On the small-time bilinear control of a nonlinear heat equation: global approximate controllability and exact controllability to trajectories

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July 15, 2024

Abstract

In this work we analyse the small-time reachability properties of a nonlinear parabolic equation, by means of a bilinear control, posed on a torus of arbitrary dimension d. Under a saturation hypothesis on the control operators, we show the small-time approximate controllability between states sharing the same sign. Moreover, in the one-dimensional case $d = 1$, we combine this property with a local exact controllability result, and prove the small-time exact controllability of any positive states towards the ground state of the evolution operator.

Keywords: bilinear control, heat equation, approximate controllability, exact controllability, moment problem, biorthogonal family

2010 MSC: 93B05, 35Q93, 35K05.

1 Introduction

In this paper we study controllability properties of the following Nonlinear Heat Equation on the d-dimensional torus $\mathbb{T}^d$

$$\begin{cases} \partial_t \psi(t,x) = \Big(\Delta - \kappa\psi(t,x)^p + \langle u(t), Q(x)\rangle\Big)\psi(t,x), & x \in \mathbb{T}^d,\ t > 0, \\ \psi(t=0,\cdot) = \psi_0(\cdot), \end{cases} \tag{NHE}$$

where $d, p \in \mathbb{N}^*$, $\kappa \in \mathbb{R}$, the operator $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the Laplacian and $Q = (Q_1, ..., Q_q, \mu_1, \mu_2) : \mathbb{T} \to \mathbb{R}^{q+2}$ is a smooth fixed function. The $\mathbb{R}^{q+2}$-valued function $u \in L^2_{loc}(\mathbb{R}^+, \mathbb{R}^{q+2})$ plays the role of a control. This means that u is the function that can be chosen to steer the solution of the problem towards a desired state. Observe that our control depends only on time.

In numerous practical problems from chemistry, neurobiology and life science, the evolution of a specific system is subjected to a control which is not external but rather a modification of the principal parameter of the evolution. In these cases, it can be appropriate to consider evolution equations in the presence of multiplicative controls. Such controls are called bilinear and take the form of $\langle u(t), Q(x)\rangle$ (as in the equation (NHE)) when only the time-depending intensities can be adjusted, while the spatial parts are fixed.

An example of a parabolic model with a dynamics governed by a multiplicative control is the distributed parameter control model studied by Lenhart and Bhat in [13]. In this work, the authors consider the wildlife damage management for controlling population of diffusive small mammal species such as beavers, raccoons and muskrats. These small mammal species often damage human interests and it is important to study the possibility to control their dispersal behaviour. The migratory habit of such animals presents obviously an additional complication. For instance, their removal from any habitat may cause the attraction of other individuals from nearby lands and a consequent increase of the trapping cost. In [13], the authors study the dynamics of the population density of one of

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this species given by a control model incorporating dispersive dynamics and a multiplicative control which represents the trapping. In such a work, the evolution is modelled by an equation of the form

$$\partial_t \varphi(t,x) = (\alpha\Delta + a - b\varphi(t,x) + p(t,x))\varphi(t,x), \qquad t>0, \tag{1.1}$$

where φ is the population density, α is constant, a and b are growth parameters and p is the rate of trapping which is used as a control. When one can control the time-dependent intensity of the trapping but not its spatial distribution, it is possible to separate the variables of function p and write

$$p(t,x) = u(t)\mu(x).$$

This choice leads to a new formulation for the evolution model where the actual control is the function u and the equation, for $a=0$, takes the form of (NHE).

Global approximate controllability on the d-dimensional torus

Let us consider the following vector space:

$$\mathcal{H}_0 := \mathrm{span}_{\mathbb{R}}\{Q \in \widehat{\mathcal{H}}\}, \quad \text{with} \quad \widehat{\mathcal{H}} := \{Q_1, \dots, Q_q\}.$$

We also introduced the vectors

$$\mathcal{K} = \Big\{(1,0,\dots,0), (0,1,\dots,0), \dots, (0,\dots,1,0), (1,\dots,1)\Big\}.$$

Assumptions I. The potential $Q = (Q_1, ..., Q_q, \mu_1, \mu_2)$ is such that $\widehat{\mathcal{H}} \subset C^\infty(\mathbb{T}^d, \mathbb{R})$ and

$$\big\{1,\ \cos\langle k,x\rangle,\ \sin\langle k,x\rangle\big\}_{k\in\mathcal{K}} \subset \mathcal{H}_0. \tag{1.2}$$

We now present the first main result of the paper, that is, a small-time global approximate controllability property of (NHE) between states sharing the same sign.

In what follows, we denote by

$$\psi(t;\psi_0,u)$$

the solution of (NHE) at time t, associated with initial condition ψ_0 and control u. Of course, it is intended that we will look at such solutions only when existence is ensured (see Proposition (2.1).

Main Theorem A. *Let $s \in \mathbb{N}^*$ be such that $s > d/2$. Assume Assumptions I be verified. Then, (NHE) verifies the following small-time approximate controllability properties.*

- *Let $\psi_0, \psi_1 \in H^s(\mathbb{T}^d, \mathbb{R})$ be such that $\mathrm{sign}(\psi_0) = \mathrm{sign}(\psi_1)$. For any $\epsilon > 0$ and $T > 0$, there exist $\tau \in [0,T]$ and $(u_1, ...u_q) \in L^2((0,\tau), \mathbb{R}^q)$ such that*
$$\big\|\psi(\tau;\psi_0,u) - \psi_1\big\|_{L^2} < \epsilon,$$
with $u = (u_1, ..., u_q, 0, 0)$.
- *Let $\psi_0, \psi_1 \in H^s(\mathbb{T}^d, \mathbb{R})$ be such that $\psi_0, \psi_1 > 0$ (or $\psi_0, \psi_1 < 0$). For any $\epsilon > 0$ and $T > 0$, there exists $(u_1, ...u_q) \in L^2((0,T), \mathbb{R}^q)$ such that*
$$\big\|\psi(T;\psi_0,u) - \psi_1\big\|_{H^s} < \epsilon,$$
with $u = (u_1, ..., u_q, 0, 0)$.

Main Theorem A is a specific case of Theorem 3.3 where approximate controllability is ensured with general control potentials Q_j verifying specific saturating assumptions.

To the best of our knowledge, the most recent works on approximate controllability for nonlinear parabolic equations via multiplicative controls are [18, 28]. In such works, the authors studied 1-dimensional problems with globally Lipschitz continuous nonlinearities. They proved approximate controllability between states with the same number of sign changes when time is sufficiently large. They considered space-time dependent controls $\alpha(t,x)$ built according to the initial and the final state. The main novelties of Main Theorem A are the following.

- The approximate controllability is achieved in arbitrarily small times.
- The approximate controllability holds on $\mathbb{T}^d$ for any $d \in \mathbb{N}$, that is, in arbitrary spatial dimensions.
- The spatial profile $Q(x)$ of the term $\langle u(t), Q(x)\rangle$ is fixed and only u plays the role of control.
- Equation (NHE) exhibits a polynomial nonlinearity.

Global exact controllability to the ground state solution for the 1-dimensional problem

The second main result of our work is the exact controllability of (NHE) to the ground state solution in the 1-dimensional flat torus $\mathbb{T}$. Let us consider the ordered eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ of the Laplacian $-\Delta$ (not counted with their multiplicity)

$$\lambda_k = k^2, \qquad \forall\, k \in \mathbb{N} \tag{1.3}$$

Note that, except for the first one $\lambda_0 = 0$, all the eigenvalues are double. We denote by $\{c_0, c_k, s_k\}_{k\in\mathbb{N}}$ the corresponding orthonormal eigenfunctions of $-\Delta$

$$c_0 = \frac{1}{\sqrt{2\pi}}, \qquad c_k(x) = \frac{1}{\sqrt{\pi}}\cos(kx), \qquad s_k(x) = \frac{1}{\sqrt{\pi}}\sin(kx), \qquad \forall\, k \in \mathbb{N}^*, \tag{1.4}$$

which form a Hilbert basis of $L^2(\mathbb{T})$. Notice that c_0 represents the free evolution ($u = 0$) of the linear ($\kappa = 0$) heat equation (NHE) with initial condition $\psi_0 = c_0$. Such a solution is usually called the *ground state solution*. Henceforth, we will denote it by Φ. To study the exact controllability of (NHE), we introduce the following additional assumption.

Assumptions II. The potential $Q = (Q_1, ..., Q_q, \mu_1, \mu_2)$ is such that $Q_1 = 1$ and $\mu_1, \mu_2 \in H^3(\mathbb{T}, \mathbb{R})$ verify

$$\begin{aligned} &\langle \mu_1, c_0\rangle_{L^2} \neq 0, \qquad \langle \mu_2, c_0\rangle_{L^2} = 0, \\ &\exists\, b_1, q_1 > 0 \,:\, \lambda_k^{q_1}\,|\langle \mu_1, c_k\rangle_{L^2}| \geq b_1, \ \text{ and } \ \langle \mu_1, s_k\rangle_{L^2} = 0, \quad \forall\, k \in \mathbb{N}^*, \\ &\exists\, b_2, q_2 > 0 \,:\, \lambda_k^{q_2}\,|\langle \mu_2, s_k\rangle_{L^2}| \geq b_2, \ \text{ and } \ \langle \mu_2, c_k\rangle_{L^2} = 0, \quad \forall\, k \in \mathbb{N}^*. \end{aligned} \tag{1.5}$$

Observe that, for any admissible potential Q in the sense of Assumption II, Φ is also a trajectory of (NHE). Indeed, denoting by $u_\kappa = \frac{\kappa}{(2\pi)^{p/2}}$ and setting $\hat{u} = (u_\kappa, 0, \dots, 0)$, one has that $\Phi \equiv \psi(t; c_0, \hat{u})$.

We now state our second main result which ensures global small-time exact controllability of the 1-dimensional (NHE) to the ground state solution, starting from a positive state.

Main Theorem B. *Let $d = 1$, $\kappa > 0$ and $p \in 2\mathbb{N}^*$. Assume Assumptions I and Assumptions II be verified. Then, (NHE) is exactly controllable in $H^3(\mathbb{T})$ to the ground state solution Φ in any positive time from any positive state. In details, for any $T > 0$ and $\psi_0 \in \{\psi \in H^3(\mathbb{T}, \mathbb{R}) :\ sgn(\psi) > 0\}$, there exists $u \in L^2((0, T), \mathbb{R}^{q+2})$ such that*

$$\psi(T; \psi_0, u) = \Phi.$$

Analogously, for any $T > 0$ and $\psi_0 \in \{\psi \in H^3(\mathbb{T}, \mathbb{R}) :\ sgn(\psi) < 0\}$, there exists $u \in L^2((0, T), \mathbb{R}^{q+2})$ such that

$$\psi(T; \psi_0, u) = -\Phi.$$

Main Theorem B yields small-time exact controllability to the ground state solution in H^3 when $\kappa > 0$ and $p \in 2\mathbb{N}^*$. Such a property is obtained by using the global approximate controllability result between positive states (Main Theorem A) together with a local exact controllability result to the ground state solution Φ in any positive time (see Theorem 4.3). The specific choice of the parameters κ and p ensures that equation (NHE) is globally well-posed in H^3 which is crucial for the methodology of the proof of the local exact controllability.

Note that the result of Main Theorem B can be achieved by means of 5 controls. Indeed, we consider a potential of the form

$$Q = (1, \cos(x), \sin(x), \mu_1(x), \mu_2(x))$$

satisfying Assumptions I and II. An example of suitable functions μ_1 and μ_2 are

$$\mu_1 = x^3(2\pi - x)^3, \qquad\qquad \mu_2 = x^3(x - \pi)^3(x - 2\pi)^3$$

(see Example 4.2 for further details). Hence, it is sufficient to consider a potential of the form

$$Q = (1,\ \cos(x),\ \sin(x),\ x^3(2\pi - x)^3,\ x^3(x - \pi)^3(x - 2\pi)^3),$$

to apply Main Theorem B and deduce the controllability of the nonlinear equation (NHE) to the ground state solution Φ in any positive time $T > 0$.

An interesting aspect of Main Theorem B is the validity of an exact controllability result on the torus where the Laplacian exhibits double eigenvalues. Indeed, the first step for proving local exact controllability of (NHE) is based on the solvability of a suitable moment problem. The method for showing local controllability for bilinear control problem of parabolic type, introduced in [5, 6], is not directly applicable in our framework. Indeed, we face two additional difficulties

- The presence of doubles eigenvalues is, a priori, an obstacle to the solvability of the moment problem at hand. We address the problem by "filtering" the spectrum of the Laplacian via the two potentials μ_1 and μ_2. We exploit hypotheses (1.5) so that μ_1 acts only on the frequencies associated to the eigenfunctions $\{c_k\}_{k\in\mathbb{N}}$ and μ_2 only on those associated to $\{s_k\}_{k\in\mathbb{N}^*}$. Hence, we decompose the moment problem into two subproblems, each characterized by simple eigenvalues, making both solvable.
- In order to adapt the technique proposed in [5, 6] the solutions of equation (NHE) need to be globally defined and unique. These properties may be trivial in the linear case, whereas a more careful analysis should be developed for a nonlinear equation like (NHE). We propose a method which require more regularity on the initial condition and on the control.

Some references

The classical controllability problem of parabolic type equations as (NHE), with bilinear control, is a delicate matter even in the linear case ($\kappa = 0$). The main reason is a structural obstacle, described in [9], which forbid from proving controllability results in $L^2(\mathbb{T},\mathbb{R})$. In details, the authors of [9] showed that the reachable set of the (NHE), with $\kappa = 0$, starting from any $\psi_0 \in L^2(\mathbb{T}^d,\mathbb{R})$, is contained in a countable union of compact subsets of $L^2(\mathbb{T}^d,\mathbb{R})$ and then it has dense complement. Therefore, this property prevents obtaining any classical exact controllability result in L^2. Hence, we shall explore different notions of controllability such as the approximate controllability or the exact controllability to trajectories.

Approximate controllability results via multiplicative controls have been obtained in [30] for 1-D linear parabolic problems in sufficiently large time $T > 0$ and between non-negative states. In [30] the control term is not scalar input type, but depends on space and time (see also [31]). Similar results were proved in [16, 17] for linear degenerate parabolic problems. Finally, approximate controllability for nonlinear problem was established in [18, 28].

The techniques leading to approximate controllability of Main Theorem A are inspired by a saturating geometric control approach introduced by Agrachev and Sarychev in [1, 2] in the case of additive controls. In such works the authors proved global approximate controllability of the 2D Navier–Stokes and Euler systems. Their approach has been extended to study different equations with additive controls (see, for instance, [3, 33, 36, 37]). The saturating control methodology has been implemented for bilinear control problems in the recent work [25]. In this paper the authors studied the non-linear Schrödinger equation, proving small-time approximate controllability between eigenmodes. Main Theorem A is inspired by the approach developed in [25]. Nevertheless, the saturating techniques for parabolic equation as (NHE) leads to different controllability properties like small-time approximate controllability between states with the same sign. For additional results on small-time controllability of PDEs through bilinear control, achieved using similar methods, see [22, 24, 34, 23].

Another type of controllability not ruled out by the negative findings of [9] is the exact controllability to trajectories. This property was firstly studied by Alabau-Boussouira, Cannarsa and Urbani in [5, 6] in an abstract setting for parabolic PDEs by means of scalar input bilinear controls. An example of application of such results is the local and semi-global controllability of a heat equation as (NHE) (for $\kappa = 0$) on the interval $(0, 1)$. The methodology has been later extended in [15] to network-type domains to cope with condensation phenomena of the eigenvalues of the diffusion operator. The approach of [5, 6] used for studying local exact controllability to eigensolutions relies on the solvability of a suitable moment problem related to the null controllability of a linearized version of the problem at hand. The resolution of moment problems has been intensively studied in literature over a long period, starting from the classical works [26, 27], to the more recent ones [7, 8, 11, 12, 15, 19, 20, 29, 10].

Scheme of the work

The paper is organized as follows. In Section 2 we present some preliminary results, mamely, local and global well-posedness of the non-linear heat equation (NHE) and a crucial limit adopted in the proof of Main Theorem A. In Section 3 we prove the small-time global approximate controllability and Main Theorem A. Section 4 is devoted to the exact controllability to the ground state solution and contains the proof of Main Theorem B. In the appendices A and B, we present the proofs of the local and global well-posedness results for equation (NHE), respectively. Appendix C contains a technical estimate that we use in the proof of Main Theorem B.

Acknowledgments

The authors would like to thank the colleagues Vahagn Nersesyan and Takéo Takahashi for the fruitful discussions on the saturation control method adopted in this work. The third author acknowledges support from INdAM

National Group for Mathematical Analysis, Probability and their Applications (GNAMPA) and from the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006 (postdoc position 2020-2022). This project has received financial support from the CNRS through the MITI interdisciplinary programs.

2 Preliminaries

The aim of this section is to present some preliminary results. We start by ensuring existence and uniqueness of solutions of equation (NHE). In the second part of the section we prove a limit of conjugated dynamics, which is a key point in the proof of Main Theorem A.

2.1 Local and global well-posedness

We start by stating the following local well-posedness result for the Cauchy problem (NHE).

Proposition 2.1. *Let $s > d/2$ and $Q \in H^s(\mathbb{T}^d, \mathbb{R}^{q+2})$. For any $\psi_0 \in H^s(\mathbb{T}^d, \mathbb{R})$ and $u \in L^2_{loc}(\mathbb{R}^+, \mathbb{R}^{q+2})$, there exists a maximal time $\mathcal{T} = \mathcal{T}(\psi_0, u) > 0$ and a unique mild solution ψ of (NHE). Namely, for any $T < \mathcal{T}$, $\psi \in C^0([0,T], H^s(\mathbb{T}^d, \mathbb{R}))$ and is represented by the formula*

$$\psi(t; \psi_0, u) = e^{t\Delta}\psi_0 + \int_0^t e^{(t-s)\Delta}\left(\langle u(s), Q(x)\rangle \psi(s,x) - \kappa\psi(s,x)^{p+1}\right) ds.$$

Moreover, if $\mathcal{T} < +\infty$, then $\|\psi(t)\|_{H^s} \to +\infty$ as $t \to \mathcal{T}^-$. In addition, we have the following properties.

i. Assume that $\psi_0, \phi_0 \in B_{H^s(\mathbb{T}^d,\mathbb{R})}(0,R)$ for some $R > 0$ and $u, v \in L^2_{loc}(\mathbb{R}^+, \mathbb{R}^{q+2})$. Then, for any $0 \le T \le \min\{\mathcal{T}(\psi_0, u), \mathcal{T}(\phi_0, v)\}$, there exists $C = C(u,v)$ such that

$$\sup_{0\le t\le T} \|\psi(t; \psi_0, u) - \psi(t; \phi_0, v)\|_{H^s} \le C\left(\|\psi_0 - \phi_0\|_{H^s} + \|u - v\|_{L^2}\right). \tag{2.1}$$

ii. Set $K = \|\psi\|_{C([0,T],H^s)} + \|\psi_0\|_{H^s} + \|u\|_{L^2}$. There exists $\delta = \delta(\mathcal{T}(\psi_0, u), K) > 0$ such that, for any $\hat{\psi}_0 \in H^s(\mathbb{T}^d, \mathbb{R})$ and $\hat{u} \in L^2((0,T), \mathbb{R}^{q+2})$ satisfying

$$\|\hat{\psi}_0 - \psi_0\|_{H^s} + \|\hat{u} - u\|_{L^2} < \delta, \tag{2.2}$$

problem (NHE) admits a unique mild solution $\hat{\psi} \in C\big([0,T], H^s(\mathbb{T}^d, \mathbb{R})\big)$ with initial condition $\hat{\psi}_0$ and control $\hat{u}$.

