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Entropy solutions of a diffusion equation with discontinuous hysteresis and their finite volume approximation*

Théo Belin^{1,2,3}, Pauline Lafitte^{1,2}, Vincent Lescarret^{1,2}, Jürgen Fuhrmann⁴, and Corrado Mascia⁵

¹Université Paris-Saclay

²Fédération de Mathématiques de CentraleSupélec (FR CNRS 3487)

³Laboratoire MICS CentraleSupélec

⁴Weierstraß-Institute für Angewandte Analysis und Stochastik

⁵Università Sapienza di Roma, Dipartimento di Matematica Guido Castelnuovo

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Abstract

We provide a finite volume approximation in dimension $d \geq 1$ to a quasilinear parabolic equation with discontinuous hysteresis modelling a phase change, arising as a singular limit of a pseudo-parabolic regularisation of a forward-backward diffusion equation. The convergence of the numerical solution to a suitable weak entropy solutions is shown under a parallelism assumption between the nonlinearities driving the evolution in each phase. The main challenge lies in the treatment of the discontinuous hysteresis operator in the proof of the compactness of the sequence of approximate solutions. This is achieved by regularising the hysteresis operator with a continuous one for which Hilbert inequalities are accessible and let us obtain crucial uniform translation estimates in L^1 in space. Numerical simulations, computed using a Julia-based framework for the finite volume discretisation of reaction-diffusion equations, are shown.

Keywords: Quasilinear parabolic equation, discontinuous hysteresis operator, finite volume method, entropy solutions.

1 Introduction

We consider the following quasilinear parabolic problem in the variables (u, v, λ) on the time interval $[0, T]$, with $T > 0$ fixed and on the d -dimensional torus \mathbb{T}^d :

Diffusion equation:

$$\left\{ \begin{array}{ll} \partial_t u - \Delta v = 0 & \text{in } (0, T) \times \mathbb{T}^d \\ u = \beta(v, \lambda) & \text{in } [0, T] \times \mathbb{T}^d \\ u|_{t=0}, v|_{t=0}, \lambda|_{t=0} = u^0, v^0, \lambda^0 & \in L^\infty(\mathbb{T}^d) \times L^\infty(\mathbb{T}^d) \times L^\infty(\mathbb{T}^d; [0, 1]) \end{array} \right. \quad (1)$$

where $\beta(v, \lambda) := (1 - \lambda)\kappa_0(v) + \lambda\kappa_1(v)$ is a convex combination between two continuous increasing functions κ_0, κ_1 such that $\kappa_0 < \kappa_1$ on \mathbb{R} . A hysteresis relation between λ and v closes the set of equations:

Hysteresis relation:

$$\lambda = k_P(v, \lambda^0) \quad \text{in } [0, T] \times \mathbb{T}^d \quad (2)$$

where $P = (P_0, P_1)$ is a couple of real values with $P_0 < P_1$ and the operator k_P is a *hysteresis operator* called the *completed delayed relay* operator with thresholds P : the operator $k_P(\cdot, \lambda^0)$ sets λ to 0 when v passes below P_0 and sets it to 1 when v passes above P_1 (see a complete description in [26]). This operator is *nonlocal in time*, since the value $k_P(v, \lambda^0)(t)$ cannot be deduced from the local values of v in a neighbourhood of t . This effectively introduces a memory effect in the diffusion equation (1): the variable λ indicates the progress of the transition between two phases with distinct diffusive properties (characterised by $\kappa_i, i = 0, 1$). A graphical description of the dynamics of this operator is seen on Figure 1, while its precise formulation is provided later in section 2.

We adopt a thermodynamical point of view for this model (see [11, 4]), u can be interpreted as the enthalpy density, v the temperature and λ the phase fraction.

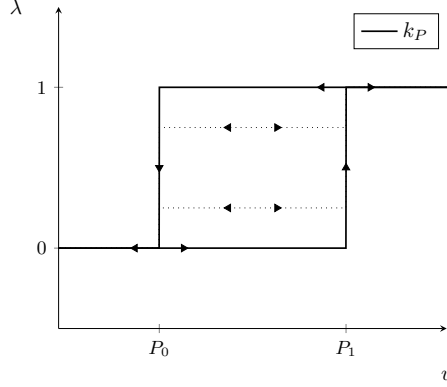


Figure 1: The graph and dynamics of the completed delayed relay operator k_P .

The set of equations (1)-(2) can arise as the singular limit as $\nu \rightarrow 0$ of a third order pseudo-parabolic equation:

$$\partial_t u_\nu - \Delta \phi(u_\nu) = \nu \Delta \partial_t u_\nu \quad (3)$$

with ϕ a coercive and *cubic-shaped* nonlinearity (see Figure 2 for a graphical representation).

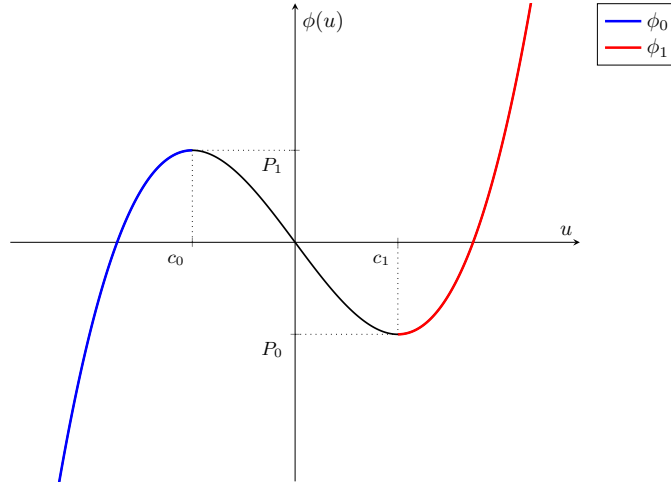


Figure 2: Cubic-shaped thermodynamical law $\phi(u) = u(u^2 - 1)$.

The properties, stability and asymptotic behaviour of the pseudo-parabolic equation (3) with fixed $\nu > 0$ have been widely studied e.g. in [17, 18]. Equations (1)-(2) and (3) have applications in a broad variety of models in ecology and population dynamics [18, 19], physical models of phase change [4] or image processing for denoising without losing contrast, in a similar spirit as for numerical use of the Perona-Malik equation [12, 9].

In the limit $\nu \rightarrow 0$ in (3), the functional relation $v = \phi(u)$ is *not* preserved due to the asymptotic

explosion of the rate of oscillations generated by the backward diffusion in the spinodal region $u \in [c_0, c_1]$. However the weak-* relative compactness of the sequence $(u_\nu)_{\nu>0}$ in $L^\infty(\mathbb{T}^d)$ gives rise to the parameter λ : the functional relation $u = \beta(v, \lambda)$ as well as *entropy inequalities* relating λ and v are proven in [20]. They take the form:

$$\partial_t \bar{G}(v) \leq g(v) \partial_t \beta(v, \lambda) \quad (4)$$

for all nondecreasing $g \in C^1(\mathbb{R})$ with $\bar{G} = (1 - \lambda)G \circ \kappa_0 + \lambda G \circ \kappa_1$ and $G' = g \circ \phi$. It turns out that the entropy inequalities (4) are a way of encoding a weak formulation of the hysteresis operator k_P (see Proposition 2.1) and are very useful for the definition of weak solutions to (1)-(2).

As explained in the initial works of Plotnikov [20, 21] ϕ determines the parameters P_0, P_1 and the functions κ_1 and κ_0 appearing in (1)-(2):

- * P_0 (resp. P_1) is the value of the local minimum (resp. maximum) of ϕ which is attained at c_1 (resp. c_0).
- * κ_0 (resp. κ_1) is some extension of the inverse of $\phi_0 := \phi|_{(-\infty, c_0]}$ (resp. $\phi_1 := \phi|_{[c_1, +\infty)}$).

Note that the choices of the extensions for κ_0 have no incidence on the original problem due to the form of the hysteresis operator k_P in (2): if $v > P_1$, then $\lambda = 1$ and $\beta(v, \lambda) = \kappa_1(v)$ and hence the definition of κ_0 above P_1 has no impact on the equations. The same comment follows for the extension of κ_1 below P_0 .

The analysis of parabolic equations involving hysteresis, either in the bulk equation [24], or on the boundary, is an important subject of investigation for the study of many physical systems (ferromagnetism, continuum mechanics). In [25] Visintin provides the continuous analysis of the diffusion model with hysteresis (1) - (2) with linear and parallel κ_0 and κ_1 . To this day, the uniqueness of solutions in higher dimensions remains an open question. A thorough answer exists in dimension 1, let us mention [16] in which the authors show the well-posedness of *pure-phase* solutions ($\lambda \in \{0, 1\}$) in dimension 1 with a Hölder-regular free boundary. The work of Terracina [23] is key in understanding that a class of solutions allowing passage in the spinodal can *not* enjoy uniqueness. If the continuous model still lacks a rigorous treatment of its well-posedness, it seems that the work of Visintin [25] paves a way to a suitable class of solutions by introducing an ϵ -regularisation of the delayed-relay operator (see Figure 3). Taking the limit $\epsilon \rightarrow 0$ selects a restricted class of solutions we may call *entropy-process solutions*. While it is claimed in [25] that the continuity of the flow in L^1 holds for the entropy-process solutions, it appears that a subtle technical flaw in the proof of this claim was found by the authors and therefore we do not rely on this result.

The goal of this work is to provide a converging numerical scheme for suitable weak solutions of Problem (1)-(2). In dimension $d = 1$, numerical experiments already exist in the literature. The 1D finite difference discretisation of the pseudo-parabolic regularisation (3) was originally investigated by Elliott in [4] and by Evans and Portilheiro in [5] and later studied more finely by Mascia and Lafitte in [13] by the use of spectral methods. It is found that the pseudo-parabolic regularisation is unsuited for efficient approximation due to the oscillations generated in the spinodal region. The semi-discrete 1D lattice regularisation of the forward-backward equation $\partial_t u = \Delta \phi(u)$ was analysed by Helmers and Herrmann in [10]. They show the convergence of this lattice regularisation to a Stefan problem with a hysteresis behaviour in the Stefan condition.

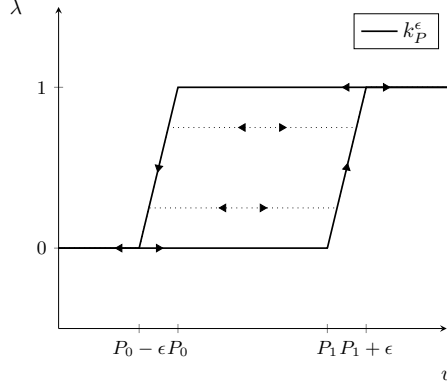


Figure 3: The graph and dynamics of the regularised delayed relay operator k_P^ϵ .

Finite volume methods are widely regarded as a natural discretisation of entropy solutions of conservation laws [15, 14]. Their structure-preserving nature makes them ideal to run simulations with discrete counterparts of properties of the continuous model. While not restricted to hyperbolic models, the physical soundness of the finite volume approximation lead to successful implementation of phase change models and degenerate advection-diffusion equations [2, 7, 8, 1].

To the authors' knowledge, there exist no converging discretisation of (1)-(2) in dimension $d > 1$. Let us be given a time-space finite volume discretisation \mathcal{D} of $(0, T) \times \mathbb{T}^d$, i.e. an *admissible mesh* $\mathcal{M} = (\mathcal{T}, \mathcal{E}, \mathcal{P})$ consisting of control volumes tiling \mathbb{T}^d , edges and center points (see Definition 3.1) and a time discretisation $(t^n)_{0 \leq n \leq N}$. In this article we provide some properties and the convergence of the following implicit finite volume scheme:

$$\left\{ \begin{array}{l} \frac{u_K^{\epsilon, n+1} - u_K^{\epsilon, n}}{\Delta t} = \frac{1}{m_K} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} (v_L^{\epsilon, n+1} - v_K^{\epsilon, n+1}) \\ u_K^{\epsilon, n} = \beta(v_K^{\epsilon, n}, \lambda_K^{\epsilon, n}) \\ \lambda_K^{\epsilon, n+1} \begin{cases} = \Gamma_\epsilon(v_K^{\epsilon, n+1}, \lambda_K^{\epsilon, n}) & \text{if } \epsilon > 0 \\ \in \Gamma_0(v_K^{0, n+1}, \lambda_K^{0, n}) & \text{if } \epsilon = 0 \end{cases} \end{array} \right. \quad (5)$$

for all $n \in \{0, \dots, N-1\}$, $K \in \mathcal{T}$. Also $u_K^{\epsilon, 0}, v_K^{\epsilon, 0}, \lambda_K^{\epsilon, 0}$ satisfy,

$$v_K^{\epsilon, 0} = \frac{1}{m_K} \int_K v^0(x) dx, \quad \lambda_K^{\epsilon, 0} = \Gamma_0(v_K^{\epsilon, 0}, \lambda_K^{\epsilon, 0}), \quad u_K^{\epsilon, 0} = \beta(v_K^0, \lambda_K^0)$$

Here Γ_0 (resp. Γ_ϵ), defined in section 2, is a discretisation of the completed delayed-relay operator (resp. its regularisation): see Figure 4. Here $K \in \mathcal{T}$ is any control volume of some admissible mesh of \mathbb{T}^d , $n \in 0, \dots, N-1$ is the time-step and $\epsilon \geq 0$ is the regularisation parameter of the hysteresis operator.

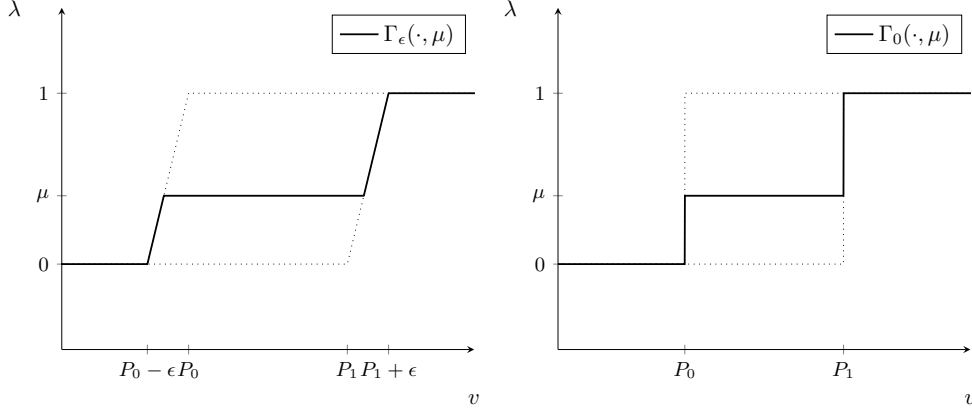


Figure 4: The graphs of $\Gamma_\epsilon(\cdot, \mu)$ ($\epsilon = 0.2$) and $\Gamma_0(\cdot, \mu)$

We denote $\text{size}(\mathcal{D}) = \max \left\{ \max_{K \in \mathcal{T}} \text{diam}(K), \max_{0 \leq n \leq N-1} t^{n+1} - t^n \right\}$. Let $(\mathcal{D}_h)_{h>0}$ be a sequence of discretisations of $(0, T) \times \mathbb{T}^d$ such that $\text{size}(\mathcal{D}_h) \rightarrow 0$ as $h \rightarrow 0$ and such that \mathcal{M}_h is a *periodic* mesh (see Definition 4.1). Our main result is the following convergence theorem.

Theorem 1.1. *Assume $\epsilon \geq 0$ and that κ_0 and κ_1 satisfy some technical assumptions (see $(\mathbf{A1})_\phi$ - $(\mathbf{A2})_\phi$ later). For $h > 0$, let $(u_h^\epsilon, v_h^\epsilon, \lambda_h^\epsilon)$ be the unique solution of (5) on \mathcal{D}_h .*

Then there exists a weak entropy solution $(u^\epsilon, v^\epsilon, \lambda^\epsilon)$ of (1)-(2) (see Definition 2.1) such that, up to a subsequence:

$$(u_h^\epsilon, v_h^\epsilon, \lambda_h^\epsilon) \xrightarrow{h \rightarrow 0} (u^\epsilon, v^\epsilon, \lambda^\epsilon) \text{ strongly in } L^p((0, T) \times \mathbb{T}^d) \text{ for all } 1 \leq p < +\infty$$

Structure of the article: In section 2 we give the mathematical formulation of the hysteresis operator k_P and its regularisation k_P^ϵ , the main assumptions on ϕ (equivalently on κ_i) and we define the notion of weak entropy solutions for (1)-(2) (see Definition 2.1). Our notion is motivated by Proposition 2.1 which rigorously establishes the entropy inequalities (4) as a suitable formulation of the dynamics of k_P .

In section 3 we show preliminary results on the scheme (5): a discrete maximum principle, the existence and uniqueness of solutions, the asymptotic preservation in the limit $\epsilon \rightarrow 0$. In particular two distinct but equivalent formulations (on v and u) are used for the analysis, the formulation on the enthalpy u is useful for the concrete implementation of the scheme. The section is concluded by the proof of discrete entropy inequalities.

We provide, in section 4, the essential discrete estimates for the convergence. We find uniform $H^1((0, T) \times \mathbb{T}^d)$ estimates for v , BV estimates in time for u and λ . The BV estimates in space needed for the compactness of approximate solutions in L^1 are only achieved under the parallaxism assumption $(\mathbf{A2})_\phi$ and on periodic meshes.

In section 5, we show the main result Theorem 1.1 using estimates of the previous section.

Last, in section 6 we describe the algorithm which uses a Julia-based finite volume framework `VoronoiFVM.jl`, developed by one of the coauthors of this work. We further provide various numerical experiments in dimensions 1 and 2, not on the torus but on a bounded domain with homogeneous Neumann boundary conditions.

2 Main assumptions, hysteresis operators and weak entropy solutions

We here make the statement of Theorem 1.1 precise by giving the main assumptions on ϕ (or κ_i equivalently) and by describing the notion of weak entropy inequalities by providing the equivalence between weak formulations of the hysteresis operator k_P (see Proposition 2.1).

2.1 Assumptions on ϕ and/or κ_i

The general assumptions we make on ϕ are the following:

$$(\mathbf{A1})_\phi \left\{ \begin{array}{l} \text{(a) } \phi : \mathbb{R} \rightarrow \mathbb{R} \text{ is locally Lipschitz continuous and for each } M > 0, \text{ we denote by } L_\phi(M) < +\infty \\ \text{the Lipschitz constant of } \phi \text{ on } [-M, M]. \\ \text{(b) There exist } c_0 < c_1 \text{ such that } \phi \text{ is } \textit{increasing} \text{ on } (-\infty, c_0] \text{ and } [c_1, +\infty), \text{ and } \textit{decreasing} \text{ on} \\ [c_0, c_1]. \\ \text{(c) } |\phi(u)| \xrightarrow{|u| \rightarrow +\infty} +\infty. \end{array} \right.$$

The example displayed on Figure 2 is $\phi(u) = u(u^2 - 1)$, associated to the double-well potential. It satisfies assumption $(\mathbf{A1})_\phi$.

While most of the analysis carried out later on a finite volume numerical scheme holds under the general assumption $(\mathbf{A1})_\phi$, an important discrete BV estimate in space is obtained under the following additional assumption on ϕ :

$(\mathbf{A2})_\phi$ There exists a positive constant $\delta > 0$ such that

$$\kappa_1(v) - \kappa_0(v) = \delta \quad \text{for all } v \in [P_0, P_1]$$

Assumption $(\mathbf{A2})_\phi$ implies that the increasing monotone branches of ϕ are *parallel* i.e. identical up to a translation of size δ in the range $[P_0, P_1]$ (see a graphical representation on Figure 5). Under assumption $(\mathbf{A2})_\phi$ and in light of the independence of the choice of extension for κ_0 and κ_1 , we can write

$$\beta(v, \lambda) = \kappa_0(v) + \delta \lambda$$

thereby the nonlinearity no longer contains multiplicative terms mixing v and λ . In such a case, if κ_0 is affine with unit slope and $\delta = 1$, we are in the setting studied by Visintin in [25].

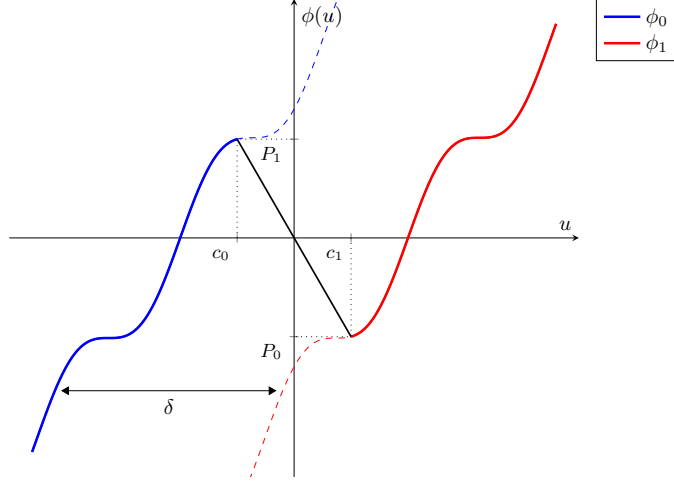


Figure 5: Example of ϕ satisfying $(\mathbf{A2})_\phi$, here $\phi_0(u) = \frac{\sin(5u)+5u}{2} - \frac{1}{2}$.

2.2 Mathematical formulations of k_P

Now we introduce the mathematical formulations of the completed delayed relay operator k_P (see [25, 25]). Given $v \in W^{1,1}(0, T; \mathbb{R}) \hookrightarrow C([0, T]; \mathbb{R})$ and $\xi \in [0, 1]$, we say that $\lambda \in k_P(v, \xi)$ if $\lambda : [0, T] \rightarrow [0, 1]$ is measurable and if it satisfies the following conditions,

$$\lambda(0) = \begin{cases} 0 & \text{if } v(0) < P_0 \\ \xi & \text{if } v(0) \in [P_0, P_1] \\ 1 & \text{if } v(0) > P_1 \end{cases} \quad (6)$$

For $t \in (0, T]$, the following *confinement conditions* are satisfied:

$$\lambda(t) \in \begin{cases} \{0\} & \text{if } v(t) < P_0 \\ [0, 1] & \text{if } v(t) \in [P_0, P_1] \\ \{1\} & \text{if } v(t) > P_1 \end{cases} \quad (7)$$

And for any $t \in (0, T]$, the following *dissipation conditions* are satisfied:

$$\begin{cases} \text{if } v(t) \neq P_0, P_1, & \text{then } \lambda \text{ is } \mathbf{constant} \text{ in a neighbourhood of } t \\ \text{if } v(t) = P_0, & \text{then } \lambda \text{ is } \mathbf{nonincreasing} \text{ in a neighbourhood of } t \\ \text{if } v(t) = P_1, & \text{then } \lambda \text{ is } \mathbf{nondecreasing} \text{ in a neighbourhood of } t \end{cases} \quad (8)$$

Weak formulations of k_P : The completed delayed relay operator $k_P : W^{1,1}(0, T) \times [0, 1] \rightarrow \mathcal{P}(BV(0, T))$ is a multi-valued operator which is closed (strong in $W^{1,1}$, weak-* in $BV(0, T)$) and

discontinuous (i.e. no continuous selection exist). The reason for using this multivalued operator instead of any other single-valued selection \tilde{k}_P such that $\tilde{k}_P(v, \xi) \in k_P(v, \xi)$ (e.g. the outer or inner relay h_P^+, h_P^-) is because $\tilde{k}_P : W^{1,1}(0, T) \times [0, 1] \rightarrow BV(0, T)$ does not enjoy useful weak formulations and is therefore inadequate for the analysis in a PDE context.

