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SELF-SIMILAR SOLUTIONS, REGULARITY AND TIME ASYMPTOTICS FOR A NONLINEAR DIFFUSION EQUATION ARISING IN GAME THEORY

MARCO ANTONIO FONTELOS, NASTASSIA POURADIER DUTEIL, AND FRANCESCO SALVARANI

ABSTRACT. In this article, we study the long-time asymptotic properties of a non-linear and non-local equation of diffusive type which describes the rock-paper-scissors game in an interconnected population. We fully characterize the self-similar solution and then prove that the solution of the initial-boundary value problem converges to the self-similar profile with an algebraic rate.

INTRODUCTION

The rock-paper-scissors game is not only one of the classical examples in game theory, but it arises also in other contexts, such as bacterial ecology and evolution, where it has been extended to the scale of an entire population. In several situations, indeed, the rock-paper-scissors game allows to model cyclic competition between species and the stabilization of bacteria populations [4, 5, 7], i.e. when three species coexist and there is cyclic domination of the first species on the second one, of the second species on the third one, and of the third species on the first one. Moreover, some applications of this game have been proposed in evolutionary game theory, for example to explain the coexistence or extinction of species [10] or male reproductive strategies [11].

This justifies the importance of having a description of the rock-paper-scissors dynamics at the mesoscopic (i.e. kinetic) and macroscopic levels, where the population is described by a density function: it allows the description of the global dynamics without needing to take into account the individual situations, and is therefore well adapted for population with a high number of individuals.

A kinetic version of the rock-paper-scissors game has been studied in [8]. This situation involves a population of players who form temporary pairs through random encounters. The two members of a pair play the game once, then look for another contestant to play with, and so on. The independent variables are the time $t \in \mathbb{R}_+$ and an individual exchange variable x (which may correspond to the wealth of individuals, if the game involves agents exchanging a certain amount of money). In the case of a fully interconnected population, assuming that there are no forbidden pairs and that players continue to play as long as their wealth allows, the corresponding kinetic model introduced in [8] has the form of an integro-differential equation on the half-line $\mathbb{R}_+ = [0, +\infty)$, with a boundary condition in $x = 0$.

By assuming that players increase the frequency of the game by a factor of ε^{-1} , (with $\varepsilon > 0$) and, at the same time, reduce the amount played in each iteration of the game by a factor of ε , in the limit $\varepsilon \rightarrow 0$ the authors of [8] obtain a non-linear and non-local partial differential equation at the classical macroscopic level.

In particular, the limiting initial-boundary value problem for the unknown $u : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$, which represents the density of agents with wealth $x \in \mathbb{R}_+$ at time $t \in \mathbb{R}_+$, is the following:

$$(1) \quad \partial_t u(t, x) = \left(\int_{\mathbb{R}_+} u(t, z) dz \right) \partial_x^2 u(t, x) \quad \text{for a.e. } (t, x) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$$

$$(2) \quad u(t, 0) = 0 \quad \text{for a.e. } t \in \mathbb{R}_+^*$$

$$(3) \quad u(0, x) = u^{\text{in}}(x) \quad \text{for a.e. } x \in \mathbb{R}_+,$$

where $u^{\text{in}} \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ and $\mathbb{R}_+^* = (0, +\infty)$.

Existence and uniqueness of a very weak solution of (1)-(3) have been proven by means of a compactness argument in [8]. However, several open questions on this problem are still waiting for an answer. In this article we study two open questions about the initial-boundary value problem (1)-(3), namely the regularity of the problem and the intermediate asymptotics with respect to a suitable self-similar solution, which we will precisely identify. We stress that the asymptotic behavior is one of the main questions on diffusion equations – see the review article [12] and the references therein.

Equation (1) has a mathematical structure that is essentially non-local. It can be interpreted as a heat equation, whose diffusivity coefficient depends on the integral of the solution itself (i.e. the total

mass, in the case of non-negative solutions), which is a typical global quantity of the system. Because of the peculiar structure of the nonlinearity in (1), our methods of proof are sometimes close to those used in the study of linear equations [2] but, in several points, the need of approaches designed for studying non-linear equations are necessary (see, for example, [3]).

More specifically, in this article we prove that, similarly to the heat equation, there exists an instantaneous gain in regularity. We moreover characterize the self-similar solutions of the problem and identify the convergence speed to the intermediate asymptotic profile under some conditions on the initial condition which we precisely characterize. We note that the algebraic convergence speed is a consequence of the non-local structure of the problem.

The structure of this article is the following. The study of the regularity of the problem, together with other basic properties of its solution, are detailed in Section 1. Then, in Section 2, we study the long-time convergence of the solution toward the self-similar solution. We illustrate our study numerically in Section 3 and, in the Appendix, we treat the convergence to the self-similar solution in the case of a bounded interval.

1. BASIC RESULTS

In this section we deduce and collect some basic results about the initial-boundary value problem (1)-(3).

1.1. Weak formulation. We first define the very weak formulation of (1)-(3) as follows:

Definition 1. Let $T > 0$. A measurable function $u \in L^1([0, T] \times \mathbb{R}_+)$ is said to be a very weak solution of the initial-boundary value problem (1)-(3) if it satisfies

$$(4) \quad \int_0^T \int_{\mathbb{R}_+} u(t, x) \partial_t \varphi(t, x) \, dx \, dt + \int_0^T \left(\int_{\mathbb{R}_+} u(t, x_*) \, dx_* \right) \int_{\mathbb{R}_+} u(t, x) \partial_x^2 \varphi(t, x) \, dx \, dt + \int_{\mathbb{R}_+} u^{\text{in}}(x) \varphi(0, x) \, dx = 0$$

for all $\varphi \in C^1([0, T]; C_c^2(\mathbb{R})) \cap L^\infty([0, T] \times \mathbb{R})$, such that $\varphi(T, x) = 0$ for all $x \in \mathbb{R}_+$, $\varphi_x(t, 0) = 0$ for all $t \in [0, T]$, where $C_c^2(\mathbb{R})$ is the space of C^2 compactly supported functions on \mathbb{R} .

Existence and uniqueness of a very weak solution to (1)-(3) was proven in [8]. Moreover, one can prove that the solution is bounded by the L^∞ norm of the initial data, and if the initial data is non-negative, the solution remains non-negative for all time. The precise results are recalled in the following theorem (see [8]):

Theorem 1. Consider the initial-boundary value problem (1)-(3), with initial condition $u^{\text{in}} \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ and such that $u^{\text{in}} \geq 0$ for a.e. $x \in \mathbb{R}_+$. Let $T > 0$. Then, it has a unique very weak solution, which belongs to $L^1((0, T) \times \mathbb{R}_+) \cap L^\infty((0, T) \times \mathbb{R}_+)$. Moreover, $\|u(t, \cdot)\|_{L^\infty(\mathbb{R}_+)} \leq \|u^{\text{in}}\|_{L^\infty(\mathbb{R}_+)}$ for a.e. $t \in (0, T)$. Lastly, the solution is non-negative, i.e. $u(t, x) \geq 0$ for a.e. $t \in (0, T)$ and for a.e. $x \in \mathbb{R}_+$.

