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Mediated communication with coarse messages

Mael Le Treust * and Tristan Tomala †

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Abstract

In this paper, we consider sender-receiver problems where the two parties communicate via a coarse set of messages. We propose a concept of mediated communication equilibrium capturing the message limitations. We give conditions under which mediation improves welfare maximisation over equilibrium outcomes, or to the contrary, can be dispensed with.

Keywords: Mediated communication, Mechanism design, Bayesian persuasion.

1 Introduction

Problem formulation and contributions. Information transmission from informed experts to decision makers is ubiquitous in modern economies. There is a number of reasons why such communications are not perfect and information is garbled, an important one being that the incentives of the involved parties are not aligned: an expert might want to persuade a decision maker to take action in a certain direction. Another common source of imperfect communication is the limitation on the size of messages. Communication is often mediated by external third parties, for instance auditors, consultants or news reporters, and typically, the reporting of information is not extensive. A news article has to abide by a word or page limit, the same applies for audit or consulting reports.

In this paper, we consider a sender-receiver environment with coarse messages. The sender is informed of a payoff relevant state parameter, communicates with the receiver via a small set of available messages, and the receiver takes an action. Our goal is to study mediated communication equilibria, constrained by the message limit. Our first contribution is to introduce a concept of mediated communication equilibrium, in the spirit of Myerson (1982, 1986) and Forges (1986), that takes into account the fixed set of available messages. To this end, we split the mediator in two entities called Mediator 1 and Mediator 2. The sender communicates without restriction with Mediator 1 whose task is to encode the information obtained from the sender into an available message, which is received by Mediator 2. In turn, Mediator 2 decodes this message and communicates with the receiver without restriction. To get an image, think of

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Mediator 1 as an auditor who summarizes their findings in a report with given number of pages, and of Mediator 2 as a consultant who reads the report and recommends a course of actions. We prove that a revelation principle holds: without loss of generality, the sender reveals the true state truthfully to Mediator 1; Mediator 2 recommends an action to the receiver who plays it obediently.

With this concept, our second contribution is to study the usefulness and optimality of mediation. We question whether both mediators are actually needed for implementing a given equilibrium outcome, or for implementing optimal outcomes, i.e. that maximize some welfare criterion. We consider two relaxed problems by ignoring the incentive compatible constraints, either of the sender or of the receiver. Ignoring the incentive constraints amounts to assuming commitment power for this player. Thus, the umbrella of mediated communication encompasses models of Bayesian persuasion with sender commitment, and, models of mechanism design (or principal-agent) with commitment from the receiver. In the case of Bayesian persuasion, we find that while mediation increases the set of equilibrium outcomes, it does not improve optimality. By contrast, in the principal-agent case, this crucially depends on the coarseness of messages: mediation does not improve optimality for at most three messages, but it does for at least four messages.

**Related literature.** Mediated communication and communication equilibria have been formalized by Myerson (1982, 1986) and Forges (1986) for Bayesian and multi-stage games. In these works is stated the revelation principle: it is without loss to restrict to canonical equilibria whereby players report whatever information they have to the mediator who recommends what to play. An earlier version is the correlated equilibrium of Aumann (1974) where the mediator only recommends actions. The relationship between information design and mediated communication is explored in Bergemann and Morris (2016). Using the revelation principle concretely, implies that there is no limit to communication: sets of messages are large enough with respect to sets of states and actions.

Sender-receiver problems have been studied in the cheap-talk context (Crawford and Sobel, 1982), in the Bayesian persuasion scenario with commitment of the sender (Kamenica and Gentzkow, 2011), and when the receiver has commitment power, these are simple instances of adverse selection. The value of mediation in sender-receiver games is studied in details in Salamanca (2021) who characterizes the best communication equilibrium for the sender with a belief based approach. In a recent paper, Corrao and Dai (2023) consider games with transparent motives and compare the sender preferred mediated equilibrium with the Bayesian persuasion benchmark and the best cheap talk equilibrium. They find conditions such that the best sender payoff with mediation coincides with either benchmark. They also characterize the games (with finite actions) where mediation has no value.

There is a number of papers which study cheap talk games with imperfect communication among them Blume and Board (2013), Giovannoni and Xiong (2019), and Hagenbach and Koessler (2020), where players have a language competence, materialized by the subset of messages they can understand. One insight from this literature is that in a setting without commitment, restricting communication is possibly welfare improving.

Another set of papers study restrictions on communication in Bayesian persuasion models.
Akyol, Langbort, and Başar (2017) and Tsakas and Tsakas (2021) consider noisy communication between the committed sender and the receiver. Le Treust and Tomala (2019) characterize the optimal sender payoff for large sequences of i.i.d. persuasion problems over a fixed communication channel. Lyu, Suen, and Zhang (2023) consider a continuous unidimensional state variable, finite set of messages, and characterize optimal disclosure policies, given the number of messages. Closer to the present paper, Aybas and Turkel (2019) consider persuasion with finite set of states and a fixed number of messages. They characterize the optimal sender payoff using the $k$-concave closure of the value function of the sender obtained by taking convex combination of at most $k$ points of the graph.

Mechanism design with limited communication is studied in Wu (2017) who gives a Blackwell ordering of mechanisms with small set of messages. Vora and Kulkarni (2020) consider a number of identical sender-receiver problems and a given set of messages. With graph-theroretic methods, they characterize the number of sequences of states that the receiver is optimally able to recover.

Finally, in one of our main results (Theorem 3), we use the geometric properties of incentive compatible mechanisms found in Lahr (2022). To our knowledge, the present paper is the first to consider limited messages for mediated communication.

**Organization of the paper.** We present the model and the mediated equilibrium concept in Section 2. Section 3 gives results and examples of usefulness or dispensability of mediation. Section 4 studies optimal communication equilibria with commitment either of the sender or of the receiver.