Relaxed operator: The confinement and dissipation conditions (7)-(8) have equivalent formulations, which are more fitting for the weak formulation of (1)-(2). Given $v \in W^{1,1}(0, T), \xi \in [0, 1]$, we have $\lambda \in k_P(v, \xi)$ if and only if the two following relations are satisfied:

The confinement condition (7) is rewritten as

$$0 \leq \lambda \leq 1, \quad \begin{cases} (\lambda - 1)(v - P_1) \geq 0 \\ \lambda(v - P_0) \geq 0 \end{cases} \quad \text{a.e in } (0, T), \quad (9)$$

and the dissipation condition (8) is equivalent to the property that for any $t \in (0, T]$,

$$\int_{[0,t]} v \, d\mu = \int_{[0,t]} P_1 \, d\mu^+ - \int_{[0,t]} P_0 \, d\mu^- =: \Psi(\lambda; [0, t]) \quad (10)$$

where $\mu = \partial_t \lambda \in \mathcal{M}_b([0, T])$ is the measure representation of the distributional derivative of λ (recall that $\lambda \in BV(0, T)$). Also μ^+, μ^- are the positive measures resulting from the Jordan decomposition $\mu = \mu^+ - \mu^-$. This formulation is found in e.g. [25] which proves useful in the formulation of the weak problem in the case of β linear.

Entropy inequalities: Another formulation of the completed delayed relay operator k_P can be expressed as a family of entropy inequalities. For a given $v \in W^{1,1}(0, T) \subset C([0, T])$ and $\lambda \in BV(0, T)$ they read:

$$\overline{G}(v)_t \leq g(v) \partial_t \beta(v, \lambda) \quad \text{in } (0, T) \quad (11)$$

for any nondecreasing $g \in C^1(\mathbb{R})$ and where $\overline{G}(v) := (1 - \lambda)G(\kappa_0(v)) + \lambda G(\kappa_1(v))$ with $G'(u) = (g \circ \phi)(u)$. The quantities $\partial_t \overline{G}(v)$ and $\partial_t \beta(v, \lambda)$ are understood in the sense of measures.

The following proposition makes the link between the dissipation conditions and the entropy inequalities rigorous:

Proposition 2.1. *Let $v \in W^{1,1}(0, T), \lambda \in BV(0, T)$ and $\xi \in \mathbb{R}$ then we have the following set of equivalences:*

- (i) $\lambda \in k_P(v, \xi)$.
- (ii) (λ, v) satisfy (6), (9) and (10).
- (iii) (λ, v) satisfy (6), (9) and (11) for any nondecreasing $g \in C^1(\mathbb{R})$.

A partial version of Proposition 2.1 is stated in [21, Theorem 1.2]. We provide a complete proof in Appendix A.

Remark 2.1. Note that the entropy inequalities (11) depend *a priori* on ϕ and the associated functions κ_0, κ_1 . However the equivalence result above shows that the inequalities in fact encode a relationship between v and λ which is independent of the specific ϕ satisfying **(A1)** $_\phi$.

Regularisation of the delayed relay operator For the purpose of the analysis (both continuous and numerical) it is important to introduce the ϵ -regularisation of the hysteresis operator. Let $\epsilon > 0$, we introduce $k_P^\epsilon : W^{1,1}(0, T) \times [0, 1] \rightarrow W^{1,1}(0, T)$ defined by an implicit relation. For $(v, \xi) \in W^{1,1}(0, T) \times [0, 1]$, we have

$$\lambda = k_P^\epsilon(v, \xi) \stackrel{\text{def}}{\iff} \lambda \in k_P(v - \epsilon(2\lambda - 1), \xi) \quad (12)$$

The operator k_P^ϵ is now single-valued and continuous from $W^{1,1}(0, T)$ onto $BV(0, T)$ (see Figure 3). The reason for introducing the regularisation (12) is because it allows to exploit specific inequalities, called Hilpert inequalities (see [22]), to be used in the proof of an L^1 contraction inequality in λ and v . In the discrete analysis, this contraction inequality passes to the limit as $\epsilon \rightarrow 0$ and allows to obtain the strong compactness of the sequence of approximate solutions in L^1 , crucial for the convergence result in this nonlinear setting.

2.3 Weak entropy solutions to the forward-backward diffusion equation

In the weak formulation of the equations, we choose the entropy inequalities to describe the dissipation condition. They allow to express a suitable notion of weak solutions under the general assumption $(\mathbf{A1})_\phi$.

Definition 2.1. Assume $\epsilon \geq 0$. Let $(u^0, v^0, \lambda^0) \in L^\infty(\mathbb{T}^d)$ be given initial conditions such that $u^0 = \beta(v^0, \lambda^0)$. We say that $(u_\epsilon, v_\epsilon, \lambda_\epsilon) \in L^\infty((0, T) \times \mathbb{T}^d)$ is a *weak entropy solution* of (1)-(2) if

(i) $v_\epsilon \in L^2((0, T), H^1(\mathbb{T}^d))$ and for any $\xi \in C_c^\infty([0, T] \times \mathbb{T}^d)$, we have,

$$\int_0^T \int_{\mathbb{T}^d} -u_\epsilon \partial_t \xi + \nabla v_\epsilon \cdot \nabla \xi \, dx \, dt + \int_{\mathbb{T}^d} u^0(x) \xi(0, x) \, dx = 0$$

(ii) Confinement condition:

$$0 \leq \lambda_\epsilon \leq 1, \quad \begin{cases} (\lambda_\epsilon - 1)(w_\epsilon - P_1) \geq 0 \\ \lambda(w - \epsilon - P_0) \geq 0 \end{cases} \quad \text{a.e in } (0, T) \times \mathbb{T}^d,$$

where $w_\epsilon := v_\epsilon - \epsilon(2\lambda_\epsilon - 1)$.

(iii) Entropy inequalities: for any nondecreasing $g \in C^1(\mathbb{R})$ such that $g, g' \in C_b(\mathbb{R})$, and any $\xi \in C_c^\infty((0, T) \times \mathbb{T}^d)$, $\xi \geq 0$, we have,

$$\int_0^T \int_{\mathbb{T}^d} -\bar{G}(w_\epsilon) \partial_t \xi + \beta(w_\epsilon, \lambda_\epsilon) \partial_t (g(w_\epsilon) \xi) \, dx \, dt \leq 0$$

Remark 2.2. In the linear case, Visintin provides a distinct formulation of weak solutions which rely on the weak dissipation condition (10). Under the slightly more general assumption $(\mathbf{A1})_\phi$ - $(\mathbf{A2})_\phi$, it is also possible to provide a notion of weak solutions where the entropy inequalities are replaced by the weak dissipation condition:

$$\int_{\mathbb{T}^d} K_0(u_\epsilon(t, x)) - K_0(u^0(x)) + \delta \Psi_\epsilon(\lambda_\epsilon(\cdot, x), [0, t]) \, dx + \iint_{(0, t) \times \mathbb{T}^d} |\nabla v_\epsilon|^2 \, dx \leq 0$$

for all $t \in [0, T]$, where $K'_0 = \kappa_0$ and $\Psi_\epsilon(\lambda(\cdot, x), [0, t]) := \Psi(\lambda(\cdot, x), [0, t]) + 2\epsilon(\lambda(t, x) - \lambda_0(x))(\lambda(t, x) + \lambda_0(x) - 1)$, recalling Ψ is defined in (10).

3 Finite volume discretisation of entropic solutions and elementary results

We give here a detailed description of the finite volume approximation scheme (5) used to approach weak *entropy* solutions of the forward-backward diffusion problem and show some of its elementary properties. We first describe the basic notations for finite volume approximation and related discrete spaces and norms. We then introduce our implicit finite volume scheme for both the original and regularised hysteresis relation (i.e. $\epsilon \geq 0$). It is possible to formulate the scheme in terms of the temperature v or the enthalpy u leading to two equivalent formulations when $\epsilon > 0$ but *a priori* distinct when $\epsilon = 0$. While the temperature formulation is well-suited for uniqueness, the enthalpy formulation naturally enjoys solvability and L^∞ -stability. In light of these results, we can actually show the equivalence between the temperature and the enthalpy formulations when $\epsilon = 0$.

3.1 Finite volume discretisation of $(0, T) \times \mathbb{T}^d$

For the space discretisation, we describe the notion of *admissible mesh* which is a natural framework for the discretisation of conservation laws. We hereafter define a standard notion (found e.g. in [6]) of admissible mesh for the torus \mathbb{T}^d .

Definition 3.1. Let $\mathcal{M} := (\mathcal{T}, \mathcal{F}, \mathcal{P})$ be a triplet satisfying the following description:

Control volumes: \mathcal{T} is a finite set of disjoint open convex bounded polygons such that,

$$\mathbb{T}^d = \bigcup_{K \in \mathcal{T}} \overline{K}.$$

Edges: \mathcal{E} is a family of closed subsets \mathbb{T}^d called *edges* contained in hyperplanes and with positive Lebesgue measure of dimension $d - 1$.

Center points: \mathcal{P} is a finite family of points in \mathbb{T}^d .

We say that \mathcal{M} is an *admissible mesh* of the polygonal domain \mathbb{T}^d if the following conditions are satisfied:

- (i) For any $K \in \mathcal{T}$, there exists a subfamily $\mathcal{E}_K \subset \mathcal{E}$ of edges such that $\partial K = \bigcup_{\sigma \in \mathcal{E}_K} \sigma$.
- (ii) For any $\sigma \in \mathcal{E}$, there exist two distinct control volumes $K, L \in \mathcal{T}$ such that $\sigma = K|L =: \overline{K} \cap \overline{L}$.
- (iii) For each $K \in \mathcal{T}$ there exists a unique center point $x_K \in \mathcal{P}$ such that $x_K \in \overline{K}$. Additionally, for any $K, L \in \mathcal{T}$, if $K|L \in \mathcal{E}$ then the line segment $[x_K, x_L]$ is orthogonal to $K|L$.

If $\mathcal{M} = (\mathcal{T}, \mathcal{E}, \mathcal{P})$ is an admissible mesh of \mathbb{T}^d , we can define the following complementary notations:

Neighbouring volumes: For any $K \in \mathcal{T}$, we define

$$\mathcal{N}(K) := \{L \in \mathcal{T} : K|L \in \mathcal{E}\}$$

Geometric quantities: For any $\sigma = K|L \in \mathcal{E}$, and for any $K \in \mathcal{T}$ we define the following positive quantities

Measure of K : $m_K := m_K$ is the d -dimensional Lebesgue measure of K

Measure of σ : $m_\sigma := \mathcal{H}^{d-1}(\sigma)$ is the $(d-1)$ -dimensional Hausdorff measure of σ .

Distance: $d_\sigma := |x_K - x_L|$.

Transmissibility: $\tau_\sigma := \frac{m_\sigma}{d_\sigma}$.

Normal vector: $\mathbf{n}_{K,\sigma} := \frac{x_L - x_K}{d_\sigma} (\in \mathbb{S}^{d-1})$ is the outward unit normal vector of the volume K on its edge σ .

Size of \mathcal{M} : $\Delta x := \max\{\text{diam}(K) \mid K \in \mathcal{T}\}$

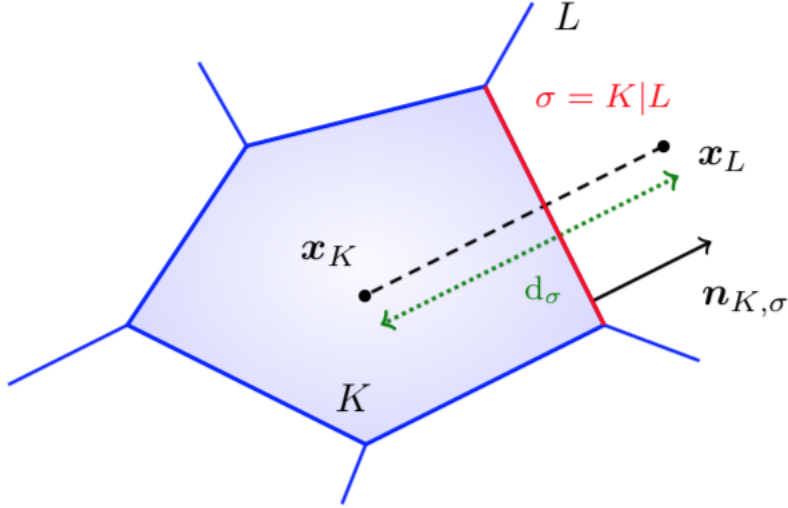


Figure 6: A control volume K in an admissible mesh.

When \mathcal{M} is an admissible mesh of \mathbb{T}^d and $(t^n)_{0 \leq n \leq N}$ is such that $0 = t^0 < t^1 < \dots < t^N = T$ we say that the couple $\mathcal{D} := (\mathcal{M}, (t^n)_{0 \leq n \leq N})$ is an *admissible discretisation* of size (h, k) where

$$\Delta t := \max\{t^{n+1} - t^n \mid 0 \leq n \leq N-1\}$$

The usual finite dimensional functional spaces associated with admissible meshes are the piecewise constant reconstructions on \mathcal{M} and \mathcal{D} :

$$X(\mathcal{M}) := \left\{ \sum_{K \in \mathcal{T}} u_K \mathbb{1}_K \mid u_K \in \mathbb{R}, \forall K \in \mathcal{T} \right\}$$

$$X(\mathcal{D}) := \left\{ \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} v_K^n \mathbb{1}_{[t^n, t^{n+1}) \times K} \mid v_K^n \in \mathbb{R}, \forall 0 \leq n \leq N-1, \forall K \in \mathcal{T} \right\}$$

For any $p \in [1, +\infty]$, $X(\mathcal{M})$, (resp. $X(\mathcal{D})$) is a finite dimensional subspace of $L^p(\mathbb{T}^d)$ (resp. $L^p((0, T) \times \mathbb{T}^d)$). Hence they are naturally equipped with the corresponding induced topology. We also equip $X(\mathcal{M})$ and $X(\mathcal{D})$ with usual topologies for the analysis of the convergence of the scheme.

Definition 3.2. Let $u \in X(\mathcal{M})$, $v \in X(\mathcal{D})$ and let $p \in [1, +\infty]$.

(i) The $H^1(\mathbb{T}^d)$ -discrete semi-norm and norm are

$$|u|_{H^1(\mathbb{T}^d), \mathcal{M}}^2 := \sum_{K|L \in \mathcal{E}} \tau_{K|L} (u_K - u_L)^2$$

$$\|u\|_{H^1(\mathbb{T}^d), \mathcal{M}}^2 := \|u\|_{L^2(\mathbb{T}^d)}^2 + |v|_{H^1(\mathbb{T}^d), \mathcal{M}}^2$$

(ii) The $L^p(0, T; H^1(\mathbb{T}^d))$ -discrete semi-norm and norm are

$$|v|_{L^p(0, T; H^1(\mathbb{T}^d)), \mathcal{D}}^p = \sum_{n=0}^{N-1} (t^{n+1} - t^n) |v(t^n, \cdot)|_{H^1(\mathbb{T}^d), \mathcal{M}}^p$$

$$\|v\|_{L^p(0, T; H^1(\mathbb{T}^d)), \mathcal{D}}^p = \|v\|_{L^p(0, T; L^2(\mathbb{T}^d))}^p + |v|_{L^p(0, T; H^1(\mathbb{T}^d)), \mathcal{D}}^p$$

(iii) The $H^1(0, T; L^p(\mathbb{T}^d))$ -discrete semi-norm and norm are

$$|v|_{H^1(0, T; L^p(\mathbb{T}^d)), \mathcal{D}}^2 = \sum_{n=0}^N \frac{1}{t^{n+1} - t^n} \|v(t^{n+1}, \cdot) - v(t^n, \cdot)\|_{L^p(\mathbb{T}^d)}^2$$

$$\|v\|_{H^1(0, T; L^p(\mathbb{T}^d)), \mathcal{D}}^2 = \|v\|_{L^2(0, T; L^p(\mathbb{T}^d))}^2 + |v|_{H^1(0, T; L^p(\mathbb{T}^d)), \mathcal{D}}^2$$

3.2 Temperature-phase formulation

Let $\epsilon \geq 0$ be fixed and let us introduce an implicit finite volume scheme for (1)-(2) on a given admissible discretisation $\mathcal{D} := ((t^n)_{0 \leq n \leq N}, \mathcal{M})$, with unknowns $(u_{\mathcal{D}}^\epsilon, v_{\mathcal{D}}^\epsilon, \lambda_{\mathcal{D}}^\epsilon) \in X(\mathcal{D})$. The variable $u_{\mathcal{D}}^\epsilon$ is called the *discrete enthalpy density*, $v_{\mathcal{D}}^\epsilon$ is called the *discrete temperature* and $\lambda_{\mathcal{D}}^\epsilon$ is called the *discrete phase fraction*.

Let us be given the initial data $(u^0, v^0, \lambda^0) \in L^\infty(\mathbb{T}^d)$ satisfying the initial hysteresis relation. To describe this relation let us introduce the region of \mathbb{R}^2

$$\mathcal{R}_0 := (-\infty, P_0] \times \{0\} \cup [P_0, P_1] \times [0, 1] \cup [P_1, +\infty) \times \{1\}$$

and for $\epsilon > 0$ we define:

$$\mathcal{R}_\epsilon := \{(v, \lambda) \in \mathbb{R}^2 \mid (v - \epsilon(2\lambda - 1), \lambda) \in \mathcal{R}_0\}$$

These regions correspond to the confinement conditions for both the regularised ($\epsilon > 0$) and non-regularised ($\epsilon = 0$) hysteresis operator. Therefore we ask that

$$(v^0, \lambda^0) \in \mathcal{R}_\epsilon \quad \text{a.e. in } \mathbb{T}^d$$

The knowledge of v^0 and λ^0 suffice to determine u^0 through the condition $u^0 = \beta(v^0, \lambda^0)$. The discrete initial data $v_{\mathcal{M}}^{0, \epsilon} \in \mathcal{M}$ is taken as the *mean value* of v^0 over the individual volumes of \mathcal{M} i.e. for each $K \in \mathcal{T}$ we define

$$v_K^{\epsilon, 0} := \frac{1}{m_K} \int_K v^0(x) \, d\mathcal{L}^d(x) \quad (13)$$

The initial value $\lambda_{\mathcal{M}}^{\epsilon,0}$ must satisfy the confinement condition (9) and $u_{\mathcal{M}}^{\epsilon,0}$ is given by the functional relation. That is we assume:

$$\lambda_K^{\epsilon,0} \in \Gamma_{\epsilon}(v_K^{\epsilon,0}, \lambda_K^{\epsilon,0}), \quad u_K^{\epsilon,0} := \beta(v_K^{\epsilon,0}, \lambda_K^{\epsilon,0}) \quad (14)$$

where for $\lambda \in [0, 1]$, $v \in \mathbb{R}$ we recall $\beta(v, \lambda) := (1 - \lambda)\kappa_0(v) + \lambda\kappa_1(v)$ and in the case $\epsilon = 0$, $\Gamma_0 : \mathbb{R} \times [0, 1] \rightarrow \mathcal{P}(\mathbb{R})$ is a multi-valued map which describes the hysteresis relation (2) and defined as follows:

$$\Gamma_0(v, \lambda) := \begin{cases} \{0\} & \text{if } v < P_0 \\ [0, \lambda] & \text{if } v = P_0 \\ \{\lambda\} & \text{if } P_0 < v < P_1 \\ [\lambda, 1] & \text{if } v = P_1 \\ \{1\} & \text{if } v > P_1 \end{cases}$$

In the case $\epsilon > 0$ then $\Gamma_{\epsilon}(v, \lambda)$ is single-valued and is defined as the unique real number z such that $z \in \Gamma_0(v - \epsilon(2z - 1), \lambda)$. Explicitly, we can write:

$$\Gamma_{\epsilon}(v, \lambda) := \begin{cases} 0 & \text{if } v < P_0 - \epsilon \\ \frac{v - P_0 + \epsilon}{\epsilon} & \text{if } v \in [P_0 - \epsilon, P_0 - (1 - \lambda)\epsilon] \\ \lambda & \text{if } v \in [P_0 - (1 - \lambda)\epsilon, P_1 + \lambda\epsilon] \\ \frac{v - P_1}{\epsilon} & \text{if } v \in [P_1 + \lambda\epsilon, P_1 + \epsilon] \\ 1 & \text{if } v > P_1 + \epsilon \end{cases} \quad (15)$$

These maps are in fact a good tool for the discretisation of the hysteresis operator as our convergence result later will show. See Figure 7 for a graphical representation of both Γ_0 and the regularisation Γ_{ϵ} .

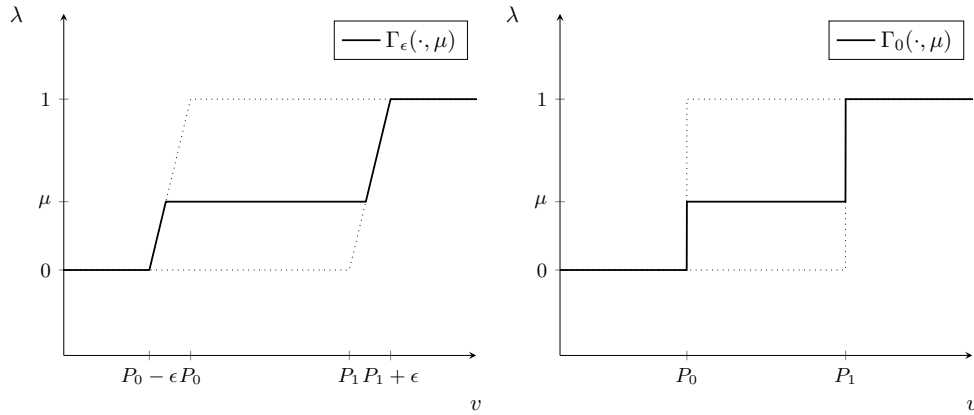


Figure 7: The graphs of $\Gamma_{\epsilon}(\cdot, \mu)$ and $\Gamma_0(\cdot, \mu)$

Hence for $\epsilon \geq 0$, the discrete problem reads:

Find $(u_{\mathcal{D}}^{\epsilon}, v_{\mathcal{D}}^{\epsilon}, \lambda_{\mathcal{D}}^{\epsilon}) \in X(\mathcal{D})$ such that for any $K \in \mathcal{T}, 0 \leq n \leq N - 1$

$$\left\{ \begin{array}{l} \frac{u_K^{\epsilon,n+1} - u_K^{\epsilon,n}}{\Delta t} = \frac{1}{m_K} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} (v_L^{\epsilon,n+1} - v_K^{\epsilon,n+1}) \\ u_K^{\epsilon,n} = \beta(v_K^{\epsilon,n}, \lambda_K^{\epsilon,n}) \\ \lambda_K^{\epsilon,n+1} \begin{cases} = \Gamma_\epsilon(v_K^{\epsilon,n+1}, \lambda_K^{\epsilon,n}) & \text{if } \epsilon > 0 \\ \in \Gamma_0(v_K^{0,n+1}, \lambda_K^{0,n}) & \text{if } \epsilon = 0 \end{cases} \end{array} \right. \quad (16)$$

For a function $v \in X(\mathcal{M})$, the term $\tau_{K|L}(v_L - v_K)$ corresponds to the two-point flux of v across the edge σ and the sum $\sum_{L \in \mathcal{N}(K)} \tau_{K|L}(v_L - v_K)$ can therefore be interpreted as an approximation of the *flux-balance* on the boundary of the volume K .