The proof of Proposition 2.1 follows from a fix point argument and from Sobolev embeddings for $s > d/2$. It can be found in Appendix A.

We further present a global well-posedness result for equation (NHE) in the case $d = 1$, $\kappa \ge 0$ and $p \in 2\mathbb{N}^*$.

Proposition 2.2. *Let $d = 1$, $p \in 2\mathbb{N}^*$, $\psi_0 \in H^3(\mathbb{T}, \mathbb{R})$, $Q \in H^3(\mathbb{T}, \mathbb{R}^{q+2})$, $u \in H^1_{loc}((0,+\infty), \mathbb{R}^{q+2})$ and $\kappa \ge 0$. Then, for any $T > 0$ there exists a unique mild solution $\psi \in C^0([0,T], H^3(\mathbb{T}, \mathbb{R}))$ of (NHE).*

The proof of Proposition 2.2 follows from some energy estimates and can be found in Appendix B.

2.2 Small-time limit of conjugated dynamics

Let us introduce the nonlinear operator

$$\mathbb{B}(\varphi)(x) = \sum_{j=1}^{d} \left(\partial_{x_j}\varphi(x)\right)^2, \qquad \forall\, \varphi \in C^1(\mathbb{T}^d, \mathbb{R}). \tag{2.3}$$

Then, the following result holds true.

Proposition 2.3. *Let $s \in \mathbb{N}^*$, $s > d/2$ and $(Q_1, ..., Q_q) \in H^{2s+1}(\mathbb{T}^d, \mathbb{R}^q)$. Assume $\psi_0 \in H^s(\mathbb{T}^d, \mathbb{R})$, $(u_1, ..., u_q) \in \mathbb{R}^q$, and $\varphi \in H^{2s+1}(\mathbb{T}^d, \mathbb{R})$ be non-negative. Then, there exists a constant $\delta_0 > 0$ such that, for any $\delta \in (0, \delta_0)$, the solution $\psi(t; e^{-\delta^{-1/2}\varphi}\psi_0, \delta^{-1}u)$ of (NHE) with $u = (u_1, ..., u_q, 0, 0)$ is well-defined in $[0, \delta]$. Furthermore, the following limit holds*

$$e^{\delta^{-1/2}\varphi}\psi(\delta; e^{-\delta^{-1/2}\varphi}\psi_0, \delta^{-1}u) \to e^{\mathbb{B}(\varphi) + \langle u, Q\rangle}\psi_0 \quad \text{in } H^s, \text{ as } \delta \to 0^+.$$

Proof. **Preliminaries.** For any $\delta>0$ we set $\phi(t):=e^{\delta^{-1/2}\varphi}\psi(t;e^{-\delta^{-1/2}\varphi}\psi_0,\delta^{-1}u)$. According to Proposition 2.1, $\phi(t)$ exists up to some maximal time $\mathcal{T}^\delta=\mathcal{T}(e^{-\delta^{-1/2}\varphi}\psi_0,\delta^{-1}u)$ and it has the same regularity of ψ. Furthermore, if $\mathcal{T}^\delta<\infty$

$$\|e^{-\delta^{-1/2}\varphi}\phi(t)\|_{H^s}\to+\infty\quad\text{as } t\to\mathcal{T}^{\delta-}.$$

We introduce the following functions

$$w(t)=e^{(\mathbb{B}(\varphi)+\langle u,Q\rangle)t}\psi_0^\delta,\qquad v(t)=\phi(\delta t)-w(t),\tag{2.4}$$

where $\psi_0^\delta=e^{\delta^{1/4}\Delta}\psi_0\in H^r(\mathbb{T}^d,\mathbb{R})$, with $r=s+2$, is such that

$$\|\psi_0-\psi_0^\delta\|_{H^s}\to0,\qquad\text{as } \delta\to0^+,\tag{2.5}$$

and there exists $C>0$ independent of $\delta>0$ such that

$$\|\psi_0^\delta\|_{H^s}\le C,\qquad\|\psi_0^\delta\|_{H^r}\le C\delta^{-1/4},\qquad\text{for } \delta\le1.\tag{2.6}$$

Our aim is to show that $\phi(\delta)\xrightarrow{\delta\to0^+}e^{(\mathbb{B}(\varphi)+\langle u,Q\rangle)}\psi_0$ in H^s. Thanks to (2.5) it is sufficient to prove that

$$\|v(t=1)\|_{H^s}\xrightarrow{\delta\to0^+}0.\tag{2.7}$$

However, before proving (2.7), we need to ensure the existence of $\delta_0>0$ small enough such that, for every $0<\delta<\delta_0$, $v(t)$ is well-defined in $[0,1]$ and moreover that

$$\delta^{-1}\mathcal{T}^\delta\ge1.\tag{2.8}$$

An intermediate inequality. In view of (2.6), there exists $C>0$ such that, for every $t\in[0,2]$,

$$\|w(t)\|_{H^s}\le C,\qquad\|w(t)\|_{H^r}\le C\delta^{-1/4}.\tag{2.9}$$

We observe that v is a solution of the following equation

$$\begin{aligned}\partial_t v=&\,\delta\Delta(v+w)-\delta\kappa(e^{-\delta^{-\frac12}\varphi}(v+w))^p(v+w)-\delta^{1/2}(v+w)\Delta\varphi\\&-2\delta^{\frac12}\nabla(v+w)\cdot\nabla\varphi+\mathbb{B}(\varphi)v+\langle u,Q\rangle v,\end{aligned}\tag{2.10}$$

with initial condition

$$v(0)=\psi_0-\psi_0^\delta.\tag{2.11}$$

Let us start by assuming that $\psi_0\in H^{2s+2}(\mathbb{T}^d,\mathbb{R})$ which implies $\psi(t)\in H^{2s+2}(\mathbb{T}^d,\mathbb{R})$ and then $v(t)\in H^{2s+2}(\mathbb{T}^d,\mathbb{R})$ for every $t\in(0,\mathcal{T}^\delta)$. Let $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{N}^d$ be such that $|\alpha|=|\alpha_1|+\ldots+|\alpha_d|\le s$. Thanks to the accreativity of the operator $-\Delta$, we get

$$\begin{aligned}\partial_t\|\partial^\alpha v\|_{L^2}^2=&-2\delta\langle(-\Delta\partial^\alpha v),\partial^\alpha v\rangle_{L^2}+2\langle\delta\Delta w+\delta\kappa(e^{-\delta^{-\frac12}\varphi}(v+w))^p(v+w)\\&-\delta^{\frac12}(v+w)\Delta\varphi+2\delta^{\frac12}\nabla(v+w)\cdot\nabla\varphi+\mathbb{B}(\varphi)v+\langle u,Q\rangle v,\partial^{2\alpha}v\rangle_{L^2}\\\le&\,2\langle\delta\Delta w+\delta\kappa e^{-p\delta^{-\frac12}\varphi}(v+w)^{p+1}-\delta^{\frac12}(v+w)\Delta\varphi\\&+2\delta^{\frac12}\nabla(v+w)\cdot\nabla\varphi+\mathbb{B}(\varphi)v+\langle u,Q\rangle v,\partial^{2\alpha}v\rangle_{L^2}\\\le&\,2\Big(\delta|\langle\partial^\alpha\Delta w,\partial^\alpha v\rangle_{L^2}|+\delta|\kappa|\big|\big\langle\partial^\alpha\big(e^{-p\delta^{-\frac12}\varphi}(v+w)^{p+1}\big),\partial^\alpha v\big\rangle_{L^2}\big|\\&+\delta^{\frac12}|\langle\partial^\alpha\big((v+w)\Delta\varphi\big),\partial^\alpha v\rangle_{L^2}|+\delta^{1/2}|\langle\partial^\alpha\big(\nabla(v+w)\cdot\nabla\varphi\big),\partial^\alpha v\rangle_{L^2}|\\&+|\langle\partial^\alpha\big(\mathbb{B}(\varphi)v\big),\partial^\alpha v\rangle_{L^2}|+|\big\langle\partial^\alpha\big(\langle u,Q\rangle v\big),\partial^{2\alpha}v\big\rangle_{L^2}|\Big).\end{aligned}\tag{2.12}$$

We observe that since all the functions involved in the above estimate are in $H^{2s+2}(\mathbb{T}^d,\mathbb{R})$, no boundary terms appear when integrating by parts.

Notice that there exist constants $C>0$, independent of δ, such that $|\langle\nabla v\cdot\nabla\varphi,\partial^{2\alpha}v\rangle_{L^2}|\le C\|\varphi\|_{H^{2s+1}}\|v\|_{H^s}^2$ and

$$\big|\big\langle\partial^\alpha\big(e^{-p\delta^{-\frac12}\varphi}(v+w)^{p+1}\big),\partial^\alpha v\big\rangle_{L^2}\big|\le C\|e^{-p\delta^{-\frac12}\varphi}\|_{L^\infty}\|\varphi\|_{H^{s+1}}^s\Big(p^s\delta^{-\frac{s}{2}}+1\Big)\|v+w\|_{H^s}^{p+1}\|v\|_{H^s}.$$

Hence, we deduce the existence of a constant $C>0$, independent of δ, such that

$$\begin{aligned}\partial_t\|\partial^\alpha v\|_{L^2}^2 \leq\ & C\delta\|w\|_{H^r}\|v\|_{H^s} + C\delta\|e^{-p\delta^{-\frac{1}{2}}\varphi}\|_{L^\infty}(\delta^{-\frac{s}{2}}+1)\|v+w\|_{H^s}^{p+1}\|v\|_{H^s} + C\delta^{\frac{1}{2}}\|v+w\|_{H^s}\|v\|_{H^s}\\ &+ C\delta^{1/2}\|v\|_{H^s}^2 + C\delta^{1/2}\|w\|_{H^r}\|v\|_{H^s} + C\|v\|_{H^s}^2 + C\|v\|_{H^s}^2.\end{aligned}$$

Now, we have that $\|e^{-\delta^{-\frac{1}{2}}\varphi}\|_{L^\infty}(\delta^{-\frac{s}{2}}+1) \xrightarrow{\delta\to 0} 0$ thanks to the positivity of φ. Using (2.9) and the Young's inequality we deduce that there exists $C>0$, independent of δ, such that

$$\begin{aligned}\partial_t\|\partial^\alpha v\|_{L^2}^2 &\leq C\delta^{3/4}\|v\|_{H^s} + C\delta\|v\|_{H^s}^{p+2} + C\delta\|v\|_{H^s} + C\delta^{\frac{1}{2}}\|v\|_{H^s}^2 + C\delta^{\frac{1}{2}}\|v\|_{H^s}\\ &\quad + C\delta^{1/2}\|v\|_{H^s}^2 + C\delta^{1/4}\|v\|_{H^s} + C\|v\|_{H^s}^2 + C\|v\|_{H^s}^2\\ &\leq C\delta^{1/2} + C(1+\delta^{1/2})\|v\|_{H^s}^2 + C\delta\|v\|_{H^s}^{p+2}.\end{aligned}$$

The last relation holds for $t\leq \delta^{-1}\mathcal{T}^\delta$. We recall that $\|\cdot\|_{H^s}^2 = \sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\leq s}}\|\partial^\alpha\cdot\|_{L^2}^2$ and therefore there exists $C>0$ such that

$$\partial_t\|v\|_{H^s}^2 \leq C\delta^{1/2} + C(1+\delta^{1/2})\|v\|_{H^s}^2 + C\delta\|v\|_{H^s}^{p+2}.$$

By the Grönwall Lemma and recalling (2.11) we obtain that

$$\|v(t)\|_{H^s}^2 \leq e^{C(1+\delta^{1/2})t}\left(C\delta^{1/2}t + \|\psi_0-\psi_0^\delta\|_{H^s}^2 + C\delta\int_0^t\|v(y)\|_{H^s}^{p+2}\mathrm{d}y\right) \tag{2.13}$$

for $t\leq\delta^{-1}\mathcal{T}^\delta$ and for every $\psi_0\in H^{2s+2}(\mathbb{T}^d,\mathbb{R})$. Finally, we can extend the validity of (2.13) for every $\psi_0\in H^s(\mathbb{T}^d,\mathbb{R})$ thanks to item *ii.* of Proposition 2.1 and to the density of $H^{2s+2}(\mathbb{T}^d,\mathbb{R})$ into $H^s(\mathbb{T}^d,\mathbb{R})$ with respect to the H^s-norm.

Properties of the maximal time. It remains to prove (2.8). We consider $\delta_0>0$ sufficiently small so that, for $0<\delta<\delta_0$, we have $\|\psi_0-\psi_0^\delta\|_{H^s}^2<1/2$ and then

$$\|v(0)\|_{H^s}^2 < 1/2.$$

Denote $\tau^\delta := \sup\left\{t<\delta^{-1}\mathcal{T}^\delta : \|v(t)\|_{H^s}<1\right\}$. The above inequality yields $\tau^\delta>0$. If $\tau^\delta=+\infty$ then (2.8) is obviously verified. Thus, let us assume that

$$\tau^\delta < +\infty.$$

To prove (2.8) we show that for $\delta_0>0$ sufficiently small and $0<\delta<\delta_0$ we have $\tau^\delta\geq 1$. Assume by contradiction that, for every $\delta_0>0$ small, there exists $0<\delta<\delta_0$ such that $\tau^\delta<1$. Thanks to (2.13), we get

$$1 = \|v(\tau^\delta)\|_{H^s}^2 < e^{C(1+\delta^{1/2})\tau^\delta}\left(C\delta^{1/2}\tau^\delta + \|\psi_0-\psi_0^\delta\|_{H^s}^2 + C\delta\int_0^{\tau^\delta}\|v(y)\|_{H^s}^{p+2}\mathrm{d}y\right). \tag{2.14}$$

We recall that, by definition, $\|v(t)\|_{H^s}<1$ in $[0,\tau^\delta)$. For δ_0 sufficiently small we have

$$e^{C(1+\delta^{1/2})\tau^\delta}\left(C\delta^{1/2}\tau^\delta + \|\psi_0-\psi_0^\delta\|_{H^s}^2\right) < \frac{1}{2}, \tag{2.15}$$

since $0<\delta<\delta_0$. Moreover,

$$e^{C(1+\delta^{1/2})\tau^\delta}\left(C\delta^{1/2}\tau^\delta + \|\psi_0-\psi_0^\delta\|_{H^s}^2 + C\delta\int_0^{\tau^\delta}\|v(y)\|_{H^s}^{p+2}\mathrm{d}y\right) < 1,$$

which contradicts (2.14). Hence, we conclude that there exists $\delta_0>0$ sufficiently small such that $\tau^\delta>1$ for every $0<\delta<\delta_0$. Thus, (2.8) holds true.

Conclusion. Finally, $1\in[0,\tau^\delta)\subset[0,\delta^{-1}\mathcal{T}^\delta)$ and thanks to (2.13) we have established the validity of (2.7) since

$$\|v(1)\|_{H^s}^2 \leq e^{C(1+\delta^{1/2})}\left(C\delta^{1/2} + \|\psi_0-\psi_0^\delta\|_{H^s}^2 + C\delta\right) \to 0 \quad \text{as } \delta\to 0^+.$$

□

A first simple consequence of Proposition 2.3, concerning the control of (NHE), is the following small-time global approximate null-controllability.

Corollary 2.4. *Let $s \in \mathbb{N}^*$ be such that $s > d/2$ and $1 \in \mathrm{span}\{Q_1, ..., Q_q\}$. Assume $\psi_0 \in H^s(\mathbb{T}^d, \mathbb{R})$. For any $\epsilon, T > 0$, there exists $(u_1, ..., u_q) \in L^2((0,T), \mathbb{R}^q)$ such that the solution $\psi(t; \psi_0, u)$ of the (NHE) in $[0,T]$ with control $u = (u_1, ..., u_q, 0, 0)$ is well-defined and*

$$\|\psi(T; \psi_0, u)\|_{H^s} < \epsilon.$$

Proof. For any $\tilde{\epsilon} > 0$, consider $c > 0$ sufficiently large so that

$$|e^{-c}| < \frac{\tilde{\epsilon}}{2\|\psi_0\|_{H^s}} \qquad \Longrightarrow \qquad \|e^{-c}\psi_0\|_{H^s} < \frac{\tilde{\epsilon}}{2}.$$

Let $-c = \sum_{j=1}^q u_j Q_j$ for some $(u_1, \dots, u_q) \in \mathbb{R}^q$. Thanks to Proposition 2.3, there exists $\delta > 0$ such that the constant control $u = (u_1, \dots, u_q)/\delta : [0, \delta] \to \mathbb{R}^q$ is such that the solution of (NHE) is well-defined in $[0, \delta]$ and

$$\|\psi(\delta; \psi_0, u/\delta) - e^{-c}\psi_0\|_{H^s} < \tilde{\epsilon}/2.$$

The triangular inequality ensures the result since, when we consider the control $v = \frac{u}{\delta}$, we have

$$\|\psi(\delta; \psi_0, v)\|_{H^s} \leq \|\psi(\delta; \psi_0, v) - e^{-c}\psi_0\|_{H^s} + \|e^{-c}\psi_0\|_{H^s} < \tilde{\epsilon}.$$

The dynamics can be obtained exactly in a time T, since 0 is stationary solution of (NHE) with control $u = 0$. Indeed, thanks to the second point of Proposition 2.1, when $\tilde{\epsilon}$ is sufficiently small the two solutions $\psi(T - \delta; \psi(\delta; \psi_0, v), 0), \psi(T - \delta; 0, 0) \equiv 0$, are defined in the same time interval $[0, T]$. Now, the first point of Proposition 2.1 yields the existence of $C > 0$, independent of $\psi(\delta; \psi_0, v)$, such that

$$\|\psi(T - \delta; \psi(\delta; \psi_0, v), 0) - \psi(T - \delta; 0, 0)\|_{H^s} \leq C\|\psi(\delta; \psi_0, v)\|_{H^s} < \tilde{\epsilon}C.$$

The proof is therefore concluded since, for every $\epsilon > 0$, it is sufficient to chose $\tilde{\epsilon} = \frac{\epsilon}{C}$. □

3 Small-time approximate controllability

The aim of this section is to prove the small-time approximate controllability results stated in Main Theorem A.