**Notations.** For a finite set $X$, the set of probability distributions over $X$ is denoted $\Delta(X)$. Given finite sets $X, Y$, $\mathcal{M}(X; Y)$ denotes the set transition probabilities from $X$ to $Y$. An element $p \in \mathcal{M}(X; Y)$ is viewed equivalently as a Markov matrix with generic entry $p(y|x)$ or as a mapping $p : X \rightarrow \Delta(Y)$. In this case, for any $x$, $p(x) := (p(y|x))_{y \in Y} \in \Delta(Y)$ denotes the distribution on $Y$ conditional on $x$. When convenient, an element $x \in X$ will be identified with the Dirac distribution on $x$. The indicator function of element $x$ is denoted $\mathbb{1}_x$.

## 2 Model

Our model has two components: a sender-receiver problem and a constraint on the number of messages available for communicating. These are formalized as follows.

**Sender-receiver environment.** There is a finite set of states $S$, a finite set of actions $A$ and a prior probability distribution $\pi \in \Delta(S)$ over states with full support. There are two players, a sender and a receiver, with respective payoff functions $u, v : S \times A \rightarrow \mathbb{R}$.

**Communication constraint.** There is finite set of messages $M = \{1, \ldots, |M|\}$ whose cardinality measures the precision, or coarseness, of communication.

We define mediated communication mechanisms that are constrained by the available messages.

**Definition 1.** A communication mechanism with message set $M$ is given by a pair of finite sets $R_1, R_2$ and two mappings $\mu_1 : R_1 \rightarrow \Delta(M)$, $\mu_2 : M \rightarrow \Delta(R_2)$. 

The interpretation is that there are two mediators, called Mediator 1 and Mediator 2 who operate the mappings $\mu_1$ and $\mu_2$ respectively. Denote $R = (R_1, R_2, \mu_1, \mu_2)$. The induced extended game $\Gamma_M(R)$ is played as follows:

- The sender observes $s$ and sends $r_1$ to Mediator 1;
- Mediator 1 chooses message $m$ with probability $\mu_1(m|r_1)$;
- Mediator 2 observes $m$ and chooses $r_2$ with probability $\mu_2(r_2|m)$;
- The receiver observes $r_2$ and chooses $a$.

The task of Mediator 1 is to “encode” the report from the sender $r_1$ into an available message $m$. This message is received by Mediator 2 who is in charge of “decoding” it into a recommendation $r_2$, which is sent to the receiver. Thus, we call $\mu_1$ the encoding function and $\mu_2$ the decoding function. A strategy of the sender is $\sigma : S \to \Delta(R_1)$ and a strategy of the receiver is $\tau : R_2 \to \Delta(A)$.

**Definition 2.** A communication equilibrium with message set $M$ is a tuple $(R, \sigma, \tau)$ such that $(\sigma, \tau)$ is a Bayes-Nash equilibrium of $\Gamma_M(R)$.

The equilibrium conditions are:

- Sender: for all $s$ and all $r'_1$,
  \[ \sum_{r_1,m,r_2,a} \sigma(r_1|s)\mu_1(m|r_1)\mu_2(r_2|m)\tau(a|r_2)u(s,a) \geq \sum_{m,r_2,a} \mu_1(m|r'_1)\mu_2(r_2|m)\tau(a|r_2)u(s,a). \]

- Receiver: for all $r_2$ and all $b$,
  \[ \sum_{s,r_1,m,a} \pi(s)\sigma(r_1|s)\mu_1(m|r_1)\mu_2(r_2|m)\tau(a|r_2)v(s,a) \geq \sum_{s,r_1,m} \pi(s)\sigma(r_1|s)\mu_1(m|r_1)\mu_2(r_2|m)v(s,b). \]

Note that if $P'(r_2) = 0$, the latter is trivially satisfied.

**Definition 3.** A communication equilibrium outcome with message set $M$ is a transition probability $p \in \mathcal{M}(S; A)$ induced by a communication equilibrium. That is:

\[ \forall (s,a) \in S \times A, \quad p(a|s) = \sum_{r_1,m,r_2} \sigma(r_1|s)\mu_1(m|r_1)\mu_2(r_2|m)\tau(a|r_2) \]

where $(\sigma, \tau)$ is an equilibrium of $\Gamma_M(R)$.

The set of communication equilibrium outcomes is denoted by $\text{CE}(u, v, |M|)$. Throughout, the set of states and actions are held fixed, the dependence on payoff functions and on number of messages is emphasized. We state a version of the revelation principle (Myerson, 1982, 1986; Forges, 1986) adapted to the present context.

**Lemma 1.** (Revelation principle) Every communication equilibrium outcome is induced by a canonical communication equilibrium where $R_1 = S$, $R_2 = A$, and $\sigma, \tau$ are the identity mappings on $S, A$ respectively.

**Proof of Lemma 1.** The proof is standard. Given a communication mechanism $(\hat{\mu}_1, \hat{\mu}_2)$ and an equilibrium $(\tilde{\sigma}, \tilde{\tau})$, define a canonical mechanism by

\[ \forall (s,m), \mu_1(m|s) = \sum_{r_1} \hat{\mu}_1(m|r_1)\tilde{\sigma}(r_1|s); \quad \forall (m,a), \mu_2(a|m) = \sum_{r_2} \hat{\mu}_2(r_2|m). \]
The equilibrium condition readily imply that the identity mappings on $S$ and $A$ form an equilibrium for the mechanism $(\mu_1, \mu_2)$.

A characterization of communication equilibrium outcomes with message set $M$ follows.

**Corollary 1.** A transition probability $p \in \mathcal{M}(S; A)$ belongs to $\text{CE}(u, v, |M|)$ if and only if it satisfies the conditions:

- **Feasibility:** There exists an encoding function $\mu_1 \in \mathcal{M}(S; M)$ and a decoding function $\mu_2 \in \mathcal{M}(M; A)$ such that $\forall (s, a), p(a|s) = \sum_m \mu_1(m|s)\mu_2(a|m)$.

- **Incentive compatibility of sender:** $\forall (s, t), \sum a p(a|s)u(s, a) \geq \sum a p(a|t)u(s, a)$.

