



**HAL**  
open science

# Hegel, Erdős-Anning theorem and the Pythagorean triangle

Daniel Parrochia

► **To cite this version:**

| Daniel Parrochia. Hegel, Erdős-Anning theorem and the Pythagorean triangle. 2024. hal-04648925

**HAL Id: hal-04648925**

**<https://hal.science/hal-04648925v1>**

Preprint submitted on 15 Jul 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# Hegel, Erdős-Anning theorem and the Pythagorean triangle

Daniel Parrochia

University of Lyon (France)

## **Abstract.**

Hegel, as we know, took as an emblem of mathematical thought the Pythagorean theorem, whose (Euclidean) demonstration seemed to him to illustrate what he rejected, namely a thought of understanding, dismembering reality instead of progressing dialectically towards synthesis. In this article, we show that Hegelian triplicity in its circular finitude results from an impossibility. A mathematical theorem, the Erdős-Anning theorem shows that points located at integer distances can only be aligned and in infinite number. If we identify concepts with discrete points, and if we want to remain in the finite, it is clear that they must be non-aligned. Therefore, their circular arrangement on a unit circle is a solution, which can still be generalized on the sphere. But this proves, contrary to Hegel's assertions, that the concepts themselves can be decomposed in a recto-triangular manner.

**Key words.** Hegel, Erdős-Anning theorem, Pythagorean triples, discrete geometry.

## **1 Hegel and Pythagoras**

On various occasions, we have already had the opportunity to discuss the relationship between philosophy and mathematics, and in particular, the Hegelian view of it (see [Parrochia 93<sup>1</sup>]; [Parrochia 18], 131). As it is known, the Hegelian conceptual triplicity, independently of the religious evocation of the Holy Trinity, can be linked to different paradigms, which range from the organic figure of a tree with

polytomal branching to the cosmological theory of epicycles today reinterpreted using the Fourier transform (see [Parrochia 23], 48-49). But, more than anything, what is striking in Hegel's relationship to science, and in particular mathematical science, is that he condemns its methodology, for him dismembering and disorganizing, as he explains in the preface to the *Phenomenology of Spirit*. We have already shown, in a previous article, that this passage – an analysis of the Pythagorean theorem based on the Euclid's proof – was particularly debatable, not only because there are multiple other proofs of this theorem, but above all because we can today reveal behind it a rich mathematical context, unknown to Hegel, and in which it is, in a way, dialectically immersed (see [Parrochia 20]). Today we just want to add a sort of codicil to this analysis.

## 2 A geometry of concepts

We will allow ourselves here a small departure from Hegelian convictions. Suppose that concepts can be represented, geometrically, by discrete points on the plane. A crucial question would be to know under what condition these concepts can form a straight line or, on the contrary, can be grouped by three in a circular figure. The Erdős-Anning theorem answers this question precisely.

### 2.1 The line and infinite sets of points: the Erdős-Anning theorem

Let's first recall some historical points. In 1945, the famous Hungarian mathematician Paul Erdős (1913-1996) and the American mathematician of Canadian origin Norman Herbert Anning (1883-1963) proved the following nice theorem since named, after them, "Erdős-Anning theorem" :

**Theorem 2.1** ([Erdős-Anning 45]). *Whenever an infinite number of points in the plane all have integer distances, the points lie on a straight line.*

We can see, in Fig. 1, an image of the integer number line, a set of infinitely many points with integer distances.

The Erdős-Anning proof is complicated, but shortly after the original publication of the theorem, Erdős provided a simpler proof (see [Erdős 45];[Solymosi 03]) whose we can give a version in 2 dimensions.

*Proof.* Let  $E$  be a set of points on the plane with integer mutual distances, containing

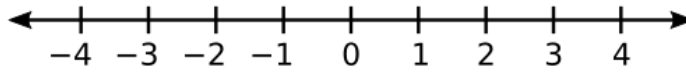


Figure 1: The integer number line

three non-aligned points  $A, B, C$ , and let  $b = d(A, B)$  and  $c = d(A, C)$ , the distances between two of these points. Let us show that  $E$  is necessarily finite, with cardinality less than or equal to  $4(b+1)(c+1)$ . For any point  $X$  of  $E$ , the integer  $|d(A, X) - d(B, X)|$  is between 0 and  $b$  according to the triangular inequality. For each of the  $b+1$  integers  $k$  between 0 and  $b$ , the locus of the points  $X$  such that  $|d(A, X) - d(B, X)| = k$  is a (possibly degenerate) hyperbola of foci  $A$  and  $B$ , and  $X$  belongs to one of these  $b+1$  hyperbolas. Likewise,  $X$  belongs to one of the  $c+1$  analogous hyperbolas of foci  $A$  and  $C$ . Each pair of hyperbolas, one of which is in the first family and the other in the second, having at most 4 points of intersection,  $E$  contains at most  $4(b+1)(c+1)$  points (see Fig. 2).  $\square$