1.2. Improved regularity and positivity. Let u be the very weak solution of the initial-boundary value problem (1)-(3). Then, it is possible to consider its antisymmetric extension v , defined for all $x \in \mathbb{R}$, such that

$$(5) \quad v(t, x) = u(t, x) \mathbf{1}_{x \geq 0} - u(t, -x) \mathbf{1}_{x \leq 0},$$

for a.e. $t \in (0, T)$.

Consequently, v solves (in the very weak sense) the following auxiliary initial value problem for the unknown $v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$

$$(6) \quad \begin{cases} \partial_t v(t, x) = \left(\int_{\mathbb{R}_+} v(t, z) \, dz \right) \partial_x^2 v(t, x) & \text{for a.e. } (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\ v(0, x) = v^{\text{in}}(x) & \text{for a.e. } x \in \mathbb{R}, \end{cases}$$

where $v^{\text{in}}(x) = u^{\text{in}}(x) \mathbf{1}_{x \geq 0} - u^{\text{in}}(-x) \mathbf{1}_{x \leq 0}$ for a.e. $x \in \mathbb{R}$. Because of the regularity conditions on the initial data, we immediately deduce that $v^{\text{in}} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. This antisymmetric extension of u will allow us to prove the following result:

Proposition 1. *Let u be the very weak solution of the initial-boundary value problem (1)-(3), with initial and boundary conditions satisfying the hypotheses of Theorem 1. Then, $u \in C^\infty((0, T) \times \mathbb{R}_+^*)$. Moreover, u admits the following semi-explicit representation:*

$$(7) \quad u(t, x) = \left(4\pi \int_0^t \int_{\mathbb{R}_+} u(\theta, z) \, dz \, d\theta \right)^{-1/2} \times \int_{\mathbb{R}_+} u^{\text{in}}(y) \left\{ \exp \left[-(x-y)^2 \left(4 \int_0^t \int_{\mathbb{R}_+} u(\theta, z) \, dz \, d\theta \right)^{-1} \right] - \exp \left[-(x+y)^2 \left(4 \int_0^t \int_{\mathbb{R}_+} u(\theta, z) \, dz \, d\theta \right)^{-1} \right] \right\} dy.$$

Proof. We consider the auxiliary problem (6). We have not yet proven that v is the unique solution of (6), but we know, by construction, that it exists and belongs to $L^1((0, T) \times \mathbb{R}) \cap L^\infty((0, T) \times \mathbb{R})$, because of the results proved in [8].

We hence introduce the spatial Fourier transform

$$\hat{v} : L^1((0, T) \times \mathbb{R}) \cap L^\infty((0, T) \times \mathbb{R}) \rightarrow L^2((0, T) \times \mathbb{R}),$$

which is meaningful because of the regularity hypotheses on v . We use the following convention for the Fourier transform of a function and for its inverse:

$$\forall \xi \in \mathbb{R}, \quad \hat{v}(t, \xi) = \int_{\mathbb{R}} v(t, x) e^{-2\pi i \xi x} \, dx,$$

and

$$\forall x \in \mathbb{R}, \quad v(t, x) = \int_{\mathbb{R}} \hat{v}(t, \xi) e^{2\pi i \xi x} \, d\xi.$$

By applying the Fourier transform with respect to the x variable to all terms in the Cauchy problem (6), we deduce a problem for the Fourier transform \hat{v} of the solution, i.e. we obtain

$$(8) \quad \begin{cases} \partial_t \hat{v}(t, \xi) = -4\pi^2 \xi^2 \left(\int_{\mathbb{R}_+} v(t, z) \, dz \right) \hat{v}(t, \xi) & \text{for a.e. } (t, \xi) \in \mathbb{R}_+ \times \mathbb{R} \\ v(0, \xi) = \hat{v}^{\text{in}}(\xi) = \mathcal{F}(v^{\text{in}})(\xi) & \text{for a.e. } x \in \mathbb{R}. \end{cases}$$

This auxiliary problem can be integrated in time, allowing to deduce the integral form of the initial-boundary value problem (6):

$$(9) \quad \hat{v}(t, \xi) = \hat{v}^{\text{in}}(\xi) \exp \left[-4\pi^2 \xi^2 \int_0^t \left(\int_{\mathbb{R}_+} v(\theta, z) \, dz \right) d\theta \right]$$

for all $\xi \in \mathbb{R}$.

Thanks to the regularity of u^{in} , we have that $\hat{v}^{\text{in}} \in L^\infty(\mathbb{R})$. Hence, by Formula (8) the decay to zero of \hat{v} when ξ tends to $+\infty$ is faster than polynomial, for any degree of the polynomial. Consequently, $v \in L^1((0, T); C^\infty(\mathbb{R}))$ (see, for example, [9]).

By applying the inverse Fourier transform to the second factor of Equation (9), we find

$$\begin{aligned} & \mathcal{F}^{-1} \left(\exp \left(-4\pi^2 \xi^2 \int_0^t \left(\int_{\mathbb{R}_+} v(\theta, z) \, dz \right) d\theta \right) \right) \\ &= \left(4\pi \int_0^t \int_{\mathbb{R}_+} v(\theta, z) \, dz \, d\theta \right)^{-1/2} \exp \left[-x^2 \left(4 \int_0^t \int_{\mathbb{R}_+} v(\theta, z) \, dz \, d\theta \right)^{-1} \right], \end{aligned}$$

so that

$$v(t, x) = \left(4\pi \int_0^t \int_{\mathbb{R}_+} v(\theta, z) \, dz \, d\theta \right)^{-1/2} \int_{\mathbb{R}} v^{\text{in}}(y) \exp \left[-(x-y)^2 \left(4 \int_0^t \int_{\mathbb{R}_+} v(\theta, z) \, dz \, d\theta \right)^{-1} \right] dy.$$

for all $x \in \mathbb{R}$.

We note that, if $v \in L^1((0, T); C^\infty(\mathbb{R}))$, then the right-hand side of the previous equation is, in fact, a quantity belonging to $C((0, T); C^\infty(\mathbb{R}))$. By a bootstrap argument [6, 9], we immediately deduce that $v \in C^\infty((0, T) \times \mathbb{R})$.

