt{\tilde{\mu}}\dot{x}(t) + \nabla_x \mathcal{L}_m(x(t), y(t)) + \mathbf{e}_x(x(t)) = 0 \\ \ddot{y}(t) + 2\sqrt{\tilde{\mu}}\dot{y}(t) - \nabla_y \mathcal{L}_m(x(t), y(t)) = 0 \end{cases} \quad (\text{ISPDS}_{m,2\sqrt{\tilde{\mu}}})$$

Systems of the form (ISPDS $_{m,2\sqrt{\tilde{\mu}}}$ ) can be found for instance in [8] or [26] in an unperturbed form, i.e., without the operator  $\mathbf{e}_x$ . To lighten the presentation, we discuss (ISPDS $_{m,2\sqrt{\tilde{\mu}}}$ ) without a Hessian damping term. And for the convenience of the reader, we present the main ingredients of the Lyapunov analysis as in (3). Notice that the existence of a global strong solution to (ISPDS $_{m,2\sqrt{\tilde{\mu}}}$ ) can be obtained as in Section 4. One can also see [8, Section 4].

Let us define the following function  $\mathcal{V} : [t_0, \infty[ \rightarrow \mathbb{R}^+$  by

$$\mathcal{V}(t) := \mathcal{L}_{m_x}(x(t), y(t)) - \mathcal{L}_{m_x}^* + \frac{1}{2}\|v(t)\|^2, \quad \text{where } v(t) := \sqrt{\tilde{\mu}}\left((x(t), y(t)) - (\bar{x}, \bar{y})\right) + (\dot{x}(t), \dot{y}(t)),$$

where  $(\bar{x}, \bar{y})$  is the equilibrium of (63) (cf. Theorem 4) and  $\mathcal{L}_{m_x}^*(\bar{x}, \bar{y}) := \mathcal{L}_{m_x}^*$ .

**Theorem 5.** *Assume that Assumption 5\* holds and let  $t \in [t_0, \infty[ \mapsto (x(t), y(t))$  be the solution of (ISPDS $_{m,2\sqrt{\tilde{\mu}}}$ ). Suppose that  $\frac{\beta_t \tau}{\tilde{\mu}} < \frac{\sqrt{2}}{4}$ . We then have:*



- for all  $t \geq t_0$

$$\mathcal{V}(t) \leq C_1 e^{-\frac{\sqrt{\mu}}{4}t}, \text{ where } C_1 = \mathcal{V}(t_0) e^{-\frac{\sqrt{\mu}}{4}t_0}.$$

In particular

$$0 \leq \mathcal{L}_{\mathbf{m}_{\bar{x}}}(x(t), y(t)) - \mathcal{L}_{\mathbf{m}_{\bar{x}}}^* \leq C e^{-\frac{\sqrt{\mu}}{4}t}. \quad (74)$$

- There exists  $C_2 > 0$  such that,

$$\|(x(t) - \bar{x}, y(t) - \bar{y})\| \leq C_2 e^{-\frac{\sqrt{\mu}}{4}t}, \quad \forall t \geq t_0.$$

*Proof.* The proof is similar to the one of [Theorem 3](#). At first, we have, using [\(ISPDS<sub>m,2</sub>√μ\)](#)

$$\dot{v}(t) = \left( \sqrt{\tilde{\mu}}\dot{x}(t) + \ddot{x}(t), \sqrt{\tilde{\mu}}\dot{y}(t) + \ddot{y}(t) \right) = \left( -\sqrt{\tilde{\mu}}\dot{x}(t) - \nabla_x \mathcal{L}_{\mathbf{m}_{\bar{x}}}(x(t), y(t)) - \mathbf{e}_{\bar{x}}(x(t)), -\sqrt{\tilde{\mu}}\dot{y}(t) - \nabla_y \mathcal{L}_{\mathbf{m}_{\bar{x}}}(x(t), y(t)) \right).$$