Formally we justify the finite volume approximation of $-\Delta v$ by applying Stokes formula:

$$\begin{aligned} -\Delta v(t^n, x_K) &= \frac{1}{m_K} \int_{\partial K} \nabla v \cdot n \, d\mathcal{H}^{d-1} + o_{\Delta x \rightarrow 0}(1) \\ &\xrightarrow{\text{discretisation}} \frac{1}{m_K} \sum_{L \in \mathcal{N}(K)} \frac{(v_K - v_L)}{d_\sigma} m_\sigma \\ &= \frac{1}{m_K} \sum_{L \in \mathcal{N}} \tau_{K|L} (v_K - v_L) \end{aligned}$$

Below we show that in the discrete problems, the limit $\epsilon \rightarrow 0$ is stable.

Lemma 3.1 (Stability as $\epsilon \rightarrow 0$). *Assume that for each $\epsilon > 0$, $(u_\mathcal{D}^\epsilon, v_\mathcal{D}^\epsilon, \lambda_\mathcal{D}^\epsilon)$ is a solution to (16) and that we have the convergence*

$$(u_\mathcal{D}^\epsilon, v_\mathcal{D}^\epsilon, \lambda_\mathcal{D}^\epsilon) \xrightarrow{\epsilon \rightarrow 0} (u_\mathcal{D}^0, v_\mathcal{D}^0, \lambda_\mathcal{D}^0) \quad \text{a.e. in } (0, T) \times \mathbb{T}^d$$

Then $(u_\mathcal{D}^0, v_\mathcal{D}^0, \lambda_\mathcal{D}^0)$ is a solution to (16) for $\epsilon = 0$.

Proof. By continuity of $\beta : \mathbb{R}^2 \rightarrow \mathbb{R}$ and of for each $K \in \mathcal{M}$ of the scheme function

$$S_K : \left\{ \begin{array}{l} \mathbb{R}^4 \longrightarrow \mathbb{R} \\ (u_1, u_2, v_1, v_2) \longmapsto \frac{u_2 - u_1}{\Delta t} - \frac{1}{m_K} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} (v_2 - v_1) \end{array} \right.$$

it is clear that the first and second equations of (16) are verified by $(u_\mathcal{D}^0, v_\mathcal{D}^0, \lambda_\mathcal{D}^0)$. For the third equation, note that the graph of Γ_ϵ is closed for any $\epsilon \geq 0$. Now the graph of Γ_ϵ converges to the graph of Γ_0 as $\epsilon \rightarrow 0$ for the Hausdorff distance in \mathbb{R}^3 ; This fact leads to the convergence for the third equation as $\epsilon \rightarrow 0$. □

Remark 3.1. The discrete problem (16) may be considered as a problem posed purely in terms of only two unknowns: the temperature v and the phase fraction λ . This is due to the functional relation $u = \beta(v, \lambda)$. Hence, we refer to this formulation as the *temperature-phase formulation*.

3.3 Enthalpy-phase formulation

We can express the problem in a different form, called the *enthalpy-phase formulation*, in which the enthalpy variable u becomes the main unknown of the problem, instead of v . This formulation is well-suited for the analysis of the existence and the L^∞ -stability of the scheme as we will see in the next section.

Let $\epsilon > 0$ and $\lambda \in [0, 1]$, observe that $\Gamma_\epsilon(\cdot, \lambda)$ is continuous non-decreasing, and because of the strict monotonicity of each κ_i , the map $\gamma_\epsilon(\cdot, \lambda) := \beta(\cdot, \Gamma_\epsilon(\cdot, \lambda))$ is continuous increasing. Therefore, we may define its inverse:

$$\Phi_\epsilon : \begin{cases} \mathbb{R} \times [0, 1] & \longrightarrow \mathbb{R} \\ (u, \lambda) & \longmapsto [\beta(\cdot, \Gamma_\epsilon(\cdot, \lambda))]^{-1}(u) \end{cases} \quad (17)$$

For a graphical representation of the graphs of Φ_0 and Φ_ϵ see Figure 8.

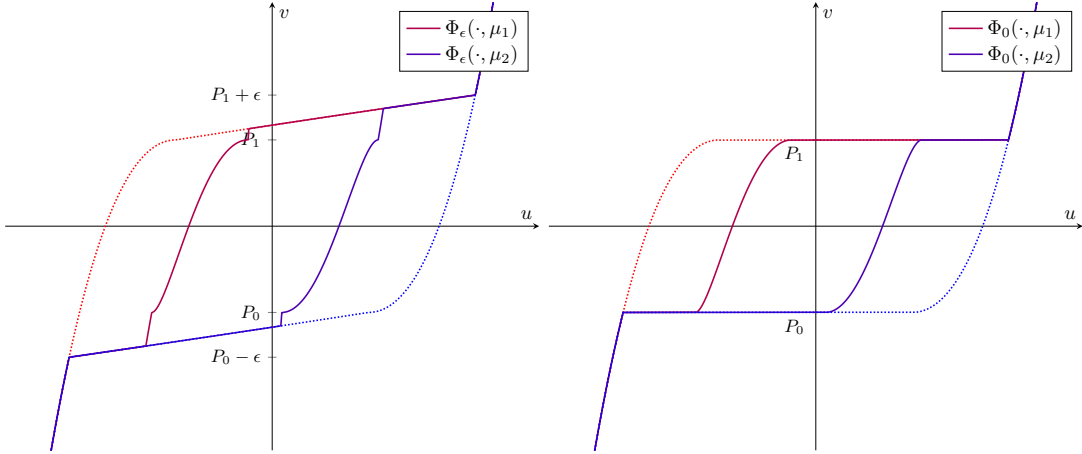


Figure 8: The graphs of $\Phi_\epsilon(\cdot, \mu_1)$, $\Phi_\epsilon(\cdot, \mu_2)$ and $\Phi_0(\cdot, \mu_1)$, $\Phi_0(\cdot, \mu_2)$ for $\mu_1 \leq \mu_2$

We gather some elementary properties of Φ_ϵ in the following lemma.

Lemma 3.2. *Assume $\epsilon > 0$, $\Phi_\epsilon : \mathbb{R} \times [0, 1] \longrightarrow \mathbb{R}$ has the following properties,*

- (i) *For any $\lambda \in [0, 1]$, $u \mapsto \Phi_\epsilon(u, \lambda)$ is nondecreasing.*
- (ii) *For any $u \in \mathbb{R}$, $\lambda \mapsto \Phi_\epsilon(u, \lambda)$ is nonincreasing.*
- (iii) *For any $u \in (-\infty, \kappa_0(P_0 - \epsilon)] \cup [\kappa_1(P_1 + \epsilon), +\infty)$, $\lambda \in [0, 1]$, we have*

$$\Phi_\epsilon(u, \lambda) = \phi(u).$$

- (iv) *For any $M > 0$ such that $[\kappa_0(P_0 - \epsilon), \kappa_1(P_1 + \epsilon)] \subset [-M, M]$, for any $u_1, u_2 \in [-M, M]$ and $\lambda_1, \lambda_2 \in [0, 1]$,*

$$|\Phi_\epsilon(u_1, \lambda_1) - \Phi_\epsilon(u_2, \lambda_2)| \leq L_\phi(M)(|u_1 - u_2| + C_\kappa(M)|\lambda_1 - \lambda_2|).$$

where $C_\kappa(M) := \sup_{v \in \phi([-M, M])} \{\kappa_1(v) - \kappa_0(v)\}$.

(v) $\Phi_\epsilon \xrightarrow{\epsilon \rightarrow 0} \Phi_0$ uniformly on $\mathbb{R} \times [0, 1]$ and $\Phi_0(\cdot, \lambda)$ is a left inverse of $\gamma_\epsilon(\cdot, \lambda)$ i.e. for any $v \in \mathbb{R}$,

$$\Phi_0(\beta(v, \Gamma_0(v, \lambda)), \lambda) = \{v\}. \quad (18)$$

(vi) For any $(u, \lambda) \in \mathbb{R} \times [0, 1]$, let $v := \Phi_\epsilon(u, \lambda)$ then

$$v = \Phi_\epsilon(u, \Gamma_\epsilon(v, \lambda)).$$

Proof. (i) & (ii): This is a direct consequence of the fact that γ_ϵ is increasing and continuous with respect to u and λ .

(iii): Let $\lambda \in [0, 1]$, and assume that $u \leq \kappa_0(P_0 - \epsilon)$. Denote $v := \Phi_\epsilon(u, \lambda)$, $\mu := \Gamma_\epsilon(v, \lambda)$; by definition of Φ_ϵ (17) we have $u = \beta(v, \mu)$, therefore

$$\begin{aligned} \kappa_0(P_0 - \epsilon) &\geq \beta(v, \mu) \\ &= (1 - \mu)\kappa_0(v) + \mu\kappa_1(v) \end{aligned} \quad (19)$$

Now because $\kappa_0 \leq \kappa_1$ on \mathbb{R} , we also have $\kappa_0(P_0 - \epsilon) \leq (1 - \mu)\kappa_0(P_0 - \epsilon) + \mu\kappa_1(P_0 - \epsilon)$. Combining this with (19) we find

$$(1 - \mu)(\kappa_0(P_0 - \epsilon) - \kappa_0(v)) + \mu(\kappa_1(P_0 - \epsilon) - \kappa_1(v)) \geq 0$$

From which we deduce that there exists $i \in \{0, 1\}$ such that $\kappa_i(v) \leq \kappa_i(P_0 - \epsilon)$ and by monotonicity of κ_i this implies $v \leq P_0 - \epsilon$. Hence from (15) we get $\mu = 0$ and therefore $u = \beta(v, 0) = \kappa_0(v)$, finally $\Phi_\epsilon(u, \lambda) = v = \kappa_0^{-1}(u) = \phi(u)$. A symmetrical reasoning is applied for the case $u \geq \kappa_1(P_1 + \epsilon)$.

(iv): By assumption on M and since κ_i are inverses of the locally Lipschitz, increasing branches of ϕ , it is not restrictive to assume that for any $v_1, v_2 \in \phi([-M, M])$,

$$|\kappa_i(v_1) - \kappa_i(v_2)| \geq \frac{1}{L_\phi(M)} |v_1 - v_2| \quad (20)$$

Lipschitz constant in u : Let $u_1, u_2 \in [-M, M]$ and $\lambda \in [0, 1]$. We have $v_j := \Phi_\epsilon(u_j, \lambda) \in \phi([-M, M])$, $j = 1, 2$. We infer from the definition of Φ_ϵ that,

$$\begin{aligned} u_1 - u_2 &= \beta(v_1, \mu_1) - \beta(v_2, \mu_2) \\ &= \left(1 - \frac{\mu_1 + \mu_2}{2}\right)(\kappa_0(v_1) - \kappa_0(v_2)) \\ &\quad + \frac{\mu_1 + \mu_2}{2}(\kappa_1(v_1) - \kappa_1(v_2)) + c(v_1, v_2)(\mu_1 - \mu_2) \end{aligned}$$

where $\mu_j := \Gamma_\epsilon(v_j, \lambda)$ and $c(v_1, v_2) := \frac{(\kappa_1 - \kappa_0)(v_1) + (\kappa_1 - \kappa_0)(v_2)}{2} > 0$. Now κ_0, κ_i and $\Gamma_\epsilon(\cdot, \lambda)$ are nondecreasing, hence the three terms appearing in the last identity have the same sign. Therefore, taking the absolute value and using (20) twice, yields

$$|u_1 - u_2| \geq \frac{1}{L_\phi(M)} |v_1 - v_2| + c(v_1, v_2) |\mu_1 - \mu_2| \geq \frac{1}{L_\phi(M)} |v_1 - v_2|$$

Which finally yields $L_\phi(M)$ as a local Lipschitz constant of Φ_ϵ on $[-M, M] \times [0, 1]$ in the variable u .

Lipschitz constant in λ : Let $u \in [-M, M]$, $\lambda_1, \lambda_2 \in [0, 1]$. We have $v_j := \Phi_\epsilon(u, \lambda_j) \in \phi([-M, M])$ and from the definition of Φ_ϵ we find this time

$$\begin{aligned} u - u &= \beta(v_1, \mu_1) - \beta(v_2, \mu_2) \\ 0 &= \left(1 - \frac{\mu_1 + \mu_2}{2}\right)(\kappa_0(v_1) - \kappa_0(v_2)) \\ &\quad + \frac{\mu_1 + \mu_2}{2}(\kappa_1(v_1) - \kappa_1(v_2)) + c(v_1, v_2)(\mu_1 - \mu_2) \end{aligned}$$

where $\mu_j := \Gamma_\epsilon(v_j, \lambda_j)$. Again the first two terms have the same sign, hence $\mu_1 - \mu_2$ must be of the opposite sign. Without loss of generality, assume that $v_1 > v_2$, hence $\Gamma_\epsilon(v_1, \lambda_1) < \Gamma_\epsilon(v_2, \lambda_2)$. By the structure of Γ_ϵ this implies that $0 < \mu_2 - \mu_1 < \lambda_2 - \lambda_1$. Finally using (20) we find,

$$c(v_1, v_2)|\lambda_1 - \lambda_2| \geq \frac{1}{L_\phi(M)}|v_1 - v_2|$$

Hence denoting $C_\kappa(M) := \sup_{v \in \phi([-M, M])} \{\kappa_1(v) - \kappa_0(v)\} < +\infty$, we can take $C_\kappa(M)L_\phi(M)$ as a local Lipschitz constant of Φ_ϵ on $[-M, M] \times [0, 1]$ in the variable λ .

(v): The Lipschitz constant of Φ_ϵ is independent of $\epsilon > 0$, hence $\{\Phi_\epsilon\}_{0 < \epsilon \leq 1}$ is equibounded and equicontinuous on the compact $K := [\kappa_0(P_0 - 1), \kappa_1(P_1 + 1)] \times [0, 1]$. By the Arzelà-Ascoli theorem, $\Phi_\epsilon \xrightarrow[\epsilon \rightarrow 0]{} \Phi$ uniformly on K , up to some subsequence and for some continuous function Φ . Property (iii) allows to infer the uniform convergence on $\mathbb{R} \times [0, 1]$. Now let $v \in \mathbb{R}$, $\lambda \in [0, 1]$ and $\mu \in \Gamma_0(v, \lambda)$. Remark that we can always find a sequence $(v_\epsilon)_{\epsilon > 0} \subset \mathbb{R}$ such that $v_\epsilon \rightarrow v$ and with $\Gamma_\epsilon(v_\epsilon, \lambda) = \mu$. Hence we find that for any $\epsilon > 0$,

$$\Phi_\epsilon(\beta(v_\epsilon, \mu), \lambda) = v_\epsilon$$

Then (18) falls from the uniform convergence of Φ_ϵ , upon taking the limit $\epsilon \rightarrow 0$. Since any limit of Φ_ϵ has to fulfill this property, we find that the limit must be unique and we denote it by Φ_0 .

(vi): Let $u \in \mathbb{R}$ and $\lambda \in [0, 1]$. Denote $v := \Phi_\epsilon(u, \lambda)$ and $\tilde{v} := \Phi_\epsilon(u, \tilde{\lambda})$ where $\tilde{\lambda} = \Gamma_\epsilon(v, \lambda)$. Note that by definition of Φ_ϵ , we have $u = \beta(v, \lambda) = \beta(\tilde{v}, \Gamma_\epsilon(\tilde{v}, \tilde{\lambda}))$. We want to show that $v = \tilde{v}$.

Assume by contradiction that

$$\tilde{v} < v \tag{21}$$

Then because $\Gamma_\epsilon(\cdot, \tilde{\lambda})$ is nondecreasing we find $\Gamma_\epsilon(\tilde{v}, \tilde{\lambda}) \leq \Gamma_\epsilon(v, \tilde{\lambda})$. However note that $\Gamma_\epsilon(v, \tilde{\lambda}) = \tilde{\lambda}$, hence we have:

$$\Gamma_\epsilon(\tilde{v}, \tilde{\lambda}) \leq \tilde{\lambda} \tag{22}$$

Again from (21), since $\beta(\cdot, \tilde{\lambda})$ is increasing, we find $\beta(\tilde{v}, \tilde{\lambda}) < \beta(v, \tilde{\lambda})$ and by combining with (22) we find $\beta(\tilde{v}, \Gamma_\epsilon(\tilde{v}, \tilde{\lambda})) < \beta(v, \tilde{\lambda})$. Which by definition of Φ_ϵ (17) implies the contradictory statement $u < u$. □

We can express a new formulation thanks to the definition of Φ_ϵ , written as follows

$$\begin{cases} \frac{u_K^{n+1} - u_K^n}{\Delta t} = \frac{1}{m_K} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} (\Phi_\epsilon(u_L^{n+1}, \lambda_L^n) - \Phi_\epsilon(u_K^{n+1}, \lambda_K^n)) \\ \lambda_K^{n+1} = \Gamma_\epsilon(\Phi_\epsilon(u_K^{n+1}, \lambda_K^n), \lambda_K^n) \end{cases}$$

Note that contrary to the temperature-phase formulation (16), the enthalpy-phase formulation allows for an explicit update in time of the phase fraction λ_K^n . Remark that therefore, in the nonregularised regime $\epsilon = 0$, this explicit update hinders the uniqueness of the discrete solution or the stability of the scheme as $\epsilon \rightarrow 0$. We can circumvent this loss of uniqueness with the following observation: when $\epsilon > 0$, Φ_ϵ is a right inverse of γ_ϵ , therefore for any $u, \lambda \in \mathbb{R} \times [0, 1]$,

$$(1 - \Gamma_\epsilon(\Phi_\epsilon(u, \lambda), \lambda))\kappa_0(\Phi_\epsilon(u, \lambda)) + \Gamma_\epsilon(\Phi_\epsilon(u, \lambda), \lambda)\kappa_1(\Phi_\epsilon(u, \lambda)) = u$$

After rearranging, we find that

$$\Gamma_\epsilon(\Phi_\epsilon(u, \lambda), \lambda) = \frac{u - \kappa_0(\Phi_\epsilon(u, \lambda))}{\kappa_1(\Phi_\epsilon(u, \lambda)) - \kappa_0(\Phi_\epsilon(u, \lambda))}$$

Thus if we have sequences of real numbers $(u_\epsilon)_{\epsilon>0}$, $(\lambda_\epsilon)_{\epsilon>0}$ such that $u_\epsilon, \lambda_\epsilon \xrightarrow{\epsilon \rightarrow 0} u, \lambda$, we find by the uniform convergence of Φ_ϵ on $\mathbb{R} \times [0, 1]$ (see (v) of Lemma 3.2):

$$\Gamma_\epsilon(\Phi_\epsilon(u_\epsilon, \lambda_\epsilon), \lambda_\epsilon) \xrightarrow{\epsilon \rightarrow 0} \frac{u - \kappa_0(\Phi_0(u, \lambda))}{\kappa_1(\Phi_0(u, \lambda)) - \kappa_0(\Phi_0(u, \lambda))}$$

This procedure allows to select the formulation which enjoys stability as $\epsilon \rightarrow 0$, written as follows:

Find $(u_{\mathcal{D}}^\epsilon, \lambda_{\mathcal{D}}^\epsilon) \in X(\mathcal{D})$ such that for all $K \in \mathcal{T}$, $0 \leq n \leq N - 1$

$$\begin{cases} \frac{u_K^{\epsilon, n+1} - u_K^{\epsilon, n}}{\Delta t} = \frac{1}{m_K} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} (\Phi_\epsilon(u_L^{\epsilon, n+1}, \lambda_L^{\epsilon, n}) - \Phi_\epsilon(u_K^{\epsilon, n+1}, \lambda_K^{\epsilon, n})) \\ \lambda_K^{\epsilon, n+1} = \frac{u_K^{\epsilon, n+1} - \kappa_0(\Phi_\epsilon(u_K^{\epsilon, n+1}, \lambda_K^{\epsilon, n}))}{\kappa_1(\Phi_\epsilon(u_K^{\epsilon, n+1}, \lambda_K^{\epsilon, n})) - \kappa_0(\Phi_\epsilon(u_K^{\epsilon, n+1}, \lambda_K^{\epsilon, n}))} \end{cases} \quad (23)$$

Below, we derive the stability of the enthalpy-phase formulation in the limit $\epsilon \rightarrow 0$.

Lemma 3.3 (Stability as $\epsilon \rightarrow 0$). *Assume that $(u_{\mathcal{D}}^\epsilon, \lambda_{\mathcal{D}}^\epsilon) \in X(\mathcal{D})$ is a solution to (23) for all $\epsilon > 0$ and that we have the convergence*

$$(u_{\mathcal{D}}^\epsilon, \lambda_{\mathcal{D}}^\epsilon) \xrightarrow{\epsilon \rightarrow 0} (u_{\mathcal{D}}^0, \lambda_{\mathcal{D}}^0) \quad \text{a.e. in } (0, T) \times \mathbb{T}^d$$

Then $(u_{\mathcal{D}}^0, \lambda_{\mathcal{D}}^0)$ fulfills (23) for $\epsilon = 0$.

The equivalence of the two formulations (16) and (23) is obvious for $\epsilon > 0$ because $\Phi_\epsilon(\cdot, \lambda)$ is a proper inverse of $\gamma_\epsilon(\cdot, \lambda)$. However in the case $\epsilon = 0$, this is no longer the case as $\Phi_0(\cdot, \lambda)$ is only a left-inverse, the equivalence of the formulations in this case still needs proper analysis. A first *partial equivalence* result is recorded below.

Lemma 3.4 (Equivalence of formulations). *Let $\epsilon > 0$ and assume $(v_{\mathcal{D}}^\epsilon, u_{\mathcal{D}}^\epsilon, \lambda_{\mathcal{D}}^\epsilon)$ is a solution of (16) with initial conditions given by (13) and (14) then $(u_{\mathcal{D}}^\epsilon, \lambda_{\mathcal{D}}^\epsilon)$ is a solution of (23) with identical initial conditions.*

Let $\epsilon \geq 0$ and assume $(u_{\mathcal{D}}^\epsilon, \lambda_{\mathcal{D}}^\epsilon)$ is a solution of (23) with initial conditions given by (13) and (14) and define $v_{\mathcal{D}} \in X(\mathcal{D})$ as the piecewise constant reconstruction of

$$v_K^n := \Phi_\epsilon(u_K^n, \lambda_K^{n-1})$$

for $n \in \{1, \dots, N\}$ and $K \in \mathcal{T}$.