- **Incentive compatibility of receiver:** $\forall (a, b), \sum_s \pi(s)p(a|s)v(s, a) \geq \sum_s \pi(s)p(a|s)v(s, b)$.

The feasibility condition materializes the restriction on communication. Without it, this is just a standard communication equilibrium (Myerson, 1982).

**Remark 1.** A Markov matrix $p \in \mathcal{M}(S; A)$ that satisfies the feasibility condition is the product $p = \mu_1 \cdot \mu_2$ with $\mu_1 \in \mathcal{M}(S; M)$ and $\mu_2 \in \mathcal{M}(M; A)$. This condition holds if and only if the rank of the matrix $p$ is no more that the number of messages.

To be able to assess the importance of each side of the mediation mechanism, we also consider the equilibrium outcomes obtained with mediation only on one side.

**Definition 4.**
- A communication mechanism $\mathcal{R} = (R_1, R_2, \mu_1, \mu_2)$ uses Mediator 1 only if Mediator 2 is passive and the receiver observes the message directly ($R_2 = M$ and $\mu_2(m) = m$). The set of all equilibrium outcomes of the game with Mediator 1 only is denoted $\text{CE}(1, \times)(u, v, |M|)$.

- A communication mechanism $\mathcal{R} = (R_1, R_2, \mu_1, \mu_2)$ uses Mediator 2 only if Mediator 1 is passive and the sender chooses the message directly ($R_1 = M$ and $\mu_1(m) = m$). The set of all equilibrium outcomes of the game with Mediator 2 only is denoted $\text{CE}(\times, 2)(u, v, |M|)$.

These are obviously particular communication equilibrium outcomes. From yet another revelation principle reasoning, the player who is provided mediation can play respectively truthful or obedient without loss of generality, while the other player may need to randomize. Also, we argue that an inactive mediator is equivalent to a deterministic mediator (who does not randomize).

**Lemma 2.** $\text{CE}(\times, 2)(u, v, |M|)$ is the set of communication equilibrium outcomes obtained from Bayes-Nash equilibria of extended games with deterministic encoding functions.

$\text{CE}(1, \times)(u, v, |M|)$ is the set of communication equilibrium outcomes obtained from Bayes-Nash equilibria of extended games with deterministic decoding functions.

**Proof of Lemma 2.** Consider first an inactive Mediator 1 and an equilibrium where the sender chooses the messages directly. Then, this is equivalent to a mechanism where the sender chooses the message, hands it to Mediator 1 who transmits it directly to Mediator 2. Conversely, if we have an equilibrium of an extended game with deterministic encoding function $\mu_1$, then the
sender can choose directly the message \( m = \mu_1(r_1) \), where \( r_1 \) is generated by his equilibrium strategy.

Second, consider an inactive Mediator 2 and an equilibrium where the receiver observe the message directly and chooses the action. This is obviously replicated by a deterministic Mediator 2 who simply forwards the message to the receiver. Conversely, suppose that Mediator 2 sends \( r_2 = f(m) \) to the receiver and that \( f \) is a deterministic function. If \( f \) is one-to-one, then the receiver deduces \( m \) and might as well observe it. Otherwise, this means that the range of \( f \) has cardinality less than \(|M|\), so we can identify \( R_2 \) with a subset of \( M \). Then, Mediator 1 can find out \( r_2 \) directly from \( r_1 \) and encode it into a message in \( M \) that Mediator 2 simply forwards to the receiver.

In the sequel, we study in details cases where one active mediator only is enough to generate all communication equilibrium outcomes. For instance, this is the case when there are sufficiently many messages.

**Lemma 3.** If \(|M| \geq \min\{|S|; |A|\} \), then \( \text{CE}(u, v, |M|) \) is the set of all communication equilibrium outcomes \( \text{CE}(u, v, \infty) \). More precisely:

- If \(|M| \geq |S| \) then \( \text{CE}^{(x, 2)}(u, v, |M|) = \text{CE}(u, v, \infty) \): the sender can report the state directly to Mediator 2.
- If \(|M| \geq |A| \), then \( \text{CE}^{(1, x)}(u, v, |M|) = \text{CE}(u, v, \infty) \): Mediator 1 can recommend directly the action to the receiver.

**3 Usefulness of mediators**

We question whether mediators are useful and whether both of them are required for implementing all communication equilibria when \(|M| < \min\{|S|; |A|\} \).

As a first result, with binary messages, only Mediator 2 is required.

**Theorem 1.**

\[
\text{CE}(u, v, 2) = \text{CE}^{(x, 2)}(u, v, 2).
\]

This is a consequence of the following lemma which gives a bound on the number of messages needed to implement a given communication equilibrium outcome.

**Lemma 4.** Let \( p \in \text{CE}(u, v, |M|) \) and let \( k \) be the number of extreme points of the convex hull of the induced mixed actions

\[
\text{conv}\{p(s), s \in S\} \subseteq \Delta(A).
\]

Then \( p \in \text{CE}^{(x, 2)}(u, v, k) \), i.e. can be implemented with Mediator 2 only, provided that at least \( k \) messages are available.

**Proof of Lemma 4.** First observe that \( k \leq \min\{|S|; |M|\} \). From its definition, \( k \leq |S| \). Also from the feasibility condition, \( p(s) \) is a convex combination of \( \{\mu_2(m), m \in M\} \), thus \( k \leq |M| \).

We assume \(|M| < |S| \) since otherwise Lemma 3 applies. Suppose that there exists \( T \subset S \) with
\(|T| = k\) such that for all \(s \in S \setminus T\),

\[
p(s) = \sum_{t \in T} \lambda_t^s p(t)
\]

with \(\lambda_t^s > 0\) and \(\sum_{t \in T} \lambda_t^s = 1\). Take \(s \in S \setminus T\). By linearity, we have \(u(s, p(s)) = \sum_{t \in T} \lambda_t^s u(s, p(t))\) and from incentive compatibility \(u(s, p(s)) \geq u(s, p(t))\). Thus, \(u(s, p(s)) = u(s, p(t))\) for all \(t \in T\).