This gives

$$\begin{aligned} \dot{\mathcal{V}}(t) &= \langle \nabla_x \mathcal{L}_{\mathbf{m}_{\bar{x}}}(x(t), y(t)), \dot{x}(t) \rangle + \langle \nabla_y \mathcal{L}_{\mathbf{m}_{\bar{x}}}(x(t), y(t)), \dot{y}(t) \rangle \\ &\quad + \langle \sqrt{\tilde{\mu}}(x(t) - \bar{x}) + \dot{x}(t), -\sqrt{\tilde{\mu}}\dot{x}(t) - \nabla_x \mathcal{L}_{\mathbf{m}_{\bar{x}}}(x(t), y(t)) - \mathbf{e}_{\bar{x}}(x(t)) \rangle \\ &\quad + \langle \sqrt{\tilde{\mu}}(y(t) - \bar{y}) + \dot{y}(t), -\sqrt{\tilde{\mu}}\dot{y}(t) - \nabla_y \mathcal{L}_{\mathbf{m}_{\bar{x}}}(x(t), y(t)) \rangle \end{aligned} \quad (75)$$

We get after some simplifications

$$\begin{aligned} \dot{\mathcal{V}}(t) + \tilde{\mu} \langle \dot{x}(t), x(t) - \bar{x} \rangle + \tilde{\mu} \langle \dot{y}(t), y(t) - \bar{y} \rangle + \sqrt{\tilde{\mu}} \|\dot{x}(t)\|^2 \\ + \sqrt{\tilde{\mu}} \|\dot{y}(t)\|^2 + \sqrt{\tilde{\mu}} \langle \nabla_x \mathcal{L}_{\mathbf{m}_{\bar{x}}}(x(t), y(t)), x(t) - \bar{x} \rangle + \sqrt{\tilde{\mu}} \langle \nabla_y \mathcal{L}_{\mathbf{m}_{\bar{x}}}(x(t), y(t)), y(t) - \bar{y} \rangle = -\langle v(t), \mathbf{E}_{\bar{x}}(x(t)) \rangle. \end{aligned} \quad (76)$$

That is

$$\begin{aligned} \dot{\mathcal{V}}(t) + \tilde{\mu} \left\langle (\dot{x}(t), \dot{y}(t)), (x(t) - \bar{x}, y(t) - \bar{y}) \right\rangle + \sqrt{\tilde{\mu}} \|\dot{x}(t), \dot{y}(t)\|^2 \\ + \sqrt{\tilde{\mu}} \langle \mathcal{T}_{\mathbf{m}_{\bar{x}}}(x(t), y(t)), (x(t) - \bar{x}, y(t) - \bar{y}) \rangle = -\langle v(t), \mathbf{E}_{\bar{x}}(x(t)) \rangle. \end{aligned} \quad (77)$$

Using  $\tilde{\mu}$ -strong monotonicity of  $\mathcal{T}_{\mathbf{m}_{\bar{x}}}$ , and the fact that  $0_{\mathcal{H} \times \mathcal{H}} \in \mathcal{T}_{\mathbf{m}_{\bar{x}}}(\bar{x}, \bar{y})$  we have

$$\langle \mathcal{T}_{\mathbf{m}_{\bar{x}}}(x(t), y(t)), (x(t) - \bar{x}, y(t) - \bar{y}) \rangle \geq \mathcal{L}_{\mathbf{m}_{\bar{x}}}(x(t), y(t)) - \mathcal{L}_{\mathbf{m}_{\bar{x}}}^* + \frac{\tilde{\mu}}{2} \|(x(t) - \bar{x}, y(t) - \bar{y})\|^2, \quad (78)$$

and using this in [\(77\)](#), we get

$$\dot{\mathcal{V}}(t) + \sqrt{\tilde{\mu}} \Theta(t) \leq \|v(t)\| \|\mathbf{E}_{\bar{x}}(x(t))\|, \quad (79)$$

where

$$\begin{aligned} \Theta(t) &:= \tilde{\mu} \left\langle (\dot{x}(t), \dot{y}(t)), (x(t) - \bar{x}, y(t) - \bar{y}) \right\rangle + \mathcal{L}_{\mathbf{m}_{\bar{x}}}(x(t), y(t)) - \mathcal{L}_{\mathbf{m}_{\bar{x}}}^* + \frac{\tilde{\mu}}{2} \|(x(t) - \bar{x}, y(t) - \bar{y})\|^2 \\ &= \mathcal{V}(t) + \sqrt{\tilde{\mu}} \|\dot{x}(t), \dot{y}(t)\|^2. \end{aligned} \quad (80)$$