In particular, when $x > 0$, we can write the previous expression in the following way:

$$v(t, x) = \left(4\pi \int_0^t \int_{\mathbb{R}_+} v(\theta, z) dz d\theta \right)^{-1/2} \times \int_{\mathbb{R}_+} u^{\text{in}}(y) \left\{ \exp \left[-(x-y)^2 \left(4 \int_0^t \int_{\mathbb{R}_+} v(\theta, z) dz d\theta \right)^{-1} \right] - \exp \left[-(x+y)^2 \left(4 \int_0^t \int_{\mathbb{R}_+} v(\theta, z) dz d\theta \right)^{-1} \right] \right\} dy.$$

Moreover, for $x = 0$, we have that $v(t, 0) = 0$ for all $t \in (0, T)$. Furthermore, v is clearly strictly positive for all $x > 0$ provided that u^{in} is non-negative for a.e. $x \in \mathbb{R}_+$. By comparing (6) and (1)-(3), we deduce that $\tilde{u} = v \mathbb{1}_{x \geq 0}$ satisfies the initial-boundary value problem (1)-(2) with initial value $\tilde{u}(0, \cdot) = v^{\text{in}} \mathbb{1}_{x \geq 0}$.

Because of the uniqueness of the very weak solution of (1)-(3) (see Theorem 1), we deduce that $\tilde{u}(t, x) = v(t, x) \mathbb{1}_{x \geq 0} = u(t, x)$ for a.e. $(t, x) \in (0, T) \times \mathbb{R}_+$. Consequently, the previous computation allows to obtain a semi-explicit representation of u :

$$u(t, x) = \left(4\pi \int_0^t \int_{\mathbb{R}_+} u(\theta, z) dz d\theta \right)^{-1/2} \times \int_{\mathbb{R}_+} u^{\text{in}}(y) \left\{ \exp \left[-(x-y)^2 \left(4 \int_0^t \int_{\mathbb{R}_+} u(\theta, z) dz d\theta \right)^{-1} \right] - \exp \left[-(x+y)^2 \left(4 \int_0^t \int_{\mathbb{R}_+} u(\theta, z) dz d\theta \right)^{-1} \right] \right\} dy.$$

Finally, we underline that $u \in C^\infty((0, T) \times \mathbb{R}_+^*)$ because u inherits the regularity properties of v . \square

1.3. Some quantitative bounds. The first step in our analysis consists in proving some uniform estimates. In all that follows, we will assume that $u^{\text{in}} \geq 0$.

Proposition 2. *Let u be the solution of the initial-boundary value problem (1)-(3) and let*

$$M : t \mapsto \int_{\mathbb{R}_+} u(t, x) dx.$$

Then M is a decreasing function of time. In particular, $M \in C^\infty((0, T))$ and, for all $t \in \mathbb{R}_+$,

$$M(t) \leq M(0) = \int_{\mathbb{R}_+} u^{\text{in}}(x) dx.$$

Proof. The result is a direct consequence of the regularity proven in Proposition 1. Integrating in x Equation (1), it holds

$$M'(t) = \left(\int_0^{+\infty} \partial_x^2 u(t, z) dz \right) M(t) = \left(\lim_{x \rightarrow +\infty} \partial_x u(t, x) - \partial_x u(t, 0) \right) M(t).$$

By differentiating both sides of Equation (7) with respect to x , we obtain

$$\begin{aligned} \partial_x u(t, x) &= -\frac{2}{\sqrt{\pi}} \left(4 \int_0^t \int_{\mathbb{R}_+} u(\theta, z) dz d\theta \right)^{-3/2} \int_{\mathbb{R}_+} u^{\text{in}}(y) (x-y) \exp \left[-(x-y)^2 \left(4 \int_0^t \int_{\mathbb{R}_+} u(\theta, z) dz d\theta \right)^{-1} \right] dy \\ &+ \frac{2}{\sqrt{\pi}} \left(4 \int_0^t \int_{\mathbb{R}_+} u(\theta, z) dz d\theta \right)^{-3/2} \int_{\mathbb{R}_+} u^{\text{in}}(y) (x+y) \exp \left[-(x+y)^2 \left(4 \int_0^t \int_{\mathbb{R}_+} u(\theta, z) dz d\theta \right)^{-1} \right] dy. \end{aligned}$$

We deduce that $\partial_x u(t, 0) \geq 0$ and $\lim_{x \rightarrow +\infty} \partial_x u(t, x) = 0$ for all $t \in (0, T)$. The thesis follows directly. \square

For future purposes, we introduce the spatial first moment $M_1 : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that, for u solution of (1)-(3), it holds

$$M_1(t) := \int_{\mathbb{R}_+} x u(t, x) dx, \quad \text{for all } t \in \mathbb{R}_+.$$

Moreover, we introduce the spatial second moment $M_2 : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that, for u solution of (1)-(3), it holds

$$M_2(t) := \int_{\mathbb{R}_+} x^2 u(t, x) dx, \quad \text{for all } t \in \mathbb{R}_+.$$

From here onward, we consider initial data with bounded spatial first moment, i.e. we suppose that the following property is satisfied.

Definition 2. *The initial condition $u^{\text{in}} \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ is admissible if and only if*

$$M_1(0) = \int_{\mathbb{R}_+} x u^{\text{in}}(x) dx < +\infty.$$

The following result holds.

Proposition 3. *Let u be a (strong) solution of (1)-(3), and suppose that u^{in} is admissible (see Definition 2). Then the spatial first moment of u is conserved, i.e. for all $t \in \mathbb{R}_+$,*

$$\int_{\mathbb{R}_+} x u(t, x) dx = \int_{\mathbb{R}_+} x u^{\text{in}}(x) dx.$$

Proof. Consider $u : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ a solution to (1)-(3), and let M be the total mass defined in Proposition 2. Let $v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be the antisymmetric extension of u defined in Equation (5) and studied in Subsection 1.2. Notice that

$$\int_{\mathbb{R}} x v(t, x) dx = \int_{\mathbb{R}_+} x u(t, x) dx - \int_{\mathbb{R}_-} x u(t, -x) dx = \int_{\mathbb{R}_+} x u(t, x) dx + \int_{\mathbb{R}_+} x u(t, x) dx = 2 \int_{\mathbb{R}_+} x u(t, x) dx.$$

Now, let $a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be defined by

$$a : t \mapsto \int_0^t M(\tau) d\tau,$$

and define $\tilde{v} : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ as follows: $\tilde{v}(a(t), x) = v(t, x)$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Then \tilde{v} is a solution to

$$\begin{cases} \partial_a \tilde{v}(a, x) = \tilde{v}_{xx}(a, x) & (a, x) \in \mathcal{T} \times \mathbb{R}, \\ \tilde{v}(0, x) = v^{\text{in}}(x) & \text{for a.e. } x \in \mathbb{R}, \end{cases}$$

where

$$\mathcal{T} = \left(0, \int_0^{+\infty} M(\tau) d\tau\right).$$

Hence, \tilde{v} satisfies the Cauchy problem for the heat equation on the real line, at least in the time interval \mathcal{T} . Since $M(0) > 0$ and M is continuous with respect to $t \in \mathbb{R}_+$, we deduce that

$$\int_0^t M(\tau) d\tau > 0 \quad \text{for all } t > 0.$$

At this point, we do not know if

$$\lim_{t \rightarrow +\infty} \int_0^t M(\tau) d\tau = +\infty \quad \text{or} \quad \lim_{t \rightarrow +\infty} \int_0^t M(\tau) d\tau < +\infty.$$

However, this is not a problem in our case. It is enough to know that the first moment of \tilde{v} , i.e.

$$\int_{\mathbb{R}} x \tilde{v}(\cdot, x) dx,$$

is conserved at least in \mathcal{T} . Hence, the first moment of v is also conserved, and

$$\int_{\mathbb{R}_+} x u(t, x) dx = \frac{1}{2} \int_{\mathbb{R}} x v(t, x) dx = \frac{1}{2} \int_{\mathbb{R}} x \tilde{v}(\tau(t), x) dx$$

is also conserved. □

Because the first moment is conserved, from here onwards, we will denote by M_1 its value, defined by $M_1 = M_1(0) = M_1(t)$ for all $t \in \mathbb{R}_+$.