Suppose that there are \(k\) messages and identify the set \(T\) with \(\{1, \ldots, k\}\). Let Mediator 2 implement \(p(t)\) for each \(t \in T\). The sender reports truthfully all states \(t \in T\). For a state \(s \notin T\), the sender reports \(t\) with probability \(\lambda_t^s\). This is incentive compatible, since the sender is indifferent. This also maintains the incentive compatibility of the receiver since the joint distribution of states and recommended actions is the same.

This lemma implies Theorem 1 since with two messages \(m_1, m_2\), \(\text{conv}\{p(s), s \in S\}\) is a subset of the one-dimensional segment \(\text{conv}\{\mu_2(m), m \in M\} = [\mu_2(m_1) ; \mu_2(m_2)]\), and therefore is also a segment.

**Example 1.** With 2 messages, dispensing with Mediator 1 is without loss of generality. Yet, there are equilibrium outcomes that cannot be obtained without Mediator 2. Consider 2 states, 2 messages, 3 actions, a uniform prior distribution and the following payoff table.

<table>
<thead>
<tr>
<th></th>
<th>(a_0)</th>
<th>(a_3)</th>
<th>(a_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s_0)</td>
<td>0, 1</td>
<td>3, 0</td>
<td>1, -4</td>
</tr>
<tr>
<td>(s_1)</td>
<td>0, -3</td>
<td>3, 0</td>
<td>1, 6</td>
</tr>
</tbody>
</table>

For the receiver, \(a_0\) is optimal when \(\mathbb{P}(s_1) \leq \frac{1}{4}\); \(a_3\) is optimal when \(\frac{1}{4} \leq \mathbb{P}(s_1) \leq \frac{1}{10}\) and \(a_1\) is optimal for \(\mathbb{P}(s_1) \geq \frac{4}{10}\). If the prior is uniform and without any mediation, it is easy to see that the sender’s payoff is 1 in all equilibria.

However, there is a communication equilibrium with sender payoff \(\frac{3}{2}\) where the sender sends message \(m_0\) in state \(s_0\), sends message \(m_1\) in state \(s_1\). Mediator 2 uses the mechanism \(\mu_2(m_0) = \frac{1}{2}a_0 + \frac{1}{2}a_3, \mu_2(m_1) = \frac{3}{4}a_1 + \frac{1}{4}a_3\). Notice that the expected sender payoff is \(\frac{3}{2}\) for both messages, so the incentive compatibility conditions are satisfied for the sender. As for the receiver, we have \(\mathbb{P}(s_1|a_0) = 0 \leq \frac{1}{4} ; \mathbb{P}(s_1|a_1) = 1 \geq \frac{4}{10}\) and \(\mathbb{P}(s_1|a_3) = \frac{1}{3} \in [\frac{1}{4}, \frac{4}{10}]\), so the action recommendation are also incentive compatible.

Next, we give an example of a communication equilibrium outcome requiring both mediators.

**Example 2.** There are 4 states \(S = \{s_1, s_2, s_3, s_4\}\) with uniform prior, there are 4 actions \(A = \{a, b, c, d\}\) and 3 messages \(M = \{m_1, m_2, m_3\}\). Consider the mechanisms given by:

\[
\begin{align*}
\mu_2(\cdot|m_1) & = \frac{3}{4}a + \frac{1}{4}b; & \mu_2(\cdot|m_2) & = \frac{1}{4}a + \frac{3}{4}d; & \mu_2(\cdot|m_3) & = \frac{1}{4}a + \frac{3}{4}c. \\
\mu_1(\cdot|s_1) & = \frac{1}{2}m_1 + \frac{1}{2}m_3; & \mu_1(\cdot|s_2) & = \frac{1}{2}m_1 + \frac{1}{2}m_2; & \mu_1(\cdot|s_3) & = m_3; & \mu_1(\cdot|s_4) & = m_2.
\end{align*}
\]

The induced actions distributions are:

\[
\begin{align*}
p(\cdot|s_1) & = \frac{1}{2}a + \frac{1}{4}b + \frac{1}{4}c; & p(\cdot|s_2) & = \frac{1}{2}a + \frac{1}{4}b + \frac{1}{4}d; & p(\cdot|s_3) & = \frac{1}{2}a + \frac{3}{4}c; & p(\cdot|s_4) & = \frac{1}{2}a + \frac{3}{4}d.
\end{align*}
\]

We choose the following utility function of the sender:
With this, it is easy to see that the $p(\cdot | s_i)$’s are incentive compatible for the sender. Next, the induced posterior distributions of states, conditional on actions, are as follows:

$$P(\cdot | a) = \frac{1}{3}s_1 + \frac{1}{6}s_2 + \frac{1}{6}s_4; \quad P(\cdot | b) = \frac{1}{2}s_1 + \frac{1}{2}s_2; \quad P(\cdot | c) = \frac{1}{3}s_2 + \frac{2}{3}s_4; \quad P(\cdot | d) = \frac{1}{3}s_1 + \frac{2}{3}s_3.$$ 

It is possible to choose payoffs of the receiver are such that $a$ is the unique optimal action at $P(\cdot | a)$, likewise for $b, c, d$. Then, $p(\cdot)$ is a communication equilibrium outcome.

We argue that no mediator can be dispensed with. With three messages and in absence of Mediator 2, there can only by three different posteriors. Mediator 2 is thus required to get four different posteriors.

With three messages, the four action distributions $p(\cdot | s_i)$ must be convex combinations of the three distributions $\mu_2(\cdot | m_i)$. Notice that these three distributions are the only possible such choice, see Figure 1. Then, we cannot dispense with the randomization of Mediator 1 since the sender in state $s_1$ prefers $\mu_2(\cdot | m_1)$ to $\mu_2(\cdot | m_3)$, and thus is not willing to randomize.

