



HAL
open science

Quaternions and rotations: applications to Minkowski's four-vectors, electromagnetic waves and polarization optics

Pierre Pellat-Finet

► **To cite this version:**

Pierre Pellat-Finet. Quaternions and rotations: applications to Minkowski's four-vectors, electromagnetic waves and polarization optics. 2024. hal-04650733

HAL Id: hal-04650733

<https://hal.science/hal-04650733v1>

Preprint submitted on 17 Jul 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

Quaternions and rotations: applications to Minkowski's four-vectors, electromagnetic waves and polarization optics

Pierre Pellat-Finet

*Université Bretagne Sud, UMR CNRS 6205, LMBA, F-56000 Vannes, France
pierre.pellat-finet@univ-ubs.fr*

Abstract. Rotations on the 3-dimensional Euclidean vector-space can be represented by real quaternions, as was shown by Hamilton. Introducing complex quaternions allows us to extend the result to elliptic and hyperbolic rotations on the Minkowski space, that is, to proper Lorentz rotations. Another generalization deals with complex rotations on the complex-quaternion algebra. Appropriate quaternionic expressions of differential operators lead to a quaternionic form of Maxwell equations; the quaternionic expressions of an electromagnetic-field in two Galilean frames in relative motion are linked by a complex rotation; some relativistic invariants of electromagnetic waves are deduced, including their polarization states and their degrees of polarization. Quaternionic forms of proper Lorentz rotations are applied to polarization optics along with illustrative examples. Equivalences between relativistic effects and crystal optics effects are mentioned. A derivation of the triangle inequality on the Minkowski space is given; the inequality is illustrated by some properties of partially polarized lightwaves.

Keywords. Complex quaternions, complex rotations, electromagnetic waves, Hamilton's quaternions, Minkowski space, polarization optics, proper Lorentz rotations, singular Lorentz transformations, special theory of relativity, Thomas precession, Wigner rotations.

Content

1. Introduction	2
2. Quaternions, rotations and Minkowski space	3
3. Electromagnetism	26
4. Electromagnetic waves	29
5. Quaternionic representation of polarization optics	37
6. Some equivalences between special theory of relativity and polarization optics ...	55
7. Triangle inequality and applications	63
8. Conclusion	68
Appendix A. Quaternionic trigonometry	68
Appendix B. Proof of $V = e_u \sin(\psi/2)$ (Sect. 2.5.8)	71
Appendix C. Mappings $q \mapsto u q u^*$ are rotations on \mathbb{H}_c	72
Appendix D. Two lemmas on unit pure quaternions	74
Appendix E. Regular rotations: eigenquaternions and eigenvalues	75
Appendix F. Singular Lorentz rotations: eigenminquats	76
Appendix G. Equivalent birefringence and optical activity	77
References	79

Notations

\mathbb{R}	Field of real numbers
\mathbb{C}	Field of complex numbers
\mathbb{E}_3	3-dimensional Euclidean vector-space
\mathbb{H}	Algebra of real quaternions (Hamilton's quaternions)
\mathbb{H}_c	Algebra of complex quaternions (Hamilton's biquaternions)
(\mathbb{R}^4, Q) $(\mathbb{R} \oplus \mathbb{E}_3, Q)$	Minkowski space (Q denotes the Lorentz quadratic form, or Lorentz metric)
\mathbb{M}	
i	Complex number such that $i^2 = -1$
$\{e_0, e_1, e_2, e_3\}$	Canonic basis of \mathbb{H}_c (also canonic basis of \mathbb{H})
e_n	Real unit pure quaternion ($e_n^2 = -e_0 = -1$)
\bar{z}	Complex conjugate of z
q^*	Quaternionic (or Hamilton) conjugate of q
$N(q)$	Norm of quaternion q ($N(q) = q q^*$)
$q \cdot r$	Dot (or scalar) product of quaternions q and r
\mathbf{V}	Vector of the 3-dimensional Euclidean vector-space \mathbb{E}_3
V	Pure quaternion representing vector \mathbf{V}
$\ \mathbf{V}\ $	Euclidean norm of vector \mathbf{V}
$\langle \mathbf{V}, \mathbf{W} \rangle$	Euclidean scalar product of vectors \mathbf{V} and \mathbf{W}
$\mathbf{V} \times \mathbf{W}$	Vector (or cross) product of vectors \mathbf{V} and \mathbf{W}
$V \times W$	Vector (or cross) product of pure quaternions V and W
\mathbf{OP}	Vector with origin O and endpoint P
∇	Nabla operator
∇	Quaternionic nabla operator
c	Light velocity (in vacuum)

For coordinates, for vector or quaternion components, a Latin index, like j in A_j or in x_j , takes the values 1, 2, 3; a Greek index, like μ in x_μ , takes the values 0, 1, 2, 3.

We use 0 as a universal symbol; for example, we generally write $\mathbf{V} = 0$ in place of $\mathbf{V} = \mathbf{0}$, to indicate that \mathbf{V} is a zero-vector.

1 Introduction

In his famous *A treatise on electricity and magnetism*, published in 1873, Maxwell used quaternions, invented by Hamilton in 1843, to represent electromagnetic fields. Nevertheless, the parallel development of vector calculus by Grassmann and Cayley led Heaviside to eventually express Maxwell equations with the help of divergence and curl operators, and utilizing quaternions for such applications was almost forgotten. Some years later, around 1912, quaternions were employed by Silberstein [1–3] for expressing the special theory of relativity; but for subsequent developments and descriptions of the theory, they were preferred by vector calculus, once more. Quantum mechanics put matrix calculus in the foreground and reinforced the use of vectors in physics, that is, linear algebra in its standard form (no quaternions!). As a result, quaternions are not very known to physicists, who generally prefer to employ vector or matrix calculus. To be complete, we should mention an original way of expressing Lorentz transformations with quaternions, proposed by Dirac [4]; and some attempts to apply quaternions to a space-time description of quantum phenomena by Kwal [5, 6] and in polarization optics in a graphical way by Cernosek [7, 8]. Nevertheless, in 1972, J. L. Synge used quaternions in Relativity again by introducing minquats [9], that is, complex quaternions suited to the Minkowski space. Independently, in 1982–1983, complex quaternions (that are equivalent to Synge's minquats) were shown to be fruitful in polarization optics too [10–12], with effective applications to polarization mode dispersion as encountered in optical communications [13, 14].

The above is focused on applications of quaternions to physics. But of course quaternions

have also been of interest in mathematics since 1843, in particular in arithmetic [15]. For a broad mathematical overview of quaternion algebras, let us mention the recent book of Voight [16].

Maybe there is a new interest today in using quaternions for various applications (other than mathematics), such as cristallography and chemistry [17], molecular modelling [18], quantum physics [19], mechanics [20] and relativistic physics [21]. Quaternions are also useful in robotics and for video game softwares: indeed, their ability to express rotations in the 3-dimensional Euclidean space has been shown to avoid the gimbal lock [22].

In this article, we will mainly focus on a subset of complex quaternions, the subset of minquats [9], which is well suited to representing 4-dimensional vectors of the Minkowski space, that is, \mathbb{R}^4 equipped with the Lorentz quadratic form. Moreover, proper Lorentz rotations form the group $\text{SO}_+(1, 3)$, which operates on the Minkowski space, and can also be represented by complex quaternions. Those structures play an important and well-known role in the special theory of relativity [9], and offer some interest in polarization optics too [23–27].

A part of the present article constitutes an attempt to show how quaternions can be used for expressing Maxwell equations in a very synthetical way and for deriving how electromagnetic fields are transformed under Lorentz boosts [21]. The results are then illustrated by proving the relativistic invariance of polarizations of lightwaves, including their degrees of polarization.

Another part of the article deals with a quaternionic representation of polarization optics and complements some previous studies on the subject [11, 12, 27]. Polarization states are represented by minquats and polarization operators by complex quaternions, which generally correspond to proper Lorentz rotations on the Minkowski space of minquats.

We eventually point out some effects that take place in Relativity and in polarization optics and that are similar to each other, as already shown in a previous article [27]. The deep reason or their similarity, indeed, is that they are physical translations in the relativistic or in the polarization optics areas, of geometrical properties of the Minkowski space. In the present article we make an effort to study the geometrical behaviours of the Minkowski space first, and then to examine how they appear in Relativity and in polarization optics.

2 Quaternions, rotations and Minkowski space

2.1 Quaternions

2.1.1 Complex quaternion algebra

The space \mathbb{C}^4 is a four-dimensional complex vector-space whose canonical basis is denoted $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where

$$\mathbf{e}_0 = (1, 0, 0, 0), \quad \mathbf{e}_1 = (0, 1, 0, 0), \quad \mathbf{e}_2 = (0, 0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 0, 1). \quad (1)$$

We introduce a **quaternionic product**, as defined in Table 1, where the first element of the product is written in the left column and the second element in the upper row (for example: $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3 = -\mathbf{e}_2\mathbf{e}_1$).

	\mathbf{e}_0	\mathbf{e}_1	\mathbf{e}_2	\mathbf{e}_3
\mathbf{e}_0	\mathbf{e}_0	\mathbf{e}_1	\mathbf{e}_2	\mathbf{e}_3
\mathbf{e}_1	\mathbf{e}_1	$-\mathbf{e}_0$	\mathbf{e}_3	$-\mathbf{e}_2$
\mathbf{e}_2	\mathbf{e}_2	$-\mathbf{e}_3$	$-\mathbf{e}_0$	\mathbf{e}_1
\mathbf{e}_3	\mathbf{e}_3	\mathbf{e}_2	$-\mathbf{e}_1$	$-\mathbf{e}_0$

Table 1: Quaternionic product.

The quaternionic product is distributive with respect to addition; it is non-commutative. When equipped with the quaternionic product, the vector space \mathbb{C}^4 becomes the **complex**

quaternion algebra, denoted by \mathbb{H}_c . (Complex quaternions were called bi-quaternions by Hamilton.) Thus, a (complex) quaternion is written as

$$q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3, \quad (2)$$

where the q_μ 's are complex numbers. An explicit expression of the product qr of quaternions q and r is given later.

If \bar{z} denotes the complex conjugate of the complex number z , the **complex conjugate** of the quaternion q above is \bar{q} , defined by

$$\bar{q} = \bar{q}_0 e_0 + \bar{q}_1 e_1 + \bar{q}_2 e_2 + \bar{q}_3 e_3. \quad (3)$$

For every quaternion q and every quaternion r , we have $\overline{qr} = \bar{q}\bar{r}$.

The **Hamilton conjugate** of q is q^* , such that

$$q^* = q_0 e_0 - q_1 e_1 - q_2 e_2 - q_3 e_3. \quad (4)$$

It can be proved that $(qr)^* = r^* q^*$, for all quaternions q and r . Complex and Hamilton conjugations commute: $(\bar{q})^* = \overline{q^*}$, for every quaternion q .

The **scalar part** of q is $q_0 e_0$ and its **vector part** is $q_1 e_1 + q_2 e_2 + q_3 e_3$. The quaternion q is a **pure quaternion**, if $q_0 = 0$. A quaternion q is a pure quaternion if, and only if, $q^* = -q$. Every quaternion q can be written

$$q = q_0 e_0 + V, \quad (5)$$

where V is a pure quaternion.

The (quaternionic) **norm** of q is $N(q)$, defined by [28]

$$N(q) = qq^* = (q_0)^2 + (q_1)^2 + (q_2)^2 + (q_3)^2. \quad (6)$$

Generally $N(q)$ is a complex number (N is not a norm in the sense of mathematical analysis). We note that $N(q) = N(q^*)$ and $N(\bar{q}) = \bar{q} \bar{q}^* = \overline{q q^*} = \overline{N(q)}$. The norm is multiplicative:

$$N(qr) = N(q)N(r), \quad (7)$$

and such that $N(qr) = N(rq)$, for all quaternions q and r .

The **dot product** (or **scalar product**) of quaternions is the **polar form** of the quaternionic norm, defined for $q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$ and $r = r_0 e_0 + r_1 e_1 + r_2 e_2 + r_3 e_3$ by

$$q \cdot r = q_0 r_0 + q_1 r_1 + q_2 r_2 + q_3 r_3, \quad (8)$$

so that $N(q) = q \cdot q$. The dot product of quaternions is commutative: $q \cdot r = r \cdot q$; it is such that

$$q \cdot r = \frac{1}{2} [N(q+r) - N(q) - N(r)]. \quad (9)$$

For μ and ν in $\{0, 1, 2, 3\}$, we have $e_\mu \cdot e_\mu = 1$, and $e_\mu \cdot e_\nu = 0$, for $\nu \neq \mu$. Quaternions q and r are said to be **orthogonal** if, and only if, $q \cdot r = 0$.

A quaternion u is **unitary** if $N(u) = 1$, that is, if $uu^* = 1$ (u is also called a **unit quaternion**). If $N(q) \neq 0$, the quaternion q has an **inverse** q^{-1} for the quaternionic product:

$$q^{-1} = \frac{q^*}{N(q)}. \quad (10)$$

Some quaternions, for example $e_0 + i e_1$, are **zero divisors** and have no inverses, since their norms are zero. They will be helpful in representing null 4-vectors in Relativity as well as completely polarized states and polarizers in polarization optics.

Remark 2.1 According to some authors [9], a quaternion u is unitary if $N(u) = \pm 1$. Then a quaternion u is a positive unit quaternion, if $N(u) = 1$, and a negative unit quaternion, if $N(u) = -1$. A negative unit quaternion u ($N(u) = -1$) takes necessarily the form $u = i v$, where v is a positive unit quaternion ($N(v) = 1$). For example e_μ ($\mu \in \{0, 1, 2, 3\}$) is a positive unit quaternion, whereas $i e_\mu$ is a negative unit quaternion.

Unit quaternions as previously defined in this article are then positive unit quaternions.

2.1.2 Practical analytic derivations

We write the product of quaternions q and r as

$$q r = (q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3)(r_0 e_0 + r_1 e_1 + r_2 e_2 + r_3 e_3), \quad (11)$$

and the result is obtained by applying Table 1 and using the distributivity of the quaternionic product with respect to addition. Explicitly we obtain

$$q r = (q_0 r_0 - q_1 r_1 - q_2 r_2 - q_3 r_3) e_0 + (q_0 r_1 + q_1 r_0 + q_2 r_3 - q_3 r_2) e_1 \\ + (q_0 r_2 + q_2 r_0 - q_1 r_3 + q_3 r_1) e_2 + (q_0 r_3 + q_3 r_0 + q_1 r_2 - q_2 r_1) e_3. \quad (12)$$

Figure 1 provides a mnemonic way to attribute the appropriate sign to a product of the form $q_i r_j e_i e_j$, when $i \neq j$ (both i and j run over 1, 2, 3).

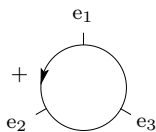


Figure 1: For $\{i, j, k\} = \{1, 2, 3\}$, we have $e_i e_j = e_k$, if we pass from e_i to e_j in a direct rotation (whose angle is $2\pi/3$, if oriented according to the arrow); and $e_i e_j = -e_k$ otherwise. For example $e_2 e_3 = e_1$, and $e_1 e_3 = -e_2$.

Nevertheless, for actual derivations, we also use the following method. We adopt the notation

$$q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}, \quad (13)$$

and we call the right part of Eq. (13) the **column form** or column representation of q . We avoid to call it “matrix representation”, since we will introduce a product of columns that is not the matrix product. (A true matrix-representation of quaternions will be introduced in the next section.)

We write the product of q and r in the form

$$q r = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} q_0 r_0 - q_1 r_1 - q_2 r_2 - q_3 r_3 \\ q_0 r_1 + q_1 r_0 + q_2 r_3 - q_3 r_2 \\ q_0 r_2 + q_2 r_0 - q_1 r_3 + q_3 r_1 \\ q_0 r_3 + q_3 r_0 + q_1 r_2 - q_2 r_1 \end{bmatrix}, \quad (14)$$

which should be understood as an operational writing for obtaining the product of two quaternions; it is not a matrix product!

The process for obtaining the product of two quaternions is as follows. The first line of the column form of $q r$, that is, the e_0 -component of $q r$, is obtained by multiplying column forms of

q and r line by line, and setting appropriate signs, as shown in Eq. (15), in which components to be multiplied are linked by a straight segment; explicitly

$$\begin{array}{c} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} \begin{array}{c} + \\ - \\ - \\ - \end{array} \end{array} \text{ provides } \begin{bmatrix} q_0 r_0 - q_1 r_1 - q_2 r_2 - q_3 r_3 \\ \text{---} \\ \text{---} \\ \text{---} \end{bmatrix}. \quad (15)$$

The second line of qr (that is, the e_1 -component of qr), is obtained from the cross product of the first and second lines of column forms of q and r , and from the cross product of the third and fourth lines, and setting appropriate signs as follows, so that

$$\begin{array}{c} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \end{array} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} \begin{array}{c} + \\ + \\ - \\ + \end{array} \end{array} \text{ provides } \begin{bmatrix} \text{---} \\ q_0 r_1 + q_1 r_0 + q_2 r_3 - q_3 r_2 \\ \text{---} \\ \text{---} \end{bmatrix}. \quad (16)$$

(Signs depend on indices, according to Table 1.)

A similar process is applied for the third and fourth lines, and

$$\begin{array}{c} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} \begin{array}{c} + \\ + \\ + \\ - \end{array} \end{array} \text{ provides } \begin{bmatrix} \text{---} \\ \text{---} \\ q_0 r_2 + q_2 r_0 - q_1 r_3 + q_3 r_1 \\ \text{---} \end{bmatrix}, \quad (17)$$

$$\begin{array}{c} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} \begin{array}{c} + \\ - \\ + \\ + \end{array} \end{array} \text{ provides } \begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \\ q_0 r_3 + q_3 r_0 + q_1 r_2 - q_2 r_1 \end{bmatrix}. \quad (18)$$

The rules to obtain the appropriate sign of a product $q_\mu r_\nu$ or $q_i r_j$ are as follows:

- Products $q_0 r_\nu$ and $q_\mu r_0$ are given the sign $+$.
- Products $q_i r_j$ ($ij \neq 0$):
 - If $i = j$, then $q_i r_j = q_i r_i$ is given the sign $-$.
 - If $i \neq j$, let $k \in \{1, 2, 3\}$ with $k \neq i$ and $k \neq j$. Then $q_i r_j$ is given the sign $+$ if $[i, j, k]$ is a circular permutation of $[1, 2, 3]$ (see also Fig. 1).

2.1.3 Matrix representation

The Pauli matrices σ_μ ($\mu = 0, 1, 2, 3$) are

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (19)$$

The following correspondence between quaternions and Pauli matrices

$$e_0 \mapsto \sigma_0, \quad e_k \mapsto -i\sigma_k, \quad k = 1, 2, 3, \quad (20)$$

defines a quadratic isomorphism (denoted σ) between the complex quaternion algebra \mathbb{H}_c and the 2×2 -complex-matrix algebra. We have

$$q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \mapsto \sigma(q) = \begin{pmatrix} q_0 - iq_1 & -q_3 - iq_2 \\ q_3 - iq_2 & q_0 + iq_1 \end{pmatrix}, \quad (21)$$

and

$$N(q) = \det \sigma(q), \quad (22)$$

where $\det \sigma(q)$ denotes the determinant of $\sigma(q)$.

The inverse isomorphism is given by

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \mapsto q = \frac{\sigma_{11} + \sigma_{22}}{2} \mathbf{e}_0 + i \frac{\sigma_{11} - \sigma_{22}}{2} \mathbf{e}_1 + i \frac{\sigma_{12} + \sigma_{21}}{2} \mathbf{e}_2 + \frac{\sigma_{21} - \sigma_{12}}{2} \mathbf{e}_3. \quad (23)$$

Actual calculations on quaternions can be developed by using the previous isomorphism. Explicitly, if $q = q_0 \mathbf{e}_0 + q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 + q_3 \mathbf{e}_3 \mapsto \sigma(q)$ and $r = r_0 \mathbf{e}_0 + r_1 \mathbf{e}_1 + r_2 \mathbf{e}_2 + r_3 \mathbf{e}_3 \mapsto \sigma(r)$, then

$$qr \mapsto \sigma(qr) = \sigma(q)\sigma(r) = \begin{pmatrix} q_0 - iq_1 & -q_3 - iq_2 \\ q_3 - iq_2 & q_0 + iq_1 \end{pmatrix} \begin{pmatrix} r_0 - ir_1 & -r_3 - ir_2 \\ r_3 - ir_2 & r_0 + ir_1 \end{pmatrix}, \quad (24)$$

and we obtain the following equations

$$\sigma(qr)_{11} = q_0 r_0 - q_1 r_1 - q_2 r_2 - q_3 r_3 - i(q_0 r_1 + q_1 r_0 + q_2 r_3 - q_3 r_2), \quad (25)$$

$$\sigma(qr)_{12} = -q_0 r_3 - q_1 r_2 + q_2 r_1 - q_3 r_0 - i(q_0 r_2 - q_1 r_3 + q_2 r_0 + q_3 r_1), \quad (26)$$

$$\sigma(qr)_{21} = q_0 r_3 + q_1 r_2 - q_2 r_1 + q_3 r_0 - i(q_0 r_2 - q_1 r_3 + q_2 r_0 + q_3 r_1), \quad (27)$$

$$\sigma(qr)_{22} = q_0 r_0 - q_1 r_1 - q_2 r_2 - q_3 r_3 + i(q_0 r_1 + q_1 r_0 + q_2 r_3 - q_3 r_2), \quad (28)$$

which lead to Eq. (12) once more, if we apply Eq. (23).

Finally we point out that quaternions can also be represented by 4×4 -complex matrices (Dirac matrices) [9, 28].

2.1.4 Real quaternion algebra

A quaternion is a **real quaternion** if its components q_μ are real numbers. Real quaternions are also called **Hamilton's quaternions**. They form a real algebra, denoted by \mathbb{H} . If \mathbb{E}_3 denotes the 3-dimensional Euclidean vector space (that is, \mathbb{R}^3 equipped with the Euclidean norm), we identify \mathbb{H} , considered as a four-dimensional real vector-space, with $\mathbb{R} \oplus \mathbb{E}_3$, so that every three-dimensional vector \mathbf{V} can be seen as a real pure quaternion V . If $\|\mathbf{V}\|$ denotes the Euclidean norm of vector \mathbf{V} in \mathbb{E}_3 , we have

$$N(V) = \|\mathbf{V}\|^2. \quad (29)$$

Sometimes we write \mathbf{V} in place of V (and vice versa), that is, we consider \mathbf{V} as a real pure quaternion: if $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis of the three-dimensional vector space \mathbb{E}_3 , the quaternion that corresponds to vector $\mathbf{V} = V_1 \mathbf{e}_1 + V_2 \mathbf{e}_2 + V_3 \mathbf{e}_3$ is $V = V_1 \mathbf{e}_1 + V_2 \mathbf{e}_2 + V_3 \mathbf{e}_3$. We also write $\|V\|$ for $\|\mathbf{V}\|$, so that $\|V\|^2 = N(V)$.

If \mathbf{V} and \mathbf{W} are two vectors of the three-dimensional Euclidean space \mathbb{E}_3 , we denote by $\langle \mathbf{V}, \mathbf{W} \rangle$ their Euclidean scalar product. If V and W are the corresponding quaternions of \mathbf{V} and \mathbf{W} , we have

$$V \cdot W = \langle \mathbf{V}, \mathbf{W} \rangle. \quad (30)$$

If $\mathbf{V} \times \mathbf{W}$ denotes the **vector product** (or cross product) of \mathbf{V} and \mathbf{W} (in \mathbb{E}_3), we define $V \times W$ as being the quaternion corresponding to $\mathbf{V} \times \mathbf{W}$. By abuse we write $V \times W = \mathbf{V} \times \mathbf{W}$, so that

$$V W = -V \cdot W \mathbf{e}_0 + V \times W = -\langle \mathbf{V}, \mathbf{W} \rangle \mathbf{e}_0 + \mathbf{V} \times \mathbf{W}, \quad (31)$$

as we deduce from Eq. (14), if we set $q_0 = r_0 = 0$, $q_i = V_i$ and $r_j = W_j$.

The identification of \mathbb{H} with $\mathbb{R} \oplus \mathbb{E}_3$ leads us to write $e_0 = 1$. For example Eq. (31) is also written

$$VW = -V \cdot W + V \times W. \quad (32)$$

A real unit pure quaternion is denoted e_n and corresponds to a unitary vector \mathbf{e}_n of the 3-dimensional Euclidean vector space. We have $e_n^2 = -e_0 = -1$, and e_n appears to be a quaternionic generalization of the complex number i . The subscript n in e_n is not a variable index: explicitly we have

$$e_n = n_1 e_1 + n_2 e_2 + n_3 e_3, \quad (n_1)^2 + (n_2)^2 + (n_3)^2 = 1. \quad (33)$$

On the other hand, we write e_i , $i = 1, 2, 3$ (or e_j , $j = 1, 2, 3$, or e_k) to designate e_1 , e_2 or e_3 . For example Eqs. (33) can be written

$$e_n = \sum_{i=1}^{i=3} n_i e_i, \quad \sum_{i=1}^{i=3} (n_i)^2 = 1. \quad (34)$$

Of course it may happen that $e_n = e_1$, etc.

Eventually, if e_m and e_n are real unit pure quaternions, then $e_m \cdot e_n = \langle \mathbf{e}_m, \mathbf{e}_n \rangle = \cos \alpha$, where α , the angle between \mathbf{e}_m and \mathbf{e}_n , may be thought of like the **angle between quaternions** e_m and e_n .

Remark 2.2 The algebra \mathbb{H} is a division algebra, that is, the set of real quaternions, equipped with addition and the quaternionic product, is a division ring: it does not contain zero divisors. The algebra \mathbb{H} is a \mathbb{R} -algebra and is not a subalgebra of \mathbb{H}_c , which is a \mathbb{C} -algebra.

2.1.5 Cross product of complex pure quaternions and intrinsic derivations

If we set $q_0 = 0 = r_0$ in Eq. (14), we obtain

$$qr = \begin{bmatrix} 0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \begin{bmatrix} 0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} -q_1 r_1 - q_2 r_2 - q_3 r_3 \\ q_2 r_3 - q_3 r_2 \\ q_3 r_1 - q_1 r_3 \\ q_1 r_2 - q_2 r_1 \end{bmatrix}. \quad (35)$$

We remark that the scalar part of qr is opposite to the dot product of q and r , already defined and denoted $q \cdot r$. If the q_i 's and the r_i 's are real numbers, the vector part of qr is identical to the quaternion associated with the vector product of (q_1, q_2, q_3) and (r_1, r_2, r_3) in \mathbb{E}_3 . We then define the **vector product** (or cross product) of complex pure quaternions V and W by

$$V \times W = VW + V \cdot W e_0. \quad (36)$$

The vector product $V \times W$ is a pure quaternion. (Since $e_0 \equiv 1$, we also write the right member of Eq. (36) as $VW + V \cdot W$.) Equation (36) is a generalization of Eq. (32), since it holds for complex pure quaternions, not only for real ones.

In particular, if V' and W' are real pure quaternions, and if $V = aV'$ and $W = bW'$, where a and b are complex numbers, we have

$$V \times W = (aV') \times (bW') = ab V' \times W'. \quad (37)$$

Intrinsic derivations. A consequence of Eq. (36) is that some derivations with quaternions can be developed without referring to the canonical basis, as done in Sect. 2.1.2. Indeed, let q and r be two complex quaternions, written $q = q_0 + V$ and $r = r_0 + W$, where q_0 and r_0 are complex numbers and V and W complex pure quaternions. Then

$$qr = (q_0 + V)(r_0 + W) = q_0 r_0 + q_0 W + r_0 V + VW = q_0 r_0 - V \cdot W + q_0 W + r_0 V + V \times W. \quad (38)$$

The result expressed by Eq. (38) is intrinsic, since it does not explicitly refer to the basis $\{e_1, e_2, e_3\}$ of pure quaternions. (Components on e_0 still play a special role.)

Examples of dot and cross products of complex quaternions.

1. Let $q = e_1 + ie_2$, and $r = e_2 + ie_3$ (both q and r are pure quaternions). Then

$$N(q) = -(e_1 + ie_2)(e_1 + ie_2) = 0 = N(r), \quad (39)$$

$$q \cdot r = (e_1 + ie_2) \cdot (e_2 + ie_3) = i \text{ (or } ie_0) \quad (40)$$

$$q \times r = (e_1 + ie_2) \times (e_2 + ie_3) = -e_1 - ie_2 + e_3, \quad (41)$$

$$qr = (e_1 + ie_2)(e_2 + ie_3) = -i - e_1 - ie_2 + e_3 = -q \cdot r + q \times r. \quad (42)$$

2. Let $q = 1 + ie_q$ and $r = 1 + ie_r$, where e_q and e_r are real unit pure quaternions (such that $e_q^2 = e_r^2 = -1$). Then

$$N(q) = (1 + ie_q)(1 - ie_q) = 0 = N(r), \quad (43)$$

$$q \cdot r = (1 + ie_q) \cdot (1 + ie_r) = 1 - e_q \cdot e_r, \quad (44)$$

$$qr = (1 + ie_q)(1 + ie_r) = 1 + e_q \cdot e_r + ie_q + ie_r - e_q \times e_r. \quad (45)$$

Since q and r are not pure quaternions, the vector product does not make sense for them. If $e_r = e_q$, then $qr = q^2 = 2(1 + ie_q)$, and $q \cdot r = q \cdot q = 0$. Since $q \cdot q = 0$ for $q \neq 0$, the dot product for complex quaternions is degenerate.

2.1.6 Quaternionic exponential

The **exponential** of a quaternion q is defined by

$$e^q = \exp q = 1 + q + \frac{q^2}{2!} + \dots + \frac{q^j}{j!} + \dots \quad (46)$$

If $q = e_n \varphi$, with φ a real number and e_n a real unit pure quaternion, since $e_n^2 = -1$, we have $e_n^{2j} = (-1)^j$ and $e_n^{2j+1} = (-1)^j e_n$, so that

$$e^{e_n \varphi} = \exp e_n \varphi = \sum_{j=0}^{+\infty} (-1)^j \frac{\varphi^{2j}}{(2j)!} + e_n \sum_{j=0}^{+\infty} (-1)^j \frac{\varphi^{2j+1}}{(2j+1)!} = \cos \varphi + e_n \sin \varphi. \quad (47)$$

If $q = ie_n \delta$ (δ a real number), then

$$e^{ie_n \delta} = \exp ie_n \delta = \cos i\delta + e_n \sin i\delta = \cosh \delta + ie_n \sinh \delta. \quad (48)$$

Equation (48) can be deduced from Eq. (47) by replacing φ with $i\delta$. Both $\exp e_n \varphi$ and $\exp ie_n \delta$ are unit quaternions.

In Appendix A we provide some quaternionic trigonometric-formulas. In particular, we point out that Eqs. (47) and (48) may take surprising forms: for exemple, if e_m and e_n are two orthogonal real unit pure quaternions ($e_m \cdot e_n = 0$), then $N(e_m + ie_n) = -(e_m + ie_n)^2 = 0$, and according to Eq. (46), for every real number φ

$$\exp[(e_m + ie_n)\varphi] = 1 + (e_m + ie_n)\varphi, \quad \text{and} \quad N[1 + (e_m + ie_n)\varphi] = 1. \quad (49)$$

Such a unitary quaternion is useful in the special theory of relativity to represent singular proper Lorentz rotations [9] and in crystal optics to represent singular-axis crystals [27] (see Sect. 6).

Polar form of a real quaternion. Let $q = q_0 + V$ be a real quaternion, with $V \neq 0$: q_0 is a real number and V is a real pure quaternion. Then $N(q) = q_0^2 + \|V\|^2 > 0$. Let $e_q = V/\|V\|$ and $\theta \in]-\pi, \pi]$ be such that

$$\cos \theta = \frac{q_0}{\sqrt{N(q)}}, \quad e_q \sin \theta = \frac{V}{\sqrt{N(q)}}. \quad (50)$$

Then q can be written

$$q = \sqrt{N(q)} (\cos \theta + e_q \sin \theta) = \sqrt{N(q)} \exp e_q \theta. \quad (51)$$

The right part of Eq. (51) is the polar form of q (to be compared with the polar form of a complex number z , written in the form $z = |z| \exp i\theta$). The quaternion e_q is a real unit pure quaternion. We remark that only the sign of $e_q \sin \theta$ is determined. If $e'_q = -e_q$ and $\theta' = -\theta$, we also have $q = \sqrt{N(q)} \exp e'_q \theta'$.

2.2 Rotations on the 3–dimensional Euclidean vector space \mathbb{E}_3

Hamilton invented quaternions to represent rotations on the 3–dimensional Euclidean vector space \mathbb{E}_3 in the same way as complex numbers of the form $\exp i\theta$ represent rotations on the 2–dimensional Euclidean plane (isomorphic to the complex plane).

2.2.1 Quaternionic expression of a rotation on \mathbb{E}_3

The result obtained by Hamilton is expressed as follows. (In this section considered quaternions are real quaternions.)

Theorem 2.1 *Let e_n be a unit vector in \mathbb{E}_3 . Let vector V' be the image of vector V under the rotation of angle φ around the axis e_n in \mathbb{E}_3 , and let V' , V and e_n be the associated quaternions. Then*

$$V' = e^{e_n \varphi / 2} V e^{-e_n \varphi / 2}. \quad (52)$$

(The angle φ is positive if the sense of rotation is the trigonometric sense for an observer who sees the vector e_n pointed towards him.)

Proof. We denote 3–dimensional vectors by their associated quaternions. Let \mathcal{P} be the plane orthogonal to e_n and passing through the origin O . Let V_\perp be the (orthogonal) projection of V on \mathcal{P} . Then

$$V = \|V\| e_n \cos \alpha + V_\perp, \quad (53)$$

where α is the angle between e_n and V , and V_\perp is perpendicular to e_n . Since e_n and $\exp(e_n \varphi / 2)$ commute, we have

$$e^{e_n \varphi / 2} (\|V\| e_n \cos \alpha) e^{-e_n \varphi / 2} = \|V\| e_n \cos \alpha. \quad (54)$$

We also derive

$$\begin{aligned} e^{e_n \varphi / 2} V_\perp e^{-e_n \varphi / 2} &= \left(\cos \frac{\varphi}{2} + e_n \sin \frac{\varphi}{2} \right) V_\perp \left(\cos \frac{\varphi}{2} - e_n \sin \frac{\varphi}{2} \right) \\ &= \left(V_\perp \cos \frac{\varphi}{2} + e_n V_\perp \sin \frac{\varphi}{2} \right) \left(\cos \frac{\varphi}{2} - e_n \sin \frac{\varphi}{2} \right) \\ &= V_\perp \cos^2 \frac{\varphi}{2} - e_n V_\perp e_n \sin^2 \frac{\varphi}{2} + (e_n V_\perp - V_\perp e_n) \cos \frac{\varphi}{2} \sin \frac{\varphi}{2}. \end{aligned} \quad (55)$$

Since V_\perp is orthogonal to e_n , we have $e_n V_\perp = e_n \times V_\perp = -V_\perp e_n$, and $V_\perp = e_n V_\perp e_n$ (see the corollary of Appendix D). If $W = \exp(e_n \varphi / 2) V_\perp \exp(-e_n \varphi / 2)$, Eq. (55) leads to

$$W = V_\perp \cos \varphi + e_n \times V_\perp \sin \varphi, \quad (56)$$

and we first conclude that W is deduced from V_\perp under the rotation of angle φ in the plane \mathcal{P} , and then that V' , defined by Eq. (52), is such that $V' = \|V\| e_n \cos \alpha + W$, and is deduced from V under the rotation of angle φ around e_n . The proof is complete. \square

2.2.2 Determinant

Let R be a rotation on \mathbb{E}_3 and let \mathcal{R} be its matrix in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The determinant of \mathcal{R} (as a matrix) is also the determinant of R (as a linear mapping on \mathbb{E}_3); since R is a rotation, it preserves the orientation of the previous basis, so that $\det R = \det \mathcal{R} = 1$.

2.3 Minkowski space. Minquats

The Minkowski space is (\mathbb{R}^4, Q) , where Q is the **Lorentz quadratic form** (or **metric**), defined by

$$Q(x_0, x_1, x_2, x_3) = (x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2. \quad (57)$$

The Minkowski space is thus isomorphic to $(\mathbb{R} \oplus \mathbb{E}_3, Q)$.

The associated bilinear form (or polar form of Q) is the pseudo-Euclidean scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3, \quad (58)$$

where $\mathbf{x} = (x_0, x_1, x_2, x_3)$ and $\mathbf{y} = (y_0, y_1, y_2, y_3)$ belong to \mathbb{R}^4 .

A quaternion x is a **minquat** if it takes the form

$$x = x_0 \mathbf{e}_0 + \mathbf{i}(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3), \quad (59)$$

where the x_μ 's are real numbers. The reason is that

$$N(x) = (x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2, \quad (60)$$

is equal to the Lorentz quadratic form associated with the real four-vector (x_0, x_1, x_2, x_3) of the Minkowski space. (The word ‘‘minquat’’, due to Synge [9], is a contraction of ‘‘Minkowski’’, or ‘‘Minkowskian’’, and ‘‘quaternion’’. The definition of Synge’s minquats is different from the definition given in this section, see Remark 2.3.)

Every minquat x is such that $x - \bar{x}^* = 0$.