3.1 An intermediate controllability result

Let us recall the definition of $\widehat{\mathcal{H}}$ and $\mathcal{H}_0$ provided in Assumption I

$$\mathcal{H}_0 = \mathrm{span}_{\mathbb{R}}\{Q \in \widehat{\mathcal{H}}\}, \qquad \widehat{\mathcal{H}} = \{Q_1, \dots, Q_q\}.$$

We define $\mathcal{H}_j$, for every $j \in \mathbb{N}^*$, iteratively as the largest vector space containing elements ψ of the form

$$\psi = \varphi_0 + \sum_{k=1}^{p} \mathbb{B}(\varphi_k), \qquad \varphi_0, \dots, \varphi_p \in \mathcal{H}_{j-1}, \quad p \in \mathbb{N}, \tag{3.1}$$

where the non-linear operator $\mathbb{B}$ is defined in (2.3). We also denote

$$\mathcal{H}_\infty = \bigcup_{j=0}^{\infty} \mathcal{H}_j.$$

Let us recall the following result from the work [25].

Proposition 3.1. *[25, Proposition 2.6] If Assumptions I are verified, then $\mathcal{H}_\infty$ is dense in $H^s(\mathbb{T}^d, \mathbb{R})$, $s \geq 0$.*

We start by ensuring the following property of small-time approximate controllability for (NHE).

Proposition 3.2. *Let $s \in \mathbb{N}^*$ be such that $s > d/2$ and $(Q_1, ..., Q_q) \in C^\infty(\mathbb{T}^d, \mathbb{R}^q)$ be such that $1 \in \mathcal{H}_0$. Assume that $\mathcal{H}_\infty$ is dense in $H^s(\mathbb{T}^d, \mathbb{R})$. Let $\psi_0 \in H^s(\mathbb{T}^d, \mathbb{R})$ and $\varphi \in H^s(\mathbb{T}^d, \mathbb{R})$. For any $\epsilon, T > 0$, there exist $\tau \in [0, T)$ and $(u_1, ..., u_q) \in L^2((0, \tau), \mathbb{R}^q)$ such that the solution $\psi(t; \psi_0, u)$ of (NHE) with control $u = (u_1, ..., u_q, 0, 0)$ is well-defined in $[0, \tau]$ and*

$$\|\psi(\tau; \psi_0, u) - e^{\varphi}\psi_0\|_{H^s} < \epsilon.$$

Proof. Let us recall that the concatenation $v * u$ of two scalar control laws $u : [0, T_1] \to \mathbb{R}^{q+2}$, $v : [0, T_2] \to \mathbb{R}^{q+2}$ is the control law defined on $[0, T_1 + T_2]$ as follows

$$(v * u)(t) = \begin{cases} u(t), & t \in [0, T_1] \\ v(t - T_1), & t \in (T_1, T_1 + T_2]. \end{cases}$$

Such definition extends to controls with values in $\mathbb{R}^q$, componentwise. We will often use the fact that

$$\psi(T_1 + t; \psi_0, v * u) = \psi(t; \psi(T_1, \psi_0, u), v), \quad t > 0.$$

Let us start by assuming that the following property holds for any $n \in \mathbb{N}$:

(P_n) for any $\psi_0 \in H^s(\mathbb{T}^d, \mathbb{R})$, $\phi \in \mathcal{H}_n$, and any $\varepsilon, T > 0$, there exist $\tau \in [0, T)$ and $(u_1, ..., u_q) : [0, \tau] \to \mathbb{R}^q$ piecewise constant such that the solution of the (NHE) associated with the initial condition ψ_0 and control $u = (u_1, ..., u_q, 0, 0)$ satisfies

$$\left\| \psi(\tau; \psi_0, u) - e^{\phi} \psi_0 \right\|_{H^s(\mathbb{T}^d)} < \varepsilon.$$

Property (P_n) combined with the density feature given in Proposition 3.1 implies at once the statement.

We are thus left to prove (P_n). An analogous property appeared in [25, Theorem 2.2] in the study of nonlinear Schrödinger equations with bilinear control. We provide the proof for completeness. Let us proceed by induction on the index n.

Inductive basis: $n = 0$
If $\phi \in \mathcal{H}_0$, there exists $(u_1, ..., u_q) \in \mathbb{R}^q$ such that $\phi(x) = \langle u, Q(x) \rangle$ with $u = (u_1, ..., u_q, 0, 0)$. Consider then the solution of (NHE) associated with the constant control $u^\tau := u/\tau \in \mathbb{R}^q$ and with the initial condition ψ_0. Applying Proposition 2.3 with $\varphi = 0$, we find $\tau \in [0, T)$ such that

$$\left\| \psi(\tau; \psi_0, u^\tau) - e^{\phi} \psi_0 \right\|_{H^s(\mathbb{T}^d)} < \varepsilon,$$

which proves the desired property.

Inductive step: $n \Rightarrow n + 1$
By assuming that (P_n) holds, we prove (P_{n+1}). If $\phi \in \mathcal{H}_{n+1}$, there exist $N \in \mathbb{N}$ and $\phi_0, \dots, \phi_N \in \mathcal{H}_n$ such that

$$\phi = \phi_0 + \sum_{j=1}^{N} \mathbb{B}(\phi_j).$$

Consider ϕ_1 and $c > 0$ such that $\tilde{\phi}_1 = \phi_1 + c \geq 0$ and note that $\mathbb{B}(\tilde{\phi}_1) = \mathbb{B}(\phi_1)$. Thanks to Proposition 2.3, we can fix $\gamma \in [0, T/3)$ small enough such that

$$\left\| e^{\gamma^{-1/2} \tilde{\phi}_1} \psi(\gamma; e^{-\gamma^{-1/2} \tilde{\phi}_1} \psi_0, 0) - e^{\mathbb{B}(\phi_1)} \psi_0 \right\|_{H^s} < \varepsilon/2.$$

Since $c \in \mathcal{H}_0$, $\phi_1 \in \mathcal{H}_n$, then $\tilde{\phi}_1 \in \mathcal{H}_n$. Thanks to the inductive hypothesis, for any $\sigma, T, \gamma > 0$ there exist $\delta \in [0, T/3)$ and a piecewise constant control $u^{\delta,\gamma} = (u_1^{\delta,\gamma}, ..., u_q^{\delta,\gamma}, 0, 0) : [0, \delta] \to \mathbb{R}^{q+2}$ such that

$$\left\| \psi(\delta; \psi_0, u^{\delta,\gamma}) - e^{-\gamma^{-1/2} \tilde{\phi}_1} \psi_0 \right\|_{H^s} < \sigma. \tag{3.2}$$

Let now the dynamics evolve freely in a time interval of leght γ, that is, we consider the control $0|_{[0,\gamma]} = (0, \dots, 0) : [0, \gamma] \to \mathbb{R}^{q+2}$. From (2.1) we deduce that there exists $C = C(\gamma)$ such that

$$\left\| \psi(\delta + \gamma; \psi_0, 0|_{[0,\gamma]} * u^{\delta,\gamma}) - \psi(\gamma; e^{-\gamma^{-1/2} \tilde{\phi}_1} \psi_0, 0) \right\|_{H^s} = \left\| \psi(\gamma; \psi(\delta, \psi_0, u^{\delta,\gamma}), 0) - \psi(\gamma; e^{-\gamma^{-1/2} \tilde{\phi}_1} \psi_0, 0) \right\|_{H^s} < C\sigma.$$

We use again the inductive hypothesis to deduce that there exist $\delta' \in [0, T/3)$ and a piecewise constant control $u^{\delta',\gamma} = (u_1^{\delta',\gamma}, ..., u_q^{\delta',\gamma}, 0, 0) : [0, \delta'] \to \mathbb{R}^{q+2}$ such that

$$\left\| \psi(\delta'; \psi(\gamma, e^{-\gamma^{-1/2} \tilde{\phi}_1} \psi_0, 0), u^{\delta',\gamma}) - e^{\gamma^{-1/2} \tilde{\phi}_1} \psi(\gamma; e^{-\gamma^{-1/2} \tilde{\phi}_1} \psi_0, 0) \right\|_{H^s} < \sigma.$$

Then, thanks to (2.1), there exists $C' = C'(\|u^{\delta',\gamma}\|_{L^2}, \delta')$ such that

$$\begin{aligned}
&\left\|\psi(\delta+\gamma+\delta';\psi_0,u^{\delta',\gamma}*0|_{[0,\gamma]}*u^{\delta,\gamma}) - e^{\mathbb{B}(\phi_1)}\psi_0\right\|_{H^s} \\
&\qquad\leq \left\|\psi(\delta';\psi(\delta+\gamma;\psi_0,0|_{[0,\gamma]}*u^{\delta,\gamma}),u^{\delta',\gamma}) - \psi(\delta';\psi(\gamma;e^{-\gamma^{-1/2}\tilde{\phi}_1}\psi_0,0),u^{\delta',\gamma})\right\|_{H^s} \\
&\qquad\qquad + \left\|\psi(\delta';\psi(\gamma;e^{-\gamma^{-1/2}\tilde{\phi}_1}\psi_0,0),u^{\delta',\gamma}) - e^{\gamma^{-1/2}\tilde{\phi}_1}\psi(\gamma;e^{-\gamma^{-1/2}\tilde{\phi}_1}\psi_0,0)\right\|_{H^s} \\
&\qquad\qquad + \left\|e^{\gamma^{-1/2}\tilde{\phi}_1}\psi(\gamma;e^{-\gamma^{-1/2}\tilde{\phi}_1}\psi_0,0) - e^{\mathbb{B}(\phi_1)}\psi_0\right\|_{H^s} \\
&\qquad\leq C'C\sigma+\sigma+\varepsilon/2.
\end{aligned}$$

Choosing $\sigma>0$ small enough such that $C'C\sigma+\sigma<\varepsilon/2$, we have then proved that the piecewise constant control $u^{\delta',\gamma}*0|_{[0,\gamma]}*u^{\delta,\gamma}$ steers the initial state ψ_0 ε-close to the state $e^{\mathbb{B}(\phi_1)}\psi_0$ in time $\tau:=\delta'+\gamma+\delta<T$. We can now repeat the same argument for ϕ_2, reasoning as if we were starting from the initial state $e^{\mathbb{B}(\phi_1)}\psi_0$, and prove that the system can be driven arbitrarily close to the state $e^{\mathbb{B}(\phi_1)+\mathbb{B}(\phi_2)}\psi_0$ in arbitrarily small time, and hence, by iteration, to $e^{\sum_{i=1}^N\mathbb{B}(\phi_i)}\psi_0$. By inductive hypothesis we conclude that there exists a piecewise constant control u leading the state $e^{\sum_{i=1}^N\mathbb{B}(\phi_i)}\psi_0$ arbitrarily close to $e^{\phi_0+\sum_{i=1}^N\mathbb{B}(\phi_i)}\psi_0$. in arbitrarily small time. This completes the proof of property (P_n). □

3.2 Small-time global approximate controllability

In this section we prove a more general results which in fact has inspired Main Theorem A.

Theorem 3.3. *Let $s\in\mathbb{N}^*$, $s>d/2$ and let $(Q_1,...,Q_q)\in C^\infty(\mathbb{T}^d,\mathbb{R}^q)$ be such that $1\in\mathcal{H}_0$ and $\mathcal{H}_\infty$ is dense in $H^s(\mathbb{T}^d,\mathbb{R})$.*

i. Let $\psi_0,\psi_1\in H^s(\mathbb{T}^d,\mathbb{R})$ be such that $\mathrm{sign}(\psi_0)=\mathrm{sign}(\psi_1)$. For any $\epsilon>0$ and $T>0$, there exist $\tau\in(0,T]$ and $(u_1,...,u_q)\in L^2((0,\tau),\mathbb{R}^q)$ for which the solution $\psi(t;\psi_0,u)$ of (NHE) with control $u=(u_1,...,u_q,0,0)$ is well-defined in $[0,\tau]$ and satisfies

$$\|\psi(\tau;\psi_0,u)-\psi_1\|_{L^2}<\epsilon.$$

ii. Let $\psi_0,\psi_1\in H^s(\mathbb{T}^d,\mathbb{R})$ be such that $\psi_0,\psi_1>0$ (or $\psi_0,\psi_1<0$). For any $\epsilon>0$ and $T>0$, there exists $(u_1,...,u_q)\in L^2((0,T),\mathbb{R}^q)$ such that the solution $\psi(t;\psi_0,u)$ of (NHE) with control $u=(u_1,...,u_q,0,0)$ is well-defined in $[0,T]$ and verifies

$$\|\psi(T;\psi_0,u)-\psi_1\|_{H^s}<\epsilon.$$

Proof. Let us start by proving *i.*. Denote by Z the set of zeroes of ψ_j, for $j=0,1$. Consider for $\eta>0$ the set

$$Z_\eta:=\{x\in\mathbb{T}^d\mid \mathrm{dist}(x,Z)<\eta\},$$

and its complement in $\mathbb{T}^d$, Z_η^c. For $\eta>0$, we define

$$\phi_\eta=\rho_\eta\log(\psi_1/\psi_0),$$

where ρ_η is a smooth function compactly supported inside $\mathbb{Z}_\eta^c$. ϕ_η is well-defined because $\psi_1/\psi_0>0$ on Z_η^c. Furthermore, ρ_η belongs to $H^s(\mathbb{T}^d)$. Notice that

$$\|e^{\phi_\eta}\psi_0-\psi_1\|_{L^2(\mathbb{T}^d)}\leq\|e^{\phi_\eta}\psi_0-\psi_1\|_{L^2(Z_\eta^c)}+\|\psi_0+\psi_1\|_{L^2(Z_\eta)}$$

Fixed any $\epsilon,T>0$, observe that we can choose $\eta>0$ small enough such that

$$\|e^{\phi_\eta}\psi_0-\psi_1\|_{L^2(\mathbb{T}^d)}<\epsilon/2.$$

We then apply Proposition 3.2 with $\varphi=\phi_\eta$ and we deduce that there exist a time $\tau\in[0,T)$ and a control $u=(u_1,...,u_q,0,0)\in L^2((0,\tau),\mathbb{R}^{q+2})$ such that the solution $\psi(t;\psi_0,u)$ of (NHE) is well-defined in $[0,\tau]$ and

$$\|\psi(\tau;\psi_0,u)-e^{\varphi}\psi_0\|_{L^2}<\epsilon/2.$$

By the triangular inequality we conclude that

$$\|\psi(\tau;\psi_0,u)-\psi_1\|_{L^2}\leq\|\psi(\tau;\psi_0,u)-e^{\varphi}\psi_0\|_{L^2}+\|e^{\varphi}\psi_0-\psi_1\|_{L^2}<\epsilon,$$

and the first item of the Proposition is then proved.

The proof of *ii.* follows from the same strategy used for proving *i.*. However, we now consider the H^s instead of the L^2-norm. Observe that Proposition 3.2 is valid for the H^s-norm. In this case, we substitute ϕ_η with two functions $\phi_1 = \log(1/\psi_0)$ and $\phi_2 = \log(\psi_1)$ which are well-defined everywhere in $\mathbb{T}^d$. Note that $\phi_1, \phi_2 \in H^s(\mathbb{T}^d, \mathbb{R})$ since $\psi_1, \psi_0 \in H^s(\mathbb{T}^d, \mathbb{R})$. Proposition 3.2 with $\varphi = \phi_1$ yields the existence of a control $u^1 = (u_1^1, ..., u_q^1, 0, 0) : [0, \tau_1) \to \mathbb{R}^{q+2}$ which steers ψ_0 close to the constant state 1 in time $\tau_1 \leq T/2$. We apply again Proposition 3.2 with $\varphi = \phi_2$. Thus, we can find a control $u^2 = (u_1^2, ..., u_q^2, 0, 0) : [0, \tau_2) \to \mathbb{R}^{q+2}$ such that the solution of (NHE) starting from a state close to 1 at time $\tau_2 \leq T/2$ reaches a neighbourhood of the final target ψ_1. Hence, the approximate controllability in H^s can be achieved in time T by exploiting that 1 is a stationary solution of (NHE) associated to the control $u^{stat} = (u_1^{stat}, ..., u_q^{stat}, 0, 0) : [0, T - \tau_1 - \tau_2] \to \mathbb{R}^{q+2}$, thanks to the assumption that $1 \in \mathcal{H}_0$. The control will therefore be defined as

$$(u^2 * u^{stat} * u^1)(t) = \mathbf{1}_{[0,\tau_1)} u^1(t) + \mathbf{1}_{[\tau_1, T-\tau_2]} u^{stat}(t - \tau_1) + \mathbf{1}_{(T-\tau_2, T]} u^2(t - T + \tau_2).$$

□

Remark 3.4. The reason why the first part of Theorem 3.3 (and, consequently, of Main Theorem A) is stated only for the L^2-norm, while the second part is valid with the stronger H^s-norm, is due to our approximation technique. More precisely, the term $\|e^{\phi_\eta}\psi_0 - \psi_1\|_{L^2(Z_\eta^c)}$ cannot be small for $\eta \to 0$ with respect to the H^s-norm, $s > 0$, as soon as $Z \neq \emptyset$.

Proof of Main Theorem A. Main Theorem A follows from Theorem 3.3. Indeed, thanks to Proposition 3.1, the space $\mathcal{H}_\infty$ is dense in $H^s(\mathbb{T}^d, \mathbb{R})$. Furthermore, the potentials Q_j belong to $H^r(\mathbb{T}^d, \mathbb{R})$ for every $r > 0$. Therefore, the proof is completed. □

4 Exact controllability to the ground state solution

We now move to the proof of Main Theorem B. Henceforth, we suppose $d = s = 1$. Let the solution of (NHE) exist for any time $T > 0$. For instance, it is enough to require $p \in 2\mathbb{N}^*$, $\kappa \geq 0$ and $\psi_0 \in H^3(\mathbb{T})$, as stated in Proposition 2.2. Main Theorem B will be a direct consequence of the following more general result.