2. SELF-SIMILAR SOLUTIONS AND LARGE-TIME ASYMPTOTICS

In this section, we consider the initial-boundary value problem (1)-(3), and always suppose that the initial data are admissible (i.e. we suppose that u^{in} satisfies Definition 2).

Note that the final time T appearing in the statement of Theorem 1 is finite, but arbitrary, so that the large-time asymptotics of the problem makes sense.

Let $\mu \in \mathbb{R}$. We look for self-similar solutions g_μ of the form

$$u(t, x) = t^{\mu-1} g_\mu(x/t^\mu),$$

so that the mass of the solution u satisfies for all $t \in \mathbb{R}_+$:

$$\int_{\mathbb{R}_+} u(t, z) dz = t^{2\mu-1} \int_{\mathbb{R}_+} g_\mu(\xi) d\xi.$$

Then $g_\mu(\eta)$ satisfies the following non-local differential equation: for all $\eta \in \mathbb{R}_+$,

$$(\mu - 1)g_\mu(\eta) - \mu\eta g'_\mu(\eta) = \left(\int_0^\infty g_\mu(s) ds \right) g''_\mu(\eta).$$

We apply the rescaling

$$\eta : \xi \mapsto \left(\int_0^{+\infty} g_\mu(s) ds \right)^{1/2} \xi,$$

and denote by

$$f_\mu : \xi \mapsto g_\mu \left(\left(\int_0^{+\infty} g_\mu(s) ds \right)^{1/2} \xi \right)$$

the solution to the simplified differential equation:

$$(10) \quad (\mu - 1)f_\mu(\xi) - \mu\xi f'_\mu(\xi) = f''_\mu(\xi).$$

Its relation with u is given by:

$$(11) \quad u(t, x) = t^{\mu-1} f_\mu \left(\left(\int_{\mathbb{R}_+} u(t, z) dz \right)^{-\frac{1}{2}} \frac{x}{t^{\frac{1}{2}}} \right) = t^{\mu-1} f_\mu \left(\left(\int_{\mathbb{R}_+} f_\mu(s) ds \right)^{-1} \frac{x}{t^\mu} \right),$$

where we used the following relations:

$$\int_{\mathbb{R}_+} g_\mu(\eta) d\eta = \left(\int_{\mathbb{R}_+} f_\mu(\xi) d\xi \right)^2 = t^{-2\mu+1} \int_{\mathbb{R}_+} u(t, x) dx.$$

The following Proposition guaranties the existence of self-similar solutions of the form (11) for any $\mu \in [\frac{1}{3}, 1)$.

Proposition 4. *For $\mu \in [\frac{1}{3}, 1)$ there exists a solution to (10) such that $f(0) = 0$ and $f(\xi) \rightarrow 0$ as $\xi \rightarrow \infty$. The solution is positive and such that if $\mu \in (\frac{1}{3}, 1)$,*

$$f_\mu(\xi) = O(\xi^{1-\frac{1}{\mu}}) \text{ as } \xi \rightarrow \infty,$$

and for $\mu = \frac{1}{3}$,

$$f_{\frac{1}{3}}(\xi) = \xi e^{-\frac{1}{6}\xi^2}.$$

Proof. Let $\mu \in [\frac{1}{3}, 1)$. We look for an analytic solution to (10), of the form

$$f_\mu(\xi) = \sum_{k=0}^{+\infty} b_k \xi^k,$$

where $b_k \in \mathbb{R}$ for all $k \in \mathbb{N}$. The boundary condition in $x = 0$ implies that $f_\mu(0) = b_0 = 0$. From (10), we obtain

$$\sum_{k=0}^{+\infty} [(\mu - 1)b_k - \mu k b_k - (k + 2)(k + 1)b_{k+2}] \xi^k = 0.$$

Thus, for any $k \in \mathbb{N}$, it holds

$$b_{k+2} = \frac{\mu - 1 - \mu k}{(k + 1)(k + 2)} b_k.$$

In particular, the condition $b_0 = 0$ implies that $b_{2n} = 0$ for all $n \in \mathbb{N}$. Thus, denoting $w_n := b_{2n+1}$ we can rewrite f as

$$f_\mu(\xi) = \sum_{n=0}^{+\infty} w_n \xi^{2n+1},$$

where $(w_n)_{n \in \mathbb{N}}$ satisfy the following relation (since $\mu > 0$):

$$a_{n+1} = -\frac{\mu}{2} \frac{n + \frac{1}{2\mu}}{(n + \frac{3}{2})(n + 1)} w_n$$

and hence

$$w_n = (-1)^n \left(\frac{\mu}{2} \right)^n \frac{\Gamma(\frac{3}{2}) \Gamma(1)}{\Gamma(\frac{1}{2\mu})} \frac{\Gamma(n + \frac{1}{2\mu})}{\Gamma(n + \frac{3}{2}) \Gamma(n + 1)} a_0.$$

The solution can be written in terms of classical hypergeometric confluent functions ${}_1F_1$:

$$f_\mu(\xi) = \xi {}_1F_1\left(\frac{1}{2\mu}, \frac{3}{2}; -\frac{\mu}{2}\xi^2\right) = \xi e^{-\frac{\mu}{2}\xi^2} {}_1F_1\left(\frac{3}{2} - \frac{1}{2\mu}, \frac{3}{2}; \frac{\mu}{2}\xi^2\right).$$

In the particular case $\mu = \frac{1}{3}$ one has

$$f_{\frac{1}{3}}(\xi) = \xi e^{-\frac{1}{6}\xi^2}$$

while, for $\mu \in (\frac{1}{3}, 1)$ (cf. [1] formula 13.7.1)

$${}_1F_1\left(\frac{1}{2\mu}, \frac{3}{2}; -\frac{\mu}{2}\xi^2\right) \sim \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2} - \frac{1}{2\mu})} \left(\frac{\mu}{2}\xi^2\right)^{-\frac{1}{2\mu}} \text{ as } \xi \rightarrow \infty,$$

that is

$$f_\mu(\xi) = O(\xi^{1-\frac{1}{\mu}}).$$

The positivity of $f_\mu(\xi)$ follows from the positivity of the integrand in the following representation formula for the confluent hypergeometric function of the first kind:

$${}_1F_1(\alpha, \beta; z) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)\Gamma(\alpha)} \int_0^1 e^{zt} t^{\alpha-1} (1-t)^{\beta-\alpha-1} dt$$

with $\alpha = \frac{1}{2\mu}$, $\beta = \frac{3}{2}$. This concludes the proof of the Lemma. \square

Remark 1. For $\mu < \frac{1}{3}$, $f_\mu(\xi)$ is not positive and has a zero, ξ_μ^0 , that comes from infinity as μ decreases and approaches $\xi_0^0(0) = \pi$ (since $f_0(\xi) = \sin(\xi)$ for $\mu = 0$).