Minquats form a 4-dimensional real vector-space, denoted \mathbb{M} , quadratically isomorphic to the Minkowski space, the correspondence being such that

$$(\mathbb{R} \oplus \mathbb{E}_3, Q) \ni \mathbf{x} = (x_0, x_1, x_2, x_3) \longmapsto x = x_0 \mathbf{e}_0 + \mathbf{i}(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3) \in \mathbb{M}, \quad (61)$$

with $Q(\mathbf{x}) = N(x)$. (Since it is a real vector-space, \mathbb{M} is not a subspace of \mathbb{H}_c , because \mathbb{H}_c is a complex vector-space.)

For example, we are led to represent four-dimensional vectors, as used in the special theory of relativity, by minquats. Thus, if x_1, x_2 and x_3 are Cartesian coordinates related to a Galilean inertial frame and if $x_0 = ct$ (c denotes light velocity in vacuum and t the time in the considered frame), the position quadrivector (x_0, x_1, x_2, x_3) is represented by the minquat $x_0 \mathbf{e}_0 + \mathbf{i}(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3)$.

In electromagnetism, if Φ denotes the scalar potential and \mathbf{a} the (three-dimensional) vector potential, in the coordinate system $\{x_\mu\} = \{x_0, x_1, x_2, x_3\}$ (in a Galilean frame), the electromagnetic-potential four-vector is $(\Phi/c, a_1, a_2, a_3)$ and the associated quaternion (a minquat) is

$$A = \frac{\Phi}{c} \mathbf{e}_0 + \mathbf{i}(a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3). \quad (62)$$

Remark 2.3 Synge [9] represents the 4-vector $\mathbf{x} = (x_0, x_1, x_2, x_3)$ (the x_μ 's are real numbers) by the quaternion $x = \mathbf{i}x_0 \mathbf{e}_0 + x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3$, which he calls a minquat. The space of Synge’s minquats is then $(\mathbf{i}\mathbb{R} \oplus \mathbb{E}_3, Q')$, where the quadratic form Q' is such that

$$Q'(\mathbf{x}) = -(x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2 = N(x). \quad (63)$$

Every Synge’s minquat x is such that $x + \bar{x}^* = 0$.

Null, scalarlike and spacelike minquats. We adopt the following vocabulary:

- A minquat x is a **null minquat** (or an **isotropic minquat**) if $N(x) = 0$. Example: $x_0\mathbf{e}_0 + ix_1\mathbf{e}_1$ is a null minquat if $|x_0| = |x_1|$. More generally, if \mathbf{e}_n is a real unit pure quaternion, then $\mathbf{e}_0 + i\mathbf{e}_n$ and $\mathbf{e}_0 - i\mathbf{e}_n$ are null minquats.
- A minquat x is a **scalarlike minquat** if $N(x) > 0$. Example: $x_0\mathbf{e}_0 + ix_1\mathbf{e}_1$ is a scalarlike minquat if $|x_0| > |x_1|$. Since minquats represent quadrivectors of the Minkowski space, which is used in the special theory of relativity under the name of Minkowski spacetime, a scalarlike minquat is also called a **timelike minquat**.
- A minquat x is a **spacelike minquat** if $N(x) < 0$. Example: $x_0\mathbf{e}_0 + ix_1\mathbf{e}_1$ is a spacelike minquat if $|x_0| < |x_1|$.

2.4 Vector spaces

We sum up the vector spaces we deal with:

- \mathbb{E}_3 denotes the **3-dimensional Euclidean vector-space**, which is a real vector-space. It is \mathbb{R}^3 equipped with the Euclidean quadratic form: $\|(x_1, x_2, x_3)\|^2 = (x_1)^2 + (x_2)^2 + (x_3)^2$. The associated scalar product (polar form) is defined for vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ by $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3$. We have $2\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2$.
- $(\mathbb{R}^4, Q) = (\mathbb{R} \oplus \mathbb{E}_3, Q)$ is the **Minkowski space**. It is a real vector-space. The quadratic form Q is the Lorentz quadratic form: $Q(x_0, x_1, x_2, x_3) = (x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2$. The associated scalar product is the pseudo-Euclidean scalar product defined for vectors $\mathbf{x} = (x_0, x_1, x_2, x_3)$ and $\mathbf{y} = (y_0, y_1, y_2, y_3)$ by $\langle \mathbf{x}, \mathbf{y} \rangle = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$. We have $2\langle \mathbf{x}, \mathbf{y} \rangle = Q(\mathbf{x} + \mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})$.
- \mathbb{H} denotes the **real algebra of Hamilton's quaternions**. If we consider only the addition of quaternions and the multiplication of quaternions by real numbers, \mathbb{H} is a real vector-space.
- $\mathbb{H}_\mathbb{C}$ denotes the **complex algebra of complex quaternions**. If we consider only the addition of quaternions and the multiplication of quaternions by complex numbers, $\mathbb{H}_\mathbb{C}$ is a complex vector-space.
- \mathbb{M} denotes the **real vector-space of minquats**. It is a quadratic space for the quaternionic norm N , such that $N(x) = xx^* = (x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2$ for the minquat $x = x_0\mathbf{e}_0 + ix_1\mathbf{e}_1 + ix_2\mathbf{e}_2 + ix_3\mathbf{e}_3$. Moreover, if $y = y_0\mathbf{e}_0 + iy_1\mathbf{e}_1 + iy_2\mathbf{e}_2 + iy_3\mathbf{e}_3$, the polar form of the norm is the dot product, defined by $x \cdot y = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$, and such that $2x \cdot y = N(x + y) - N(x) - N(y)$. The minquat space (\mathbb{M}, N) is isomorphic to the Minkowski space (\mathbb{R}^4, Q) . The isomorphism is quadratic: N and Q are such that $N(x) = N(x_0\mathbf{e}_0 + ix_1\mathbf{e}_1 + ix_2\mathbf{e}_2 + ix_3\mathbf{e}_3) = Q(x_0\mathbf{e}_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) = Q(\mathbf{x})$.

2.5 Proper Lorentz rotations

A rotation on a quadratic space is an automorphism whose determinant is equal to 1. A rotation R_3 on \mathbb{E}_3 can naturally be extended to the Minkowski space. For that purpose we write an element of the Minkowski space $\mathbb{R} \oplus \mathbb{E}_3$ in the form (x_0, \mathbf{V}) , where x_0 is a real number and \mathbf{V} a vector in \mathbb{E}_3 . Let R be the mapping defined on $\mathbb{R} \oplus \mathbb{E}_3$ by $R(1, \mathbf{0}) = 1$, and $R(0, \mathbf{V}) = R_3(\mathbf{V})$, for every 3-dimensional vector \mathbf{V} of \mathbb{E}_3 . If \mathcal{R}_3 denotes the matrix of R_3 in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

(in \mathbb{E}_3), the matrix of R in the basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is

$$\mathcal{R} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \mathcal{R}_3 & \\ 0 & & & \end{array} \right), \quad (64)$$

and $\det R = \det \mathcal{R} = \det \mathcal{R}_3 = 1$, which means that R is a rotation. Moreover R separately preserves the orientations of both \mathbb{R} and \mathbb{E}_3 , and is called a **proper rotation**.

A transformation on the Minkowski space that preserves the Lorentz quadratic form is called a **Lorentz transformation**. If its determinant is equal to 1, the transformation is a rotation. On the Minkowski space, a rotation that **separately** preserves the orientation of \mathbb{R} and that of \mathbb{E}_3 is a **proper Lorentz rotation**. (Consequently every proper Lorentz rotation preserves the orientation of the Minkowski space; the converse does not hold.) In the special theory of relativity, the transformations that preserve the orientation of \mathbb{R} are often called **orthochronous**; proper Lorentz rotations are thus orthochronous Lorentz rotations.

The previous rotation R , as mentioned above, is a proper Lorentz rotation, but every proper Lorentz rotation is not necessarily written, in matrix form, as in Eq. (64), as we shall see. Proper Lorentz rotations form a (non-commutative) group, denoted $SO_+(1, 3)$ [28], generated by hyperbolic rotations and elliptic rotations (that will be introduced in the next sections).

2.5.1 Elliptic rotations on the Minkowski space

On the Minkowski space, the effect of an **elliptic** (or pure, or spatial) rotation of angle φ around the axis \mathbf{e}_1 is expressed, in matrix form, by

$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \varphi & -\sin \varphi \\ 0 & 0 & \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}. \quad (65)$$

The 4-vector $\mathbf{x} = (x_0, x_1, x_2, x_3)$ is transformed into the 4-vector $\mathbf{x}' = (x'_0, x'_1, x'_2, x'_3)$.

With minquats we have

$$\mathbf{x}' = \exp\left(\mathbf{e}_1 \frac{\varphi}{2}\right) \mathbf{x} \exp\left(-\mathbf{e}_1 \frac{\varphi}{2}\right), \quad (66)$$

that is

$$\begin{bmatrix} x'_0 \\ ix'_1 \\ ix'_2 \\ ix'_3 \end{bmatrix} = \begin{bmatrix} \cos(\varphi/2) \\ \sin(\varphi/2) \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} x_0 \\ ix_1 \\ ix_2 \\ ix_3 \end{bmatrix} \begin{bmatrix} \cos(\varphi/2) \\ -\sin(\varphi/2) \\ 0 \\ 0 \end{bmatrix}. \quad (67)$$

Displaying quaternions as in Eq. (67) affords a technical way to actually check that Eq. (66) is equivalent to Eq. (65): quaternion products are performed according to Eq. (14).

If the rotation is around the axis $\mathbf{e}_n = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$ (\mathbf{e}_n is a real unit pure quaternion: $n_1^2 + n_2^2 + n_3^2 = 1$), then

$$\mathbf{x}' = \exp\left(\mathbf{e}_n \frac{\varphi}{2}\right) \mathbf{x} \exp\left(-\mathbf{e}_n \frac{\varphi}{2}\right), \quad (68)$$

that is

$$\begin{bmatrix} x'_0 \\ ix'_1 \\ ix'_2 \\ ix'_3 \end{bmatrix} = \begin{bmatrix} \cos(\varphi/2) \\ n_1 \sin(\varphi/2) \\ n_2 \sin(\varphi/2) \\ n_3 \sin(\varphi/2) \end{bmatrix} \begin{bmatrix} x_0 \\ ix_1 \\ ix_2 \\ ix_3 \end{bmatrix} \begin{bmatrix} \cos(\varphi/2) \\ -n_1 \sin(\varphi/2) \\ -n_2 \sin(\varphi/2) \\ -n_3 \sin(\varphi/2) \end{bmatrix}. \quad (69)$$

Remark 2.4 Passing from Eq. (66) to Eq. (68) illustrates the efficiency of quaternions in writing a proper Lorentz rotation around an arbitrary axis. Moreover Eq. (69) shows that such a writing is not merely a theoretical notation but truly a practical one, allowing us to actually manage derivations. Equation (68) can also be used for intrinsic derivations (see Sect. 2.1.5).

2.5.2 Hyperbolic rotations on the Minkowski space

The hyperbolic rotation of direction x_1 and parameter δ ($\delta \in \mathbb{R}$) is the following transformation, on the Minkowski space

$$x'_0 = x_0 \cosh \delta + x_1 \sinh \delta, \quad (70)$$

$$x'_1 = x_0 \sinh \delta + x_1 \cosh \delta, \quad (71)$$

$$x'_2 = x_2, \quad (72)$$

$$x'_3 = x_3, \quad (73)$$

and its matrix expression is

$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \cosh \delta & \sinh \delta & 0 & 0 \\ \sinh \delta & \cosh \delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}. \quad (74)$$

With minquats $x = x_0 e_0 + i(x_1 e_1 + x_2 e_2 + x_3 e_3)$ and $x' = x'_0 e_0 + i(x'_1 e_1 + x'_2 e_2 + x'_3 e_3)$, the quaternionic form of Eqs. (70–73) is

$$x' = \exp\left(i e_1 \frac{\delta}{2}\right) x \exp\left(i e_1 \frac{\delta}{2}\right), \quad (75)$$

and also, if we display quaternions as in Eq. (14),

$$\begin{bmatrix} x'_0 \\ i x'_1 \\ i x'_2 \\ i x'_3 \end{bmatrix} = \begin{bmatrix} \cosh(\delta/2) \\ i \sinh(\delta/2) \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} x_0 \\ i x_1 \\ i x_2 \\ i x_3 \end{bmatrix} \begin{bmatrix} \cosh(\delta/2) \\ i \sinh(\delta/2) \\ 0 \\ 0 \end{bmatrix}. \quad (76)$$

We also say that the direction of the previous hyperbolic rotation is along e_1 , or by comparison with elliptic rotations (see Sect. 2.5.1), that e_1 is the **axis** of the hyperbolic rotation. If the direction is along e_n , Eq. (75) is replaced with

$$x' = \exp\left(i e_n \frac{\delta}{2}\right) x \exp\left(i e_n \frac{\delta}{2}\right), \quad (77)$$

and Eq. (76) with

$$\begin{bmatrix} x'_0 \\ i x'_1 \\ i x'_2 \\ i x'_3 \end{bmatrix} = \begin{bmatrix} \cosh(\delta/2) \\ i n_1 \sinh(\delta/2) \\ i n_2 \sinh(\delta/2) \\ i n_3 \sinh(\delta/2) \end{bmatrix} \begin{bmatrix} x_0 \\ i x_1 \\ i x_2 \\ i x_3 \end{bmatrix} \begin{bmatrix} \cosh(\delta/2) \\ i n_1 \sinh(\delta/2) \\ i n_2 \sinh(\delta/2) \\ i n_3 \sinh(\delta/2) \end{bmatrix}. \quad (78)$$

with $e_n = n_1 e_1 + n_2 e_2 + n_3 e_3$.

Remark 2.4 is also appropriate here.

Remark 2.5 From a strict geometrical point of view, there is some abuse in calling e_n the axis of the hyperbolic rotation, but the abuse makes sense in polarization optics by reference to the Poincaré sphere, on which e_n defines an axis that intercepts the sphere at two points. Those two points represent the (polarization) eigenstates of a dichroic device, exactly as for a birefringent device, represented by an elliptic rotation for which the notion of rotation axis makes sense (see Section 5). We also abusively call $i\delta$ the angle of an hyperbolic rotation of parameter δ .

2.5.3 Hyperbolic rotations and Lorentz boosts

In the special theory of relativity, the Lorentz boost in the direction x_1 and velocity $v e_1$ is the following transformation

$$x'_0 = \gamma(x_0 - \beta x_1), \quad (79)$$

$$x'_1 = \gamma(x_1 - \beta x_0), \quad (80)$$

$$x'_2 = x_2, \quad (81)$$

$$x'_3 = x_3, \quad (82)$$

where $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$. Equations (79–82) link coordinates of two Galilean systems of inertia \mathcal{S} and \mathcal{S}' in relative translatory motion, that is, coordinates $\{x_\mu\}$ in \mathcal{S} to coordinates $\{x'_\mu\}$ in \mathcal{S}' . Spatial axes are chosen parallel ($x'_j \parallel x_j$, $j = 1, 2, 3$). If t and t' denote the times in \mathcal{S} in \mathcal{S}' respectively, time origins are chosen so that spatial origins coincide at $x_0 = 0 = x'_0$ (with $x_0 = ct$ and $x'_0 = ct'$). The relative velocity of \mathcal{S} with respect to \mathcal{S}' is v and is along the x_1 -axis.

The link with a hyperbolic rotation is obtained by writing $\tanh \delta = -\beta$, so that

$$\cosh \delta = \gamma, \quad \text{and} \quad \sinh \delta = -\gamma\beta, \quad (83)$$

and Eqs. (79–82) become Eqs. (70–73). (The parameter δ is sometimes called **rapidity** [29,30].)

Equation (77) is the quaternionic representation of a Lorentz boost, in the form of a hyperbolic rotation. It holds if the considered Lorentz boost is applied to 4-vectors of the Minkowski space. We may suppose, however, that physical quantities other than 4-vectors of the Minkowski space are transformed when passing from a Galilean inertial frame to another. We also say that such physical quantities undergo a Lorentz boost, but the boost is not expressed by a hyperbolic rotation on the Minkowski space. In electromagnetism, for example, the scalar potential and the vector potential (a 3-vector) form together a 4-vector, called 4-potential vector and element of an appropriate Minkowski space (see Sect. 3.2). In a boost, a 4-potential vector transforms according to Eq. (77). On the other hand, both the electric field \mathbf{E} and the magnetic field \mathbf{B} are transformed under a Lorentz boost between two Galilean inertial frames. Since they are not 4-vectors, Eq. (77) can be applied separately neither to \mathbf{E} nor to \mathbf{B} . The electric field \mathbf{E} and the magnetic field \mathbf{B} can be mixed to form a complex vector and we shall show in Section 3.4 that the effect of a Lorentz boost on the quaternion representative of such a vector is an elliptic rotation that operates on \mathbb{H}_c , and not a hyperbolic rotation (that should operate on \mathbb{M}).

In conclusion we are led to make a difference between Lorentz boosts and hyperbolic rotations. Hyperbolic rotations are proper Lorentz rotations on the Minkowski space, operating on 4-vectors, whereas Lorentz boosts may also apply to sets of physical quantities distincts from Minkowski 4-vectors.

2.5.4 Composition of an elliptic and a hyperbolic rotations with a common axis

A proper Lorentz rotation is not necessarily an elliptic rotation, nor a hyperbolic rotation, that is, its representative quaternion does not necessarily take the form $\exp(e_n \varphi/2)$ or $\exp(i e_n \delta/2)$ as in Eqs. (68) or (77). Nevertheless the group of proper Lorentz rotations on the Minkowski space, denoted $\text{SO}_+(1, 3)$, is generated by elliptic rotations and hyperbolic rotations: every proper Lorentz rotation can be shown to be the product (or composition) of elliptic and hyperbolic rotations [26]. Before we give the most general forms of proper Lorentz rotations on the Minkowski space (see Section 2.5.6), we examine the composition of an elliptic rotation and a hyperbolic one, the two rotations sharing a common axis.

If ψ is a complex number and e_n a real unit pure quaternion, the quaternion $u = \exp(e_n \psi/2)$ represents an elliptic rotation if ψ is real, a hyperbolic rotation if ψ is purely imaginary. If

$\psi = \varphi + i\delta$, we obtain

$$u = \exp\left(e_n \frac{\psi}{2}\right) = \exp\left(e_n \frac{\varphi}{2}\right) \exp\left(i e_n \frac{\delta}{2}\right), \quad (84)$$

which shows that u represents the product (or composition) of an elliptic rotation with a hyperbolic one, that is, a proper Lorentz rotation. (A proof is as follows: if R_1 and R_2 are proper Lorentz rotations, then $\det R_1 = \det R_2 = 1$, so that $\det R_1 \circ R_2 = \det R_1 \det R_2 = 1$; and $R_1 \circ R_2$ preserves the orientation of both \mathbb{R} and \mathbb{E}_3 , because both R_1 and R_2 do.)

The effect on a minquat of the proper Lorentz rotation represented by u above, is deduced from Eqs. (68) and (77) and is expressed by

$$x' = u x \bar{u}^*. \quad (85)$$

Remark 2.6 The representation of proper Lorentz rotations by complex unit quaternions is twofold since both u and $-u$ represent the same rotation, as a consequence of

$$u x \bar{u}^* = (-u) x (-\bar{u}^*). \quad (86)$$

2.5.5 Infinitesimal rotations

The quaternion u in Eq. (84), with ψ a complex number and e_n a real unit pure quaternion, is not a representative of the most general proper Lorentz rotation, as we shall show in this section.

Complex unit pure quaternions. Let e_m and e_n be two real unit pure quaternions and let φ and δ be two real numbers; we assume $\varphi\delta \neq 0$ and denote α the angle between e_m and e_n , so that $e_m \cdot e_n = \cos \alpha$. The norm of $e_m\varphi + i e_n\delta$ is

$$N(e_m\varphi + i e_n\delta) = -(e_m\varphi + i e_n\delta)^2 = \varphi^2 - \delta^2 + 2i\varphi\delta e_m \cdot e_n = \varphi^2 - \delta^2 + 2i\varphi\delta \cos \alpha. \quad (87)$$

Generally, the norm $N(e_m\varphi + i e_n\delta)$ is a complex number. It is equal to 0 only if $\delta = \pm\varphi$ and $\alpha = \pi/2 \pmod{\pi}$. Let us assume $N(e_m\varphi + i e_n\delta) \neq 0$, and let us introduce a complex number ψ , such that $\psi^2 = N(e_m\varphi + i e_n\delta)$. The number ψ takes the form $\psi = |\psi| \exp i\chi$ with

$$|\psi| = \sqrt[4]{\varphi^4 + 2\varphi^2\delta^2 \cos 2\alpha + \delta^4}, \quad (88)$$

and

$$\tan 2\chi = \frac{2\varphi\delta \cos \alpha}{\varphi^2 - \delta^2}. \quad (89)$$

(We remark that since $-1 \leq \cos 2\alpha$, we have $0 \leq (\varphi^2 - \delta^2)^2 \leq \varphi^4 + 2\varphi^2\delta^2 \cos 2\alpha + \delta^4$, so that taking the fourth root in Eq. (88) makes sense.)

Since $N(e_m\varphi + i e_n\delta) \neq 0$, we have $\psi \neq 0$ and we define e_u by

$$e_u = \frac{1}{\psi}(e_m\varphi + i e_n\delta) = \frac{e^{-i\chi}}{|\psi|}(e_m\varphi + i e_n\delta). \quad (90)$$

The quaternion e_u is a complex unit pure quaternion ($N(e_u) = 1$). It is such that $e_u^2 = -1$, and

$$\exp(e_m\varphi + i e_n\delta) = \exp e_u \psi = \cos \psi + e_u \sin \psi, \quad (91)$$

so that $N[\exp(e_m\varphi + i e_n\delta)] = 1$ (see Eqs. (416) and (417) of Appendix A).

One-parameter subgroup. Let $g = e_u \xi$, where e_u is a unit pure quaternion (it may be complex) and let ξ be a complex number. Let s be a real parameter belonging to an interval $[0, S]$. We have

$$\frac{d}{ds} \exp gs = g \exp gs, \quad (92)$$

and

$$\left. \frac{d}{ds} \exp gs \right|_{s=0} = g. \quad (93)$$

The mapping $s \mapsto \exp gs$ is a one-parameter subgroup of the multiplicative group of quaternions and g is called its **infinitesimal generator**. We have $\exp(gs') \exp(gs'') = \exp[g(s' + s'')]$, and in particular

$$\exp(gs) \exp(g ds) = \exp[g(s + ds)]. \quad (94)$$

Elliptic and hyperbolic infinitesimal rotations. If e_m is a real unit pure quaternion and η a real number, the quaternion $v = \exp(e_m \eta ds/2)$ represents an infinitesimal elliptic rotation (axis e_m ; angle ηds), which operates on minquats according to $x' = v x \bar{v}^*$. Similarly, if e_n is a real unit pure quaternion and ζ a real number, the quaternion $w = \exp(i e_n \zeta ds/2)$ represents an infinitesimal hyperbolic rotation (axis e_n ; parameter ζds or angle $i\zeta ds$), which operates on minquats according to $x' = w x \bar{w}^*$. Since infinitesimal rotations commute, the product of the two rotations is an infinitesimal proper Lorentz rotation (as product of two proper Lorentz rotations), represented by

$$u = vw = \exp \frac{e_m \eta ds}{2} \exp \frac{i e_n \zeta ds}{2} = \exp \frac{(e_m \eta + i e_n \zeta) ds}{2}, \quad (95)$$

and operating on minquats according to $x' = v(w x \bar{w}^*) \bar{v}^* = v w x \bar{v} \bar{w}^* = u x \bar{u}^*$.

We assume that the norm of $e_m \eta + i e_n \zeta$ is not zero, and we define $\xi = |\xi| \exp i\chi$ and e_u with respect to η and ζ as ψ and e_u with respect to φ and δ in Eqs. (88–90). If α is the angle between e_m and e_n , we obtain

$$|\xi| = \sqrt[4]{\eta^4 + 2\eta^2 \zeta^2 \cos 2\alpha + \zeta^4}, \quad \tan 2\chi = \frac{2\eta \zeta \cos \alpha}{\eta^2 - \zeta^2}. \quad (96)$$

and

$$e_u = \frac{e^{-i\chi}}{|\xi|} (e_m \eta + i e_n \zeta). \quad (97)$$

The unit quaternion $u = \exp(e_u \xi ds/2)$ represents an infinitesimal proper Lorentz rotation whose axis is the complex unit pure quaternion e_u and “angle” is ξds .

The mapping $s \mapsto \exp(e_u \xi s/2)$ is a one-parameter subgroup, whose generator is $e_u \xi/2$. As a consequence of Eq. (94), the product of such infinitesimal rotations for s varying from 0 to S is given by

$$\exp \left(e_u \frac{\xi}{2} \int_0^S ds \right) = \exp \left(e_u \frac{\xi S}{2} \right) = \exp \left(e_u \frac{\psi}{2} \right), \quad (98)$$

where $\psi = \xi S$.

The quaternion $u = \exp(e_u \psi/2)$ represents a proper Lorentz rotation more general than that represented by u in Eq. (84), because both e_u and ψ are complex. The rotation operates on minquats according to $x' = u x \bar{u}^*$, or more explicitly

$$x' = \exp \left(e_u \frac{\psi}{2} \right) x \exp \left(-\bar{e}_u \frac{\bar{\psi}}{2} \right). \quad (99)$$

2.5.6 General form of proper Lorentz rotations

We write the representative quaternion of a proper Lorentz rotation in the general form

$$u = \exp\left(e_u \frac{\psi}{2}\right), \quad (100)$$

where ψ is a complex number and e_u a complex pure quaternion. (We recall that $-u$ also represents the same rotation, see Remark 2.6). The rotation operates on minquats according to $x' = u x \bar{u}^*$ (see Eq. (99) for a more explicit expression).

In the previous section e_u is a complex pure quaternion, but also a unit quaternion. If $N(e_u) = 0$, we shall see, however, that u , given by Eq. (100), still represents a proper Lorentz rotation. Hence we have to consider two kinds of proper Lorentz rotations:

- **Regular** proper Lorentz rotations, for which $N(e_u) = 1$. The quaternion u can be written $u = \cos(\psi/2) + e_u \sin(\psi/2)$. The rotation is an elliptic rotation if both ψ and e_u are real; it is a hyperbolic rotation if ψ is an imaginary number and if e_u is real.
- **Singular** proper Lorentz rotations, for which $N(e_u) = 0$. (See below for examples and properties.)

Eigenvalues and eigenminquats (regular rotations). We consider the linear mapping $q \mapsto u q \bar{u}^*$, defined on $\mathbb{H}_\mathbb{C}$, where u is given by Eq. (100). The following quaternions are eigenquaternions [23, 26] (an eigenquaternion q is such that $u q \bar{u}^* = \Lambda q$):

$$x' = (e_0 + i e_u)(e_0 + i \bar{e}_u), \quad (101)$$

$$x'' = (e_0 - i e_u)(e_0 - i \bar{e}_u), \quad (102)$$

$$q' = (e_0 - e_u)(e_0 - \bar{e}_u), \quad (103)$$

$$q'' = (e_0 + e_u)(e_0 + \bar{e}_u). \quad (104)$$

(See Appendix E for a proof.) Only x' and x'' are minquats (and are denoted as such). They are eigenminquats of the rotation given by Eq. (99). The corresponding eigenvalues (for minquats x' and x'') are

$$\Lambda' = (C - iS)(\bar{C} + i\bar{S}) = C\bar{C} + S\bar{S} + iC\bar{S} - i\bar{C}S, \quad (105)$$

$$\Lambda'' = (C + iS)(\bar{C} - i\bar{S}) = C\bar{C} + S\bar{S} - iC\bar{S} + i\bar{C}S, \quad (106)$$

where $C = \cos(\psi/2)$ and $S = \sin(\psi/2)$.

Here are some examples, with e_n a real unit pure quaternion, φ and δ real numbers:

- $u = \exp(e_n \varphi/2) = \cos(\varphi/2) + e_n \sin(\varphi/2)$. Null eigenminquats are $x' = 1 + i e_n$ and $x'' = 1 - i e_n$. They define eigendirections in \mathbb{M} . The corresponding eigenvalues are $\Lambda' = \Lambda'' = 1$.
- $u = \exp(i e_n \delta/2) = \cosh(\delta/2) + i e_n \sinh(\delta/2)$. Null eigenminquats are $x' = 1 + i e_n$ and $x'' = 1 - i e_n$. The corresponding eigenvalues are $\Lambda' = \exp \delta$ and $\Lambda'' = \exp(-\delta)$.

2.5.7 Singular proper Lorentz rotations

In Section 2.5.5, real unit pure quaternions e_m and e_n are assumed to be not orthogonal to each other. We now assume that they are: $e_m \cdot e_n = 0$, so that $N(e_m \pm i e_n) = 0$.

Let $e_u = e_m + i e_n$, let η be a real number, and let s be a real parameter in the interval $[0, S]$. Both $\exp(e_m \eta ds/2)$ and $\exp(i e_n \eta ds/2)$ represent infinitesimal proper Lorentz rotations and their commutative product $\exp(e_u \eta ds/2) = \exp(e_m \eta ds/2) \exp(i e_n \eta ds/2)$ also does. The mapping $s \mapsto \exp(e_u \eta s/2)$ is a one-parameter subgroup whose generator is $e_u \eta/2$. If $\varphi = \eta S$,

the product of infinitesimal rotations, represented by $\exp(\mathbf{e}_u \eta ds/2)$, for s varying from 0 to S is a proper Lorentz rotation, represented by

$$u_+ = \exp \frac{\mathbf{e}_u \varphi}{2} = \exp \left[(\mathbf{e}_m + i \mathbf{e}_n) \frac{\varphi}{2} \right] = \mathbf{e}_0 + (\mathbf{e}_m + i \mathbf{e}_n) \frac{\varphi}{2}. \quad (107)$$

The mapping $x \mapsto u_+ x \bar{u}_+^*$ is a proper Lorentz rotation; the minquat $x = \mathbf{e}_0 + i \mathbf{e}_m \mathbf{e}_n$ is an eigenminquat, the eigenvalue being equal to 1 (see Appendix F for a proof). There is no other null-minquat eigendirection.

The result holds if we replace $\mathbf{e}_u = \mathbf{e}_m + i \mathbf{e}_n$ with $\mathbf{e}_u = \mathbf{e}_m - i \mathbf{e}_n$, and u_+ with

$$u_- = \exp \frac{\mathbf{e}_u \varphi}{2} = \mathbf{e}_0 + (\mathbf{e}_m - i \mathbf{e}_n) \frac{\varphi}{2}, \quad (108)$$

the minquat $x = \mathbf{e}_0 - i \mathbf{e}_m \mathbf{e}_n$ providing the unique null-minquat eigendirection of the proper Lorentz rotation $x \mapsto u_- x \bar{u}_-^*$.

Quaternions u_+ and u_- represent **singular** proper Lorentz rotations. Such a rotation has only one null-minquat eigendirection in \mathbb{M} .

Remark 2.7 Here, “singular” means that the considered Lorentz rotation has only one null-minquat eigendirection and is represented by u as in Eq. (100), with $N(\mathbf{e}_u) = 0$. As mentioned by Synge [9], it is not singular in the usual sense of possessing no inverse (the inverse of $\exp(\mathbf{e}_u \psi/2) = 1 + (\mathbf{e}_u \psi/2)$ is $\exp(-\mathbf{e}_u \psi/2) = 1 - (\mathbf{e}_u \psi/2)$, if $\mathbf{e}_u^2 = 0$). Singular proper Lorentz rotations are also called **null** rotations [29, 30]. “Regular”, as used above, means that \mathbf{e}_u in Eq. (100) is such that $N(\mathbf{e}_u) = 1$. The corresponding rotation admits two null-minquat eigendirections.

2.5.8 Some compositions of proper Lorentz rotations

Composition of two elliptic rotations. We consider the elliptic rotation of angle φ around \mathbf{e}_m followed by the elliptic rotation of angle θ around the axis \mathbf{e}_n . We assume $-\pi < \varphi < \pi$ and $\varphi \neq 0$; also $-\pi < \theta < \pi$ and $\theta \neq 0$. The composition of the two rotations is an elliptic rotation represented by

$$u = \exp \left(\mathbf{e}_n \frac{\theta}{2} \right) \exp \left(\mathbf{e}_m \frac{\varphi}{2} \right). \quad (109)$$

We shall show that u takes the form $\exp(\mathbf{e}_q \psi/2)$, where ψ is a real number and \mathbf{e}_q a real pure quaternion.

We first remark that if $\mathbf{e}_m = \pm \mathbf{e}_n$, we obtain $u = \exp[\mathbf{e}_n(\theta \pm \varphi)/2]$, that is $\mathbf{e}_q = \mathbf{e}_n$ and $\psi = \theta \pm \varphi$.

We then assume $\mathbf{e}_m \neq \pm \mathbf{e}_n$. We have

$$\begin{aligned} u &= \left(\cos \frac{\theta}{2} + \mathbf{e}_n \sin \frac{\theta}{2} \right) \left(\cos \frac{\varphi}{2} + \mathbf{e}_m \sin \frac{\varphi}{2} \right) \\ &= \cos \frac{\theta}{2} \cos \frac{\varphi}{2} - \mathbf{e}_n \cdot \mathbf{e}_m \sin \frac{\theta}{2} \sin \frac{\varphi}{2} \\ &\quad + \mathbf{e}_n \cos \frac{\theta}{2} \sin \frac{\varphi}{2} + \mathbf{e}_m \cos \frac{\varphi}{2} \sin \frac{\theta}{2} + \mathbf{e}_n \times \mathbf{e}_m \sin \frac{\theta}{2} \sin \frac{\varphi}{2} \\ &= u_0 + V. \end{aligned} \quad (110)$$

where u_0 is a real number and V a real pure quaternion.

Since $\exp(\mathbf{e}_m \varphi/2)$ and $\exp(\mathbf{e}_n \theta/2)$ are unit quaternions, their product u is also a unit quaternion. From $1 = N(u) = u_0^2 + \|V\|^2$ we conclude that $|u_0| \leq 1$, and there is a number ψ ($-2\pi \leq \psi \leq 2\pi$) such that

$$\cos \frac{\psi}{2} = u_0 = \cos \frac{\theta}{2} \cos \frac{\varphi}{2} - \mathbf{e}_n \cdot \mathbf{e}_m \sin \frac{\theta}{2} \sin \frac{\varphi}{2}. \quad (111)$$

(So far ψ is defined up to sign.)

In Appendix B, we prove that V takes the form $V = e_q \sin(\psi/2)$. We eventually obtain that u takes the form

$$u = \exp\left(e_q \frac{\psi}{2}\right) = \cos \frac{\psi}{2} + e_q \sin \frac{\psi}{2}, \quad (112)$$

with

$$e_q = \frac{1}{\sqrt{1 - \cos^2 \frac{\psi}{2}}} \left(e_n \cos \frac{\theta}{2} \sin \frac{\varphi}{2} + e_m \cos \frac{\varphi}{2} \sin \frac{\theta}{2} + e_n \times e_m \sin \frac{\theta}{2} \sin \frac{\varphi}{2} \right). \quad (113)$$

Equation (111) provides the angle of the rotation, up to sign; Eq. (113) provides the axis. In fact the sign of $e_q \sin(\psi/2)$ is perfectly defined by Eq. (110): once e_q is defined by Eq. (113), the sign of ψ is perfectly defined, according to Eq. (110). (We obtain the same rotation if we change ψ into $-\psi$ and e_q into $-e_q$, the sign of $e_q \sin(\psi/2)$ being unchanged.)

The effect of the compound rotation on a minquat x is given by

$$x' = e^{e_q \psi/2} x e^{-e_q \psi/2} = e^{e_n \theta/2} e^{e_m \varphi/2} x e^{-e_m \varphi/2} e^{-e_n \theta/2}. \quad (114)$$

Remark 2.8 The interest of quaternions in representing elliptic rotations on the Minkowski space is twofold:

1. The most representative (most “natural”) parameters of a rotation are: its axis, given by the unit vector e_n , and the angle of rotation φ around e_n . The rotation is represented by $\exp(e_n \varphi/2)$, which explicitly includes the rotation angle as well as the axis. Moreover such a notation has an additional operational virtue: under the considered rotation an arbitrary minquat x is transformed into

$$x' = e^{e_n \varphi/2} x e^{-e_n \varphi/2}, \quad (115)$$

and Eq. (115) can actually be applied to derive x' from x .

2. The exponential representation of a rotation is preserved in the product of rotations. That is, the product of quaternions $\exp(e_m \varphi/2)$ and $\exp(e_n \theta/2)$ “naturally” takes the form $\exp(e_q \psi/2) = \cos(\psi/2) + e_n \sin(\psi/2)$, which explicitly highlights the angle and the axis of the compound rotation.

The two previous items hold for elliptic rotations on the 3-dimensional Euclidean vector-space \mathbb{E}_3 (see Sect. 2.2).