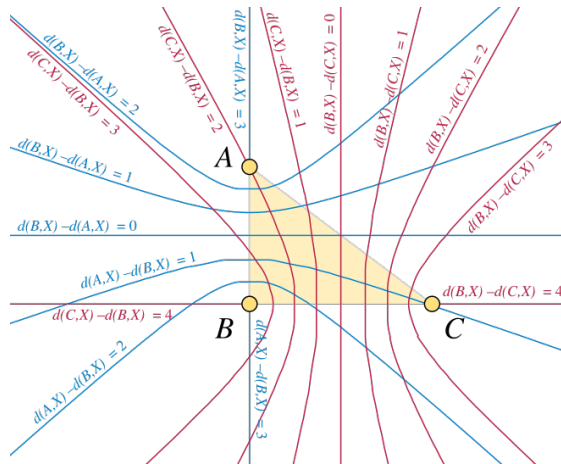


Figure 2: Integer points of a right triangle and hyperbolas

The same result holds in higher dimensions: one cannot have infinitely many points in  $n$ -dimensional space not all on a line, with all the distances being integral.

## 2.2 Finite sets of points and the circle

Another way of stating the theorem 2.1 is that a non-collinear set of points in the plane with integer distances can only be extended by adding a *finite* number of additional points, before it is no longer possible to add only one point. A set of points with both integer coordinates and integer distances, to which nothing can be added while preserving both properties, forms an Erdős-Diophantine graph. The second proof from Erdős can be used to check whether a set of points forms an Erdős-Diophantine graph, i.e. an inextensible system of integer points with integer distances<sup>1</sup>. In other words, we get the following definition:

**Definition 2.1** (Erdős-Diophantine graph). An Erdős-Diophantine graph is a maximal set of points with integer mutual distances. We can reformulate the Erdős-Anning theorem by saying that such a set, if it is unaligned, is necessarily finite.

Fig. 3 shows an example of such a graph (see [Kohnert-Kurz 07]).

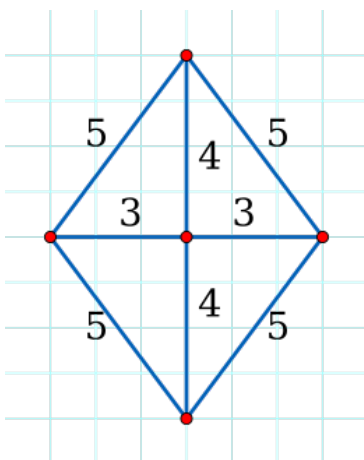


Figure 3: Five-vertex Erdős-Diophantine graph

We can no longer conclude that the points are aligned if we only ask that all their distances be rational. On the contrary, we have known since Euler (see [Euler 62], 229, theorem 5) that this is not the case.

For example, on the unit circle, the points of affix  $e^{i\alpha}$  ( $-\pi < \alpha \leq \pi$ ) with  $\cos(\alpha/2)$

---

<sup>1</sup>This proof of the Erdős-Anning theorem can be used in an algorithm to check whether a given set of integer points with integer distances forms an Erdős-Diophantine graph: this involves simply finding all the crossing points of the hyperbolas used in the proof and to check whether any of the resulting points also have integer coordinates and integer distances from the given set.

and  $\sin(\alpha/2)$  rational, that is to say  $\tan(\alpha/4)$  rational, are at rational distances from each other because  $|e^{i\alpha} - e^{i\beta}| = 2|\sin(\alpha/2 - \beta/2)|$  (see also [Eckert 84]).

More generally, a circle contains a dense set of points at rational distances from each other if and only if the square of its radius is rational.

Any finite set of points at rational distances from each other can be transformed into a set of points at integer mutual distances, by any similarity whose ratio is a common denominator of these rational distances. This allows one to construct arbitrarily large finite sets of unaligned points with integer mutual distances. This process, which does not scale to an infinite set<sup>2</sup>, allows however the formation of dense sets.

For example, the subset of points on a unit circle obtained as the even multiples of one of the acute angles of an integer-sided right triangle (such as the triangle with side lengths 3, 4, and 5) has this property. This construction forms exactly a dense set in the circle.

