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ON A BRANCHED TRANSPORT MODEL FOR TYPE-I SUPERCONDUCTORS

MICHAEL GOLDMAN

ABSTRACT. This is an extended abstract of the talk I gave in Cortona for the conference Geometric Measure Theory and applications, 2024.

A few years ago we derived in [3] a branched transport model from the Ginzburg-Landau functional for type I superconductors in the regime of very small applied external magnetic fields. Let us first describe the limit model. We let $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ be the two-dimensional flat torus and consider for $T > 0$ the domain $Q_T = \mathbb{T}^2 \times (-T, T)$. The unknown is a measure $\mu = \mu_t \otimes dt \in \mathcal{M}(Q_T)$ such that for a.e. $t \in (-T, T)$ we have

$$\mu_t = \sum_i \varphi_i \delta_{X_i}$$

where $\varphi_i \geq 0$ are such that $\sum_i \varphi_i = 1$ and $X_i \in \mathbb{T}^2$. We then define the internal energy

$$I(\mu) = \int_{-T}^T \sum_i \sqrt{\varphi_i} + \varphi_i |\dot{X}_i|^2 dt. \quad (0.1)$$

Here \dot{X}_i denotes the derivative with respect to t of the curve $t \mapsto X_i(t)$. For $\lambda > 0$ we then consider the energy

$$E_\lambda(\mu) = I(\mu) + \lambda \|\mu_{\pm T} - 1\|_{H^{-1/2}}^2. \quad (0.2)$$

Here we used the short-hand notation

$$\|\mu_{\pm T} - 1\|_{H^{-1/2}}^2 = \|\mu_{-T} - 1\|_{H^{-1/2}}^2 + \|\mu_T - 1\|_{H^{-1/2}}^2.$$

Remark 0.1. *Let us point out that in [3], the case $\lambda = \infty$ i.e. imposing the constraint $\mu_{\pm T} = 1$ was obtained. This corresponds to the so-called uniform branching regime, see below. The derivation of the model in the case $\lambda \in (0, \infty)$ is work in progress.*

Let us notice that the first term in the definition (0.1) of I corresponds to isoperimetric effects. Since it penalizes a concave power of φ_i , this term forces concentration. Using the Benamou-Brenier formula, the second term in (0.1) can be seen to be a transport term, see [3]. This makes the model a cousin of the more common branched transport energies studied for instance in [1]. A boiled-down version of the scaling laws obtained in [2, 4] yields that

$$\min_\mu E_\lambda(\mu) \simeq \min(T^{\frac{1}{3}}, \lambda^{\frac{2}{7}} T^{\frac{3}{7}}). \quad (0.3)$$

The first regime corresponds to the so-called uniform branching regime where we expect that $\mu_{\pm T} \simeq 1$ while the second regime corresponds to the non-uniform branching regime where we expect the boundary measure to be quite far from the Lebesgue measure.

In the non-uniformly branching regime, based on the construction leading to the upper bound estimate in (0.3), Conti Otto and Serfaty conjectured that

$$\dim \mu_{\pm T} = 8/5.$$

For $\varepsilon > 0$, let

$$I(\mu, \varepsilon) = \int_{T-\varepsilon}^T \sum_i \sqrt{\varphi_i} + \varphi_i |\dot{X}_i|^2 dt.$$

That is $I(\mu, \varepsilon)$ measures how much energy there is in the slab of height ε close to $\{t = T\}$. In [6] we proved that dimensional bounds on μ_T are equivalent to decay rates for $I(\mu, \varepsilon)$.

Theorem 0.2. *Let μ be a minimizer of E_λ . If there exists $\beta > 0$ such that $\limsup_{\varepsilon \rightarrow 0} I(\mu, \varepsilon)/\varepsilon^\beta < \infty$ then*

$$f(\beta) = \frac{1 + 3\beta}{1 + \beta} \leq \dim \mu_T \leq g(\beta) = \frac{4(1 - \beta)}{1 + \beta}. \quad (0.4)$$

Notice that we have

$$f(\beta) = g(\beta) = \frac{8}{5} \quad \text{for} \quad \beta = \frac{3}{7}.$$

In particular this means that if we could show that $I(\mu, \varepsilon) \lesssim \varepsilon^{3/7}$ then the conjecture of Conti-Otto-Serfaty would be proved. Let us point out that, recalling (0.3), this is in line with the fact that for a self-similar minimizer, the global and local scaling laws should coincide. The proof of (0.4) is made of two parts. The estimate $\dim \mu_T \geq f(\beta)$ may be seen as a regularity statement. Indeed, we prove through a variation argument that if $I(\mu, \varepsilon)$ is small then $\mu_T \in H^{-\gamma}(\mathbb{T}^2)$ for some $\gamma < 1/2$. Moreover, the smaller $I(\mu, \varepsilon)$ is (that is the larger β), the smaller is γ . The proof of $\dim \mu_T \leq g(\beta)$ is instead a refinement of the interpolation argument leading to the proof of the lower bound in (0.3).

When considering a one-dimensional variant of the problem, that is replacing $\mathbb{T}^2 \times (-T, T)$ by $\mathbb{T}^1 \times (-T, T)$, where $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ more information can be obtained on the minimizers. In this case it is also natural to replace the perimeter term $\sum_i \sqrt{\varphi_i}$ by $\#\{\varphi_i \neq 0\}$. The main advantage in this setting is that by the no-loop property, the transport plan induced by the branched transport problem is monotone and thus coincides with the classical optimal transport map. In particular, it is possible to exactly compute the irrigation tree connecting a Dirac mass to the Lebesgue measure, see [7]. This is one of the very rare examples where explicit minimizers for such non-convex variational problems can be found. Moreover, assuming that the optimal measure is Ahlfors regular (which is a strong hypothesis!) we obtained in [5] its precise dimension.

Theorem 0.3. *Let μ be a minimizer of E_λ in $\mathbb{T}^1 \times (-T, T)$ and assume that for some $\alpha \in [0, 1]$, $\mu_T(B_r(x)) \simeq r^\alpha$ for every $x \in \text{Spt } \mu_T$ and every $r \in (0, 1)$, then*

$$\alpha = \frac{2}{3}.$$

In order to prove that $\alpha \leq 2/3$, we combine the interpolation argument leading to $\dim \mu_T \leq g(\beta)$ (recall that this is similar to the lower bound in (0.3)) in (0.4) together with a first variation argument. In turn, the proof of $\alpha \geq 2/3$ combines a construction (as in the proof of the upper bound in (0.3)) together with another first variation argument.

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