Then $(v_{\mathcal{D}}^\epsilon, u_{\mathcal{D}}^\epsilon, \lambda_{\mathcal{D}}^\epsilon)$ is a solution of (16) with identical initial conditions.

Remark 3.2. (i) In what follows we will see that the equivalence indeed holds for $\epsilon = 0$ as a consequence of the stability as $\epsilon \rightarrow 0$ and of the unique solvability of both schemes.

(ii) The property (vi) of Lemma 3.2 allows to write (23) as an implicit scheme in Λ . Indeed, we observe that

$$\Phi_\epsilon(u_K^{n+1}, \lambda_K^n) = \Phi_\epsilon(u_K^{n+1}, \lambda_K^{n+1})$$

for all $n \in \{0, \dots, N-1\}$ (it also holds for $n = 0$) This is an important property of the scheme for the derivation of H^1 and BV estimates (see Proposition 4.1, Proposition 4.3 later). Informally, the update of the phase at the next time-step has no incidence on the value of the temperature.

3.4 L^∞ -stability, unique solvability and equivalence of the discrete problems

This paragraph is dedicated to the L^∞ -stability and existence of a solution for the enthalpy formulation for $\epsilon \geq 0$. We then show the uniqueness of solutions in the temperature formulation for $\epsilon \geq 0$. These inquiries, along with the stability results (Lemma 3.1, Lemma 3.3), then allow to prove the equivalence of the two formulations (16) and (23) in the case $\epsilon = 0$.

Proposition 3.5 (Maximum Principle). *Assume $\epsilon \geq 0$ and that $(u_{\mathcal{D}}^\epsilon, \lambda_{\mathcal{D}}^\epsilon)$ is a solution to (23). Then we have the following bounds for any $n \in \{0, \dots, N\}$, $K \in \mathcal{T}$:*

$$\min \left\{ \kappa_0(P_0 - \epsilon), \min_{\mathbb{T}^d} u_{\mathcal{M}}^{\epsilon,0} \right\} \leq u_{\mathcal{D}}^\epsilon(t^n) \leq \max \left\{ \kappa_1(P_1 + \epsilon), \max_{\mathbb{T}^d} u_{\mathcal{D}}^{\epsilon,0} \right\}.$$

Proof. We prove the result by induction on $0 \leq n \leq N$ and for the first inequality only, the other inequality being symmetric.

The base case $n = 0$ is obvious. Now assume the stability estimate holds up until time step t^n and denote by $J \in \mathcal{T}$ the volume on which $u_{\mathcal{D}}^\epsilon(t^{n+1}, \cdot) \in X(\mathcal{M})$ is minimal, i.e. $u_J^{\epsilon, n+1} = \min_{\mathbb{T}^d} u_{\mathcal{D}}^\epsilon(t^{n+1})$.

— If $u_J^{\epsilon, n+1} \geq \kappa_0(P_0 - \epsilon)$ then we are done.

— Otherwise if $u_j^{\epsilon, n+1} < \kappa_0(P_0 - \epsilon)$, then by the expression of the scheme we have

$$\begin{aligned}
u_j^{\epsilon, n+1} &= u_j^{\epsilon, n} + \frac{\Delta t}{m_K} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} \left(\Phi_\epsilon(u_L^{\epsilon, n+1}, \lambda_L^{\epsilon, n}) - \Phi_\epsilon(u_j^{\epsilon, n+1}, \lambda_j^{\epsilon, n}) \right) \\
&= u_j^{\epsilon, n} + \frac{\Delta t}{m_K} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} \left(\Phi_\epsilon(u_L^{\epsilon, n+1}, \lambda_L^{\epsilon, n}) - \phi(u_j^{\epsilon, n+1}) \right) \\
&= u_j^n + \frac{\Delta t}{m_K} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} \left(\Phi_\epsilon(u_L^{\epsilon, n+1}, \lambda_L^{\epsilon, n}) - \Phi_\epsilon(u_j^{\epsilon, n+1}, \lambda_L^{\epsilon, n}) \right) \\
&\geq u_j^{\epsilon, n} \\
&\geq \min \left\{ \kappa_0(P_0 - \epsilon), \min_{K \in \mathcal{T}} u_K^0 \right\}
\end{aligned}$$

By noting first from (iii) of Lemma 3.2 that $\Phi_\epsilon(u_j^{\epsilon, n+1}, \lambda) = \phi(u_j^{\epsilon, n+1})$ for all $\lambda \in [0, 1]$ because $u_j^{\epsilon, n+1} < \kappa_0(P_0 - \epsilon)$ and second that $u \mapsto \Phi(u, \lambda)$ is nondecreasing. The conclusion then falls by minimality of $u_j^{\epsilon, n+1}$. □

Proposition 3.6 (Existence for the enthalpy formulation). *For any $\epsilon \geq 0$ there exists a solution $(u_{\mathcal{D}}^\epsilon, \lambda_{\mathcal{D}}^\epsilon)$ of (23).*

Proof. The topological degree $\deg(f, W, y) \in \mathbb{Z}$ of $f \in C(\overline{W}, \mathbb{R}^M)$, $W \subset \mathbb{R}^M$ open and $y \notin f(\partial W) \subset \mathbb{R}^M$ has the three following properties:

- (d1) $\deg(f, W, y) \neq 0$ implies that $f(x) = y$ admits a solution $x \in W$.
- (d2) $\deg(Id, W, y) = 1$ if $y \in W$.
- (d3) $\lambda \mapsto \deg(f(\lambda, \cdot), W, y)$ is constant on $[0, 1]$ if $f : [0, 1] \times \overline{W} \rightarrow \mathbb{R}^M$ is continuous and $y \notin f(\lambda, \partial W)$ for any $\lambda \in [0, 1]$.

The construction of such a degree and its properties can be readily found in [3, Theorem 3.1].

We reason by induction. Let $0 \leq n \leq N - 1$ and let $(u_K^{\epsilon, n}, \lambda_K^{\epsilon, n})_{K \in \mathcal{T}}$ be a given solution at time step n . Let us introduce the scheme function,

$$S : \begin{cases} [0, 1] \times \mathbb{R}^{|\mathcal{T}|} & \longrightarrow \mathbb{R}^{|\mathcal{T}|} \\ \left(\mu, (u_K^{\epsilon, n+1})_{K \in \mathcal{T}} \right) & \longmapsto \left(u_K^{\epsilon, n+1} + \mu \frac{\Delta t}{m_K} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} \left(\Phi_\epsilon(u_K^{\epsilon, n+1}, \lambda_K^{\epsilon, n}) - \Phi_\epsilon(u_L^{\epsilon, n+1}, \lambda_L^{\epsilon, n}) \right) \right)_{K \in \mathcal{T}} \end{cases}$$

Any solution $(u_K^{\epsilon, n+1})_{K \in \mathcal{T}}$ to the nonlinear equation,

$$S(1, (u_K^{\epsilon, n+1})_{K \in \mathcal{T}}) = (u_K^{\epsilon, n})_{K \in \mathcal{T}}$$

is a solution of (23) for time step $n + 1$.

The continuity of S with respect to its variables comes from the continuity of Φ_ϵ and of the basic operations in the vector space $\mathbb{R}^{|\mathcal{T}|}$. Now let $r := \max\{\max_{K \in \mathcal{T}} |u_K^{\epsilon, n}|, |\kappa(P_0 - \epsilon)|, |\kappa(P_1 + \epsilon)|\} + 1$ and define W as the L^∞ -ball of $\mathbb{R}^{|\mathcal{T}|}$ with radius r .

From (d2), since $(u_K^{\epsilon, n})_{K \in \mathcal{T}} \in W$, we have $\deg(S(0, \cdot), W, (u_K^{\epsilon, n})_{K \in \mathcal{T}}) \neq 0$. Also for any $\mu \in [0, 1]$, $(u_K^{\epsilon, n})_{K \in \mathcal{T}} \notin S(\mu, \partial W)$ by virtue of Proposition 3.5.

Now from (d3) we deduce that $\deg(S(1, \cdot), W, (u_K^{\epsilon, n})_{K \in \mathcal{T}}) = 1$ and (d1) yields the existence of a solution. \square

The uniqueness of the temperature formulation is found by an energy-type argument which is performed by induction. However because of the hysteretic nature of the problem, we cannot obtain a more general L^2 -contraction result with two distinct initial conditions. The following result crucially relies on the hypothesis that the two considered solutions start at the same initial condition.

Proposition 3.7 (Uniqueness for temperature-phase formulation). *For any $\epsilon \geq 0$ and any given initial conditions (u^0, v^0, λ^0) there is at most one solution of (16)*

Proof. Assume there exists two solutions $(u_{\mathcal{D}}^\epsilon, v_{\mathcal{D}}^\epsilon, \lambda_{\mathcal{D}}^\epsilon)$ and $(\bar{u}_{\mathcal{D}}^\epsilon, \bar{v}_{\mathcal{D}}^\epsilon, \bar{\lambda}_{\mathcal{D}}^\epsilon)$. Denote $w := u_{\mathcal{D}}^\epsilon - \bar{u}_{\mathcal{D}}^\epsilon$, $z := v_{\mathcal{D}}^\epsilon - \bar{v}_{\mathcal{D}}^\epsilon \in X(\mathcal{D})$ and $\mu := \lambda_{\mathcal{D}}^\epsilon - \bar{\lambda}_{\mathcal{D}}^\epsilon$.

We show the result by induction on the time step n , the base case $n = 0$ being given by the initial condition. Hence we assume that $w(t^n, \cdot) = z(t^n, \cdot) = \mu(t^n, \cdot) = 0$ on \mathbb{T}^d . Remark that the conclusion $w(t^{n+1}, \cdot) = 0$ will follow directly from the fact $z(t^{n+1}, \cdot) = \mu(t^{n+1}, \cdot) = 0$. Let us now show this last claim.

First subtract both equations (16) at time step n and volume $K \in \mathcal{T}$ to find,

$$\beta(v_K^{n+1}, \lambda_K^{n+1}) - \beta(\bar{v}_K^{n+1}, \bar{\lambda}_K^{n+1}) = \frac{\Delta t}{m_K} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} (z_L^{n+1} - z_K^{n+1}) \quad (24)$$

There exists $M > 0$ such that $v_{\mathcal{D}}, \bar{v}_{\mathcal{D}} \in \phi([-M, M])$, hence our observation:

$$\begin{aligned} \left(\beta(v_K^{n+1}, \lambda_K^{n+1}) - \beta(\bar{v}_K^{n+1}, \bar{\lambda}_K^{n+1}) \right) z_K^{n+1} &= \left(\beta(v_K^{n+1}, \lambda_K^{n+1}) - \beta(\bar{v}_K^{n+1}, \lambda_K^{n+1}) \right) z_K^{n+1} \\ &\quad + \left(\beta(\bar{v}_K^{n+1}, \lambda_K^{n+1}) - \beta(\bar{v}_K^{n+1}, \bar{\lambda}_K^{n+1}) \right) z_K^{n+1} \\ &\geq \frac{1}{L_\phi(M)} (z_K^{n+1})^2 + \mu_K^{n+1} (\kappa_1(\bar{v}_K^{n+1}) - \kappa_0(\bar{v}_K^{n+1})) z_K^{n+1} \\ &\geq \frac{1}{L_\phi(M)} (z_K^{n+1})^2 \end{aligned}$$

because $\kappa_1 \geq \kappa_0$ on \mathbb{R} and the monotonicity of $v \mapsto \Gamma_\epsilon(v, \lambda)$. Hence multiplying (24) by $m_K z_K^{n+1}$ and summing over $K \in \mathcal{T}$ we find,

$$\frac{1}{L_\phi(M)} \|z(t^{n+1})\|_{L^2(\mathbb{T}^d)}^2 \leq \Delta t \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} (z_L^{n+1} - z_K^{n+1}) z_K^{n+1}$$

$$\begin{aligned}
&= \Delta t \sum_{K|L \in \mathcal{E}} \tau_{K|L} (z_L^{n+1} - z_K^{n+1}) z_K^{n+1} + (z_K^{n+1} - z_L^{n+1}) z_L^{n+1} \\
&= -\Delta t \sum_{K|L} \tau_{K|L} (z_L^{n+1} - z_K^{n+1})^2 \\
&\leq 0
\end{aligned}$$

We conclude that $z(t^{n+1}, \cdot) = 0$ and consequently from (24) we find $\mu(t^{n+1}, \cdot) = 0$ by injectivity of $\lambda \mapsto \beta(v, \lambda)$ for $v \in \mathbb{R}$. This concludes our proof. \square

In the following proposition, concluding this first section, we show the equivalence between the temperature-phase and the enthalpy phase formulation in every regime of regularisation $\epsilon \geq 0$.

Proposition 3.8. *For any $\epsilon \geq 0$ the formulations (16) and (23) are equivalent. Consequently both formulations satisfy the maximum principle Proposition 3.5 and enjoy unique solvability (Proposition 3.6 and Proposition 3.7).*

Proof. There only remains to show that any solution of (16) must also be a solution of (23) in the case $\epsilon = 0$.

Let $(u_{\mathcal{D}}, v_{\mathcal{D}}, \lambda_{\mathcal{D}})$ be any solution to (16) with $\epsilon = 0$. For each $\epsilon > 0$, there exists a unique solution $(u_{\mathcal{D}}^{\epsilon}, v_{\mathcal{D}}^{\epsilon}, \lambda_{\mathcal{D}}^{\epsilon})$ of (16), and thanks to the maximum principle Proposition 3.5 and the stability result Lemma 3.1, we find that this family of solutions must converge (up to subsequence) to some solution of (16) as $\epsilon \rightarrow 0$, which by uniqueness Proposition 3.7 must be our solution $(u_{\mathcal{D}}, v_{\mathcal{D}}, \lambda_{\mathcal{D}})$.

Additionally, we know from Lemma 3.4 that $(u_{\mathcal{D}}^{\epsilon}, \lambda_{\mathcal{D}}^{\epsilon})$ is a solution of (23) for $\epsilon > 0$. Hence from the stability of the enthalpy form Lemma 3.3 indeed is $(u_{\mathcal{D}}, \lambda_{\mathcal{D}})$ a solution to (23). Hence the conclusion. \square

3.5 Discrete entropy inequalities

In this last part we show the discrete counterpart to the entropy inequalities (11) between the unknowns v and λ , in preparation for the proof of the convergence.

Proposition 3.9. *Assume that $\epsilon \geq 0$ and that $(u_{\mathcal{D}}^{\epsilon}, v_{\mathcal{D}}^{\epsilon}, \lambda_{\mathcal{D}}^{\epsilon})$ is a discrete solution to (16). Then for any $K \in \mathcal{T}$, $n \in \{0, \dots, N-1\}$ and $g \in C^1(\mathbb{R})$ increasing, we have*

$$\overline{G}_K^{\epsilon, n+1} - \overline{G}_K^{\epsilon, n} \leq g(w_K^{\epsilon, n+1}) (\beta_K^{\epsilon, n+1} - \beta_K^{\epsilon, n}) \quad (25)$$

where $w_K^{\epsilon, n+1} := v_K^{\epsilon, n+1} - \epsilon(2\lambda_K^{\epsilon, n+1} - 1)$, and

$$\overline{G}_K^{\epsilon, n} = (1 - \lambda_K^{\epsilon, n}) G(\kappa_0(w_K^{\epsilon, n+1})) + \lambda_K^{\epsilon, n} G(\kappa_1(w_K^{\epsilon, n+1}))$$

with $G(u) = \int_{\cdot}^u g \circ \phi(s) ds$ and

$$\beta_K^{\epsilon, n} := \beta(w_K^{\epsilon, n}, \lambda_K^{\epsilon, n})$$

Proof. Remark that, by definition of Γ_{ϵ} the target inequality (25) is independent of the choice of continuations of κ_0 outside $(-\infty, P_1]$ and of κ_1 outside $[P_0, +\infty)$. Hence we can always assume that

κ_0 is continued by c_0 above P_1 and κ_1 is continued by c_1 below P_0 .

Let us write the first order Taylor expansion with integral remainder of G_0 around $\kappa_0(w_K^{n+1})$, we have for any $v \in \mathbb{R}$,

$$\begin{aligned} G_0(v) &= G(\kappa_0(w_K^{n+1})) + g(\phi(\kappa_0(w_K^{n+1}))) (\kappa_0(v) - \kappa_0(w_K^{n+1})) \\ &\quad + (\kappa_0(v) - \kappa_0(w_K^{n+1}))^2 \int_0^1 (1-\theta) G''(\theta \kappa_0(w_K^{n+1}) + (1-\theta)\kappa_0(v)) d\theta \\ &= G(\kappa_0(w_K^{n+1})) + g(P_1 \wedge w_K^{n+1}) (\kappa_0(v) - \kappa_0(w_K^{n+1})) \\ &\quad + (u - \kappa_0(w_K^{n+1}))^2 \int_0^1 (1-\theta) [g' \circ \phi \times \phi'] (\theta \kappa_0(w_K^{n+1}) + (1-\theta)\kappa_0(v)) d\theta \end{aligned} \quad (26)$$

Now remark that since $\kappa_0(v), \kappa_0(w_K^{n+1}) \in (-\infty, c_0]$, by convexity for all $\theta \in [0, 1]$, $\theta \kappa_0(w_K^{n+1}) + (1-\theta)\kappa_0(v) \in (-\infty, c_0]$, which is an interval where ϕ is increasing. Since g is also increasing, the integral remainder in (26) is positive. Let us apply this to $v = w_K^n$ to find,

$$G_0(w_K^n) \geq G_0(w_K^{n+1}) + g(P_1 \wedge w_K^{n+1}) (\kappa_0(w_K^n) - \kappa_0(w_K^{n+1}))$$

Similarly for G_1 we find,

$$G_1(w_K^n) \geq G_1(w_K^{n+1}) + g(P_0 \vee w_K^{n+1}) (\kappa_1(w_K^n) - \kappa_1(w_K^{n+1}))$$

Hence we find that

$$\begin{aligned} \overline{G}_K^{\epsilon, n+1} - \overline{G}_K^{\epsilon, n} &= (1 - \lambda_K^{\epsilon, n+1}) G_0(w_K^{n+1}) + \lambda_K^{\epsilon, n+1} G_1(w_K^{n+1}) - (1 - \lambda_K^{\epsilon, n}) G_0(w_K^n) - \lambda_K^{\epsilon, n} G_1(w_K^n) \\ &\leq (\lambda_K^{\epsilon, n+1} - \lambda_K^{\epsilon, n}) (G_1(w_K^{n+1}) - G_0(w_K^{n+1})) + g(P_1 \wedge w_K^{n+1}) (1 - \lambda_K^{\epsilon, n}) (\kappa_0(w_K^{n+1}) - \kappa_0(w_K^n)) \\ &\quad + g(P_0 \vee w_K^{n+1}) \lambda_K^{\epsilon, n} (\kappa_1(w_K^{n+1}) - \kappa_1(w_K^n)) \end{aligned}$$

Remark that we have,

$$\begin{aligned} g(P_1 \wedge w_K^{n+1}) (1 - \lambda_K^{\epsilon, n}) (\kappa_0(w_K^{n+1}) - \kappa_0(w_K^n)) &\leq g(w_K^{n+1}) (1 - \lambda_K^{\epsilon, n}) (\kappa_0(w_K^{n+1}) - \kappa_0(w_K^n)) \\ g(P_0 \vee w_K^{n+1}) \lambda_K^{\epsilon, n} (\kappa_1(w_K^{n+1}) - \kappa_1(w_K^n)) &\leq g(w_K^{n+1}) \lambda_K^{\epsilon, n} (\kappa_1(w_K^{n+1}) - \kappa_1(w_K^n)) \end{aligned}$$

Therefore

$$\overline{G}_K^{\epsilon, n+1} - \overline{G}_K^{\epsilon, n} \leq (\lambda_K^{\epsilon, n+1} - \lambda_K^{\epsilon, n}) (G_1(w_K^{n+1}) - G_0(w_K^{n+1}) - g(w_K^{n+1}) \delta(w_K^{n+1})) + g(w_K^{n+1}) (\beta_K^{\epsilon, n+1} - \beta_K^{\epsilon, n})$$

where we denote $\delta(w) := \kappa_1(w) - \kappa_0(w)$. Let us show now that $(\lambda_K^{\epsilon, n+1} - \lambda_K^{\epsilon, n}) (G_1(w_K^{n+1}) - G_0(w_K^{n+1}) - g(w_K^{n+1}) \delta(w_K^{n+1})) \leq 0$, this will finish the proof.

Assume that $(\lambda_K^{\epsilon, n+1} - \lambda_K^{\epsilon, n}) \geq 0$. Since $\lambda_K^{\epsilon, n+1} = \Gamma_0(w_K^{n+1}, \lambda_K^{\epsilon, n})$, this implies that $w_K^{n+1} \geq P_1$ and therefore that the maximum of $g \circ \phi$ on $[\kappa_0(w_K^{n+1}), \kappa_1(w_K^{n+1})]$ is $g(w_K^{n+1})$. Thus:

$$\begin{aligned} G_1(w_K^{n+1}) - G_0(w_K^{n+1}) - g(w_K^{n+1}) \delta(w_K^{n+1}) &= \int_{\kappa_0(w_K^{n+1})}^{\kappa_1(w_K^{n+1})} (g \circ \phi)(s) ds - g(w_K^{n+1}) \delta(w_K^{n+1}) \\ &\leq 0 \end{aligned}$$

which concludes our claim and the proof. \square

4 Discrete estimates of $v_{\mathcal{D}}$, $u_{\mathcal{D}}$ and $\lambda_{\mathcal{D}}$

In the following section we obtain discrete estimates for $v_{\mathcal{D}}$, $u_{\mathcal{D}}$ and $\lambda_{\mathcal{D}}$ which are uniform w.r.t. \mathcal{D} in order to obtain the compactness in the limit $\Delta t, \Delta x \rightarrow 0$. If v^0 is $H^1(\mathbb{T}^d)$, then the temperature $v_{\mathcal{D}}$ enjoys H^1 regularity both in time and space. This ensures the strong compactness in L^2 of any sequence of approximates. However the L^2 framework is not suitable for the discrete estimates for $u_{\mathcal{D}}$ and $\lambda_{\mathcal{D}}$. We instead show BV estimates in time and space to obtain in the next section the strong compactness of the sequence in L^1 . The BV estimate in space requires a regular mesh and the additional hypothesis **(A2)** _{ϕ} .

4.1 Full $H^1((0, T) \times \mathbb{T}^d)$ estimates of $v_{\mathcal{D}}$

This paragraph deals with uniform H^1 estimates of $v_{\mathcal{D}}$. These estimates rely on the monotonicity and the Lipschitz regularity of Φ_ϵ and on the maximum principle Proposition 3.5.

The full time-space discrete H^1 estimate of $v_{\mathcal{D}}$ reads as follows.