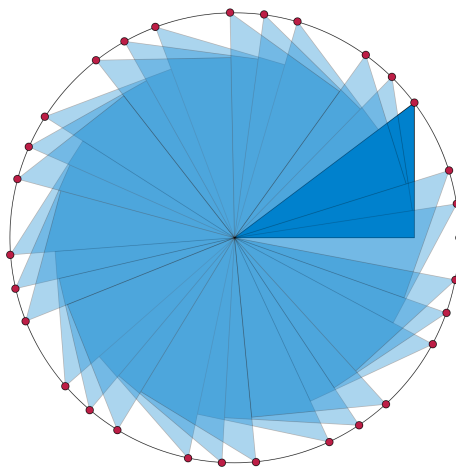


Figure 4: The integer multiples of the angle of a 3-4-5 right triangle.

In Fig. 4, all pairwise distances among the even multiples (every other point from this set) are rational numbers. Scaling any finite subset of these points by the least

---

<sup>2</sup>In fact, it is not known whether there exists a dense set of points on the Euclidean plane with rational mutual distances. It is the famous (and still unsolved) Erdős-Ulam problem.

common denominator of their distances produces an arbitrarily large finite set of points at integer distances from each other.

If we therefore represent Hegelian concepts by discrete points, it is clear that a way to avoid infinity (the Hegelian "bad infinity") is to operate on a circle, not on a line. But there is a paradoxical consequence of this requirement. The points in question, which are each time dialectical syntheses, can be decomposed geometrically into the sides of a right triangle of which the hypotenuse carries out the synthesis. In other words, we can argue with a little provocation that Hegelian philosophy, which accused Euclid's proof of dismembering the Pythagorean triangle in fact implicitly renews this triangle if we identify, as we have done, the concepts at points on the plan. Under this hypothesis, each concept is an implicit right triangle of which it is the highest point, and the concepts are distributed in groups of three on a circle of which they constitute the rational, dense points in the integral whole. Moral: when you throw mathematics out the door, it comes back out the window. But we can say a little more, because mathematical structures become generalized.

### 3 From triangles and circles to sphere and other figures

#### 3.1 Generalization of objects

As Heiko Harborth said, "first results on geometrical objects with integral sides go back to the time of Pythagoreans" (see [Harborth 98]). But regardless of known negative results, there are many still open problems. Some of them come from a generalization of Pythagorean triples.

Consider for example what we call "Heron triangles". A triangle is called a Heron triangle (or Heronian triangle) if each of the lengths of its sides as well as its area are expressed in non-zero natural integers. In summary, it is an entire triangle of entire area. According to Heron's formula, it is a question of determining the solutions  $(a, b, c, s)$  in natural numbers of the Diophantine equation  $s^2 = (a + b + c)(a + b - c)(b + c - a)(c + a - b)$ , taking  $a \leq b \leq c$ . Heron of Alexandria is credited with the solution (13, 14, 15, 84) (see [Sastry 01]).

Here, one question is: do Heron triangles exist with integral medians? But there are many others: for instance, does a perfect box exist, i.e. a cuboid with integral sides face diagonals, and body diagonals? Or again do there exist points in the plane with integral distances to the four vertex points of an integral sided square?

Harborth reports different problems (and results) of integral distances dating back to the last decades, showing that these are not just theoretical questions but very real ones, whose different applications are imaginable : in radio-astronomy (wave length), chemistry (molecules), physics (energy quanta), robotics, architecture and other fields.

He starts from Pythagorean triangles – of the type  $(3, 4, 5)$  – but immediately passes to the case of simple triangles in integers – of the type  $(1, 2, 2)$  – whose circumscribed circle, at cause of Ptolemy’s theorem, allows us to easily determine, for a circular quadrangle where the sum of the products of the opposite pairs of sides is equal to the product of the diagonals, all the additional distances which all turn out to be rational.

We move from there to higher dimensions where, inside a set of circular integer points of radius  $r$ ,  $D$  – 2-dimensional tetrahedra meet, with sides of length  $a$  where  $a$  and  $r$  are the legs of a Pythagorean triangle. It also seeks to define the smallest set of points with integer distances between them.

We go from there to more complicated rational polygons, in the case of integer distances to or on figures, including Fibonacci triangles and the ancient Platonic solids. Each time, non-trivial problems arise, not always resolved today.

But the most interesting generalizations from our point of view appear in a recent article by David Eppstein (see [Eppstein 24]).