Thus, [\(79\)](#) becomes

$$\dot{\mathcal{V}}(t) + \sqrt{\tilde{\mu}} \mathcal{V}(t) + \frac{\sqrt{\tilde{\mu}}}{2} \|\dot{x}(t), \dot{y}(t)\|^2 \leq \|v(t)\| \|\mathbf{E}_{\bar{x}}(x(t))\|, \quad (81)$$

Again, using  $\tilde{\mu}$ -strong monotonicity of  $\mathcal{T}_{\mathbf{m}_{\bar{x}}}$  we get

$$\dot{\mathcal{V}}(t) + \sqrt{\tilde{\mu}} \mathcal{V}(t) + \frac{\sqrt{\tilde{\mu}}}{2} \|\dot{x}(t), \dot{y}(t)\|^2 + \frac{\tilde{\mu}^{3/2}}{2} \|(x(t) - \bar{x}, y(t) - \bar{y})\|^2 \leq \|v(t)\| \|\mathbf{E}_{\bar{x}}(x(t))\|. \quad (82)$$

We have, using Young's inequality

$$\|v(t)\| \|\mathbf{E}_{\bar{x}}(x(t))\| \leq \frac{\sqrt{\tilde{\mu}}}{8} \|v(t)\|^2 + \frac{2}{\sqrt{\tilde{\mu}}} \|\mathbf{E}_{\bar{x}}(x(t))\|^2 \leq \frac{\sqrt{\tilde{\mu}}}{4} \mathcal{V}(t) + \frac{2\beta_f^2 \tau^2}{\sqrt{\tilde{\mu}}} \|(x(t) - \bar{x}, y(t) - \bar{y})\|^2.$$

Using [Lemma 7-\(iii\)](#), we get after rearranging the terms

$$\dot{\mathcal{V}}(t) + \frac{\sqrt{\tilde{\mu}}}{2}\mathcal{V}(t) + \frac{\sqrt{\tilde{\mu}}}{4}\|\dot{x}(t), \dot{y}(t)\|^2 + \sqrt{\tilde{\mu}}\left(\frac{\tilde{\mu}}{4} - \frac{2\beta_f^2\tau^2}{\tilde{\mu}}\right)\|(x(t) - \bar{x}, y(t) - \bar{y})\|^2 \leq 0.$$

Since  $\frac{\beta_f\tau}{\tilde{\mu}} < \frac{\sqrt{2}}{4}$ , we have

$$\dot{\mathcal{V}}(t) + \frac{\sqrt{\tilde{\mu}}}{2}\mathcal{V}(t) \leq 0,$$

which gives after integration

$$\mathcal{V}(t) \leq \mathcal{V}(t_0)e^{-\frac{\sqrt{\tilde{\mu}}}{4}(t-t_0)}, \quad \forall t \geq t_0. \quad (83)$$

By definition of  $\mathcal{V}$ , we have

$$\mathcal{L}_{m_{\bar{x}}}(x(t), y(t)) - \mathcal{L}_{m_{\bar{x}}}^* \leq \mathcal{V}(t_0)e^{-\frac{\sqrt{\tilde{\mu}}}{4}(t-t_0)}. \quad (84)$$

Coming back to [\(83\)](#), we have

$$\|v(t)\|^2 = \|\dot{x}(t), \dot{y}(t)\|^2 + 2\sqrt{\tilde{\mu}}\langle \dot{x}(t), \dot{y}(t), (x(t) - \bar{x}, y(t) - \bar{y}) \rangle + \tilde{\mu}\|(x(t) - \bar{x}, y(t) - \bar{y})\|^2 \leq C_1e^{-\frac{\sqrt{\tilde{\mu}}}{4}t}, \quad (85)$$

with  $C_1 = 2e^{\frac{\sqrt{\tilde{\mu}}}{4}t_0}\mathcal{V}(t_0)$ . Setting  $U(t) = \sqrt{\tilde{\mu}}\|(x(t) - \bar{x}, y(t) - \bar{y})\|^2$ , [\(85\)](#) gives