Theorem 4.1. *Let $\kappa \geq 0$ and $p \in 2\mathbb{N}^*$. Assume Assumptions II be verified and $\mathcal{H}_\infty$ be dense in $H^3(\mathbb{T}, \mathbb{R})$. Then, (NHE) is exactly controllable to the ground state solution Φ in any positive time from any positive state. In other words, for any $T > 0$ and*

$$\psi_0 \in \{\psi \in H^3(\mathbb{T}, \mathbb{R}) : \ sgn(\psi) > 0\},$$

there exists $u \in L^2((0,T), \mathbb{R}^{q+2})$ such that

$$\psi(T; \psi_0, u) = \Phi.$$

Analogously, for any $T > 0$ and

$$\psi_0 \in \{\psi \in H^3(\mathbb{T}, \mathbb{R}) : \ sgn(\psi) < 0\},$$

there exists $u \in L^2((0,T), \mathbb{R}^{q+2})$ such that

$$\psi(T; \psi_0, u) = -\Phi.$$

Example 4.2. Examples of suitable functions μ_1 and μ_2 verifying hypotheses (1.5) in Assumptions II are

$$\mu_1 = x^3(2\pi - x)^3, \qquad \mu_2 = x^3(x - \pi)^3(x - 2\pi)^3.$$

Indeed, both functions belong to $H^3(\mathbb{T}, \mathbb{R})$, μ_1 is symmetric with respect to $x = \pi$ and μ_2 is antisymmetric. Thus, $\langle \mu_1, c_0 \rangle_{L^2} \neq 0$ and $\langle \mu_2, c_0 \rangle_{L^2} = 0$. Moreover,

$$\langle \mu_1, s_k \rangle_{L^2} = 0, \qquad \langle \mu_2, c_k \rangle_{L^2} = 0.$$

By further computations, the remaining hypotheses in (1.5) are verified since

$$\langle \mu_1, c_k \rangle_{L^2} = \frac{96\pi(k^2\pi^2 - 15)}{k^6}, \qquad \langle \mu_2, s_k \rangle_{L^2} = \frac{-864\pi(840 - 105k^2\pi^2 + 2k^4\pi^4)}{k^9}.$$

Before proving Theorem 4.1, and consecutively Main Theorem B, we need to ensure the following local exact controllability result to the ground state solution $\Phi = \frac{1}{\sqrt{2\pi}}$.

Theorem 4.3. *Let $\kappa \geq 0$ and $p \in 2\mathbb{N}^*$. Assume Assumptions II be verified. Then, (NHE) is locally exactly controllable to the ground state solution Φ in any positive time. In other words, for any $T > 0$ there exists $R_T > 0$ such that, for any*

$$\psi_0 \in \{\psi \in H^3(\mathbb{T}, \mathbb{R}) : \|\psi - \Phi\|_{H^1} < R_T\},$$

there exists $(u_1, u_2) \in H^1((0,T), \mathbb{R}^2)$ such that $\psi(T; \psi_0, u) = \Phi$, where $u = (\frac{\kappa}{\Phi^p}, 0, ..., 0, u_1, u_2)$. Furthermore,

$$\|u\|_{H^1(0,T)} \leq \frac{e^{-\pi^2 \Gamma_0 / T}}{e^{2\pi^2 \Gamma_0/(3T)} - 1}, \tag{4.1}$$

where

$$\Gamma_0 := 2\nu + (\max\{\ln \gamma_1, 0\} + \max\{\ln C_Q^2, 0\} + \gamma_2 + \ln 8)/2 \tag{4.2}$$

with

$$\gamma_1 := 2\kappa(p+1)^2 \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j}, \qquad \gamma_2 := 2\kappa(p+1)\Phi^p + \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} + 1,$$

and

$$R_T = e^{-6\Gamma_0/T_1}, \tag{4.3}$$

with

$$T_1 := \min\left\{\frac{6}{\pi^2} T, 1, T_0\right\}.$$

The constant T_0 is defined in (4.5) and C_Q in (C.4).

4.1 Control of the linearized system

The result of Theorem 4.3 follows from the null-controllability of an associated linear system and the iteration of a control procedure on a clever choice of time steps, as proposed in [6].

We first observe that the ground state $\Phi = 1/\sqrt{2\pi}$ is solution of (NHE) in any time interval $[0, T]$ for $u = (u_\kappa, 0, 0, \ldots, 0)$, with $u_\kappa = \frac{\kappa}{(2\pi)^{p/2}}$ and $\psi_0 = \Phi$. Indeed, we recall that $Q_1 = 1$, thanks to Assumptions II.

Let $s_1, s_0 > 0$. Consider the following linear system

$$\begin{cases} \partial_t \xi(t) - \partial_x^2 \xi(t) + \kappa p \xi(t) = \langle v(t), Q \rangle \Phi, \quad t \in (s_0, s_1). \\ \xi(s_0) = \xi_0, \end{cases} \tag{4.4}$$

with $Q = (Q_1, ..., Q_q, \mu_1, \mu_2) \in H^3(\mathbb{T}, \mathbb{R}^{q+2})$ and $q \in \mathbb{N}^*$. We denote by $\xi(\cdot; s_0, \xi_0, v)$ the solution of (4.4) with initial condition ξ_0 at time s_0 and control v.

Definition 4.4. The pair $(-\partial_x^2, Q)$ is said to be 1-*null controllable* in time $T > s_0$ if there exists a constant $N(T) > 0$ such that for any $\xi_0 \in L^2(\mathbb{T}, \mathbb{R})$, there exists a control $v \in L^2((s_0, T), \mathbb{R}^{q+2})$ such that $\xi(T; s_0, \xi_0, v) = 0$ and moreover $\|v\|_{L^2(0,T)} \leq N(T) \|\xi_0\|_{L^2}$. The best constant, that is,

$$N(T) := \sup_{\|\xi_0\|_{L^2}=1} \inf \{\|v\|_{L^2} : \xi(T; s_0, \xi_0, v) = 0\}$$

is called the *control cost*. If $v \in H_0^1((s_1, T), \mathbb{R}^{q+2})$, the pair $(-\partial_x^2, Q)$ is called *smoothly* 1-null controllable and

$$N(T) := \sup_{\|\xi_0\|_{L^2}=1} \inf \{\|v\|_{H^1} : \xi(T; s_0, \xi_0, v) = 0\}$$

Remark 4.5. Let us observe that, fixed $v \in L^2((s_0, s_1), \mathbb{R}^{q+2})$ and $Q \in H^1(\mathbb{T}, \mathbb{R}^{q+2})$, for any $\xi_0 \in L^2(\mathbb{T}, \mathbb{R})$, there exists a unique mild solution

$$\xi \in C([s_0, s_1], L^2(\mathbb{T}))$$

of (4.4) (see [6, Proposition 2.1] adapted to the current case). Furthermore, given a smoother initial condition of (4.4), for instance $\xi_0 \in H^1(\mathbb{T})$, the solution ξ belongs to the space

$$C([s_0, s_1], H^1(\mathbb{T})) \cap H^1((s_0, s_1), L^2(\mathbb{T})) \cap L^2((s_0, s_2), H^2(\mathbb{T})).$$

Such regularity is called *maximal regularity*, and it is due to the analiticity of the semigroup generated by the operator ∂_x^2 (see, for instance, [6, Proposition 4] and [4, Corollary 3] that can be adapted to (4.4)).

The result that follows shows that we use the last two components of the control to drive the linear system (4.4) to rest. Henceforth, where constants C come with no specific index, they may vary from line to line.

Proposition 4.6. *Let Assumptions II be verified. Then, $(-\partial_x^2, Q)$ is smoothly null controllable in any time $T > 0$. Namely, for any $\xi_0 \in L^2(\mathbb{T}, \mathbb{R})$, there exists $(v_1, v_2) \in H_0^1((0,T), \mathbb{R}^2)$ such that the solution of (4.4) with $(s_0, s_1) = (0, T)$ and $v = (0, ..., 0, v_1, v_2)$ satisfies*

$$\xi(T; \xi_0, v) = 0.$$

Furthermore, there exist $\nu, T_0 > 0$ such that

$$N(\tau) \leq e^{\nu/\tau}, \quad \forall\, 0 < \tau \leq T_0. \tag{4.5}$$

Proof. For all $T > 0$, we want to prove the existence of $N(T) > 0$ such that for any initial condition $\xi_0 \in L^2(\mathbb{T}, \mathbb{R})$ there exists a control $v = (0, ..., 0, v_1, v_2)$ so that $\xi(T; \xi_0, v) = 0$. To this purpose, we first note that the operator $-\partial_x^2 + p\kappa$ exhibits double eigenvalues $\tilde{\lambda}_j = \lambda_j + p\kappa$ with $j \in \mathbb{N}$ (see definition (1.3) of λ_j), with the exception of the first one, associated to the eigenfunctions $\{c_0, c_j, s_j\}_{j \in \mathbb{N}^*}$, defined in (1.4).

Let us consider the reduced problem (4.4) with $\hat{Q} = (\mu_1, \mu_2)$ and $(s_0, s_1) = (0, T)$. We look for a control of the form $\hat{v} = (v_1, v_2)$. We decompose the solution with respect to the Hilbert basis $\{c_0, c_k, s_k\}_{k \in \mathbb{N}^*}$. Recalling that $\langle \mu_1, s_k \rangle_{L^2} = \langle \mu_2, c_k \rangle_{L^2} = 0$, the smooth-null controllability property is equivalent to finding $\hat{v} \in H_0^1((0,T); \mathbb{R}^2)$ such that

$$\begin{aligned} 0 = \xi(T; \xi_0, \hat{v}) = \sum_{k \in \mathbb{N}} e^{-\tilde{\lambda}_k T} \langle \xi_0, c_k \rangle_{L^2} c_k + \sum_{k \in \mathbb{N}^*} e^{-\tilde{\lambda}_k T} \langle \xi_0, s_k \rangle_{L^2} s_k \\ - \int_0^T v_1(s) \sum_{k \in \mathbb{N}} e^{-\tilde{\lambda}_k (T-s)} \langle \mu_1 \Phi, c_k \rangle_{L^2} c_k \, ds - \int_0^T v_2(s) \sum_{k \in \mathbb{N}^*} e^{-\tilde{\lambda}_k (T-s)} \langle \mu_2 \Phi, s_k \rangle_{L^2} s_k \, ds. \end{aligned}$$

Notice that $\Phi = c_0$ is a constant. The above relation is verified when the following infinite number of identities are satisfied for a control function $\hat{v} = (v_1, v_2)$

$$\begin{aligned} \langle \xi_0, c_k \rangle_{L^2} &= c_0 \int_0^T e^{\tilde{\lambda}_k s} v_1(s) \langle \mu_1, c_k \rangle_{L^2} \, ds, \quad \forall\, k \in \mathbb{N}, \\ \langle \xi_0, s_k \rangle_{L^2} &= c_0 \int_0^T e^{\tilde{\lambda}_k s} v_2(s) \langle \mu_2, s_k \rangle_{L^2} \, ds, \quad \forall\, k \in \mathbb{N}^*, \end{aligned}$$

that can be rewritten in compact form as

$$\int_0^T e^{\lambda_k s} \tilde{v}_1(s) \, ds = d_k^1, \quad \forall\, k \in \mathbb{N}, \tag{4.6}$$

$$\int_0^T e^{\lambda_k s} \tilde{v}_2(s) \, ds = d_k^2, \quad \forall\, k \in \mathbb{N}^*, \tag{4.7}$$

where

$$d_k^1 := \frac{\langle \xi_0, c_k \rangle_{L^2}}{c_0 \langle \mu_1, c_k \rangle_{L^2}}, \qquad d_k^2 := \frac{\langle \xi_0, s_k \rangle_{L^2}}{c_0 \langle \mu_2, s_k \rangle_{L^2}} \quad \forall\, k \in \mathbb{N}^*,$$

are well-defined thanks to Assumptions II and

$$\tilde{v}_1(s) = e^{p\kappa s} v_1(s), \qquad \tilde{v}_2(s) = e^{p\kappa s} v_2(s).$$

We treat (4.6) an (4.7) as two separate moment problems. Let us start solving (4.6). We seek for a function $\tilde{v}_1$ such that

$$\tilde{v}_1'(t) = r_1 \left(\frac{T}{2} - t \right) e^{-(\frac{T}{2} - t)}, \qquad r_1 \in L^2 \left(-\frac{T}{2}, \frac{T}{2} \right).$$

Therefore, it should hold

$$\tilde{v}_1(t) - \tilde{v}_1(0) = \int_0^t \tilde{v}_1'(s) ds = \int_0^t r_1 \left(\frac{T}{2} - s \right) e^{-(\frac{T}{2} - s)} ds.$$

We require that $\tilde{v}_1(0) = \tilde{v}_1(T) = 0$, which implies that r_1 must satisfy

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} r_1(s) e^{-s} ds = 0. \tag{4.8}$$

Integrating by parts (4.6) and taking into account that $\tilde{v}_1(0) = \tilde{v}_1(T) = 0$, we get for any $k \in \mathbb{N}^*$

$$\begin{aligned} d_k^1 &= \int_0^T e^{\lambda_k s} \tilde{v}_1(s) ds = \frac{1}{\lambda_k} \left(e^{\lambda_k t} \tilde{v}_1(t) \Big|_0^T - \int_0^T e^{-(\frac{T}{2}-t)+\lambda_k t} r_1 \left(\frac{T}{2} - t \right) dt \right) \\ &= \frac{1}{\lambda_k} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-s+\lambda_k(\frac{T}{2}-s)} r_1(s) ds. \end{aligned}$$

The above identities can be rewritten as follows

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-(1+\lambda_k)s} r_1(s) ds = \lambda_k d_k^1 e^{-\frac{T}{2}\lambda_k}, \qquad k \in \mathbb{N}^*.$$

For $k = 0$ we have that

$$\begin{aligned} d_0^1 &= \int_0^T \tilde{v}_1(s) ds = s\tilde{v}_1(s) \Big|_0^T - \int_0^T s r_1 \left(\frac{T}{2} - s \right) e^{-(\frac{T}{2}-s)} ds = \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\frac{T}{2} - s \right) r_1(s) e^{-s} ds \\ &= -\int_{-\frac{T}{2}}^{\frac{T}{2}} s e^{-s} r_1(s) ds, \end{aligned}$$

where we have used (4.8). Define the family

$$\omega_k := 1 + \lambda_k, \qquad k \in \mathbb{N}^*,$$

and the sequence $\tilde{d}_{kj}^1$

$$\begin{aligned} \tilde{d}_{01}^1 = -d_0, \qquad \tilde{d}_{k1}^1 = 0, \qquad k \in \mathbb{N}^*, \\ \tilde{d}_{00}^1 = 0, \qquad \tilde{d}_{k0}^1 = \lambda_k d_k^1 e^{-\frac{T}{2}\lambda_k}, \qquad k \in \mathbb{N}^*. \end{aligned}$$

Thus, we look for $r_1 \in L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ such that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} s^j e^{-\omega_k s} r_1(s) ds = \tilde{d}_{kj}^1, \qquad j = 0, 1, \quad k \in \mathbb{N}. \tag{4.9}$$

As proved in [11, Theorem 1.5], there exists $T_0 > 0$ such that for every $0 < T < T_0$ there exists a family

$$\{\sigma_{k,j}\}_{k,j \in \mathbb{N}} \subset L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$$

which is biorthogonal to

$$e_{kj}(s) = s^j e^{-\omega_k s}, \qquad k, j \in \mathbb{N},$$

and moreover

$$\|\sigma_{k,j}\|_{L^2(-T/2,T/2)} \leq C e^{C\sqrt{\omega_k} + \frac{C}{T}}, \qquad j = 0, 1, \quad k \in \mathbb{N}^*.$$

Therefore, by defining

$$r_1(s) := \tilde{d}_{01}^1 \sigma_{01} + \sum_{k=1}^{\infty} \tilde{d}_{k0}^1 \sigma_{k0}(s),$$

we deduce that r_1 solves the moment problem (4.9). Let us finally show that $r_1 \in L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$:

$$\begin{aligned}
\|r_1\|_{L^2(-T/2,T/2)} &\leq |\tilde{d}^1_{00}|\,\|\sigma_{01}\|_{L^2(-T/2,T/2)} + \sum_{k=1}^{\infty} |\tilde{d}^1_{k0}|\,\|\sigma_{k0}\|_{L^2(-T/2,T/2)} \\
&\leq Ce^{C/T}|\tilde{d}^1_{00}| + C\sum_{k=1}^{\infty} \lambda_k |d^1_k| e^{-\frac{T}{2}\lambda_k + C\sqrt{1+\lambda_k}+\frac{C}{T}} \\
&\leq Ce^{C/T}\left(\frac{|\langle \xi_0, c_0\rangle_{L^2}|}{c_0|\langle \mu_1, c_0\rangle_{L^2}|} + \left(\sum_{k=1}^{\infty} \frac{\lambda_k^2 e^{-T\lambda_k + C\sqrt{\lambda_k+1}}}{c_0^2|\langle \mu_1, c_k\rangle_{L^2}|^2}\right)^{1/2} \|\xi_0\|_{L^2}\right) \\
&\leq Ce^{C/T}\left(1 + \left(\sum_{k=1}^{\infty} \lambda_k^{2(q_1+1)} e^{-T\lambda_k + C\sqrt{\lambda_k+1}}\right)^{1/2}\right)\|\xi_0\|_{L^2}\,,
\end{aligned}$$

where q_1 is the parameter introduced in Assumptions II. Let us analyse the behaviour with respect to T of the following series:

$$S(T) := \sum_{k=1}^{\infty} \lambda_k^{2(q_1+1)} e^{-T\lambda_k + C\sqrt{\lambda_k+1}} = \sum_{k=1}^{\infty} \left(\lambda_k^{2(q_1+1)} e^{-\frac{T}{2}\lambda_k}\right) e^{-\frac{T}{2}\lambda_k + C\sqrt{\lambda_k+1}}. \tag{4.10}$$

For any $\lambda \geq 0$, we introduce the function $f(\lambda) := e^{-\frac{T}{2}\lambda + C\sqrt{\lambda+1}}$. Its derivative is given by

$$f'(\lambda) = e^{-\frac{T}{2}\lambda + C\sqrt{\lambda}}\left(-\frac{T}{2} + \frac{C}{2\sqrt{\lambda+1}}\right), \tag{4.11}$$

and its maximum is attended at $\lambda = \left(\frac{C}{T}\right)^2 - 1$. Hence, for every $0 < T \leq 1$ we have

$$S(T) \leq e^{\frac{T}{2}+\frac{C}{T}} \sum_{k=1}^{\infty} \left(\lambda_k^{2(q_1+1)} e^{-\frac{T}{2}\lambda_k}\right) \leq e^{\frac{C}{T}} \sum_{k=1}^{\infty} \left(\lambda_k^{2(q_1+1)} e^{-\frac{T}{2}\lambda_k}\right). \tag{4.12}$$

Now, for any $\lambda \geq 0$, we consider the function $g(\lambda) := \lambda^{2(q_1+1)} e^{-\frac{T}{2}\lambda}$. From its derivative $g'(\lambda) = \lambda^{2q_1+1} e^{-\frac{T}{2}\lambda}\left(2(q_1+1) - \frac{T}{2}\lambda\right)$, we deduce that

$$g(\lambda) \text{ is } \begin{cases} \text{increasing} & \text{if } 0 \leq \lambda < \frac{4(q_1+1)}{T} \\ \\ \text{decreasing} & \text{if } \lambda \geq \frac{4(q_1+1)}{T} \end{cases} \tag{4.13}$$

and g has a maximum for $\lambda = \frac{4(q_1+1)}{T}$. We define the index

$$k_1 := k_1(T) = \sup\left\{k \in \mathbb{N}^* \,:\, \lambda_k \leq \frac{4(q_1+1)}{T}\right\}. \tag{4.14}$$