Proposition 4.1. *Assume $\epsilon \geq 0$ and that $v^0 \in H^1(\mathbb{T}^d)$. Then there exists $C > 0$ independent of \mathcal{D} and ϵ such that,*

$$|v_{\mathcal{D}}|_{H^1(0, T; L^2(\mathbb{T}^d)), \mathcal{D}}^2 := \sum_{n=0}^N \sum_{K \in \mathcal{T}} m_K \frac{(v_K^{n+1} - v_K^n)^2}{\Delta t} \leq C$$

$$|v_{\mathcal{D}}|_{L^2(0, T; H^1(\mathbb{T}^d)), \mathcal{D}}^2 := \sum_{n=0}^N \Delta t \sum_{K|L} \tau_{K|L} (v_K^n - v_L^n)^2 \leq C$$

Proof. **$H^1(0, T; L^2(\mathbb{T}^d))$ estimate:** By the equivalence of the two formulations given by Proposition 3.8, we have for any $n \in \{1, \dots, N-1\}$ and $K \in \mathcal{T}$

$$\begin{aligned} (v_K^{n+1} - v_K^n)^2 &= (v_K^{n+1} - v_K^n)(\Phi_\epsilon(u_K^{n+1}, \lambda_K^n) - \Phi_\epsilon(u_K^n, \lambda_K^{n-1})) \\ &= (v_K^{n+1} - v_K^n)(\Phi_\epsilon(u_K^{n+1}, \lambda_K^n) - \Phi_\epsilon(u_K^n, \lambda_K^n)) \\ &\leq L_\phi(M)(v_K^{n+1} - v_K^n)(u_K^{n+1} - u_K^n) \end{aligned} \quad (27)$$

where we used (ii) of Remark 3.2, the monotonicity and the local Lipschitz continuity of $u \mapsto \Phi_\epsilon(u, \lambda)$ (see (i)-(iv) of Lemma 3.2) and where M is given by Proposition 3.5 and depends on u^0 . Note that the inequality also holds for $n = 0$ by compatibility of the initial conditions (14). Now plugging (16) in (27), multiplying by m_K and summing over $K \in \mathcal{T}$ we find:

$$\begin{aligned} \sum_{K \in \mathcal{T}} m_K \frac{(v_K^{n+1} - v_K^n)^2}{\Delta t} &\leq L_\phi(M) \sum_{K \in \mathcal{T}} m_K \frac{(u_K^{n+1} - u_K^n)(v_K^{n+1} - v_K^n)}{\Delta t} \\ &= L \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} (v_L^{n+1} - v_K^{n+1})(v_K^{n+1} - v_K^n) \\ &= L_\phi(M) \sum_{K|L \in \mathcal{E}} \tau_{K|L} (v_L^{n+1} - v_K^{n+1})(v_K^{n+1} - v_K^n) + \tau_{K|L} (v_K^{n+1} - v_L^{n+1})(v_L^{n+1} - v_L^n) \end{aligned}$$

$$\begin{aligned}
&= -L_\phi(M) \sum_{K|L \in \mathcal{E}} \tau_{K|L} \left((v_L^{n+1} - v_K^{n+1})^2 - (v_L^{n+1} - v_K^{n+1})(v_L^n - v_K^n) \right) \\
&\leq -\frac{L_\phi(M)}{2} \left(|v_{\mathcal{D}}(t^{n+1}, \cdot)|_{H^1(\mathbb{T}^d), \mathcal{M}}^2 - |v_{\mathcal{D}}(t^n, \cdot)|_{H^1(\mathbb{T}^d), \mathcal{M}}^2 \right)
\end{aligned}$$

where we used Young's inequality for the last inequality. Summing over $n \in \{0, \dots, N-1\}$ we end up with:

$$|v_{\mathcal{D}}|_{H^1(0, T; L^2(\mathbb{T}^d)), \mathcal{D}}^2 \leq \frac{L_\phi(M)}{2} |v^0|_{H^1(\mathbb{T}^d), \mathcal{M}}^2$$

$L^2(0, T; H^1(\mathbb{T}^d))$ **estimate:** Multiply (16) by v_K^{n+1} and integrate over $K \in \mathcal{T}$ and $n \in \{0, \dots, N-1\}$ to obtain,

$$\begin{aligned}
&\sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} m_K \frac{u_K^{n+1} - u_K^n}{\Delta t} v_K^{n+1} = \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} (v_L^{n+1} - v_K^{n+1}) v_K^{n+1} \\
&= \sum_{n=0}^{N-1} \Delta t \sum_{K|L \in \mathcal{F}_{int}} \tau_{K|L} (v_L^{n+1} - v_K^{n+1}) v_K^{n+1} + \tau_{K|L} (v_K^{n+1} - v_L^{n+1}) v_L^{n+1} \\
&= -\sum_{n=0}^{N-1} \Delta t \sum_{K|L \in \mathcal{F}_{int}} \tau_{K|L} (v_L^{n+1} - v_K^{n+1})^2 \\
&= -|v_{\mathcal{D}}|_{L^2(0, T; H^1(\mathbb{T}^d)), \mathcal{D}}^2
\end{aligned} \tag{28}$$

Observe that,

$$\begin{aligned}
&\sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} m_K \frac{u_K^{n+1} - u_K^n}{\Delta t} v_K^{n+1} = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} m_K u_K^{n+1} v_K^{n+1} - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} m_K u_K^n v_K^{n+1} \\
&= \sum_{n=1}^N \sum_{K \in \mathcal{T}} m_K u_K^n v_K^n - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} m_K u_K^n v_K^{n+1} \\
&= \sum_{K \in \mathcal{T}} m_K (u_K^N v_K^N - u_K^0 v_K^0) - \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} m_K u_K^n \frac{v_K^{n+1} - v_K^n}{\Delta t}
\end{aligned} \tag{29}$$

Now we know from Proposition 3.5 that for any $n \in \{0, \dots, N\}$, $K \in \mathcal{T}$,

$$v_K^n \in \phi([-M, M])$$

Therefore $\|u_{\mathcal{D}}\|_{L^\infty((0, T) \times \mathbb{T}^d)}, \|v_{\mathcal{D}}\|_{L^\infty((0, T) \times \mathbb{T}^d)} \leq C \left(\|u^0\|_{L^\infty(\mathbb{T}^d)} \right)$.

Therefore we can conclude, assembling (28) and (29) and using the Cauchy-Schwarz inequality, that

$$\begin{aligned}
|v_{\mathcal{D}}|_{L^2(0, T; H^1(\mathbb{T}^d)), \mathcal{D}}^2 &\leq 2^{\frac{1}{2}} C \left(\|u^0\|_{L^\infty(\mathbb{T}^d)} \right) + T^{\frac{1}{2}} C \left(\|u^0\|_{L^\infty(\mathbb{T}^d)} \right) |v_{\mathcal{D}}|_{H^1(0, T; L^2(\mathbb{T}^d)), \mathcal{D}} \\
&\leq C \left(\|u^0\|_{L^\infty(\mathbb{T}^d)}, \|v^0\|_{H^1(\mathbb{T}^d)}, T \right)
\end{aligned}$$

□

4.2 Full $BV((0, T) \times \mathbb{T}^d)$ estimates

In this second paragraph, we show BV estimates in time for $u_{\mathcal{D}}$ on any admissible mesh. In the case where ϕ satisfies $(\mathbf{A1})_{\phi}$ - $(\mathbf{A2})_{\phi}$ and on periodic meshes, we obtain a BV estimate in space for $\lambda_{\mathcal{D}}$.

A question of interest, still open, to the knowledge of the authors, would first be to remove assumption $(\mathbf{A2})_{\phi}$ and also to treat the case of unstructured meshes.

The obtained BV estimates, which rely on a discrete version of the Hilbert inequality [22, 26] classical for hysteresis operators, provide the strong compactness in L^1 of the sequence of approximations. The nonlinear nature of the problem calls for this compactness result for the convergence analysis.

4.2.1 BV estimate in time for $u_{\mathcal{D}}$ and $\lambda_{\mathcal{D}}$

Let us derive the BV estimate in time related to the enthalpy formulation of Problem (23). As a first step, we describe a technical lemma on the sign function before deriving our estimate. The lemma will also prove to be useful in the derivation of the space BV estimates.

Denote by $s : \mathbb{R} \mapsto \{-1, 0, 1\}$ the sign function $s(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$ we also denote \bar{s} the multivalued function $\bar{s}(x) = \begin{cases} \{-1\} & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ \{1\} & \text{if } x > 0 \end{cases}$.

Lemma 4.2. *Let $F, G : \mathbb{R} \mapsto \mathbb{R}$ be two nondecreasing functions. Then for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $s_x \in \bar{s}(x_1 - x_2)$, $s_y \in \bar{s}(y_1 - y_2)$, we have*

$$((F(x_1) - F(x_2)) - (G(y_1) - G(y_2)))(s_x - s_y) \geq 0$$

Proof. Assume by contradiction and without loss of generality that there exist $x_1, x_2, y_1, y_2 \in \mathbb{R}$ such that

$$F(x_1) - F(x_2) > G(y_1) - G(y_2)$$

and

$$s_x < s_y \tag{30}$$

By definition of \bar{s} , this implies that $x_2 \geq x_1$ and since F is nondecreasing $0 \geq F(x_1) - F(x_2) > G(y_1) - G(y_2)$. Now G is nondecreasing hence $y_1 < y_2$ and therefore $s_y = -1$. But this contradicts (30) because $s_x \geq -1$. \square

Proposition 4.3 ($BV(0, T, L^1(\mathbb{T}^d))$). *Assume that $\epsilon \geq 0$ and that there exists $C \geq 0$, independent of \mathcal{D} such that*

$$\|\Delta_{\mathcal{M}} v_{\mathcal{M}}^0\|_{L^1(\mathbb{T}^d)} := \sum_{K \in \mathcal{T}} \left| \sum_{L \in \mathcal{N}(K)} \tau_{K|L} (v_L^0 - v_K^0) \right| \leq C$$

Then we have

$$\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} m_K |u_K^{n+1} - u_K^n| \leq C$$

and also there exists $\omega : [0, +\infty) \rightarrow [0, +\infty)$, continuous, nonincreasing and such that $\omega(0) = 0$ and independent of \mathcal{D} such that:

$$\sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} m_K |\lambda_K^{n+1} - \lambda_K^n| \leq \omega(\Delta t)$$

Proof. Estimate for $u_{\mathcal{D}}$: Let $n \in \{1, \dots, N-1\}$, write the scheme (16) at time steps n and $n-1$, subtract both and multiply by $\sigma_K^{n+1} := s(u_K^{n+1} - u_K^n)$ and then integrate over \mathcal{T} .

$$\begin{aligned} \sum_{K \in \mathcal{T}} m_K \frac{(u_K^{n+1} - u_K^n) - (u_K^n - u_K^{n-1})}{\Delta t} \sigma_K^{n+1} &= \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} ((v_L^{n+1} - v_K^{n+1}) - (v_L^n - v_K^n)) \sigma_K^{n+1} \\ &= \sum_{K|L \in \mathcal{E}} \tau_{K|L} ((v_L^{n+1} - v_K^{n+1}) - (v_L^n - v_K^n)) (\sigma_K^{n+1} - \sigma_L^{n+1}) \\ &= - \sum_{K|L \in \mathcal{E}} \tau_{K|L} ((v_K^{n+1} - v_L^{n+1}) - (v_K^n - v_L^n)) (\sigma_K^{n+1} - \sigma_L^{n+1}) \end{aligned}$$

Remember that $v_K^{n+1} = \Phi_\epsilon(u_K^{n+1}, \lambda_K^n)$ and from Lemma 3.2 (vi) that $v_K^n = \Phi_\epsilon(u_K^n, \lambda_K^n)$. By applying Lemma 4.2 to the non-decreasing functions $F := \Phi_\epsilon(\cdot, \lambda_K^n)$ and $G := \Phi_\epsilon(\cdot, \lambda_L^n)$ we find,

$$((v_K^{n+1} - v_K^n) - (v_L^{n+1} - v_L^n)) (\sigma_K^{n+1} - \sigma_L^{n+1}) \geq 0$$

Therefore,

$$\sum_{K \in \mathcal{T}} m_K \frac{(u_K^{n+1} - u_K^n) - (u_K^n - u_K^{n-1})}{\Delta t} \sigma_K^{n+1} \leq 0$$

And we also have $((u_K^{n+1} - u_K^n) - (u_K^n - u_K^{n-1})) \sigma_K^{n+1} \geq |u_K^{n+1} - u_K^n| - |u_K^n - u_K^{n-1}|$, hence we find,

$$\sum_{K \in \mathcal{T}} m_K \frac{|u_K^{n+1} - u_K^n|}{\Delta t} \leq \sum_{K \in \mathcal{T}} m_K \frac{|u_K^n - u_K^{n-1}|}{\Delta t}$$

After integrating on $(0, T)$, with a basic induction on n we then find:

$$\begin{aligned} \sum_{n=0}^N \sum_{K \in \mathcal{T}} m_K |u_K^{n+1} - u_K^n| &\leq \sum_{K \in \mathcal{T}} m_K \frac{|u_K^1 - u_K^0|}{\Delta t} \\ &= \sum_{K \in \mathcal{T}} m_K \frac{u_K^1 - u_K^0}{\Delta t} \sigma_K^1 \\ &= \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} (v_L^1 - v_K^1) \sigma_K^1 \\ &= \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} ((v_L^1 - v_K^1) - (v_L^0 - v_K^0)) \sigma_K^1 + \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} (v_L^0 - v_K^0) \sigma_K^1 \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{K|L \in \mathcal{E}} \tau_{K|L} \underbrace{(v_L^1 - v_L^0 - (v_K^1 - v_K^0))(\sigma_K^1 - \sigma_L^1)}_{\leq 0} + \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{L}(K)} \tau_{K|L} (v_L^0 - v_K^0) \sigma_K^1 \\
&\leq \sum_{K \in \mathcal{T}} \left| \sum_{L \in \mathcal{L}(N)} \tau_{K|L} (v_K^0 - v_L^0) \right| \tag{31}
\end{aligned}$$

where we used Lemma 4.2 again to obtain (31).

Estimate for $\lambda_{\mathcal{D}}$: Denote by $\omega_{\kappa} : [0, +\infty) \rightarrow [0, +\infty)$ a common, nondecreasing and concave modulus of continuity for κ_0 and κ_1 on a big enough compact $K \subset \mathbb{R}$ such that $v_{\mathcal{D}}$ has values within K (K may be taken independent of \mathcal{D} thanks to Proposition 3.5). For any $n \in \{0, \dots, N-1\}$, $K \in \mathcal{T}$, remark,

$$u_K^{n+1} - u_K^n = a_K^n (\kappa_0(v_K^{n+1}) - \kappa_0(v_K^n)) + b_K^n (\kappa_1(v_K^{n+1}) - \kappa_1(v_K^n)) + c(v_K^{n+1}, v_K^n) (\lambda_K^{n+1} - \lambda_K^n) \tag{32}$$

where $a_K^n := 1 - \frac{\lambda_K^{n+1} + \lambda_K^n}{2} \in [0, 1]$, $b_K^n := \frac{\lambda_K^{n+1} + \lambda_K^n}{2} \in [0, 1]$ and for $v, w \in \mathbb{R}$, $c(v, w) := \frac{\kappa_1(v) + \kappa_1(w)}{2} - \frac{\kappa_0(v) + \kappa_0(w)}{2} > 0$. Since $\kappa_1 > \kappa_0$ on \mathbb{R} , by continuity there exists $\delta_K > 0$ such that for $v, w \in K$ we have,

$$c(v, w) \geq \delta_K$$

Now take the absolute value in (32), integrate in time and space and use (31) to find,

$$\begin{aligned}
C\Delta t &\geq \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} m_K |u_K^{n+1} - u_K^n| \\
&\geq \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} m_K \delta_K |\lambda_K^{n+1} - \lambda_K^n| - \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} m_K (|\kappa_0(v_K^{n+1}) - \kappa_0(v_K^n)| + |\kappa_1(v_K^{n+1}) - \kappa_1(v_K^n)|)
\end{aligned}$$

After rearranging we obtain, using the uniform continuity of κ_i on K , $i = 0, 1$:

$$\begin{aligned}
\delta_K \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} m_K |\lambda_K^{n+1} - \lambda_K^n| &\leq C\Delta t + 2 \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} m_K \omega_{\kappa}(|v_K^{n+1} - v_K^n|) \\
&\leq C\Delta t + 2Tm(\mathbb{T}^d) \omega_{\kappa} \left(\frac{1}{Tm(\mathbb{T}^d)} \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} m_K |v_K^{n+1} - v_K^n| \right) \\
&\leq C\Delta t + 2Tm(\mathbb{T}^d) \omega_{\kappa}(C\Delta t)
\end{aligned}$$

This last inequality comes from the uniform H^1 estimate in time of $v_{\mathcal{D}}$ Proposition 4.1. A valid modulus is then $\omega(r) := \frac{C}{\delta_K} r + \frac{2Tm(\mathbb{T}^d)}{\delta_K} \omega_{\kappa}(Cr)$. \square

4.2.2 BV estimates in space for $\lambda_{\mathcal{D}}$ on admissible periodic meshes and under $(\mathbf{A2})_{\phi}$

In the two following sections we assume that ϕ satisfies assumption $(\mathbf{A2})_{\phi}$.

For now, the BV estimate in space of $\lambda_{\mathcal{D}}$ is only accessible under a strong assumption on the mesh: we assume that \mathcal{M} is an admissible *periodic mesh*.

Definition 4.1. We say that a mesh is *admissible and periodic* if it is admissible and there exists a finite generating family Ξ of vectors of \mathbb{R}^d , called *characteristic vectors* of \mathcal{M} and such that for any $K \in \mathcal{T}$, $\xi \in \Xi$

$$\begin{aligned}\xi + x_K &\in \mathcal{P} \\ \xi + K &\in \mathcal{T}\end{aligned}$$

$$\mathcal{N}(K) = \{\xi + K \mid \xi \in \Xi\}$$

An admissible and periodic mesh satisfies the following properties:

1. For any $K, L \in \mathcal{T}$, $m_K = m_L$.
2. For all $K \in \mathcal{T}$, $\xi \in \Xi$,

$$\mathcal{N}(\xi + K) = \{\xi + L \mid L \in \mathcal{N}(K)\}$$

3. For all $\sigma \in \mathcal{E}$, $\xi \in \Xi$, we have $\xi + \sigma \in \mathcal{E}$ and $\tau_\sigma = \tau_{\xi + \sigma}$.
4. $\Delta x = \max_{\xi \in \Xi} |\xi|$.

For practicality, when $\xi \in \Xi$ we will write $K_\xi := \xi + K$ and $\sigma_\xi := \xi + \sigma$ whenever $K \in \mathcal{T}$, $\sigma \in \mathcal{E}$.

Any cartesian grid in any dimension d is an admissible periodic mesh with characteristic vectors of the form $\{\pm h_i e_i\}_{1 \leq i \leq d}$ with $h_i > 0$ for $i \in \{1, \dots, d\}$. Regular hexagonal meshes in \mathbb{R}^2 are other examples of admissible periodic meshes.

The BV estimate in space relies on a discrete version of Hilpert's inequality: an important tool in the study of continuous hysteresis operators to obtain L^1 contraction properties. We state and show the lemma below. We follow along the lines of [22] (continuous case) and [26] (where the discrete case is treated).

Lemma 4.4 (Discrete Hilpert's Inequality). *Let $\epsilon > 0$, for any $\lambda_1, \lambda_2 \in [0, 1]$, $v_1, v_2 \in \mathbb{R}$, we have:*

$$((\Gamma_\epsilon(v_1, \lambda_1) - \Gamma_\epsilon(v_2, \lambda_2)) - (\lambda_1 - \lambda_2))s(v_1 - v_2) \geq |\Gamma_\epsilon(v_1, \lambda_1) - \Gamma_\epsilon(v_2, \lambda_2)| - |\lambda_1 - \lambda_2|$$

where s is the sign function.

Proof. Denote $\mu_1 := \Gamma_\epsilon(v_1, \lambda_1)$, $\mu_2 := \Gamma_\epsilon(v_2, \lambda_2)$. We divide the analysis into several cases,

- * $\mu_1 > \mu_2$ and $v_1 > v_2$, the result comes at once from $(\lambda_1 - \lambda_2)s(v_1 - v_2) \leq |\lambda_1 - \lambda_2|$
- * $\mu_1 < \mu_2$ and $v_1 < v_2$, the result comes at once for the same reason.
- * $\mu_1 > \mu_2$ and $v_1 \leq v_2$. But this implies that $\lambda_1 > \lambda_2$ and therefore $\mu_1 - \mu_2 \leq \lambda_1 - \lambda_2$, i.e. $|\mu_1 - \mu_2| - |\lambda_1 - \lambda_2| \leq 0$. Since $s(v_1 - v_2) \leq 0$, we infer that

$$((\mu_1 - \mu_2) - (\lambda_1 - \lambda_2))s(v_1 - v_2) \geq 0 \geq |\mu_1 - \mu_2| - |\lambda_1 - \lambda_2|$$

- * The case $\mu_1 < \mu_2$ and $v_1 \geq v_2$ is treated symmetrically.

□

We next show the wanted BV estimate for admissible and periodic meshes covering \mathbb{R}^d or the torus \mathbb{T}^d (i.e. periodic boundary conditions).