$$\dot{U}(t) + \sqrt{\tilde{\mu}}U(t) \leq C_1e^{-\frac{\sqrt{\tilde{\mu}}}{4}t},$$

which implies after integrating between  $t_0$  and  $t$  that

$$\|(x(t) - \bar{x}, y(t) - \bar{y})\| \leq C_2e^{-\frac{\sqrt{\tilde{\mu}}}{4}t},$$

where the constant  $C_2$  depends only on  $C_1$  and  $U(t_0) = \|(x(t_0) - \bar{x}, y(t_0) - \bar{y})\|$ . This finishes the proof.  $\blacksquare$

## 7. COMMENTS, EXTENSIONS AND FUTURE WORK

In this paper, we adopted a dynamical system approach to study some stochastic optimization problems with state-dependent distributions. We investigated the existence and uniqueness of equilibrium points, well-posedness as well as convergence properties of the trajectories, for both first and second-order dynamics. We highlighted some dynamical and geometrical properties of the state-dependent distributions suggesting that the natural framework to study problems of the form [\(5\)](#) is the one of *metric random walk spaces*. More particularly, the notion of coarse Ricci curvature gives a new insight on the geometrical hidden structure of this kind of problems. Finally, we discussed as an application the inertial primal-dual algorithm. We present here some ongoing works, possible extensions as well as some open problems.

**7.1. Inertial algorithms.** Relying on the discretization of the dynamics studied in [Section 4](#) and [Section 6](#), more specifically,  $(\text{ISEHD}_{m,2\sqrt{\tilde{\mu}}})$  and  $(\text{ISPDS}_{m,2\sqrt{\tilde{\mu}}})$ , we obtain new inertial algorithms with Hessian-driven damping for stochastic optimization problems with decision-dependent distributions. These algorithms exhibit rapid convergence properties and can also be adapted to the nonsmooth case. This is being addressed in ongoing work.

**7.2. Implicit Hessian damping.** We focused in [Section 4](#) on the explicit Hessian damping in the smooth case. Yet, it is possible to consider implicit damping as in [\[9, 14\]](#)

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla G_{\mathfrak{m}_{\bar{x}}}(x(t) + \omega\dot{x}(t)) + \mathbf{e}_{\bar{x}}(x(t)) = 0, \quad (\text{ISIHD}_{\mathfrak{m},\gamma})$$

The dynamics  $(\text{ISIHD}_{\mathfrak{m},\gamma})$  is referred to as an Inertial System with Implicit Hessian damping, since one can observe, using Taylor expansion

$$\nabla G_{\mathfrak{m}_{\bar{x}}}(x(t) + \omega\dot{x}(t)) \approx \nabla G_{\mathfrak{m}_{\bar{x}}}x(t) + \nabla^2 G_{\mathfrak{m}_{\bar{x}}}(x(t))\dot{x}(t).$$

As it was observed in [\[14\]](#), higher-order moments of the perturbation  $\mathbf{e}_{\bar{x}}$  are required to get fast convergence guarantees in the implicit case compared to the explicit one. Since in our analysis (see [Theorem 3](#)) no integrability assumption on  $\mathbf{e}_{\bar{x}}$  is needed, it is interesting to investigate the effect of implicit Hessian damping, both in the smooth and nonsmooth cases.

**7.3. Tikhonov-regularization.** In [Section 4](#) we restricted ourselves, for sake of simplicity, the analysis to the case where the operator  $F_{\mathfrak{m}_{\bar{x}}}$  is smooth. However, it is possible to consider second-order dynamics for general (and possibly nonpotential) operators, by considering, for  $\lambda > 0$  the following dynamic