### 4 Optimal mediation

The previous examples show that in general, mediation increases the set of equilibrium outcomes and that both sides of mediation may be required. In this section, we study whether each side of mediation is required for implementing the equilibrium outcomes that are optimal for some welfare criterion. Given a welfare function $f : S \times A \to \mathbb{R}$ and $p \in \mathcal{M}(S; A)$, denote $f(p) = \sum_{s,a} \pi(s)p(a|s)f(s,a)$.

**Definition 5.** A communication equilibrium outcome $p^* \in \text{CE}(u,v,|M|)$ is optimal if:

$$f(p^*) = \max \{ f(p) : p \in \text{CE}(u,v,|M|) \}$$

for some welfare function $f$. If $p^*$ is the unique maximiser for some $f$, we say that $p^*$ is extremal.

**Remark 2.** An extremal $p^*$ is an extreme point of the convex hull of $\text{CE}(u,v,|M|)$. With no message restriction, or $|M| \geq \min\{|S|, |A|\}$, this is simply an extreme point of the polytope $\text{CE}(u,v,\infty)$. When $|M| < \min\{|S|, |A|\}$, $\text{CE}(u,v,|M|)$ is not necessarily convex. For instance, a convex combination of two equilibrium outcomes implementable with two messages, may induce
distributions \( p(s) \) that does not lie on a line segment. Put otherwise, the convex combination of two matrices of rank 2 can have a higher rank.

In the next sections, we will study two particular cases of problems where either the sender or the receiver has commitment power. To formalize and motivate these cases, denote \( 0 \) the payoff function constantly equal to 0 and observe the following:

**Observation 1.** For any \( u, v \) and \( M \),

\[
\text{CE}(u, v, |M|) = \text{CE}(0, v, |M|) \cap \text{CE}(u, 0, |M|).
\]

The sets \( \text{CE}(0, v, |M|) \) and \( \text{CE}(u, 0, |M|) \) are defined by considering in isolation the incentive compatibility constraints, either of the sender or of the receiver. In the sequel we study optimal and extremal equilibria in both cases. Notice that there are three types of extremal equilibria, the extremal points of \( \text{CE}(0, v, |M|) \) that belong to \( \text{CE}(u, 0, |M|) \), the extremal points of \( \text{CE}(u, 0, |M|) \) that belong to \( \text{CE}(0, v, |M|) \), and the ones that are extremal neither in \( \text{CE}(0, v, |M|) \) nor in \( \text{CE}(u, 0, |M|) \).

**Remark 3.** Any strategy is optimal for a player whose payoff function is constantly 0, and thus, no matter the mechanism, this player can be assumed to play truthful (for the sender) or obedient (for the receiver). The outcomes \( \text{CE}(0, v, |M|) \) are therefore the same as in the Bayesian persuasion model where the sender commits to the mapping from states to distribution over messages. Similarly, the outcomes \( \text{CE}(u, 0, |M|) \) are the same as in the mechanism design model where the receiver commits to the mapping from messages to distributions over actions.

### 4.1 Sender committed

When the sender has constant payoff, we are in the Bayesian persuasion scenario, the sender is merged with Mediator 1 and chooses \( \mu_1 : S \rightarrow \Delta(M) \). Given a payoff function \( u \) for the sender, we call Bayesian persuasion payoff, the maximal sender payoff subject to feasibility and to the incentive compatibility conditions of the receiver. This is,

\[
U^{BP}(|M|) := \max \left\{ u(p) = \sum_{s,a} \pi(s)p(a|s)u(s,a) : p \in \text{CE}(0, v, |M|) \right\}.
\]

Note that \( u \) can be replaced by any welfare function \( f \).

The next result shows the connection with the concave closure approach to Bayesian persuasion. For \( q \in \Delta(S) \), let \( \text{BR}_v(q) = \operatorname{argmax} \{ \sum_s q(s)v(s,a) : a \in A \} \) be the set of actions that are optimal for the receiver when its belief is \( q \), and denote

\[
U(q) = \max \left\{ \sum_s q(s)u(s,a) : a \in \text{BR}_v(q) \right\}
\]

the expected payoff of the sender when the belief is \( q \) and the receiver chooses a best-reply, breaking ties in favor of the sender. For each integer \( |M| \), we let

\[
\text{cav}_{|M|}(\pi) = \max \left\{ \sum_{m=1}^{\lfloor |M| \rfloor} \lambda_m U(q_m) : \forall m, \lambda_m \geq 0, q_m \in \Delta(S), \sum_{m=1}^{\lfloor |M| \rfloor} \lambda_m = 1, \sum_{m=1}^{\lfloor |M| \rfloor} \lambda_m q_m = \pi \right\}.
\]
If $|M|$ is unbounded, or simply large enough, this quantity is the concave closure of $U$, i.e. the smallest concave function $\text{cav} U$, pointwise greater or equal to $U$. This restricted concavification is found in Aybas and Turkel (2019). We have then,

**Theorem 2.** For all $|M|$, $\text{CE}(1, x)(0, v, |M|) \subset \text{CE}(0, v, |M|)$ with possible strict inclusion whenever $|M| < |A|$. However,

$$U^{BP}(|M|) = \max \{ u(p) : p \in \text{CE}(1, x)(0, v, |M|) \} = \text{cav}_{|M|} U(\pi).$$

The implication of this result is that Mediator 2 is not needed for implementing optimal equilibrium outcomes: the payoff $\text{cav}_{|M|} U(\pi)$ can be obtained with a receiver that observes the message $m$ directly. Notice that Mediator 2 can be used to generate a richer distribution of posteriors, but this is not useful for optimality. Intuitively, all Mediator 2 can do is garbling the information contained in the message, and if such a garbling was optimal, then Mediator 1 (here the sender) could have improved by targeting those posteriors. This is formally related to the number of posteriors required to achieve the concave closure of a function.

**Proof of Theorem 2.** To see the possible strict inclusion of $\text{CE}(1, x)(0, v, |M|) \subset \text{CE}(0, v, |M|)$ when $|M| < |A|$, note that if the receiver observes the message directly, then there are $|M|$ posterior beliefs. By contrast, if Mediator 2 randomises the recommended action conditional on the message, then this generates possibly $|A|$ posteriors. Generating such a conditional distribution requires Mediator 2, see Example 2. However, Mediator 2 is not required for optimality as we show now.