Composition of two hyperbolic rotations. The composition of the hyperbolic rotation of parameter δ_m along e_m followed by the hyperbolic rotation of parameter δ_n along e_n (we assume $\delta_m \delta_n \neq 0$) is represented by

$$u = \exp\left(i e_n \frac{\delta_n}{2}\right) \exp\left(i e_m \frac{\delta_m}{2}\right). \quad (116)$$

Then

$$\begin{aligned} u &= \left(\cosh \frac{\delta_n}{2} + i e_n \sinh \frac{\delta_n}{2} \right) \left(\cosh \frac{\delta_m}{2} + i e_m \sinh \frac{\delta_m}{2} \right) \\ &= \cosh \frac{\delta_n}{2} \cosh \frac{\delta_m}{2} + e_n \cdot e_m \sinh \frac{\delta_n}{2} \sinh \frac{\delta_m}{2} \\ &\quad + i e_n \sinh \frac{\delta_n}{2} \cosh \frac{\delta_m}{2} + i e_m \sinh \frac{\delta_m}{2} \cosh \frac{\delta_n}{2} - e_n \times e_m \sinh \frac{\delta_n}{2} \sinh \frac{\delta_m}{2}. \end{aligned} \quad (117)$$

(Equation (117) is no more than Eq. (110) in which φ is replaced with $i\delta_m$ and θ by $i\delta_n$.)

If e_m and e_n are collinear ($e_m = \pm e_n$), then u is a hyperbolic rotation along e_m . If e_m and e_n are not collinear, then $e_n \times e_m$ is not zero and u does not represent a pure hyperbolic rotation anymore, because of the term $e_n \times e_m \sinh(\delta_n/2) \sinh(\delta_m/2)$, which is not imaginary. There is an additional elliptic rotation (see Section 6.2).

Application to Relativity. The composition of two hyperbolic rotations is used in the special theory of relativity to obtain the relativistic composition-law of velocities. Let us consider three Galilean frames \mathcal{S} , \mathcal{S}' and \mathcal{S}'' . The relative velocity of \mathcal{S}' with respect \mathcal{S} is $v_m e_m$ and that of \mathcal{S}'' with respect to \mathcal{S}' is $v_n e_n$. The corresponding rapidities are δ_m and δ_n with $\tanh \delta_m = -\beta_m = -v_m/c$ and $\tanh \delta_n = -\beta_n = -v_n/c$ (see Section 2.5.3 for a definition).

If $e_n = e_m$, we have $u = \exp[i e_m (\delta_m + \delta_n)/2] = \exp(i e_m \delta/2)$, with $\delta = \delta_m + \delta_n$, so that u is also a hyperbolic rotation, corresponding to the relative velocity of \mathcal{S}'' with respect to \mathcal{S} in the form $v e_m$, with $v = c\beta = -c \tanh \delta$. The equality $\delta = \delta_m + \delta_n$ constitutes the **addition law of rapidities** in that case. From $\delta = \delta_m + \delta_n$, we deduce

$$\beta = -\tanh \delta = -\tanh(\delta_m + \delta_n) = -\frac{\tanh \delta_m + \tanh \delta_n}{1 + \tanh \delta_m \tanh \delta_n} = \frac{\beta_m + \beta_n}{1 + \beta_m \beta_n}, \quad (118)$$

that is

$$v = \frac{v_m + v_n}{1 + \frac{v_m v_n}{c^2}}, \quad (119)$$

which is the relativistic composition-law for collinear velocities.

If e_m and e_n are not collinear, the product of the two hyperbolic rotations includes a pure rotation in addition to a boost and leads to the effect known as Thomas precession (see Section 6.2).

2.5.9 Superposition of finite rotations

The infinitesimal rotations represented by $\exp(e_v \eta ds/2)$ and $\exp(e_w \zeta ds/2)$ commute and

$$\exp\left(e_v \frac{\eta}{2} ds\right) \exp\left(e_w \frac{\zeta}{2} ds\right) = \exp\left[\frac{1}{2}(e_v \eta + e_w \zeta) ds\right]. \quad (120)$$

The succession of such rotations for s varying from 0 to S is represented by

$$u = \exp\left[(e_v \eta + i e_w \zeta) \frac{S}{2}\right], \quad (121)$$

which takes the form $u = \exp(e_u \psi/2)$, where ψ is a complex number and e_u a complex pure quaternion.

We will say that u represents the **superposition** of finite rotations represented by quaternions $\exp(e_v \eta S/2)$ and $\exp(e_w \zeta S/2)$. In particular, the superposition of finite proper Lorentz rotations represented by $\exp(e_m \varphi/2)$ and $\exp(i e_n \delta/2)$ is represented by $u = \exp(e_u \psi/2)$, where e_u is given by Eq. (90) and ψ is defined by Eqs. (88) and (89); it is the product of infinitesimal rotations of the form $\exp[(e_m \eta + i e_n \zeta) ds/2]$, with $\eta = \varphi/S$ and $\zeta = \delta/S$, that are generated by infinitesimal elliptic rotations $\exp(e_m \eta ds/2)$ and infinitesimal hyperbolic rotations $\exp(i e_n \zeta ds/2)$.

The complex quaternion u represents a proper Lorentz rotation and operates on minquats according to Eq. (85): $x' = u x \bar{u}^*$.

Remark 2.9 The superposition of the rotations represented by $\exp(e_m \varphi/2)$ and $\exp(i e_n \delta/2)$ is distinct from their composition, because for finite rotations, in general

$$\exp \frac{e_m \varphi + i e_n \delta}{2} \neq \exp \frac{e_m \varphi}{2} \exp \frac{i e_n \delta}{2}. \quad (122)$$

The left member of Inequality (122) represents the superposition of the two rotations, and the right member their composition (or product).

A physical example of a superposition of two rotations is given in Section 5.4.4.

2.6 The four-nabla and its transforms

2.6.1 The four-nabla differential form

We use the notation

$$\partial_\mu = \frac{\partial}{\partial x_\mu}, \quad (123)$$

where the x_μ 's are four coordinates in the Minkowski space. (In Relativity they are spatio-temporal coordinates with $x_0 = ct$.) The quaternionic representation of the four-nabla differential operator is defined by

$$\nabla = e_0 \partial_0 + i(e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3). \quad (124)$$

2.6.2 Transformation under a hyperbolic rotation

We consider a hyperbolic rotation on the Minkowski space, given by Eqs. (70–73), linking coordinates $\{x'_\mu\}$ to $\{x_\mu\}$. We denote

$$\partial'_\mu = \frac{\partial}{\partial x'_\mu}. \quad (125)$$

We apply the “chain rule”

$$\partial_\mu = \frac{\partial}{\partial x_\mu} = \sum_{\nu=0}^{\nu=3} \frac{\partial}{\partial x'_\nu} \frac{\partial x'_\nu}{\partial x_\mu} = \sum_{\nu=0}^{\nu=3} \partial_\mu x'_\nu \partial'_\nu, \quad (126)$$

and use Eqs. (70–73) to obtain

$$\partial_0 = \partial_0 x'_0 \partial'_0 + \partial_0 x'_1 \partial'_1 + \partial_0 x'_2 \partial'_2 + \partial_0 x'_3 \partial'_3 = \cosh \delta \partial'_0 + \sinh \delta \partial'_1. \quad (127)$$

We also obtain

$$\partial_1 = \partial_1 x'_0 \partial'_0 + \partial_1 x'_1 \partial'_1 + \partial_1 x'_2 \partial'_2 + \partial_1 x'_3 \partial'_3 = \cosh \delta \partial'_1 + \sinh \delta \partial'_0, \quad (128)$$

and

$$\partial_2 = \partial'_2, \quad \partial_3 = \partial'_3. \quad (129)$$

Then Eqs. (70–71) lead us to write

$$\partial'_0 = \cosh \delta \partial_0 - \sinh \delta \partial_1, \quad (130)$$

$$\partial'_1 = -\sinh \delta \partial_0 + \cosh \delta \partial_1, \quad (131)$$

$$\partial'_2 = \partial_2, \quad (132)$$

$$\partial'_3 = \partial_3. \quad (133)$$

The comparison of Eqs. (130–133) and Eqs. (70–73) shows that partial derivatives transform according to the inverse hyperbolic rotation of the previous one (δ is replaced with $-\delta$). With quaternions, we write

$$\nabla' = \exp\left(-i e_1 \frac{\delta}{2}\right) \nabla \exp\left(-i e_1 \frac{\delta}{2}\right). \quad (134)$$

If the hyperbolic rotation is along e_n we have

$$\nabla' = \exp\left(-i e_n \frac{\delta}{2}\right) \nabla \exp\left(-i e_n \frac{\delta}{2}\right). \quad (135)$$

2.6.3 Transformation under an elliptic rotation

The same method is applied to the rotation of angle φ around the axis e_1 , given by Eq. (65). We obtain

$$\partial'_0 = \partial_0, \quad (136)$$

$$\partial'_1 = \partial_1, \quad (137)$$

$$\partial'_2 = \cos \varphi \partial_2 - \sin \varphi \partial_3, \quad (138)$$

$$\partial'_3 = \sin \varphi \partial_2 + \cos \varphi \partial_3, \quad (139)$$

that is

$$\nabla' = \exp\left(e_1 \frac{\varphi}{2}\right) \nabla \exp\left(-e_1 \frac{\varphi}{2}\right). \quad (140)$$

We conclude that under an elliptic rotation the four-nabla transforms as coordinates do.

If the rotation is around e_n , we have

$$\nabla' = \exp\left(e_n \frac{\varphi}{2}\right) \nabla \exp\left(-e_n \frac{\varphi}{2}\right). \quad (141)$$

2.6.4 General transformation

Let u be a proper Lorentz rotation on the Minkowski space. From Eqs. (135) and (141) we deduce that under this rotation, the four-nabla is transformed according to

$$\nabla' = \bar{u} \nabla u^*. \quad (142)$$

Equation (142) includes both Eqs. (135) and (141).

2.7 Elliptic rotations on \mathbb{H}_c

An elliptic rotation on minquats is represented by a quaternion of the form $u = \exp(e_n \psi/2)$, where e_n is a real unit pure quaternion (the rotation axis) and ψ a real number (the rotation angle). A minquat x is transformed into the minquat x' such that

$$x' = u x u^*. \quad (143)$$

The rotation can be extended to complex quaternions: we consider the mapping defined on \mathbb{H}_c by

$$q \mapsto q' = u q u^*, \quad (144)$$

where u is a unit quaternion, so that $N(q') = N(q)$.

If $u = \exp(e_n \psi/2)$, with ψ a real number, we prove in Appendix C that the mapping R_r , defined by $R_r(q) = u q u^*$, is a rotation on \mathbb{H}_c , called a **real elliptic** rotation.

We may also consider imaginary values of ψ : if $\psi = i\theta$, we have $u = \exp(i e_n \theta/2)$, and we define a mapping R_i on \mathbb{H}_c by Eq. (144) once more. We prove in Appendix C that R_i is a rotation, called an **imaginary elliptic** rotation. A more general elliptic rotation is obtained for $\psi = \varphi + i\theta$ and corresponds to the notion of a **real-axis complex elliptic** rotation, represented by

$$u = \exp\left(e_n \frac{\psi}{2}\right) = \exp\left(e_n \frac{\varphi + i\theta}{2}\right) = \exp\left(e_n \frac{\varphi}{2}\right) \exp\left(i e_n \frac{\theta}{2}\right), \quad (145)$$

and operating according to Eq. (144), that is

$$q' = \exp\left(e_n \frac{\varphi + i\theta}{2}\right) q \exp\left(-e_n \frac{\varphi + i\theta}{2}\right). \quad (146)$$

Finally the most general elliptic rotation is represented by

$$u = \exp\left(e_u \frac{\psi}{2}\right), \quad (147)$$

where e_u is a complex unit pure quaternion and ψ a complex number (see Appendix C). We call it a **complex-axis elliptic** rotation; it operates on quaternions according to

$$q' = u q u^* = \exp\left(e_u \frac{\psi}{2}\right) q \exp\left(-e_u \frac{\psi}{2}\right). \quad (148)$$

A property of the previous elliptic rotations is the following. Let $q = q_0 e_0 + V$ (V is the vector part of q ; it is a pure quaternion) and let $q' = u q u^* = q'_0 e_0 + V'$. Since $u u^* = 1$, we have $u q_0 e_0 u^* = q_0 e_0$. We also have $(u V u^*)^* = u V^* u^* = -u V u^*$, which means that $u V u^*$ is a pure quaternion. Then necessarily $V' = u V u^*$ and $q'_0 = q_0$. An elliptic rotation on \mathbb{H}_c operates separately on scalar and vector parts of complex quaternions: the scalar part is preserved, and the vector part is transformed into the vector part of the image. (The same property holds for real elliptic rotations, so that it is legitimate to extend the notion of elliptic rotation to rotations with complex angles and to designate them as “complex elliptic rotations”.)

Remark 2.10 Formally, a complex elliptic rotation on \mathbb{H}_c is no more than a usual elliptic rotation whose angle is a complex number, because its representative quaternion can be written

$$u = \exp \frac{e_n \psi}{2} = \cos \frac{\psi}{2} + e_n \sin \frac{\psi}{2}, \quad (149)$$

where ψ is a complex number.

It can also be understood as a rotation on \mathbb{C}^3 , and is formally similar to a usual elliptic rotation on \mathbb{E}_3 , whose angle becomes a complex number. For example, the matrix form of the rotation of angle ψ and axis e_1 on \mathbb{C}^3 is similar to the submatrix \mathcal{R}_3 in Eq. (64) or Eq. (65) and is written

$$\begin{pmatrix} z'_1 \\ z'_2 \\ z'_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & -\sin \psi \\ 0 & \sin \psi & \cos \psi \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \quad (150)$$

where both (z_1, z_2, z_3) and (z'_1, z'_2, z'_3) belong to \mathbb{C}^3 , and $\psi = \varphi + i\theta$.

Finally, we indicate that u may also be written

$$\begin{aligned} u = \exp e_n \frac{\psi}{2} &= \exp\left(e_n \frac{\varphi + i\theta}{2}\right) = \cos\left(\frac{\varphi + i\theta}{2}\right) + e_n \sin\left(\frac{\varphi + i\theta}{2}\right) \\ &= \cos \frac{\varphi}{2} \cosh \frac{\theta}{2} - i \sin \frac{\varphi}{2} \sinh \frac{\theta}{2} + e_n \left(\sin \frac{\varphi}{2} \cosh \frac{\theta}{2} + i \cos \frac{\varphi}{2} \sinh \frac{\theta}{2} \right). \end{aligned} \quad (151)$$

2.8 Classification of rotations

At this point, it is useful to recapitulate all the rotations that have been introduced and to inspect whether they preserve the scalar product of pure quaternions. Those rotations are represented by unitary quaternions in the form

$$u = \exp\left(e_u \frac{\psi}{2}\right), \quad (152)$$

where ψ is a complex number and e_u is generally a complex unit pure quaternion or possibly a zero-norm pure quaternion.

We point out, however, that knowing the representative quaternion of a rotation is not enough to characterize the rotation. The knowledge of the vector space on which the rotation is defined is also necessary and induces the way the rotation operates. For example a quaternion of the form $\exp(i e_n \theta/2)$ (θ a real number) may represent a hyperbolic rotation (a proper Lorentz rotation) on \mathbb{M} and it operates according to $x' = \exp(i e_n \theta/2) x \exp(i e_n \theta/2)$; it may also represent a complex elliptic rotation on \mathbb{H}_c operating according to $q' = \exp(i e_n \theta/2) q \exp(-i e_n \theta/2)$.

Of course all the mentioned rotations are represented by unit quaternions and preserve the dot products and the norms of quaternions.

The rotations we are dealing with are the followings:

1. *Elliptic rotations on \mathbb{H}_c* : by definition, they operate on complex quaternions, according to Eq. (144), that is

$$q' = u q u^*, \quad (153)$$

and they preserve the dot products and the norms of complex quaternions. They preserve the scalar parts of quaternions and they transform a pure quaternion into a pure quaternion. They preserve the scalar product of real pure quaternions (that is, they preserve the scalar product in \mathbb{E}_3).

Particular elliptic rotations on \mathbb{H}_c are (e_n denotes a real unit pure quaternion):

- (a) *Real (elliptic) rotations*, represented by $u = \exp(e_n \varphi/2)$, with φ a real number. They are extension of elliptic rotations (defined on \mathbb{E}_3) to \mathbb{H}_c .
 - (b) *Imaginary (elliptic) rotations*, represented by $u = \exp(i e_n \theta/2)$, with θ a real number.
2. *Proper Lorentz rotations*: they operate on minquats, that is, on quaternions of the form

$$x = x_0 e_0 + i(x_1 e_1 + x_2 e_2 + x_3 e_3), \quad x_\mu \in \mathbb{R}, \quad (154)$$

according to

$$x' = u x \bar{u}^*. \quad (155)$$

Particular proper Lorentz rotations are (e_n is a real unit pure quaternion):

- (a) *Elliptic rotations*, represented by quaternions of the form $u = \exp(e_n \varphi/2)$, where φ is a real number. Then $\bar{u} = u$ and Eq. (155) reduces to

$$x' = u x \bar{u}^* = u x u^* = \exp\left(e_n \frac{\varphi}{2}\right) x \exp\left(-e_n \frac{\varphi}{2}\right). \quad (156)$$

Elliptic rotations transform vector parts of minquats into vector parts of their images (and scalar parts into scalar parts). They preserve both the dot product of minquats and the scalar product of their vector parts.

- (b) *Hyperbolic rotations*, represented by $u = \exp(i e_n \delta/2)$, where δ is a real number. Then $\bar{u}^* = u$ and Eq. (155) reduces to

$$x' = u x \bar{u}^* = u x u = \exp\left(i e_n \frac{\delta}{2}\right) x \exp\left(i e_n \frac{\delta}{2}\right). \quad (157)$$

A hyperbolic rotation preserves the dot product of minquats, but not the scalar product of their vector parts. If $x = x_0 e_0 + iV$ is a minquat (with $x_0 \neq 0$ and V a pure real quaternion) and if x' is its image by a hyperbolic rotation, with $\delta \neq 0$, then $x_0 e_0$ is not transformed into the scalar part of x' , and iV is not necessarily transformed into the vector part of x' .

- (c) *Singular rotations*, represented by $u = \exp[(e_m \pm i e_n) \varphi/2] = e_0 + (e_m \pm i e_n) \varphi/2$, where φ is a real number and e_m and e_n two orthogonal real unit pure quaternions. They operate according to

$$x' = u x \bar{u}^* = \left[e_0 + (e_m \pm i e_n) \frac{\varphi}{2} \right] x \left[e_0 - (e_m \mp i e_n) \frac{\varphi}{2} \right]. \quad (158)$$

3 Electromagnetism

3.1 Classical expressions of the fields

Both the electric and magnetic fields are described as vector fields that are deduced from the scalar and vector potentials, respectively denoted Φ and \mathbf{a} . The three-dimensional nabla operator is denoted by ∇ . If c denotes light velocity, the electric-field vector is

$$\mathbf{E} = -\nabla\Phi - c\partial_0\mathbf{a}, \quad (159)$$

and the magnetic-field vector (also called magnetic-induction vector, to distinguish it from the magnetic-field \mathbf{H})

$$\mathbf{B} = \nabla \times \mathbf{a}. \quad (160)$$

At every point (x_1, x_2, x_3) and time t , by Eq. (159) the electric field is

$$\mathbf{E}(x_0, x_1, x_2, x_3) = -\nabla\Phi(x_0, x_1, x_2, x_3) - c\partial_0\mathbf{a}(x_0, x_1, x_2, x_3), \quad (161)$$

where $x_0 = ct$. A similar coordinate dependence can be established for the magnetic field from Eq. (160).

3.2 Expression from the four-potential

Four-potential. In the coordinate system $\{x_\mu\}$, attached to a Galilean inertial frame \mathcal{S} , the 4-potential is $\mathbf{A} = (\Phi/c, \mathbf{a}) = (\Phi/c, a_1, a_2, a_3)$, where Φ and the a_j 's are functions of the x_μ 's. The 4-potential is represented by the minquat $A = (\Phi/c)\mathbf{e}_0 + \mathbf{ia}$, where $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$.

If A' denotes the quaternionic representation of the 4-potential in the coordinate system $\{x'_\mu\}$ of another Galilean inertial frame \mathcal{S}' in uniform translatory motion with respect to \mathcal{S} , since A and A' are minquats, they are linked by a proper Lorentz rotation, represented by u , and $A' = uA\bar{u}^*$. The proper rotation is hyperbolic only if the spatial-coordinate axes are parallel to each other ($x_j \parallel x'_j, j = 1, 2, 3$). More generally, the proper rotation may be the compound of hyperbolic and elliptic rotations.

Field complex-vector. The calculation of ∇A gives

$$\nabla A = \begin{bmatrix} \partial_0 \\ \mathbf{i}\partial_1 \\ \mathbf{i}\partial_2 \\ \mathbf{i}\partial_3 \end{bmatrix} \begin{bmatrix} \Phi/c \\ \mathbf{ia}_1 \\ \mathbf{ia}_2 \\ \mathbf{ia}_3 \end{bmatrix} = \begin{bmatrix} (1/c)\partial_0\Phi + \partial_1a_1 + \partial_2a_2 + \partial_3a_3 \\ \mathbf{i}\partial_0a_1 + (i/c)\partial_1\Phi - \partial_2a_3 + \partial_3a_2 \\ \mathbf{i}\partial_0a_2 + (i/c)\partial_2\Phi + \partial_1a_3 - \partial_3a_1 \\ \mathbf{i}\partial_0a_3 + (i/c)\partial_3\Phi - \partial_1a_2 + \partial_2a_1 \end{bmatrix}, \quad (162)$$

that is

$$\nabla A = \left(\frac{1}{c}\partial_0\Phi + \nabla \cdot \mathbf{a}\right)\mathbf{e}_0 + \mathbf{i}\partial_0\mathbf{a} + \frac{\mathbf{i}}{c}\nabla\Phi - \nabla \times \mathbf{a}. \quad (163)$$

We adopt the Lorentz gauge, according to which

$$\frac{1}{c}\partial_0\Phi + \nabla \cdot \mathbf{a} = 0, \quad (164)$$

and we obtain

$$-\nabla A = \mathbf{B} + \frac{\mathbf{i}}{c}\mathbf{E}. \quad (165)$$

Equation (165) is the quaternionic form of Eqs. (159) and (160). The magnetic field is opposite to the real part of ∇A , and the electric field is the imaginary part multiplied by ic .

We remark that ∇A is not a minquat; it is a complex vector, sometimes called a bivector, an element of \mathbb{C}^3 .

A more explicit form of Eq. (165) is

$$-\nabla A = B_1 \mathbf{e}_1 + B_2 \mathbf{e}_2 + B_3 \mathbf{e}_3 + \frac{i}{c}(E_1 \mathbf{e}_1 + E_2 \mathbf{e}_2 + E_3 \mathbf{e}_3), \quad (166)$$

where the B_j 's and the E_j 's ($j = 1, 2, 3$) are functions of the x_μ 's ($\mu = 0, 1, 2, 3$).

3.3 Maxwell equations

3.3.1 Quaternionic expression of Maxwell equations

Maxwell equations in vacuum are classically expressed as follows

$$\nabla \times \mathbf{E} + c \partial_0 \mathbf{B} = 0, \quad (167)$$

$$\nabla \times \mathbf{B} - \frac{1}{c} \partial_0 \mathbf{E} = \mu_0 \mathbf{j}, \quad (168)$$

$$\langle \nabla, \mathbf{E} \rangle = \frac{\rho}{\varepsilon_0}, \quad (169)$$

$$\langle \nabla, \mathbf{B} \rangle = 0, \quad (170)$$

where ρ denotes the electric charge density and \mathbf{j} the current density at the considered point.

We use Eq. (166) and calculate

$$\begin{aligned} -\nabla^* (\nabla A) &= \begin{bmatrix} \partial_0 \\ -i\partial_1 \\ -i\partial_2 \\ -i\partial_3 \end{bmatrix} \begin{bmatrix} 0 \\ B_1 + (i/c)E_1 \\ B_2 + (i/c)E_2 \\ B_3 + (i/c)E_3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{c}(\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3) + i(\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3) \\ \partial_0 B_1 + \frac{i}{c}\partial_0 E_1 - i\partial_2 B_3 + \frac{1}{c}\partial_2 E_3 + i\partial_3 B_2 - \frac{1}{c}\partial_3 E_2 \\ \partial_0 B_2 + \frac{i}{c}\partial_0 E_2 + i\partial_1 B_3 - \frac{1}{c}\partial_1 E_3 - i\partial_3 B_1 + \frac{1}{c}\partial_3 E_1 \\ \partial_0 B_3 + \frac{i}{c}\partial_0 E_3 - i\partial_1 B_2 + \frac{1}{c}\partial_1 E_2 + i\partial_2 B_1 - \frac{1}{c}\partial_2 E_1 \end{bmatrix}, \end{aligned} \quad (171)$$

that is

$$\nabla^* \nabla A = \left(\frac{1}{c} \nabla \cdot \mathbf{E} - i \nabla \cdot \mathbf{B} \right) \mathbf{e}_0 - \partial_0 \mathbf{B} - \frac{1}{c} \nabla \times \mathbf{E} - \frac{i}{c} \partial_0 \mathbf{E} + i \nabla \times \mathbf{B}. \quad (172)$$

We introduce the current density four-vector, represented by

$$\mathbf{J} = c \rho \mathbf{e}_0 + i \mathbf{j} = c \rho \mathbf{e}_0 + i(j_1 \mathbf{e}_1 + j_2 \mathbf{e}_2 + j_3 \mathbf{e}_3). \quad (173)$$

Since $\varepsilon_0 \mu_0 c^2 = 1$, Maxwell equations—Eqs. (167–170)—reduce to

$$\nabla^* \nabla A = \mu_0 \mathbf{J}. \quad (174)$$

Equation (174) is the quaternionic form of Maxwell equations. (We recall that A is the quaternionic representation of the 4-potential; both A and \mathbf{J} are minquats.)

3.3.2 Relativistic invariance of Maxwell equations

Maxwell equations are invariant under every proper Lorentz rotation as shown by the following derivation. Let u be the representing quaternion of a proper Lorentz rotation that links two Galilean inertial frames \mathcal{S} and \mathcal{S}' . Then, according to Eqs. (85) and (142), and since u is unitary ($u u^* = 1$), we have

$$\nabla'^* \nabla' A' = (\bar{u} \nabla u^*)^* (\bar{u} \nabla u^*) (u A \bar{u}^*) = u \nabla^* \bar{u}^* \bar{u} \nabla u^* u A \bar{u}^* = u \nabla^* \nabla A \bar{u}^*. \quad (175)$$

Since J is a 4-vector, it transforms according to

$$J' = u J \bar{u}^*. \quad (176)$$

Then $\nabla'^* \nabla' A' = \mu_0 J'$ is equivalent to $\nabla^* \nabla A = \mu_0 J$, and we conclude that Maxwell equations are invariant under every proper Lorentz rotation.

3.3.3 Wave equation

Since $\nabla^* \nabla = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2$, Maxwell equations are equivalent to the set of the two equations

$$\begin{cases} (\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2) \Phi = \frac{\rho}{\epsilon_0}, & (177a) \\ (\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2) \mathbf{a} = \mu_0 \mathbf{j}. & (177b) \end{cases}$$

Equations (177a) and (177b) are wave equations for the scalar potential Φ and for the vector potential \mathbf{a} . Hence the notion of an electromagnetic wave.

3.4 Electromagnetic-field complex-vector and its transformation under a boost

3.4.1 Definition and effect of a boost

To represent the electromagnetic field, we introduce the complex vector \mathbf{F} such that

$$\mathbf{F} = \mathbf{B} + \frac{i}{c} \mathbf{E}, \quad (178)$$

or in quaternionic form

$$F = B + \frac{i}{c} E = -\nabla A. \quad (179)$$

Let us consider a boost, represented by the quaternion u , connecting two Galilean inertial frames \mathcal{S} and \mathcal{S}' . The 4-vector A transforms according to $A' = u A \bar{u}^*$, and the 4-nabla transforms according to $\nabla' = \bar{u} \nabla u^*$, so that ∇A transforms according to

$$\nabla' A' = (\bar{u} \nabla u^*) (u A \bar{u}^*) = \bar{u} \nabla A \bar{u}^*, \quad (180)$$

because u is unitary ($u^* u = 1$). Since u is a boost, we have $\bar{u} = u^*$, so that

$$\nabla' A' = u^* \nabla A u, \quad (181)$$

and by Eq. (179) we conclude

$$F' = u^* F u, \quad (182)$$

which can be written $F' = u^* F (u^*)^*$ and which means that when passing from \mathcal{S} to \mathcal{S}' , the field complex-vector F is transformed under the elliptic rotation on \mathbb{H}_c represented by u^* , according to Eq. (153). More precisely, u represents a boost and takes the form $u = \exp(i e_n \delta / 2)$, where both e_n and δ are real, so that Eq. (182) corresponds to the imaginary elliptic rotation represented by $u^* = \exp(-i e_n \delta / 2)$ (which operates on \mathbb{H}_c).

3.4.2 Effect of a boost on the electric and magnetic fields

We now consider a Lorentz boost and we shall obtain the usual expressions of the transformations of the electric and magnetic fields. According to Eq. (182) the boost operates on F as an imaginary elliptic rotation. For explicit derivations, we use coordinates in \mathcal{S} , so that

$$F = B + \frac{i}{c}E = \begin{bmatrix} 0 \\ B_1 + (i/c)E_1 \\ B_2 + (i/c)E_2 \\ B_3 + (i/c)E_3 \end{bmatrix}, \quad (183)$$

and in \mathcal{S}'

$$F' = B' + \frac{i}{c}E' = \begin{bmatrix} 0 \\ B'_1 + (i/c)E'_1 \\ B'_2 + (i/c)E'_2 \\ B'_3 + (i/c)E'_3 \end{bmatrix}. \quad (184)$$

We consider a Lorentz boost, represented by $u = \exp(i\mathbf{e}_1\delta/2)$, and operating on \mathbb{H}_c . Then

$$\begin{aligned} u^* F u &= \exp\left(-i\mathbf{e}_1\frac{\delta}{2}\right) F \exp\left(i\mathbf{e}_1\frac{\delta}{2}\right) \\ &= \begin{bmatrix} \cosh(\delta/2) \\ -i\sinh(\delta/2) \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ B_1 + (i/c)E_1 \\ B_2 + (i/c)E_2 \\ B_3 + (i/c)E_3 \end{bmatrix} \begin{bmatrix} \cosh(\delta/2) \\ i\sinh(\delta/2) \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} i[B_1 + (i/c)E_1]\sinh(\delta/2) & & & \\ [B_1 + (i/c)E_1]\cosh(\delta/2) & & & \\ [B_2 + (i/c)E_2]\cosh(\delta/2) + i[B_3 + (i/c)E_3]\sinh(\delta/2) & & & \\ [B_3 + (i/c)E_3]\cosh(\delta/2) - i[B_2 + (i/c)E_2]\sinh(\delta/2) & & & \end{bmatrix} \begin{bmatrix} \cosh(\delta/2) \\ i\sinh(\delta/2) \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ B_1 + (i/c)E_1 \\ [B_2 + (i/c)E_2]\cosh\delta + i[B_3 + (i/c)E_3]\sinh\delta \\ [B_3 + (i/c)E_3]\cosh\delta - i[B_2 + (i/c)E_2]\sinh\delta \end{bmatrix}. \end{aligned} \quad (185)$$

The comparison of Eqs. (184) and (185) leads to

$$\begin{cases} E'_1 = E_1, & B'_1 = B_1, \\ E'_2 = E_2 \cosh\delta + cB_3 \sinh\delta, & B'_2 = B_2 \cosh\delta - (1/c)E_3 \sinh\delta, \\ E'_3 = E_3 \cosh\delta - cB_2 \sinh\delta, & B'_3 = B_3 \cosh\delta + (1/c)E_2 \sinh\delta. \end{cases} \quad (186)$$

Equations (186) provide relations between electric and magnetic field components in two Galilean inertial frames in relative translatory motion. Under a boost, the complex pure vector \mathbf{F} is transformed under an imaginary rotation [31], as already mentioned.

For a boost along \mathbf{e}_n we have

$$F' = \exp\left(-i\mathbf{e}_n\frac{\delta}{2}\right) F \exp\left(i\mathbf{e}_n\frac{\delta}{2}\right). \quad (187)$$

4 Electromagnetic waves

4.1 Some relativistic invariants

We first estimate $(\nabla A)^2$ in the sense of quaternions:

$$(\nabla A)^2 = (\nabla A)(\nabla A) = \left(B + \frac{i}{c}E\right) \left(B + \frac{i}{c}E\right) = \frac{\|E\|^2}{c^2} - \|B\|^2 - 2\frac{i}{c}E \cdot B, \quad (188)$$

which means that $(\nabla A)^2$ is a scalar. (Hint for the above derivation: since both E and B are real pure quaternions, we have $-E^2 = N(E) = \|E\|^2$, and the same for B . We also have $EB = -E \cdot B + E \times B$ and $BE = -E \cdot B + B \times E$, so that $EB + BE = -2E \cdot B$.)

Let us consider a proper Lorentz rotation, represented by u , that transforms A into A' and ∇ into ∇' . We apply Eq. (181) and write

$$(\nabla' A')^2 = (\nabla' A')(\nabla' A') = (\bar{u} \nabla A \bar{u}^*)(\bar{u} \nabla A \bar{u}^*), \quad (189)$$

and since u is unitary (i.e. $\bar{u}^* u = \overline{u^* u} = 1$), we obtain

$$(\nabla' A')^2 = \bar{u} (\nabla A)^2 \bar{u}^*. \quad (190)$$

Since $(\nabla A)^2$ is scalar, the product on the right side of Eq. (190) is commutative, and we deduce

$$(\nabla' A')^2 = (\nabla A)^2, \quad (191)$$

that is

$$\frac{\|E'\|^2}{c^2} - \|B'\|^2 - 2\frac{i}{c} E' \cdot B' = \frac{\|E\|^2}{c^2} - \|B\|^2 - 2\frac{i}{c} E \cdot B. \quad (192)$$

Eventually, the isomorphism between 3-vectors in \mathbb{E}_3 and real pure quaternions allows us to write $\|E\| = \|\mathbf{E}\|$, $\|B\| = \|\mathbf{B}\|$ and $E \cdot B = \langle \mathbf{E}, \mathbf{B} \rangle$, so that Eq. (192) provides the two following relativistic invariants

$$\frac{\|\mathbf{E}'\|^2}{c^2} - \|\mathbf{B}'\|^2 = \frac{\|\mathbf{E}\|^2}{c^2} - \|\mathbf{B}\|^2, \quad (193)$$

$$\langle \mathbf{E}', \mathbf{B}' \rangle = \langle \mathbf{E}, \mathbf{B} \rangle. \quad (194)$$

Equations (193) and (194) hold true for the most general proper Lorentz rotation, and in particular for pure rotations as well as for hyperbolic rotations of the Minkowski space.

4.2 Application to electromagnetic waves

A progressive monochromatic electromagnetic plane wave, propagating in free isotropic space, possesses the following properties:

1. It is a transverse wave.
2. Field vectors \mathbf{E} and \mathbf{B} at a same point are such that

$$\|\mathbf{E}\| = c\|\mathbf{B}\|. \quad (195)$$

(Equation (195) can be deduced from Maxwell equations). Equation (193) shows that this property holds true in every Galilean inertial frame.

3. Vectors \mathbf{E} and \mathbf{B} are orthogonal to each other, that is $\langle \mathbf{E}, \mathbf{B} \rangle = 0$. According to Eq. (194), this holds true in every Galilean inertial frame.

Since ∇A is a pure quaternion, we have $N(\nabla A) = -(\nabla A)^2$, and we deduce from Eqs. (188) and (195) and from $E \cdot B = 0$ that $N(\nabla A) = 0$, that is, $F = -\nabla A$ is an isotropic (or null) vector.

We now prove that property of item 1 above holds true in every Galilean inertial frame. We consider a Galilean frame \mathcal{S} . The energy-momentum of a photon, whose frequency is ν and direction of propagation is along the unit vector \mathbf{e}_n (or pure quaternion e_n), is a 4-vector represented by the minquat

$$P = \frac{h\nu}{c}(e_0 + i e_n), \quad (196)$$

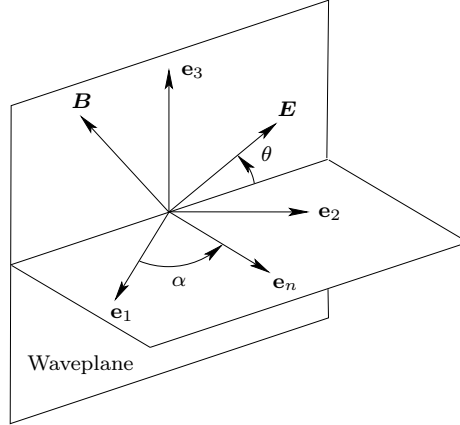


Figure 2: Plane lightwave propagating along \mathbf{e}_n in a Galilean reference frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The electric field \mathbf{E} and the magnetic field \mathbf{B} are transverse and orthogonal to each other.

where h denotes the Planck constant. The transversality of vectors \mathbf{E} and \mathbf{B} in \mathcal{S} is expressed by $\mathbf{e}_n \cdot \mathbf{E} = \langle \mathbf{e}_n, \mathbf{E} \rangle = 0$, and $\mathbf{e}_n \cdot \mathbf{B} = \langle \mathbf{e}_n, \mathbf{B} \rangle = 0$, so that

$$0 = \langle \mathbf{e}_n, \mathbf{B} \rangle + \frac{i}{c} \langle \mathbf{e}_n, \mathbf{E} \rangle = \mathbf{e}_n \cdot \left(\mathbf{B} + \frac{i}{c} \mathbf{E} \right) = \mathbf{e}_n \cdot \mathbf{F} = -\mathbf{e}_n \cdot \nabla A. \quad (197)$$

If \mathcal{S}' is another Galilean frame, in which the lightwave direction of propagation is \mathbf{e}'_n and the electromagnetic field is \mathbf{F}' , we have to prove that $\mathbf{e}'_n \cdot \mathbf{F}' = 0$. It would be tempting to say that the dot product $\mathbf{e}_n \cdot \mathbf{F}'$ is preserved under the boost. But we may not proceed this way, because \mathbf{F}' is transformed under an elliptic rotation on \mathbb{H}_c , whereas \mathbf{e}_n is not. To obtain the result, we will proceed as follows.