Proposition 4 thus provides us with a family of solutions f_μ to equation (10), for $\mu \in [\frac{1}{3}, 1)$. Recall that by definition, the solution to (1)-(3) must have finite mass. The relation between u and f_μ given by (11) implies that for all $t \in \mathbb{R}_+$,

$$\int_{\mathbb{R}_+} u(t, x) dx = t^{2\mu-1} \left(\int_{\mathbb{R}_+} f_\mu(\xi) d\xi \right)^2.$$

From Lemma 4, f_μ is not integrable for any $\mu \in (\frac{1}{3}, 1)$, which means that the only admissible solution to (10) giving a self-similar solution to (1)-(3) with finite mass is given by $\mu = \frac{1}{3}$.

We then postulate that the self-similar solution $f_{\frac{1}{3}}$ for $\mu = \frac{1}{3}$ is an attractor, in the sense that solutions tend to the self-similar solution in a suitable norm as $t \rightarrow \infty$, for all initial data that decay sufficiently fast.

The remainder of this article aims to prove that this is indeed the case. We begin by defining a quantity that plays an important role in the definition of u and in the analysis of its asymptotic behavior. Given a solution u to (1)-(3), and its first moment $M : t \rightarrow M(t)$, we define

$$(12) \quad a(t) := \int_0^t M(s) ds.$$

The question now is the identification of $a(t)$ given by (12). Note that from (7),

$$u(t, x) = \frac{1}{2\sqrt{\pi a(t)}} \int_0^{+\infty} \left(e^{-\frac{(x-s)^2}{4a(t)}} - e^{-\frac{(x+s)^2}{4a(t)}} \right) u^{\text{in}}(s) ds$$

so that

$$\partial_x u(t, 0) = \frac{1}{2\sqrt{\pi}} a^{-\frac{3}{2}}(t) \int_0^{+\infty} s e^{-\frac{s^2}{4a(t)}} u^{\text{in}}(s) ds.$$

Integrating Equation (1) in \mathbb{R}_+ , as seen in the proof of Proposition 2, it holds

$$\frac{dM(t)}{dt} = -M(t) \partial_x u(t, 0),$$

which allows us to conclude that $a(t)$ satisfies the following integro-differential equation:

$$(13) \quad a''(t) = -a'(t) \frac{1}{2\sqrt{\pi}} a^{-\frac{3}{2}}(t) \int_0^{+\infty} s e^{-\frac{s^2}{4a(t)}} u^{\text{in}}(s) ds.$$

We define now

$$G(a) := \frac{1}{2\sqrt{\pi}} a^{-\frac{3}{2}} \int_0^{+\infty} s e^{-\frac{s^2}{4a}} u^{\text{in}}(s) ds.$$

The quantity $G(a)$ is bounded provided that u^{in} is linear at the origin and has its first moment $M_1(0)$ bounded.

Lemma 1. *Let $u^{\text{in}} \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ a positive and admissible initial condition. The integro-differential equation (13), with initial condition $a(0) = 0$ and $a'(0) = M(0)$, has a solution $a : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that*

$$a(t) \sim \left(\frac{3}{2\sqrt{\pi}} M_1 \right)^{\frac{2}{3}} t^{\frac{2}{3}} \text{ as } t \rightarrow \infty$$

and

$$a'(t) \sim \frac{2}{3} \left(\frac{3}{2\sqrt{\pi}} M_1 \right)^{\frac{2}{3}} t^{-\frac{1}{3}} \text{ as } t \rightarrow \infty.$$

Proof. Since

$$a''(t) = -\frac{d}{dt} \int_0^{a(t)} \left(\frac{1}{2\sqrt{\pi}} (a^*)^{-3/2} \int_0^{+\infty} s e^{-\frac{s^2}{4a^*}} u^{\text{in}}(s) ds \right) da^*$$

Integrating once in time and using that $a'(0) = M(0)$, it holds

$$a'(t) + \int_0^{a(t)} \left(\frac{1}{2\sqrt{\pi}} (a^*)^{-3/2} \int_0^{+\infty} s e^{-\frac{s^2}{4a^*}} u^{\text{in}}(s) ds \right) da^* = M(0),$$

which we rewrite as

$$(14) \quad a'(t) + F(a(t)) = M(0),$$

denoting $F(a) := \int_0^a G(a^*) da^*$. But now

$$\begin{aligned} F(a) &= \int_0^a G(a^*) da^* = \int_0^a \left(\frac{1}{2\sqrt{\pi}} (a^*)^{-3/2} \int_0^{+\infty} s e^{-\frac{s^2}{4a^*}} u^{\text{in}}(s) ds \right) da^* \\ &= \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \left(\int_0^a (a^*)^{-3/2} e^{-\frac{s^2}{4a^*}} da^* \right) s u^{\text{in}}(s) ds = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \left(\int_0^{as^{-2}} (a^*)^{-3/2} e^{-\frac{1}{4a^*}} da^* \right) u^{\text{in}}(s) ds \end{aligned}$$

and, since $\int_0^{+\infty} a^{-\frac{3}{2}} e^{-\frac{1}{4a}} da = 2\sqrt{\pi}$, we obtain

$$\lim_{a^* \rightarrow +\infty} F(a^*) = M(0).$$

We conclude that as t tends to infinity, if $a(t) \rightarrow +\infty$, then $a'(t) \rightarrow 0$.

Notice that from its definition, a is an increasing function, hence it has a limit when t goes to infinity. Let $a_\infty := \lim_{t \rightarrow +\infty} a(t)$, and suppose that $a_\infty < +\infty$. Then $\lim_{t \rightarrow +\infty} a'(t) = 0$, from which we get $F(a_\infty) = M(0)$. However, $F(a)$ is the primitive of a strictly positive function and hence is strictly growing as a function of a , which contradicts $\lim_{a^* \rightarrow +\infty} F(a^*) = M(0)$.