We can rewrite the sum in (4.12) as

$$\sum_{k=1}^{\infty} \lambda_k^{2(q_1+1)} e^{-\frac{T}{2}\lambda_k} = \sum_{1\leq k\leq k_1-1} \lambda_k^{2(q_1+1)} e^{-\frac{T}{2}\lambda_k} + \sum_{k_1\leq k\leq k_1+1} \lambda_k^{2(q_1+1)} e^{-\frac{T}{2}\lambda_k} + \sum_{k\geq k_1+2} \lambda_k^{2(q_1+1)} e^{-\frac{T}{2}\lambda_k}. \tag{4.15}$$

For any $1 \leq k \leq k_1 - 1$, we have

$$\int_{\lambda_k}^{\lambda_{k+1}} \lambda^{2(q_1+1)} e^{-\frac{T}{2}\lambda} d\lambda \geq (\lambda_{k+1} - \lambda_k)\lambda_k^{2(q_1+1)} e^{-\frac{T}{2}\lambda_k} \geq \lambda_k^{2(q_1+1)} e^{-\frac{T}{2}\lambda_k}, \tag{4.16}$$

and for any $k \geq k_1 + 2$, it holds that

$$\int_{\lambda_{k-1}}^{\lambda_k} \lambda^{2(q_1+1)} e^{-\frac{T}{2}\lambda} d\lambda \geq (\lambda_k - \lambda_{k-1})\lambda_k^{2(q_1+1)} e^{-\frac{T}{2}\lambda_k} \geq \lambda_k^{2(q_1+1)} e^{-\frac{T}{2}\lambda_k}. \tag{4.17}$$

Therefore, we obtain that

$$\sum_{k=1}^{\infty}\lambda_k^{2(q_1+1)}e^{-\frac{T}{2}\lambda_k}\leq 2\int_0^{\infty}\lambda^{2(q_1+1)}e^{-\frac{T}{2}\lambda}d\lambda+\sum_{k_1\leq k\leq k_1+1}\lambda_k^{2(q_1+1)}e^{-\frac{T}{2}\lambda_k}. \tag{4.18}$$

Recalling that g has a maximum at $\lambda=\frac{4(q_1+1)}{T}$, we have that

$$\lambda_k^{2(q_1+1)}e^{-\frac{T}{2}\lambda_k}\leq\left(\frac{4(q_1+1)}{T}\right)^{2(q_1+1)}e^{-2(q_1+1)},\quad \text{for } k=k_1,k_1+1. \tag{4.19}$$

Moreover, we can rewrite the integral term in (4.18) as

$$\begin{aligned}\int_0^{\infty}\lambda^{2(q_1+1)}e^{-\frac{T}{2}\lambda}d\lambda&=\frac{2}{T}\int_0^{\infty}\left(\frac{2s}{T}\right)^{2(q_1+1)}e^{-s}ds\\&=\left(\frac{2}{T}\right)^{2q_1+3}\int_0^{\infty}s^{2(q_1+1)}e^{-s}ds=\Gamma(2q_1+3)\left(\frac{2}{T}\right)^{2q_1+3},\end{aligned} \tag{4.20}$$

where by $\Gamma(\cdot)$ we indicate the Euler integral of the second kind. Therefore, thanks to (4.19) and (4.20) we conclude that there exist two positive constants $C_{q_1}^1, C_{q_1}^2$ such that

$$\sum_{k=1}^{\infty}\lambda_k^{2(q_1+1)}e^{-\frac{T}{2}\lambda_k}\leq\frac{C_{q_1}^1}{T^{2q_1+2}}+\frac{C_{q_1}^2}{T^{2q_1+3}}. \tag{4.21}$$

From the above estimate we can prove that there exist positive constants C for which

$$\|r_1\|_{L^2(-T/2,T/2)}\leq Ce^{C/T}\|\xi_0\|_{L^2},\quad \forall\, 0<T<\min\{T_0,1\}, \tag{4.22}$$

and thus

$$\|(\tilde{v}_1)'\|_{L^2(0,T)}\leq e^{T/2}\|r_1\|_{L^2(-T/2,T/2)}\leq Ce^{C/T}\|y_0\|,\quad \forall\, 0<T<\min\{T_0,1\}. \tag{4.23}$$

From the Poincaré inequality, there exists a constant $C>0$ such that for any $\tilde{v}\in H_0^1(0,T)$ it holds that $\|\tilde{v}_1\|_{H^1(0,T)}\leq C\|(\tilde{v}_1)'\|_{L^2(0,T)}$, and we deduce that

$$\|\tilde{v}_1\|_{H^1(0,T)}\leq C\|(\tilde{v}_1)'\|_{L^2(0,T)}\leq Ce^{C/T}\|\xi_0\|_{L^2},\quad \forall\, 0<T<\min\{T_0,1\}. \tag{4.24}$$

Finally, recalling that $v_1(t)=e^{-p\kappa t}\tilde{v}_1(t)$, we conclude that

$$\begin{aligned}\|v_1\|_{H^1(0,T)}^2&=\int_0^T e^{-2p\kappa t}|\tilde{v}_1(t)|^2dt+\int_0^T\left(\left(e^{-2pkt}|\tilde{v}_1(t)|\right)'\right)^2dt\\&\leq C\|\tilde{v}_1\|_{H^1(0,T)}^2\leq Ce^{C/T}\|\xi_0\|_{L^2}^2,\end{aligned}$$

for all $0<T\leq\min\{T_0,1\}$. The same computations are also valid for the second component of the control v_2 and the proof is then completed. □

4.2 Proof of Main Theorem B

We now prove Theorem 4.3.

Proof of Theorem 4.3. (1) **Time decomposition and preliminaries.** Let $T>0$ and define

$$T_f:=\min\{T,\frac{\pi^2}{6},\frac{\pi^2}{6}T_0\},$$

where T_0 is the constant in (4.5). Let T_1 be defined as follows

$$T_1:=\frac{6}{\pi^2}T_f.$$

Observe that, with this choice $0 < T_1 \leq 1$. We now defined the sequences

$$T_j := \frac{T_1}{j^2}, \quad j \geq 1 \qquad \text{and} \qquad \tau_n := \sum_{j=1}^{n} T_j, \quad n \geq 0,$$

with the convention $\sum_{j=1}^{0} T_j = 0$. It is easy to prove that

$$\sum_{j=1}^{\infty} T_j = T_f$$

and thus we will perform an iterative control procedure on the consecutive time intervals $[\tau_n, \tau_{n+1}]$, $n \geq 0$ so that at the limit $n \to \infty$ we will prove exact controllability of (NHE) to the ground state solution in time T_f.

(2) **Estimates in the first time step: inductive basis.** Let us set $y = \psi - \Phi$, where ψ and Φ are solutions of (NHE) associated with the initial conditions $\psi_0, \Phi \in H^3(\mathbb{T})$ and controls $u \in H^1_{loc}(\mathbb{R}^+, \mathbb{R}^{q+2})$ and $\hat{u} := (u_\kappa, 0, 0, \dots, 0)$ with $u_\kappa = \frac{\kappa}{\Phi^p}$, respectively. Consider the equation satisfied by y in a general time interval (s_0, s_1):

$$\begin{cases} \partial_t y(t) - \partial_x^2 y(t) + \kappa (y(t) + \Phi)^{p+1} = \langle u(t), Q \rangle y(t) + \langle u(t), Q \rangle \Phi, \quad t \in (s_0, s_1) \\ y(s_0) = y_{s_0} = \psi(s_0) - \Phi. \end{cases} \tag{4.25}$$

Our aim is to prove that system (4.25) is null controllable in time T_f by means of a bilinear control u. This would imply that $\psi(T_f; \psi_0, 0, u) = \Phi$. For this purpose, we shall first consider problem (4.25) and its linearization (4.4), with initial condition $\xi_0 = y_0 := \psi_0 - \Phi$, in the time interval $[s_0, s_1] = [\tau_0, \tau_1] = [0, T_1]$. Observe that, thanks to the regularity of ψ_0 and u, the solution of (NHE) is globally in time well-defined (see Proposition 2.2), as well as the solution of the associated linear system

$$\begin{cases} \partial_t \xi(t) - \partial_x^2 \xi(t) + \kappa p \xi(t) = \langle v^1(t), Q \rangle \Phi, \quad t \in (\tau_0, \tau_1) \\ \xi(\tau_0) = \xi_0, \end{cases} \tag{4.26}$$

(see Remark 4.5).

From Proposition 4.6 we deduce that there exists a control, defined as $v^1 = (0, \dots, 0, v_1^1, v_2^1)$, with $(v_1^1, v_2^1) \in H_0^1((\tau_0, \tau_1), \mathbb{R}^2)$ such that

$$\xi(\tau_1; \tau_0, \xi_0, v^1) = 0,$$

and

$$\left\| v^1 \right\|_{H^1(\tau_0, \tau_1)} \leq N(\tau_1) \left\| \xi_0 \right\|_{L^2} = N(\tau_1) \left\| y_0 \right\|_{L^2} \leq N(\tau_1) \left\| y_0 \right\|_{H^1}, \tag{4.27}$$

with $N(\tau_1)$ that satisfies (4.5) (because $\tau_1 \leq T_0$). We now define

$$u^1 = (u_\kappa, 0, \dots, 0, v_1, v_2) \in H^1((\tau_0, \tau_1), \mathbb{R}^{q+2}).$$

Using such control in equation (NHE) (and so in (4.25) in the time interval $[\tau_0, \tau_1]$), one easily finds that (4.25) reads as

$$\begin{cases} \partial_t y(t) - \partial_x^2 y(t) + \kappa \displaystyle\sum_{j=2}^{p+1} \binom{p+1}{j} y^j(t) \Phi^{p+1-j} = \langle v^1(t), Q \rangle y(t) + \langle v^1(t), Q \rangle \Phi, \quad t \in (\tau_0, \tau_1) \\ y(\tau_0) = y_0. \end{cases} \tag{4.28}$$

We recall that $y_0 \in H^3(\mathbb{T}, \mathbb{R})$ and that $\mu_1, \mu_2 \in H^3(\mathbb{T}, \mathbb{R})$, thanks to Assumptions II. Hence, from the definition of $y = \psi - \Phi$ and since ψ and Φ are both solutions of (NHE), we deduce from *i.* of Proposition 2.1 that

$$\sup_{\tau_0 \leq t \leq \tau_1} \left\| y(t) \right\|_{H^1} = \sup_{\tau_0 \leq t \leq \tau_1} \left\| \psi(t) - \Phi \right\|_{H^1} \leq C \left(\left\| \psi_0 - \Phi \right\|_{H^1} + \left\| u^1 - \hat{u} \right\|_{L^2(\tau_1, \tau_1)} \right) \leq C \left(\left\| y_0 \right\|_{H^1} + \left\| v^1 \right\|_{H^1(\tau_1, \tau_1)} \right). \tag{4.29}$$

Thanks to (4.27), we conclude that

$$\sup_{\tau_0 \leq t \leq \tau_1} \left\| y(t) \right\|_{H^1} \leq C \left(1 + N(\tau_1) \right) \left\| y_0 \right\|_{H^1}.$$

Now, we introduce $w := y - \xi$ (we use the control u^1 for y and v^1 for ξ on the time interval $[\tau_0, \tau_1]$) and we observe that w solves the following problem

$$\begin{cases} \partial_t w(t) - \partial_x^2 w(t) - \kappa(p+1)\Phi^p w(t) + \kappa \displaystyle\sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} y^j(t) = \langle v^1(t), Q\rangle y(t), \quad t \in (\tau_0, \tau_1) \\ w(\tau_0) = 0. \end{cases} \tag{4.30}$$

Thanks to estimate (C.3) of Proposition C.1, we have that

$$\sup_{t\in[\tau_0,\tau_1]} \|w(t)\|_{H^1} \le A_4(\tau_1, \|y_0\|_{H^1}) \|y_0\|_{H^1}^2 . \tag{4.31}$$

Observe that our initial condition ψ_0 satisfies $\psi_0 \in \{\psi \in H^3(\mathbb{T}, \mathbb{R}) : \|\psi - \Phi\|_{H^1} < R_T\}$, with R_T defined in (4.3). Therefore, since $R_T < 1$, we have that $\|y_0\|_{H^1} < 1$ and moreover

$$N(\tau_1) \|y_0\|_{H^1} \le e^{\nu/\tau_1} e^{-6\Gamma_0/\tau_1} < 1$$

since $\Gamma_0 > \nu$ (see definition (4.2)). Hence, we obtain that

$$A_4(\tau_1, \|y_0\|_{H^1}) \le K(\tau_1) \tag{4.32}$$

where

$$K^2(\tau) := 2\left(2\kappa\tau(p+1)^2 \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \left(1 + N(\tau)^4\right) + C_Q^2 N(\tau)^2 \left(1 + N(\tau)^2\right)\right) \cdot e^{\tau\left(2\kappa(p+1)\Phi^p + \kappa(p+1)\sum_{j=2}^{p+1} \Phi^{p+1-j} + 1\right)}. \tag{4.33}$$

Remark 4.7. Notice that for any $\tau \le 1$ it is possible to prove that

$$K(\tau) \le e^{\Gamma_0/\tau}. \tag{4.34}$$

Since $\tau_1 \le 1$, we deduce that (4.34) holds for $\tau = \tau_1$. Using (4.32) and (4.34) (for $\tau = \tau_1$) in (4.31) and recalling that $\xi(\tau_1; \tau_0, \xi_0, v^1) = 0$, we conclude that

$$\left\|y(\tau_1; \tau_0, y_0, u^1)\right\|_{H^1} \le e^{\Gamma_0/\tau_1} e^{-12\Gamma_0/\tau_1} = e^{-11\Gamma_0/\tau_1} < 1. \tag{4.35}$$

(3) **Induction argument.** Inequality (4.36) enables us to apply an iterative argument. In fact, we have just proved the first step of an induction procedure which consists in building in consecutive time intervals of the form $[\tau_{n-1}, \tau_n]$, $n \ge 1$, a control $u^n = (u_\kappa, 0, ..., 0, u_1^n, u_2^n) \in H^1((\tau_{n-1}, \tau_n), \mathbb{R}^{q+2})$ such that

$$\begin{array}{ll} 1. & \|u^n\|_{H^1(\tau_{n-1},\tau_n)} \le N(T_n) \|y_{n-1}\|_{H^1} , \\ 2. & \xi(\tau_n; \tau_{n-1}, y_{n-1}, v^n) = 0, \\ 3. & \|y(\tau_n; \tau_{n-1}, y_{n-1}, u^n)\|_{H^1} \le e^{\left(\sum_{j=1}^n 2^{n-j} j^2 - 2^n 6\right)\Gamma_0/T_1}, \\ 4. & \|y(\tau_n; \tau_{n-1}, y_{n-1}, u^n)\|_{H^1} \le \prod_{j=1}^n K(T_j)^{2^{n-j}} \|y_0\|_{H^1}^{2^n} , \end{array} \tag{4.36}$$

where $v^n := u^n - \hat{u}$ and, thanks to Proposition 2.2,

$$y_{n-1} := y(\tau_{n-1}, 0, y_0, q^{n-1}) \in H^3(\mathbb{T}, \mathbb{R}), \tag{4.37}$$

$$q^{n-1}(t) = \sum_{j=1}^{n-1} u^j(t) \chi_{[\tau_{j-1},\tau_j]}(t) \qquad \text{(component-wise)}. \tag{4.38}$$

Observe that, by construction

$$y_n = y(\tau_n; \tau_{n-1}, y_{n-1}, u^n) \in H^3(\mathbb{T}, \mathbb{R}), \qquad \forall\, n \ge 1.$$

We underline that, thanks to the global well-posedness of the solution of (NHE) (and so of (4.25)), we do not meet any problem of existence when changing at each step of the proof the initial condition, the control and the time interval.

Let us now prove the iterative step of the induction argument. We suppose properties 1.–4. of (4.36) hold for each $j = 1, \dots, n-1$. Hence, suppose we have built controls $u^j = (u_\kappa, 0, ..., u_1^j, u_2^j) \in H^1((\tau_{j-1}, \tau_j), \mathbb{R}^{q+2})$ such that 1.–4. of (4.36) are satisfied. In particular, for $j = n-1$, assume that

$$\begin{aligned} &1. \quad \left\|u^{n-1}\right\|_{H^1(\tau_{n-2},\tau_{n-1})} \leq N(T_{n-1}) \left\|y_{n-2}\right\|_{H^1} \\ &2. \quad \xi(\tau_{n-1}; \tau_{n-2}, y_{n-2}, v^{n-1}) = 0 \\ &3. \quad \left\|y(\tau_{n-1}; \tau_{n-2}, y_{n-2}, u^{n-1})\right\|_{H^1} \leq e^{(\sum_{j=1}^{n-1} 2^{n-1-j} j^2 - 2^{n-1} 6)\Gamma_0/T_1} \\ &4. \quad \left\|y(\tau_{n-1}; \tau_{n-2}, y_{n-2}, u^{n-1})\right\|_{H^1} \leq \prod_{j=1}^{n-1} K(T_j)^{2^{n-1-j}} \left\|y_0\right\|_{H^1}^{2^{n-1}} . \end{aligned} \tag{4.39}$$

We now prove the existence of $u^n = (u_\kappa, 0, ..., 0, u_1^n, u_2^n) \in H^1((\tau_{n-1}, \tau_n), \mathbb{R}^{q+2})$ such that 1.–4. of (4.36) are fulfilled. Define q^{n-1} and y_{n-1} as in (4.38) and (4.37). We apply Proposition 4.6 to the linear system (4.4) with $(s_0, s_1) = (0, T_n)$ and $\xi(0) = y_{n-1}$. The result ensures the existence of a control $\tilde{v}^n = (0, ..., 0, \tilde{u}_1^n, \tilde{u}_2^n) \in H_0^1((0, T_n), \mathbb{R}^{q+2})$ such verifies

$$\left\|\tilde{v}^n\right\|_{H^1(0,T_n)} \leq N(T_n) \left\|y_{n-1}\right\|_{H^1} ,$$

with $N(T_n) \leq e^{\nu/T_n}$ and

$$\xi(T_n; 0, y_{n-1}, \tilde{v}^n) = 0.$$

We set $\tilde{u}^n(s) := \hat{u} + \tilde{v}^n(s) = (u_\kappa, 0, ..., 0, \tilde{u}_1^n, \tilde{u}_2^n)$ with $s \in [0, T_n]$. Consider problem (4.25) for $(s_0, s_1) = (0, T_n)$ with $y(0) = y_{n-1}$. By using the control $\tilde{u}^n$ in (4.25), we get (4.28) in $[0, T_n]$ with control $\tilde{v}^n$ instead of v^1. We denote by $y(s; 0, y_{n-1}, \tilde{u}^n)$ with $s \in [0, T_n]$ its solution. Observe that, since

$$y_{n-1} = y(\tau_{n-1}; 0, y_0, q_{n-1}) = y(\tau_{n-1}; \tau_{n-2}, y_{n-2}, u^{n-1}) \in H^3(\mathbb{T}, \mathbb{R}),$$

from 3. of (4.39) we deduce that

$$N(T_n) \left\|y_{n-1}\right\|_{H^1} \leq e^{\nu n^2/T_1} e^{(\sum_{j=1}^{n-1} 2^{n-1-j} j^2 - 2^{n-1} 6)\Gamma_0/T_1} \leq e^{-(2n+3)\Gamma_0/T_1} < 1, \tag{4.40}$$

where we have used that $\nu < \Gamma_0$ and the identity

$$\sum_{j=0}^{n} \frac{j^2}{2^j} = 2^{-n}(-n^2 - 4n + 6(2^n - 1)), \qquad \forall\, n \geq 0.$$

Now, for $s \in [0, T_n]$, we define

$$w(s) := y(s; 0, y_{n-1}, \tilde{u}^n) - \xi(s; 0, y_{n-1}, \tilde{v}^n).$$

Thus, we apply Proposition C.1, and we deduce that

$$\left\|y(T_n; 0, y_{n-1}, \tilde{u}^n)\right\|_{H^1} = \left\|w(T_n; 0, \tilde{v}^n)\right\|_{H^1} \leq A_4(T_n; \left\|y_{n-1}\right\|_{H^1}) \left\|y_{n-1}\right\|_{H^1}^2 .$$

We shift forward time interval into $[\tau_{n-1}, \tau_n]$ and we define

$$u^n(t) := \tilde{u}^n(t - \tau_{n-1}), \qquad v^n(t) := \tilde{v}^n(t - \tau_{n-1}), \qquad t \in (\tau_{n-1}, \tau_n),$$

and we obtain

$$\left\|u^n\right\|_{H^1(\tau_{n-1},\tau_n)} \leq N(T_n) \left\|y_{n-1}\right\|_{H^1} ,$$

and

$$\xi(\tau_n; \tau_{n-1}, y_{n-1}, v^n) = \xi(T_n; 0, y_{n-1}, \tilde{v}^n) = 0.$$

Thus, 1. and 2. of (4.36) are fulfilled.