We consider a boost, represented by the quaternion u , between the Galilean frames \mathcal{S} (coordinates $\{x_\mu\}$) and \mathcal{S}' (coordinates $\{x'_\mu\}$). We assume $u = \exp(i\mathbf{e}_1\delta/2)$.

We first assume $\mathbf{e}_n = \mathbf{e}_1$ (in \mathcal{S} , the lightwave propagates along \mathbf{e}_1 , or axis x_1). In \mathcal{S}' the energy-momentum of the previous photon is represented by the minquat

$$P' = \frac{h\nu'}{c}(\mathbf{e}_0 + i\mathbf{e}'_n) = u P \bar{u}^* = \frac{h\nu}{c} \exp \frac{i\mathbf{e}_1\delta}{2} (\mathbf{e}_0 + i\mathbf{e}_1) \exp \frac{i\mathbf{e}_1\delta}{2} = \frac{h\nu}{c} e^\delta (\mathbf{e}_0 + i\mathbf{e}_1), \quad (198)$$

because $\mathbf{e}_0 + i\mathbf{e}_1$ is an eigenminquat of the hyperbolic rotation $x \mapsto u x \bar{u}^*$ (with eigenvalue $\exp \delta$). From Eq. (198) we conclude that $\mathbf{e}'_n = \mathbf{e}_1$, which means that in \mathcal{S}' , the lightwave propagates along \mathbf{e}_1 (axis x'_1 , identical to x_1). Since both the electric field \mathbf{E} and the magnetic field \mathbf{B} are transverse, their components on \mathbf{e}_1 are $E_1 = 0$ and $B_1 = 0$. According to Eqs. (186) we have $E'_1 = E_1 = 0$, which means that \mathbf{E}' is transverse. And \mathbf{B}' is also transverse, because $B'_1 = B_1 = 0$.

We now assume that \mathbf{e}_n belongs to the $(\mathbf{e}_1, \mathbf{e}_2)$ plane and that $\mathbf{e}_n \neq \mathbf{e}_1$ (see Fig. 2). Then $\mathbf{e}_n = \mathbf{e}_1 \cos \alpha + \mathbf{e}_2 \sin \alpha$ ($\alpha \neq 0$).

In the coordinate system $\{x'_\mu\}$ the energy-momentum of the previous photon is represented by the minquat

$$P' = \frac{h\nu'}{c}(\mathbf{e}_0 + i\mathbf{e}'_n) = u P \bar{u}^* = \frac{h\nu}{c} \exp \frac{i\mathbf{e}_1\delta}{2} (\mathbf{e}_0 + i\mathbf{e}_n) \exp \frac{i\mathbf{e}_1\delta}{2}, \quad (199)$$

that is

$$P' = \frac{h\nu}{c} \begin{bmatrix} \cosh \delta + \sinh \delta \cos \alpha \\ i \sinh \delta + i \cosh \delta \cos \alpha \\ i \sin \alpha \\ 0 \end{bmatrix}. \quad (200)$$

We note that e'_n is proportional to the pure quaternion $N' = e_1(\sinh \delta + \cosh \delta \cos \alpha) + e_2 \sin \alpha$. To lighten the notations we use Eq. (83) and write $\cosh \delta = \gamma$ and $\sinh \delta = -\gamma\beta$, so that $N' = \gamma e_1(\cos \alpha - \beta) + e_2 \sin \alpha$.

If θ denotes the angle between the electric field \mathbf{E} (represented by the pure quaternion E) and the plane (e_1, e_2) , then (see Fig. 2)

$$E = \tilde{E}(-e_1 \cos \theta \sin \alpha + e_2 \cos \theta \cos \alpha + e_3 \sin \theta), \quad (201)$$

where \tilde{E} denotes the electric field amplitude (it is a scalar complex-valued function depending on the x_μ 's).

The magnetic field is represented by the pure quaternion

$$B = \frac{\tilde{E}}{c}(e_1 \sin \theta \sin \alpha - e_2 \sin \theta \cos \alpha + e_3 \cos \theta). \quad (202)$$

(We may check that $e_n \cdot E = 0 = e_n \cdot B$ and $E \cdot B = 0$.)

According to Eqs. (186) we have

$$E' = E'_1 e_1 + E'_2 e_2 + E'_3 e_3 \quad (203)$$

$$\begin{aligned} &= E_1 e_1 + \gamma(E_2 - c\beta B_3) e_2 + \gamma(E_3 + c\beta B_2) e_3 \\ &= \tilde{E}[-e_1 \cos \theta \sin \alpha + e_2 \gamma(\cos \theta \cos \alpha - \beta \cos \theta) + e_3 \gamma(\sin \theta - \beta \sin \theta \cos \alpha)] \end{aligned} \quad (204)$$

and we obtain

$$\begin{aligned} N' \cdot E' &= \tilde{E}[-\gamma \cos \theta \sin \alpha \cos \alpha + \gamma\beta \cos \theta \sin \alpha + \gamma \cos \theta \cos \alpha \sin \alpha - \gamma\beta \cos \theta \sin \alpha] \\ &= 0. \end{aligned} \quad (205)$$

Similarly we have

$$cB' = cB'_1 e_1 + cB'_2 e_2 + cB'_3 e_3 \quad (206)$$

$$\begin{aligned} &= cB_1 e_1 + \gamma(cB_2 + \beta E_3) e_2 + \gamma(cB_3 - \beta E_2) e_3 \\ &= \tilde{E}[e_1 \sin \theta \sin \alpha + e_2 \gamma(-\sin \theta \cos \alpha + \beta \sin \theta) + e_3 \gamma(\cos \theta - \beta \cos \theta \cos \alpha)] \end{aligned} \quad (207)$$

and eventually

$$\begin{aligned} N' \cdot B' &= (\tilde{E}/c)[\gamma \sin \theta \sin \alpha \cos \alpha - \gamma\beta \sin \theta \sin \alpha - \gamma \sin \theta \cos \alpha \sin \alpha + \gamma\beta \sin \theta \sin \alpha] \\ &= 0. \end{aligned} \quad (208)$$

Since N' is collinear to e'_n , we conclude that $e'_n \cdot F' = 0$, so that both \mathbf{E}' and \mathbf{B}' are transverse in \mathcal{S}' .

To complete the proof, indeed, we eventually point out that the previous configuration is the most general one, because given two Galilean frames, we may choose the x_1 and x'_1 -axes along the relative velocity, that is, e_1 collinear to the relative velocity, and then, by an appropriate rotation around e_1 , make e_2 be coplanar with e_1 and e_n (if $e_n \neq e_1$).

4.3 Doppler effect and aberration of light [29, 32, 33]

To show how quaternion algebra works, when applied to electromagnetic waves, we consider the following problem. Two Galilean inertial frames \mathcal{S} and \mathcal{S}' are in relative motion with velocity $v\mathbf{e}_m$ (v is a real number and $|v| < c$). A light ray is propagating along \mathbf{e}_n in \mathcal{S} . What is its propagating direction in \mathcal{S}' ? What is the relation between frequencies ν in \mathcal{S} and ν' in \mathcal{S}' ?

The relative motion of \mathcal{S}' with respect to \mathcal{S} corresponds to a hyperbolic rotation, represented by a unit quaternion u such that

$$u = \exp \frac{i\delta \mathbf{e}_m}{2} = \cosh \frac{\delta}{2} + i \mathbf{e}_m \sinh \frac{\delta}{2}, \quad (209)$$

with the rapidity δ defined according to Eqs. (83).

The energy-momentum of a photon propagating along \mathbf{e}_n in \mathcal{S} is a 4-vector whose minquat expression is

$$P = \frac{h\nu}{c}(\mathbf{e}_0 + i \mathbf{e}_n), \quad (210)$$

and in \mathcal{S}' it is

$$P' = \frac{h\nu'}{c}(\mathbf{e}_0 + i \mathbf{e}'_n). \quad (211)$$

We have (hyperbolic rotation)

$$P' = u P \bar{u}^* = \frac{h\nu}{c} \left(\cosh \frac{\delta}{2} + i \mathbf{e}_m \sinh \frac{\delta}{2} \right) (\mathbf{e}_0 + i \mathbf{e}_n) \left(\cosh \frac{\delta}{2} + i \mathbf{e}_m \sinh \frac{\delta}{2} \right). \quad (212)$$

By applying quaternion-algebra rules, we obtain

$$\begin{aligned} P' &= \frac{h\nu}{c} \left(\cosh \frac{\delta}{2} + i \mathbf{e}_m \sinh \frac{\delta}{2} + i \mathbf{e}_n \cosh \frac{\delta}{2} - \mathbf{e}_m \mathbf{e}_n \sinh \frac{\delta}{2} \right) \left(\cosh \frac{\delta}{2} + i \mathbf{e}_m \sinh \frac{\delta}{2} \right) \\ &= \frac{h\nu}{c} \left(\cosh^2 \frac{\delta}{2} + \sinh^2 \frac{\delta}{2} + 2i \mathbf{e}_m \cosh \frac{\delta}{2} \sinh \frac{\delta}{2} + i \mathbf{e}_n \cosh^2 \frac{\delta}{2} - \mathbf{e}_n \mathbf{e}_m \cosh \frac{\delta}{2} \sinh \frac{\delta}{2} \right. \\ &\quad \left. - \mathbf{e}_m \mathbf{e}_n \cosh \frac{\delta}{2} \sinh \frac{\delta}{2} - i \mathbf{e}_m \mathbf{e}_n \mathbf{e}_m \sinh^2 \frac{\delta}{2} \right) \\ &= \frac{h\nu}{c} \left(\cosh \delta + \mathbf{e}_m \cdot \mathbf{e}_n \sinh \delta + i \mathbf{e}_m \sinh \delta + i \mathbf{e}_n \cosh^2 \frac{\delta}{2} - i \mathbf{e}_m \mathbf{e}_n \mathbf{e}_m \sinh^2 \frac{\delta}{2} \right). \quad (213) \end{aligned}$$

From $(\mathbf{e}_m \mathbf{e}_n \mathbf{e}_m)^* = \mathbf{e}_m^* \mathbf{e}_n^* \mathbf{e}_m^* = -\mathbf{e}_m \mathbf{e}_n \mathbf{e}_m$, we conclude that $\mathbf{e}_m \mathbf{e}_n \mathbf{e}_m$ is a pure quaternion, so that the scalar part of P' is

$$\frac{h\nu'}{c} = \frac{h\nu}{c} (\cosh \delta + \mathbf{e}_m \cdot \mathbf{e}_n \sinh \delta). \quad (214)$$

Equation (214) expresses the Doppler effect. Since $\cosh \delta = \gamma$ and $\sinh \delta = -\gamma\beta$, Eq. (214) also writes

$$\frac{\nu'}{\nu} = \gamma(1 - \beta \mathbf{e}_m \cdot \mathbf{e}_n) = \frac{1 - \frac{v}{c} \mathbf{e}_m \cdot \mathbf{e}_n}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (215)$$

The longitudinal Doppler effect is obtained when $\mathbf{e}_m = \pm \mathbf{e}_n$; the transverse Doppler effect (a relativistic effect) when $\mathbf{e}_m \cdot \mathbf{e}_n = 0$ (\mathbf{e}_m and \mathbf{e}_n are orthogonal to each other).

Remark 4.1 The Doppler effect is usually referred to an observer at rest in \mathcal{S} who is receiving light from a moving light source, attached to \mathcal{S}' . In Eq. (215) the frequency ν' is the frequency measured in the frame attached to the moving light source (proper frequency of the source) and ν is the frequency measured by the observer at rest. If the source moves radially away from the observer, then $v > 0$. Moreover since light travels towards the observer, we have $e_n = -e_m$ and we obtain

$$\frac{\nu'}{\nu} = \frac{1 + \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} > 1. \quad (216)$$

There is a “red shift”: the frequency in \mathcal{S} is measured smaller than in \mathcal{S}' . If the light source approaches the observer radially ($v < 0$), then $\nu > \nu'$, and there is a “blue shift.”

Aberration of light (stellar aberration). According to Eq. (213), the light ray direction in \mathcal{S}' is given by e'_n such that

$$e'_n = \frac{e_m \sinh \delta + e_n \cosh^2 \frac{\delta}{2} - e_m e_n e_m \sinh^2 \frac{\delta}{2}}{\cosh \delta + e_m \cdot e_n \sinh \delta}. \quad (217)$$

Equation (217) expresses the “aberration of light”.

If $e_m = \pm e_n$, then $e_m e_n e_m = -e_n$, so that $e'_n = e_n$ (no aberration). If $e_m \cdot e_n = 0$, then $e_m e_n e_m = e_n$ and

$$e'_n = \frac{e_m \sinh \delta + e_n}{\cosh \delta}. \quad (218)$$

In general the unit pure quaternion e_n is chosen oriented from the star to the observer in \mathcal{S} , and e'_n from the star to the observer in \mathcal{S}' . Let us assume that $e_m \neq \pm e_n$. Then—in the subspace of real pure quaternions, isomorphic to \mathbb{E}_3 —the unit pure quaternion $-e_m e_n e_m$ is deduced from e_n under the rotation of angle π around e_m and it belongs to the plane (e_m, e_n) . Eq. (217) shows that e'_n also belongs to that plane. Moreover we have $e_m \cdot e_n = -e_m \cdot (e_m e_n e_m)$ (see Fig. 11 in Appendix D and replace e_p with e_m). Let θ be the angle between e_m and $-e_n$, taken from e_m to $-e_n$ (since e_n is oriented from the star to the observer, θ is the angle under which the observer in \mathcal{S} see the light coming from the star). Let θ' be the equivalent of θ for the observer in \mathcal{S}' . We have $\cos \theta = -e_m \cdot e_n = e_m \cdot (e_m e_n e_m)$ and $\cos \theta' = -e_m \cdot e'_n$. Since $\cosh \delta = \gamma$ and $\sinh \delta = -v/c$, Eq. (217) becomes

$$\cos \theta' = -e_m \cdot e'_n = \frac{\frac{v}{c} + \cos \theta}{1 + \frac{v}{c} \cos \theta}, \quad (219)$$

from which we deduce

$$\tan^2 \frac{\theta'}{2} = \frac{1 - \cos \theta'}{1 + \cos \theta'} = \frac{1 - \frac{v}{c}}{1 + \frac{v}{c}} \cdot \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{1 - \frac{v}{c}}{1 + \frac{v}{c}} \tan^2 \frac{\theta}{2}. \quad (220)$$

Since θ and θ' belong to $[0, \pi]$ and since $|v| < c$, we eventually obtain the general light aberration in the form [29, 30]

$$\tan \frac{\theta'}{2} = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \tan \frac{\theta}{2}. \quad (221)$$

If e_m and e_n are orthogonal, then $\theta = \pi/2$; and we obtain

$$\tan \frac{\theta'}{2} = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}, \quad (222)$$

that is, $\cos \theta' = v/c$, which can also be deduced from Eq. (218).

4.4 Light polarization as a relativistic invariant

We consider a Lorentz boost along e_1 once more, corresponding to two Galilean frames \mathcal{S} and \mathcal{S}' : light is propagating along e_n in \mathcal{S} , and e'_n in \mathcal{S}' . For the sake of simplicity, we choose e_n in the (e_1, e_2) plane (see Fig. 2), that is

$$e_n = e_1 \cos \alpha + e_2 \sin \alpha. \quad (223)$$

(We may assume $-\pi/2 \leq \alpha \leq \pi/2$.)

The energy-momentum 4-vector of a photon is represented by P in \mathcal{S} and by P' in \mathcal{S}' according to Eqs. (196) and (199). The general form of the quaternion representing the electric-field vector associated with a rectilinearly polarized plane wave is (\tilde{E} denotes the field complex-amplitude)

$$E = \tilde{E}(-e_1 \cos \theta \sin \alpha + e_2 \cos \theta \cos \alpha + e_3 \sin \theta), \quad (224)$$

where θ is the angle of E with the (e_1, e_2) plane (see Fig. 2). The corresponding magnetic-field vector is

$$B = \frac{\tilde{E}}{c}(e_1 \sin \theta \sin \alpha - e_2 \sin \theta \cos \alpha + e_3 \cos \theta). \quad (225)$$

We use Eqs. (186) and obtain, in \mathcal{S}' ,

$$E' = \tilde{E}[-e_1 \cos \theta \sin \alpha + e_2(\cos \theta \cos \alpha \cosh \delta + \cos \theta \sinh \delta) + e_3(\sin \theta \cosh \delta + \sin \theta \cos \alpha \sinh \delta)]. \quad (226)$$

We first assume that the considered lightwave is horizontally polarized in \mathcal{S} : we may set $\theta = 0$, in Eq. (226), that is

$$E' = \tilde{E}[-e_1 \sin \alpha + e_2(\cos \alpha \cosh \delta + \sinh \delta)]. \quad (227)$$

We conclude that the polarization is also horizontal in \mathcal{S}' .

Moreover, from Eq. (227) we deduce (we use $\|E\| = |\tilde{E}|$)

$$\|E'\| = \|E\|(\sinh \delta \cos \alpha + \cosh \delta). \quad (228)$$

We now assume that the lightwave is vertically polarized in \mathcal{S}' : $\theta = \pi/2$. Then Eq. (226) gives

$$E' = \tilde{E} e_3(\cosh \delta + \cos \alpha \sinh \delta), \quad (229)$$

and the polarization is also vertical in \mathcal{S}' ; Equation (228) still holds true in the vertical case.

We eventually conclude as follows.

1. A rectilinear polarization with an arbitrary orientation (in the waveplane) is the superposition (in phase) of horizontal and vertical polarizations (linear combination). Since both horizontal and vertical polarizations are preserved with homothetic moduli when passing from \mathcal{S} to \mathcal{S}' , as shown by Eq. (228), which holds for both polarizations, the resulting polarization direction is also preserved: the polarization is the same whether observed in \mathcal{S} or in \mathcal{S}' .

2. If the considered lightwave is elliptically polarized, the electric field is the superposition of horizontal and vertical polarized fields with homothetic moduli and with a phase delay between them. Phase delays in \mathcal{S} and \mathcal{S}' are equal, and the polarization ellipse in \mathcal{S}' is homothetic to the ellipse in \mathcal{S} . (A proof of the invariance of the phase of a plane wave is given in the book of Møller [32], see §3, p. 6–7.)

The wave polarization is the same in \mathcal{S} and in \mathcal{S}' (this property of polarization is mentioned by Misner *et al.* in their book *Gravitation* [34], see Exercice 22.12 p. 577 and 581). Only the lightwave power is different, due to the Doppler effect.

Remark 4.2 Eq. (228) is no more than the relativistic longitudinal Doppler effect (see Eq. (214) for e_m and e_n collinear).

Remark 4.3 Light polarization is a relativistic invariant, but polarization states are not, because the polarization state of a lightwave includes the polarization but also the power of the lightwave (we anticipate Sections 4.5 and 5.1). Two observers in relative motion agree about the polarization of a lightwave in the sense that they measure homothetic oriented-countour ellipses, but the areas of the ellipses are not equal because of the Doppler effect.

4.5 Stokes parameters, degree of polarization

4.5.1 Stokes parameters [35–40]

We consider a polarized plane-lightwave with narrow spectrum around the frequency $\tilde{\nu}$. If $E_x(M, t)$ and $E_y(M, t)$ are the orthogonal components of the electric field in a plane wavefront (that is, an equiphase plane) at point M and time t , the **Stokes parameters** of the polarized wave are defined by

$$X_0 = \langle E_x \bar{E}_x + E_y \bar{E}_y \rangle_t, \quad (230)$$

$$X_1 = \langle E_x \bar{E}_x - E_y \bar{E}_y \rangle_t, \quad (231)$$

$$X_2 = \langle E_x \bar{E}_y + E_y \bar{E}_x \rangle_t, \quad (232)$$

$$X_3 = i \langle E_x \bar{E}_y - E_y \bar{E}_x \rangle_t, \quad (233)$$

where the bracket $\langle \rangle_t$ denotes a temporal average (taken on a time length much more longer than $1/\tilde{\nu}$).

Stokes parameters are observables: they are homogeneous to irradiances (up to a dimensional factor related to the propagation-medium impedance). Stokes parameters of a given lightwave are obtained by measuring light irradiances after the wave has passed through polarizers (also called analyzers in that case) with appropriate orientations (including circular polarizers). According to Section 4.4, two observers related to Galilean frames \mathcal{S} and \mathcal{S}' in relative motion measure Stokes parameters of a given lightwave that are proportionnal to each other (Stokes parameters are subject to Doppler effect). With Stokes parameters X_μ in \mathcal{S} ($X_0 \neq 0$) and X'_μ in \mathcal{S}' , for $X_j X'_j \neq 0$ we have

$$\frac{X'_0}{X_0} = \frac{X'_j}{X_j}, \quad j = 1, 2, 3. \quad (234)$$

4.5.2 Stokes vectors

The four Stokes parameters X_μ ($\mu = 0, 1, 2, 3$) form a 4-dimensional real vector belonging to \mathbb{R}^4 and written in matrix form as $\mathbf{X} = {}^t(X_0, X_1, X_2, X_3)$, where upperscript t means “transpose”, and called a **Stokes vector**. A vector \mathbf{X} in \mathbb{R}^4 is a Stokes vector, associated with a lightwave, only if $(X_0)^2 \geq (X_1)^2 + (X_2)^2 + (X_3)^2$, and $X_0 \geq 0$. The set of Stokes vector is thus a half cone \mathcal{C} included in \mathbb{R}^4 : if $\mathbf{X} \in \mathcal{C}$, then $\alpha \mathbf{X} \in \mathcal{C}$, for every $\alpha \geq 0$.

If $(X_0)^2 = (X_1)^2 + (X_2)^2 + (X_3)^2$, the Stokes vector corresponds to a **completely polarized** lightwave. The lightwave is **partially polarized** if $(X_0)^2 > (X_1)^2 + (X_2)^2 + (X_3)^2 > 0$; it is **unpolarized** (or natural) if $X_0 > 0$ and $X_1 = X_2 = X_3 = 0$. The Stokes vector of a completely polarized lightwave lies on the surface of \mathcal{C} . The interior of \mathcal{C} corresponds to partially polarized lightwaves.

In the previous meaning, Stokes vectors do not form a vector space (for example if \mathbf{X} is a Stokes vector, associated with a physical lightwave, the mathematical opposite $-\mathbf{X}$ is not a Stokes vector, in the sense that it does not correspond to a physical lightwave). Nevertheless the sum of two Stokes vectors makes sense. If \mathbf{X} and \mathbf{Y} are two Stokes vectors associated with lightwaves \mathcal{X} and \mathcal{Y} , then the Stokes vector $\mathbf{X} + \mathbf{Y}$ represents the polarization of the **incoherent superposition** of \mathcal{X} and \mathcal{Y} [38]. More generally, for every \mathbf{X} and \mathbf{Y} in \mathcal{C} and for every positive number α and β , a linear combination such as $\alpha\mathbf{X} + \beta\mathbf{Y}$ makes sense and actually represents a (partially) polarized lightwave. As a consequence, the half cone \mathcal{C} is convex. (A complete analysis of partially coherent superpositions of partially polarized lightwaves is given by Ramachandran and Ramaseshan [38].)

Example. Let us consider two completely polarized lightwaves, whose electric fields (in the respective planewaves) are $\mathbf{E}_1 = (E_0, 0)$ and $\mathbf{E}_2 = (0, E_0)$. The two lightwave polarizations are orthogonal to each other and their respective Stokes vectors are $\mathbf{X}_1 = {}^t(|E_0|^2, |E_0|^2, 0, 0)$ and $\mathbf{X}_2 = {}^t(|E_0|^2, -|E_0|^2, 0, 0)$. The Stokes vector of the incoherent superposition of the two lightwaves is $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 = {}^t(2|E_0|^2, 0, 0, 0)$ and corresponds to natural light.

4.5.3 Degree of polarization

The **degree of polarization** of a partially-polarized wave is defined by

$$\rho = \frac{\sqrt{X_1^2 + X_2^2 + X_3^2}}{X_0}, \quad (235)$$

where the X_μ 's are the Stokes parameters of the lightwave ($X_0 > 0$).

If $\rho = 1$, the considered lightwave is completely polarized, and if $\rho = 0$, the lightwave is unpolarized (natural light). As a consequence of Eq. (234), the degree of polarization of a given wave is a relativistic invariant: two observers in two Galilean inertial frames receiving a light beam measure identical degrees of polarizations (that is $\rho = \rho'$).

5 Quaternionic representation of polarization optics

5.1 Poincaré sphere and Stokes parameters

A harmonic plane wave generates an electromagnetic field at a given point O in the physical space. Over time, the end point of the electric field at O describes an ellipse in the plane tangent to the wave surface at O : the wave is said to be **elliptically polarized**. The ellipse is **left-handed** if the electric field rotates in the trigonometric sense (anticlockwise) for an observer looking towards the source of light (Fig. 3); it is **right-handed** if the electric field rotates in the opposite sense. A **oriented-contour ellipse** defines the **polarization state** of the harmonic wave. We say that the polarization state is left-handed (resp. right-handed) or levorotatory (resp. dextrorotatory) if the ellipse is left-handed (resp. right-handed). Particular polarization states are **circularly polarized** states and **rectilinearly polarized** states.

An ellipse is characterized by its orientation, given by the angle α its major axis makes with an arbitrary predetermined direction in the plane wave (which is generally the “horizontal” direction) and by its ellipticity, related to the angle χ (see Fig. 3). By abuse, we shall call χ the **ellipticity** of the ellipse, and also the ellipticity of the corresponding polarization state.

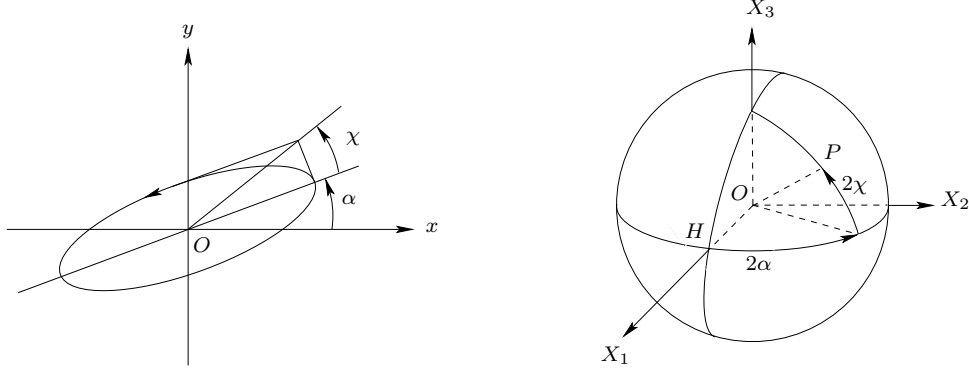


Figure 3: On the left: in a waveplane, the polarization ellipse is characterized by its orientation (given by α) with respect to the horizontal axis, and by its ellipticity (given by χ). On the right: on the Poincaré sphere, the corresponding polarization state is represented by the point P whose longitude is 2α and latitude is 2χ . The latitude is positive for levogyre states (the ellipse on the left diagram is left-handed). Point H on the X_1 -axis represents a horizontal rectilinear polarization state.

We will adopt the following rule: $\chi > 0$, for left-handed ellipses, and $\chi < 0$, for right-handed ellipses; we have $|\chi| \leq \pi/4$. The power of a polarization state is proportional to $a^2 + b^2$, where a and b are the lengths of the major and minor axes of the corresponding oriented-contour ellipse. Two oriented-contour ellipses with the same α and the same χ correspond to a same polarization state, up to a power (or energy) factor. We shall abusively say “the ellipse (α, χ) ”, assuming the power of the polarization state being equal to 1 (in S. I. units for example).

A polarization state is also represented by a point on the **Poincaré sphere** [35,38], which is a sphere of unit radius, referred to orthogonal axes X_1 , X_2 and X_3 (Fig. 3). The **equator** is the intersection of the sphere with the plane X_1 - X_2 , so that the X_3 -axis intercepts the sphere at the North and South poles. The **longitude** is taken along the equator, with origin in the plane X_1 - X_3 , and is positive if taken from X_1 to X_2 . The **latitude** is taken from the equator, along a great circle passing through the poles; it is positive for $X_3 \geq 0$. The ellipse (α, χ) is represented on the sphere by the point P of longitude 2α and latitude 2χ (Fig. 3); we write $P(2\alpha, 2\chi)$. The North pole corresponds to **left-circular** polarizations, the South pole to **right-circular** polarizations.

Two oriented-contour ellipses are said to be **orthogonal** if their major axes are orthogonal to each other and if they have opposite ellipticities (with equal absolute values): one ellipse is left-handed, the other right-handed. (Two orthogonal ellipses are similar: their major and minor axes are in the same ratio. They have the same shape and may differ by a scaling factor.) The corresponding polarization states are also said to be orthogonal. Orthogonality of elliptically-polarized states constitutes a generalization of the notion of orthogonal rectilinear-polarizations. Right-circular polarization states are orthogonal to left-circular polarization states. Let (α, χ) be an oriented-contour ellipse, represented on the Poincaré sphere by the point $P(2\alpha, 2\chi)$. The orthogonal ellipse is $(\alpha_\perp, \chi_\perp)$ such that $(\alpha_\perp, \chi_\perp) = (\alpha + \pi/2, -\chi)$ and is represented on the Poincaré sphere by the point $P_\perp(2\alpha + \pi, -2\chi)$. Points P and P_\perp are diametrically opposite and they define an axis of the sphere (see Fig. 4).

The Poincaré sphere may also be seen like a geometrical representation of Stokes vectors (defined in Section 4.5) [35,38]. Since the Stokes parameter X_0 of a lightwave is an irradiance, it is proportional to the lightwave power and the Poincaré sphere axes may be calibrated in irradiances. The Poincaré-sphere radius is generally taken equal to unity and a completely polarized lightwave with unitary irradiance X_0 is represented on the sphere by the point whose coordinates are (X_1, X_2, X_3) , where the X_μ 's are the Stokes parameters of the polarized wave.

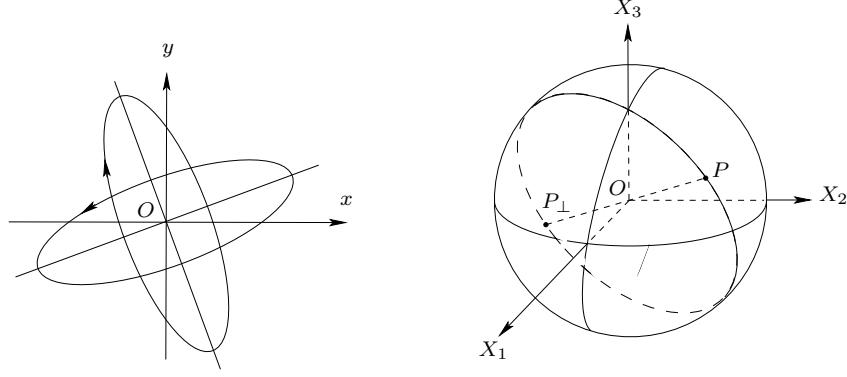


Figure 4: Two (unitary) orthogonal oriented-contour ellipses (on the left) are represented by two diametrically opposite points (P and P_\perp) on the Poincaré sphere (on the right).

Partially polarized waves with $X_0 < 1$ are represented by points inside the sphere.

The X_1 -axis is associated with horizontal and vertical rectilinear-polarizations. More precisely the horizontal rectilinear-polarization state is represented by the point $(1, 0, 0)$ and the vertical polarization by $(-1, 0, 0)$.

For a completely polarized lightwave, we have

$$X_0^2 = X_1^2 + X_2^2 + X_3^2. \quad (236)$$

If the lightwave is partially polarized, then

$$X_0^2 > X_1^2 + X_2^2 + X_3^2, \quad (237)$$

and the corresponding degree of polarization is given by Eq. (235).

5.2 Polarization states. Birefringence and dichroism. Polarizers

Basic properties of polarizers, birefringent and dichroic devices are assumed to be known, as well as their associated Mueller matrices [36–40].

5.2.1 Polarization states and minquats

With the Stokes vector $\mathbf{X} = {}^t(X_0, X_1, X_2, X_3)$, we associate the minquat [11, 12, 25, 26]

$$X = X_0 e_0 + i(X_1 e_1 + X_2 e_2 + X_3 e_3). \quad (238)$$

The norm of X is

$$N(X) = X_0^2 - X_1^2 - X_2^2 - X_3^2, \quad (239)$$

so that completely polarized waves are represented by isotropic (or null) minquats (whose norms are zero), of the form $X_0(1 + i e_p)$, where e_p is a real unit pure quaternion. If we refer to the Poincaré sphere (whose radius is equal to 1), we have $\mathbf{e}_p = \mathbf{OP}$, see Fig. 4. By abuse we say that the minquat $X_0(1 + i e_p)$ is polarized along e_p .

A partially polarized wave, with polarization degree ρ , is represented by a minquat of the form $X_0(1 + i \rho e_p)$.

Two orthogonal unitary polarization states are represented by opposite points, say P and P_\perp on the Poincaré sphere: $\mathbf{OP}_\perp = -\mathbf{OP}$. If $\mathbf{OP} = \mathbf{e}_p$, the representative minquats are $1 + i e_p$ and $1 - i e_p$. More generally, minquats $X_0(1 + i e_p)$ and $Y_0(1 - i e_p)$ represent orthogonal states of polarization.

Proposition 5.1 *Two null minquats with equal (positive) scalar parts represent two orthogonal completely-polarized states if, and only if, their vector parts are opposite to each other.*

5.2.2 Polarized components and unpolarized components

A minquat of the form $X = X_0(1 + i e_p)$ represents a completely-polarized state to which $X_\perp = X_0(1 - i e_p)$ is orthogonally polarized.

The (completely) **polarized component** of $X = X_0(1 + i \rho e_p)$ —where ρ is the degree of polarization—is $X_c = \rho X_0(1 + i e_p)$; its power is ρX_0 . The **unpolarized component** of X is $X_u = (1 - \rho)X_0$. We have $X = X_c + X_u$.

If $X = X_0(1 + i \rho e_p)$ and $X' = X_0(1 - i \rho e_p)$, then $X'_c = \rho X_0(1 - i e_p) = X_{c\perp}$, and $X'_u = \rho X_0 = X_u$: the polarized components of X and X' are orthogonal to each other, whereas their unpolarized components are equal.

5.2.3 Dichroic devices

A dichroic device absorbs different polarized waves by different amounts. It owns two orthogonal unitary polarization-eigenstates that are represented by two opposite points on the Poincaré sphere. Those two points define an axis on the sphere, oriented from the more absorbed state to the less absorbed one.

If we replace x_μ with X_μ in Eq. (74), we obtain the matrix expression of a pure dichroic device. More precisely, let $\mathbf{X} = {}^t(X_0, X_1, X_2, X_3)$ be the Stokes vector associated with a lightwave incident on a pure dichroic device whose eigenstates are horizontal and vertical polarization states, and whose dichroism is δ . The corresponding emerging lightwave polarization is represented by the vector $\mathbf{X}' = {}^t(X'_0, X'_1, X'_2, X'_3)$ such that

$$\begin{pmatrix} X'_0 \\ X'_1 \\ X'_2 \\ X'_3 \end{pmatrix} = \begin{pmatrix} \cosh \delta & \sinh \delta & 0 & 0 \\ \sinh \delta & \cosh \delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}, \quad (240)$$

where the square matrix is the Mueller matrix of the dichroic device.

According to Eqs. (74) and (75), the quaternionic form of Eq. (240) is

$$X' = \exp\left(i e_1 \frac{\delta}{2}\right) X \exp\left(i e_1 \frac{\delta}{2}\right), \quad (241)$$

where e_1 is the axis of the dichroic. The unit eigenstates of polarization are represented by $1 + i e_1$ and $1 - i e_1$ (we take $X_0 = 1$).

A more general pure-dichroic device has two orthogonal (unit) eigenstates represented by $1 + i e_n$ and $1 - i e_n$, and e_n is its axis on the Poincaré sphere. The dichroic device is represented by $u = \exp(i e_n \delta/2)$ and operates on minquats according to

$$X' = \exp\left(i e_n \frac{\delta}{2}\right) X \exp\left(i e_n \frac{\delta}{2}\right). \quad (242)$$

Equation (242) shows that pure dichroics operate on polarization minquats as hyperbolic rotations.