### 3.2 Generalization of geometry

The Erdős-Ahning theorem, as we have seen, states that every point set in the Euclidean plane with integer distances must be either collinear or finite. But there are many metrics other than Euclidean distance for which we might ask the same questions. According to [Eppstein 24], two obstacles, however, have long prevented such a generalization:

1. The first is that the usual proof of the Erdős-Ahning theorem is heavily based on algebraic geometry. In the Euclidean plane, the locus of points whose distances to two given points differ by a fixed integer is, as we have seen, a branch of a hyperbola. Therefore, the points with integer distances to three non-collinear points lie at the intersections of two finite families of hyperbolas. Hyperbolas are algebraic curves of degree two and, by Bezout’s theorem, two hyperbolas have at most four crossings, so these two finite families have finitely many intersection points. But it is not obvious how to extend this argument to metrics for which the corresponding curves are more



complicated or non-algebraic. Erdős and Anning originally used a different argument using trigonometric inequalities, but its generalization is also non-obvious.

2. The second obstacle is that analogous statements are untrue for some metrics. For the  $L_1$  or  $L_\infty$  distances in the plane, for instance, the integer lattice provides a familiar example of an infinite set that is not collinear and has only integer distances. Thus, generalizations of the theorem to other families of metrics must avoid families that include  $L_1$  or  $L_\infty$ .

In his paper, Eppstein, however, proves some generalized version of the Erdős-Anning theorem which takes the following form (we quote the passage in full) :

A. "For any strictly convex distance function on  $\mathbb{R}^2$ , every point set with integer distances must be either collinear or finite. For any (non-degenerate) triangle of diameter  $\delta$ , at most  $O(\delta^2)$  points can have integer distances from all three triangle vertices. If a set with integer distances has diameter  $D$ , it has at most  $O(D)$  points".

B. "For any two-dimensional complete Riemannian manifold of bounded genus  $g$ , every point set with integer distances must either be finite or have shortest curves through each triple of points that all lie on a single geodesic. If some three of the points do not lie on a single shortest curve, and have diameter  $\delta$ , at most  $O((g+1)\delta^2)$  points can have integer distances from these three points."

C. "For geodesic distance on the boundary of a convex set in  $\mathbb{R}^3$ , every point set with integer distances must either be finite or have shortest curves through each triple of points that all lie on a single geodesic. If some three of the points do not lie on a single shortest curve, and have diameter  $\delta$ , at most  $O(\delta^2)$  points can have integer distances from these three points. If a set with integer distances has diameter  $D$ , it has at most  $O(D^{4/3})$  points."

(A) is the most immediate reinforcement that can be found of the Erdős-Anning theorem. (B) opens the doors to Riemannian geometry and its non-Euclidean metric. (C) moves from plane to space.

The proofs from Eppstein generally replace the intersection properties of low-degree algebraic curves with the geometric and topological properties of additively weighted Voronoi diagrams. For Riemannian manifolds of equilateral dimension, there is a maximum number of points at unit distance of these manifolds. For complete Riemannian 2-manifolds of bounded genus, the equilateral dimension is bounded by a function of the genus, but for locally Euclidean incomplete Riemannian 2-manifolds,

for complete Riemannian 2-manifolds of unbounded genus, and for complete Riemannian metrics on  $\mathbb{R}^3$ , the equilateral dimension is unbounded.

Among many other results, the question of dense sets posed by Harborth is reviewed and modified within the framework of this generalization. Euclidean distance is a convex distance function, Eppstein observes, and, as Harborth has shown after Euler, there exist infinite dense sets of points on the unit circle with rational Euclidean distances. The construction of Fig. 5 for such a system of points using complex number coordinates for the points of the plane is renewed. It consists, as we have seen, of the complex numbers  $\{q^{2j} \in \mathbb{Z}\}$  where  $q = \frac{a}{c} + i\frac{c}{b}$  is a rational point on the unit circle derived from an integer right triangle with side lengths  $a, b$ , and  $c$ . Geometrically, this point set is obtained from a single point by repeatedly rotating by twice the angle of the triangle. But this construction, essentially due to Harborth, used all integer multiples of the angle rather than only the even multiples. Eppstein corrects Harborth and shows that the points are distributed on a unit sphere whose angles are even multiples of the angles of a 3-4-5 right triangle having pairwise rational distances (see Fig. 5).

