



**HAL**  
open science

# On a branched transport model for type-I superconductors

Michael Goldman

► **To cite this version:**

| Michael Goldman. On a branched transport model for type-I superconductors. 2024. hal-04644995

**HAL Id: hal-04644995**

**<https://hal.science/hal-04644995v1>**

Preprint submitted on 11 Jul 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON A BRANCHED TRANSPORT MODEL FOR TYPE-I SUPERCONDUCTORS

MICHAEL GOLDMAN

ABSTRACT. This is an extended abstract of the talk I gave in Cortona for the conference Geometric Measure Theory and applications, 2024.

A few years ago we derived in [3] a branched transport model from the Ginzburg-Landau functional for type I superconductors in the regime of very small applied external magnetic fields. Let us first describe the limit model. We let  $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$  be the two-dimensional flat torus and consider for  $T > 0$  the domain  $Q_T = \mathbb{T}^2 \times (-T, T)$ . The unknown is a measure  $\mu = \mu_t \otimes dt \in \mathcal{M}(Q_T)$  such that for a.e.  $t \in (-T, T)$  we have

$$\mu_t = \sum_i \varphi_i \delta_{X_i}$$

where  $\varphi_i \geq 0$  are such that  $\sum_i \varphi_i = 1$  and  $X_i \in \mathbb{T}^2$ . We then define the internal energy

$$I(\mu) = \int_{-T}^T \sum_i \sqrt{\varphi_i} + \varphi_i |\dot{X}_i|^2 dt. \quad (0.1)$$

Here  $\dot{X}_i$  denotes the derivative with respect to  $t$  of the curve  $t \mapsto X_i(t)$ . For  $\lambda > 0$  we then consider the energy

$$E_\lambda(\mu) = I(\mu) + \lambda \|\mu_{\pm T} - 1\|_{H^{-1/2}}^2. \quad (0.2)$$

Here we used the short-hand notation

$$\|\mu_{\pm T} - 1\|_{H^{-1/2}}^2 = \|\mu_{-T} - 1\|_{H^{-1/2}}^2 + \|\mu_T - 1\|_{H^{-1/2}}^2.$$

**Remark 0.1.** *Let us point out that in [3], the case  $\lambda = \infty$  i.e. imposing the constraint  $\mu_{\pm T} = 1$  was obtained. This corresponds to the so-called uniform branching regime, see below. The derivation of the model in the case  $\lambda \in (0, \infty)$  is work in progress.*

Let us notice that the first term in the definition (0.1) of  $I$  corresponds to isoperimetric effects. Since it penalizes a concave power of  $\varphi_i$ , this term forces concentration. Using the Benamou-Brenier formula, the second term in (0.1) can be seen to be a transport term, see [3]. This makes the model a cousin of the more common branched transport energies studied for instance in [1]. A boiled-down version of the scaling laws obtained in [2, 4] yields that

$$\min_\mu E_\lambda(\mu) \simeq \min(T^{\frac{1}{3}}, \lambda^{\frac{2}{7}} T^{\frac{3}{7}}). \quad (0.3)$$

The first regime corresponds to the so-called uniform branching regime where we expect that  $\mu_{\pm T} \simeq 1$  while the second regime corresponds to the non-uniform branching regime where we expect the boundary measure to be quite far from the Lebesgue measure.

In the non-uniformly branching regime, based on the construction leading to the upper bound estimate in (0.3), Conti Otto and Serfaty conjectured that

$$\dim \mu_{\pm T} = 8/5.$$

For  $\varepsilon > 0$ , let

$$I(\mu, \varepsilon) = \int_{T-\varepsilon}^T \sum_i \sqrt{\varphi_i} + \varphi_i |\dot{X}_i|^2 dt.$$

That is  $I(\mu, \varepsilon)$  measures how much energy there is in the slab of height  $\varepsilon$  close to  $\{t = T\}$ . In [6] we proved that dimensional bounds on  $\mu_T$  are equivalent to decay rates for  $I(\mu, \varepsilon)$ .