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + F_{\mathfrak{m}_{\bar{x}(t)}}^{\lambda(t)}(x(t)) + \omega \frac{d}{dt} \left( F_{\mathfrak{m}_{\bar{x}(t)}}^{\lambda(t)}(x(t)) \right) + \mathbf{e}_{\bar{x}}(x(t)) = 0. \quad (\text{ISEHD}_{\mathfrak{m},\lambda,\gamma})$$

where  $F_{\mathfrak{m}_{\bar{x}}}^{\lambda}$  is the so-called Yosida approximation of  $F_{\mathfrak{m}_{\bar{x}}}$  defined by  $F_{\mathfrak{m}_{\bar{x}}}^{\lambda} = \frac{1}{\lambda}(\text{id} - J_{\lambda F_{\mathfrak{m}_{\bar{x}}}})$  and  $J_{\lambda F_{\mathfrak{m}_{\bar{x}}}} = (\text{id} + JF_{\mathfrak{m}_{\bar{x}}})^{-1}$  is the resolvent of  $F_{\mathfrak{m}_{\bar{x}}}$ . This approach comes with several advantages. First, the Yosida approximation is single valued so that the monotone inclusion ([p-ISMI](#)) reduces to the classical differential equation  $(\text{ISEHD}_{\mathfrak{m},\lambda,\gamma})$ . In addition one can exploit the  $\lambda$ -cocoercivity of  $F_{\mathfrak{m}_{\bar{x}}}^{\lambda}$  and the fact that  $\text{zer } F_{\mathfrak{m}_{\bar{x}}} = \text{zer } F_{\mathfrak{m}_{\bar{x}}}^{\lambda}$ . The approach was used in [\[17\]](#) for  $\omega = 0$  and in the recent work [\[12\]](#) for Newton-like dynamics. We are exploring the adaption of this techniques to stochastic monotone inclusions with state dependent distribution in an ongoing work.

**7.4. Weaker Assumptions.** We have seen that one of the crucial assumptions in the analysis is [Assumption 2](#), which concerns the Lipschitz behavior of the distribution  $(\mathfrak{m}_x)_x$ . A natural question that arises is what happens under a weaker assumption. For example, when  $x \mapsto \mathfrak{m}_x$  is Hölder continuous. We are not aware of any existing results in this direction. We plan to investigate this in future work.

#### ACKNOWLEDGEMENT

The work of H.E was supported by the ANR grant, reference: ANR-20-CE92-0037.

#### A. GRONWALL INEQUALITIES

In this section we list several auxiliary results that we make use of in the paper.

**Lemma 9** (Gronwall's lemma: differential form). *Let  $u, v$  be two  $C^0$  (resp.  $C^1$ ) nonnegative function on  $[0, T]$  and let  $w$  be a continuous function on  $[0, T]$ . We assume that*

$$\frac{1}{2} \frac{d}{dt} u^2(t) \leq w(t)u^2(t) + u(t)v(t) \text{ on } (0, T), \quad (86)$$

then, for any  $t \in [0, T]$

$$u(t) \leq u(0)e^{K(t)} + \int_0^t v(s)e^{K(t)-K(s)} ds, \quad (87)$$

where  $K(t) = \int_0^t w(s) ds$ .

**Lemma 10.** [27, Lemma A.5] Let  $v \in L^1(t_0, T; \mathbb{R}^+)$  and  $u \in C^0(t_0, T)$  such that

$$\frac{1}{2}u^2(t) \leq \frac{1}{2}c^2 + \int_{t_0}^t u(s)v(s)ds,$$

for some  $c \geq 0$  for all  $t \in [t_0, T]$ . Then

$$|u(t)| \leq c + \int_{t_0}^t v(s)ds.$$

## B. BANACH FIXED POINT THEOREM & PICARD ITERATIVE METHOD

**Theorem 6** (see, e.g., [6, 28]). Let  $(\mathcal{X}, d)$  be a complete metric space and  $S : \mathcal{X} \rightarrow \mathcal{X}$  be a strict contraction, i.e., there exists a constant  $\rho < 1$  such that

$$d(S(x), S(y)) \leq \rho d(x, y), \forall x, y \in \mathcal{X}.$$

Then, there exists a unique  $\bar{x} \in \mathcal{X}$  such that  $S(\bar{x}) = \bar{x}$ . Moreover, for any  $x_0 \in \mathcal{X}$ , the sequence starting from  $x_0$  with  $x_{n+1} = S(x_n)$  for all  $n \in \mathbb{N}$  converges to  $\bar{x}$  as  $n$  goes to  $\infty$ .

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