Proposition 4.5 ($L^\infty(0, T; BV(\mathbb{T}^d))$). *Assume $\epsilon \geq 0$, that $(\mathbf{A1})_\phi$ - $(\mathbf{A2})_\phi$ hold and that \mathcal{M} is an admissible periodic mesh on \mathbb{T}^d with $\mathbb{T}^d = \mathbb{R}^d$ or $\mathbb{T}^d = \mathbb{T}^d$. Let $\xi \in \Xi$ be a characteristic vector of \mathcal{M} . Assume that $(u_{\mathcal{D}}, v_{\mathcal{D}}, \lambda_{\mathcal{D}}) \in X(\mathcal{D})^3$ is a solution to (16) on $\mathcal{D} = ((t^n)_{0 \leq n \leq N}, \mathcal{M})$ with initial conditions $u^0, v^0 \in L^1(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$, then we have*

$$\sup_{0 \leq n \leq N} \sum_{K \in \mathcal{T}} m_K \left(\left| \kappa_0(v_K^n) - \kappa_0(v_{K_\xi}^n) \right| + \delta \left| \lambda_K^n - \lambda_{K_\xi}^n \right| \right) \leq \sum_{K \in \mathcal{T}} m_K \left(\left| \kappa_0(v_K^0) - \kappa_0(v_{K_\xi}^0) \right| + \delta \left| \lambda_K^0 - \lambda_{K_\xi}^0 \right| \right) \quad (33)$$

Proof. **Case $\epsilon > 0$:** Let $K \in \mathcal{M}$ be a control volume, $n \in \{0, \dots, N-1\}$ and $\xi \in \Xi$. Let us write the first equation of the scheme on K and K_ξ :

$$m_K(u_K^{n+1} - u_K^n) = \Delta t \sum_{L \in \mathcal{N}(K)} \tau_{K|L}(v_L^{n+1} - v_K^{n+1}) \quad (34)$$

$$m_K(u_{K_\xi}^{n+1} - u_{K_\xi}^n) = \Delta t \sum_{L \in \mathcal{N}(K_\xi)} \tau_{K|L}(v_L^{n+1} - v_{K_\xi}^{n+1}) \quad (35)$$

By periodicity of the mesh \mathcal{M} we know that $\mathcal{N}(K_\xi) = \{L_\xi \mid L \in \mathcal{N}(K)\}$. Now subtract (34) with (35), multiply by the quantity $\sigma_K^{n+1} := s(v_K^{n+1} - v_{K_\xi}^{n+1})$ and sum over $K \in \mathcal{T}$ to obtain:

$$\begin{aligned} T &:= \sum_{K \in \mathcal{T}} m_K \left((u_K^{n+1} - u_{K_\xi}^{n+1}) - (u_K^n - u_{K_\xi}^n) \right) \sigma_K^{n+1} \\ &= \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} \tau_{K|L}(v_L^{n+1} - v_K^{n+1}) \sigma_K^{n+1} - \tau_{K_\xi|L_\xi}(v_{L_\xi}^{n+1} - v_{K_\xi}^{n+1}) \sigma_K^{n+1} \\ &= \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} \left((v_L^{n+1} - v_K^{n+1}) - (v_{L_\xi}^{n+1} - v_{K_\xi}^{n+1}) \right) \sigma_K^{n+1} \\ &= \sum_{K|L \in \mathcal{E}} \tau_{K|L} \left((v_L^{n+1} - v_K^{n+1}) - (v_{L_\xi}^{n+1} - v_{K_\xi}^{n+1}) \right) (\sigma_K^{n+1} - \sigma_L^{n+1}) \\ &= - \sum_{K|L \in \mathcal{E}} \tau_{K|L} \left((v_L^{n+1} - v_{L_\xi}^{n+1}) - (v_K^{n+1} - v_{K_\xi}^{n+1}) \right) (\sigma_L^{n+1} - \sigma_K^{n+1}) \end{aligned}$$

And now Lemma 4.2 allows us to conclude that $T \leq 0$. Also recall from the Hilpert inequality Lemma 4.4 and the fact that κ_0 is increasing, that

$$T \geq \sum_{K \in \mathcal{T}} m_K \left(\left| \kappa_0(v_K^{n+1}) - \kappa_0(v_{K_\xi}^{n+1}) \right| - \left| \kappa_0(v_K^n) - \kappa_0(v_{K_\xi}^n) \right| + \delta \left| \lambda_K^{n+1} - \lambda_{K_\xi}^{n+1} \right| - \delta \left| \lambda_K^n - \lambda_{K_\xi}^n \right| \right)$$

This last observation yields the wanted result.

Case $\epsilon = 0$: Inequality (33) falls at once from the fact that $(v_{\mathcal{D}}^\epsilon, \lambda_{\mathcal{D}}^\epsilon) \rightarrow (v_{\mathcal{D}}^0, \lambda_{\mathcal{D}}^0)$ strongly in $L^1((0, T) \times \mathbb{T}^d)$ as a consequence of the stability property Lemma 3.1 and the maximum principle Proposition 3.5. □

Remark 4.1. In the general case (even on a submesh of a peridiodic mesh on a bounded domain) and with homogeneous Neumann boundary conditions, the difficulty lies in the the fact that for a given volume $K \in \mathcal{T}$, it is no longer clear that $K_\xi \in \mathcal{T}$ and also that the neighbours $\mathcal{N}(K_\xi)$ can be described in terms of the neighbours $\mathcal{N}(K)$. This leads to boundary terms which cannot be controlled properly.

5 Convergence to a weak entropy solution

Let us be given a sequence of admissible meshes $(\mathcal{D}_h)_{h>0}$ such that $\Delta x(\mathcal{D}_h), \Delta t(\mathcal{D}_h) \xrightarrow{h \rightarrow 0} 0$. We will show the following convergence result:

Theorem 5.1. *Assume $\epsilon \geq 0$ and that $(\mathbf{A1})_\phi$ - $(\mathbf{A2})_\phi$ hold. for $h > 0$, let $(u_h^\epsilon, v_h^\epsilon, \lambda_h^\epsilon)$ be the solution to (16) on \mathcal{D}_h with initial condition (u^0, v^0, λ^0) satisfying (13) - (14).*

Then there exists a weak entropy solution $(u^\epsilon, v^\epsilon, \lambda^\epsilon)$ according to Definition 2.1 such that, up to a subsequence:

$$(u_h^\epsilon, v_h^\epsilon, \lambda_h^\epsilon) \xrightarrow{h \rightarrow 0} (u^\epsilon, v^\epsilon, \lambda^\epsilon) \text{ strongly in } L^p((0, T) \times \mathbb{T}^d) \text{ for all } 1 \leq p < +\infty$$

Proof. Relative compactness: The compactness is due to translation estimates stemming from the uniform discrete estimates for v_h^ϵ and λ_h^ϵ . The uniform H^1 estimates given by Proposition 4.1 for v_h^ϵ yield that

$$(v_h^\epsilon)_{h>0} \text{ is relatively strongly compact in } L^2((0, T) \times \mathbb{T}^d)$$

We refer to [6, Lemmatas 18.3, 18.6] for the translation estimates. Analogously, the uniform discrete BV estimates for λ_h^ϵ through Proposition 4.3 and Proposition 4.5 allow for a similar compactness result in $L^1((0, T) \times \mathbb{T}^d)$. We find

$$(\lambda_h^\epsilon)_{h>0} \text{ is relatively strongly compact in } L^1((0, T) \times \mathbb{T}^d)$$

Now the boundedness of \mathbb{T}^d , the continuity of $(v, \lambda) \mapsto \beta(v, \lambda)$ and the maximum principle Proposition 3.5 lets us also conclude that

$$(u_h^\epsilon)_{h>0} \text{ is relatively strongly compact in } L^1((0, T) \times \mathbb{T}^d)$$

From now on, we select a subsequence in h , still denoted h , such that $(u_h^\epsilon, v_h^\epsilon, \lambda_h^\epsilon) \xrightarrow{h \rightarrow 0} (u^\epsilon, v^\epsilon, \lambda^\epsilon)$ strongly in $L^1((0, T) \times \mathbb{T}^d)$. Since $(u_h^\epsilon, v_h^\epsilon, \lambda_h^\epsilon)_{h>0}$ are also uniformly bounded in $L^\infty((0, T) \times \mathbb{T}^d)$ by Proposition 3.5, the strong convergence also holds in any $L^p((0, T) \times \mathbb{T}^d)$ for $1 \leq p < \infty$.

Convergence for the volume equation: From the strong convergence in $L^p((0, T) \times \mathbb{T}^d)$ of the trilpet $(u_h^\epsilon, v_h^\epsilon, \lambda_h^\epsilon)$ it is enough to show the convergence towards the weak formulation of the volume equation. The arguments are classical and can be found in [6, Theorem 18.1].

Convergence for the entropy inequalities: The convergence for the entropy inequalities is performed by passing to the limit in the discrete entropy inequalities Proposition 3.9.

Let $g \in C^1(\mathbb{R})$ be nondecreasing. Let $K \in \mathcal{T}_h$, $n \in \{0, \dots, N-1\}$, denote as before $w_K^n := v_K^{\epsilon, n} - \epsilon(2\lambda_K^{\epsilon, n} - 1)$. Remark that since $(v_h^\epsilon)_{h>0}$ and $(\lambda_h^\epsilon)_{h>0}$ converge strongly in $L^2(\mathbb{T}_T^d)$ towards v^ϵ and λ^ϵ respectively, we know that

$$w_h^\epsilon \xrightarrow{h \rightarrow 0} v^\epsilon - \epsilon(2\lambda^\epsilon - 1) =: w^\epsilon \quad \text{strongly in } L^2(\mathbb{T}_T^d)$$

Since $(v_h^\epsilon)_{h>0}$ and $(\lambda_h^\epsilon)_{h>0}$ are also uniformly bounded in $L^\infty((0, T) \times \mathbb{T}^d)$, the dominated convergence theorem yields by continuity of G_i , g , g' and β :

$$G_0(w_h^\epsilon), G_1(w_h^\epsilon), g(w_h^\epsilon), g'(w_h^\epsilon), \beta(w_h^\epsilon, \lambda_h^\epsilon) \longrightarrow G_0(w^\epsilon), G_1(w^\epsilon), g(w^\epsilon), g'(w^\epsilon), \beta(w^\epsilon, \lambda^\epsilon) \quad \text{strongly in } L^2(\mathbb{T}_T^d)$$

For any $f \in X(\mathcal{D})$, denote its discrete time-derivative $\partial_{t,h} f := \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_h} \frac{f_K^{n+1} - f_K^n}{\Delta t} \mathbb{1}_{[t^n, t^{n+1}) \times K}$. We claim that for fixed $\epsilon \geq 0$,

$$(\partial_{t,h} g(w_h^\epsilon))_{h>0} \text{ is uniformly bounded in } L^2((0, T) \times \mathbb{T}^d) \quad (36)$$

Indeed, in the case $\epsilon = 0$, this follows easily from Proposition 4.1, the fact that $g \in C^1(\mathbb{R})$ and the maximum principle Proposition 3.5. In the case $\epsilon > 0$, observe that, for any $K \in \mathcal{T}_h$, $n \in \{0, \dots, N-1\}$, the $\frac{1}{\epsilon}$ -Lipschitz continuity of Γ_ϵ in v gives:

$$\begin{aligned} |\lambda_K^{n+1} - \lambda_K^n| &\leq |\Gamma_\epsilon(v_K^{n+1}, \lambda_K^n) - \Gamma_\epsilon(v_K^n, \lambda_K^n)| \\ &\leq \frac{1}{\epsilon} |v_K^{n+1} - v_K^n| \end{aligned}$$

which leads to

$$\|\partial_{t,h} \lambda_h^\epsilon\|_{L^2((0, T) \times \mathbb{T}^d)} \leq \frac{1}{\epsilon} \|\partial_{t,h} v_h^\epsilon\|_{L^2((0, T) \times \mathbb{T}^d)}$$

Hence $\partial_{t,h} w_h^\epsilon$ is also uniformly bounded in $L^2((0, T) \times \mathbb{T}^d)$ and from the same arguments as in the case $\epsilon = 0$ follows (36).

From this, it follows that $(\partial_{t,h} g(w_h^\epsilon))_{h>0}$ converges weakly in $L^2((0, T) \times \mathbb{T}^d)$, up to subsequence, to an L^2 function which is $\partial_t g(w^\epsilon)$ by consistency of the discrete derivative with ∂_t .

We are now ready to pass to the limit in the discrete inequalities. Let $h > 0$, $\xi \in \mathcal{D}(\mathbb{T}_T^d)$ with $\xi \geq 0$, denote for $n \in \{0, \dots, N\}$, $K \in \mathcal{T}_h$, $\xi_K^n := \xi(t^n, x_K)$. Multiply (25) by $m_K \xi_K^{n+1}$ and sum over $n \in \{0, \dots, N-1\}$, $K \in \mathcal{T}_h$ to find:

$$\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_h} m_K (\overline{G}_K^{\epsilon, n+1} - \overline{G}_K^{\epsilon, n}) \xi_K^{n+1} \leq \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_h} m_K g(w_K^n) (\beta_K^{\epsilon, n+1} - \beta_K^{\epsilon, n}) \xi_K^{n+1}$$

which we summarize by

$$T_1^h \leq T_2^h \quad (37)$$

We have:

$$T_1^h = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_h} m_K (\overline{G}_K^{\epsilon, n+1} - \overline{G}_K^{\epsilon, n}) \xi_K^{n+1}$$

$$\begin{aligned}
&= - \sum_{K \in \mathcal{T}_h} m_K \overline{G}_K^{\epsilon,0} \xi_K^1 - \sum_{n=1}^{N-1} \sum_{K \in \mathcal{T}_h} m_K (\xi_K^{n+1} - \xi_K^n) \overline{G}_K^{\epsilon,n} \\
&\xrightarrow{h \rightarrow 0} - \int_{\mathbb{T}_T^d} \overline{G}(v^\epsilon) \partial_t \xi \, dx \, dt
\end{aligned}$$

And we have on the other hand:

$$\begin{aligned}
T_2^h &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_h} m_K g(w_K^n) (\beta_K^{\epsilon,n+1} - \beta_K^{\epsilon,n}) \xi_K^{n+1} \\
&= - \sum_{K \in \mathcal{T}_h} m_K \beta_K^{\epsilon,0} g(w_K^0) \xi_K^1 - \sum_{n=1}^{N-1} \sum_{K \in \mathcal{T}_h} m_K (\xi_K^{n+1} g(w_K^n) - \xi_K^n g(w_K^{n-1})) \beta_K^{\epsilon,n} \\
&= - \sum_{K \in \mathcal{T}_h} m_K \beta_K^{\epsilon,0} g(w_K^0) \xi_K^1 - \sum_{n=1}^{N-1} \sum_{K \in \mathcal{T}_h} m_K \beta_K^{\epsilon,n} ((\xi_K^{n+1} - \xi_K^n) g(w_K^n) + \xi_K^n (g(w_K^n) - g(w_K^{n-1}))) \\
&\xrightarrow{h \rightarrow 0} - \int_{\mathbb{T}_T^d} \beta(w^\epsilon, \lambda^\epsilon) (\partial_t \xi g(w^\epsilon) + \xi \partial_t g(w^\epsilon)) \tag{38}
\end{aligned}$$

The convergence (38) being justified by the weak convergence in $L^2(\mathbb{T}_T^d)$ of $\partial_{t,h} g(w_h^\epsilon)$. Hence, passing to the limit in (37) yields the wanted inequality. \square

6 Numerical experiments

We here describe the implementation of an efficient algorithm to compute the numerical approximations and show numerical experiments on Riemann problems.

6.1 Description of the algorithm

Our simulations rely on the finite volume Julia package `VoronoiFVM.jl` developed by a co-author of this work. It is based on a Voronoi meshing algorithm and has demonstrated its efficiency on reaction-diffusion equations arising e.g. in semi-conductor models. The Voronoi meshing algorithm provides admissible meshes in the sense of Definition 3.1 for 2D geometries.

The basic machinery of the algorithm relies on the implicit finite volume discretisation of a flux $\mathbf{J}(u)$ in reaction-diffusion equations of the form:

$$\partial_t s(u) = \operatorname{div}(\mathbf{J}(u)) + r(u)$$

where s is a storage term, r is the reaction term. With this machinery, it is *almost* straightforward to implement the enthalpy-phase formulation (23) where $s = \operatorname{Id}$, $r = 0$ and the flux is $\mathbf{J}(u) := \nabla \Phi(u, \lambda)$ which depends on the phase parameter λ . Since in (23), the update of λ is done explicitly in time, at each time step t^n the flux can be determined explicitly. Newton's method is used to compute the numerical solution at each time step. It is performed with an automatic differentiation method

based on dual numbers, implemented in the Julia package `ForwardDiff.jl`. Even if the scheme is implicit and no CFL condition appears in the analysis, the degeneracy of the parabolic problem solved at each time-step yields instabilities in the Newton method. The computations may actually *fail* to converge if Δt is not chosen *small enough* in comparison with Δx and the spatial variations of the initial conditions.

Another challenge of the implementation is the computation of Φ_ϵ for $\epsilon \geq 0$ which we recall is defined in (17) as the inverse function of $v \mapsto \beta(v, \Gamma_\epsilon(v, \lambda))$. Under the assumption $(\mathbf{A1})_\phi$ - $(\mathbf{A2})_\phi$, there is a simple expression for Φ_ϵ in terms of ϕ . However under the general assumption $(\mathbf{A1})_\phi$, this computation requires to invert a whole family of functions $(\gamma(\cdot, \lambda))_{\lambda \in [0, 1]}$. We have chosen to sample 100 evenly spaced values of $\lambda \in [0, 1]$ to compute single inverses and then construct an approximate Φ_ϵ with a piecewise constant reconstruction.

6.2 Riemann problems in 1D and 2D under $(\mathbf{A1})_\phi$ - $(\mathbf{A2})_\phi$

Here we investigate solutions of the numerical scheme (16) for $\epsilon = 0$ with various Riemann problems both in 1D and 2D. We comment the behaviour of the arising free boundary in the simulations.

Expected behaviour of the solutions When two distinct phase values are considered initially in the Riemann problem, we should observe the formation of a *free boundary* I identified as a persistent jump discontinuity on the enthalpy u and the phase fraction λ . If the free boundary I is moving, we are in a situation similar to a two-phase Stefan problem with a specific *latent heat* proportional to the jump of enthalpy $\llbracket u \rrbracket_I$. This phenomenon is theoretically described by the Rankine-Hugoniot condition (see [5]):

$$\llbracket u \rrbracket_I \nu_t^I = \llbracket \nabla v \rrbracket_I \cdot \nu_x^I \quad (39)$$

where $\llbracket u \rrbracket_I$ and $\llbracket \nabla v \rrbracket_I$ are the jumps of u and ∇v across the free boundary I and $\nu^I = (\nu_t^I, \nu_x^I) \in \mathbb{S}^d$ is some normal unit vector on I in $\mathbb{R}_+ \times \mathbb{R}^d$. The absolute normal velocity of I is therefore given by $V := \frac{|\nu_t^I|}{|\nu_x^I|} = \frac{|\llbracket \nabla v \rrbracket_I \cdot n_x^I|}{\llbracket u \rrbracket_I}$, where $n_x^I := \frac{\nu_x^I}{|\nu_x^I|}$. Hence if everything else is fixed, a lower enthalpy jump leads to a higher absolute velocity.

On the other hand, if the free boundary is stationary, then the equations reduce to a diffusion equation in v (with possibly inhomogeneous and degenerate diffusion coefficient in space). Note it is possible to prepare an initial data for which the heat transfer is directed from the liquid phase to the solid phase (see Figure 11).

In the 1D setting, the Riemann problem yields a single-point free boundary and the expected behaviours can be thoroughly explored.

In 2D the possible situations are more complex. In particular the free boundary may exhibit topological changes over time, as is the case for two expanding disks seen on Figure 13. We also investigate a situation where the free boundary is steady on a given subset and moving on the other subset.

6.2.1 1D experiments

We perform our simulations on the interval $\Omega = (0, 1)$ with Neumann boundary conditions. While this setting does not exactly match our theoretical assumptions, in all our experiments we can recover the periodic boundary conditions by symetrizing around the point $x = 1$. We are under the assumptions $(\mathbf{A1})_\phi$ - $(\mathbf{A2})_\phi$ with $\kappa_0 : v \mapsto v - 1$ and $\delta = 2$. Denoting $x_0 = \frac{1}{2}$, we consider the following structure of 1D initial conditions:

$$u^0(x) = \begin{cases} u_l & \text{if } x < x_0 \\ u_r & \text{if } x \geq x_0 \end{cases} \quad v^0(x) = \begin{cases} v_l & \text{if } x < x_0 \\ v_r & \text{if } x \geq x_0 \end{cases} \quad \lambda^0(x) = \begin{cases} \lambda_l & \text{if } x < x_0 \\ \lambda_r & \text{if } x \geq x_0 \end{cases}$$

with the condition $u_0 = \beta(v_0, \lambda_0)$ being satisfied. For the following experiments we choose a regular cartesian grid on $[0, 1]$ of step Δx and a time step Δt , provided on each figure.

Moving free boundary with saturated phase fraction In this experiment shown in Figure 9, we fix $\lambda_l = 1$ and $\lambda_r = 0$ and we take $v_l = P_1 + \delta\theta$, $v_r = P_1$. In this situation the free boundary moves from left to right: the pure solid phase melts as the heat transfers through the boundary.

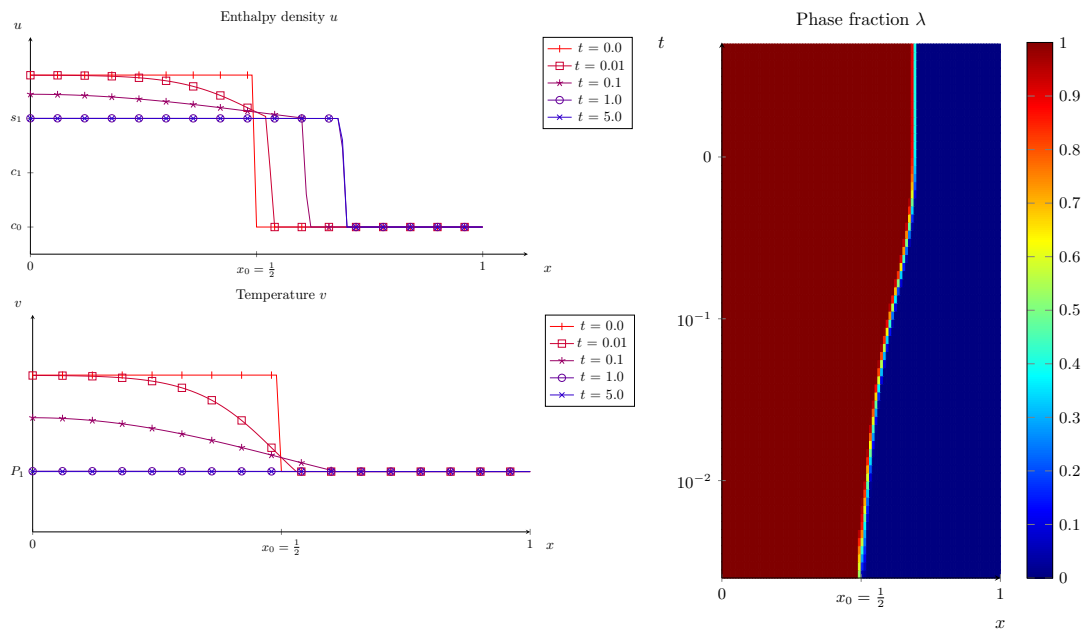


Figure 9: Free boundary evolution over time with pure phase initial condition. $\Delta x = 0.001$, $\Delta t = 0.002$.

Moving free boundary with unsaturated phase fraction In this experiment, shown in Figure 10, we fix $\lambda_l = 1$ and $\lambda_r = 0.5$ and we take as before $v_l = P_1 + \delta\theta$, $v_r = P_1$. In this situation the free boundary moves from left to right at a higher rate than in the pure phase situation.

We observe that, with identical initial temperature profiles, the interface in the unsaturated case stabilizes further away in the positive direction than the one in the saturated case. This observation is in compliance with the Rankine-Hugoniot condition (39).