Pure dichroics and physical dichroics. If $\delta > 0$ and $X_1 > 0$, Eq. (240) shows that $X'_0 > X_0$, a result that cannot be obtained with a passive device. To represent a physical dichroic device we have to introduce an **isotropic absorption factor** κ , and we write

$$\begin{pmatrix} X'_0 \\ X'_1 \\ X'_2 \\ X'_3 \end{pmatrix} = \kappa \begin{pmatrix} \cosh \delta & \sinh \delta & 0 & 0 \\ \sinh \delta & \cosh \delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}. \quad (243)$$

The isotropic factor κ is a positive number. Since the power of \mathbf{X}' should be less than or equal to the power of \mathbf{X} , we have, for every X_0 ($X_0 \geq 0$) and X_1 ($|X_1| \leq X_0$)

$$X'_0 = \kappa(X_0 \cosh \delta + X_1 \sinh \delta) \leq X_0. \quad (244)$$

Since $\max(X_0 \cosh \delta + X_1 \sinh \delta) = X_0(\cosh \delta + \sinh |\delta|) = X_0 \exp |\delta|$, we conclude that κ is such that

$$0 \leq \kappa \exp |\delta| \leq 1. \quad (245)$$

With quaternions, Eq. (243) becomes

$$X' = \kappa \exp\left(\mathbf{i} e_n \frac{\delta}{2}\right) X \exp\left(\mathbf{i} e_n \frac{\delta}{2}\right), \quad (246)$$

and since $\kappa > 0$, the dichroic device is represented by the quaternion $\sqrt{\kappa} \exp(\mathbf{i} e_n \delta/2)$.

If $X_+ = X_0(1 + \mathbf{i} e_n)$, we obtain $X'_+ = \kappa \exp(\delta) X_+$, and if $X_- = X_0(1 - \mathbf{i} e_n)$, we obtain $X'_- = \kappa \exp(-\delta) X_-$. The polarization state X_+ is less absorbed than the state X_- ($\delta > 0$). On the Poincaré sphere the axis of the dichroic is $e_n = \mathbf{OP}$ and P represents the less absorbed state.

A pure dichroic ($\kappa = 1$) can be obtained if the device is active: the polarization state $X_0(1 + \mathbf{i} e_n)$ is amplified, whereas the state $X_0(1 - \mathbf{i} e_n)$ is mitigated (for $\delta > 0$). Such a result may be achieved in a laser cavity. More generally, we may have $\kappa > 1$ in a laser cavity in which an eigenstate is more amplified than the orthogonal eigenstate.

5.2.4 Birefringent devices

A birefringent device has two orthogonal unitary polarization-eigenstates that are represented by two opposite points on the Poincaré sphere, which define an axis; the device introduces a phase shift (or phase delay) φ —also called birefringence, with some abuse—between the eigenstates. If the fast axis corresponds to the horizontal polarization state and the slow axis to the vertical polarization, the Mueller matrix of the birefringent is as in Eq. (65), that is

$$\begin{pmatrix} X'_0 \\ X'_1 \\ X'_2 \\ X'_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \varphi & -\sin \varphi \\ 0 & 0 & \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}. \quad (247)$$

The quaternionic representation of the birefringent is $u = \exp(\mathbf{e}_1 \varphi/2)$ and according to Eq. (66) we have

$$X' = \exp\left(\mathbf{e}_1 \frac{\varphi}{2}\right) X \exp\left(-\mathbf{e}_1 \frac{\varphi}{2}\right). \quad (248)$$

On the Poincaré sphere, the birefringent is represented by the rotation of angle φ around \mathbf{e}_1 . For a general birefringence device, the rotation is around an axis defined by a unit vector \mathbf{e}_n and associated with the orthogonal eigenstates of the device; the axis (that is \mathbf{e}_n) is oriented from the slow-axis representative point to the fast one. The birefringent device is represented by $\exp(\mathbf{e}_n \varphi/2)$ and operates according to

$$X' = \exp\left(\mathbf{e}_n \frac{\varphi}{2}\right) X \exp\left(-\mathbf{e}_n \frac{\varphi}{2}\right). \quad (249)$$

Birefringents operate on polarization minquats as elliptic rotations.

5.2.5 Polarizers

A polarizer also admits two orthogonal unitary polarization-eigenstates that define an axis on the Poincaré sphere (one eigenstate is transmitted, the other is totally absorbed or reflected). If the axis is along e_p (corresponding to the transmitted eigenstate) on the Poincaré sphere, the polarizer is represented by

$$u = \frac{1}{2}(1 + ie_p), \quad (250)$$

and it operates on minquats according to $x' = ux\bar{u}^*$. Explicitly, if the incident state is represented by $X = X_0(1 + ie_n)$, the emerging state is

$$X' = \frac{X_0}{4}(1 + ie_p)(1 + ie_n)(1 + ie_p). \quad (251)$$

If $e_n = e_p$, from $(1 + ie_p)^2 = 2(1 + ie_p)$, we deduce $X' = X$; if $e_n = -e_p$, we obtain $X' = 0$, because $(1 + ie_p)(1 - ie_p) = 0$.

Malus Law. If $e_p = e_1$ (polarizer transmitting the horizontal polarization state) and if the incident wave is polarized along $e_n = n_1e_1 + n_2e_2 + n_3e_3$ (with $(n_1)^2 + (n_2)^2 + (n_3)^2 = 1$), then

$$X' = \frac{X_0}{4} \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ in_1 \\ in_2 \\ in_3 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} = \frac{X_0}{4} \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 + n_1 \\ i + in_1 \\ in_2 - n_3 \\ in_3 + n_2 \end{bmatrix} = \frac{X_0}{2} \begin{bmatrix} 1 + n_1 \\ i(1 + n_1) \\ 0 \\ 0 \end{bmatrix}, \quad (252)$$

that is,

$$X' = \frac{(1 + n_1)X_0}{2} (1 + ie_1). \quad (253)$$

The output state is completely polarized along e_1 (horizontal polarization), if $n_1 \neq -1$. The transmitted power is proportional to $(1 + n_1)/2$ (Malus law).

Pure polarizers and real polarizers. Equations (250) and (253) hold for pure polarizers. Physical (real) polarizers generally exhibit an isotropic absorption, as dichroic device do. We introduce an isotropic absorption factor κ ($0 < \kappa \leq 1$) and represent a real polarizer by $(\sqrt{\kappa}/2)(1 + ie_p)$. A pure polarizer corresponds to $\kappa = 1$.

Remark 5.1 (Polarizer as limit of a dichroic.) A polarizer may be thought of as the limit of a dichroic device. We consider a dichroic device represented by

$$u = \sqrt{\kappa} \exp \frac{ie_p \delta}{2} = \sqrt{\kappa} \left(\cosh \frac{\delta}{2} + ie_p \sinh \frac{\delta}{2} \right). \quad (254)$$

Let us make δ tend to infinity and κ to 0, under the constraint

$$\lim_{\substack{\kappa \rightarrow 0 \\ \delta \rightarrow +\infty}} \sqrt{\kappa} \exp \frac{\delta}{2} = \sqrt{\kappa_0} \leq 1, \quad (255)$$

Then

$$\lim_{\substack{\kappa \rightarrow 0 \\ \delta \rightarrow +\infty}} \sqrt{\kappa} \exp \frac{ie_p \delta}{2} = \frac{\sqrt{\kappa_0}}{2} (1 + ie_p). \quad (256)$$

If $\kappa_0 = 1$, the limit is a pure polarizer.

Many polarizers used in experiments are ‘‘Polaroid’’ also called sheet polarizers, initially and commercially developed by the Polaroid Corporation (created by Edwin Land, who invented those components) [36, 37, 39]. Strictly speaking, polaroids are not polarizers but dichroic devices (i.e. partial polarizers) with very large δ . Typical values are $\kappa_0 \sim 40\%$ and $\exp \delta \sim 10^5$.

Remark 5.2 (Analyzer) A polarizer, as a physical device, may be used to produce a polarization state—it acts as a polarizer in the strict sense—but also to analyze a polarization state—it acts then as an analyzer. Most often we will use the word “polarizer” for both functions.

5.2.6 Pure dephasors

Birefringent devices and pure dichroic devices are special cases of what we call **pure dephasors**, represented by unit quaternions in the form $u = \exp(\mathbf{e}_u \psi / 2)$, with \mathbf{e}_u a complex pure quaternion and ψ a complex number, and operating on polarization minquats according to

$$X' = u X' \bar{u}^*, \quad (257)$$

where X is the quaternionic representation of the polarization state incident on the pure dephaser, and X' represents the emerging state. The complex quaternion \mathbf{e}_u may be unitary or a null quaternion (an example is provided in Section 6.3.3).

Since u is a unit quaternion, we have $N(X') = N(X)$. A pure dephaser preserves the norm of quaternions associated with polarization states: the quantity $(X_0)^2 - (X_1)^2 - (X_2)^2 - (X_3)^2$ is invariant under a pure dephaser. The degree of polarization is a relativistic invariant; it is generally not invariant under a pure dephaser (it is invariant under a birefringent device, but not under a dichroic device).

Pure dephasors form a group isomorphic to $\text{SO}_+(1, 3)$ and generated by birefringent and dichroic devices [23, 26]. Pure dephasors operate on polarization minquats as proper Lorentz rotations. As already mentioned, dichroic devices are equivalent in crystal optics to Lorentz boosts in Relativity; birefringents are equivalent to elliptic (or pure) rotations.

Polarization eigenstates. We consider $u = \exp(\mathbf{e}_u \psi / 2)$ as above, with \mathbf{e}_u a complex pure quaternion that may be unitary or a null quaternion.

If \mathbf{e}_u is unitary, eigenminquats x' and x'' of the mapping $q \mapsto u q \bar{u}^*$ are given by Eqs. (101) and (102). We begin with real \mathbf{e}_u . Thus $\bar{\mathbf{e}}_u = \mathbf{e}_u$, and $x' = (1 + \mathbf{i} \mathbf{e}_u)^2 = 2(1 + \mathbf{i} \mathbf{e}_u)$. Similarly $x'' = (1 - \mathbf{i} \mathbf{e}_u)^2 = 2(1 - \mathbf{i} \mathbf{e}_u)$. According to Proposition 5.1, minquats x' and x'' represent orthogonal polarization states (they are not unitary).

We now assume that \mathbf{e}_u is a complex unit pure quaternion: $\mathbf{e}_u = \mathbf{e}_m + \mathbf{i} \mathbf{e}_n$, where \mathbf{e}_m and \mathbf{e}_n are real unit pure quaternions. We have $\mathbf{e}_u + \bar{\mathbf{e}}_u = 2 \mathbf{e}_m$, and

$$\mathbf{e}_u \bar{\mathbf{e}}_u = (\mathbf{e}_m + \mathbf{i} \mathbf{e}_n)(\mathbf{e}_m - \mathbf{i} \mathbf{e}_n) = 2\mathbf{i} \mathbf{e}_m \times \mathbf{e}_n, \quad (258)$$

so that eigenminquats with equal scalar parts are

$$x' = (1 + \mathbf{i} \mathbf{e}_u)(1 + \mathbf{i} \bar{\mathbf{e}}_u) = \mathbf{e}_0 + 2\mathbf{i} \mathbf{e}_m + 2\mathbf{i} \mathbf{e}_m \times \mathbf{e}_n, \quad (259)$$

and

$$x'' = (1 - \mathbf{i} \mathbf{e}_u)(1 - \mathbf{i} \bar{\mathbf{e}}_u) = \mathbf{e}_0 - 2\mathbf{i} \mathbf{e}_m + 2\mathbf{i} \mathbf{e}_m \times \mathbf{e}_n. \quad (260)$$

Minquats x' and x'' represent polarization states that are not orthogonal to each other, because the vector parts of x' and x'' are not opposite to each other (see Proposition 5.1).

If \mathbf{e}_u is a complex null pure quaternion the mapping $x \mapsto u x \bar{u}^*$ is a singular rotation on the minquat space. As proved in Appendix F, there is only one null-minquat eigendirection, the corresponding eigenvalue being equal to 1, that is, there is only one unitary polarization eigenstate (see Section 6.3.3 for an example).

Remark 5.3 (Linear and rectilinear operators) An operator \mathcal{L} on a complex vector-space is **linear** if for all vectors $|a\rangle$ and $|b\rangle$, and all complex numbers α and β , we have

$$\mathcal{L}(\alpha|a\rangle + \beta|b\rangle) = \alpha\mathcal{L}|a\rangle + \beta\mathcal{L}|b\rangle. \quad (261)$$

All birefringent and dichroic devices, as well as polarizers and their compounds, operating on polarization states, are linear operators in the previous mathematical meaning. On the other hand, a polarization operator is **rectilinear** if its eigenstates are rectilinear polarization states (the electric field oscillates along a straight line). For example an optically active medium or a circular polarizer are linear operators but not rectilinear ones. Thus a birefringent, which is a linear operator, may be a rectilinear, elliptic or circular birefringent.

Remark 5.4 The quaternionic representations of polarization states are introduced here from the usual definitions of Stokes vectors and the representations of birefringent and dichroic devices from their Mueller matrices. Doing so affords the advantage to refer to very known formalisms and to promote—if possible!—a rapid assimilation of the proposed calculus. Another advantage is to link polarization optics to conventional expressions of proper Lorentz transformations (in their matrix forms) as used in the special theory of relativity. We point out, however, that another way of introducing quaternions in polarization optics can be established from fundamental geometrical properties of polarized light, without referring to usual Stokes vectors nor to Mueller matrices [23–26]. That can be achieved in two steps: (i) Basic geometrical properties of polarized light lead us to regard polarization operators as proper Lorentz rotations, independently of any actual representation [23–25]; (ii) A quaternionic representation of proper Lorentz rotations can directly be deduced from abstract considerations, as shown by Synge [9]. Linking the two items leads us to intrinsically formulate the quaternionic representation of polarized light [23–26].

5.3 Consistency of representations

In the proposed representations of polarized light, some conventions are chosen: definitions of left and right-handed polarizations; left-handed states located on the northern hemisphere of the Poincaré sphere; birefringent axis on the Poincaré sphere orientated from the slow eigen-vibration representative point to the fast one; positive rotation angle for an observer to which

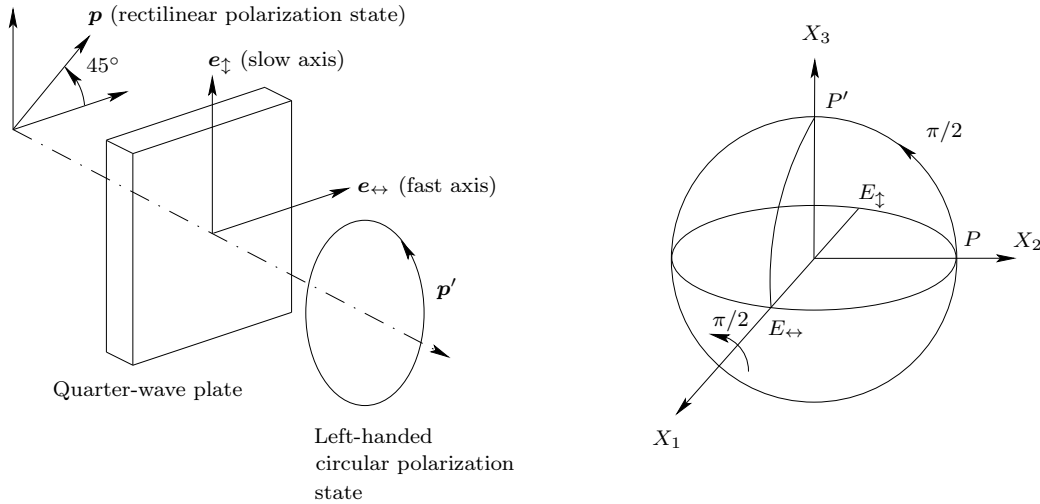


Figure 5: Checking the consistency of the representations of polarized light. If the incident state of polarization \mathbf{p} is rectilinear at 45° from the fast axis of the quarter-wave plate ($\mathbf{e}_{\leftrightarrow}$), the emerging state \mathbf{p}' is left-handed circular. On the Poincaré sphere, the quarter-wave plate axis is orientated from the slow-vibration representative-point E_{\dagger} to the fast vibration E_{\leftrightarrow} , and is along X_1 . The incident vibration is at point P , which is transformed into P' (North pole) under the rotation of angle $\pi/2$ around the X_1 -axis.

the birefringent axis is pointing. Although arbitrary, those conventions have to be consistent with physical reality.

We check the consistency of our conventions with the following example, illustrated on Fig. 5. A completely polarized lightwave is assumed to be incident on a quarter-wave plate whose fast axis is horizontal and slow axis vertical, in the waveplane (the quarter-wave plate is a birefringent device). The lightwave is polarized at 45° from the horizontal direction. Physically, the emerging lightwave is circularly polarized and left-handed [41].

On the Poincaré sphere, the birefringent axis is the X_1 -axis, and the effect of the waveplate is a $\pi/2$ rotation around X_1 , the sense of rotation being the trigonometric one (anticlockwise) for an observer to which X_1 is pointing. The input state is along X_2 , which goes to X_3 under the previous rotation. The emerging polarization state is thus circularly polarized and left-handed, in accordance with physical reality.

With quaternions, the input state is $X = X_0(1 + i e_2)$, and since the quarter-wave plate is represented by $u = \exp(e_1 \pi/4) = (\sqrt{2}/2)(1 + e_1)$, the emerging state is

$$\begin{aligned} X' &= u X \bar{u}^* = \frac{X_0}{2}(1 + e_1)(1 + i e_2)(1 - e_1) = \frac{X_0}{2}(1 + e_1 + i e_2 + i e_3)(1 - e_1) \\ &= X_0(1 + i e_3). \end{aligned} \quad (262)$$

The emerging state is circularly polarized and left-handed, in accordance with the previous results.

Another feature to be checked is related to optical activity, or rotatory power, as shown on Fig. 6. A propagation medium is optically active if the polarization state of a wave has been rotated after propagation through the medium. The rotation angle is the rotatory power, say ρ . Fresnel explained the phenomenon by regarding the medium as a circular birefringent medium whose eigenstates are circularly polarized. Those states may be regarded as fast and low vibrations of the medium. The phase shift between left-handed and right-handed circular states is twice the rotatory power, namely $\theta = 2\rho$. If the medium is left-handed, the fast vibration is the left-handed one, and the right-handed vibration is the slow one, so that $\rho > 0$.

On the Poincaré sphere, the medium is a circular birefringent whose axis is oriented from the slow axis (southern pole) to the fast one (northern pole), and is along X_3 . The effect of the medium is a rotation of angle $\theta = 2\rho > 0$ around the X_3 -axis, the positive sense being anticlockwise for an observer to which the axis X_3 is pointing. On the Poincaré sphere, the point P is transformed into P' under the rotation of angle $\theta = 2\rho$ around the X_3 -axis (Fig. 6). Since the longitude on the Poincaré sphere is twice the angle between the horizontal and the

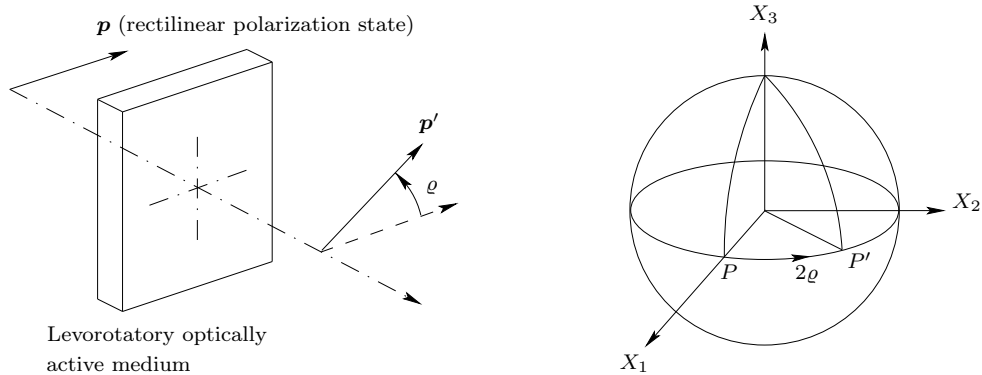


Figure 6: After passing through the optically active medium, the incident rectilinear state of polarization \mathbf{p} has turned an angle ρ , which is positive for a levorotatory medium. On the Poincaré sphere, the representative point P has turned an angle 2ρ , around the X_3 -axis.

polarization direction (in the waveplane), the angle between \mathbf{p} and \mathbf{p}' is ϱ .

With quaternions the input state is $X = X_0(1 + i e_1)$ and the medium is represented by $u = \exp(e_3\theta/2) = \exp e_3\theta$. The emerging state is obtained as follows

$$\begin{aligned} X' &= u X \bar{u}^* = X_0(\cos \varrho + e_3 \sin \varrho)(1 + i e_1)(\cos \varrho - e_3 \sin \varrho) \\ &= X_0(\cos \varrho + i e_1 \cos \varrho + i e_2 \sin \varrho + e_3 \sin \varrho)(\cos \varrho - e_3 \sin \varrho) \\ &= X_0[\cos^2 \varrho + \sin^2 \varrho + i e_1(\cos^2 \varrho - \sin^2 \varrho) + 2i e_2 \cos \varrho \sin \varrho] \\ &= X_0(1 + i e_1 \cos 2\varrho + i e_2 \sin 2\varrho). \end{aligned} \quad (263)$$

Then for $X_0 = 1$, we obtain $X'_1 = X_0 \cos 2\varrho = \cos 2\varrho$, $X'_2 = X_0 \sin 2\varrho = \sin 2\varrho$, and $X'_3 = 0$, so that on the Poincaré sphere $P'(\cos 2\varrho, \sin 2\varrho, 0)$ is deduced from $P(1, 0, 0)$ under the rotation of angle $\theta = 2\varrho$ around X_3 , which is consistent with the previous results.

5.4 Intrinsic derivations

The derivation in Eq. (252) is analytic in the sense that we explicitly use coordinates. Intrinsic derivations—that do not refer to coordinates—may be performed as illustrated by the following.

5.4.1 Composition of two half-wave plates

Since its birefringence (considered as a phase shift) is equal to π , a half-wave plate is represented by $\exp(e_n\pi/2) = e_n$, where e_n is the plate axis (on the Poincaré sphere). The succession of a half-wave plate of axis e_m (first plate to be crossed by light) and a half-wave plate of axis e_n ($e_n \neq \pm e_m$) is represented by the product $e_n e_m$, that is, by

$$u = e_n e_m = -e_n \cdot e_m + e_n \times e_m = -\cos \alpha - e_q \sin \alpha = -\exp e_q \alpha, \quad (264)$$

where α is the angle between e_m and e_n (on the Poincaré sphere), oriented from e_m to e_n ($\alpha \neq 0$) and where e_q , defined by $e_q \sin \alpha = e_m \times e_n$, is orthogonal to both e_m and e_n . Since u and $-u$ represent the same birefringent, we may represent the composition of the two half-wave plates by $u = \exp e_q \alpha$. Generally u represents an elliptic birefringent whose birefringence is equal to 2α .

If both wave plates are rectilinear, then $e_q = e_3$, and u represents a rotatory power equal to α (a well-known result), that is, a circular-birefringence phase-shift equal to 2α . (We note that in the “physical” space, the angle between the respective fast axes of the two half-wave plates is $\alpha/2$.)

5.4.2 Composition of two quarter-wave plates

A quarter-wave plate whose axis is e_n is represented by

$$\exp\left(e_n \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(1 + e_n), \quad (265)$$

because the birefringence is $\varphi = \pi/2$.

The succession of a quarter-wave plate, whose axis is e_m , and a quarter-wave plate, whose axis is e_n , is thus represented by

$$\begin{aligned} u &= \exp\left(e_q \frac{\psi}{2}\right) = \exp\left(e_n \frac{\pi}{4}\right) \exp\left(e_m \frac{\pi}{4}\right) = \frac{1}{2}(1 + e_n)(1 + e_m) \\ &= \frac{1}{2}(1 - e_n \cdot e_m + e_n + e_m + e_n \times e_m). \end{aligned} \quad (266)$$

In general u represents an elliptic birefringent. The birefringence ψ is given by

$$\cos \frac{\psi}{2} = \frac{1}{2}(1 - e_n \cdot e_m) = \frac{1}{2}(1 - \cos \alpha), \quad (267)$$

where α is the angle between e_m and e_n (on the Poincaré sphere). The axis is given by

$$e_q \sin \frac{\psi}{2} = \frac{1}{2}(e_n + e_m + e_n \times e_m). \quad (268)$$

If both wave plates are rectilinear birefringents, then $e_n \times e_m = -e_3 \sin \alpha$, where the angle α is taken from e_m to e_n . Let e_s be defined by

$$e_s = \frac{e_n + e_m}{2 \cos(\alpha/2)}. \quad (269)$$

The quaternion e_s is a real unit pure quaternion orthogonal to e_3 . We eventually obtain

$$e_q = \frac{1}{\sqrt{4 \cos^2(\alpha/2) + \sin^2 \alpha}} \left(2 e_s \cos \frac{\alpha}{2} - e_3 \sin \alpha \right). \quad (270)$$

Both ψ and e_q only depends on the relative positions of e_m and e_n and not on their coordinates.

5.4.3 Composition of rectilinear birefringence and optical activity

Let \mathcal{B} be a birefringent of axis e_n and birefringence φ ($-\pi < \varphi \leq \pi$), assumed to be rectilinear, which means that e_n takes the form $e_n = n_1 e_1 + n_2 e_2$. The birefringent \mathcal{B} is followed by an optically active device \mathcal{R} . Since the work of Fresnel, we know that the eigenstates of an optically active medium are the circularly polarized states, so that the axis of \mathcal{R} on the Poincaré sphere is e_3 . We also know that the phase shift θ between levorotatory and dextrorotatory circular polarizations is twice the rotatory power. We assume $-2\pi < \theta \leq 2\pi$. If $\theta > 0$, the rotatory power is levorotatory. If $\theta < 0$, the rotatory power is dextrorotatory, the axis still being e_3 . (Equivalently we may change both the axis e_3 into $-e_3$ and θ into $-\theta$, the optically active device does not change.)

The birefringent \mathcal{B} is represented by $\exp(e_n \varphi/2)$ and \mathcal{R} by $\exp(e_3 \theta/2)$. If $\varphi \theta = 0$ the composition of \mathcal{B} and \mathcal{R} is trivial: it reduces to \mathcal{B} , if $\theta = 0$; to \mathcal{R} , if $\varphi = 0$; to identity, if $\theta = \varphi = 0$. If $\varphi \theta \neq 0$, the composition $\mathcal{R} \circ \mathcal{B}$ is represented by

$$\begin{aligned} u &= \exp \frac{e_3 \theta}{2} \exp \frac{e_n \varphi}{2} = \left(\cos \frac{\theta}{2} + e_3 \sin \frac{\theta}{2} \right) \left(\cos \frac{\varphi}{2} + e_n \sin \frac{\varphi}{2} \right) \\ &= \cos \frac{\theta}{2} \cos \frac{\varphi}{2} + e_n \cos \frac{\theta}{2} \sin \frac{\varphi}{2} + e_q \sin \frac{\theta}{2} \sin \frac{\varphi}{2} + e_3 \sin \frac{\theta}{2} \cos \frac{\varphi}{2}, \end{aligned} \quad (271)$$

where $e_q = e_3 e_n = e_3 \times e_n$, because e_3 and e_n are orthogonal. We remark that $e_q = -n_2 e_1 + n_1 e_2$.

The quaternion u takes the form $\exp(e_p \psi/2)$ and it represents an elliptic birefringent whose birefringence is ψ such that

$$\cos \frac{\psi}{2} = \cos \frac{\theta}{2} \cos \frac{\varphi}{2}. \quad (272)$$

The birefringent axis is e_p given by

$$e_p = \frac{1}{\sqrt{1 - \cos^2 \frac{\theta}{2} \cos^2 \frac{\varphi}{2}}} \left(e_n \cos \frac{\theta}{2} \sin \frac{\varphi}{2} + e_q \sin \frac{\theta}{2} \sin \frac{\varphi}{2} + e_3 \sin \frac{\theta}{2} \cos \frac{\varphi}{2} \right). \quad (273)$$

5.4.4 Superposition of rectilinear birefringence and optical activity

There are crystals that mix birefringence and optical activity at the same time: every infinitesimal layer of a crystal sample exhibits rectilinear birefringence and optical activity. Such are photorefractive BSO or BGO crystals ($\text{Bi}_{12}\text{SiO}_{20}$ or $\text{Bi}_{12}\text{GeO}_{20}$) [10, 42–45], when an externally

electric field is applied to them. If no electric field is applied to it, a BSO-crystal sample exhibits optical activity, which depends on the crystal thickness (for example the rotatory power is about $22^\circ/\text{mm}$ at $\lambda = 633 \text{ nm}$). When an uniform electric field is applied, the crystal also exhibits rectilinear birefringence, due to the electro-optic effect (Pockels effect). If n_0 is the refractive index at wavelength λ , if r_{41} is the electro-optic coefficient of BSO-crystals and E the applied electric-field amplitude, the birefringence is $\Delta n = (1/2) n_0^3 r_{41} E$, for a transverse-field, perpendicular to $[100]$ crystallographic planes (light rays perpendicular to $[0\bar{1}1]$ planes) [44].

If ℓ is the optical thickness of a BSO-crystal sample, L the distance between the electrodes and V the applied voltage ($V = EL$), the birefringence Δn results in a phase shift (also abusively called birefringence) which is, at wavelength λ [44]

$$\varphi = \frac{2\pi\ell\Delta n}{\lambda} = \frac{\pi n_0^3 r_{41} V}{\lambda L} \ell = \eta\ell. \quad (274)$$

We call φ the **intrinsic rectilinear birefringence** of the crystal sample. It is the birefringence (phase shift) the crystal would exhibit under an applied electric-field if there were no optical activity.

If ρ denotes the rotatory power per unit length of BSO-crystals, the optical activity of the crystal sample is $\rho\ell$, and in phase, it is $\theta = 2\rho\ell$. We call θ the **intrinsic circular-birefringence** of the crystal ($\rho\ell$ would be the intrinsic optical activity).

Each infinitesimal layer of the crystal, whose thickness is $d\ell$, exhibits both intrinsic rectilinear birefringence $\eta d\ell$ and intrinsic circular birefringence $2\rho d\ell$ (or optical activity $\rho d\ell$). The whole crystal is the superposition of intrinsic rectilinear birefringence $\varphi = \eta\ell$ and intrinsic circular birefringence $\theta = 2\rho\ell$, in the sense of Section 2.5.9. The rectilinear birefringence axis can be taken along e_1 , whereas the circular birefringence axis is e_3 . The crystal (elliptic) birefringence is represented by

$$u = \exp \frac{(\eta e_1 + 2\rho e_3)\ell}{2} = \exp \frac{\varphi e_1 + \theta e_3}{2}, \quad (275)$$

and is the superposition of intrinsic rectilinear birefringence, represented by $\exp(e_1\varphi/2)$, and intrinsic circular birefringence, represented by $\exp(e_3\theta/2)$.

The quaternion u in Eq. (275) takes the form $\exp(e_n\psi/2)$, where ψ is a real number and e_n a real unit pure quaternion. We have

$$\psi = \sqrt{N(e_n\psi)} = \sqrt{\varphi^2 + \theta^2}, \quad (276)$$

and

$$e_n = \frac{1}{\sqrt{\varphi^2 + \theta^2}}(\varphi e_1 + \theta e_3) = e_1 \cos \chi + e_3 \sin \chi, \quad \tan \chi = \frac{\theta}{\varphi}. \quad (277)$$

(Only the sign of $e_n\psi$ is defined, so that e_n and ψ may be replaced with $-e_n$ and $-\psi$.)

On the Poincaré sphere the vector e_n is represented by the point of longitude 0 and latitude χ with $\tan \chi = \theta/\varphi$, i.e. the point $E(0, \chi)$ (see Section 5.1). It represents an elliptic polarization; the oriented-contour ellipse is $(0, \chi/2)$. The crystal sample is an elliptic birefringent.

Equivalent parameters. The previous elliptic birefringent may be thought of as the product (or composition) of a rectilinear birefringent, represented by $\exp(e_m\varphi'/2)$, and an optically active device, represented by $\exp(e_3\theta'/2)$, where φ' and θ' are called **equivalent rectilinear birefringence** and **equivalent circular birefringence** (the equivalent optical activity being $\rho' = \theta'/2$). The axis e_m of the equivalent rectilinear birefringent corresponds to a rectilinear polarization, so that $e_m = e_1 \cos \alpha' + e_2 \sin \alpha'$. The equivalent parameters are such that

$$u = \exp \frac{e_m\varphi'}{2} \exp \frac{e_3\theta'}{2}, \quad (278)$$

which means that the equivalent optical activity operates first. (Of course the choice of equivalent birefringence operating first is possible.)

In Appendix G we prove the following relations between intrinsic parameters and equivalent ones:

$$\begin{cases} \sin \frac{\varphi'}{2} = \cos \chi \sin \frac{\psi}{2} = \frac{\varphi}{\sqrt{\varphi^2 + \theta^2}} \sin \frac{\sqrt{\varphi^2 + \theta^2}}{2}, & (279a) \\ \tan \frac{\theta'}{2} = \sin \chi \tan \frac{\psi}{2} = \frac{\theta}{\sqrt{\varphi^2 + \theta^2}} \tan \frac{\sqrt{\varphi^2 + \theta^2}}{2}, & (279b) \\ \alpha' = \frac{\theta'}{2}. & (279c) \end{cases}$$

From an experimental point of view, it is interesting to reverse those equations and to express intrinsic parameters as functions of equivalent ones. We first remark that the intrinsic optical activity can be obtained by measuring the rotation of a rectilinear polarization state after propagation across the crystal sample, when no electric field is applied, that is, when the crystal is free of rectilinear birefringence. But the intrinsic birefringence, which appears when an electric field is applied to the crystal sample, is not directly accessible to experiment, because it is always mixed with the intrinsic optical activity. On the other hand, equivalent parameters can be measured according to ellipsometric methods. They thus offer us an indirect means of measuring the intrinsic birefringence. The method has been applied to the determination of the electro-optic coefficient of BSO crystals [10, 44].

Application to an elliptic birefringent. The method of the previous section may be applied to express every elliptic birefringent as the product of an equivalent optical activity and an equivalent rectilinear birefringent. The only difference is that the elliptic-birefringent axis is not necessarily in the X_1 - X_3 plane (on the Poincaré sphere) but in a plane of longitude α : if the elliptical birefringent is represented by $u = \exp(\mathbf{e}_n \psi/2)$, then Eq. (277) is replaced with $\mathbf{e}_n = \mathbf{e}_1 \cos \alpha \cos \chi + \mathbf{e}_2 \sin \alpha \cos \chi + \mathbf{e}_3 \sin \chi$. The equivalent parameters φ' , θ' and α' are given by

$$\begin{cases} \sin \frac{\varphi'}{2} = \cos \chi \sin \frac{\psi}{2}, & (280a) \\ \tan \frac{\theta'}{2} = \sin \chi \tan \frac{\psi}{2}, & (280b) \\ \alpha' - \alpha = \frac{\theta'}{2}. & (280c) \end{cases}$$

5.4.5 General Malus law

Equation (251) can also be treated intrinsically. Let $X = X_0(1 + i\mathbf{e}_n)$ be incident on a polarizer whose axis is \mathbf{e}_p . The outgoing state is represented by

$$\begin{aligned} X' &= \frac{X_0}{4}(1 + i\mathbf{e}_p)(1 + i\mathbf{e}_n)(1 + i\mathbf{e}_p) = \frac{X_0}{4}(1 + i\mathbf{e}_n + i\mathbf{e}_p - \mathbf{e}_p\mathbf{e}_n)(1 + i\mathbf{e}_p) \\ &= \frac{X_0}{4}(2 + 2\mathbf{e}_n \cdot \mathbf{e}_p + 2i\mathbf{e}_p + i\mathbf{e}_n - i\mathbf{e}_p\mathbf{e}_n\mathbf{e}_p). \end{aligned} \quad (281)$$

We apply Lemma 2 of Appendix D and obtain from Eq. (281)

$$X' = \frac{X_0}{2}(1 + \mathbf{e}_n \cdot \mathbf{e}_p)(1 + i\mathbf{e}_p) = \frac{X_0}{2}(1 + \cos \alpha)(1 + i\mathbf{e}_p), \quad (282)$$

where α is the angle between \mathbf{e}_n and \mathbf{e}_p , on the Poincaré sphere.

The Malus law takes the form

$$X'_0 = \frac{X_0}{2}(1 + \mathbf{e}_n \cdot \mathbf{e}_p). \quad (283)$$

If $X = X_0(1 + i\rho \mathbf{e}_n)$ —where ρ denotes the degree of polarization—we obtain

$$X' = \frac{X_0}{2}(1 + \rho \mathbf{e}_n \cdot \mathbf{e}_p)(1 + i \mathbf{e}_p) = \frac{X_0}{2}(1 + \rho \cos \alpha)(1 + i \mathbf{e}_p), \quad (284)$$

and

$$X'_0 = \frac{X_0}{2}(1 + \rho \mathbf{e}_n \cdot \mathbf{e}_p), \quad (285)$$

which is a general form of Malus law.

Equations (282), (283), (284) and (285) are intrinsic: only the relative positions of \mathbf{e}_n and \mathbf{e}_p matter.