First, we argue that $\text{cav}_{|M|} U(\pi) \leq \max \{ u(p) : p \in \text{CE}(1, x)(0, v, |M|) \}$, that is to say, $\text{cav}_{|M|} U(\pi)$ can be obtained without Mediator 2. Consider a distribution of posteriors $(\lambda_m, q_m)_{m=1}^{|M|}$, $\lambda_m \geq 0$, $q_m \in \Delta(S)$, $\sum_{m=1}^{|M|} \lambda_m = 1$, $\sum_{m=1}^{|M|} \lambda_m q_m = \pi$. This distribution can be induced by a mechanism as follows. The sender (or Mediator 1) selects message $m$ conditional on state $s$ with probability $\lambda_m q_m(s)/\pi(s)$. Conditional on message $m$, Mediator 2 recommends an action $a_m \in \text{BR}_v(q_m)$ that achieves $U(q_m)$. The resulting $p : S \rightarrow \Delta(A)$ is feasible by construction and is incentive compatible for the receiver since recommended actions are best replies. Observe that Mediator 2 is not needed: message $m$ can be sent directly to the receiver who is willing to choose action $a_m \in \text{BR}_v(q_m)$.

Second, we show that $U^{BP}(|M|) \leq \text{cav}_{|M|} U(\pi)$. Take $p \in \text{CE}(0, v, |M|)$ and a corresponding mechanism $(\mu_1, \mu_2)$ that satisfies feasibility and incentive compatibility for the receiver, and consider the expected payoff of the sender. The main argument comes from the following observation. A strategy of the sender generates $|M|$ possible posteriors beliefs $\tilde{q}_m = (\mathbb{P}(s|m))_{s \in S}$, $m = 1, \ldots, |M|$. The strategy of the mediator $\mu_2 : M \rightarrow R_2$ generates possibly more posteriors $q_{r_2} = (\mathbb{P}(s|r_2))_{s \in S}$ since the cardinality of $R_2$ is arbitrary. But since $r_2$ depends only on $m$, those posteriors are convex combinations of the posteriors given $m$. Precisely, for all $s$ and $r_2$, $\mathbb{P}(s|r_2) = \sum_m \mathbb{P}(s, m|r_2) = \sum_m \mathbb{P}(m|r_2)\mathbb{P}(s|m, r_2) = \sum_m \mathbb{P}(m|r_2)\mathbb{P}(s|m)$,

equivalently for all $r_2$, $q_{r_2} = \sum_m \mathbb{P}(m|r_2)\tilde{q}_m$. It follows that when the receiver chooses an
optimal action given posterior \(q_r\) for each \(r\), the sender’s expected payoff is no more than \(\sum_r \mathbb{P}(r_2)U(q_{r_2})\). Define now \(\hat{U}_{\mid M} : \text{conv}\{\tilde{q}_1, \ldots, \tilde{q}_M\} \to \mathbb{R}\) as the restriction of \(U\) to the convex hull of the posteriors \(\tilde{q}_m, m = 1, \ldots, |M|\). Since all \(q_r\) belong to this convex hull, we have,

\[
\sum_r \mathbb{P}(r_2)\hat{U}_{\mid M}(q_{r_2}) \leq \sum_r \mathbb{P}(r_2)\text{cav} \hat{U}_{\mid M}(\pi) \leq \text{cav} \hat{U}_{\mid M}(\pi).
\]

The first inequality holds because \(\text{cav} \hat{U}_{\mid M} \leq \text{cav} \tilde{U}_{\mid M}\), the second since \(\text{cav} \tilde{U}_{\mid M}\) is concave and \(\pi = \sum_r \mathbb{P}(r_2)q_{r_2}\). Now, \(\hat{U}_{\mid M}\) is defined on a polytope of dimension at most \(|M| - 1\), thus its concave closure is obtained by taking at most \(|M|\) posteriors. Therefore \(\text{cav} \hat{U}_{\mid M}(\pi) \leq \text{cav}_{\mid M}U(\pi)\), which concludes the proof.

4.2 Receiver committed

When the receiver has constant payoff power, we are in the mechanism design scenario, the receiver is merged with Mediator 2 and chooses \(\mu_2 : M \to \Delta(A)\). Given a payoff function \(v\) for the receiver, we call mechanism design payoff, the maximal receiver payoff subject to feasibility and to the incentive compatibility conditions of the sender. This is,

\[
V^{MD}(|M|) = \max \left\{ v(p) = \sum_{s,a} \pi(s)p(a|s)v(s,a) : p \in \text{CE}(u, 0, |M|) \right\}.
\]

As before, \(v\) can be replaced by any welfare function \(f\).

If \(|M| \geq |S|\), we know from the revelation principle that we get all the incentive compatible mechanisms whereby the sender reports the state truthfully to the committed receiver. In particular, there is no need for Mediator 1. That is,

\[
\text{CE}(u, 0, |S|) = \text{CE}^{(x,2)}(u, 0, |S|) = \left\{ p \in \mathcal{M}(S; A) : \forall (s, t) \in S \times S, u(s, p(s)) \geq u(s, p(t)) \right\}.
\]

The following result shows that the importance of Mediator 1 crucially depends on the number of available messages.