We then conclude that $a(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. Then, as $a(t) \rightarrow +\infty$, $G(a(t)) \sim \frac{1}{2\sqrt{\pi}} a(t)^{-\frac{3}{2}} M_1$ and

$$a''(t) \sim -a'(t) a^{-\frac{3}{2}}(t) \frac{1}{2\sqrt{\pi}} M_1,$$

so that $a(t) \sim ct^{\frac{2}{3}}$ as $t \rightarrow \infty$, with $-\frac{2}{9}c = -\frac{1}{3}c^{-\frac{1}{2}} \sqrt{\frac{1}{\pi}} M_1$, that is

$$c = \left(\frac{3}{2\sqrt{\pi}} M_1 \right)^{\frac{2}{3}}.$$

This concludes the proof of the Lemma. \square

On the other hand, if u^{in} does not have its first moment bounded but

$$u^{\text{in}}(x) = O(x^{-\delta}), \text{ as } x \rightarrow +\infty, \quad 1 < \delta < 2$$

then

$$G(a) = \frac{1}{2\sqrt{\pi}} a^{-\frac{3}{2}} \int_0^{+\infty} s e^{-\frac{s^2}{4a}} u^{\text{in}}(s) ds \sim \frac{1}{\sqrt{\pi}} a^{-\frac{1}{2}} \int_0^{+\infty} s e^{-\frac{s^2}{2}} u^{\text{in}}(\sqrt{2a} \frac{1}{2} s) ds \sim C a^{-\frac{1}{2} - \frac{\delta}{2}} \int_0^{+\infty} e^{-\frac{s^2}{2}} s^{1-\delta} ds$$

and hence,

$$a''(t) \sim -C a' a^{-\frac{1}{2} - \frac{\delta}{2}}$$

implying

$$a(t) = O(t^{\frac{2}{\delta+1}})$$

so that

$$\mu = \delta + 1.$$

Lemma 1 thus gives us the asymptotic behavior of a as t goes to infinity. It has two consequences. In Proposition 5, we prove that the L^∞ -norm of the solution to (1)-(3) decays like $t^{-\frac{2}{3}}$. We can then

compare the asymptotic behavior of u to that of the candidate self-similar profile. Proposition 6 shows that this profile indeed also decays like $t^{-\frac{2}{3}}$.

Proposition 5. *Let $u^{\text{in}} \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ be a positive and admissible initial condition. Then for all $x \in \mathbb{R}_+$,*

$$(15) \quad u(t, x) \leq \frac{M_1}{\sqrt{2e\pi a(t)}} \sim \frac{CM_1^{\frac{1}{3}}}{t^{\frac{2}{3}}} \quad \text{as } t \rightarrow +\infty,$$

where the value of C can be computed explicitly and does not depend on the initial data.

Proof. From Equation (7),

$$u(t, x) = \frac{1}{2\sqrt{\pi a(t)}} \int_0^{+\infty} f_a(s) u^{\text{in}}(s) ds,$$

where

$$f_a(s) := e^{-\frac{(x-s)^2}{4a}} - e^{-\frac{(x+s)^2}{4a}}.$$

Notice that

$$f_a(s) = g_a(s) - g_a(-s) = \int_{-s}^s g'_a(\xi) d\xi,$$

where $g_a(\xi) = e^{-\frac{(x-\xi)^2}{4a}}$. One easily sees that

$$g'_a(\xi) = \frac{1}{\sqrt{a}} \left(\frac{x-\xi}{\sqrt{4a}} e^{-\frac{(x-\xi)^2}{4a}} \right) \leq \frac{1}{\sqrt{2ea}},$$

in which we used the property: $|ze^{-z^2}| \leq (2e)^{-\frac{1}{2}}$ for all $z \in \mathbb{R}_+$. Hence, $f_a(s) \leq \frac{2s}{\sqrt{2ea}}$, which implies that

$$u(t, x) \leq \frac{1}{\sqrt{2e\pi a(t)}} M_1.$$

Lemma 1 allows us to conclude. \square

Proposition 6. *Let $u^{\text{in}} \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ be a positive and admissible initial condition. Let u be the solution to (1)-(3), and let*

$$a : t \mapsto \int_0^{+\infty} u(t, x) dx.$$

Then for all $x \in \mathbb{R}_+$,

$$(16) \quad \frac{M_1 x}{2\sqrt{\pi a^{\frac{3}{2}}(t)}} e^{-\frac{x^2}{4a(t)}} \sim \frac{C}{t^{\frac{2}{3}}} \quad \text{as } t \rightarrow +\infty,$$

for some positive constant C .

Proof. For all $x \in \mathbb{R}_+$,

$$\frac{M_1 x}{2\sqrt{\pi a^{\frac{3}{2}}(t)}} e^{-\frac{x^2}{4a(t)}} = \frac{M_1}{\sqrt{\pi a(t)}} \frac{x}{2\sqrt{a(t)}} e^{-\left(\frac{x}{\sqrt{4a(t)}}\right)^2} \leq \frac{M_1}{\sqrt{2e\pi a(t)}} \sim \frac{C}{t^{\frac{2}{3}}},$$

using the asymptotic behavior of a shown in Lemma 1. \square

Propositions 5 and 6 show that the L^∞ -norms of the solution u to (1)-(3) and of the candidate self-similar solution decay with the same order. In the final theorem of this article we show that u asymptotically approaches the self-similar solution as soon as the second moment of the initial data is bounded.

Theorem 2. *If the initial data u^{in} has a bounded second moment $M_2(0)$, then there exists $C > 0$ such that*

$$\left| u(t, x) - M_1 \frac{x}{2\sqrt{\pi a^{\frac{3}{2}}(t)}} e^{-\frac{x^2}{4a(t)}} \right| \leq \frac{CM_2(0)}{t}$$

for $t > 1$.

Proof. Since

$$u(t, x) = \frac{1}{2\sqrt{\pi a(t)}} \int_0^{+\infty} \left(e^{-\frac{(x-y)^2}{4a(t)}} - e^{-\frac{(x+y)^2}{4a(t)}} \right) u^{\text{in}}(y) dy,$$

denoting

$$v(x) = \frac{1}{2\sqrt{\pi a}} \int_0^{+\infty} \left(\frac{e^{-\frac{(x-y)^2}{4a}} - e^{-\frac{(x+y)^2}{4a}}}{y} \right) y v^{\text{in}}(y) dy,$$

we have

$$\begin{aligned} & v(x) - M_1 \frac{x}{2\sqrt{\pi a^{\frac{3}{2}}}} e^{-\frac{x^2}{4a}} \\ &= \frac{1}{2\sqrt{\pi a}} \int_0^{+\infty} \left(\frac{e^{-\frac{(x-y)^2}{4a}} - e^{-\frac{(x+y)^2}{4a}}}{y} - \frac{x}{a} e^{-\frac{x^2}{4a}} \right) y v^{\text{in}}(y) dy. \end{aligned}$$