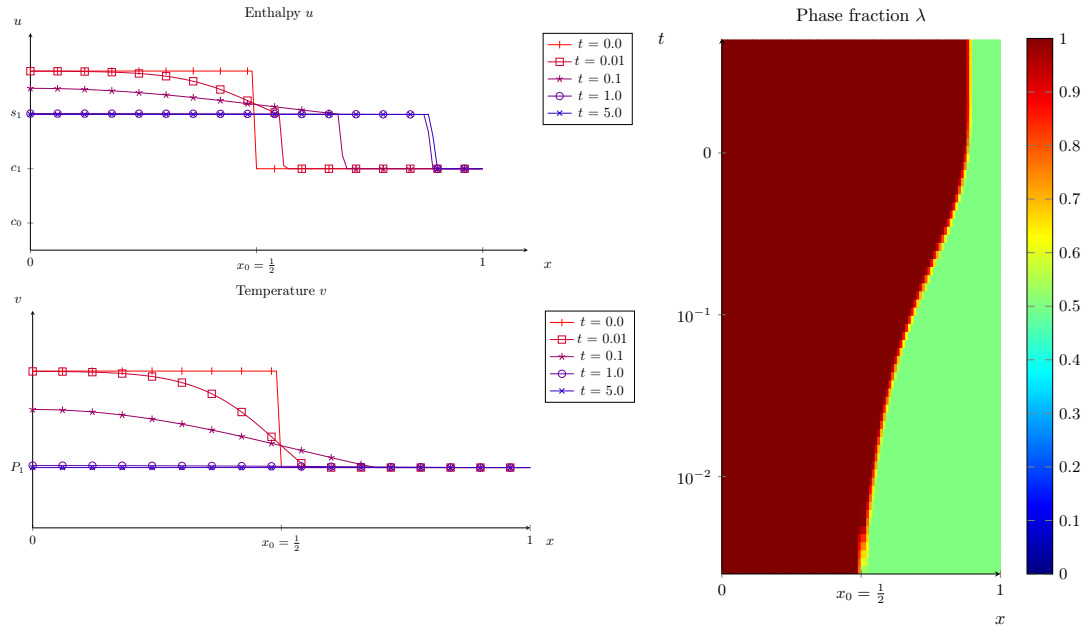


Figure 10: Free boundary evolution over time with partially melted material on the right. $\Delta x = 0.001$, $\Delta t = 0.002$.

Steady free boundary In this experiment, shown in Figure 11 we fix $\lambda_l = 1$ and $\lambda_r = 0$ and we take $v_l = P_0$, $v_r = P_1$. The liquid phase is at the freezing temperature and the solid phase is at the melting temperature. Since $P_1 > P_0$ we observe an *anomalous* heat transfer directed *from* the solid *to* the liquid: the solid phase cools down, the liquid phase heats up and no phase change occurs.

6.2.2 2D experiments

In the 2D experiments, we choose a square domain $\Omega = [-1, 1]^2$ with Neumann boundary conditions. Again we choose the basic assumptions $(\mathbf{A1})_\phi$ - $(\mathbf{A2})_\phi$ with $\kappa_0 : v \mapsto v - 1$ and $\delta = 2$. A subset $S \subset \Omega$ is used to prepare the Riemann initial conditions:

$$u^0(x, y) = \begin{cases} u_i & \text{if } (x, y) \in S \\ u_o & \text{if } (x, y) \notin S \end{cases} \quad v^0(x, y) = \begin{cases} v_i & \text{if } (x, y) \in S \\ v_o & \text{if } (x, y) \notin S \end{cases} \quad \lambda^0(x, y) = \begin{cases} \lambda_i & \text{if } (x, y) \in S \\ \lambda_o & \text{if } (x, y) \notin S \end{cases}$$

with the additional conditions $\lambda^0 \in \Gamma_0(v^0)$ and $u^0 = \beta(v^0, \lambda^0)$.

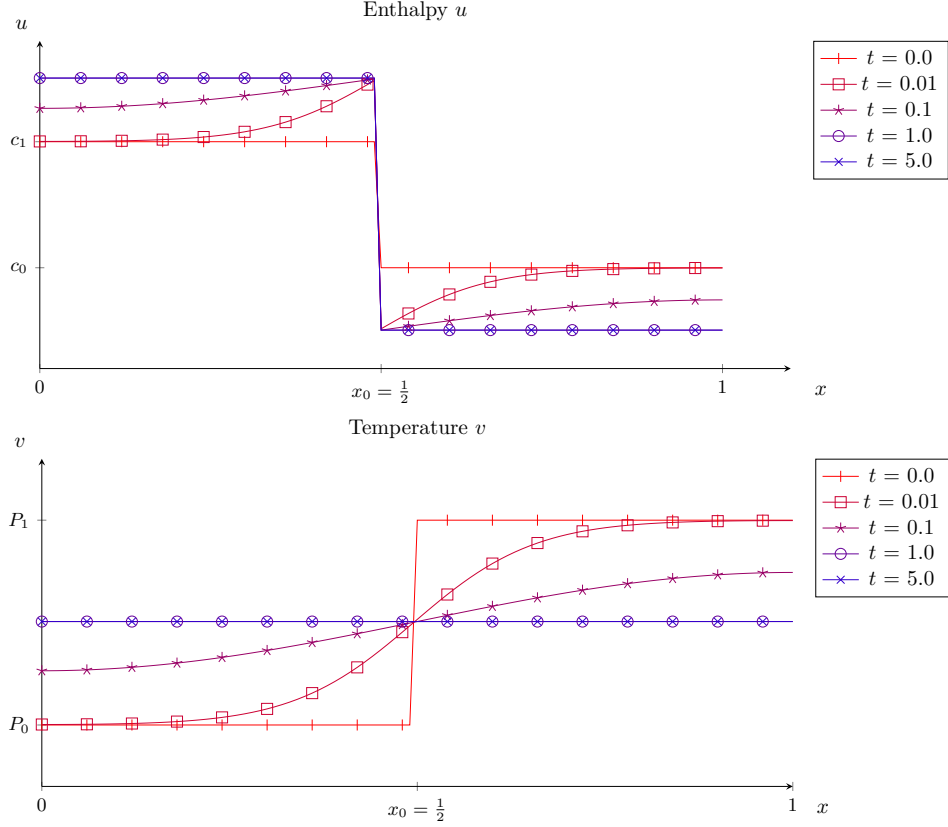


Figure 11: Steady free boundary with anomalous heat transfer. $\Delta x = 0.001$, $\Delta t = 0.002$.

Moving free boundary Here we show a single expanding disk (with $S = D((0, 0), \frac{1}{2})$, Figure 12) and two expanding disks (with $S = D((-\frac{1}{2}, 0), \frac{2}{5}) \cup D((\frac{1}{2}, 0), \frac{2}{5})$, Figure 13). We fix $\lambda_i = 1$, $\lambda_o = 0$ and choose $v_i = P_1 + \delta\theta$, $v_o = P_0$ for some $\delta\theta > 0$.

As for the classical Stefan problem, we observe the self-similarity of the radially symmetric solution on Figure 12. The temperature v reaches the constant equilibrium temperature $v_\infty = P_0$, while the phase λ stays pure (0 or 1) except on the volumes located at the boundary.

The topological change of the boundary in time is shown on Figure 13.

Mixed behaviour Here we consider a 2D Riemann problem with a piecewise linear perturbation. Let $S = [-1, 0] \times [-1, 1]$ be the left half of the domain Ω and we consider the perturbed initial temperature:

$$\tilde{v}^0(x, y) = v^0(x, y) + k(y - y_0)_+$$

where $k > 0$ is the coefficient of the linear perturbation and $y_0 \in [-1, 1]$ prescribes the support of the perturbation. Of course, the initial phase fraction is left unchanged and $u^0 = \beta(v^0, \lambda^0)$.

Corresponding numerical results are shown on Figure 14.

Random initial conditions We here display the evolution of the discrete solution with a random initial condition. The randomness is done with a uniform sampling in the range $[-10, 10]$ for v^0 and in the range $[0, 1]$ for λ^0 before the projection on the confinement domain. As usual u^0 is defined with $u^0 = \beta(v^0, \lambda^0)$. Results are shown on Figure 15

6.3 Additional experiments and comments

To conclude our numerical experiments we show some 1D experiments under the general assumption $(\mathbf{A1})_\phi$ with $\phi(u) = u^3 - u$. We investigate whether the L^1 control of norms is attained. In this case, an homologous result to Proposition 4.5 between two solutions (u_1, v_1, λ_1) and (u_2, v_2, λ_2) can be obtained for the continuous model under the additional $W^{1,1}(0, T; L^\infty(\Omega))$ estimate for $c(v_1, v_2) := \frac{\kappa_1(v_1) + \kappa_1(v_2)}{2} - \frac{\kappa_0(v_1) + \kappa_0(v_2)}{2}$:

$$\|c\|_{W^{1,1}(0, T; L^\infty(\Omega))} \leq C(T, v_1^0, v_2^0)$$

where the constant C is independent of the solutions v_1 and v_2 . Note that in the parallel case $(\mathbf{A2})_\phi$ we always have $c = 0$. Let us write:

$$\begin{aligned} u_1 - u_2 &= (1 - \lambda_1)\kappa_0(v_1) + \lambda_1\kappa_1(v_1) - (1 - \lambda_2)\kappa_0(v_2) - \lambda_2\kappa_1(v_2) \\ &= a(\lambda_1, \lambda_2)(\kappa_0(v_1) - \kappa_0(v_2)) + b(\lambda_1, \lambda_2)(\kappa_1(v_1) - \kappa_1(v_2)) + (\lambda_1 - \lambda_2)c(v_1, v_2) \end{aligned} \quad (40)$$

where $a(\lambda_1, \lambda_2) = 1 - \frac{\lambda_1 + \lambda_2}{2}$, $b(\lambda_1, \lambda_2) = \frac{\lambda_1 + \lambda_2}{2}$ are nonnegative functions. Now take the partial time derivative in (40) and multiply by $s(v_1 - v_2)$ where s is the sign function. We obtain:

$$\begin{aligned} \partial_t(u_1 - u_2)s(v_1 - v_2) &= \partial_t|a(\kappa_0(v_1) - \kappa_0(v_2))| + \partial_t|b(\kappa_1(v_1) - \kappa_1(v_2))| \\ &\quad + \partial_t(c|\lambda_1 - \lambda_2|) - 2\left|\frac{\partial_t c}{c}\right|c|\lambda_1 - \lambda_2| \end{aligned}$$

Now remark that $\int_{\Omega} \partial_t(u_1 - u_2)s(v_1 - v_2) dx = \int_{\Omega} \Delta(v_1 - v_2)s(v_1 - v_2) dx \leq 0$, hence we obtain after integration over Ω , and denoting $\zeta_v(t, x) := |a(\kappa_0(v_1) - \kappa_0(v_2))| + |b(\kappa_1(v_1) - \kappa_1(v_2))|$ and $\zeta_\lambda := c|\lambda_1 - \lambda_2|$,

$$0 \geq \int_{\Omega} \partial_t \zeta_v(t, x) dx + \int_{\Omega} \partial_t \zeta_\lambda(t, x) dx - 2 \int_{\Omega} \left| \frac{\partial_t c}{c} \right| \zeta_\lambda(t, x) dx$$

Therefore we find,

$$\begin{aligned} \frac{d}{dt} \left(\|\zeta_v(t)\|_{L^1(\Omega)} + \|\zeta_\lambda(t)\|_{L^1(\Omega)} \right) &\leq 2 \int_{\Omega} \left| \frac{\partial_t c}{c} \right| \zeta_\lambda(t, x) dx \\ &\leq 2 \left\| \frac{\partial_t c}{c} \right\|_{L^\infty(\Omega)} \|\zeta_\lambda(t)\|_{L^1(\Omega)} \end{aligned}$$

Now an application of Gronwall's lemma to the function $t \mapsto \|\zeta_v(t)\|_{L^1(\Omega)} + \|\zeta_\lambda(t)\|_{L^1(\Omega)}$ leads to the inequality,

$$\|\zeta_v(t)\|_{L^1(\Omega)} + \|\zeta_\lambda(t)\|_{L^1(\Omega)} \leq \exp\left(2 \int_0^t \left\| \frac{\partial_t c}{c}(s) \right\|_{L^\infty(\Omega)} ds\right) \left(\|\zeta_v(0)\|_{L^1(\Omega)} + \|\zeta_\lambda(0)\|_{L^1(\Omega)} \right)$$

We here show the numerical decrease in time of the quantity $N(t) - \exp\left(2 \int_0^t \left\| \frac{\partial_t c}{c} \right\|_{L^\infty} ds\right) N(0)$ where $N(t) = \|\zeta_\lambda(t)\|_{L^1} + \|\zeta_v(t)\|_{L^1}$ for two couples of solutions. On Figure 16 we display this decrease for two couples of solutions.

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A Equivalence of weak formulations of the hysteresis operator

Proposition A.1. *Let $v \in W^{1,1}(0,T)$, $\lambda \in BV(0,T)$, $\xi \in \mathbb{R}$ then we have the following set of equivalences:*

(i) $\lambda \in k_P(v, \xi)$.

(ii) (λ, v) satisfy (6), (9) and (10).

(iii) (λ, v) satisfy (6), (9) and (11) for any nondecreasing $g \in C^1(\mathbb{R})$.

Proof. (i) \Rightarrow (ii) It is clear that (7) implies (9). Now using (8), we see that μ^+ must be supported on the set $\{t \in [0, T] \mid v(t) = P_1\}$ otherwise there would exist $t^* \in [0, T]$ such that $v(t^*) \neq P_1$ and for any $\epsilon > 0$ two points $t_1^\epsilon < t_2^\epsilon \in [t^* - \epsilon, t^* + \epsilon]$ such that $\lambda(t_1^\epsilon) < \lambda(t_2^\epsilon)$ thereby contradicting the fact that λ is nonincreasing in a neighbourhood of t^* . Hence,

$$\int_0^t v \, d\mu^+ = \int_0^t P_1 \, d\mu^+$$

Similarly,

$$\int_0^t v \, d\mu^- = \int_0^t P_0 \, d\mu^-$$

The result then follows immediately after summing the two identities.

(ii) \Rightarrow (iii) Nothing has to be done for (6) and (9). Assume that λ and v satisfy (10) holds, we will show that (11) holds.

Claim 1 For any $\eta : [0, T] \mapsto \mathbb{R}$ bounded and measurable, there holds,

$$\int_0^T v \eta \, d\mu = \int_0^T P_1 \eta \, d\mu^+ - \int_0^T P_0 \eta \, d\mu^- \quad (41)$$

The claim obviously holds for $\eta = \mathbf{1}_{[s,t]}$, $s, t \in [0, T]$. Since $\{[s, t], s, t \in [0, T]\}$ is a Π -system generating $\mathcal{B}([0, T])$ we obtain that (41) holds for $\eta = \mathbf{1}_A$, with $A \in \mathcal{B}([0, T])$. It then also holds for simple functions and by a standard density argument the claim follows.

Claim 2 μ^+ (resp. μ^-) is concentrated on $\{t \in [0, T] : v(t, x) = P_1\}$ (resp. P_0).

We only prove the part of the claim concerning μ^+ , because symmetrical arguments can be applied to μ^- to infer the wanted result. By the Jordan decomposition $\mu = \mu^+ - \mu^-$, the positive measures μ^+ and μ^- have disjoint support up to $|\mu|$ -null sets of $[0, T]$. Denote by $S^+ \subset [0, T]$, a measurable set which satisfies $\mu^-(S^+) = 0$ and on which μ^+ is concentrated.

Now for any $r > 0$, denote $A_r := \{t \in [0, T] : v(t) \geq P_1 + r\}$. Assume by contradiction that there exists $r_0 > 0$ such that $\mu^+(A_{r_0}) > 0$. Then we have $\mu^-(A_{r_0}) = 0$ and we find from **Claim 1** with $\eta = \mathbf{1}_{A_{r_0} \cap S^+}$,

$$0 = \int_{[0, T]} (v - P_1) \mathbf{1}_{A_{r_0} \cap S^+} \, d\mu^+$$

$$\begin{aligned}
&\geq r_0 \mu^+(A_{r_0} \cap S^+) \\
&= r_0 \mu^+(A_{r_0}) \\
&> 0
\end{aligned}$$

By contradiction, we showed that $\mu^+(A_r) = 0$ for all $r > 0$. By monotonicity, this implies that $\mu^+(\{t \in [0, T] : v(t) > P_1\}) = 0$.

In exactly the same manner, if $B_r := \{t \in [0, T] : v(t) \leq P_1 - r\}$, we also show by contradiction that $\mu_+(B_r) = 0$ for all $r > 0$, thereby showing our initial claim upon combining the monotonicity of μ^+ with the identity:

$$\{t \in [0, T] : v(t) \neq P_1\} = \bigcup_{m \geq 1} \left(A_{\frac{1}{m}} \cup B_{\frac{1}{m}} \right)$$

From **Claim 2**, we can infer that for any real continuous function $h \in C(\mathbb{R})$ and any bounded measurable function η , we have,

$$\begin{aligned}
\int_{[0, T]} h(v) \eta \, d\mu &= \int_{[0, T]} h(v) \eta \, d\mu^+ - \int_{[0, T]} h(v) \eta \, d\mu^- \\
&= \int_{[0, T]} h(v) \eta \mathbf{1}_{\{v=P_1\}} \, d\mu^+ - \int_{[0, T]} h(v) \eta \mathbf{1}_{\{v=P_0\}} \, d\mu^- \\
&= \int_{[0, T]} h(P_1) \eta \, d\mu^+ - \int_{[0, T]} h(P_0) \eta \, d\mu^- \tag{42}
\end{aligned}$$

To show the entropy inequalities, we first need some C^1 regularity for κ_i , then the general case will follow by a density argument. Hence we consider for now that κ_i are C^1 functions. Take any $\eta \in C_c^\infty((0, T))$, $\eta \geq 0$ and $g \in C^1(\mathbb{R})$ such that $g(a) = 0$ for some $a \in (P_0, P_1)$ and write

$$\begin{aligned}
-\int_0^T \overline{G}(v) \partial_t \eta \, dt &= -\int_0^T \lambda \underbrace{(G(\kappa_1(v)) - G(\kappa_0(v)))}_{=h_1(v)} \partial_t \eta \, dt - \int_0^T G(\kappa_0(v)) \partial_t \eta \, dt \\
&= \int_0^T h_1(v) \eta \, d\mu + \int_0^T \lambda \partial_t h_1(v) \eta \, dt - G(\kappa_0(v)) \partial_t \eta \, dt \\
&= \int_0^T h_1(P_1) \eta \, d\mu^+ - \int_0^T h_1(P_0) \eta \, d\mu^- \\
&\quad + \int_0^T \lambda h_{1,t}(v) \eta \, dt + \int_0^T \partial_t G(\kappa_0(v)) \eta \, dt
\end{aligned}$$

where we used (42) with $h = h_1$ and η . Let us denote $\delta(v) := \kappa_1(v) - \kappa_0(v)$ and on the other hand consider that

$$\begin{aligned}
-\int_0^T u \partial_t (g(v) \eta) \, dt &= -\int_0^T \lambda \delta(v) \partial_t (g(v) \eta) \, dt - \int_0^T \kappa_0(v) \partial_t (g(v) \eta) \, dt \\
&= \int_0^T \underbrace{\delta(v) g(v)}_{=h_2(v)} \eta \, d\mu
\end{aligned}$$

$$\begin{aligned}
& + \int_0^T \lambda \partial_t \delta(v) g(v) \eta \, dt - \int_0^T \kappa_0(v) \partial_t (g(v) \eta) \, dt \\
& = \int_0^T h_2(P_1) \eta \, d\mu^+ - \int_0^T h_2(P_0) \eta \, d\mu^- \\
& \quad + \int_0^T \lambda \partial_t \delta(v) g(v) \eta \, dt + \int_0^T \partial_t \kappa_0(v) g(v) \eta \, dt \\
& = \int_0^T h_2(P_1) \eta \, d\mu^+ - \int_0^T h_2(P_0) \eta \, d\mu^- \\
& \quad + \int_0^T \lambda h_1(v) \eta \, dt + \int_0^T \partial_t G(\kappa_0(v)) \eta \, dt
\end{aligned}$$

We used (28) and the fact that $\partial_t G(\kappa_i(v)) = \partial_t \kappa_i(v) g(v)$ a.e. $i = 0, 1$ since $v \in W^{1,1}(0, T)$. Therefore there only remains to show the two inequalities $h_2(P_1) - h_1(P_1) \geq 0$ and $h_2(P_0) - h_1(P_0) \leq 0$. This is done following those lines,

$$\begin{aligned}
h_2(P_1) - h_1(P_1) & = \delta(P_1) g(P_1) - (G(\kappa_1(P_1)) - G(\kappa_0(P_1))) \\
& = \delta(P_1) g(P_1) - \int_a^{P_1} \delta'(w) g(w) \, dw \\
& = \delta(P_1) g(P_1) - [\delta(w) g(w)]_a^{P_1} + \int_a^{P_1} g'(w) \delta(w) \, dw \\
& = g(a) \delta(a) + \int_a^{P_1} g'(w) \delta(w) \, dw \\
& = \int_a^{P_1} g'(w) \delta(w) \, dw \\
& \geq 0
\end{aligned}$$

And in a similar manner for P_0 we obtain,

$$\begin{aligned}
h_2(P_0) - h_1(P_0) & = \int_a^{P_0} g'(w) \delta(w) \, dw \\
& \leq 0
\end{aligned}$$

since $a \in (P_0, P_1)$. Now for general g , reduce to the case $g(a) = 0$ by considering $\tilde{g} := g - g(a)$, since the quantities $\overline{G}(v)$ and $g(v) \partial_t u$ are linear with respect to g and the entropy inequalities are trivially verified for constant functions g . Therefore we obtain, for $\kappa_i \in C^1(\mathbb{R})$,

$$- \int_0^T \overline{G}(v) \partial_t \eta \, dt \leq - \int_0^T u \partial_t (g(v) \eta) \, dt$$

Finally, choose sequences of C^1 approximations $(\kappa_i^n)_{n \in \mathbb{N}}$ such that $\kappa_i^n \rightarrow \kappa_i$ locally uniformly on \mathbb{R} , and observe that $G(\kappa_i^n(v)) \rightarrow G_i(v)$ and $\beta_n(v, \lambda) := (1 - \lambda) \kappa_i^n(v) + \lambda \kappa_i^n(v) \rightarrow \beta(v, \lambda)$ uniformly on $[0, T]$ as $n \rightarrow +\infty$ to find the wanted result for continuous κ_i .

(iii) \Rightarrow (i) If we assume that κ_i are regular, we can show that for any $\eta \geq 0$, $\eta \in C_c^\infty((0, T))$,

$$\int_0^T h_g(v) \eta \, d\mu \leq 0 \quad (43)$$

where $h_g : v \mapsto G_1(v) - G_0(v) - \delta(v)g(v)$ and $\mu = \partial_t \lambda$. Again by a density argument we can show that (43) also holds for continuous κ_i . Now note that:

$$h_g(v) = -\delta(a)g(a) - \int_a^v \delta(w)g'(w) \, dw$$

for any $a \in \mathbb{R}$. For a fixed $r > 0$, let us consider g such that $\text{supp } g' = (-\infty, P_1 - r]$ with $g(P_1) = 0$. Now take $a = P_1$ and observe that $h_g(v) > 0$ for all $v \leq P_1 - r$. Then taking a sequence $(\eta_n)_{n \in \mathbb{N}}$ such that η_n increases towards $\frac{\mathbb{1}_{v \leq P_1 - r}}{h_g(v)}$ a.e. on $[0, T]$, we infer by passing to the limit in (43) that $\mu(\{v \leq P_1 - r\}) \leq 0$. Since this holds for any $r > 0$ we infer that $\mu(\{v < P_1\}) \leq 0$.