5.5 Transforming the axis of a rectilinear birefringent

We examine the following issue: how to transform a given rectilinear birefringent \mathcal{B}_1 , of axis \mathbf{e}_1 , into a birefringent \mathcal{B}_p of pre-defined axis \mathbf{e}_p , with the help of rectilinear phase-plates, without changing the birefringence (phase delay)? The fast vibration of \mathcal{B}_1 is represented by P_0 , which is on the X_1 -axis, on the Poincaré sphere. The axis \mathbf{e}_p is along OP , where P should represent the fast vibration of the transformed birefringent \mathcal{B}_p . The longitude of P is 2α and its latitude is 2χ , so that (Fig. 7)

$$\mathbf{e}_p = \mathbf{e}_1 \cos 2\alpha \cos 2\chi + \mathbf{e}_2 \sin 2\alpha \cos 2\chi + \mathbf{e}_3 \sin 2\chi. \quad (286)$$

We first solve the following problem: how to transform the polarization state P_0 into P ? A classical solution (non-unique) is obtained by using a half-wave plate, say \mathcal{L}_1 , and a quarter-wave plate \mathcal{L}_2 , whose axes are OL_1 and OL_2 respectively, on the Poincaré sphere. The half-wave plate \mathcal{L}_1 transforms first the state P_0 into P' , such that $\angle L_2 OP' = 2\chi$, and the quarter-wave plate \mathcal{L}_2 transforms P' into P , which is deduced from P' in the rotation of angle $\pi/2$ around the axis OL_2 . The longitude of L_2 is 2α , and since $\angle P_0 OP' = 2\alpha + 2\chi$, the longitude of L_1 is $\alpha + \chi$ (Fig. 7).

The axis \mathbf{e}_m of \mathcal{L}_1 (in the sense of quaternions) is then given by

$$\mathbf{e}_m = \mathbf{e}_1 \cos(\alpha + \chi) + \mathbf{e}_2 \sin(\alpha + \chi), \quad (287)$$

and the axis \mathbf{e}_n of \mathcal{L}_2 by

$$\mathbf{e}_n = \mathbf{e}_1 \cos 2\alpha + \mathbf{e}_2 \sin 2\alpha. \quad (288)$$

We now use the quaternion calculus to check that P is actually deduced from P_0 in the product of the rotation of angle π around \mathbf{e}_m , followed by the rotation of angle $\pi/2$ around \mathbf{e}_n .

We denote \mathbf{e}'_m the unit quaternion along OP' , deduced from \mathbf{e}_1 in the rotation of angle π around \mathbf{e}_m . We have

$$\begin{aligned} \mathbf{e}'_m &= e^{e_m \pi/2} \mathbf{e}_1 e^{-e_m \pi/2} = -\mathbf{e}_m \mathbf{e}_1 \mathbf{e}_m \\ &= -[\mathbf{e}_1 \cos(\alpha + \chi) + \mathbf{e}_2 \sin(\alpha + \chi)] \mathbf{e}_1 [\mathbf{e}_1 \cos(\alpha + \chi) + \mathbf{e}_2 \sin(\alpha + \chi)] \\ &= [\cos(\alpha + \chi) + \mathbf{e}_3 \sin(\alpha + \chi)] [\mathbf{e}_1 \cos(\alpha + \chi) + \mathbf{e}_2 \sin(\alpha + \chi)] \\ &= \mathbf{e}_1 \cos(2\alpha + 2\chi) + \mathbf{e}_2 \sin(2\alpha + 2\chi). \end{aligned} \quad (289)$$

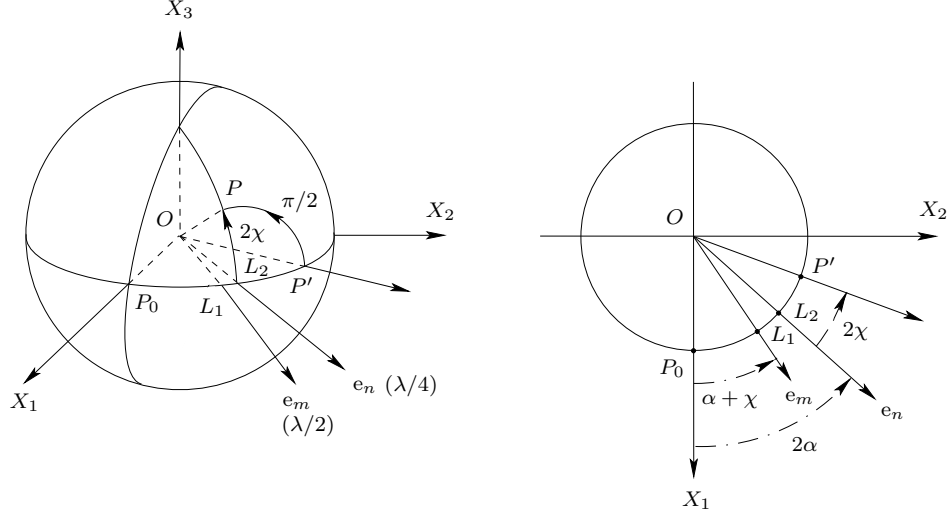


Figure 7: The point P_0 is transformed into P' under the rotation of angle π around the axis e_m : the incident polarization state becomes the state P' , with the help of a half-wave plate whose axis is e_m . Then P' is transformed into P under the rotation of angle $\pi/2$ around the axis e_n : the polarization P' is transformed into P , with the help of a quarter-wave plate.

The rotation of angle $\pi/2$ around e_n is expressed according to

$$\begin{aligned}
e^{e_n \pi/4} e'_m e^{-e_n \pi/4} &= \frac{1}{2}(1 + e_n) [e_1 \cos(2\alpha + 2\chi) + e_2 \sin(2\alpha + 2\chi)] (1 - e_n) \\
&= \frac{1}{2} \begin{bmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \cos(2\alpha + 2\chi) \\ \sin(2\alpha + 2\chi) \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -\cos 2\alpha \\ -\sin 2\alpha \\ 0 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} -\cos 2\alpha \cos(2\alpha + 2\chi) - \sin 2\alpha \sin(2\alpha + 2\chi) \\ \cos(2\alpha + 2\chi) \\ \sin(2\alpha + 2\chi) \\ \cos 2\alpha \sin(2\alpha + 2\chi) - \sin 2\alpha \cos(2\alpha + 2\chi) \end{bmatrix} \begin{bmatrix} 1 \\ -\cos 2\alpha \\ -\sin 2\alpha \\ 0 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} -\cos 2\chi \\ \cos(2\alpha + 2\chi) \\ \sin(2\alpha + 2\chi) \\ \sin 2\chi \end{bmatrix} \begin{bmatrix} 1 \\ -\cos 2\alpha \\ -\sin 2\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cos 2\alpha \cos 2\chi \\ \sin 2\alpha \cos 2\chi \\ \sin 2\chi \end{bmatrix} = e_p. \quad (290)
\end{aligned}$$

In conclusion we have

$$e_p = e^{e_n \pi/4} e^{e_m \pi/2} e_1 e^{-e_m \pi/2} e^{-e_n \pi/4}, \quad (291)$$

and then

$$1 \pm i e_p = e^{e_n \pi/4} e^{e_m \pi/2} (1 \pm i e_1) e^{-e_m \pi/2} e^{-e_n \pi/4} = u (1 \pm i e_1) u^*, \quad (292)$$

where

$$u = \exp e_n \frac{\pi}{4} \exp e_m \frac{\pi}{2} = \frac{\sqrt{2}}{2} (1 + e_n) e_m. \quad (293)$$

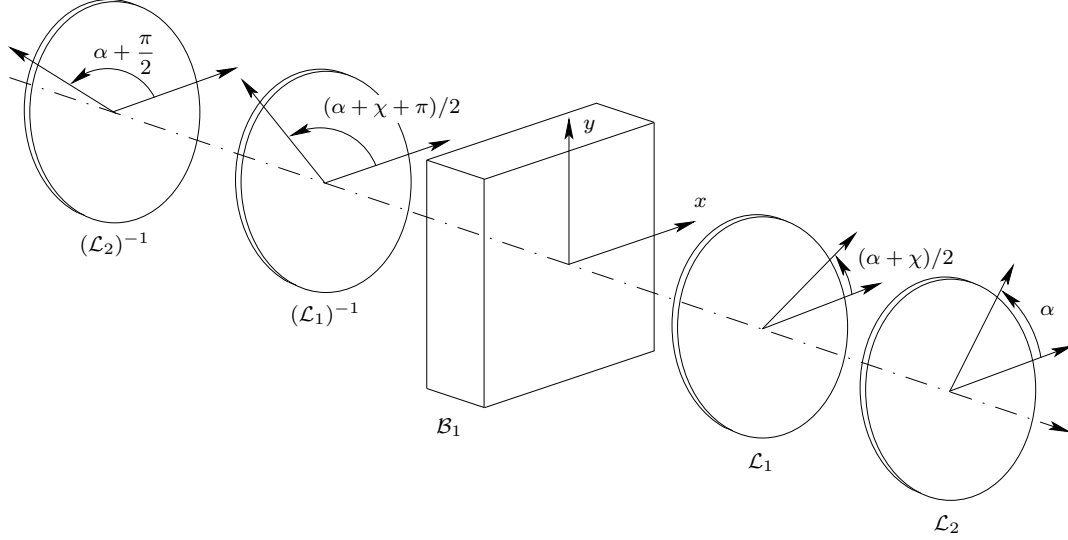


Figure 8: Optical setup to transform the rectilinear birefringent \mathcal{B}_1 into a birefringent \mathcal{B}_p with a predefined axis. In general the eigenstates of \mathcal{B}_p are elliptically polarized. If the fast vibration of \mathcal{B}_1 is along the x -axis (horizontal polarization), the major axis of the ellipse corresponding to the fast vibration of \mathcal{B}_p makes the angle α with the x -axis, and the ellipticity is equal to χ . Waveplates \mathcal{L}_1 and \mathcal{L}_1^{-1} are half-wave plates, and \mathcal{L}_2 and \mathcal{L}_2^{-1} are quarter-wave plates. Only the fast vibration of each wave-plate is drawn, and its angle with the x -axis is given. Light propagates from left to right.

The quaternion u is the representation of a polarization operator, say \mathcal{U} , which operates on polarization states. Since \mathcal{U} is the product of a half-wave plate \mathcal{L}_1 followed by a quarter-wave plate \mathcal{L}_2 , we write $\mathcal{U} = \mathcal{L}_2 \circ \mathcal{L}_1$, where \circ denotes the composition law of operators.

Let us go back now to the initial issue. To say that e_p is the axis of \mathcal{B}_p means that $1 + ie_p$ and $1 - ie_p$ represent the eigenstates of \mathcal{B}_p (they respectively represent the fast and the slow vibrations). If a lightwave whose polarization state is $1 + ie_p$ is incident on \mathcal{B}_p , the polarization of the emerging lightwave should also be $1 + ie_p$. Such a result is obtained in three steps that are as follows.

1. The incident state $1 + ie_p$ is transformed into the state $1 + ie_1$. According to Eq. (292), since $u^{-1} = u^*$, we have $1 + ie_1 = u^*(1 + ie_p)u$. That means that the incident lightwave first goes across the inverse operator $\mathcal{U}^{-1} = (\mathcal{L}_2 \circ \mathcal{L}_1)^{-1} = \mathcal{L}_1^{-1} \circ \mathcal{L}_2^{-1}$.
2. The lightwave then goes across \mathcal{B}_1 . The birefringence is φ . Since $1 + ie_1$ is an eigenstate, we obtain

$$e^{e_1\varphi/2} (1 + ie_1) e^{-e_1\varphi/2} = 1 + ie_1. \quad (294)$$

3. Finally, we have to transform the state $1 + ie_1$ into $1 + ie_p$. The light wave emerging from \mathcal{B}_1 goes across \mathcal{U} , according to Eq. (292).

We eventually obtain $\mathcal{B}_p = \mathcal{U} \circ \mathcal{B}_1 \circ \mathcal{U}^{-1}$ and, since $\mathcal{U} = \mathcal{L}_2 \circ \mathcal{L}_1$, we have

$$\mathcal{B}_p = \mathcal{L}_2 \circ \mathcal{L}_1 \circ \mathcal{B}_1 \circ \mathcal{L}_1^{-1} \circ \mathcal{L}_2^{-1}. \quad (295)$$

The result is illustrated by Fig. 8.

Remark 5.5 The result obtained for a birefringent device can be applied to transform a rectilinear dichroic device or a rectilinear polarizer into a dichroic device or polarizer with arbitrary (orthogonal) eigenstates. Only \mathcal{B}_1 has to be replaced by the appropriate rectilinear dichroic or rectilinear polarizer, whose axes are e_1 .

5.6 Depolarizers [46]

In this section we use greek letters β and γ with a distinct meaning of that of Section 2.5.3 (special theory of relativity).

5.6.1 Perfect depolarizers

A perfect depolarizer is such that for every incident state of polarization, the emerging state is unpolarized [38, 46]. If $X = X_0(1 + i\rho e_n)$ represents the incident state, the emerging state should take the form $X' = X'_0$, where X'_0 is a positive number.

Let Ψ be the linear mapping defined on quaternions by

$$\Psi(q) = \frac{1}{4}(q - e_1 q e_1 - e_2 q e_2 - e_3 q e_3). \quad (296)$$

We have $\Psi(1) = 1$ and $\Psi(e_1) = \Psi(e_2) = \Psi(e_3) = 0$, so that for every minquat $X = X_0(1 + i\rho e_n)$, we obtain $\Psi(X) = X_0$. We conclude that the restriction of Ψ to minquats, denoted by D , represents a perfect depolarizer. It is such that

$$D : X \mapsto D(X) = \frac{1}{4}(X - e_1 X e_1 - e_2 X e_2 - e_3 X e_3). \quad (297)$$

More generally let $\{e_u, e_v, e_w\}$ be a direct orthonormal basis of the subspace of pure quaternions. The mapping D' defined on minquats by

$$D'(X) = \frac{1}{4}(X - e_u X e_u - e_v X e_v - e_w X e_w), \quad (298)$$

is such that $D'(X) = X_0$ for every minquat $X = X_0(1 + i\rho e_n)$; thus D' also represents a perfect depolarizer.

Remark 5.6 For $X = X_0(1 + i\rho e_n)$ we obtain $-e_1 X e_1 - e_2 X e_2 - e_3 X e_3 = X_0(3 - i\rho e_n)$, whose polarized component $\rho X_0(1 - i e_n)$ is orthogonal to the polarized component of X , with the same power. According to Eq. (297) the depolarizing effect is obtained by **incoherently** adding two lightwaves whose polarized components are orthogonally polarized (and have the same power).

5.6.2 Partial depolarizers

A partial depolarizer depolarizes by a factor $1 - \gamma$ (with $0 \leq \gamma \leq 1$), if the outgoing polarization state corresponding to the incident state $X = X_0(1 + i\rho e_n)$ is $X' = X'_0(1 + i\gamma\rho e_n)$. In other words, the degree of polarization ρ is changed into $\gamma\rho$.

According to Eq. (298) the vectorial part of the polarized component of X is balanced by the vectorial part of $-e_u X e_u - e_v X e_v - e_w X e_w$. For the superposition to provide a partially polarized lightwave, not necessarily an unpolarized wave, we reduce the weight of $-e_u X e_u - e_v X e_v - e_w X e_w$ by introducing a factor β ($0 \leq \beta \leq 1$) and we consider the mapping

$$D_p : X \mapsto D_p(X) = \frac{1}{1 + 3\beta}[X - \beta(e_u X e_u - e_v X e_v - e_w X e_w)], \quad (299)$$

where $0 \leq \beta \leq 1$. We have

$$D_p(1 + i\rho e_n) = 1 + \frac{1 - \beta}{1 + 3\beta} i\rho e_n. \quad (300)$$

Then $D_p(1 + i\rho e_n)$ is a minquat and its degree of polarization is $(1 - \beta)\rho/(1 + 3\beta)$. To reduce the degree of polarization from ρ to $\gamma\rho$, we choose β such that

$$\gamma = \frac{1 - \beta}{1 + 3\beta}, \quad (301)$$

that is

$$\beta = \frac{1 - \gamma}{1 + 3\gamma}, \quad (302)$$

and Eq. (299) becomes

$$D_p(X) = \frac{1 + 3\gamma}{4}X - \frac{1 - \gamma}{4}(e_u X e_u - e_v X e_v - e_w X e_w). \quad (303)$$

Equation (303) is the general expression of a partial depolarizer that depolarizes by a factor $1 - \gamma$.

5.6.3 Actual design of a depolarizer

We deal with $\{e_u, e_v, e_w\} = \{e_1, e_2, e_3\}$ and begin with a perfect depolarizer. We notice that $-e_1 X e_1 = \exp(e_1\pi/2) X \exp(-e_1\pi/2)$, so that the mapping $X \mapsto -e_1 X e_1$ expresses the effect of a half-wave plate on the polarization state represented by the minquat X .

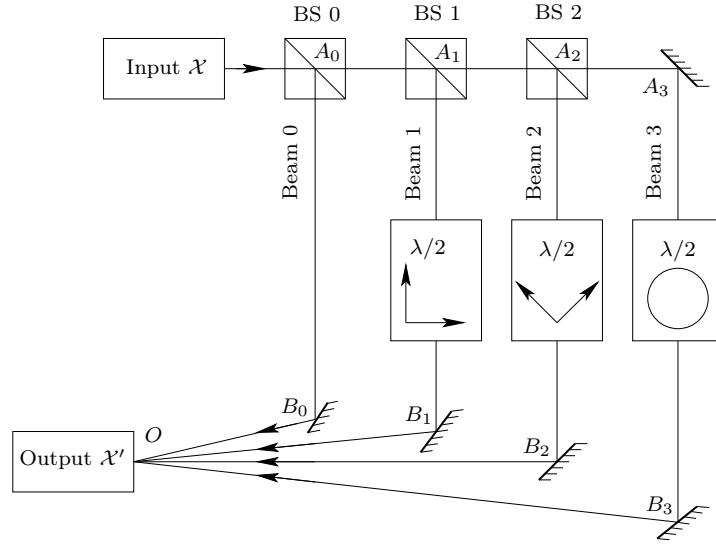


Figure 9: Experimental setup for depolarizing a polarized lightwave \mathcal{X} , represented by a minquat $X = X_0(1 + i\rho e_n)$. Beam splitter BS 0 transmission factor is $T_0 = 0.75$, and its reflection factor is $R_0 = 0.25$. For BS 1, we have $T_1 = 2/3 \approx 0.67$ and $R_1 = 1/3 \approx 0.33$; for BS 2, $T_2 = R_2 = 0.5$. The four beams 0, 1, 2, 3 thus have equal powers (equal to $X_0/4$). Beam 1 passes through a half-wave plate whose eigenvibrations are horizontal and vertical; beam 2 passes through a half-wave plate whose eigenvibrations are at $\pm 45^\circ$ from the horizontal; beam 3 passes through an optically active medium whose rotatory power is $\pi/2$ (which is a half-wave plate whose eigenvibrations correspond to circularly polarized states). To make the wave superposition at O incoherent, the optical path differences $[A_2 A_3 B_3 O] - [A_2 B_2 O]$, $[A_1 A_2 B_2 O] - [A_1 B_1 O]$ and $[A_0 A_1 B_1 O] - [A_0 B_0 O]$ must be larger than the coherence length of the light source. For every input polarized wave \mathcal{X} , the output is an unpolarized lightwave \mathcal{X}' , represented by a minquat of the form $X' = X_0$ (scalar quaternion).

The axis of the plate is X_1 on the Poincaré sphere, which means that the fast eigenvibration of the plate is horizontal, whereas its slow eigenvibration is vertical.

Similarly the mapping $X \mapsto -e_2 X e_2$ expresses the effect of a half-wave plate, whose eigenvibrations are at $\pm 45^\circ$ of the horizontal (X_2 -axis on the Poincaré sphere); and the mapping $X \mapsto -e_3 X e_3$ expresses the effect of a half-wave plate, whose eigenvibrations are circularly polarized (X_3 -axis on the Poincaré sphere), that is, a rotaroy power equal to $\pi/2$.

A setup is shown on Fig. 9. Explanations are given in the caption. To obtain a partial depolarizer we replace the beam splitter 0 with a beam splitter whose transmission factor is $T_0 = 3(1 - \gamma)/4$ and reflection factor is $R_0 = (1 + 3\gamma)/4$. For example, if we want $\gamma = 0.5$, then we have to set $T_0 = 0.375$ and $R_0 = 0.625$.

6 Some equivalences between special theory of relativity and polarization optics

6.1 Effect of a hyperbolic rotation

6.1.1 Analysis in the Minkowski space

We consider a hyperbolic rotation of axis e_n and parameter δ , represented by $\exp(i e_n \delta/2)$, and a minquat $x = x_0(e_0 + i a e_m)$, with $x_0 > 0$ and $|a| \leq 1$. Under the rotation, x is changed into x' such that

$$x' = \exp\left(i e_n \frac{\delta}{2}\right) x \exp\left(i e_n \frac{\delta}{2}\right). \quad (304)$$

Then

$$\begin{aligned} \frac{x'}{x_0} &= \left(e_0 \cosh \frac{\delta}{2} + i e_n \sinh \frac{\delta}{2}\right) (e_0 + i a e_m) \left(e_0 \cosh \frac{\delta}{2} + i e_n \sinh \frac{\delta}{2}\right) \\ &= \left(e_0 \cosh \frac{\delta}{2} + i e_n \sinh \frac{\delta}{2} + i a e_m \cosh \frac{\delta}{2} - a e_n e_m \sinh \frac{\delta}{2}\right) \left(e_0 \cosh \frac{\delta}{2} + i e_n \sinh \frac{\delta}{2}\right) \\ &= e_0 \left(\cosh^2 \frac{\delta}{2} + \sinh^2 \frac{\delta}{2}\right) + 2i e_n \cosh \frac{\delta}{2} \sinh \frac{\delta}{2} - a (e_n e_m + e_m e_n) \cosh \frac{\delta}{2} \sinh \frac{\delta}{2} \\ &\quad + i a e_m \cosh^2 \frac{\delta}{2} - i a e_n e_m e_n \sinh^2 \frac{\delta}{2} \\ &= e_0 (\cosh \delta + a e_m \cdot e_n \sinh \delta) + i e_n \sinh \delta + i a e_m \cosh^2 \frac{\delta}{2} - i a e_n e_m e_n \sinh^2 \frac{\delta}{2}. \end{aligned} \quad (305)$$

We examine two particular cases:

1. We assume $e_n = e_m$. Then $e_n e_m e_n = -e_m$ and Eq. (305) becomes

$$x' = x_0 e_0 (\cosh \delta + a \sinh \delta) + i x_0 e_m (\sinh \delta + a \cosh \delta). \quad (306)$$

2. We assume $e_n \cdot e_m = 0$. According to the corollary of Appendix D we have $e_n e_m e_n = e_m$ and Eq. (305) becomes

$$x' = x_0 e_0 \cosh \delta + i x_0 e_n \sinh \delta + i x_0 a e_m. \quad (307)$$

6.1.2 Application to the special theory of relativity

The result of Section 6.1.1 is applied to the change of the components of a 4-vector when passing from a Galilean frame to another. The 4-vector under consideration will be the 4-velocity of a particle.

In the present section u does not represent a rotation, but the algebraic value of the 3-velocity of a particle.

The hyperbolic rotation is a Lorentz boost that gives the change of coordinates between two inertial frames of reference, say \mathcal{S} and \mathcal{S}' , in relative uniform translatory motion. The relative 3-velocity of \mathcal{S}' with respect to \mathcal{S} is $v \mathbf{e}_n$, and it is usual to denote $\beta = v/c$ and

$$\gamma_v = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (308)$$

The link with the rapidity δ is given by

$$\cosh \delta = \gamma_v, \quad \text{and} \quad \sinh \delta = -\gamma_v \beta. \quad (309)$$

that is, $\tanh \delta = -\beta = -v/c$.

We consider a particle whose 3-velocity in \mathcal{S} is $u \mathbf{e}_m$, so that the minquat representation of its 4-velocity [29, 32], with respect to \mathcal{S} is

$$U = \gamma_u (c \mathbf{e}_0 + i u \mathbf{e}_m), \quad (310)$$

where

$$\gamma_u = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}. \quad (311)$$

The result of Section 6.1.1 is applied by taking the 4-velocity of the particle as x : we set $x = U$, $x_0 = \gamma_u c$ and $a = u/c$.

The two cases of Section 6.1.1 are the followings.

1. If $\mathbf{e}_n = \mathbf{e}_m$. According to Eq. (306), x' takes the form $x' = x'_0 (\mathbf{e}_0 + i a' \mathbf{e}_m)$ with

$$x'_0 = \gamma_u \gamma_v c \left(1 - \frac{uv}{c^2} \right), \quad (312)$$

and

$$x'_0 a' = \gamma_u \gamma_v (u - v). \quad (313)$$

(Since $|u| < c$ and $|v| < c$, we have $x'_0 > 0$.) We deduce

$$a' = \frac{u - v}{c \left(1 - \frac{uv}{c^2} \right)}, \quad (314)$$

so that the relative 3-velocity of the particle with respect to \mathcal{S}' takes the form $u' \mathbf{e}_m$ with

$$u' = a' c = \frac{u - v}{1 - \frac{uv}{c^2}}. \quad (315)$$

Since $|u| < c$, we remark that there is a real number α such that $a = u/c = \tanh \alpha$. Equation (314) becomes

$$a' = \frac{\tanh \alpha + \tanh \delta}{1 + \tanh \alpha \tanh \delta} = \tanh(\alpha + \delta). \quad (316)$$

We write $a' = \tanh \alpha'$, and obtain $\alpha' = \alpha + \delta$, which is the composition law of rapidities. (See Eq. 118 in Section 2.5.8.)

Let $\gamma_{u'}$ be defined by

$$\gamma_{u'} = \frac{1}{\sqrt{1 - \frac{u'^2}{c^2}}}. \quad (317)$$

We have

$$\frac{1}{\gamma_{u'}^2} = 1 - \frac{(u-v)^2}{c^2 \left(1 - \frac{uv}{c^2}\right)^2} = \frac{c^2 + (u^2v^2/c^2) - u^2 - v^2}{c^2 \left(1 - \frac{uv}{c^2}\right)^2} = \frac{1}{\gamma_u^2 \gamma_v^2 \left(1 - \frac{uv}{c^2}\right)^2}, \quad (318)$$

so that ($x'_0 > 0$)

$$x'_0 = \gamma_{u'} c. \quad (319)$$

With respect to \mathcal{S}' , the four-velocity vector of the particle takes the form

$$U' = \gamma_{u'} c e_0 + i \gamma_{u'} u' e_m, \quad (320)$$

with is similar to Eq. (310).

Equation (315) is the relativistic composition law of 3-velocities, when the particle velocity is along the relative velocity of \mathcal{S} and \mathcal{S}' . It may be inverted in the form

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}, \quad (321)$$

which is exactly Eq. (119), in which v_m is replaced with v , velocity v_n is replaced with u' , and v is eventually replaced with u .

2. If $e_n \cdot e_m = 0$, Eq. (307) gives

$$x' = \gamma_u \gamma_v c e_0 - i \gamma_u \gamma_v v e_n + i \gamma_u u e_m = x'_0 e_0 + i x'_0 a' e_u, \quad (322)$$

that is

$$x'_0 = \gamma_u \gamma_v c = \gamma_v x_0, \quad (323)$$

$$x'_0 a' e_u = \frac{x'_0 u'}{c} e_u = \gamma_u (u e_m - \gamma_v v e_n), \quad (324)$$

and we deduce

$$u' e_u = \frac{u}{\gamma_v} e_m - v e_n. \quad (325)$$

Since e_m , e_n and e_u are unit pure quaternions and since e_m and e_n are orthogonal to each other, we obtain

$$u' = \sqrt{v^2 + \frac{u^2}{\gamma_v^2}}, \quad (326)$$

that is

$$a' = \frac{u'}{c} = \sqrt{\tanh^2 \delta + \frac{a^2}{\cosh^2 \delta}}. \quad (327)$$

6.1.3 Application to polarization optics

The quaternion $\exp(i e_n \delta/2)$ represents a dichroic device, whose dichroism is δ ($\delta > 0$) and whose eigenvibrations are represented by the points $-\mathbf{e}_n$ and \mathbf{e}_n on the Poincaré sphere. The minquat x represents a partially polarized wave, and with the previous notations for polarization states we write $x = X = X_0(1 + i\rho \mathbf{e}_n)$, where $x_0 = X_0$ is the lightwave power and $a = \rho$ is its degree of polarization.

We obtain the following results.

1. If $\mathbf{e}_m = \mathbf{e}_n$, the polarized component of X is polarized along an eigenvibration of the dichroic. According to Eq. (306), the emerging state takes the form $X' = X'_0(1 + i\rho' \mathbf{e}_m)$, with

$$X'_0 = X_0(\cosh \delta + \rho \sinh \delta), \quad (328)$$

$$\rho' = \frac{\sinh \delta + \rho \cosh \delta}{\cosh \delta + \rho \sinh \delta}. \quad (329)$$

- (a) If $\rho = 1$, we obtain $\rho' = 1$. The minquat $X = X_0(\mathbf{e}_0 + i \mathbf{e}_m)$ represents a completely polarized eigenstate and also does X' . Since $\delta > 0$, we have $X'_0 > X_0$ (the dichroic is a pure dichroic, so that the lightwave power is amplified; for a real case an isotropic absorption factor should be taken into account, see Section 5.2.3).
- (b) For $0 \leq \rho < 1$, the degree of polarization ρ can be written $\rho = \tanh \alpha$, with $\alpha \geq 0$. Then

$$\rho' = \frac{\rho + \tanh \delta}{1 + \rho \tanh \delta} = \frac{\tanh \alpha + \tanh \delta}{1 + \tanh \alpha \tanh \delta} = \tanh(\alpha + \delta). \quad (330)$$

Equation (330) is equivalent in polarization optics to Eq. (316) in Relativity.

Let us assume $\delta > 0$. Then $\tanh(\alpha + \delta) > \tanh \alpha$, which means $\rho' > \rho$. The degree of polarization is increased under a dichroic (with positive dichroism). That is why dichroics sometimes are called **partial polarizers**.

2. If \mathbf{e}_m and \mathbf{e}_n are orthogonal, according to Eq. (307) we obtain

$$X' = X_0 \mathbf{e}_0 \cosh \delta \left(\mathbf{e}_0 + i \mathbf{e}_n \tanh \delta + \frac{i\rho \mathbf{e}_m}{\cosh \delta} \right), \quad (331)$$

which can be written $X' = X'_0(\mathbf{e}_0 + i\rho' \mathbf{e}_u)$. Since \mathbf{e}_m and \mathbf{e}_n are orthogonal, we obtain

$$\rho' = \sqrt{\tanh^2 \delta + \frac{\rho^2}{\cosh^2 \delta}}, \quad (332)$$

which is equivalent to Eq. (327). We also have

$$\rho' \mathbf{e}_u = \mathbf{e}_n \tanh \delta + \frac{\rho \mathbf{e}_m}{\cosh \delta}, \quad (333)$$

equivalent to Eq. (325); and $X'_0 = X_0 \cosh \delta$, equivalent to Eq. (323).

- (a) If $\rho = 1$, then $\rho' = 1$: both X and X' correspond to completely polarized waves.
- (b) If $0 \leq \rho < 1$, Eq. (332) gives $\rho'^2 \cosh^2 \delta = \sinh^2 \delta + \rho^2$, that is, $1 - \rho'^2 = (1 - \rho^2) \cosh^2 \delta$. Since $\rho < 1$ and $\cosh^2 \delta > 1$, we obtain $0 < 1 - \rho'^2 < 1 - \rho^2$ and eventually $\rho < \rho' < 1$. The degree of polarization is increased by the dichroic, which operates as a partial polarizer.

6.2 Wigner rotation and Thomas precession

6.2.1 Geometric Wigner rotation

We will show that the product of two hyperbolic rotations, with distinct axes, does not reduce to a hyperbolic rotation, but involves an elliptic rotation, known as Wigner rotation.

Proposition 6.1 (Wigner rotation) *Let $\exp(i e_\ell \delta'/2)$ and $\exp(i e_m \delta''/2)$ represent two hyperbolic rotations, with $\delta' \delta'' \neq 0$ and let θ be the angle between e_ℓ and e_m , taken from e_ℓ to e_m (we assume $0 \leq \theta \leq \pi$). Then $u = \exp(i e_\ell \delta'/2) \exp(i e_m \delta''/2)$ takes the form $u = \exp(e_n \varepsilon/2) u'$, where u' is a unit quaternion representing a hyperbolic rotation and, if $\theta \neq 0 \pmod{\pi}$, where $\exp(e_n \varepsilon/2)$ represents an elliptic rotation (called Wigner rotation) whose axis is e_n , with $e_n \sin \theta = e_\ell \times e_m$, and whose angle is ε such that*

$$\tan \frac{\varepsilon}{2} = \frac{\tanh \frac{\delta'}{2} \tanh \frac{\delta''}{2} \sin \theta}{1 + \tanh \frac{\delta'}{2} \tanh \frac{\delta''}{2} \cos \theta}. \quad (334)$$

Proof. Since $\delta' \delta'' \neq 0$, if $\theta \neq 0 \pmod{\pi}$, we have $\tan(\varepsilon/2) \neq 0$ and $\varepsilon \neq 0 \pmod{\pi}$, so that $\exp(e_n \varepsilon/2)$ represents an actual elliptic rotation (not the identity).

To lighten notations we write

$$r = \cos \frac{\varepsilon}{2}, \quad s = \sin \frac{\varepsilon}{2}, \quad r' = \cosh \frac{\delta'}{2}, \quad s' = \sinh \frac{\delta'}{2}, \quad r'' = \cosh \frac{\delta''}{2}, \quad s'' = \sinh \frac{\delta''}{2}, \quad (335)$$

so that Eq. (334) becomes

$$\frac{s}{r} = \frac{\frac{s' s''}{r' r''} \sin \theta}{1 + \frac{s' s''}{r' r''} \cos \theta}, \quad (336)$$

and gives

$$s r' r'' + s s' s'' \cos \theta - r s' s'' \sin \theta = 0. \quad (337)$$

For $\theta \neq 0$, the quaternion $e_n = e_\ell \times e_m / \sin \theta$ is unitary and orthogonal to the plane (e_ℓ, e_m) . We obtain a direct basis $\{e_\ell, e_q, e_n\}$ of pure quaternions by introducing $e_q = e_n e_\ell = e_n \times e_\ell$, so that $e_m = e_\ell \cos \theta + e_q \sin \theta$. We define $u' = \exp(-e_n \varepsilon/2) u$ and then derive

$$\begin{aligned} u' &= \exp\left(-e_n \frac{\varepsilon}{2}\right) u \\ &= \exp\left(-e_n \frac{\varepsilon}{2}\right) \exp\left(i e_m \frac{\delta''}{2}\right) \exp\left(i e_\ell \frac{\delta'}{2}\right) \\ &= (r - s e_n)(r'' + i s'' e_m)(r' + i s' e_\ell) \\ &= (r - s e_n)[r'' + i s''(e_\ell \cos \theta + e_q \sin \theta)](r' + i s' e_\ell) \\ &= i s'' \cos \theta (r - s e_n) e_\ell (r' + i s' e_\ell) + (r - s e_n)(r'' + i s'' e_q \sin \theta)(r' + i s' e_\ell) \\ &= r s' s'' \cos \theta + i r r' s'' e_\ell \cos \theta - i s r' s'' e_q \cos \theta - s s' s'' e_n \cos \theta \\ &\quad + r r' r'' + s s' s'' \sin \theta + i(r s' r'' + s r' s'' \sin \theta) e_\ell + i(r r' s'' \sin \theta - s s' r'') e_q \\ &\quad + (r s' s'' \sin \theta - s r' r'') e_n. \end{aligned} \quad (338)$$

According to Eq. (337), the coefficient of e_n in Eq. (338) is zero, so that we obtain

$$\begin{aligned} u' &= r s' s'' \cos \theta + r r' r'' + s s' s'' \sin \theta + i(r r' s'' \cos \theta + r s' r'' + s r' s'' \sin \theta) e_\ell \\ &\quad + i(r r' s'' \sin \theta - s s' r'' - s r' s'' \cos \theta) e_q. \end{aligned} \quad (339)$$

The quaternion u' is a unit quaternion as product of three unit quaternions. Moreover, Eq. (339) shows that its vector part is imaginary: then u' represents a hyperbolic rotation on the Minkowski space. The proof is complete. \square

Remark 6.1 In Proposition 6.1, we assume $0 \leq \theta \leq \pi$. If $-\pi < \theta < 0$, we bring back to the previous assumption by changing e_m into $-e_m$, that is, θ into $\pi + \theta$. We also change δ'' into $-\delta''$, so that u is not changed, because $\exp(i e_m \delta''/2) = \exp[i(-e_m)(-\delta'')/2]$.

6.2.2 Application to Relativity: Thomas precession [29, 32–34]

The Thomas precession is a relativistic effect that is no more than a physical achievement of the Wigner rotation. It is a consequence of the fact that, according to the previous section, the product of two boosts is not a boost, in general, but a boost plus an elliptic rotation (Wigner rotation).

We will only consider a particular case: quaternions e_ℓ and e_m are orthogonal, that is $\theta = \pi/2$. We have $e_n = e_\ell \times e_m$ and $e_q = e_m$. We deduce

$$\tan \frac{\varepsilon}{2} = \tanh \frac{\delta'}{2} \tanh \frac{\delta''}{2}, \quad (340)$$

the axis of the Wigner rotation being e_n .

We consider three Galilean inertial systems \mathcal{R} , \mathcal{R}' and \mathcal{R}'' with three respective coordinates frames Ω , Ω' and Ω'' . The system \mathcal{R}' has a uniform translatory motion with respect to \mathcal{R} ; and \mathcal{R}'' has a uniform translatory motion with respect to \mathcal{R}' . Contrary to what would be expected according to the laws of classical mechanics, the connection between coordinates in Ω'' and coordinates in Ω does not correspond to a hyperbolic rotation: an additional rotation has to be taken into account.