We write now

$$\frac{e^{-\frac{(x-y)^2}{4a}} - e^{-\frac{(x+y)^2}{4a}}}{y} - \frac{x}{a} e^{-\frac{x^2}{4a}} \equiv \frac{1}{a^{\frac{1}{2}}} \Phi \left(\frac{x}{a^{\frac{1}{2}}}, \frac{y}{a^{\frac{1}{2}}} \right)$$

with

$$\Phi(X, Y) = \frac{e^{-\frac{(X-Y)^2}{4}} - e^{-\frac{(X+Y)^2}{4}}}{Y} - X e^{-\frac{X^2}{4}}.$$

It is simple to show that there exists a constant C such that

$$|\Phi(X, Y)| \leq CY$$

so that

$$\left| v(x) - M_1 \frac{x}{2\sqrt{\pi a^{\frac{3}{2}}}} e^{-\frac{x^2}{4a}} \right| \leq \frac{C}{a^{\frac{3}{2}}} \int_0^{+\infty} y^2 u^{\text{in}}(y) dy.$$

□

Remark 2. Note that the previous result can be rewritten as

$$t^{2/3} \left| u(t, x) - M_1 \frac{x}{2\sqrt{\pi a^{\frac{3}{2}}}(t)} e^{-\frac{x^2}{4a(t)}} \right| \leq \frac{CM_2(0)}{t^{1/3}},$$

which means that the convergence of the solution to the self-similar profile takes place at a faster rate than the decay of their L^∞ -norms, which is to be expected.

3. NUMERICAL TESTS

In this section we perform some numerical experiments in order to verify the theoretical results obtained above.

At the numerical level, we worked with the finite space interval $[0, 400]$, which is sufficiently wide to minimize the boundary effects on the numerical solution, especially for initial data having a fast decay when $x \rightarrow +\infty$.

We have introduced a fixed space step $\Delta x > 0$ and a time step $\Delta t > 0$. Then, we have divided the interval $[0, 400]$ in N sub-intervals of measure $\Delta x = 400/N$.

We have then used an explicit finite differences scheme where the diffusion coefficient (i.e. the mass) at each time step is taken as the mass in the previous time step. The method is stable under the standard stability condition $\Delta t \leq M(0)(\Delta x)^2/2$.

We have taken, as initial data,

$$(17) \quad u_0(x) = \chi_{[1,2]} = \begin{cases} 1 & x \in [1, 2] \\ 0 & \text{otherwise.} \end{cases}$$

In Figure 1, we plot the numerical approximation of the solution u for $t = 50, 500, 5000, 50000$ and, in Figure 2, we show the time evolution of the quantity $\log(M_0(t))$ (i.e. the logarithm of the mass of u). As we can see, $\log(M_0(t))$ tends to follow a straight line with slope $-\frac{1}{3}$, which indicates an asymptotics of type $M_0(t) = O(t^{-1/3})$ as $t \rightarrow \infty$. Finally, in Figure 3 we rescale the profiles in Figure 1 by multiplying them by $a(t)$ and representing them as a function of $x/a^{\frac{1}{2}}(t)$. As we can see, they approach the self-similar profile $f(\eta) = \frac{M_1}{\sqrt{4\pi}} \eta e^{-\frac{\eta^2}{4}}$ (dashed line).

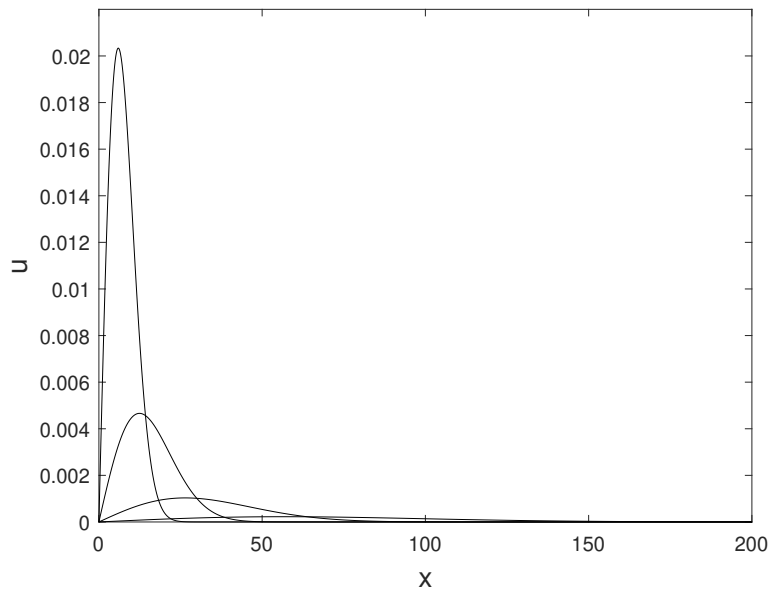


FIGURE 1. Numerical profiles of the solution of (1)-(3) at times $t = 50, 500, 5000, 50000$, with initial condition given in (17).

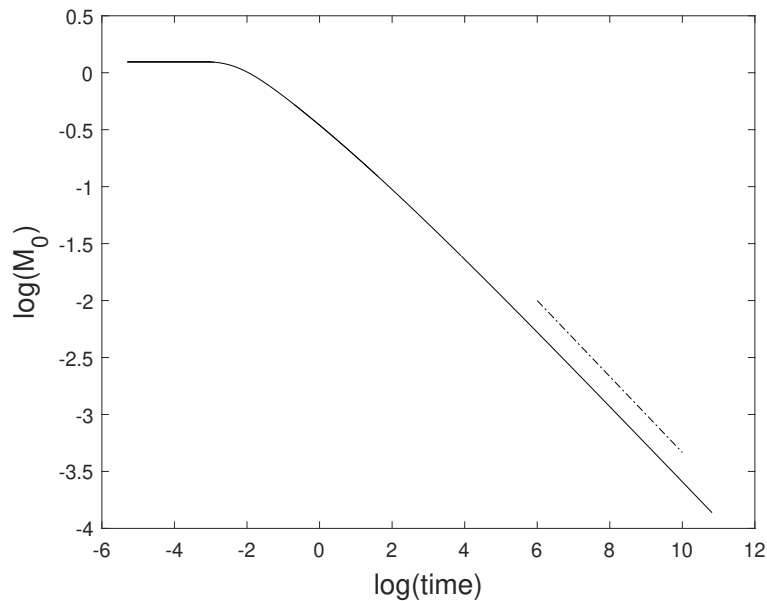


FIGURE 2. Logarithm of the mass vs logarithm of time and comparison with a $-1/3$ slope (dotted-dashed line).

APPENDIX

We consider here the case of the finite domain $\Omega = (0, \pi)$, with homogeneous Dirichlet boundary conditions. This setting describes the diffusive limit of the kinetic rock-paper-scissors game by supposing that only individuals with wealth $x \in \Omega$ play the game, and can be deduced from the kinetic model described in [8] by adapting the same arguments.