Recalling definition (4.33), property (4.40) and $\nu < \Gamma_0$, we deduce that

$$\left\|y(\tau_n; \tau_{n-1}, y_{n-1}, u^n)\right\|_{H^1} = \left\|y(T_n; 0, y_{n-1}, \tilde{u}^n)\right\|_{H^1} \leq K(T_n) \left\|y_{n-1}\right\|_{H^1}^2 . \tag{4.41}$$

Thus, using 3. of (4.39) and the estimate above, we obtain

$$\left\|y(\tau_n; \tau_{n-1}, y_{n-1}, u^n)\right\|_{H^1} \leq e^{n^2\Gamma_0/T_1} \left[e^{(\sum_{j=1}^{n-1} 2^{n-1-j} j^2 - 2^{n-1} 6)\Gamma_0/T_1}\right]^2 = e^{(\sum_{j=1}^{n} 2^{n-1-j} j^2 - 2^{n-1} 6)\Gamma_0/T_1},$$

that is, 3. of (4.36) is satisfied. Finally, using again (4.41) and thanks to 4. of (4.39) we conclude that

$$\|y(\tau_n;\tau_{n-1},y_{n-1},u^n)\|_{H^1} \le K(T_n)\left[\prod_{j=1}^{n-1} K(T_j)^{2^{n-1-j}}\,\|y_0\|_{H^1}^{2^{n-1}}\right]^2 = \prod_{j=1}^{n} K(T_j)^{2^{n-j}}\,\|y_0\|_{H^1}^{2^n},$$

which is exactly 4. of (4.36). The induction argument is therefore concluded.

(4) **Conclusion.** Let us now observe that, from 4. of (4.36), for every $n\in\mathbb{N}$ it holds that

$$\begin{aligned}\|y(\tau_n;\tau_{n-1},y_{n-1},u^n)\|_{H^1} &\le \prod_{j=1}^{n} K(T_j)^{2^{n-j}}\,\|y_0\|_{H^1}^{2^n} \le \prod_{j=1}^{n}\left(e^{\Gamma_0 j^2/T_1}\right)^{2^{n-j}}\|y_0\|_{H^1}^{2^n}\\ &= e^{(\Gamma_0 2^n/T_1)\sum_{j=1}^{n} j^2/2^j}\,\|y_0\|_{H^1}^{2^n} \le e^{(\Gamma_0 2^n/T_1)\sum_{j=1}^{\infty} j^2/2^j}\,\|y_0\|_{H^1}^{2^n}\\ &\le \left(e^{6\Gamma_0/T_1}\,\|y_0\|_{H^1}\right)^{2^n},\end{aligned}$$

where in the last inequality we have used that $\sum_{j=1}^{\infty}\frac{j^2}{2^j}=6$. By definition, the above estimate reads as

$$\|y(\tau_n;0,y_0,q^n)\|_{H^1} \le \left(e^{6\Gamma_0/T_1}\,\|y_0\|_{H^1}\right)^{2^n},$$

with

$$q^n(t)=\sum_{j=1}^{n} u^j(t)\chi_{(\tau_{j-1},\tau_j)}(t) \qquad \text{(component-wise)}.$$

Taking the limit as $n\to+\infty$, we deduce that

$$\left\|\psi(\pi^2T_1/6;0,\psi_0,q^\infty)-\Phi\right\|_{H^1} = \left\|y(\pi^2T_1/6;0,\psi_0,q^\infty)\right\|_{H^1} = \|y(T_f;0,\psi_0,q^\infty)\|_{H^1} \le 0$$

thanks to the definition (4.3) of R_T. Thus, we construct a control $u\in H^1_{loc}(\mathbb{R}^+,\mathbb{R}^{q+2})$

$$u(t):=\begin{cases}\sum_{j=1}^{\infty} u^j(t)\chi_{(\tau_{j-1},\tau_j)}(t), & 0<t\le T_f,\\ 0 & t>T_f,\end{cases} \qquad \text{(component-wise)},$$

such that, at time $T_f\le T$, the solution of (NHE) reaches exactly the ground state, that is,

$$\psi(T_f;0,\psi^0,u)=\Phi.$$

Furthermore, we can derive a bound for the H^1-norm of the control

$$\|u\|^2_{H^1(0,T_f)} \le \sum_{n=1}^{\infty}\|u^n\|^2_{H^1(0,T_f)} \le \sum_{n=1}^{\infty}\left(N(T_n)\,\|y_{n-1}\|_{H^1}\right)^2 \le \sum_{n=1}^{\infty} e^{-2(2n+3)\Gamma_0/T_1} = \frac{e^{-\pi^2\Gamma_0/T_f}}{e^{2\pi^1\Gamma_0/(3T_f)}-1}.$$

where we have used (4.40). Since $T_f\le T$, we easily obtain (4.1). □

We are finally ready to prove Theorem 4.1 and Main Theorem B.

Proof of Theorem 4.1. The result is obtained by combining Theorem 3.3 with $s=3$ and Theorem 4.3. In details, for every $\psi_0\in H^3(\mathbb{T},\mathbb{R})$ strictly positive, Theorem 3.3 allows defining a dynamics steering ψ_0 close to Φ in the H^3 metric in an arbitrarily small time. If we are close enough to Φ, we can use the local controllability result of Theorem 4.3. Thus, the solution of (NHE) reaches the target Φ in any finite time. □

Proof of Main Theorem B. Main Theorem B is a direct consequence of Theorem 4.1 and the density property of $\mathcal{H}_\infty$ in $H^1(\mathbb{T},\mathbb{R})$ ensured by Proposition 3.1. □

Appendices

A Proof of Proposition 2.1

In this section, we shall prove the existence and uniqueness of solutions of (NHE).

Proof. (1) **Existence and uniqueness of solutions.** Let $s > d/2$ and $\psi_0 \in H^s(\mathbb{T}^d, \mathbb{R})$. For the sake of shortness, we consider $\kappa = 1$. However, the proof remains valid in the general case. We are going to show that there exists $t_1 > 0$ such that the Cauchy problem (NHE) admits a unique solution $\psi \in C^0([0, t_1], H^s(\mathbb{T}^d, \mathbb{R}))$. We define the following quantities

$$M := \sup\{\left\|e^{t\Delta} - I\right\|_{\mathcal{L}(H^s(\mathbb{T}^d,\mathbb{R}))}, 0 \le t \le 1\}, \qquad r(\psi_0) := 2M \left\|\psi_0\right\|_{H^s}$$

$$C(Q) := \max_{1\le i\le d} \|Q_i\|_{H^s}, \qquad C(d, u, Q) := \sqrt{2d}\, \|u\|_{L^2(0,1)}\, C(Q).$$

Observe that, since $s > d/2$, we deduce that the embedding $H^s(\mathbb{T}^d, \mathbb{R}) \hookrightarrow C^0(\mathbb{T}^d, \mathbb{R})$ is continuous, that is, there exists a constant $C(\mathbb{T}^d)$ such that

$$\sup_{x\in\mathbb{T}^d} |y(x)| \le C(\mathbb{T}^d) \|y\|_{H^s}, \quad \forall\, y \in H^s(\mathbb{T}^d, \mathbb{R}). \tag{A.1}$$

We now define

$$\delta := \min\left\{1, \frac{(r(\psi_0))^2}{4C(\mathbb{T}^d)^2 \left(C(d, u, Q)(r(\psi_0) + \|\psi_0\|_{H^s}) + 2^p((r(\psi_0))^{p+1} + \|\psi_0\|_{H^s}^{p+1})\right)^2}\right\}, \tag{A.2}$$

and set $t_1 = \delta$.

We denote $B := B_{C^0([0,t_1],H^s))}(\psi_0, r(\psi_0))$ the ball in the space $C^0([0, t_1], H^s(\mathbb{T}^d, \mathbb{R}))$ of center ψ_0 and radius $r(\psi_0)$. For every $\psi \in B$ we define the following function

$$\Phi(\psi)(t) := e^{t\Delta}\psi_0 + \int_0^t e^{(t-s)\Delta} \left(\langle u(s), Q(x)\rangle \psi(s, x) + \psi(s, x)^{p+1}\right) ds. \tag{A.3}$$

Let us show that Φ maps B into itself:

$$\begin{aligned}
\|\Phi(\psi)(t) - \psi_0\|_{H^s} &\le \left\|e^{t\Delta}\psi_0 - \psi_0\right\|_{H^s} + \left\|\int_0^t e^{(t-s)\Delta} \left(\langle u(s), Q\rangle \psi(s) + \psi(s)^{p+1}\right) ds\right\|_{H^s} \\
&\le M \|\psi_0\|_{H^s} + \int_0^t \|\langle u(s), Q\rangle \psi(s)\|_{H^s} + \left\|\psi(s)^{p+1}\right\|_{H^s} ds \\
&\le M \|\psi_0\|_{H^s} + C(\mathbb{T}^d) \left(C(Q) \int_0^t \sum_{i=1}^q |u_i(s)| \|\psi(s)\|_{H^s} ds + \int_0^t \|\psi(s)\|_{H^s}^{p+1} ds\right) \\
&\le M \|\psi_0\|_{H^s} + C(\mathbb{T}^d)\sqrt{d}C(Q) \left(\int_0^t \sum_{i=1}^q |u_i(s)|^2 ds\right)^{1/2} \left(2\int_0^t \|\psi(s) - \psi_0\|_{H^s}^2 + \|\psi_0\|_{H^s}^2 ds\right)^{1/2} \\
&\quad + C(\mathbb{T}^d)2^p \int_0^t \|\psi(s) - \psi_0\|_{H^s}^{p+1} + \|\psi_0\|_{H^s}^{p+1} ds.
\end{aligned}$$

Thus, we deduce that $\Phi(\psi) \in B$ since

$$\begin{aligned}
\sup_{t\in[0,t_1]} \|\Phi(\psi)(t) - \psi_0\|_{H^s} &\le \frac{r(\psi_0)}{2} + C(\mathbb{T}^d)\sqrt{2d}C(Q) \|u\|_{L^2(0,1)} \left(\sup_{t\in[0,t_1]} \|\psi(t) - \psi_0\|_{H^s} + \|\psi_0\|_{H^s}\right) \sqrt{t_1} \\
&\quad + C(\mathbb{T}^d)2^p \left(\sup_{t\in[0,t_1]} \|\psi(t) - \psi_0\|_{H^s}^{p+1} + \|\psi_0\|_{H^s}^{p+1}\right) \sqrt{t_1} \le \frac{r(\psi_0)}{2} + \frac{r(\psi_0)}{2} = r(\psi_0).
\end{aligned}$$

Now, we show that Φ is a contraction over B. Let $\psi, \phi \in B$, then

$$\begin{aligned}
\|\Phi(\psi)(t) - \Phi(\phi)(t)\|_{H^s} &= \left\| \int_0^t \langle u(s), Q(x)\rangle(\psi(s) - \phi(s)) + \psi(s)^{p+1} - \phi(s)^{p+1} ds \right\|_{H^s} \\
&\leq \int_0^t \|\langle u(s), Q(x)\rangle(\psi(s) - \phi(s))\|_{H^s} \, ds \\
&\quad + C(\mathbb{T}^d) \int_0^t \|\psi(s) - \phi(s)\|_{H^s} \sum_{j=0}^{p} \|\psi(s)\|_{H^s}^j \, \|\phi(s)\|_{H^s}^{p-j} \, ds \\
&\leq C(Q)\sqrt{2d} \, \|u\|_{L^2((0,1),\mathbb{R}^{q+2})} \left(\int_0^t \|\psi(s) - \phi(s)\|_{H^s}^2 \, ds \right)^{1/2} \\
&\quad + C(\mathbb{T}^d) 2^{p-2} \int_0^t \|\psi(s) - \phi(s)\|_{H^s} \, L(\phi, \phi) ds
\end{aligned}$$

where

$$L(\psi, \phi) := \sum_{j=0}^{p} \left(\|\psi(s) - \psi_0\|_{H^s}^j + \|\psi_0\|_{H^s}^j \right) \left(\|\phi(s) - \psi_0\|_{H^s}^{p-j} + \|\psi_0\|_{H^s}^{p-j} \right)$$

Therefore, we get that

$$\begin{aligned}
\sup_{0\leq t\leq t_1} \|\Phi(\psi)(t) - \Phi(\phi)(t)\|_{H^s} &\leq C(Q)\sqrt{2d} \, \|u\|_{L^2((0,1)} \sqrt{t_1} \sup_{0\leq t\leq t_1} \|\psi(t) - \phi(t)\|_{H^s} \\
&\quad + C(\mathbb{T}^d) 2^{p-2} \sqrt{t_1} \sup_{0\leq t\leq t_1} L(\psi, \phi) \sup_{0\leq t\leq t_1} \|\psi(t) - \phi(t)\|_{H^s} \\
&\leq \left(C(d, u, Q) + C(\mathbb{T}^d) 2^{p-2} \tilde{L}(\psi_0) \right) \sqrt{t_1} \sup_{0\leq t\leq t_1} \|\psi(t) - \phi(t)\|_{H^s} ,
\end{aligned}$$

where

$$\tilde{L}(\psi_0) := \sum_{j=0}^{p} \left(r(\psi_0)^j + \|\psi_0\|_{H^s}^j \right) \left(r(\psi_0)^{p-j} + \|\psi_0\|_{H^s}^{p-j} \right).$$

With the same kind of computation one can prove that

$$\sup_{0\leq t\leq t_1} \left\|\Phi^2(\psi)(t) - \Phi^2(\phi)(t)\right\|_{H^s} \leq \left(C(d, u, Q) + C(\mathbb{T}^d) 2^{p-2} \tilde{L}(\psi_0) \right)^2 \frac{(\sqrt{t_1})^2}{\sqrt{2}} \sup_{0\leq t\leq t_1} \|\psi(t) - \phi(t)\|_{H^s} ,$$

and, iterating the procedure, one shows that

$$\sup_{0\leq t\leq t_1} \|\Phi^n(\psi)(t) - \Phi^n(\phi)(t)\|_{H^s} \leq \left(C(d, u, Q) + C(\mathbb{T}^d) 2^{p-2} \tilde{L}(\psi_0) \right)^n \frac{(\sqrt{t_1})^n}{\sqrt{n!}} \sup_{0\leq t\leq t_1} \|\psi(t) - \phi(t)\|_{H^s} .$$

For n large enough, it holds that

$$\left(C(d, u, Q) + C(\mathbb{T}^d) 2^{p-2} \tilde{L}(\psi_0) \right)^n \frac{(\sqrt{t_1})^n}{\sqrt{n!}} < 1.$$

Hence, form a well-known corollary of the Banach fixed point Theorem, we deduce that Φ is a contraction over B and so there exists a unique fixed point $\psi \in B$ which is the solution of (NHE). Furthermore, it holds that

$$\sup_{t\in[0,t_1]} \|\psi(t)\|_{H^s} \leq (2M + 1) \, \|\psi_0\|_{H^s} .$$

We have just proved that if ψ is solution of (NHE) in $[0, \tau]$, then we can extend it into $[0, \tau + \delta(\tau)]$. Indeed, by defining the quantities

$$M(\tau) := \sup_{\tau\leq t\leq \tau+1} \left\{ \left\| e^{t\Delta} - I \right\|_{\mathcal{L}(H^s(\mathbb{T}^d,\mathbb{R}))} \right\}, \quad r(\tau, \psi(\tau)) := 2M(\tau) \, \|\psi(\tau)\|_{H^s} , \quad C(d, u, Q) := \sqrt{2d} \, \|u\|_{L^2(\tau,\tau+1)} \, C(Q),$$

and

$$\delta(\tau) := \min \left\{ 1, \frac{(r(\psi(\tau))^2}{4C(\mathbb{T}^d)^2 \left(C(d, u, Q)(r(\psi(\tau)) + \|\psi(\tau)\|_{H^s}) + 2^p((r(\psi(\tau)))^{p+1} + \|\psi(\tau)\|_{H^s}^{p+1}) \right)^2} \right\},$$

then one just sets $\psi(t) = \zeta(t)$ for $t \in [\tau, \tau + \delta(\tau)]$, where

$$\zeta(t) = e^{(t-\tau)\Delta}\psi(\tau) + \int_\tau^t e^{(t-s)\Delta}\left(\langle u(s), Q(x)\rangle\psi(s,x) + \psi(s,x)^{p+1}\right) ds.$$