We now assume that the relative velocity of \mathcal{R}' with respect to \mathcal{R} is $v' e_1$ and that the relative velocity of \mathcal{R}'' with respect to \mathcal{R}' is $v'' e_2$. The axis of the Wigner rotation is e_3 .

In place of velocities, we use rapidities, that is δ' and δ'' such that

$$\tanh \delta' = -\frac{v'}{c}, \quad \tanh \delta'' = -\frac{v''}{c}. \quad (341)$$

We assume $|v'| \ll c$ and $|v''| \ll c$, so that at first order

$$\tanh \frac{\delta'}{2} \approx -\frac{v'}{2c}, \quad \tanh \frac{\delta''}{2} \approx -\frac{v''}{2c}. \quad (342)$$

Equation (340) gives

$$\varepsilon \approx \frac{v' v''}{2c^2}. \quad (343)$$

Generally v'' is regarded like a variation of v' , namely $v'' = dv'$, and eventually

$$\frac{d\varepsilon}{dt} \approx \frac{1}{2c^2} v' \frac{dv'}{dt}, \quad (344)$$

where t denotes proper time in \mathcal{R} .

6.2.3 Application to polarization optics

In optics, the Wigner rotation causes the product of two dichroics not to be reduced to a dichroic device, in general, but to include a birefringent part. To provide an example we consider two rectilinear dichroics whose eigenvibrations are at 45° to each other. On the Poincaré sphere

the axes of the two dichroics are along orthogonal unit pure quaternions e_ℓ and e_m (their representative points are on the equator). Their respective dichroisms are denoted δ' and δ'' .

We use notations of Section 6.2.1 and assume $\theta = \pi/2$. The product of the two dichroics involves a rotatory power, because $e_n = e_\ell \times e_m = e_\ell e_m = e_3$, so that the Wigner rotation is around e_3 . According to Proposition 6.1 the angle of the rotation is ε such that

$$\tan \frac{\varepsilon}{2} = \tanh \delta' \tanh \delta'', \quad (345)$$

which corresponds to a rotatory power $\varrho = \varepsilon/2$.

The product of the two dichroic devices is represented by $u = \exp(i e_3 \varrho) u'$, with

$$\begin{aligned} u' = & \cos \varrho \cosh \frac{\delta'}{2} \cosh \frac{\delta''}{2} + \sin \varrho \sinh \frac{\delta'}{2} \sinh \frac{\delta''}{2} \\ & + i e_\ell \left(\cos \varrho \sinh \frac{\delta'}{2} \cosh \frac{\delta''}{2} + \sin \varrho \cosh \frac{\delta'}{2} \sinh \frac{\delta''}{2} \right) \\ & + i e_m \left(\cos \varrho \cosh \frac{\delta'}{2} \sinh \frac{\delta''}{2} - \sin \varrho \sinh \frac{\delta'}{2} \cosh \frac{\delta''}{2} \right), \end{aligned} \quad (346)$$

because e_q in Eq. (339) is equal to e_m (we have $e_q = e_3 e_\ell = e_\ell e_m e_\ell = e_m$). The unit quaternion u' represents a rectilinear dichroic. It can be written

$$u' = \cosh \frac{\delta}{2} + i e_p \sinh \frac{\delta}{2}, \quad (347)$$

where

$$\cosh \frac{\delta}{2} = \cos \varrho \cosh \frac{\delta'}{2} \cosh \frac{\delta''}{2} + \sin \varrho \sinh \frac{\delta'}{2} \sinh \frac{\delta''}{2}. \quad (348)$$

6.3 Singular proper Lorentz rotations in Relativity and in polarization optics

6.3.1 Geometrical analysis

The composition of a hyperbolic rotation represented by $\exp(i e_n \delta/2)$ and an elliptic rotation rerepresented by $\exp(e_m \varphi/2)$ is represented by

$$\begin{aligned} u = & \exp(e_m \varphi/2) \exp(i e_n \delta/2) \\ = & \left(\cos \frac{\varphi}{2} + e_m \sin \frac{\varphi}{2} \right) \left(\cosh \frac{\delta}{2} + i e_n \sinh \frac{\delta}{2} \right) \\ = & \cos \frac{\varphi}{2} \cosh \frac{\delta}{2} - i e_m \cdot e_n \sin \frac{\varphi}{2} \sinh \frac{\delta}{2} + e_m \sin \frac{\varphi}{2} \cosh \frac{\delta}{2} + i e_n \cos \frac{\varphi}{2} \sinh \frac{\delta}{2} \\ & + i e_m \times e_n \sin \frac{\varphi}{2} \sinh \frac{\delta}{2}, \end{aligned} \quad (349)$$

which takes the form $u = \exp(e_u \psi/2)$, where ψ is a complex number and e_u a complex pure quaternion. (We assume $\varphi \delta \neq 0$.)

In general, the quaternion e_u is a complex unit pure quaternion ($N(e_u) = 1$), so that $u = \cos(\psi/2) + e_u \sin(\psi/2)$, and the corresponding proper Lorentz rotation is regular (in the meaning of Section 2.5.6): it admits two isotropic eigenquats given by Eqs. (101) and (102).

But e_u may also be a null quaternion ($N(e_u) = 0$), so that u represents a singular proper Lorentz rotation. If $N(e_u) = 0$, we have $u = \exp(e_u \psi/2) = 1 + (e_u \psi/2)$, and for this to happen it is necessary that $\cos(\varphi/2) \cosh(\delta/2) = \pm 1$ and $e_m \cdot e_n = 0$ (because $\varphi \delta \neq 0$).

We conclude that if e_m and e_n are orthogonal to each other and if φ and δ are such that

$$\cos \frac{\varphi}{2} = \pm \frac{1}{\cosh \frac{\delta}{2}}, \quad (350)$$

then u represents a singular proper Lorentz rotation. That rotation admits only one null-minquat eigendirection, which is along $e_0 + i e_m e_n$.

6.3.2 Singular direction in Relativity

Let \mathcal{S} and \mathcal{S}' be two Galilean inertial frames in uniform translatory motion. Let Ω be a spatial frame in \mathcal{S} (coordinates x_μ) and Ω' in \mathcal{S}' (coordinates x'_μ). In general the previous frames are such that the x'_j -axis is parallel to the x_j -axis; moreover $x'_0 = 0$, if and only if, $x_0 = 0$; the connection between those coordinates is given by an hyperbolic rotation, in the form $x' = \exp(i e_n \delta / 2) x \exp(i e_n \delta / 2)$, where δ is the rapidity (the relative velocity of \mathcal{S}' with respect to \mathcal{S} is $v e_n$ and $\tan \delta = -v/c$).

Let us assume now that the axes in \mathcal{S}' are rotated with respect to the x'_j -axes. We denote by x''_j the rotated axes. The passage from Ω' to Ω'' is a pure rotation and takes the form $x'' = \exp(e_m \varphi / 2) x' \exp(-e_m \varphi / 2)$. The passage from Ω to Ω'' is then

$$x'' = \exp(e_m \varphi / 2) \exp(i e_n \delta / 2) x \exp(i e_n \delta / 2) \exp(-e_m \varphi / 2), \quad (351)$$

and corresponds to a proper Lorentz rotation, represented by u of the form $u = \exp(e_u \psi / 2)$.

In general the proper Lorentz rotation u admits two null-minquat eigendirections. That means that an observer in \mathcal{S} and an observer in \mathcal{S}' will find two directions in space that are identical for them [9]. In the singular case, the two directions reduce to a single one [9].

6.3.3 Singular axis in polarization optics

In polarization optics, the quaternion u in Eq. (349) represents the succession (or product, or composition) of a dichroic (dichroism δ , axis e_n) and a birefringent (birefringence φ , axis e_m), in this order. In general the resulting device admits two unit polarization eigenstates (they are not orthogonal, see Section 5.2.6). If the birefringent and the dichroic are rectilinear, however, the birefringent eigenvibrations being at 45° of the dichroic eigenvibrations, so that e_m and e_n are orthogonal, and if moreover φ and δ satisfy Eq. (350), the device represented by u corresponds to a singular case: there is only one unit polarization eigenstate, that is, only one null-minquat eigendirection. (Not only rectilinear devices may generate singular cases, elliptical ones also may.)

A singular-axis device is also obtained through the superposition of birefringence and dichroism. We consider a crystal of optical length ℓ that exhibits birefringence and dichroism at the same time (each infinitesimal sample of the crystal exhibits both properties). We use notations of Section 2.5.5 and denote η the birefringence and ζ the dichroism by unit length, the birefringence axis being e_m and the dichroism axis being e_n . We define $\varphi = \eta \ell$ and $\delta = \zeta \ell$, so that the crystal is thus the superposition of a birefringent represented by $\exp(e_m \varphi / 2)$ and a dichroic, represented by $\exp(i e_n \delta / 2)$. The crystal is a pure dephaser represented by

$$u = \exp \frac{e_m \varphi + i e_n \delta}{2}. \quad (352)$$

The singular case occurs if $N(e_m \varphi + i e_n \delta) = 0$, that is if $\delta = \pm \varphi$ and $e_m \cdot e_n = 0$. Only one polarization is unchanged under propagation in the crystal. Such an effect has been observed on iolite crystals and the single eigenpolarization has been experimentally revealed by Pancharatnam [47–49] (the result is also reported by Ramachandran and Ramaseshan in *Crystal*

optics [38] p. 143–144). Iolite is an absorbing biaxial crystal that exhibits rectilinear birefringence and rectilinear dichroism at the same time. Near an optical axis there is a direction for which birefringence and dichroism are equal with rectilinear eigenvibrations at 45° : conditions for having a singular axis are fulfilled. The rectilinear eigenvibrations are represented by e_m (birefringence) and e_n (dichroism) on the equator (Poincaré sphere), so that the unit eigenminquat is $X = 1 - i e_m e_n = 1 + i e_3$ (or $1 - i e_3$, depending on the orientations of e_m and e_n), which means that the eigenpolarization is left-circular (or right-circular).

7 Triangle inequality and applications

7.1 Triangle inequality

A geometric property of Minkowski space is that the triangle inequality takes a form inverse from its (well known) Euclidean form [33, 50].

Proposition 7.1 (Triangle inequality) *Let $\mathbf{x} = (x_0, x_1, x_2, x_3)$ and $\mathbf{y} = (y_0, y_1, y_2, y_3)$ be two vectors of the Minkowski space that may independently be isotropic or timelike. We assume $x_0 y_0 > 0$. Then (triangle inequality)*

$$\sqrt{Q(\mathbf{x} + \mathbf{y})} \geq \sqrt{Q(\mathbf{x})} + \sqrt{Q(\mathbf{y})}, \quad (353)$$

where Q is the Lorentz quadratic form. The equality holds if, and only if, \mathbf{x} and \mathbf{y} are collinear.

A nice and concise proof is given by Manoukian [50]. Here we will prove the triangle inequality for minquats. We replace the Lorentz form Q with N , the norm of quaternions, and since $Q(\mathbf{x}) = N(x)$, if $\mathbf{x} = (x_0, x_1, x_2, x_3)$ and $x = x_0 + i(x_1 e_1 + x_2 e_2 + x_3 e_3)$, Proposition 7.1 is equivalent to the following proposition:

Proposition 7.2 (Triangle inequality for minquats) *Let x and y be two minquats that may independently be isotropic (null) or scalarlike (timelike). We denote x_0 and y_0 the respective scalar components of x and y , and assume $x_0 y_0 > 0$. Then (triangle inequality)*

$$\sqrt{N(x + y)} \geq \sqrt{N(x)} + \sqrt{N(y)}, \quad (354)$$

where N denotes the norm of quaternions. The equality holds if, and only if, x and y are collinear.

Proof. (i) *Inequality.* We write $x = x_0 + i x_m e_m$ and $y = y_0 + i y_n e_n$, where $q e_m$ and e_n are real unit pure quaternions. Since x and y are isotropic or scalarlike, we have $N(x) \geq 0$ and $N(y) \geq 0$; then $|x_0| \geq |x_m|$, and $|y_0| \geq |y_n|$, and since $x_0 y_0 > 0$, we have $x_0 y_0 \geq |x_m| |y_n|$.

Since x and y are minquats, $x + y$ is also a minquat, and

$$\begin{aligned} N(x + y) &= (x + y)(x^* + y^*) = x x^* + y y^* + x y^* + y x^* \\ &= N(x) + N(y) + (x_0 + i x_m e_m)(y_0 - i y_n e_n) + (y_0 + i y_n e_n)(x_0 - i x_m e_m) \\ &= N(x) + N(y) + 2x_0 y_0 + x_m y_n e_m e_n + x_m y_n e_n e_m \\ &= N(x) + N(y) + 2x_0 y_0 - 2x_m y_n e_m \cdot e_n. \end{aligned} \quad (355)$$

Since $x_m y_n e_m \cdot e_n \leq |x_m| |y_n|$ (which also holds if $x_m y_n e_m \cdot e_n \leq 0$), we obtain

$$N(x + y) \geq N(x) + N(y) + 2x_0 y_0 - 2|x_m| |y_n|. \quad (356)$$

From $x_0 y_0 \geq |x_m| |y_n|$ and Eq. (356) we deduce $N(x + y) \geq 0$, so that $x + y$ is a scalarlike or isotropic minquat, and $\sqrt{N(x + y)}$ makes sense.

From $(x_0 |y_n| - y_0 |x_m|)^2 \geq 0$, we deduce

$$(x_0 y_0)^2 + (x_m y_n)^2 - 2x_0 y_0 |x_m| |y_n| \geq (x_0 y_0)^2 + (x_m y_n)^2 - (x_0 y_n)^2 - (y_0 x_m)^2, \quad (357)$$

that is

$$(x_0 y_0 - |x_m| |y_n|)^2 \geq [(x_0)^2 - (x_m)^2] [(y_0)^2 - (y_n)^2] = N(x) N(y). \quad (358)$$

Since $x_0 y_0 \geq |x_m| |y_n| \geq 0$, we obtain

$$x_0 y_0 - |x_m| |y_n| \geq \sqrt{N(x)} \sqrt{N(y)}, \quad (359)$$

and Eq. (356) leads to

$$N(x + y) \geq N(x) + N(y) + 2\sqrt{N(x)} \sqrt{N(y)} = \left(\sqrt{N(x)} + \sqrt{N(y)} \right)^2, \quad (360)$$

which gives Eq. (354).

(ii) *Equality.* We first assume x and y collinear. There exists a real number α such that $y = \alpha x$. Then $N(y) = \alpha^2 N(x)$. Since $x_0 y_0 > 0$ and $y_0 = \alpha x_0$, necessarily $\alpha > 0$, and we obtain

$$\sqrt{N(x + y)} = \sqrt{N[(1 + \alpha)x]} = (1 + \alpha)\sqrt{N(x)} = \sqrt{N(x)} + \sqrt{N(y)}. \quad (361)$$

Conversely, if Eq. (361) holds, we have

$$\begin{aligned} x \cdot y &= \frac{1}{2} [N(x + y) - N(x) - N(y)] = \sqrt{N(x)N(y)} \\ &= \sqrt{[(x_0)^2 - (x_m)^2] [(y_0)^2 - (y_n)^2]}. \end{aligned} \quad (362)$$

We also have

$$x \cdot y = (x_0 + ix_m e_m) \cdot (y_0 + iy_n e_n) = x_0 y_0 - x_m y_n e_m \cdot e_n, \quad (363)$$

and Eq. (359) gives

$$x_0 y_0 - |x_m| |y_n| \geq x_0 y_0 - x_m y_n e_m \cdot e_n, \quad (364)$$

so that, since $x_0 y_0 \geq |x_m| |y_n|$, we necessarily have

$$|x_m| |y_n| \leq x_m y_n e_m \cdot e_n. \quad (365)$$

But in general $|x_m| |y_n| \geq x_m y_n e_m \cdot e_n$, and we conclude

$$|x_m| |y_n| = x_m y_n e_m \cdot e_n, \quad (366)$$

which can be obtained only if e_m and e_n are collinear ($e_m \cdot e_n = 1$).

Then Eq. (357) becomes an equality so that

$$(x_0 |y_n| - y_0 |x_m|)^2 = 0, \quad (367)$$

Since $x_0 y_0 > 0$, the number x_0 is not zero and we write

$$|y_n| = \frac{y_0}{x_0} |x_m|, \quad (368)$$

and since $|x_m| |y_n| \geq 0$, Eq. (366) leads to

$$y_n e_n = \frac{y_0}{x_0} x_m e_m, \quad (369)$$

and finally

$$y = y_0 + iy_n e_n = \frac{y_0}{x_0} (x_0 + ix_m e_m) = \frac{y_0}{x_0} x, \quad (370)$$

which means that x and y are collinear. \square

7.2 Application to Relativity: twin paradox [29, 51–53]

The so called “twin paradox” can be explained by applying the triangle inequality to two observers, one of whom is making a round trip. We assume that the notions of proper time and worldline are known.

The effect is analyzed on Fig. 10, where only one spatial direction is considered. An observer \mathcal{O} stays at abscissa $x_1 = 0$, while another observer \mathcal{O}' travels from abscissa $x_1 = 0$ to abscissa $x_1 = b$ on the x_1 -axis and then travels back to abscissa $x_1 = 0$. A Galilean inertial frame \mathcal{S} is attached to \mathcal{O} . Both motions of \mathcal{O}' are accomplished in uniform translation with respect to \mathcal{O} .

The worldline of \mathcal{O} is \mathbf{AC} . The spatio-temporal coordinates of A in \mathcal{S} are $(a_0, 0, 0, 0)$ and those of C are $(c_0, 0, 0, 0)$. Then

$$\mathbf{AC} = (c_0 - a_0, 0, 0, 0). \quad (371)$$

The proper time-length of \mathcal{O} between events A and C is (Q denotes the Lorentz quadratic form)

$$\Delta t = \frac{1}{c} \sqrt{Q(\mathbf{AC})}, \quad (372)$$

which is the time-length of the \mathcal{O}' -travel as seen by \mathcal{O} in \mathcal{S} .

The worldline of \mathcal{O}' is $\mathbf{AB} + \mathbf{BC}$. For the traveling observer the duration of the travel is

$$\Delta t' = \frac{1}{c} \left(\sqrt{Q(\mathbf{AB})} + \sqrt{Q(\mathbf{BC})} \right). \quad (373)$$

The triangle inequality leads to

$$\sqrt{Q(\mathbf{AC})} > \sqrt{Q(\mathbf{AB})} + \sqrt{Q(\mathbf{BC})}, \quad (374)$$

that is $\Delta t > \Delta t'$. The travel duration measured by \mathcal{O} (in \mathcal{S}) is longer than the travel duration measured by \mathcal{O}' (on his own clock). If the two observers are twins, \mathcal{O} is older than \mathcal{O}' when they meet again.

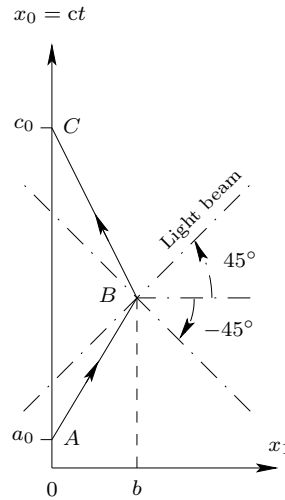


Figure 10: A schematic diagram for analyzing the twin paradox. Both \mathbf{AB} and \mathbf{BC} are timelike vectors: angles of \mathbf{AB} and \mathbf{BC} with the x_0 -axis are less than 45° .

7.3 Application to polarization optics

7.3.1 Upper bounds of the degrees of polarization of incoherent superpositions of waves

The incoherent superposition of two partially polarized states, represented by minquats X and Y , is represented by the sum $X + Y$. Since polarization states are represented by null or scalarlike minquats, we conclude that $\sqrt{N(X+Y)} \geq \sqrt{N(X)} + \sqrt{N(Y)}$. The result may be expressed with the corresponding degrees of polarization. For that, we consider a partially polarized state X , whose degree of polarization is ρ_x . The state can be written in the form $X = X_0(1 + i\rho_x e_m)$ ($X_0 > 0$), so that $N(X) = (X_0)^2(1 - \rho_x^2)$. For physical states, we have $0 \leq \rho_x \leq 1$, then $N(X) \geq 0$, and

$$\sqrt{N(X)} = X_0\sqrt{1 - \rho_x^2}. \quad (375)$$

If we consider the incoherent superposition of $X = X_0(1 + i\rho_x e_m)$ and $Y = Y_0(1 + i\rho_y e_n)$, we obtain

$$X + Y = X_0 + Y_0 + i(X_0\rho_x e_m + Y_0\rho_y e_n) = (X_0 + Y_0)(1 + i\rho e_\ell), \quad (376)$$

where ρ is the degree of polarization of $X + Y$, and e_ℓ is a real unit pure quaternion. The triangle inequality leads to

$$\sqrt{1 - \rho^2} \geq \frac{X_0\sqrt{1 - \rho_x^2} + Y_0\sqrt{1 - \rho_y^2}}{X_0 + Y_0}. \quad (377)$$

Examples. We consider two waves \mathcal{X} and \mathcal{Y} whose polarization states are represented by minquats X and Y . We symbolically write $\mathcal{X} + \mathcal{Y}$ the incoherent superposition of waves \mathcal{X} and \mathcal{Y} , whose polarization state is represented by $X + Y$.

1. X and Y are collinear: $X = X_0(1 + i\rho_x e_m)$ and $Y = (Y_0/X_0)X = Y_0(1 + i\rho_x e_m)$, that is $\rho_x = \rho_y$. The triangle inequality becomes an equality. Thus:
 - (a) Since $\rho_x = \rho_y$, by Eq. (377) we obtain $\rho = \rho_x$.
 - (b) In particular, if $\rho_x = \rho_y = 1$ (\mathcal{X} and \mathcal{Y} are completely polarized), then $\sqrt{1 - \rho^2} = 0$, so that $\rho = 1$. The wave $\mathcal{X} + \mathcal{Y}$ is completely polarized.
 - (c) If \mathcal{X} and \mathcal{Y} are unpolarized, then $\rho_x = \rho_y = 0$ and $\sqrt{1 - \rho^2} = 1$, so that $\mathcal{X} + \mathcal{Y}$ is unpolarized.
2. X and Y are not collinear.
 - (a) If \mathcal{X} and \mathcal{Y} have equal degrees of polarization (not zero): $\rho_x = \rho_y \neq 0$, we obtain $\sqrt{1 - \rho^2} > \sqrt{1 - \rho_x^2}$: the degree of polarization of $X + Y$ is smaller than the degree of polarization of both X and Y . In particular, if $\rho_x = \rho_y = 1$, we obtain $\sqrt{1 - \rho^2} > 0$, which provides no additional information on ρ , since we a priori knew that $\rho < 1$. A more precise result is obtained if we take the angle between e_m and e_n into account, as done in Section 7.3.2.
 - (b) For $\rho_x = 0.80$, $\rho_y = 0.60$ and $X_0 = Y_0$, we obtain $\sqrt{1 - \rho^2} > 0.70$ and $\rho < 0.714$.

7.3.2 Explicit expression of the degree of polarization of the incoherent superposition of two waves

We write $X = X_0(1 + i\rho_x e_m)$ and $Y = Y_0(1 + i\rho_y e_n)$, and then

$$\begin{aligned} N(X + Y) &= (X + Y)(X^* + Y^*) = XX^* + YY^* + XY^* + YX^* \\ &= N(X) + N(Y) + X_0Y_0[(1 + i\rho_x e_m)(1 - i\rho_y e_n) + (1 + i\rho_y e_n)(1 - i\rho_x e_m)] \\ &= N(X) + N(Y) + 2X_0Y_0(1 - \rho_x \rho_y e_m \cdot e_n), \end{aligned} \quad (378)$$

that is

$$1 - \rho^2 = \frac{X_0^2(1 - \rho_x^2) + Y_0^2(1 - \rho_y^2) + 2X_0Y_0(1 - \rho_x \rho_y e_m \cdot e_n)}{(X_0 + Y_0)^2}. \quad (379)$$

Examples.

1. Both \mathcal{X} and \mathcal{Y} are completely polarized ($\rho_x = \rho_y = 1$) and $X_0 = Y_0$. Then

$$1 - \rho^2 = \frac{1}{2}(1 - e_m \cdot e_n). \quad (380)$$

We obtain the following results:

- (a) If moreover the waves \mathcal{X} and \mathcal{Y} are orthogonally polarized, that is if $e_n = -e_m$, then $\rho = 0$, which means that $\mathcal{X} + \mathcal{Y}$ is unpolarized. The incoherent superposition of two orthogonal completely polarized waves of equal powers is unpolarized.
 - (b) If X and Y are collinear, that is $e_n = e_m$, then $\rho = 1$, and $\mathcal{X} + \mathcal{Y}$ is completely polarized (along e_m on the Poincaré sphere). This is example 1-b of Section 7.3.1.
 - (c) If \mathcal{X} and \mathcal{Y} are rectilinearly polarized at 45° to each other (in the wave plane), then e_m and e_n are orthogonal on the Poincaré sphere and $e_m \cdot e_n = 0$. We obtain $\rho = \sqrt{2}/2 \approx 0.707$. The same result is obtained if \mathcal{X} is rectilinearly polarized and \mathcal{Y} circularly polarized. The incoherent superposition of the two waves is partially polarized.
2. We assume $\rho_x = \rho_y = 0$. Then $1 - \rho^2 = 1$ and $\rho = 0$. This is example 1-c of Section 7.3.1.
 3. Let us assume $\rho_x = 0.80$, $\rho_y = 0.60$ and $X_0 = Y_0$.
 - (a) If $e_n = -e_m$, we obtain $1 - \rho^2 = 0.99$ and then $\rho = 0.10$.
 - (b) If $e_m \cdot e_n = 0$, we obtain $1 - \rho^2 = 0.75$ and then $\rho = 0.50$.
 - (c) If $e_n = e_m$, we obtain $1 - \rho^2 = 0.51$ and then $\rho = 0.70$.

The first two examples above show that the estimate of the example 2-b of Section 7.3.1 ($\rho < 0.714$) may be a rather rough estimate. Moreover, the value $\rho = 0.714$ cannot be reached, whatever e_m and e_n . To obtain $\rho = 0.714$, we should have $1 - \rho^2 = 0.49$, and then, with $X_0 = Y_0$,

$$0.49 = \frac{1}{4}(0.36 + 0.64 + 2 - 0.96 e_m \cdot e_n), \quad (381)$$

that is

$$0.24 e_m \cdot e_n = 0.26, \quad (382)$$

which is not possible, because e_m and e_n are unit pure quaternions (for which $|e_m \cdot e_n| \leq 1$).

8 Conclusion

Without doubt the special theory of relativity constitutes the main application of the Minkowski space. Moreover the fundamental role of the special theory of relativity in physics in defining the framework of space and time leads physicists to speak of Minkowski spacetime. But Minkowski space also plays a role in another branch of physics, explicitly, in polarization optics, involving physical magnitudes not directly related to time nor to space. There is then an interest in studying the Minkowski space structure from a geometrical point of view, without referring to time and space. The present article is an attempt in such a way.

Quaternion algebra appears like an appropriate algebraic structure in representing both Minkowski space and proper Lorentz rotations, regardless of the intended application. That is the point of view developed in the present article. Thus both polarization optics and special theory of relativity should appear like concrete and physical implementations of an abstract geometrical structure, that of the Minkowski space.

Appendix A Quaternionic trigonometry

Usual trigonometry. Before we give some trigonometric formulas that include quaternions, we recall usual relations between elliptical sine and cosine functions and their hyperbolic counterparts.

If x is a real number, we start with

$$\exp x = \cosh x + \sinh x, \quad (383)$$

where

$$\cosh x = \frac{\exp x + \exp(-x)}{2}, \quad \text{and} \quad \sinh x = \frac{\exp x - \exp(-x)}{2}. \quad (384)$$

We also have

$$\exp ix = \cos x + i \sin x, \quad (385)$$

where

$$\cos x = \frac{\exp ix + \exp(-ix)}{2}, \quad \text{and} \quad \sin x = \frac{\exp ix - \exp(-ix)}{2i}. \quad (386)$$

From Eqs. (384) and (386) we deduce

$$\cosh ix = \cos x, \quad \sinh ix = i \sin x, \quad (387)$$

and

$$\cos ix = \cosh x, \quad \sin ix = i \sinh x. \quad (388)$$

We also have

$$1 = \cos^2 x + \sin^2 x = \cosh^2 ix - \sinh^2 ix, \quad (389)$$

and

$$1 = \cosh^2 x - \sinh^2 x = \cos^2 ix + \sin^2 ix. \quad (390)$$

Hyperbolic functions with quaternion arguments. If q is a quaternion, the general formula for the exponential is

$$\exp q = 1 + q + \frac{q^2}{2!} + \frac{q^3}{3!} + \dots + \frac{q^j}{j!} + \dots = \sum_{j=0}^{+\infty} \frac{q^j}{j!}. \quad (391)$$

We define hyperbolic functions by

$$\cosh q = \frac{\exp q + \exp(-q)}{2} = \sum_{j=0}^{+\infty} \frac{q^{2j}}{(2j)!}, \quad (392)$$

$$\sinh q = \frac{\exp q - \exp(-q)}{2} = \sum_{j=0}^{+\infty} \frac{q^{2j+1}}{(2j+1)!}, \quad (393)$$

so that

$$\exp q = \cosh q + \sinh q. \quad (394)$$

Quaternions $\exp q$, $\cosh q$ and $\sinh q$ commute, because they involve only powers of q in their series expansions. From $\exp q \exp(-q) = 1$, we deduce

$$\cosh^2 q - \sinh^2 q = (\cosh q + \sinh q)(\cosh q - \sinh q) = \exp q \exp(-q) = 1. \quad (395)$$

Example. Let $q = 1 + i e_1$. (We note that $N(q) = 0$.) Then

$$\begin{aligned} q^2 &= (1 + i e_1)(1 + i e_1) = 2(1 + i e_1) = 2q, \\ q^3 &= q q^2 = 4q, \\ q^4 &= q q^3 = 8q, \\ &\dots \quad \dots \\ q^j &= q q^{j-1} = 2^{j-1} q. \end{aligned} \quad (396)$$

We obtain (with $e = \exp 1$)

$$\exp q = 1 + q \sum_{j=1}^{+\infty} \frac{2^{j-1}}{j!} = 1 + \frac{q}{2} \sum_{j=1}^{+\infty} \frac{2^j}{j!} = 1 + \frac{q}{2}(e^2 - 1). \quad (397)$$

We also have

$$\begin{aligned} (-q)^2 &= (1 + i e_1)(1 + i e_1) = 2(1 + i e_1) = 2q, \\ (-q)^3 &= -q q^2 = -4q, \\ (-q)^4 &= (-q)(-q)^3 = 8q, \\ &\dots \quad \dots \\ (-q)^j &= (-q)(-q)^{j-1} = (-1)^j 2^{j-1} q, \end{aligned} \quad (398)$$

so that

$$\exp(-q) = 1 + q \sum_{j=1}^{+\infty} (-1)^j \frac{2^{j-1}}{j!} = 1 + \frac{q}{2} \sum_{j=1}^{+\infty} (-1)^j \frac{2^j}{j!} = 1 + \frac{q}{2}(e^{-2} - 1). \quad (399)$$

Since $q^2 = 2q$, we check

$$\exp q \exp(-q) = \left(1 + \frac{q}{2}e^2 - \frac{q}{2}\right) \left(1 + \frac{q}{2}e^{-2} - \frac{q}{2}\right) = 1. \quad (400)$$

From Eqs. (397) and (399), we deduce

$$\cosh q = 1 + \frac{q}{2}(\cosh 2 - 1), \quad \text{and} \quad \sinh q = \frac{q}{2} \sinh 2, \quad (401)$$

or more explicitly

$$\cosh(1 + i e_1) = 1 + \frac{1 + i e_1}{2}(\cosh 2 - 1), \quad \text{and} \quad \sinh(1 + i e_1) = \frac{1 + i e_1}{2} \sinh 2. \quad (402)$$

We remark that $\cosh(1 + i e_1)$ and $\sinh(1 + i e_1)$ may also be written as follows

$$\cosh(1 + i e_1) = 1 + (1 + i e_1) \sinh^2 1 = \cosh^2 1 + i e_1 \sinh^2 1, \quad (403)$$

$$\sinh(1 + i e_1) = (1 + i e_1) \cosh 1 \sinh 1. \quad (404)$$

We use $q^2 = 2q$ and eventually check

$$\cosh^2 q - \sinh^2 q = 1 + \frac{q^2}{4}(1 + \cosh^2 2 - 2 \cosh 2) + q(\cosh 2 - 1) - \frac{q^2}{4} \sinh^2 2 = 1. \quad (405)$$

Pure-quaternion trigonometry [21]. We assume that $q = e_q \varphi$, where e_q is a real pure unit quaternion and φ a real number. Since $e_q^2 = -1$, we have

$$\cosh e_q \varphi = \sum_{j=0}^{+\infty} \frac{(e_q \varphi)^{2j}}{(2j)!} = \sum_{j=0}^{+\infty} (-1)^j \frac{\varphi^{2j}}{(2j)!} = \cos \varphi, \quad (406)$$

and

$$\sinh e_q \varphi = \sum_{j=0}^{+\infty} \frac{(e_q \varphi)^{2j+1}}{(2j+1)!} = e_q \sum_{j=0}^{+\infty} (-1)^j \frac{\varphi^{2j+1}}{(2j+1)!} = e_q \sin \varphi, \quad (407)$$

so that

$$\exp e_q \varphi = \cosh e_q \varphi + \sinh e_q \varphi = \cos \varphi + e_q \sin \varphi. \quad (408)$$

If we replace φ with $e_q \varphi$ in Eqs. (406) and (407) we obtain

$$\cos e_q \varphi = \cosh(-\varphi) = \cosh \varphi, \quad (409)$$

$$\sin e_q \varphi = -e_q \sinh(-\varphi) = e_q \sinh \varphi. \quad (410)$$

Equations (406), (407), (409) and (410) are similar to Eqs. (387) and (388) where i has been changed into e_q .

If we replace φ with $i\varphi$ in Eq. (408), we obtain

$$\exp i e_q \varphi = \cosh i e_q \varphi + \sinh i e_q \varphi = \cos e_q \varphi + i \sin e_q \varphi = \cosh \varphi + i e_q \sinh \varphi, \quad (411)$$

that is

$$\cosh i e_q \varphi = \cosh \varphi, \quad \sinh i e_q \varphi = i e_q \sinh \varphi, \quad (412)$$

and then

$$\cosh^2 i e_q \varphi - \sinh^2 i e_q \varphi = \cosh^2 \varphi - \sinh^2 \varphi = 1. \quad (413)$$

If we replace φ with $i\varphi$ in Eqs. (409) and (410), we obtain

$$\cos i e_q \varphi = \cosh i \varphi = \cos \varphi, \quad \sin i e_q \varphi = e_q \sinh i \varphi = i e_q \sin \varphi, \quad (414)$$

and

$$\cos^2 i e_q \varphi + \sin^2 i e_q \varphi = \cos^2 \varphi + \sin^2 \varphi = 1. \quad (415)$$

Eventually we point out that similar results are obtained for $\exp e_u \psi$, where ψ is a complex number and e_u a complex unit pure quaternion ($N(e_u) = 1$ and $e_u^2 = -1$). The reason is that Eqs. (406) and (407) hold if we replace e_q (a real unit pure quaternion) with e_u , and φ (a real number) with ψ (a complex number). Then

$$\exp e_u \psi = \cos \psi + e_u \sin \psi, \quad (416)$$

and

$$N(\exp e_u \psi) = (\cos \psi + e_u \sin \psi)(\cos \psi - e_u \sin \psi) = 1. \quad (417)$$

(Equation (90) provides an example of a complex unit pure quaternion).

Example. We apply Eq. (412) and obtain

$$\cosh(1 + i e_1) = \cosh 1 \cosh i e_1 + \sinh 1 \sinh i e_1 = \cosh^2 1 + i e_1 \sinh^2 1, \quad (418)$$

$$\sinh(1 + i e_1) = \sinh 1 \cosh i e_1 + \cosh 1 \sinh i e_1 = (1 + i e_1) \cosh 1 \sinh 1, \quad (419)$$

which are Eqs. (403) and (404) once more.

Zero-square quaternions. Equation (391) holds for every quaternion q . Let us assume now that q is such that $q^2 = 0$. Then $\exp q = 1 + q$, by Eq. (391). We apply Eqs. (392) and (393) and obtain

$$\cosh q = 1, \quad \sinh q = q, \quad (420)$$

so that

$$\exp q = 1 + q = \cosh q + \sinh q. \quad (421)$$

Since $q^2 = 0$, we obtain $\cosh^2 q - \sinh^2 q = 1$.

For example, let $q = e_1 + i e_2$. Then $q^2 = (e_1 + i e_2)(e_1 + i e_2) = 0$, and

$$\cosh(e_1 + i e_2) = 1, \quad \sinh(e_1 + i e_2) = e_1 + i e_2, \quad (422)$$

so that

$$\exp(e_1 + i e_2) = 1 + e_1 + i e_2. \quad (423)$$

More generally, let $e_u = e_m + i e_n$, where e_m and e_n are real unit pure quaternion, with $e_m \cdot e_n = 0$, so that $N(e_u) = -e_u^2 = 0$. Let $q = e_u \psi$, where ψ is a complex number. Then $q^2 = e_u^2 \psi^2 = 0$, as above, and we obtain

$$\exp e_u \psi = \exp q = 1 + q = \cosh q + \sinh q = \cosh e_u \psi + \sinh e_u \psi = 1 + e_u \psi. \quad (424)$$

We remark that $\exp e_u \psi$ cannot be written as in Eq. (416), if $N(e_u) = 0$.