The problem studied in this Appendix is hence the following. We consider the equation

$$(18) \quad w_t = \left[\int_0^\pi w(t, \xi) d\xi \right] w_{xx}, \quad (t, x) \in \mathbb{R}_+ \times \Omega$$

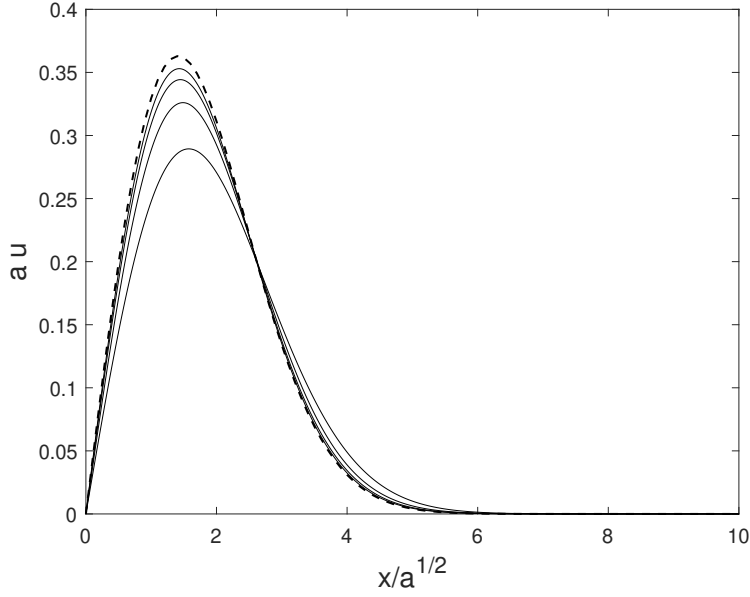


FIGURE 3. Rescaled numerical profiles and comparison with the self-similar profile (dashed line).

with initial data

$$(19) \quad w(0, x) = w^{\text{in}} \in L^2(0, \pi) \quad x \in \Omega$$

and boundary conditions

$$(20) \quad w(t, 0) = w(t, \pi) = 0, \quad t \in \mathbb{R}_+,$$

where $w^{\text{in}} \geq 0$ for a.e. $x \in (0, \pi)$. Note that, by parabolic theory [6], $w(t, x) \geq 0$.

An explicit solution of (18)-(20), when $w^{\text{in}} = M \sin(x)$, is the following:

$$w^*(t, x) = \frac{M \sin(x)}{2(1 + Mt)}, \quad (t, x) \in \mathbb{R}_+ \times \Omega,$$

where $M > 0$ is a given constant. The function w^* also turns out to be a self-similar solution with the similarity exponent $\mu = 0$. Its initial mass is

$$\int_0^\pi w^*(t, \xi) d\xi = M.$$

We will show that, indeed, if the initial data w^{in} is in $L^2(\Omega)$, the solution will tend to the explicit solution, i.e.

$$w(t, x) \sim \frac{M \sin x}{2(1 + Mt)} \text{ as } t \rightarrow +\infty,$$

and the rate of convergence is $\mathcal{O}(t^{-2})$. Clearly, $w \in C(\mathbb{R}_+; L^2(\Omega))$ and we can write $w(t, x)$ in terms of Fourier series which, because of the boundary conditions, takes the form

$$(21) \quad w(t, x) = \sum_{n=1}^{+\infty} w_n(t) \sin(nx).$$

By simple inspection in (18)-(20), we deduce that w solves (18), with initial condition

$$w^{\text{in}} = \sum_{n=1}^{+\infty} w_n(0) \sin(nx),$$

and boundary conditions (20). Let

$$M(t) \equiv \int_0^\pi w(t, x) dx.$$

Then

$$M(t) = \sum_{n \text{ odd}} \frac{2w_n(t)}{n}$$

so that

$$\frac{dw_n}{dt} = -n^2 M(t) w_n.$$

Hence

$$w_n(t) = w_n(0) e^{-n^2 \int_0^t M(t') dt'},$$

and we can compute

$$M(t) = \sum_{n \text{ odd}} \frac{2w_n(0)}{n} e^{-n^2 \int_0^t M(t') dt'}.$$

Denoting, as before,

$$a(t) = \int_0^t M(t') dt',$$

we have then the ordinary differential equation

$$a'(t) = \sum_{n \text{ odd}} \frac{2w_n(0)}{n} e^{-n^2 a(t)}$$

so that, since $w(t, x) \geq 0$ and hence $M(t) > 0$, we can integrate explicitly to obtain

$$G(a) \equiv \int_0^a \frac{e^{a'}}{\sum_{n \text{ odd}} \frac{2w_n(0)}{n} e^{(1-n^2)a'}} da' = t.$$

Note that

$$\begin{aligned} G(a) &= \frac{1}{2w_1(0)} (e^a - 1) - \frac{1}{2w_1(0)} \int_0^a \frac{\sum_{n=3,5,\dots} \frac{w_n(0)}{nw_1(0)} e^{(2-n^2)a'}}{1 + \sum_{n=3,5,\dots} \frac{w_n(0)}{nw_1(0)} e^{(1-n^2)a'}} da' \\ &= \frac{1}{2w_1(0)} e^a - K + O(e^{-7a}), \text{ as } a \rightarrow +\infty \end{aligned}$$

with

$$K = \frac{1}{2w_1(0)} + \frac{1}{2w_1(0)} \int_0^\infty \frac{\sum_{n=3,5,\dots} \frac{w_n(0)}{nw_1(0)} e^{(2-n^2)a'}}{1 + \sum_{n=3,5,\dots} \frac{w_n(0)}{nw_1(0)} e^{(1-n^2)a'}} da'$$

We have then

$$a \sim \log(2w_1(0)(t + K) + O(t^{-7})), \text{ as } t \rightarrow +\infty,$$

and hence

$$M(t) = a' \sim \frac{1}{t + K}, \text{ as } t \rightarrow +\infty.$$

Therefore

$$(22) \quad w_n(t) \sim \frac{w_n(0)}{(2w_1(0)(t + K))^{n^2}} \text{ as } t \rightarrow +\infty.$$

We can prove then the following Lemma:

Lemma 2. *Let w^{in} be the initial condition of the initial-boundary value problem (18)-(20) and suppose that $w^{\text{in}} \in L^1(\Omega) \cap L^\infty(\Omega)$. Then there exists a constant C , depending on w^{in} , and a time $T > 0$ such that, for any $t > T$,*

$$\left| w(t, x) - \frac{M}{2} \frac{\sin(x)}{1 + Mt} \right| \leq \frac{C}{t^2}.$$

Proof. We note

$$w_n(0) = \frac{2}{\pi} \int_0^\pi w^{\text{in}}(x) \sin(nx) dx,$$

so that

$$|w_n(0)| \leq \frac{2}{\pi} \int_0^\pi |w^{\text{in}}(x)| dx = \frac{2}{\pi} \|w^{\text{in}}\|_{L^1(\Omega)}$$

and use (21), (22). □

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