Let $[0, \mathcal{T})$ be the maximal interval of existence of ψ, solution of the (NHE). If $\mathcal{T} < +\infty$, then $\|\psi(t)\|_{H^s} \to +\infty$ as $t \to \mathcal{T}^-$, otherwise ψ could be extended, which contradicts the maximality of $\mathcal{T}$. Observe that, for any $0 < T < \mathcal{T}$,

$$\sup_{t\in[0,T]} \|\psi(t)\|_{H^s} \leq C \|\psi_0\|_{H^s}. \tag{A.4}$$

(2) **Proof of the continuity (2.1) and the stability (2.2).** Let $\psi, \phi \in C^0([0,T], H^s(\mathbb{T}^d, \mathbb{R}))$, with $0 \leq T \leq \min\{\mathcal{T}(\psi_0), \mathcal{T}(\phi_0)\}$, be the solutions of the (NHE) corresponding to the initial conditions ψ_0 and ϕ_0 and controls u and v, respectively. Then,

$$\begin{aligned}
\|\psi(t) - \phi(t)\|_{H^s} &\leq \|\psi_0 - \phi_0\|_{H^s} \\
&\quad + \int_0^t (\|\langle u(s), Q\rangle\psi(s) - \langle v(t), Q\rangle\phi(s)\|_{H^s} + \left\|\psi(s)^{p+1} - \phi(s)^{p+1}\right\|_{H^s})ds \\
&\leq \|\psi_0 - \phi_0\|_{H^s} + \int_0^t \|\langle u(s), Q\rangle(\psi(s) - \phi(s))\|_{H^s} ds + \\
&\quad + \int_0^t \|(\langle u(s), Q\rangle - \langle v(s), Q\rangle)\phi(s)\|_{H^s} ds \\
&\quad + C(\mathbb{T}^d)\int_0^t \|\psi(s) - \phi(s)\|_{H^s} \sum_{j=0}^p \|\psi(s)\|_{H^s}^j \|\phi(s)\|_{H^s}^{p-j} ds \\
&\leq \|\psi_0 - \phi_0\|_{H^s} + C(Q)\sqrt{2d}\|u\|_{L^2(0,t)}\sqrt{t}\sup_{0\leq s\leq t}\|\psi(s) - \phi(s)\|_{H^s} \\
&\quad + C(Q)\sqrt{2d}\|u - v\|_{L^2(0,t)}\sqrt{t}\sup_{0\leq s\leq t}\|\phi(s)\|_{H^s} \\
&\quad + C(\mathbb{T}^d)\sqrt{t}\left(\sum_{j=0}^p \sup_{0\leq s\leq t}\|\psi(s)\|_{H^s}^j \sup_{0\leq s\leq t}\|\phi(s)\|_{H^s}^{p-j}\right)\sup_{0\leq s\leq t}\|\psi(s) - \phi(s)\|_{H^s}
\end{aligned}$$

Therefore, we get the existence of $C_1, C_2, C_3 > 0$, only depending on the parameters of the problem, such that

$$\begin{aligned}
\sup_{0\leq t\leq T}\|\psi(t) - \phi(t)\|_{H^s} &\leq \|\psi_0 - \phi_0\|_{H^s} + \left(C(Q)\sqrt{2d}\sqrt{T}\sup_{0\leq t\leq T}\|\phi(t)\|_{H^s}\right)\|u - v\|_{L^2(0,T)} \\
&\quad + \left(C(d,u,Q) + C(\mathbb{T}^d)\sum_{j=0}^p \sup_{0\leq t\leq T}\|\psi(t)\|_{H^s}^j \sup_{0\leq t\leq T}\|\phi(t)\|_{H^s}^{p-j}\right)\sqrt{T}\sup_{0\leq t\leq T}\|\psi(t) - \phi(t)\|_{H^s} \\
&\leq \|\psi_0 - \phi_0\|_{H^s} + C_1 R\sqrt{T}\|u - v\|_{L^2(0,T)} + (C_2 + C_3 R^p)\sqrt{T}\sup_{0\leq t\leq T}\|\psi(t) - \phi(t)\|_{H^s}
\end{aligned}$$

where we have used (A.4) and that $\psi_0, \phi_0 \in B_{H^s}(0, R)$. If $(C_2 + C_3 R^p)\sqrt{T} < 1$, then we obtain the validity of (2.1) and then

$$\sup_{0\leq t\leq T}\|\psi(t) - \phi(t)\|_{H^s} \leq C\left(\|\psi_0 - \phi_0\|_{H^s} + \|u - v\|_{L^2(0,T)}\right).$$

Otherwise, we can subdivide the interval $[0, T]$ into subintervals where inequality $(C_2 + C_3 R^p)\sqrt{T} < 1$ holds and obtain the result. Finally, the same techniques also imply the validity of (2.2). □

B Global well-posedness and proof of Proposition 2.2

This appendix aims to prove Proposition 2.2. We show that for more regular initial conditions, $\psi_0 \in H^3(\mathbb{T}, \mathbb{R})$, the solution of the unidimensional problem (NHE) is global in time, that is, the maximum time of existence is $\mathcal{T}(\psi_0) = +\infty$. First, we observe that if $\psi_0 \in H^1(\mathbb{T}, \mathbb{R})$ then, from Proposition 2.1, there exists $\mathcal{T}(\psi_0)$ such that for

every $0 < T < \mathcal{T}(\psi_0)$ there exists a unique solution $\psi \in C^0([0,T], H^1(\mathbb{T},\mathbb{R}))$. Second, if $\mathcal{T}(\psi_0) = +\infty$ then the solution is global in time, otherwise

$$\|\psi(t)\|_{H^1} \to +\infty \qquad \text{as } t \to \mathcal{T}(\psi_0).$$

It can be proved, thanks to the analiticity of the semigroup generated by Δ and applying [14, Theorem 3.1, p. 143] to the fixed point argument used in the proof of Proposition 2.1, that

$$\psi \in H^1((0,T), L^2(\mathbb{T},\mathbb{R})) \cap L^2((0,T), H^2(\mathbb{T},\mathbb{R})).$$

Let us consider $\Psi := \psi_t$ that formally solves the problem

$$\begin{cases} \partial_t \Psi - \Delta\Psi + \kappa(p+1)\psi^p\Psi = \langle u'(t), Q(x)\rangle\psi + \langle u(t), Q(x)\rangle\Psi \\ \Psi(0) = \Psi_0 := \Delta\psi_0 - \kappa\,\psi_0^{p+1} + \langle u(0), Q(x)\rangle\psi_0. \end{cases} \tag{B.1}$$

Let us now proved the following result which is necessary for the proof of Proposition 2.2.

Proposition B.1. *Let $\psi_0 \in H^3(\mathbb{T},\mathbb{R})$, $Q \in H^3(\mathbb{T},\mathbb{R}^{q+2})$, $u \in H^1_{loc}((0,+\infty),\mathbb{R}^{q+2})$, $0 < T < \mathcal{T}(\psi_0, u)$ and $\kappa \geq 0$. Then, there exists a unique solution $\Psi \in L^2((0,T), H^1(\mathbb{T},\mathbb{R}))$ of (B.1) given by*

$$\Psi(t) = e^{t\Delta}\Psi_0 + \int_0^t e^{(t-s)\Delta}\left(\langle u'(s), Q(x)\rangle\psi(s) + \langle u(s), Q(x)\rangle\Psi(s) - \kappa(p+1)\psi(s)^p\Psi(s)\right) ds,$$

where ψ is the unique solution of (NHE) with initial condition ψ_0. Furthermore, it holds that $\Psi = \psi_t$.

Proof. (1) **Existence and uniqueness of solutions.** We already know from the local well-posedness of (NHE) that $\Psi = \psi_t \in L^2((0,T), L^2(\mathbb{T},\mathbb{R}))$. We now consider equation (B.1). In order to apply a fix point argument, for every $\xi \in C([0,T], H^1(\mathbb{T},\mathbb{R}))$, we consider the following map

$$\Phi(\xi)(t) := e^{t\Delta}\Psi_0 + \int_0^t e^{(t-s)\Delta}\Big(\langle u'(s), Q(x)\rangle\psi(s) + \langle u(s), Q(x)\rangle\xi(s) - \kappa(p+1)\psi(s)^p\xi(s)\Big) ds,$$

where $\psi \in H^1((0,T), L^2(\mathbb{T},\mathbb{R})) \cap L^2((0,T), H^2(\mathbb{T},\mathbb{R})) \cap C([0,T], H^3(\mathbb{T},\mathbb{R}))$ is the solution of (NHE) with initial condition $\psi_0 \in H^3(\mathbb{T},\mathbb{R})$. We first observe that $\Psi_0 = \Delta\psi_0 - \kappa\psi_0^p\psi_0 + \langle u(0), Q(x)\rangle\psi_0$ is well-defined and is in $H^1(\mathbb{T},\mathbb{R})$. Let us first prove that Φ maps $C([0,T], H^1(\mathbb{T},\mathbb{R}))$ into itself. Since

$$f(\cdot) := \langle u'(\cdot), Q\rangle\psi + \langle u(\cdot), Q\rangle\xi(\cdot) - \kappa(p+1)\psi(\cdot)^p\xi(\cdot) \in L^2((0,T), L^2(\mathbb{T},\mathbb{R})),$$

from [14, Theorem 3.1, p. 143] we deduce that $\Phi(\xi) \in H^1([0,T], L^2(\mathbb{T},\mathbb{R})) \cap L^2((0,T), H^2(\mathbb{T},\mathbb{R}))$ for every $\xi \in C([0,T], H^1(\mathbb{T},\mathbb{R}))$. Thus, we deduce from [32, Proposition 2.1, p. 22 and Theorem 3.1, p. 23] that Φ maps $C([0,T], H^1(\mathbb{T},\mathbb{R}))$ into itself. We now prove that Φ is a contraction. Let $\xi, \tilde{\xi} \in C([0,T], H^1(\mathbb{T},\mathbb{R}))$. Then,

$$\begin{aligned}
&\sup_{t\in[0,T]} \left\|\Phi(\xi)(t) - \Phi(\tilde{\xi})\right\|_{H^1} = \sup_{t\in[0,T]} \left\|\int_0^t e^{(t-s)\Delta}\left(\langle u(s), Q(x)\rangle(\xi(s) - \tilde{\xi}(s)) - \kappa(p+1)\psi(s)^p(\xi(s) - \tilde{\xi}(s))\right) ds\right\|_{H^1} \\
&\leq C\left\|(\langle u(\cdot), Q\rangle - \kappa(p+1)|\psi(s)|^p)\,(\xi(\cdot) - \tilde{\xi}(\cdot))\right\|_{L^2((0,T),L^2(\mathbb{T}))} \leq C\left(\int_0^t \left\|\xi(s) - \tilde{\xi}(s)\right\|_{L^2}^2 ds\right)^{1/2} \\
&\leq C\sqrt{T}\sup_{t\in[0,T]}\left\|\xi(t) - \tilde{\xi}(t)\right\|_{L^2} \leq C\sqrt{T}\sup_{t\in[0,T]}\left\|\xi(t) - \tilde{\xi}(t)\right\|_{H^1}
\end{aligned}$$

If $C\sqrt{T} < 1$ then Φ is a contraction. Otherwise, one can divide the interval $[0,T]$ in a finite number of sub-intervals where Φ is a contraction and conclude the argument. So, we deduce that Φ admits a unique fix point Ψ in the space $C([0,T], H^1(\mathbb{T},\mathbb{R}))$ which is the solution of (B.1).

(1) **Proof of the identity $\Psi = \psi_t$.** We now prove that the unique solution of (B.1) is indeed ψ_t. Let $t \in [0,T]$, $\tau > 0$ such that $0 < t + \tau < T$, and consider the difference

$$\frac{\psi(t+\tau) - \psi(t)}{\tau} - \Psi(t).$$

Our aim is to prove that the above quantity converges to 0 as $\tau \to 0$. Thanks to the expression of ψ and Ψ as mild solutions, we get

$$
\begin{aligned}
\frac{\psi(t+\tau)-\psi(t)}{\tau} - \Psi(t) &= \frac{1}{\tau}\left\{e^{(t+\tau)\Delta}\psi_0 - e^{t\Delta}\psi_0\right\} + \frac{1}{\tau}\int_0^\tau e^{(t+\tau-s)\Delta}\left[-\kappa\,\psi(s)^{p+1} + \langle u(s), Q\rangle\psi(s)\right] ds \\
&+ \frac{1}{\tau}\int_0^t e^{(t-s)\Delta}\left[-\kappa\,\psi(s+\tau)^{p+1} + \langle u(s+\tau), Q\rangle\psi(s+\tau) + \kappa\,\psi(s)^{p+1} - \langle u(s), Q\rangle\psi(s)\right] ds \\
&- e^{t\Delta}(\Delta\psi_0 - \kappa\,\psi_0^{p+1} + \langle u(0), Q\rangle\psi_0) - \int_0^t e^{(t-s)\Delta}\left(\langle u'(s), Q(x)\rangle\psi(s) + \langle u(s), Q(x)\rangle\xi(s) - \kappa(p+1)\psi(s)^p\xi(s)\right) ds \\
&= \frac{1}{\tau}\left\{e^{(t+\tau)\Delta}\psi_0 - e^{t\Delta}\psi_0\right\} - e^{t\Delta}\psi_0 - e^{t\Delta}(-\kappa\,\psi_0^{p+1} + \langle u(0), Q\rangle\psi_0)) \\
&+ \frac{1}{\tau}\int_0^\tau e^{(t+\tau-s)\Delta}\left[-\kappa\,\psi(s)^{p+1} + \langle u(s), Q\rangle\psi(s)\right] ds + \int_0^t e^{(t-s)\Delta}\left[-\kappa\,\psi(s+\tau)^p\left(\frac{\psi(s+\tau)-\psi(s)}{\tau} - \Psi(s)\right)\right] ds \\
&+ \int_0^t e^{(t-s)\Delta}\left[-\kappa\left[p\,\psi(s)^p\left(\frac{\psi(s+\tau)-\psi(s)}{\tau} - \Psi(s)\right) + o\left(\frac{\psi(s+\tau)-\psi(s)}{\tau}\right)\right]\right] ds \\
&+ \int_0^t e^{(t-s)\Delta}\left[\langle\frac{u(s+\tau)-u(s)}{\tau}, Q\rangle\psi(s+\tau) + \langle u(s+\tau), Q\rangle\left(\frac{\psi(s+\tau)-\psi(s)}{\tau} - \Psi(s)\right)\right] ds \\
&+ \int_0^t e^{(t-s)\Delta}\left[-\kappa\,\psi(s+\tau)^p\Psi(s) - p\kappa\,\psi(s)^p\Psi(s) + \langle u(s+\tau), Q\rangle\Psi(s)\right] ds \\
&- \int_0^t e^{(t-s)\Delta}\left(\langle u'(s), Q(x)\rangle\psi(s) + \langle u(s), Q(x)\rangle\Psi(s) - \kappa(p+1)\psi(s)^p\Psi(s)\right) ds.
\end{aligned}
$$

Now, by taking the absolute value of the above identity, we obtain

$$
\begin{aligned}
\left|\frac{\psi(t+\tau)-\psi(t)}{\tau} - \Psi(t)\right| &\le \left|\frac{1}{\tau}\left\{e^{(t+\tau)\Delta}\psi_0 - e^{t\Delta}\psi_0\right\} - e^{t\Delta}\psi_0\right| \\
&+ \left|\frac{1}{\tau}\int_0^\tau e^{(t+\tau-s)\Delta}\left[-\kappa\psi(s)^{p+1} + \langle u(s), Q\rangle\psi(s)\right] ds - e^{t\Delta}(-\kappa\,\psi_0^{p+1} + \langle u(0), Q\rangle\psi_0))\right| \\
&+ \int_0^t \left|e^{(t-s)\Delta}\Big[\langle\frac{u(s+\tau)-u(s)}{\tau}, Q\rangle\psi(s+\tau) - \langle u'(s), Q(x)\rangle\psi(s) - \kappa\,\psi(s+\tau)^p\Psi(s)\right. \\
&\qquad \left. - p\kappa\,\psi(s)^p\Psi(s) + \langle u(s+\tau), Q\rangle\Psi(s) - \langle u(s), Q(x)\rangle\Psi(s) + \kappa(p+1)\psi(s)^p\Psi(s)\Big]\right| ds \\
&+ \int_0^t \left|-\kappa\,\psi(s+\tau)^p - \kappa p\,\psi(s)^p + \langle u(s+\tau), Q\rangle\right|\left|\frac{\psi(s+\tau)-\psi(s)}{\tau} - \Psi(s)\right| ds \\
&+ \int_0^t \left|e^{(t-s)\Delta}o\left(\frac{\psi(s+\tau)-\psi(s)}{\tau}\right)\right| ds,
\end{aligned}
$$

which is of the type $f(t) \le \alpha(t) + \int_0^t \beta(s)f(s)ds$. We apply the Grönwall's Lemma, and we deduce that $f(t) \le \alpha(t) + \int_0^t \alpha(s)\beta(s)e^{\int_s^t \beta(r)dr}ds$, and then, by taking the limit as $\tau \to 0$, we finally obtain that

$$
\left|\frac{\psi(t+\tau)-\psi(t)}{\tau} - \Psi(t)\right| \to 0.
$$

Hence, we have proved that the unique solution Ψ of (B.1) coincides with ψ_t. □

We are now ready to prove the global well-posedness of (NHE) with $d = 1$ stated in Proposition 2.2.