**Theorem 3.**

1. When there are two messages, then we can dispense with Mediator 1:

\[
\text{CE}(u, 0, 2) = \text{CE}^{(x,2)}(u, 0, 2).
\]

2. When there are three messages:

   (a) Mediator 1 generates more communication equilibrium outcomes:

   \[
   \text{CE}^{(x,2)}(u, 0, 3) \subset \text{CE}(u, 0, 3), \text{ with possible strict inclusion}.
   \]

   (b) Yet, Mediator 1 does not improve the mechanism design payoff:

   \[
   V^{MD}(3) = \max \left\{ v(p) : p \in \text{CE}^{(x,2)}(u, 0, 3) \right\}.
   \]

3. With \(|M| \geq 4\) messages, it is possible that Mediator 1 improves the mechanism design payoff:
Let $p \in \mathcal{M}(S; M)$ be the mechanism of Mediator 1 and $\mu_2(m_i) \in \Delta(A)$ the distribution of Mediator 2, equivalently the strategy of the receiver. Denote $\mu_2(m_i) \in \Delta(A)$ the distribution of actions conditional on message $m_i, i = 1, 2, 3$. For each $s$, the distribution $p(s) = \sum_{m_i} \mu_1(m_i) \mu_2(m_i)$ belongs to the triangle $\mathcal{T} = \text{conv}\{\mu_2(m_1), \mu_2(m_2), \mu_2(m_3)\} \subset \Delta(A)$. Let $X = \text{conv}\{p(s), s \in S\} \subset \mathcal{T}$. We will prove that there is an optimal $p$ such that the number of extreme points of $X$ is no more than 3.

First, it is without loss of optimality to assume that for each $s$, $p(s)$ is an extreme point of $X$. Assume to the contrary that $p(s) = \lambda p(s') + (1 - \lambda)p(s'')$ with $\lambda \in (0, 1)$, $p(s'), p(s'')$ are distinct extreme points of $X$. Incentive compatibility implies $u(s, p(s)) = u(s, p(s')) = u(s, p(s''))$. Define $p', p''$ such that $p'(s) = p(s')$, $p''(s) = p(s'')$, and for all $\tilde{s} \neq s$, $p'(\tilde{s}) = p''(\tilde{s}) = p(\tilde{s})$. Since those distributions belong to $\mathcal{T}$, they satisfy feasibility. Also, since $u(s, p(s)) = u(s, p(s')) = u(s, p(s''))$, they satisfy incentive compatibility. We have $p = \lambda p' + (1 - \lambda)p''$ and by linearity $v(p) = \lambda v(p') + (1 - \lambda)v(p'')$. Thus $\max\{v(p'), v(p'')\} \geq v(p)$ and we can replace $p$ by $p'$ or $p''$ without decreasing $v$. We iterate this reasoning until each $p(s)$ is an extreme point of $X$.

Second, we assume that $\mathcal{T}$ is indeed a triangle, i.e. of dimension 2. Otherwise, it is a line segment and Theorem 1 applies. We have then a useful geometric claim. In the statement below, for two subsets $X', X''$ of $\mathbb{R}^4$ and $\lambda \in [0, 1]$, we denote,

$$\lambda X' + (1 - \lambda)X'' = \{\lambda x' + (1 - \lambda)x'' : x' \in X', x'' \in X''\}.$$  

**Claim 1.** Let $X$ be a convex polytope in $\mathcal{T}$ that has $k > 3$ extreme points.

(a) Then $X = \lambda X' + (1 - \lambda)X''$ where $\lambda \in (0, 1)$, $X', X''$ are distinct polytopes in $\mathcal{T}$, each of them having strictly less than $k$ extreme points.

(b) Any extreme point $x$ of $X$ can be written $x = \lambda x' + (1 - \lambda)x''$ with $x', x''$ extreme points of respectively $X', X''$.

(c) For any $s$, if $u(s, x) = \max\{u(s, y) : y \in X\}$, then $x = \lambda x' + (1 - \lambda)x''$ with $u(s, x') = \max\{u(s, y') : y' \in X'\}$ and $u(s, x'') = \max\{u(s, y'') : y'' \in X''\}$.

See Figure 2 for an illustration.

**Proof of Claim 1.** (a) This is a direct consequence of Theorem 2 in Lahr (2022). See references therein, see also Yaglom and Boltyanskii (1961), Problem 4-12, p.177, for geometrical constructions.
Thus it can be decomposed non-trivially as $x = \frac{1}{2}y + \frac{1}{2}z$ for distinct points $y, z \in X'$. This gives a non-trivial decomposition of $x$,

$$x = \frac{1}{2}(\lambda y + (1 - \lambda)x'') + \frac{1}{2}(\lambda z + (1 - \lambda)x''),$$

a contradiction.

(c) We have

$$u(s, x) = \lambda u(s, x') + (1 - \lambda)u(s, x'') = \max \left\{ u(s, \lambda y + (1 - \lambda)z) : y \in X', z \in X'' \right\}$$

$$= \lambda \max \left\{ u(s, y) : y \in X' \right\} + (1 - \lambda)u(s, x'').$$

Thus $u(s, x') \geq \max \left\{ u(s, y) : y \in X' \right\}$ and similarly $u(s, x'') \geq \max \left\{ u(s, z) : z \in X'' \right\}$. \qed

Now, we apply this claim to $X = \text{conv}\{p(s), s \in S\} \subset \mathbb{T}$. Suppose that $X$ has more than 3 extreme points and consider $X', X''$ given by the claim. Define then $p', p''$ such that for each $s$, $p'(s)$ is an extreme point of $X'$ that maximises $u(s, \cdot)$ over $X'$, $p''(s)$ is an extreme point of $X''$ that maximises $u(s, \cdot)$ over $X''$. These distributions are feasible and satisfy incentive compatibility. Then $p = \lambda p' + (1 - \lambda)p''$ and from linearity $\max\{v(p'), v(p'')\} \geq v(p)$, so again we can replace $p$ by $p'$ or $p''$ without decreasing $v$. We iterate this reasoning until the number of extreme points is no more than 3.

Thus, it is without loss of optimality to assume that the polytope $\text{conv}\{p(s), s \in S\}$ has at most three extreme points in $\mathbb{T}$, which we denote $\alpha_i, i = 1, 2, 3$. Then $\alpha_i$ induces the mixed action $\bar{\alpha}_i = \sum_m \alpha_i(m)\mu_2(m) \in \Delta(A)$. Also, denote $S_i = \{ s \in S : u(s, \bar{\alpha}_i) = \max_j u(s, \bar{\alpha}_j) \}$ (breaking ties in favor of the receiver’s payoff). Let then the receiver choose $\bar{\mu}_2(m_i) = \bar{\alpha}_i$, $i = 1, 2, 3$. The sender has an incentive to choose message $m_i$ for all states in $S_i$. It is thus without loss of optimality to dispense with Mediator 1.