Symmetrically, if we consider instead for $r > 0$, g such that $\text{supp } g' = [P_0 + r, +\infty)$, with $g(P_0) = 0$, we are lead to the conclusion: $\mu(\{v > P_0\}) \geq 0$. From this we deduce that μ must be supported in $\{v \leq P_0 \text{ or } v \geq P_1\}$. Now due to the confinement condition (9), we know that $\mu(\{v < P_0\}) = \mu(\{v > P_1\}) = 0$. Hence the Jordan decomposition of μ yields that $\text{supp } \mu^+ \subset \{v = P_1\}$ and $\text{supp } \mu^- \subset \{v = P_0\}$ which precisely expresses condition (8). \square

References

- [1] ANGELINI, O., BRENNER, K., AND HILHORST, D. A finite volume method on general meshes for a degenerate parabolic convection–reaction–diffusion equation. *Numerische Mathematik* 123, 2 (2013), 219–257.
- [2] CANCÈS, C. *Two-phase flows in heterogeneous porous media: modeling and analysis of the flows of the effects involved by the discontinuities of the capillary pressure*. PhD thesis, Université de Provence - Aix-Marseille I, 2008.
- [3] DEIMLING, K. *Nonlinear functional analysis*. Springer Verlag, 1985.
- [4] ELLIOTT, C. M. The stefan problem with a non-monotone constitutive relation. *IMA Journal of Applied Mathematics* 35, 2 (1985), 257–264.
- [5] EVANS, L., AND PORTILHEIRO, M. Irreversibility and hysteresis for a forward-backward diffusion equation. *Mathematical Models and Methods in Applied Sciences* 14 (11 2004).
- [6] EYMARD, R., GALLOUËT, T., AND HERBIN, R. Finite volume methods. In *Solution of Equation in \mathbb{R}^n (Part 3), Techniques of Scientific Computing (Part 3)*, vol. 7 of *Handbook of Numerical Analysis*. Elsevier, 2000, pp. 713 – 1018.
- [7] EYMARD, R., GALLOUËT, T., HERBIN, R., AND MICHEL, A. Convergence of a finite volume scheme for nonlinear degenerate parabolic equations. *Numerische Mathematik* 92 (07 2002), 41 – 82.

- [8] GALLOUËT, T. Discrete functional analysis tools for some evolution equations. *Computational Methods in Applied Mathematics* (2018).
- [9] GUIDOTTI, P. A backward–forward regularization of the perona–malik equation. *Journal of Differential Equations* 252, 4 (2012), 3226–3244.
- [10] HELMERS, M., AND HERRMANN, M. Hysteresis and phase transitions in a lattice regularization of an ill-posed forward–backward diffusion equation. *Archive for Rational Mechanics and Analysis* 230, 1 (2018), 231–275.
- [11] JÄCKLE, J., AND FRISCH, H. Relaxation of chemical potential and a generalized diffusion equation. *Journal of Polymer Science* 23 (1985), 675–682.
- [12] KICHENASSAMY, S. The perona–malik paradox. *SIAM Journal on Applied Mathematics* 57, 5 (1997), 1328–1342.
- [13] LAFITTE, P., AND MASCIA, C. Numerical exploration of a forward-backward diffusion equation. *Mathematical Models and Methods in Applied Sciences* 22 (08 2010).
- [14] LEVEQUE, R. J. *Finite volume methods for hyperbolic problems*, vol. 31. Cambridge university press, 2002.
- [15] LEVEQUE, R. J., AND LEVEQUE, R. J. *Numerical methods for conservation laws*, vol. 214. Springer, 1992.
- [16] MASCIA, C., TERRACINA, A., AND TESEI, A. Two-phase entropy solutions of a forward–backward parabolic equation. *Arch. Rational Mech. Anal.* 194 (2009), 887–925.
- [17] NOVICK-COHEN, A., AND PEGO, R. L. Stable patterns in a viscous diffusion equation. *Trans. AMS* 324 (1991), 331–351.
- [18] PADRÒN, V. Sobolev regularization of a nonlinear ill-posed parabolic problem as a model for aggregating populations. *Communications in Partial Differential Equations* (1998), 457–486.
- [19] PADRÒN, V. Effect of aggregation on population recovery modeled by a forward-backward pseudoparabolic equation. *Transactions of the American Mathematical Society* 356, 7 (2004), 2739–2756.
- [20] PLOTNIKOV, P. Passing to the limit with respect to viscosity in an equation with variable parabolicity direction. *Differential Equations* 30, 4 (1994), 614–622.
- [21] PLOTNIKOV, P. Forward-backward parabolic equations and hysteresis. *Journal of Mathematical Sciences* 93 (02 1999).
- [22] RODRIGUES, J. F. Mathematical models for phase change problems. In *Proc. European Workshop held at Obidos, Portugal* (1988), vol. 88, Springer.
- [23] TERRACINA, A. Non-uniqueness results for entropy two-phase solutions of forward-backward parabolic problems with unstable phase. *Journal of Mathematical Analysis and Applications* 413, 2 (2013), 963–975.

- [24] VISINTIN, A. Forward-backward parabolic equations and hysteresis. *Calculus of Variations and Partial Differential Equations* 15, 1 (2002), 115–132.
- [25] VISINTIN, A. Quasilinear parabolic pdes with discontinuous hysteresis. *Annali di Matematica Pura ed Applicata (1923-)* 4, 185 (2006), 487–519.
- [26] VISINTIN, A. *Differential models of hysteresis*, vol. 111. Springer Science & Business Media, 2013.

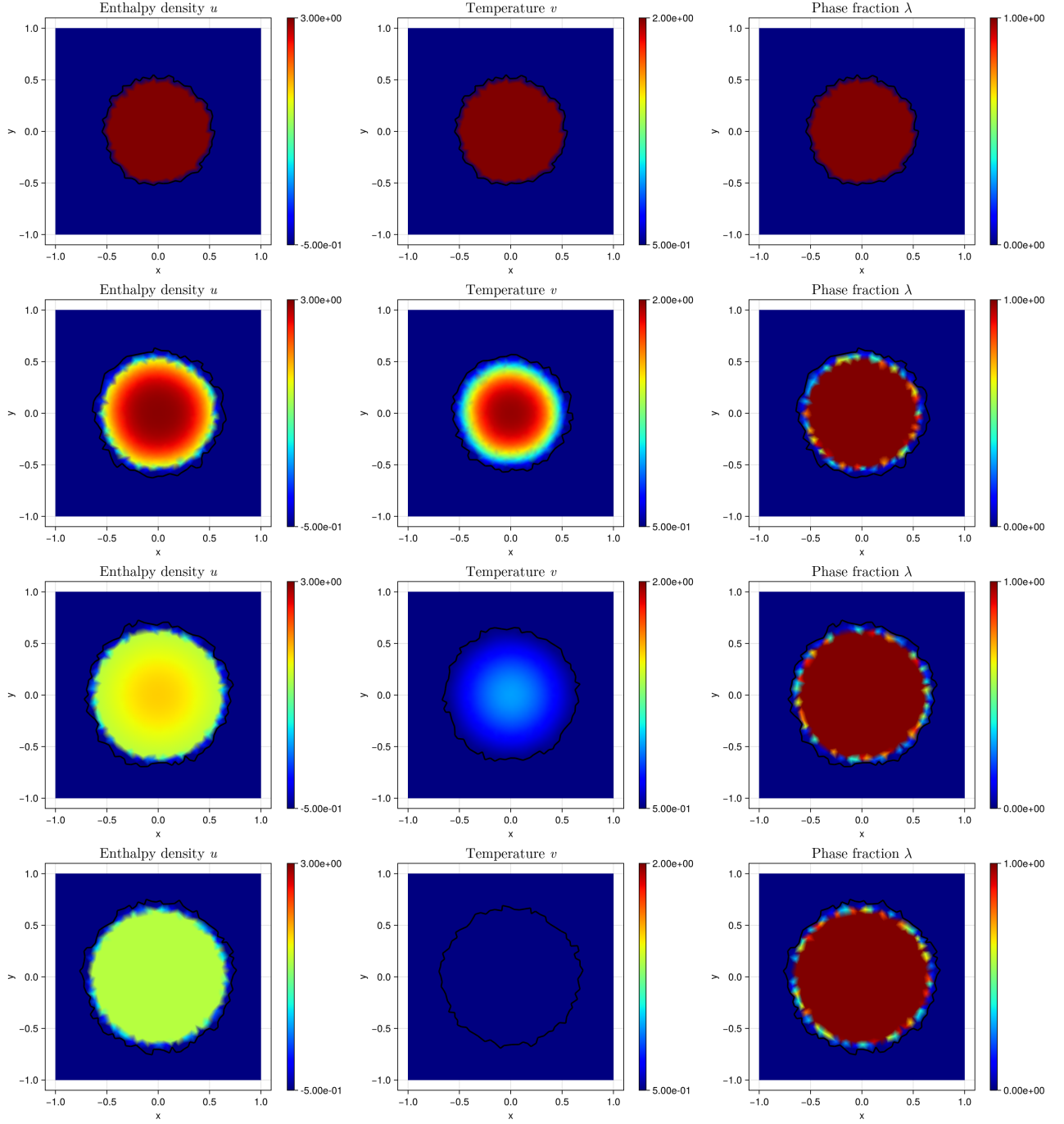


Figure 12: Expanding disk with a melting front. $\Delta t = 0.004$, $\Delta x = 0.002$. From top to bottom, solution at $t = 0$, $t = 0.01$, $t = 0.1$ and $t = 1$.

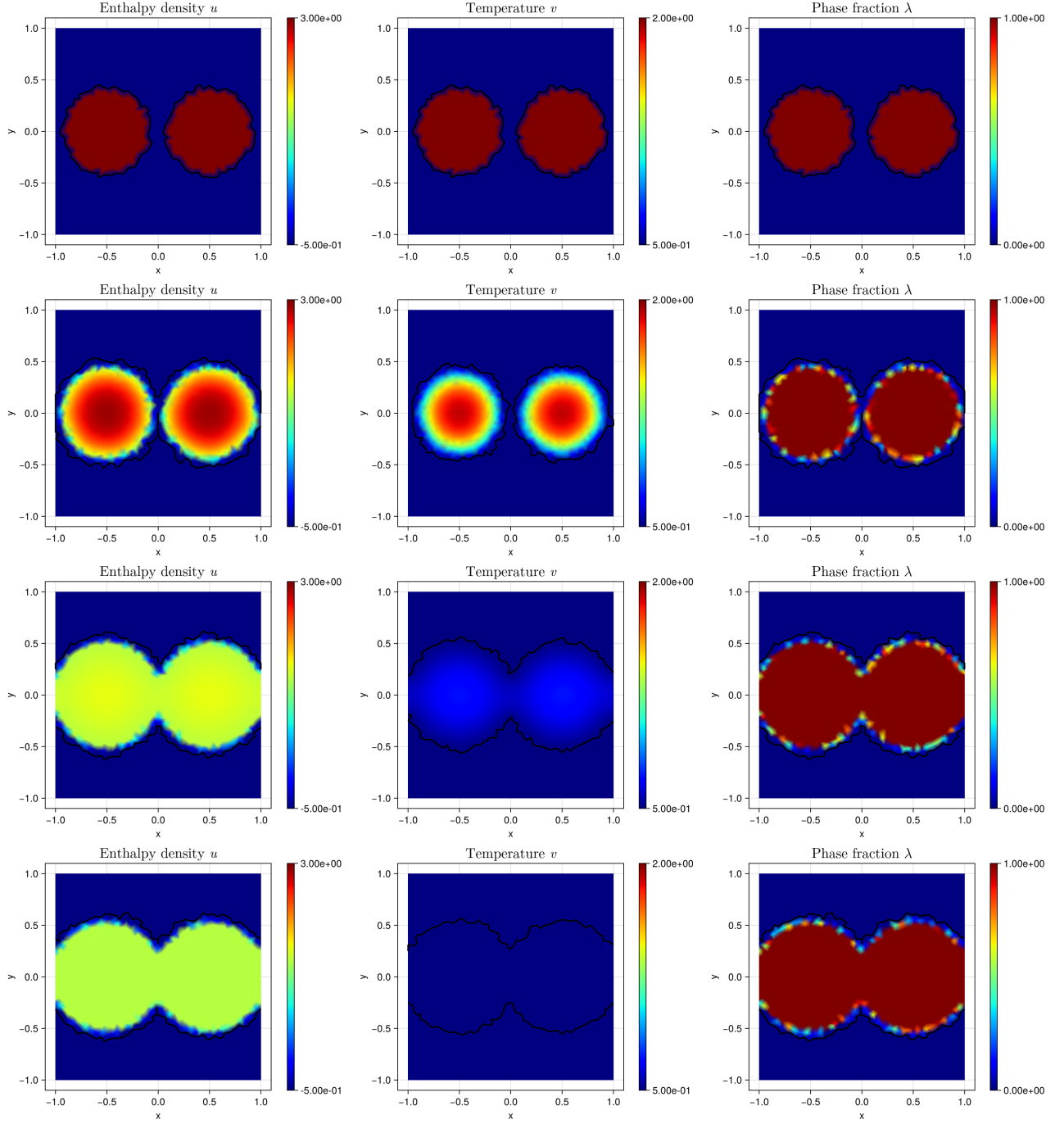


Figure 13: Two disks expanding and merging. $\Delta t = 0.004$, $\Delta x = 0.002$. From top to bottom, solution at $t = 0$, $t = 0.01$, $t = 0.1$ and $t = 1$.

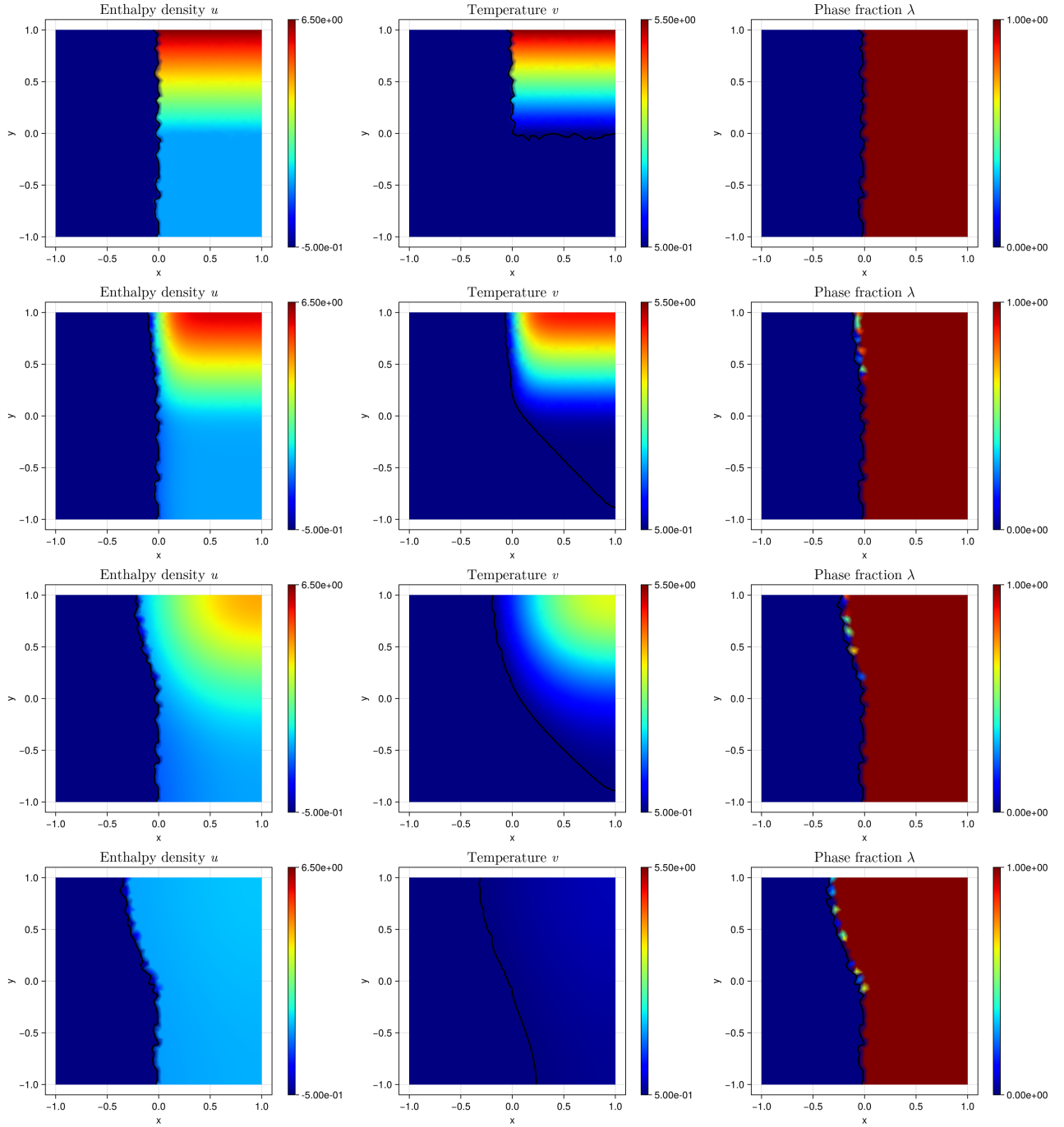


Figure 14: Mixed behaviour on the boundary with $\Delta t = 0.004$, $\Delta x = 0.002$. From top to bottom, numerical solution at $t = 0$, $t = 0.01$, $t = 0.1$ and $t = 1$.

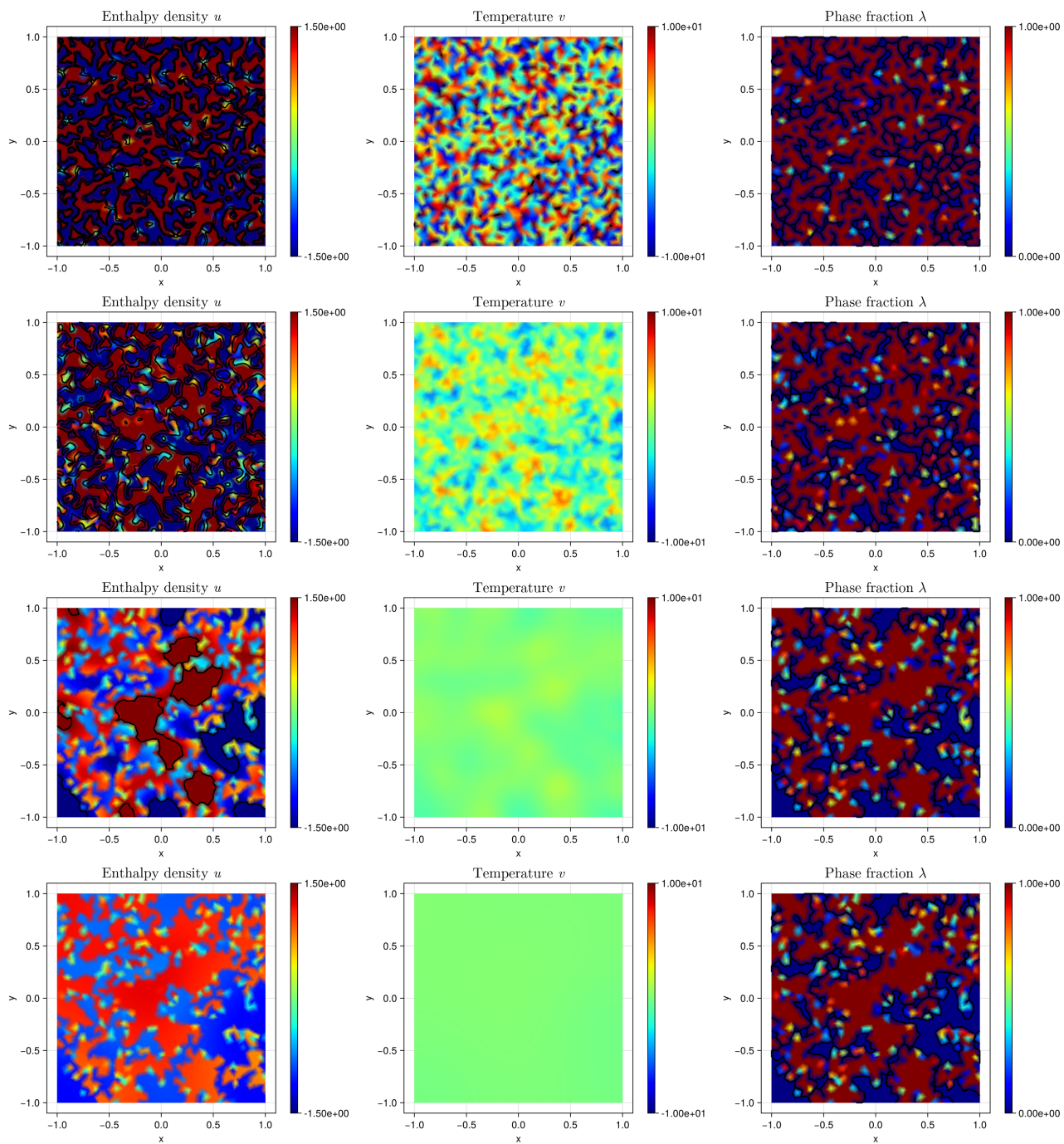


Figure 15: Random initial condition with $\Delta t = 0.0002$, $\Delta x = 0.002$. From top to bottom, numerical solution at $t = 0$, $t = 0.001$, $t = 0.01$ and $t = 0.1$.

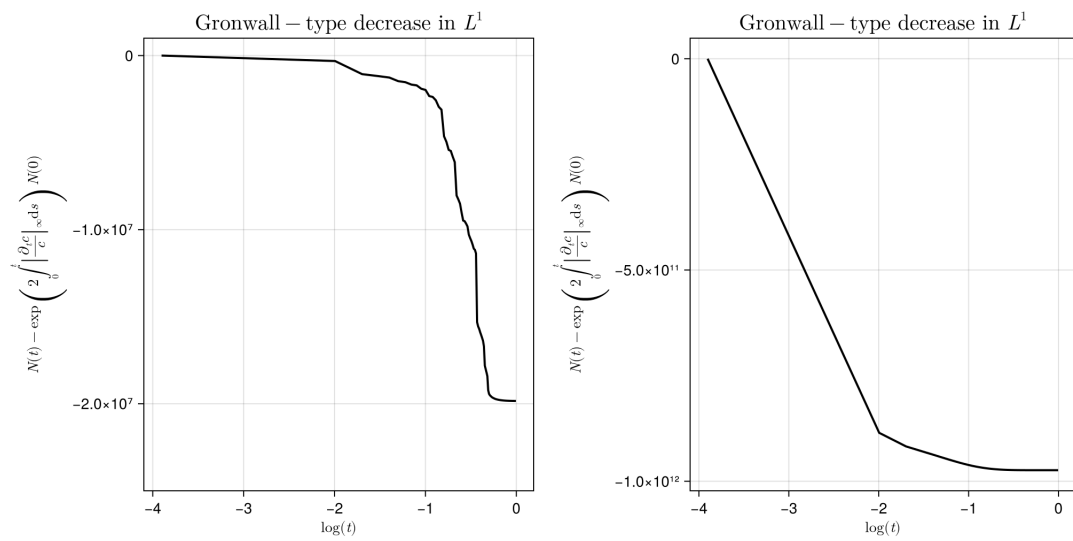


Figure 16: Gronwall-type decrease of $N(t)$. Left: Riemann initial conditions differing by a translation of size 0.1. Right: Randomly generated initial conditions.