Appendix B Proof of $V = e_u \sin(\psi/2)$ (Sect. 2.5.8)

Notations are those of Section 2.5.8, related to the composition of two elliptic rotations. We assume $e_n \neq \pm e_m$. We recall that θ, φ belong to the open interval $]-\pi, \pi[$.

Let us assume $\psi = 0 \pmod{2\pi}$. Then $u_0 = \pm 1$ and since $u_0^2 + \|V\|^2 = N(u) = 1$, we conclude that $\|V\| = 0$ and then $V = 0$. Since $e_n \neq \pm e_m$, vectors e_m, e_n and $e_n \times e_m$ are linearly independant, and if $V = 0$, then necessarily $\varphi = 0 = \theta$, according to Eq. (110). Since $\varphi \neq 0 \neq \theta$, we conclude that $\cos(\psi/2) \neq \pm 1$ and then $\sin(\psi/2) \neq 0$.

The angle α between e_m and e_n is such that $\cos \alpha = e_m \cdot e_n$ ($\alpha \neq 0 \pmod{\pi}$, because $e_m \neq \pm e_n$). Let e_p be the unit pure quaternion defined by $e_p \sin \alpha = e_m \times e_n$, and let e_q be the unit pure quaternion defined by $e_q = e_m \times e_p$. The unit quaternion e_n is written $e_n = e_m \cos \alpha - e_q \sin \alpha$. According to Eq. (110), we have

$$\begin{aligned} V &= e_n \cos \frac{\theta}{2} \sin \frac{\varphi}{2} + e_m \cos \frac{\varphi}{2} \sin \frac{\theta}{2} + e_n \times e_m \sin \frac{\theta}{2} \sin \frac{\varphi}{2} \\ &= e_m \left(\cos \frac{\varphi}{2} \sin \frac{\theta}{2} + \cos \alpha \cos \frac{\theta}{2} \sin \frac{\varphi}{2} \right) + e_p \sin \alpha \sin \frac{\theta}{2} \sin \frac{\varphi}{2} - e_q \sin \alpha \cos \frac{\theta}{2} \sin \frac{\varphi}{2}. \end{aligned} \quad (425)$$

Since e_m, e_p and e_q form an orthogonal direct basis of \mathbb{E}_3 , we have

$$\begin{aligned} \|V\|^2 &= \cos^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} + \cos^2 \alpha \cos^2 \frac{\theta}{2} \sin^2 \frac{\varphi}{2} + 2 \cos \alpha \cos \frac{\varphi}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin \frac{\varphi}{2} \\ &\quad + \sin^2 \alpha \sin^2 \frac{\theta}{2} \sin^2 \frac{\varphi}{2} + \sin^2 \alpha \cos^2 \frac{\theta}{2} \sin^2 \frac{\varphi}{2} \\ &= \cos^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} \sin^2 \frac{\varphi}{2} + \sin^2 \alpha \sin^2 \frac{\theta}{2} \sin^2 \frac{\varphi}{2} + 2 \cos \alpha \cos \frac{\varphi}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin \frac{\varphi}{2} \\ &= \cos^2 \frac{\varphi}{2} - \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2} + \sin^2 \frac{\varphi}{2} - \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} \\ &\quad + \sin^2 \alpha \sin^2 \frac{\theta}{2} \sin^2 \frac{\varphi}{2} + 2 \cos \alpha \cos \frac{\varphi}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin \frac{\varphi}{2} \\ &= 1 - \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2} - \cos^2 \alpha \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} + 2 \cos \alpha \cos \frac{\varphi}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin \frac{\varphi}{2} \\ &= 1 - \left(\cos \frac{\varphi}{2} \cos \frac{\theta}{2} - e_m \cdot e_n \sin \frac{\varphi}{2} \sin \frac{\theta}{2} \right)^2 \\ &= 1 - \cos^2 \frac{\psi}{2} = \sin^2 \frac{\psi}{2}. \end{aligned} \quad (426)$$

Since $\sin(\psi/2) \neq 0$, we conclude that $V/\sin(\psi/2)$ is a unit pure quaternion: it can be written e_u . Eventually we obtain

$$V = e_u \sin \frac{\psi}{2}. \quad (427)$$

We remark that Eq. (427) leads to $V = 0$, if $\psi = 0$, so that if both θ and φ are 0, Eq. (427) still holds true.

Appendix C Mappings $q \mapsto u q u^*$ are rotations on \mathbb{H}_c

Let R be the mapping $R : \mathbb{H}_c \rightarrow \mathbb{H}_c$ defined by $q \mapsto R(q) = u q u^*$, where u is a unit quaternion. Since $N(u q u^*) = N(q)$, we conclude that R is an isometry on \mathbb{H}_c . To prove that R is a rotation, we have to prove that $\det R = 1$.

i- Real elliptic rotations. Let $u = \exp e_n \varphi$, where φ is a real number and e_n a real unit pure quaternion. If $e_n = e_1$, we obtain the matrix of R by replacing φ with 2φ in Eq. (65). More generally the matrix of R in the canonical basis $\{e_0, e_1, e_2, e_3\}$ of \mathbb{H}_c is given by Eq. (64). Then $\det R = 1$, and R is a rotation. The rotation angle is 2φ .

ii- Imaginary elliptic rotations of axis e_1 . We assume first that u takes the form $u = \exp i e_1 \theta$, where θ is a real number. We have $R(e_0) = e_0$, and

$$\begin{aligned} R(e_1) &= (\cosh \theta + i e_1 \sinh \theta) e_1 (\cosh \theta - i e_1 \sinh \theta) \\ &= (-i \sinh \theta + e_1 \cosh \theta) (\cosh \theta - i e_1 \sinh \theta) \\ &= e_1 (\cosh^2 \theta - \sinh^2 \theta) = e_1. \end{aligned} \quad (428)$$

We also have

$$\begin{aligned}
R(e_2) &= (\cosh \theta + i e_1 \sinh \theta) e_2 (\cosh \theta - i e_1 \sinh \theta) \\
&= (e_2 \cosh \theta + i e_3 \sinh \theta) (\cosh \theta - i e_1 \sinh \theta) \\
&= e_2 (\cosh^2 \theta + \sinh^2 \theta) + 2i e_3 \cosh \theta \sinh \theta \\
&= e_2 \cosh 2\theta + i e_3 \sinh 2\theta,
\end{aligned} \tag{429}$$

and

$$\begin{aligned}
R(e_3) &= (\cosh \theta + i e_1 \sinh \theta) e_3 (\cosh \theta - i e_1 \sinh \theta) \\
&= (e_3 \cosh \theta - i e_2 \sinh \theta) (\cosh \theta - i e_1 \sinh \theta) \\
&= -2i e_2 \cosh \theta \sinh \theta + e_3 (\cosh^2 \theta + \sinh^2 \theta) \\
&= -i e_2 \sinh 2\theta + e_3 \cosh 2\theta.
\end{aligned} \tag{430}$$

The matrix of R is then

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh 2\theta & -i \sinh 2\theta \\ 0 & 0 & i \sinh 2\theta & \cosh 2\theta \end{pmatrix}, \tag{431}$$

and $\det R = \det \mathcal{R} = 1$: the mapping R is a rotation on \mathbb{H}_c . The rotation angle is $2i\theta$.

iii– Imaginary elliptic rotations of arbitrary axis e_n . We assume $u = \exp i e_n \theta$ with θ a real number and e_n a real unit pure quaternion ($e_n \neq e_1$). Let e_p be the unit pure quaternion such that

$$e_1 \times e_n = e_p \sin \alpha, \tag{432}$$

where α is the angle between e_1 and e_n ($-\pi < \alpha < \pi$, $\alpha \neq 0$). Then e_n can be deduced from e_1 under the rotation R_α of angle α around e_p , that is

$$e_n = e^{e_p \alpha / 2} e_1 e^{-e_p \alpha / 2}. \tag{433}$$

Let $w = \exp(e_p \alpha / 2)$. Then (this is Lemma 1 of Appendix D)

$$\begin{aligned}
w \exp(i e_1 \theta) w^* &= w (e_0 \cosh \theta + i e_1 \sinh \theta) w^* = e_0 \cosh \theta + i w e_1 w^* \sinh \theta \\
&= \exp(i w e_1 w^* \theta) = \exp(i e_n \theta),
\end{aligned} \tag{434}$$

that is

$$\exp(i e_n \theta) = \exp\left(e_p \frac{\alpha}{2}\right) \exp(i e_1 \theta) \exp\left(-e_p \frac{\alpha}{2}\right), \tag{435}$$

and

$$\exp(-i e_n \theta) = \exp\left(e_p \frac{\alpha}{2}\right) \exp(-i e_1 \theta) \exp\left(-e_p \frac{\alpha}{2}\right). \tag{436}$$

Let R_1 be the rotation represented by $\exp i e_1 \theta$, and $R_{-\alpha}$ be the rotation of angle $-\alpha$ around e_p . We derive

$$\begin{aligned}
R(q) &= u q u^* = \exp(i e_n \theta) q \exp(-i e_n \theta) \\
&= \exp\left(\frac{e_p \alpha}{2}\right) \exp(i e_1 \theta) \exp\left(-\frac{e_p \alpha}{2}\right) q \exp\left(\frac{e_p \alpha}{2}\right) \exp(-i e_1 \theta) \exp\left(-\frac{e_p \alpha}{2}\right),
\end{aligned} \tag{437}$$

so that

$$R(q) = R_\alpha \circ R_1 \circ R_{-\alpha}(q), \tag{438}$$

and R is a rotation as product of rotations on \mathbb{H}_c .

iv- Complex elliptic rotations. The unit quaternion u takes the form $u = \exp e_n \psi$, with e_n a real unit pure quaternion and $\psi = \varphi + i\theta$. We write $\exp e_n \psi = \exp e_n \varphi \exp i e_n \theta$ and deduce that R is the (commutative) product of a real rotation R_r and an imaginary rotation R_i so that $\det R = \det R_r \det R_i = 1$.

v- Complex-axis elliptic rotations. The unit quaternion u takes the form $u = \exp e_u \psi$, where e_u is a complex unit pure quaternion and ψ a complex number.

We consider the group $\text{SO}_+(1,3)$ of proper Lorentz rotations. Those rotations are represented by quaternions as u above. Since $\text{SO}_+(1,3)$ is generated by elliptic and hyperbolic rotations, the quaternion u is a product of the form $u = vw$ (or $u = wv$) with $v = \exp e_m \varphi$ and $w = \exp e_n \theta$. But v and w also represent elliptic rotations on \mathbb{H}_c : their products (vw or wv) then represent rotations on \mathbb{H}_c . These rotations operates, on \mathbb{H}_c , according to $q' = w q w^*$ and $q'' = v q' v^*$, so that

$$q'' = v q' v^* = v(w q w^*) v^* = (vw) q (vw)^* = u q u^*. \quad (439)$$

Appendix D Two lemmas on unit pure quaternions

Lemma 1. *Let e_u be a complex unit pure quaternion and ψ a complex number. Then for every real unit quaternion w*

$$w \exp(e_u \psi) w^* = \exp(w e_u \psi w^*). \quad (440)$$

Proof. Since w is a unit quaternion, we have $w w^* = 1$, so that

$$w \exp(e_u \psi) w^* = w (e_0 \cos \psi + e_u \sin \psi) w^* = e_0 \cos \psi + w e_u w^* \sin \psi. \quad (441)$$

To conclude, we remark that $(w e_u w^*)^* = (w^*)^* e_u^* w^* = -w e_u w^*$, which means that $w e_u w^*$ is a unit pure quaternion, so that

$$w \exp(e_u \psi) w^* = \exp(w e_u w^* \psi) = \exp(w e_u \psi w^*). \quad (442)$$

The proof is complete. \square

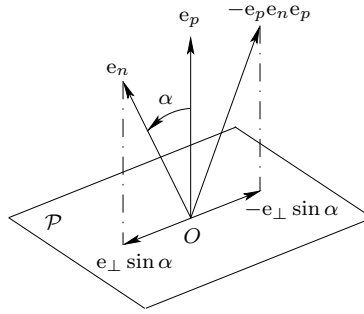


Figure 11: Elements for the proof of Lemma 2. The plane \mathcal{P} is orthogonal to e_p . The orthogonal projection of e_n on \mathcal{P} is $e_{\perp} \sin \alpha$, and $-e_{\perp} \sin \alpha$ is the orthogonal projection of $-e_p e_n e_p$.

Most often in this article Lemma 1 is applied for e_u a real unit pure quaternion (generally denoted e_n).

Lemma 2. *Let e_n and e_p be two real unit pure quaternions and let α be the angle between them. Then*

$$e_n - e_p e_n e_p = 2(e_n \cdot e_p) e_p = 2 e_p \cos \alpha. \quad (443)$$

Proof. We consider real pure quaternions as vectors of \mathbb{E}_3 , with origin at O : we write e_n in place of e_n , and e_j in place of e_j ($j = 1, 2, 3$).

Let \mathcal{P} be the plane orthogonal to e_p and passing through O (see Fig. 11). Then e_n can be written $e_n = e_p \cos \alpha + e_\perp \sin \alpha$, where $e_\perp \sin \alpha$ is the (orthogonal) projection of e_n on \mathcal{P} and is perpendicular to e_p . The pure quaternion (or vector) e_\perp becomes $-e_\perp$ under the rotation of angle π around e_p (Fig. 11), so that

$$-e_\perp = \exp\left(e_p \frac{\pi}{2}\right) e_\perp \exp\left(-e_p \frac{\pi}{2}\right) = -e_p e_\perp e_p, \quad (444)$$

that is

$$e_p e_\perp e_p = e_\perp. \quad (445)$$

We deduce then

$$\begin{aligned} e_n - e_p e_n e_p &= e_p \cos \alpha + e_\perp \sin \alpha - e_p (e_p \cos \alpha + e_\perp \sin \alpha) e_p \\ &= 2e_p \cos \alpha + (e_\perp - e_p e_\perp e_p) \sin \alpha \\ &= 2e_p \cos \alpha. \end{aligned} \quad (446)$$

The proof is complete. \square

The following corollary is a straightforward consequence of Lemma 2.

Corollary. *If e_n and e_p are orthogonal real unit pure quaternions, then $e_p e_n e_p = e_n$. Moreover, if $V = \alpha e_n$ (α a real number), then $e_p V e_p = V$.*

Remark D.1 [Another proof of Lemma 2] We begin with a direct proof of the previous corollary. If e_n and e_p are orthogonal we have $e_n e_p = -e_p e_n$ and then $e_p e_n e_p = -(e_p)^2 e_n = e_n$, that is $e_n - e_p e_n e_p = 0$. The proof of the corollary is complete.

To prove Lemma 2, we first consider than e_n and e_p are orthogonal. Then $e_n - e_p e_n e_p = 0$, as we have just proved, and Lemma 2 is proved for that case.

If e_n is not orthogonal to e_p , let e_\perp be such that $e_\perp \sin \alpha = e_n - e_p \cos \alpha$. We have $e_p \cdot e_\perp = 0$, and e_\perp is orthogonal to e_p ; we may write $e_\perp - e_p e_\perp e_p = 0$, which is Eq. (445). The end of the proof is given by Eq. (446).

Appendix E Regular rotations: eigenquaternions and eigenvalues

Eigenquaternions and eigenvalues. We will only prove Eqs. (101) and (105). Let us denote $C = \cos(\psi/2)$ and $S = \sin(\psi/2)$, so that $u = C + S e_u$ and $\bar{u}^* = \bar{C} - \bar{S} \bar{e}_u$. Let $x' = (1 + i e_u)(1 + i \bar{e}_u)$. We have

$$\left(\overline{1 + i e_u}\right)^* = (1 - i \bar{e}_u)^* = 1 + i \bar{e}_u, \quad (447)$$

and

$$\left(\overline{1 + i\bar{e}_u}\right)^* = (1 - ie_u)^* = 1 + ie_u, \quad (448)$$

so that

$$\bar{x}'^* = \left[\overline{(1 + ie_u)(1 + i\bar{e}_u)}\right]^* = \left(\overline{1 + i\bar{e}_u}\right)^* \left(\overline{1 + ie_u}\right)^* = (1 + ie_u)(1 + i\bar{e}_u) = x', \quad (449)$$

and x' is a minquat.

Then

$$\begin{aligned} u x' \bar{u}^* &= (C + S e_u)(1 + ie_u)(1 + i\bar{e}_u)(\bar{C} - \bar{S} \bar{e}_u) \\ &= [C - iS + (S + iC) e_u] [\bar{C} + i\bar{S} - (\bar{S} - i\bar{C}) \bar{e}_u] \\ &= (C - iS)(1 + ie_u)(\bar{C} + i\bar{S})(1 + i\bar{e}_u) \\ &= (C - iS)(\bar{C} + i\bar{S})(1 + ie_u)(1 + i\bar{e}_u) \\ &= (C - iS)(\bar{C} + i\bar{S}) x', \end{aligned} \quad (450)$$

and x' is an eigenminquat of the rotation represented by u . The corresponding eigenvalue is

$$\Lambda = (C - iS)(\bar{C} + i\bar{S}) = C\bar{C} + S\bar{S} + iC\bar{S} - i\bar{C}S. \quad (451)$$

Appendix F Singular Lorentz rotations: eigenminquats

We shall prove that $e_0 + ie_m e_n$ is an eigenminquat of the mapping $x \mapsto u_+ x \bar{u}_+^*$. For the sake of simplicity we set $\phi = \varphi/2$ in Eq. (107). Since e_m and e_n are orthogonal, we have $e_m e_n e_m = e_n$, and $e_n e_m e_n = e_m$, according to the corollary of Appendix D. The following derivation shows that $e_0 + ie_m e_n$ is an eigenminquat of the proper Lorentz rotation $x \mapsto u_+ x \bar{u}_+^*$:

$$\begin{aligned} u_+ (e_0 + ie_m e_n) \bar{u}_+^* &= (e_0 + e_m \phi + ie_n \phi)(e_0 + ie_m e_n)(e_0 - e_m \phi + ie_n \phi) \\ &= (e_0 + ie_m e_n + e_m \phi - ie_n \phi + ie_n \phi - e_n e_m e_n \phi)(e_0 - e_m \phi + ie_n \phi) \\ &= (e_0 + ie_m e_n)(e_0 - e_m \phi + ie_n \phi) \\ &= e_0 - e_m \phi + ie_n \phi + ie_m e_n - ie_m e_n e_m \phi + e_m \phi \\ &= e_0 + ie_m e_n. \end{aligned} \quad (452)$$

The corresponding eigenvalue is 1.

A similar derivation holds for $u_- = e_0 + (e_m - ie_n)\phi$ and shows that $e_0 - ie_m e_n$ is an eigenminquat for the mapping $x \mapsto u_- x \bar{u}_-^*$ (just replace i with $-i$ in the previous derivation).

That there is only one eigendirection for null minquats may be understood as follows. We consider the case $e_m = e_1$ and $e_n = e_2$, so that $e_m e_n = e_3$. Let $x = e_0 + ie_p$ be a null minquat, with $x_0 = 1$ and

$$e_p = x_1 e_1 + x_2 e_2 + x_3 e_3, \quad (x_1)^2 + (x_2)^2 + (x_3)^2 = 1. \quad (453)$$

The image of x by the considered proper Lorentz rotation is

$$x' = x'_0 e_0 + ix'_1 e_1 + ix'_2 e_2 + ix'_3 e_3 = u_+ x \bar{u}_+^*, \quad (454)$$

that is

$$\begin{bmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \phi \\ i\phi \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 \\ -\phi \\ i\phi \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - i\phi x_1 + \phi x_2 \\ ix_1 + \phi - \phi x_3 \\ ix_2 + i\phi - i\phi x_3 \\ ix_3 + i\phi x_2 + \phi x_1 \end{bmatrix} \begin{bmatrix} 1 \\ -\phi \\ i\phi \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + 2\phi x_2 + 2\phi^2(1 - x_3) \\ x_1 \\ (1 + 2\phi^2)x_2 \\ x_3 + 2\phi x_2 + 2\phi^2(1 - x_3) \end{bmatrix}. \quad (455)$$

For $1 + i e_p$ to be an eigenvector, we should have

$$x'_0 = \frac{x'_1}{x_1} = \frac{x'_2}{x_2} = \frac{x'_3}{x_3}, \quad (456)$$

with $x'_j = 0$ if $x_j = 0$.

The reasoning as follows.

- Let us assume $x_1 x_2 \neq 0$. Then $x'_1/x_1 = 1$ does not depend on ϕ , so that x'_2/x_2 should not depend on ϕ , which would be possible only if $x_2 = 0$.

We conclude that necessarily $x_1 x_2 = 0$.

- Let us assume $x_1 \neq 0$ and $x_2 = 0$. Then $x'_1/x_1 = 1$ and x'_3/x_3 should not depend on ϕ : necessarily $x_3 = 1$, which leads to $(x_1)^2 + (x_3)^2 = (x_1)^2 + 1 > 1$, a contradiction with Eq. (453).
- Let us assume $x_1 = 0$ and $x_2 \neq 0$.
 - * If $x_3 \neq 1$, Eq. (456) gives

$$x'_0 = \frac{x'_3}{x_3} = \frac{x'_0 - x'_3}{1 - x_3} = \frac{1 - x_3}{1 - x_3} = 1, \quad (457)$$

and necessarily x'_2/x_2 should not depend on ϕ , that is, x_2 should be zero, a contradiction with the assumption.

- * If $x_3 = 1$, then $(x_2)^2 + (x_3)^2 = (x_2)^2 + 1 > 1$, a contradiction with Eq. (453).
- * We conclude that $x_1 = 0$ and $x_2 \neq 0$ is not admissible.

- Eventually, necessarily $x_1 = x_2 = 0$, and $x_3 = 1$. There is only one null-minquat eigendirection.

The reasoning holds in the more general orthonormal basis $\{e_m, e_n, e_m e_n\}$ in place of $\{e_1, e_2, e_3\}$.

Remark F.1 To establish the second equality in Eq. (457) we use the following identity (with $abcd \neq 0$)

$$\frac{a}{b} = \frac{c}{d} = \frac{a - c}{b - d}. \quad (458)$$

To establish the third equality we report the values of x'_0 and x'_3 given by Eq. (455).

Appendix G Equivalent birefringence and optical activity

The superposition of the rectilinear birefringent represented by $\exp(e_1\varphi/2)$ and the circular birefringent (optical activity) represented by $\exp(e_3\theta/2)$ is an elliptical birefringent represented by $u = \exp[(e_1\varphi + e_3\theta/2)]$ (we assume $\varphi\theta \neq 0$). The birefringence is $\psi = (\varphi^2 + \theta^2)^{1/2} \neq 0$.

The elliptical birefringent may be seen as the succession (in the order) of an equivalent optical activity, represented by $\exp(e_3\theta'/2)$, and an equivalent rectilinear birefringent, represented by $\exp(e_m\varphi'/2)$. We write then

$$\exp \frac{e_m\varphi'}{2} \exp \frac{e_3\theta'}{2} = \exp \frac{e_n\psi}{2} = \exp \frac{e_1\varphi + e_3\theta}{2}. \quad (459)$$

The unit quaternion e_m takes the form

$$e_m = e_1 \cos \alpha' + e_2 \sin \alpha', \quad (460)$$

because the equivalent birefringent is rectilinear.

The equivalent parameters are φ' , θ' and α' (which provides e_m). To obtain the link between them and the intrinsic parameters (φ and θ), we proceed as follows.

We first derive

$$\begin{aligned} \exp \frac{e_m \varphi'}{2} \exp \frac{e_3 \theta'}{2} &= \left(\cos \frac{\varphi'}{2} + e_1 \cos \alpha' \sin \frac{\varphi'}{2} + e_2 \sin \alpha' \sin \frac{\varphi'}{2} \right) \left(\cos \frac{\theta'}{2} + e_3 \sin \frac{\theta'}{2} \right) \\ &= \begin{bmatrix} \cos(\varphi'/2) \\ \cos \alpha' \sin(\varphi'/2) \\ \sin \alpha' \sin(\varphi'/2) \\ 0 \end{bmatrix} \begin{bmatrix} \cos(\theta'/2) \\ 0 \\ 0 \\ \sin(\theta'/2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\varphi'/2) \cos(\theta'/2) \\ \cos \alpha' \sin(\varphi'/2) \cos(\theta'/2) + \sin \alpha' \sin(\varphi'/2) \sin(\theta'/2) \\ \sin \alpha' \sin(\varphi'/2) \cos(\theta'/2) - \cos \alpha' \sin(\varphi'/2) \sin(\theta'/2) \\ \cos(\varphi'/2) \sin(\theta'/2) \end{bmatrix}, \end{aligned}$$

so that Eq. (459) is equivalent to the system

$$\begin{cases} \cos \frac{\varphi'}{2} \cos \frac{\theta'}{2} = \cos \frac{\psi}{2}, & (461a) \\ \sin \frac{\varphi'}{2} \cos \left(\alpha' - \frac{\theta'}{2} \right) = \cos \chi \sin \frac{\psi}{2}, & (461b) \\ \sin \frac{\varphi'}{2} \sin \left(\alpha' - \frac{\theta'}{2} \right) = 0, & (461c) \\ \cos \frac{\varphi'}{2} \sin \frac{\theta'}{2} = \sin \chi \sin \frac{\psi}{2}. & (461d) \end{cases}$$

According to Eq. (461d), we remark that $\theta' \neq 0$ (because $\chi\psi \neq 0$). Equations (461a) and (461d) give $\cos^2(\varphi'/2) = \cos^2(\psi/2) + \sin^2 \chi \sin^2(\psi/2) < 1$, so that $\varphi' \neq 0$. Equation (461c) gives $\alpha' - (\theta'/2) = 0 \pmod{\pi}$, so that $\cos[\alpha' - (\theta'/2)] = \pm 1$. If we choose $\alpha' - (\theta'/2) = 0$, we obtain

$$\begin{cases} \cos \frac{\varphi'}{2} \cos \frac{\theta'}{2} = \cos \frac{\psi}{2}, & (462a) \\ \sin \frac{\varphi'}{2} = \cos \chi \sin \frac{\psi}{2}, & (462b) \\ \alpha' - \frac{\theta'}{2} = 0, & (462c) \\ \cos \frac{\varphi'}{2} \sin \frac{\theta'}{2} = \sin \chi \sin \frac{\psi}{2}, & (462d) \end{cases}$$

and if we choose $\alpha' - (\theta'/2) = \pi$, the system becomes

$$\begin{cases} \cos \frac{\varphi'}{2} \cos \frac{\theta'}{2} = \cos \frac{\psi}{2}, & (463a) \\ \sin \frac{\varphi'}{2} = -\cos \chi \sin \frac{\psi}{2}, & (463b) \\ \alpha' - \frac{\theta'}{2} = \pi, & (463c) \\ \cos \frac{\varphi'}{2} \sin \frac{\theta'}{2} = \sin \chi \sin \frac{\psi}{2}. & (463d) \end{cases}$$

In both cases we have $\tan(\theta'/2) = \sin \chi \tan(\psi/2)$, so that θ' is defined modulo 2π .

We remark that the sign of φ' is given by Eqs. (462b) and (463b), so that if φ' is solution of the system (462), then $-\varphi'$ is solution of (463). And if α' is solution of (462) then $\alpha' - \pi$ is solution of (463). Then if e_m is the axis for the solution of (462), the unit quaternion $-e_m$ is the axis for (463). The equivalent birefringent is represented by $\exp(e_m\varphi'/2)$ and is the same for both systems. There is a unique solution for both (462) and (463).

Eventually, the equivalent parameters are as follows

$$\begin{cases} \sin \frac{\varphi'}{2} = \cos \chi \sin \frac{\psi}{2} = \frac{\varphi}{\sqrt{\varphi^2 + \theta^2}} \sin \frac{\sqrt{\varphi^2 + \theta^2}}{2}, & (464a) \\ \tan \frac{\theta'}{2} = \sin \chi \tan \frac{\psi}{2} = \frac{\theta}{\sqrt{\varphi^2 + \theta^2}} \tan \frac{\sqrt{\varphi^2 + \theta^2}}{2}, & (464b) \\ \alpha' = \frac{\theta'}{2}. & (464c) \end{cases}$$

As explained in Section 5.4.4 expressing the intrinsic parameters as functions of the equivalent ones is interesting for their measurements. The elliptic birefringence ψ is given by Eq. (461a)

$$\cos \frac{\psi}{2} = \cos \frac{\theta'}{2} \cos \frac{\varphi'}{2}, \quad (465)$$

and from Eqs. (461b) and (461d) we deduce

$$\tan \chi = \sin \frac{\theta'}{2} \cot \frac{\varphi'}{2}. \quad (466)$$

Remark G.1 If the equivalent rectilinear birefringent is chosen to be crossed by light before the equivalent circular birefringent (optical activity), that is, if u is

$$u = \exp \frac{e_3\theta'}{2} \exp \frac{e_m\varphi'}{2}, \quad (467)$$

the only change in the result is that Eq. (464c) is replaced with $\alpha' = -\theta'/2$. Only the direction of e_m is changed.

References

- [1] SILBERSTEIN (L.), Quaternionic form of relativity, *Phil. Mag.* **23** (1912) 790–809.
- [2] CONWAY (A. W.), The quaternionic form of relativity, *Phil. Mag.* **24** (1912) 208–208.
- [3] SILBERSTEIN (L.), Second memoir on quaternionic relativity, *Phil. Mag.* **25** (1913) 135–144.
- [4] DIRAC (P. A. M.), Application of quaternions to Lorentz transformations, *Proc. Irish Acad.* **A 50** (1945) 261–270.
- [5] KWAL (B.), La théorie des équations de Maxwell et le calcul des opérateurs matriciels, *Journal de physique* **8**, (1934) 445–448.
- [6] KWAL (B.), Sur la description spatio-temporelle des phénomènes quantiques, *Le Journal de physique et le radium*, **VII** Tome VIII (1937) 81–87.
- [7] CERNOSEK (J.), Simple geometrical method for analysis of elliptical polarization, *J. Opt. Soc. Am.* **61** (1970) 324–327.
- [8] THEOCARIS (P. S.), GDOUTOS (E. E.), *Matrix theory of photoelasticity*, Springer Verlag, Berlin, 1979.
- [9] SYNGE (J. L.), Quaternions, Lorentz transformations, and the Conway-Dirac-Eddington matrices, Dublin Institute for Advanced Studies, Serie **A 21** (1972) 1–67.

- [10] PELLAT-FINET (P.), *De la biréfringence elliptique*, Thèse de Docteur-ingénieur, Université d'Aix-Marseille III and Université de Toulon et du Var, 1983.
- [11] PELLAT-FINET (P.), Représentation des états et des opérateurs de polarisation de la lumière par des quaternions, *Optica Acta*, **31** (1984) 415–434.
- [12] PELLAT-FINET (P.), An introduction to a vectorial calculus for polarization optics, *Optik*, **84** (1990) 169–175.
- [13] BRUYÈRE (F.), Effets de polarisation dans les systèmes à amplification optique de longue distance, Thesis, Université Paris-Sud, Orsay, 1994.
- [14] PELLAT-FINET (P.), Iterative experimental method for generating eigenstates and principal states of polarization, *Appl. Opt.* **51** (2012) 4403–4408.
- [15] VIGNÉRAS (M.-F.), *Arithmétique des algèbres de quaternions*, Springer, Berlin, 1980.
- [16] VOIGHT (J.), *Quaternion Algebras*, Springer, 2021.
- [17] ALTMANN (S. L.), *Rotations, quaternions and double groups*, Clarendon Press, Oxford, 1986.
- [18] KARNEY (C. F. F.), Quaternions in molecular modelling, *J. Molecular Graphics and Modelling*, **25** (2007) 595–604.
- [19] ADLER (S. L.), *Quaternionic quantum mechanics and quantum fields*, Oxford University Press, New York, 1995.
- [20] MAHECHA (J.), *Mecánica clásica avanzada*, Editorial Universidad de Antioquia, Medellín, 2006.
- [21] GIRARD (P. R.), *Quaternions, Clifford Algebras and Relativistic Physics*, Birkhäuser, Basel, 2007.
- [22] ÖZCAN (B.), KINLI (F.), KIRAÇ (F.), Quaternion Capsule Networks, arxiv 2007.04389v1 (2020) 1–8.
- [23] PELLAT-FINET (P.), *Représentation géométrique de la lumière polarisée*, Thèse de doctorat d'État ès Sciences physiques, Université de Toulon et du Var, 1987.
- [24] PELLAT-FINET (P.), Sur la géométrie de la lumière polarisée, *C. R. Ac. Sc. Paris*, **308** II (1989) 827–830.
- [25] PELLAT-FINET (P.), Geometrical approach to polarization optics. I– Geometrical structure of polarized light, *Optik*, **87** (1991) 27–33.
- [26] PELLAT-FINET (P.), Geometrical approach to polarization optics. II– Quaternionic representation of polarized light, *Optik*, **87** (1991) 68–76.
- [27] PELLAT-FINET (P.), BAUSSET (M.), What is common to both polarization optics and relativistic kinematics?, *Optik*, **90** (1992) 101–106.
- [28] DEHEUVELS (R.), *Formes quadratiques et groupes classiques*, P.U.F., Paris, 1981.
- [29] GOURGOULHON (É.), *Special Relativity in General Frames. From Particles to Astrophysics*, Springer, Berlin, 2013.
- [30] PENROSE (R.), RINDLER (W.), *Spinors and space-time*, Cambridge University Press, Cambridge, 1984.
- [31] LANDAU (L.), LIFCHITZ (E.), *Théorie des champs in Physique théorique*, Vol. 2, 4th edn, Mir, Moscow, 1989. (Translated from Russian. Published in English: *The classical theory of fields*, in *Course of Theoretical physics*, Vol. 2, Pergamon Press, Oxford.)
- [32] MØLLER (C.), *The theory of relativity*, Oxford University Press, London, 1962.
- [33] ROUGÉ (A.), *Introduction à la relativité*, Éditions de l'École polytechnique, Palaiseau, 2008.
- [34] MISNER (C. W.), THORNE (K. S.), WHEELER (J. A.), *Gravitation*, W. H. Freeman and Company, New York, 1973.
- [35] BORN (M.), WOLF (E.), *Principles of Optics*, 7th edn., Cambridge University Press, Cambridge, 1999.
- [36] CLARKE (D.), GRAINGER (J. F.), *Polarized light and optical measurement*, Pergamon Press, Oxford, 1971.
- [37] GOLDSTEIN (D.), *Polarized Light*, 2nd edn., CRC Press, Boca Raton, 2003.
- [38] RAMACHANDRAN (G. N.), RAMASESHAN (S.), *Crystal Optics*, in Handbuch der Physik XXV/1, S. Flügge editor, Springer Verlag, Berlin, 1961.

- [39] SHURCLIFF (W.), *Polarized light*, Harvard University Press, Cambridge, 1962.
- [40] SIMMONS (J. W.), GUTMANN (M. J.), *States, Waves and Photons: a modern introduction to light*, Addison Wesley, Reading, 1970.
- [41] BRUHAT (G.), KASTLER (A.), *Optique*, 6th edn., Masson, Paris, 1992.
- [42] HERRIAU (J.-P.), HUIGNARD (J.-P.), Some polarization properties of volume holograms in $\text{Bi}_{12}\text{SiO}_{20}$ crystals and applications, *Appl. Opt.* **17** (1978) 1851–1852.
- [43] HUIGNARD (J.-P.), GÜNTER (P.) (EDS.), *Photorefractive materials and their applications. I: Fundamental phenomena*, and *II: Surveys and applications*, Springer, Berlin, 1988, 1989.
- [44] PELLAT-FINET (P.), Measurement of the electro-optic coefficient of BSO crystals, *Opt. Comm.* **50** (1984) 275–280.
- [45] PETROV (M. P.), STEPANOV (S. L.), KHOMENKO (A. V.), *Photorefractive crystals in coherent optical systems*, Springer, Berlin, 1991.
- [46] PELLAT-FINET (P.), Quaternionic representation and design of depolarizers, arxiv 2405.04196 (2024) 1–9.
- [47] PANCHARATNAM (S.), The propagation of light in absorbing biaxial crystals. I—Theoretical, *Proc. Ind. Acad. Sci. A*, **42** (1955) 86–109.
- [48] PANCHARATNAM (S.), The propagation of light in absorbing biaxial crystals. II—Experimental, *Proc. Ind. Acad. Sci. A*, **42** (1955) 235–248.
- [49] PANCHARATNAM (S.), General theory of interference and its applications. Part IV—Interference figures in absorbing biaxial crystals, *Proc. Ind. Acad. Sci. A*, **46** (1957) 1–18.
- [50] MANOUKIAN (E. B.), On the reversal of the triangle inequality in Minkowski spacetime in relativity, *Eur. J. Phys.* **14** (1993) 43–44.
- [51] BRILLOUIN (L.), *Relativity Reexamined*, Academic Press, New York, 1970.
- [52] CHOQUET-BRUHAT (Y.), *Introduction to General Relativity, Black Holes and Cosmology*, Oxford University Press, New York, 2015.
- [53] SCHUTZ (B.), *A First Course in General Relativity*, Cambridge University Press, Cambridge, 1985.