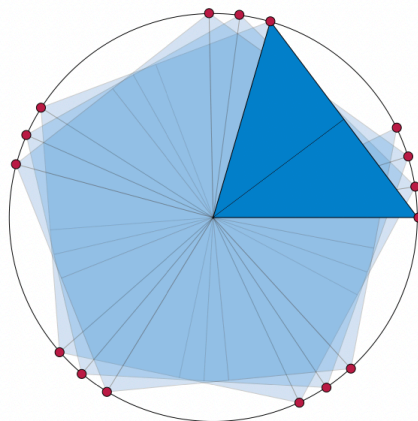


Figure 5: The sphere of even multiples of the angle of a 3-4-5 right triangle (after [Eppstein 24], 5).

This passage from the circle to the sphere thus brings a new generalization of the famous Hegelian circle with its conceptual triplicity, now possibly projected into three-dimensional space and onto a Riemannian manifold. But there are, of course, many other solutions including the techniques of Voronoi diagrams to be performed.

## 4 Some final philosophical views

If we were to look for general philosophical applications of the preceding considerations, we would have to show that a philosophical system is precisely the equivalent of an Euler-Diophantine graph. By assimilating its fundamental concepts to discrete points placed at integer or rational distances, we could say that it is made up of an inextensible set of categories, any new possible category falling back on one of the already existing categories. Philosophical systems assumed as such (for example, those of Hegel, Whitehead or Eric Weil) are of this nature. But other models are obviously possible, including if we take into account non-Euclidean metrics.

The meaning of the metric should obviously be explained. But it is clear that philosophical systems, in attempting to bring order to reality (or thought), often present themselves in the form of hierarchical classifications (see [Parrochia 93<sup>2</sup>]), which allows the existence of ultrametric distances between pairs of concepts. There is therefore a certain relevance to mobilizing, in order to understand their structure, the Erdős-Anning theorem and the Erdős-Diophantine graphs.

## References

- [Anning 16] Anning, N. H., "Note On Triangles Whose Sides Are Whole Numbers", *School Science and Mathematics*, vol. 16, no 1,,Äé 82-83, 1916.
- [Eckert 84] Eckert, E. J., "The Group of Primitive Pythagorean Triangles", *Mathematics Magazine*, Vol. 57, 22-27, No. 1, Jan., 1984.
- [Eppstein 24] Eppstein, D., "Non-Euclidean Erdős-Anning theorems", *arXiv:2401.06328v2 [math.MG]*, 22 Jan 2024.
- [Erdős-Anning 45] Anning, N.H. and Erdős, P., "Integral distances", *Bull. Amer. Math. Soc.*, vol. 51, (8),Äé 598-600, 1945.
- [Erdős 45] Erdős, P., "Integral distances", *Bull. Amer. Math. Soc.*, 51 (12): 996, 1945.
- [Euler 62] Euler, L., "Fragmenta arithmetica ex Adversariis mathematicis deprompta, C: Analysis Diophantea", *Opera postuma* (in Latin), vol. I, Petropolis: Eggers, 204-263, 1862

- [Harborth 98] Harborth, H., "Integral distances in Point Sets", in *Karl der Grosse und sein Nachwirken, 1200 Jahre Kultur und Wissenschaft in Europa: Band II, Mathematisches Wissen*, Brepols Publishers, 213-224, Turnhout, Belgium, 1998
- [Kohnert-Kurz 07] Kohnert, A., Kurz, S., "A note on Erdős-Diophantine graphs and Diophantine carpets", *Mathematica Balkanica*, New Series, 21 (1-2): 1-5, 2007. See also, arXiv:math/0511705, MR 2350714.
- [Parrochia 93<sup>1</sup>] Parrochia, D., "Hegel: logique spéculative et mathématiques pseudo-synthétiques", *Kairos*, No 4, 113-150, 1993.
- [Parrochia 93<sup>2</sup>] Parrochia, D., *La Raison Systématique*, Vrin, Paris, 1993.
- [Parrochia 18] Parrochia, D., *Mathematics and Philosophy*, Iste-Wiley, London, 2018.
- [Parrochia 20] Parrochia, D., "Mathematical physics vs Philosophy: Hegel, Pythagorean triples, Spinors and Clifford Algebras", hal-02613438, 2020.
- [Parrochia 23] Parrochia, D., *Graphs, Orders, Infinites and Philosophy*, Iste-Wiley, London, 2023.
- [Sastry 01] Sastry, K. R.S., "Heron triangles: A Gergonne-Cevian-and-median perspective", *Forum Geometricorum*, vol. 1, 17-24, 2001.
- [Solymosi 03] Solymosi, J., "Note on integral distances", *Discrete & Computational Geometry*, 30 (2), 337-342, 2003.