3. By example for $|M| = 4$. Consider 4 actions $\{a, b, c, d\}$, 5 states $\{s_1, s_2, s_3, s_4, s_5\}$ and 4 messages $\{m_a, m_b, m_c, m_d\}$. The utility function of the sender is:
Consider \( p \in \mathcal{M}(S; A) \) such that:
\[
p(s_1) = a, \quad p(s_2) = b, \quad p(s_3) = c, \quad p(s_4) = \frac{1}{2}b + \frac{1}{2}d, \quad p(s_5) = \frac{1}{2}a + \frac{1}{2}d.
\]
It is easy to check that this is incentive compatible. Notice that in state \( s_5 \), the sender is indifferent between \( p(s_3) \), \( p(s_4) \) and \( p(s_5) \) and would prefer to increase the probability of action \( d \), see Figure 3.

To implement this distribution, let the receiver commit to \( \mu_2(m_a) = a, \mu_2(m_b) = b, \mu_2(m_c) = c, \mu_2(m_d) = d \). The mechanism \( \mu_1 \) is:
\[
\mu_1(s_1) = m_a, \quad \mu_1(s_2) = m_b, \quad \mu_1(s_3) = m_c, \quad \mu_1(s_4) = \frac{1}{2}m_b + \frac{1}{2}m_d, \quad \mu_1(s_5) = \frac{1}{2}m_a + \frac{1}{2}m_d.
\]
This outcome cannot be implemented without Mediator 1. First, none of the five mixed actions is a convex combination of the others, so the randomization of Mediator 1 is required. Second, in states \( s_4 \) and \( s_5 \), the sender would deviate to \( m_d \) rather than randomizing.

Now, this example is chosen such that \( p \) is an extreme point of the set of communication equilibria \( \text{CE}(u, 0, \infty) \). To see this, suppose to the contrary that \( p = \frac{1}{2}q + \frac{1}{2}r \) with \( q \neq r \). Necessarily,
\[
q(s_1) = r(s_1) = a, \quad q(s_2) = r(s_2) = b, \quad q(s_3) = r(s_3) = c, \quad q(s_4) = \nu b + (1 - \nu)d, \quad q(s_5) = \nu c + (1 - \nu)d, \\
r(s_4) = (1 - \nu)b + \nu d, \quad r(s_5) = (1 - \lambda)a + \lambda d,
\]
with either \( \lambda \) or \( \nu \) different from \( \frac{1}{2} \). If \( \lambda \neq \frac{1}{2} \), then either \( q \) (if \( \lambda > \frac{1}{2} \)) or \( r \) (if \( \lambda < \frac{1}{2} \)) is not incentive compatible since the sender in state \( s_5 \) would prefer to report \( s_3 \) to get action \( c \). If \( \lambda = \frac{1}{2} \) and \( \nu \neq \frac{1}{2} \), then either \( q \) (if \( \nu < \frac{1}{2} \)) or \( r \) (if \( \nu > \frac{1}{2} \)) is not incentive compatible since the sender in state \( s_5 \) would prefer to report \( s_4 \).

Since \( p \) is an extreme point of \( \text{CE}(u, 0, \infty) \subset \mathbb{R}^{S \times A} \), there exists a non-zero vector \( \hat{v} = (\hat{v}(s, a))_{s,a} \) in \( \mathbb{R}^{S \times A} \) such that for all \( q \in \text{CE}(u, 0, \infty) \setminus \{p\} \),
\[
\sum_{s,a} p(a | s) \hat{v}(s, a) > \sum_{s,a} q(a | s) \hat{v}(s, a).
\]
Denoting \( v(s, a) = \hat{v}(s, a)/\pi(s) \) we get that for all \( q \in \text{CE}(u, 0, \infty) \setminus \{p\} \),
\[
\sum_{s,a} \pi(s)p(a | s)v(s, a) > \sum_{s,a} \pi(s)q(a | s)v(s, a).
\]
For the utility function of the receiver given by this \( v \), we get that \( V^{MD}(4) = \sum_{s,a} \pi(s)p(a | s)v(s, a) \) is strictly bigger than whatever is obtained by a deterministic mechanism.

Finally, it is easy to extend the example to any \( |M| \geq 4 \) by adding states, messages and actions, provided that there are \( |M| \) actions and \( |M| + 1 \) states.
5 Concluding remarks.

Recap. In this paper, we consider various sets of equilibrium outcomes, we summarize here the various comparisons that we found. These sets depend on whether we consider both mediators, or only one of them. If there is no mediator at all, then we get the set of Bayes Nash equilibrium outcomes of the cheap talk game with $|M|$ messages denoted by $\text{NE}(u, v, |M|)$. We have the following inclusions:

$$\text{NE}(u, v, |M|) \subseteq \text{CE}^{(1, \times)}(u, v, |M|) \subseteq \text{CE}(u, v, |M|);$$

$$\text{NE}(u, v, |M|) \subseteq \text{CE}^{(\times, 2)}(u, v, |M|) \subseteq \text{CE}(u, v, |M|).$$

Example 1 shows that the inclusion $\text{NE}(u, v, |M|) \subset \text{CE}^{(\times, 2)}(u, v, |M|)$ can be strict. Example 2 shows that the inclusion $\text{CE}^{(1, \times)}(u, v, |M|) \cup \text{CE}^{(\times, 2)}(u, v, |M|) \subset \text{CE}(u, v, |M|)$ can also be strict. Theorem 2 compares $\text{CE}^{(1, \times)}(0, v, |M|)$ with $\text{CE}(0, v, |M|)$, Theorem 3 compares $\text{CE}(u, 0, |S|)$ with $\text{CE}^{(\times, 2)}(u, 0, |S|)$.

References