Proof of Proposition 2.2. From the local well-posedness result, we know that for any $\psi_0 \in H^1(\mathbb{T}, \mathbb{R})$ there exists $\mathcal{T}(\psi_0, u) > 0$ such that for any $0 < t < \mathcal{T}(\psi_0, u)$ there exists a unique solution $\psi \in H^1((0,t), L^2(\mathbb{T}, \mathbb{R})) \cap L^2((0,t), H^2(\mathbb{T}, \mathbb{R})) \cap C([0,t], H^1(\mathbb{T}, \mathbb{R}))$ of problem (NHE). Furthermore, if $\mathcal{T}(\psi_0, u) < +\infty$, then $\|\psi(t)\|_{H^1} \to +\infty$ as $t \to \mathcal{T}(\psi_0, u)$. We shall prove that

$$
\|\psi(t)\|_{H^1} \le C \qquad \text{as } t \to \mathcal{T}(\psi_0, u),
$$

and we would deduce that $\mathcal{T}(\psi_0,u)=+\infty$, that is, the solution is globally well-defined. We recall that $\|\psi\|_{H^1}^2 := \|\psi\|_{L^2}^2 + \|\partial_x\psi\|_{L^2}^2$. For almost every $0<t<\mathcal{T}(\psi_0,u)$, we multiply the equation in (NHE) by ψ and we obtain

$$\langle \partial_t\psi,\psi\rangle_{L^2} - \langle \Delta\psi,\psi\rangle_{L^2} + \kappa\langle \psi^{p+1},\psi\rangle_{L^2} = \big\langle \langle u(t),Q(x)\rangle\psi,\psi\big\rangle_{L^2}.$$

Recalling that $\kappa\geq 0$ and $p\in 2\mathbb{N}^*$, we get $\kappa\langle \psi^{p+1},\psi\rangle_{L^2}>0$ and then $\frac{1}{2}\frac{d}{dt}\|\psi(t)\|_{L^2}^2 \leq C(t)\|\psi(t)\|_{L^2}^2$ thanks to the accreativity of $-\Delta$. Therefore, for a.e. $t\in(0,\mathcal{T}(\psi_0,u))$, we have the inequality

$$\|\psi(t)\|_{L^2}^2 \leq \|\psi_0\|_{L^2}^2\, e^{2\int_0^t C(s)ds} \leq \|\psi_0\|_{L^2}^2\, e^{2C\sqrt{\mathcal{T}(\psi_0,u)}\|u\|_{L^2(0,\mathcal{T}(\psi_0,u))}}. \tag{B.2}$$

Now we multiply equation (NHE) by $-\Delta\psi$ and we get

$$-\langle \partial_t\psi,\Delta\psi\rangle_{L^2} + \langle \Delta\psi,\Delta\psi\rangle_{L^2} - \kappa\langle \psi^{p+1},\Delta\psi\rangle_{L^2} = -\big\langle \langle u(t),Q(x)\rangle\psi,\Delta\psi\big\rangle_{L^2}. \tag{B.3}$$

Let us observe that thanks to the properties of Δ, for every $\xi\in H^2(\mathbb{T})$, it holds that

$$t\mapsto \langle \Delta\xi(t),\xi(t)\rangle_{L^2}$$

is absolutely continuous and

$$\frac{d}{dt}\langle \Delta\xi(t),\xi(t)\rangle_{L^2} = 2\langle \partial_t\xi(t),\Delta\xi(t)\rangle_{L^2}.$$

We rewrite (B.3) in the equivalent form

$$-\int_0^{2\pi}\partial_t\psi\Delta\psi dx + \int_0^{2\pi}(\Delta\psi)^2dx - \kappa\int_0^{2\pi}\psi^{p+1}\Delta\psi dx = -\int_0^{2\pi}\langle u(t),Q(x)\rangle\psi\Delta\psi dx$$

and, integrating by parts, we obtain

$$\begin{aligned}-\partial_t\psi\partial_x\psi\big|_0^{2\pi} + \int_0^{2\pi}\partial_x(\partial_t\psi)\partial_x\psi dx + \int_0^{2\pi}(\Delta\psi)^2dx - \kappa\psi^{p+1}\partial_x\psi\big|_0^{2\pi} + \kappa(p+1)\int_0^{2\pi}\psi^p(\partial_x\psi)^2dx \\ \leq C(t)\left|\psi\partial_x\psi|_0^{2\pi} - \int_0^{2\pi}(\partial_x\psi)^2dx\right|,\end{aligned}$$

and as above we get an inequality of the form $\int_0^{2\pi}\partial_x(\partial_t\psi)\partial_x\psi dx \leq C(t)\int_0^{2\pi}(\partial_x\psi)^2dx$, where we have used that the first integral on the left-hand side is well-defined since $\psi_t\in L^2((0,T),H^1(\mathbb{T},\mathbb{R}))$ thanks to the fact that $\psi_0\in H^3(\mathbb{T},\mathbb{R})$ and to Proposition B.1. Hence, we deduce that

$$\frac{1}{2}\partial_t\|\partial_x\psi(t)\|_{L^2}^2 \leq C(t)\|\partial_x\psi_0\|_{L^2}^2,$$

and by the Grönwall's inequality we get

$$\|\partial_x\psi(t)\|_{L^2}^2 \leq \|\partial_x\psi_0\|_{L^2}^2\, e^{2\int_0^t C(s)ds} \leq \|\partial_x\psi_0\|_{L^2}^2\, e^{2C\sqrt{\mathcal{T}(\psi_0,u)}\|u\|_{L^2(0,\mathcal{T}(\psi_0,u))}}. \tag{B.4}$$

Therefore, for almost every $t\in(0,\mathcal{T}(\psi_0,u))$ we have proved that

$$\|\psi(t)\|_{H^1}^2 = \|\psi(t)\|_{L^2}^2 + \|\partial_x\psi(t)\|_{L^2}^2 \leq (\|\psi_0\|_{L^2}^2 + \|\partial_x\psi_0\|_{L^2}^2)e^{2C\sqrt{\mathcal{T}(\psi_0,u)}\|u\|_{L^2(0,\mathcal{T}(\psi_0,u))}},$$

where we have used (B.2) and (B.4). Since the right-hand side does not depend on t, we can take the limit as $t\to\mathcal{T}(\psi_0,u)$ and conclude that $\|\psi(t)\|_{H^1}^2\leq C$ as $t\to\mathcal{T}(\psi_0,u)$. □

C An estimate for the exact controllability to the ground state solution

In this section we derive an estimate for the solution w of (4.30) which is useful for the proof of Theorem 4.3. Recall that $\Phi = 1/\sqrt{2\pi}$. We shall take advantage of estimate (4.29) that we have obtained for the solution of

$$\begin{cases}\partial_t y(t) - \partial_x^2 y(t) + \kappa\displaystyle\sum_{j=2}^{p+1}\binom{p+1}{j}y^j(t)\Phi^{p+1-j} = \langle v^1(t),Q\rangle y(t) + \langle v^1(t),Q\rangle\Phi, \quad t\in(s_0,s_1),\\ y(s_0) = y_{s_0} := \psi(s_0) - \Phi.\end{cases} \tag{C.1}$$

Proposition C.1. *Let $\psi(s_0) \in H^3(\mathbb{T},\mathbb{R})$ and $v^1 \in H^1((s_0,s_1),\mathbb{R}^{q+2})$ with $0 \le s_0 < s_1$. Consider $Q \in H^3(\mathbb{T},\mathbb{R}^{q+2})$ and $y \in C([s_0,s_1],H^3(\mathbb{T},\mathbb{R})) \cap C^1([s_0,s_1],H^1(\mathbb{T},\mathbb{R}))$ be the solution of (C.1). Let v^1 satisfy (4.27). Then, the solution $w \in C([s_0,s_1],H^1(\mathbb{T},\mathbb{R})) \cap H^1([s_0,s_1],L^2(\mathbb{T},\mathbb{R}))$ of*

$$\begin{cases} \partial_t w(t) - \partial_x^2 w(t) - \kappa(p+1)\Phi^p w(t) + \kappa \displaystyle\sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} y^j(t) = \langle v^1(t), Q\rangle y(t), \quad t \in (s_0,s_1) \\ w(s_0) = 0 \end{cases} \tag{C.2}$$

satisfies the following inequality with $\sigma = s_1 - s_0$ and $A_4(\sigma, \|y_{s_0}\|_{H^1})$ defined in (C.7):

$$\sup_{t\in[s_0,s_1]} \|w(t)\|_{H^1} \le A_4(\sigma, \|y_{s_0}\|_{H^1}) \|y_{s_0}\|_{H^1}^2 . \tag{C.3}$$

Proof. We recall that w has been defined as $w := y - \xi$ with $\xi \in C([s_0,s_1],H^1(\mathbb{T},\mathbb{R})) \cap H^1(s_0,s_1,L^2(\mathbb{T},\mathbb{R})) \cap L^2(s_0,s_1,H^2(\mathbb{T},\mathbb{R}))$ solution of the linear system (4.4) with control $v^1 \in H_0^1((s_0,s_1),\mathbb{R}^{q+2})$ such that

$$\xi(s_1;s_0,\xi_0,v^1) = 0, \quad \text{with} \quad \xi(s_0) = y_{s_0} = \psi(s_0) - \Phi$$

and $\|v^1\|_{H^1(s_0,s_1)} \le N(\sigma)\|y_{s_0}\|_{H^1}$, with $\sigma = s_1 - s_0$. The existence of such control has been established in Proposition 4.6. Let us estimate the norm of w at time s_1. We first multiply the equation in (C.2) by w and we obtain

$$\langle \partial_t w(t), w(t)\rangle_{L^2} - \langle \partial_x^2 w(t), w(t)\rangle_{L^2} - \kappa(p+1)\Phi^p \langle w(t), w(t)\rangle_{L^2} + \kappa \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \langle y^j(t), w(t)\rangle_{L^2}$$
$$= \big\langle \langle v^1(t), Q\rangle y(t), w(t)\big\rangle_{L^2}.$$

Using the accreativity of $-\partial_x^2$ and that $H^1(\mathbb{T}) \hookrightarrow C(\mathbb{T})$, we get

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 \le \kappa(p+1)\Phi^p \|w(t)\|_{L^2}^2 +$$
$$+ \kappa \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \left(\frac{\|y(t)\|_{H^1}^{2j}}{2} + \frac{\|w(t)\|_{L^2}^2}{2} \right) + C_Q^2 \sum_{j=1}^{q+2} |v_j^1(t)|^2 \frac{\|y(t)\|_{L^2}^2}{2} + \frac{\|w(t)\|_{L^2}^2}{2},$$

where

$$C_Q := \sup_{i=1,\dots,q+2} \|Q^i\|_{C^0} \tag{C.4}$$

By the Grönwall's Lemma, for any $t \in (s_0,s_1)$ it holds that

$$\|w(t)\|_{L^2}^2 \le \left(\int_{s_0}^t \kappa \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \|y(s)\|_{H^1}^{2j} + C_Q^2 \sum_{j=1}^{q+2} |v_j^1(s)|^2 \|y(s)\|_{H^1}^2 \, ds \right) \cdot$$
$$e^{\int_{s_0}^t \left(2\kappa(p+1)\Phi^p + \sum_{j=2}^{p+1}\binom{p+1}{j}\Phi^{p+1-j} + 1\right) ds}.$$

Therefore, by taking the supremum over $[s_0,s_1]$ we obtain

$$\sup_{t\in[s_0,s_1]} \|w(t)\|_{L^2}^2 \le \left(\kappa\sigma \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \sup_{t\in[s_0,s_1]} \|y(t)\|_{H^1}^{2j} + C_Q^2 \|v^1\|_{L^2(s_0,s_1)}^2 \sup_{t\in[\tau_0,\tau_1]} \|y(t)\|_{H^1}^2 \right) e^{A_1(\sigma)}$$
$$\le \left(2\kappa\sigma(p+1) \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \left(\|y_{s_0}\|_{H^1}^{2j} + \|v^1\|_{L^2(s_0,s_1)}^{2j} \right) + C_Q^2 \|v^1\|_{L^2(s_0,s_1)}^2 \left(\|y_{s_0}\|_{H^1}^2 + \|v^1\|_{L^2(s_0,s_1)}^2 \right) \right) e^{A_1(\sigma)}$$
$$\le \left(2\kappa\sigma(p+1) \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \left(1 + N(\sigma)^{2j}\right) \|y_{s_0}\|_{H^1}^{2j} + C_Q^2 N(\sigma)^2 \left(1 + N(\sigma)^2\right) \|y_{s_0}\|_{H^1}^4 \right) e^{A_1(\sigma)}$$
$$\le \left(2\kappa\sigma(p+1) \|y_{s_0}\|_{H^1}^4 \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \left(1 + N(\sigma)^{2j}\right) \|y_{s_0}\|_{H^1}^{2(j-2)} + C_Q^2 N(\sigma)^2 \left(1 + N(\sigma)^2\right) \|y_{s_0}\|_{H^1}^4 \right) e^{A_1(\sigma)}$$
$$\le A_2(\sigma, \|y_{s_0}\|_{H^1})^2 \|y_{s_0}\|_{H^1}^4 \tag{C.5}$$

where

$$A_1(\sigma) := \sigma \left(2\kappa(p+1)\Phi^p + \kappa \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} + 1 \right)$$

$$A_2(\sigma, \|y_{s_0}\|_{H^1}) := \left(2\kappa\sigma(p+1) \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \left(1 + N(\sigma)^{2j}\right) \|y_{s_0}\|_{H^1}^{2(j-2)} + C_Q^2 N(\sigma)^2 \left(1 + N(\sigma)^2\right) \right)^{1/2} e^{A_1(\sigma)/2}.$$

Let us now multiply the equation in (4.30) by $-\partial_x^2 w(t)$

$$- \langle \partial_t w(t), \partial_x^2 w(t) \rangle_{L^2} + \langle \partial_x^2 w(t), \partial_x^2 w(t) \rangle_{L^2} + \kappa(p+1)\Phi^p \langle w(t), \partial_x^2 w(t) \rangle_{L^2} - \kappa \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \langle y^j(t), \partial_x^2 w(t) \rangle_{L^2} = -\big\langle \langle v^1(t), Q \rangle y(t), \partial_x^2 w(t) \big\rangle_{L^2}.$$

We now perform integrations by parts, and we get

$$\langle \partial_t(\partial_x w(t)), \partial_x w(t) \rangle_{L^2} + \left\| \partial_x^2 w(t) \right\|_{L^2}^2 - \kappa(p+1)\Phi^p \langle \partial_x w(t), \partial_x w(t) \rangle_{L^2} + \kappa \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} j \langle y^{j-1}(t) \partial_x y(t), \partial_x w(t) \rangle_{L^2} = -\langle \langle v^1(t), Q \rangle y(t), \partial_x^2 w(t) \rangle_{L^2}$$

and therefore

$$\frac{1}{2}\frac{d}{dt} \|\partial_x w(t)\|_{L^2}^2 + \left\| \partial_x^2 w(t) \right\|_{L^2}^2 \leq \kappa(p+1)\Phi^p \|\partial_x w(t)\|_{L^2}^2 + \kappa(p+1) \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \left(\frac{\|y(t)\|_{H^1}^{2j}}{2} + \frac{\|\partial_x w(t)\|_{L^2}^2}{2} \right) + C_Q^2 \sum_{j=1}^{q+2} |v^{1,j}(t)|^2 \|y(t)\|_{H^1}^2 + \frac{\left\| \partial_x^2 w(t) \right\|_{L^2}^2}{2},$$

where we have used that $\partial_t w = \partial_t y - \partial_t \xi \in L^2((s_0, s_1), H^1(\mathbb{T}, \mathbb{R}))$ (see Proposition B.1 for y and Remark 4.5 for ξ). We apply the Grönwall's Lemma, and we obtain that, for any $t \in [s_0, s_1]$,

$$\|\partial_x w(t)\|_{L^2}^2 \leq \left(\int_{s_0}^t \kappa(p+1) \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \|y(s)\|_{H^1}^{2j} + C_Q^2 \sum_{j=1}^{q+2} |v^{1,j}(s)|^2 \|y(s)\|_{H^1}^2 \, ds \right) e^{A_3(s_0,t)}$$

with

$$A_3(s_0, t) := \int_{s_0}^t \left(2\kappa(p+1)\Phi^p + \kappa(p+1) \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \right) ds.$$

Taking the supremum over the interval $[s_0, s_1]$, we obtain

$$\begin{aligned} \sup_{t \in [s_0,s_1]} \|\partial_x w(t)\|_{L^2}^2 &\leq \left(\int_{s_0}^{s_1} \kappa(p+1) \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \|y(s)\|_{H^1}^{2j} + C_Q^2 \sum_{j=1}^{q+2} |v^{1,j}(s)|^2 \|y(s)\|_{H^1}^2 \, ds \right) e^{A_3(s_0,s_1)} \\ &\leq \left(\kappa\sigma(p+1) \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \sup_{t \in [s_0,s_1]} \|y(t)\|_{H^1}^{2j} + C_Q^2 \left\| v^1 \right\|_{L^2(s_0,s_1)}^2 \sup_{t \in [s_0,s_1]} \|y(t)\|_{H^1}^2 \right) e^{A_3(s_0,s_1)} \\ &\leq \left(2\kappa\sigma(p+1)^2 \sum_{j=2}^{p+1} \binom{p+1}{j} \Phi^{p+1-j} \left(\|y_{s_0}\|_{H^1}^{2j} + \left\| v^1 \right\|_{L^2(s_0,s_1)}^{2j} \right) + C_Q^2 \left\| v^1 \right\|_{L^2(s_0,s_1)}^2 \left(\|y_{s_0}\|_{H^1}^2 + \left\| v^1 \right\|_{L^2(s_0,s_1)}^2 \right) \right) e^{A_3(s_0,s_1)}. \end{aligned}$$

Thanks to the assumption on y_0 we deduce

$$\begin{aligned}
&\sup_{t\in[s_0,s_1]} \|\partial_x w(t)\|_{L^2}^2 \\
&\quad\le \left(2\kappa\sigma(p+1)^2\sum_{j=2}^{p+1}\binom{p+1}{j}\Phi^{p+1-j}\left(1+N^{2j}(\sigma)\right)\|y_{s_0}\|_{H^1}^{2j}+C_Q^2N^2(\sigma)\left(1+N^2(\sigma)\right)\|y_{s_0}\|_{H^1}^4\right)e^{A_3(s_0,s_1)} \\
&\quad= \left(2\kappa\sigma(p+1)^2\sum_{j=2}^{p+1}\binom{p+1}{j}\Phi^{p+1-j}\left(1+N^{2j}(\sigma)\right)\|y_{s_0}\|_{H^1}^{2(j-2)}+C_Q^2N^2(\sigma)\left(1+N^2(\sigma)\right)\right)e^{A_3(s_0,s_1)}\|y_{s_0}\|_{H^1}^4 \\
&\quad= A_3(\sigma,\|y_{s_0}\|_{H^1})^2\|y_{s_0}\|_{H^1}^4\,,
\end{aligned} \tag{C.6}$$

where

$$\begin{aligned}
&A_3(\sigma,\|y_{s_0}\|_{H^1}) := \\
&\qquad\left(2\kappa\sigma(p+1)^2\sum_{j=2}^{p+1}\binom{p+1}{j}\Phi^{p+1-j}\left(1+N^{2j}(\sigma)\right)\|y_{s_0}\|_{H^1}^{2(j-2)}+C_Q^2N^2(\sigma)\left(1+N^2(\sigma)\right)\right)^{1/2}e^{A_3(s_0,s_1)/2}.
\end{aligned}$$

Finally, from (C.5) and (C.6), we conclude that

$$\begin{aligned}
\sup_{t\in[s_0,s_1]}\|w(t)\|_{H^1}^2 &\le 2\left(\sup_{t\in[s_0,s_1]}\|w(t)\|_{L^2}^2+\sup_{t\in[s_0,s_1]}\|\partial_x w(t)\|_{L^2}^2\right) \\
&\le 2\left(A_2(\sigma,\|y_{s_0}\|_{H^1})^2+A_3(\sigma,\|y_{s_0}\|_{H^1})^2\right)\|y_{s_0}\|_{H^1}^4 \le A_4^2(\sigma,\|y_{s_0}\|_{H^1})^2\|y_{s_0}\|_{H^1}^4\,,
\end{aligned}$$

with

$$\begin{aligned}
A_4(\sigma,\|y_{s_0}\|_{H^1}) := \sqrt{2}\left(2\kappa\sigma(p+1)^2\sum_{j=2}^{p+1}\binom{p+1}{j}\Phi^{p+1-j}\left(1+N(\sigma)^{2j}\right)\|y_{s_0}\|_{H^1}^{2(j-2)}+C_Q^2N^2(\sigma)\left(1+N(\sigma)^2\right)\right)^{1/2}\cdot \\
e^{\sigma(2\kappa(p+1)\Phi^p+\kappa(p+1)\sum_{j=2}^{p+1}\binom{p+1}{j}\Phi^{p+1-j}+1))/2}.
\end{aligned} \tag{C.7}$$

The proof is therefore concluded. □

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