**Theorem 0.2.** *Let  $\mu$  be a minimizer of  $E_\lambda$ . If there exists  $\beta > 0$  such that  $\limsup_{\varepsilon \rightarrow 0} I(\mu, \varepsilon)/\varepsilon^\beta < \infty$  then*

$$f(\beta) = \frac{1 + 3\beta}{1 + \beta} \leq \dim \mu_T \leq g(\beta) = \frac{4(1 - \beta)}{1 + \beta}. \quad (0.4)$$

Notice that we have

$$f(\beta) = g(\beta) = \frac{8}{5} \quad \text{for} \quad \beta = \frac{3}{7}.$$

In particular this means that if we could show that  $I(\mu, \varepsilon) \lesssim \varepsilon^{3/7}$  then the conjecture of Conti-Otto-Serfaty would be proved. Let us point out that, recalling (0.3), this is in line with the fact that for a self-similar minimizer, the global and local scaling laws should coincide. The proof of (0.4) is made of two parts. The estimate  $\dim \mu_T \geq f(\beta)$  may be seen as a regularity statement. Indeed, we prove through a variation argument that if  $I(\mu, \varepsilon)$  is small then  $\mu_T \in H^{-\gamma}(\mathbb{T}^2)$  for some  $\gamma < 1/2$ . Moreover, the smaller  $I(\mu, \varepsilon)$  is (that is the larger  $\beta$ ), the smaller is  $\gamma$ . The proof of  $\dim \mu_T \leq g(\beta)$  is instead a refinement of the interpolation argument leading to the proof of the lower bound in (0.3).

When considering a one-dimensional variant of the problem, that is replacing  $\mathbb{T}^2 \times (-T, T)$  by  $\mathbb{T}^1 \times (-T, T)$ , where  $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$  more information can be obtained on the minimizers. In this case it is also natural to replace the perimeter term  $\sum_i \sqrt{\varphi_i}$  by  $\#\{\varphi_i \neq 0\}$ . The main advantage in this setting is that by the no-loop property, the transport plan induced by the branched transport problem is monotone and thus coincides with the classical optimal transport map. In particular, it is possible to exactly compute the irrigation tree connecting a Dirac mass to the Lebesgue measure, see [7]. This is one of the very rare examples where explicit minimizers for such non-convex variational problems can be found. Moreover, assuming that the optimal measure is Ahlfors regular (which is a strong hypothesis!) we obtained in [5] its precise dimension.

**Theorem 0.3.** *Let  $\mu$  be a minimizer of  $E_\lambda$  in  $\mathbb{T}^1 \times (-T, T)$  and assume that for some  $\alpha \in [0, 1]$ ,  $\mu_T(B_r(x)) \simeq r^\alpha$  for every  $x \in \text{Spt } \mu_T$  and every  $r \in (0, 1)$ , then*

$$\alpha = \frac{2}{3}.$$

In order to prove that  $\alpha \leq 2/3$ , we combine the interpolation argument leading to  $\dim \mu_T \leq g(\beta)$  (recall that this is similar to the lower bound in (0.3)) in (0.4) together with a first variation argument. In turn, the proof of  $\alpha \geq 2/3$  combines a construction (as in the proof of the upper bound in (0.3)) together with another first variation argument.

## REFERENCES

- [1] BERNOT, M., CASELLES, V., AND MOREL, J.-M. *Optimal transportation networks*, vol. 1955 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. Models and theory.

- [2] CHOKSI, R., CONTI, S., KOHN, R. V., AND OTTO, F. Ground state energy scaling laws during the onset and destruction of the intermediate state in a type-I superconductor. *Comm. Pure Appl. Math.* 61 (2008), 595–626.
- [3] CONTI, S., GOLDMAN, M., OTTO, F., AND SERFATY, S. A branched transport limit of the Ginzburg-Landau functional. *Journal de l'École polytechnique–Mathématiques* 5 (2018), 317–375.
- [4] CONTI, S., OTTO, F., AND SERFATY, S. Branched microstructures in the Ginzburg-Landau model of type-I superconductors. *SIAM J. Math. Anal.* 48, 4 (2016), 2994–3034.
- [5] COSENZA, A., GOLDMAN, M., AND KOSER, M. News dimensional bounds for a branched transport problem.
- [6] DE PHILIPPIS, G., GOLDMAN, M., AND RUFFINI, B. From energy bounds to dimensional estimates in a branched transport model for type-I superconductors. *arXiv preprint arXiv:2304.12715* (2023).
- [7] GOLDMAN, M. Self-similar minimizers of a branched transport functional. *Indiana Univ. Math. J.* 69 (2020), 1073–1104.

CMAP, CNRS, ÉCOLE POLYTECHNIQUE, INSTITUT POLYTECHNIQUE DE PARIS, 91120 PALAISEAU, FRANCE  
Email address: michael.goldman@cnrs.fr